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GATE

(Graduate Aptitude Test in Engineering)

Electrical Engineering

TRISHNA KNOWLEDGE SYSTEMS

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Preface

Graduate Aptitude Test in Engineering (GATE) is one of the preliminary tests for undergraduate subjects in Engineering/ Technology/Architecture and postgraduate subjects in Science stream only.

The number of aspirants appearing for the GATE examination is increasing significantly every year, owing to multifaceted opportunities open to any good performer. Apart from giving the aspirant a chance to pursue an M.Tech. from institutions like the IITs /NITs, a good GATE score can be highly instrumental in landing the candidate a plush public sector job, as many PSUs are recruiting graduate engineers on the basis of their performance in GATE. The GATE examination pattern has undergone several changes over the years—sometimes apparent and sometimes subtle. It is bound to continue to do so with changing technological environment.

GATE Electrical Engineering, as a complete resource helps the aspirants be ready with conceptual understanding, and enables them to apply these concepts in various applications, rather than just proficiency with questions type. Topics are handled in a comprehensive manner, beginning with the basics and progressing in a step-by-step manner along with a bottom-up approach. This allows the student to better understand the concept and to practice applicative techniques in a focused manner. The content has been systematically organized to facilitate easy understanding of all topics. The given examples will not only help the students to understand the concepts involved in the problems but also help to get a good idea about the different models of problems on that particular topic. Due care has also been taken to cover a very wide range of problems including questions that have been appearing over the last few years in GATE examination.

The practice exercises in every chapter, contain questions ranging simple to moderate to difficult level. These exercises are meant to hone the examination readiness over a period of time. At the end of each unit, practice tests have been placed. These tests will help the student assess their level of learning on a regular interval.

This book has been prepared by a group of faculty who are highly experienced in training GATE preparations and are also subject matter experts. As a result, this book would serve as an effective tool for GATE aspirant to crack the examination.

Salient Features

1. Elaborate question bank covering previous 12 years' GATE question papers
2. 5 free online mock tests for practice
3. Detailed coverage of key topics
4. Complete set of solved 2017 GATE online papers with chapter-wise analysis
5. Exhaustive pedagogy:
   (a) More than 1300 Solved Examples
   (b) More than 6000 Practice Questions
   (c) Unit-wise time-bound tests
   (d) Modular approach for easy understanding

We would like to thank the below mentioned reviewers for their valuable feedback and suggestions which has helped in shaping this book.

R. Marudhachalam	Professor (Sr. Grade), Kumaraguru College of Technology Coimbatore, Tamil Nadu
Daya Gupta	Professor, Delhi Technological University, Main Bawana Road, Delhi
Manoj Kumar Gupta	Associate Professor, Delhi Technological University Main Bawana Road, Delhi
Gurpreet Kour	Lecturer, Lovely Professional University Phagwara, Punjab
Pinaki Chakraborty	Assistant Professor, Netaji Subhas Institute of Technology Dwarka, Delhi
Gunit Kaur	Lecturer, Lovely Professional University Phagwara, Punjab

Despite of our best efforts, some errors may have inadvertently crept into the book. Constructive comments and suggestions to further improve the book are welcome and shall be acknowledged gratefully.

Wishing you all the very best..!!!

—Trishna Knowledge Systems

Key Pedagogical Features

Learning Objectives

List of important topics which are covered in chapter.

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Sampling
- Sampling theorem
- The *z*-transform
- Region of convergence
- *z*-transform properties
- Inverse *z*-transform
- Analysis of discrete-time LTI systems
- Causal and stable systems
- The unilateral *z*-transform
- Relationship between *z* and Laplace transform
- Direct form-I structure
- Direct form-II structure of IIR system

Solved Example

Solved problems given topic-wise to learn to apply the concepts learned in a particular section as per exam pattern.

Solved Examples

Example 1: Consider the following signals $x(t) = \sin\pi t + \cos 2\pi t$.

$$y(t) = \sin 2t + \sin 5t, z(t) = \sin t + \cos\pi t$$

Periodic signals are?

(A) $x(t)$ and $z(t)$ (B) $x(t)$ and $y(t)$
(C) $y(t)$ and $z(t)$ (D) Only $y(t)$

Solution: (B)

$x(t)$ is periodic with period $T = 2$ sec and $y(t)$ is periodic with period $T = 2\pi$ sec. For $z(t)$ fundamental time period can't be determined.

Example 2: Consider the following signals with $z > 0$.

(a) $e^{-zt}\,u(t)$ (b) $e^{zt}u(t)$ (c) $tu(t)$

The energy signals are?

(A) a and b (B) b only
(C) a only (D) all

Exercises

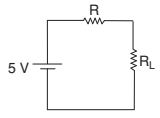
Practice problems for students to master the concepts studied in chapter. Exercises consist of two levels of problems "**Practice Problem I**" and "**Practice Problem II**" based on increasing difficulty level.

EXERCISE

Practice Problems I

Directions for questions 1 to 28: Select the correct alternative from the given choices.

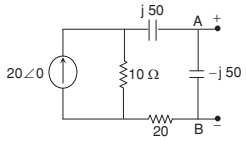
1. The maximum power transferred to the load in the circuit is given as 0.5 W. Get the values of R and R_L.



(A) 15 Ω, 10 Ω (B) 12.5 Ω, 12.5 Ω
(C) 10 Ω, 15 Ω (D) 10 kΩ, 10 kΩ

(A) 353.5 W (B) 291.5 W
(C) 250 W (D) 176.7 W

4. In the circuit shown below the Norton equivalent current in amps across A – B is

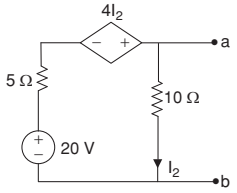


(A) 19.45 + j3.24 (B) 6.48 – j1.08
(C) 12.97 – j2.16 (D) 20 + j0

Practice Problems 2

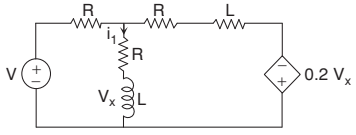
Directions for questions 1 to 16: Select the correct alternative from the given choices.

1. Find the Thevenin's equivalent of the Circuit given below:



(A) $V_{TH} = 20$ V, $R_{TH} = 3.3\ \Omega$
(B) $V_{TH} = 16$ V, $R_{TH} = 5\ \Omega$
(C) $V_{TH} = 20$ V, $R_{TH} = 5\ \Omega$
(D) $V_{TH} = 4$ V, $R_{TH} = 10\ \Omega$

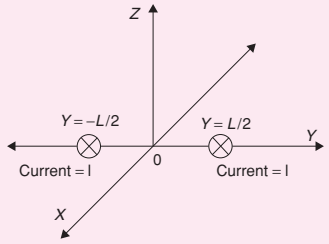
2. Find the state equation for the circuit given.



Previous Years' Questions

1. Two identical coupled inductors are connected in series. The measured inductances for the two possible series connections are 380 μH and 240 μH. Their mutual inductance in μH is _____. **[2014]**

2. A steady current I is flowing in the $-x$ direction through each of two infinitely long wires at $y = \pm\frac{L}{2}$ as shown in the figure. The permeability of the medium is μ_0. The $\vec{B}$-field at (0, L, 0) is **[2015]**



(A) $-\frac{4\mu_0 t}{3\pi L}\hat{Z}$ (B) $+\frac{4\mu_0 I}{3\pi L}\hat{Z}$

(C) 0 (D) $-\frac{3\mu_0 I}{4\pi L}\hat{Z}$

3. The self inductance of the primary winding of a single phase, 50 Hz, transformer is 800 mH, and that of the secondary winding is 600 mH. The mutual inductance between these two windings is 480 mH. The secondary winding of this transformer is short circuited and the primary winding is connected to a 50 Hz, single phase, sinusoidal voltage source. The current flowing in both the windings is less than their respective rated currents. The resistance of both windings can be neglected. In this condition, what is the effective inductance (in mH) seen by the source? **[2015]**

(A) 416 (B) 440

(C) 200 (D) 920

4. A symmetrical square wave of 50% duty cycle has amplitude of ±15 V and time period of 0.4π ms. This square wave is applied across a series RLC circuit with $R = 5\ \Omega$, $L = 10$ mH, and $C = 4$ μF. The amplitude of the 5000 rad/s component of the capacitor voltage (in Volt) is _____. **[2015]**

5. Two identical coils each having inductance L are placed together on the same core. If an overall inductance of αL is obtained by interconnecting these two coils, the minimum value of α is _____. **[2015]**

6. The flux linkage (λ) and current (i) relation for an electromagnetic system is $\lambda = \left(\sqrt{i}\right)$ /g. when i = 2A and g(air - gap length) = 10 cm, the magnitude of mechanical force on the moving part, in N, is _____. **[2016]**

GATE Previous Years' Questions

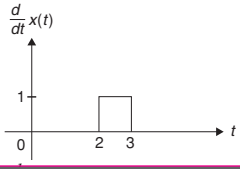
Contains previous 10 years GATE Questions at end of every chapter which help students to get an idea about the type of problems asked in GATE and prepare accordingly.

Hints/Solutions

Practice Problems I

1. $\omega_0 = \text{G.C.D. } \{\omega_1, \omega_2\}$.
$= \{1, \sqrt{2}\} = \text{Not exist}$
So it is not periodic
Hence, the correct option is (D).

2. $\frac{d}{dt}x(t)$



$$y(t) = \begin{cases} 1 \text{ for } -1 < t < 0 \\ -1 \text{ for } 0 < t < 1 \\ 0 \text{ Otherwise} \end{cases}$$

$$\text{Energy} = \int_{-1}^{0} (1)^2\,dt + \int_{0}^{1} (-1)^2\,dt$$
$$= 1 + 1 = 2$$

Hence, the correct option is (A).

7. $\text{Energy } E = \int_0^2 (x(t))^2\,dt$

$$= 2\left|\int_0^1 t^2\,dt + \int_1^2 (2-t)^2\,dt\right|$$

Hints/Solutions

This section gives complete solutions of all the unsolved questions given in the chapter.

Practice Tests

Time-bound Test provided at end of each unit for assessment of topics leaned in the unit.

Test
Signals and Systems

Time: 60 min.

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. The system represented by the input-output relationship $y(t) = \int_{-\infty}^{4t} x(\tau)d\tau, t > 0$ is

(A) Linear and causal
(B) Linear but not causal
(C) Causal but not linear
(D) Neither linear nor causal

2. Which of the following cannot be the Fourier series expansion of a periodic signal?
(A) $x(t) = 2\cos t + 3\cos 3t$
(B) $x(t) = 2\cos\pi t + 7\cos t$
(C) $x(t) = \cos t + 0.5$
(D) $x(t) = \cos t + \cos\sqrt{2}t$

3. The fundamental period of the signal $x(t) = \cos\left(\frac{2\pi}{T}t\right)$ is

(A) $\frac{T}{8}$ (B) $\frac{T}{4}$

(C) $\frac{T}{2}$ (D) $2T$

8. A signal $x(n) = \sin(\omega_0 n + \phi)$ is the input to a linear time-invariant system having a frequency response $H(e^{j\omega})$. If the output of the system is $Ax(n - n_0)$, then the most general form of $\angle H(e^{j\omega})$ will be
(A) $-n_0\omega_0 + \beta$ for any arbitrary real β
(B) $-n_0\omega_0 + 2\pi k$ for any arbitrary integer k
(C) $n_0\omega_0 + 2\pi k$ for any arbitrary integer k
(D) $-n_0\omega_0 + \phi$

9. A system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$, where $x[n]$ is causal is an example of

Syllabus: Electrical Engineering

Electric Circuits and Fields: Network graph, KCL, KVL, node and mesh analysis, transient response of dc and ac networks; sinusoidal steady-state analysis, resonance, basic filter concepts; ideal current and voltage sources, Thevenin's, Norton's and Superposition and Maximum Power Transfer theorems, two-port networks, three-phase circuits; Gauss theorem, electric field and potential due to point, line, plane and spherical charge distributions; Ampere's and Biot-Savart's laws; inductance; dielectrics; capacitance.

Signals and Systems: Representation of continuous and discrete-time signals; shifting and scaling operations; linear, time-invariant and causal systems; Fourier series representation of continuous periodic signals; sampling theorem; Fourier, Laplace and Z transforms.

Electrical Machines: Single-phase transformer—equivalent circuit, phasor diagram, tests, regulation and efficiency; three-phase transformers—connections, parallel operation; autotransformer; energy conversion principles; DC machines—types, windings, generator characteristics, armature reaction and commutation, starting and speed control of motors; three, phase induction motors—principles, types, performance characteristics, starting and speed control; single-phase induction motors; synchronous machines—performance, regulation and parallel operation of generators, motor starting, characteristics and applications; servo and stepper motors.

Power Systems: Basic power generation concepts; transmission line models and performance; cable performance, insulation; corona and radio interference; distribution systems; per-unit quantities; bus impedance and admittance matrices; load flow; voltage control; power factor correction; economic operation; symmetrical components; fault analysis; principles of overcurrent, differential and distance protection; solid state relays and digital protection; circuit breakers; system stability concepts, swing curves and equal area criterion; HVDC transmission and FACTS concepts.

Control Systems: Principles of feedback; transfer function; block diagrams; steady-state errors; Routh and Niquist techniques; Bode plots; root loci; lag, lead and lead-lag compensation; state space model; state transition matrix, controllability and observability.

Electrical and Electronic Measurements: Bridges and potentiometers; PMMC, moving iron, dynamometer and induction type instruments; measurement of voltage, current, power, energy and power factor; instrument transformers; digital voltmeters and multimeters; phase, time and frequency measurement; Q-meters; oscilloscopes; potentiometric recorders; error analysis.

Analog and Digital Electronics: Characteristics of diodes, BJT, FET; amplifiers—biasing, equivalent circuit and frequency response; oscillators and feedback amplifiers; operational amplifiers—characteristics and applications; simple active filters; VCOs and timers; combinational and sequential logic circuits; multiplexer; Schmitt trigger; multi-vibrators; sample and hold circuits; A/D and D/A converters; 8-bit microprocessor basics, architecture, programming and interfacing.

Power Electronics and Drives: Semiconductor power diodes, transistors, thyristors, triacs, GTOs, MOSFETs and IGBTs—static characteristics and principles of operation; triggering circuits; phase control rectifiers; bridge converters—fully controlled and half controlled; principles of choppers and inverters; basis concepts of adjustable speed DC and AC drives.

Chapter-wise Analysis of GATE Previous Years' Papers

Subject	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
General Aptitude																
1 Mark										5	5	5	5	5	7	5
2 Marks										5	5	5	5	5	4	5
Total Marks										**15**	**15**	**15**	**15**	**15**	**15**	**15**
Engineering Maths																
1 Mark				4	0	1	3	1	3	2	3	4	4	3	5	5
2 Marks				6	5	8	5	4	5	3	5	4	5	4	4	4
Total Marks				**16**	**10**	**17**	**13**	**9**	**13**	**8**	**13**	**12**	**14**	**11**	**13**	**13**
Electric Circuits and Fields																
1 Mark	5	3	1	4	2	1	2	2	3	2	4	4	3	3	5	3
2 Marks	8	6	7	6	6	7	6	5	4	4	7	3	4	5	4	4
Total Marks	**21**	**15**	**15**	**16**	**14**	**15**	**14**	**12**	**11**	**10**	**18**	**10**	**11**	**13**	**13**	**11**
Signals and Systems																
1 Mark	1	0	0	0	3	1	2	1	2	3	2	3	3	2	2	2
2 Marks	0	0	2	4	4	6	7	3	4	1	2	1	2	2	2	2
Total Marks	**1**	**0**	**4**	**8**	**11**	**13**	**16**	**7**	**10**	**5**	**6**	**5**	**7**	**6**	**6**	**6**
Electrical Machines																
1 Mark	4	5	6	4	3	3	4	3	2	3	2	2	4	3	4	2
2 Marks	3	11	13	10	12	8	10	8	4	2	4	1	4	3	5	5
Total Marks	**10**	**27**	**32**	**24**	**27**	**19**	**24**	**19**	**10**	**7**	**10**	**4**	**12**	**9**	**14**	**12**
Power Systems																
1 Mark	3	5	6	4	4	4	3	2	6	4	2	2	1	3	2	2
2 Marks	2	11	10	8	10	9	8	5	4	3	1	3	4	2	3	3
Total Marks	**7**	**27**	**26**	**20**	**24**	**22**	**19**	**12**	**14**	**10**	**4**	**8**	**9**	**7**	**8**	**8**
Control Systems																
1 Mark	4	3	3	3	1	1	1	4	2	5	2	3	2	3		3
2 Marks	5	7	7	6	3	7	7	4	3	3	5	4	4	3	2	4
Total Marks	**14**	**17**	**17**	**15**	**7**	**15**	**15**	**12**	**8**	**11**	**12**	**11**	**10**	**9**	**2**	**11**
Electrical and Electronic Measurements																
1 Mark	1	3	3	3	2	1	1	2	2	4	3	2	2	2		2
2 Marks	0	8	7	5	4	1	2	2	1	1	1	1	3	3	2	2
Total Marks	**1**	**19**	**17**	**13**	**10**	**3**	**5**	**6**	**4**	**6**	**5**	**4**	**8**	**8**	**2**	**6**
Analog and Digital Electronics																
1 Mark	3	5	5	4	2	4	1	4	3	3	3	3	3	4	3	4
2 Marks	1	9	10	9	8	4	8	4	4	5	2	5	3	5	2	4
Total Marks	**5**	**23**	**25**	**22**	**18**	**12**	**17**	**12**	**11**	**13**	**7**	**13**	**9**	**14**	**7**	**12**
Power Electronics and Drives																
1 Mark	3	4	4	3	2	4	2	1	2	2	4	0	3	2	2	2
2 Marks	5	5	6	5	8	7	7	4	0	4	3	6	1	3	4	2
Total Marks	**13**	**14**	**16**	**13**	**18**	**18**	**16**	**9**	**2**	**10**	**10**	**12**	**5**	**8**	**6**	**6**

General Information about GATE

Structure of GATE

The GATE examination consists of a single online paper of 3-hour duration, in which there will be a total of 65 questions carrying 100 marks out of which 10 questions carrying a total of 15 marks are in General Aptitude (GA).

Section Weightage and Marks

70% of the total marks is given to the technical section while 15% weightage is given to General Aptitude and Engineering Mathematics each.

Weightage	Questions (Total 65)	
Respective Engineering Branch	70 Marks	25—1 mark questions 30—2 mark questions
Engineering Maths	15 Marks	
General Aptitude	15 Marks	5—1 mark questions 5—2 mark questions

Particulars

For 1-mark multiple-choice questions, 1/3 marks will be deducted for a wrong answer. Likewise, for 2-mark multiple-choice questions, 2/3 marks will be deducted for a wrong answer. There is no negative marking for numerical answer type questions.

Question Types

1. **Multiple Choice Questions (MCQ)** carrying 1 or 2 marks each in all papers and sections. These questions are objective in nature, and each will have a choice of four answers, out of which the candidate has to mark the correct answer.
2. **Numerical Answer/Fill in the blank Questions** carrying 1 or 2 marks each in all papers and sections. For numerical answer questions, choices will not be given. Candidates have to enter the answer (which will be a real number, signed or unsigned, e.g., 25.06, −25.06, 25, −25 etc.) using a virtual keypad. An appropriate range will be considered while evaluating the numerical answer-type questions so that the candidate is not penalized due to the usual round-off errors.

Design of Questions

The fill in the blank questions usually consist of 35%– 40% of the total weightage.

The questions in a paper may be designed to test the following abilities:

1. **Recall:** These are based on facts, principles, formulae or laws of the discipline of the paper. The candidate is expected to be able to obtain the answer either from his/her memory of the subject or at most from a one-line computation.
2. **Comprehension:** These questions will test the candidate's understanding of the basics of his/her field, by requiring him/her to draw simple conclusions from fundamental ideas.
3. **Application:** In these questions, the candidate is expected to apply his/her knowledge either through computation or by logical reasoning.
4. **Analysis and Synthesis:** In these questions, the candidate is presented with data, diagrams, images etc. that require analysis before a question can be answered. A Synthesis question might require the candidate to compare two or more pieces of information. Questions in this category could, for example, involve candidates in recognising unstated assumptions, or separating useful information from irrelevant information.

About Online Pattern

The examination for all the papers will be carried out in an ONLINE Computer Based Test (CBT) mode where the candidates will be shown the questions in a random sequence on a computer screen. The candidates are required to either select the answer (for MCQ type) or enter the answer for numerical answer-type question using a mouse on a virtual keyboard (keyboard of the computer will be disabled). The candidates will also be allowed to use a calculator with which the online portal is equipped with.

Important Tips for GATE

The followings are some important tips which would be helpful for students to prepare for GATE exam

1. Go through the pattern (using previous year GATE paper) and syllabus of the exam and start preparing accordingly.
2. Preparation time for GATE depends on many factors, such as, individual's aptitude, attitude, fundamentals, concentration level etc., Generally rigorous preparation for 4 to 6 months is considered good but it may vary from student to student.
3. Make a list of books which cover complete syllabus, contains solved previous year questions and mock tests for practice based on latest GATE pattern. Purchase these books and start your preparation.
4. Make a list of topics which needs to be studied and make priority list for study of every topic based upon the marks for which that particular topic is asked in GATE exam. Find out the topics which fetch more marks and give more importance to those topics. Make a timetable for study of topics and follow the timetable strictly.
5. An effective way to brush up your knowledge about technical topics is group study with your friends. During group study you can explore new techniques and procedures.
6. While preparing any subject highlight important points (key definitions, equations, derivations, theorems and laws) which can be revised during last minute preparation.
7. Pay equal attention to both theory and numerical problems. Solve questions (numerical) based on latest exam pattern as much as possible, keeping weightage of that topic in mind. Whatever topics you decide to study, make sure that you know everything about it.
8. Try to use short-cut methods to solve problems instead of traditional lengthy and time consuming methods.
9. Go through previous year papers (say last ten years), to check your knowledge and note the distribution of different topics. Also analyze the topics in which you are weak and concentrate more on those topics. Always try to solve papers in given time, to obtain an idea how many questions you are able to solve in the given time limit.
10. Finish the detail study of topics one and a half month before your exam. During last month revise all the topics once again and clear leftover doubts.

GATE 2017 Solved Paper
Electrical Engineering
Set – I

Number of Questions: 65 **Total Marks:100.0**

Wrong answer for MCQ will result in negative marks, (–1/3) for 1 Mark Questions and (–2/3) for 2 Marks Question.

Electrical Engineering

Number of Questions: 55 **Section Marks: 85**

Q.1 to Q.25 carry 1 mark each and Q.26 to Q.55 carry 2 marks each.

Question Number: 1 **Question Type: MCQ**

The matrix $\mathbf{A}=\begin{bmatrix}\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & -1 & 0\\ \frac{1}{2} & 0 & \frac{3}{2}\end{bmatrix}$ has three distinct eigenvalues and one of its eigenvectors is $\begin{bmatrix}1\\0\\1\end{bmatrix}$. Which one of the following can be another eigenvector of **A**?

(A) $\begin{bmatrix}0\\0\\-1\end{bmatrix}$ (B) $\begin{bmatrix}-1\\0\\0\end{bmatrix}$

(C) $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ (D) $\begin{bmatrix}1\\-1\\1\end{bmatrix}$

Solution: Given matrix is $A=\begin{bmatrix}\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & -1 & 0\\ \frac{1}{2} & 0 & \frac{3}{2}\end{bmatrix}$

Given that $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is an eigen vector of A.

As A is a symmetric matrix and it was given that the matrix A has distinct eigen values,

Eigen vectors corresponding to two distinct eigen values of A are orthogonal.

Let $X=\begin{bmatrix}1\\0\\1\end{bmatrix}$

From the options, the vector given in option (C),

$\left(i.e.,\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$ is orthogonal to X.

For, $X^{T}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=1+0-1=0$

$\therefore$ Another eigen vector of A is $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

Hence, the correct option is (C).

Question Number: 2 **Question Type: MCQ**

For a complex number z, $\lim\limits_{z\to i}\dfrac{z^2+1}{z^3+2z-i\left(z^2+2\right)}$

(A) $-2i$ (B) $-i$

(C) i (D) $2i$

Solution: $\lim\limits_{z\to i}\dfrac{z^2+1}{z^3+2z-i(z^2+2)}=\lim\limits_{z\to i}\dfrac{(z-i)(z+i)}{(z-i)(z^2+2)}$

$=\lim\limits_{z\to i}\dfrac{z+i}{z^2+2}$

$=2i$

Hence, the correct option is (D).

Question Number: 3 **Question Type: MCQ**

Let $z(t)=x(t)*y(t)$, where "*" denotes convolution. Let c be a positive real-valued constant. Choose the correct expression for $z(ct)$.

(A) $c\cdot x(ct)*y(ct)$ (B) $x(ct)*y(ct)$

(C) $c\cdot x(t)*y(ct)$ (D) $c\cdot x(ct)*y(t)$

Solution: From the scaling property of convolution

If $x(t)*y(t)=z(t)$, then

$$x(ct)*y(ct)=\frac{1}{|C|}Z(ct)$$

So, $$z(ct)=c\,x(ct)*y(ct)$$

Hence, the correct option is (A).

Question Number: 4 **Question Type: MCQ**

A solid iron cylinder is placed in a region containing a uniform magnetic field such that the cylinder axis is parallel to the magnetic field direction. The magnetic field lines inside the cylinder will

(A) bend closer to the cylinder axis
(B) bend farther away from the axis
(C) remain uniform as before
(D) cease to exist inside the cylinder

Solution: As the iron cylinder provides a relatively lower reluctance path of the flee flow, the magnetic field lines will bend closer to the cylinder axis.

Hence, the correct option is (A).

Question Number: 5 **Question Type: MCQ**

Consider an election, a neutron and a proton initially at rest and placed along a straight line such that the neutron is exactly at the center of the line joining the electron and proton. At $t = 0$, the particles are released but are constrained to move along the same straight line. Which of these will collide first?

(A) the particles will never collide
(B) all will collide together
(C) proton and neutron
(D) electron and neutron

Ans: (B)*.

Question Number: 6 **Question Type: MCQ**

The transfer function of a system is given by,

$$\frac{V_o(s)}{V_i(s)} = \frac{1-s}{1+s}$$

Let the output of the system be $v_o(t) = V_m \sin(\omega t + \varphi)$ for the input, $v_i(t) = V_m \sin(\omega t)$ .Then the minimum and maximum values of φ (in radians) are respectively

(A) $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ (B) $\frac{-\pi}{2}$ and 0
(C) 0 and $\frac{\pi}{2}$ (D) $-\pi$ and 0

Solution:

$$M(s) = \frac{V_0(s)}{V_i(s)} = \frac{1-s}{1+s}$$

$$M(j\omega) = \frac{1-j\omega}{1+j\omega}$$

$$\phi = /M(j\omega) = -tan^{-1}\omega - tan^{-1}\omega = -2\ tan^{-1}(\omega)$$

$\omega = 0 \rightarrow \phi = -2 \tan^{-1}(0) = 0° \rightarrow$ maximum phase angle

$\omega = \infty \rightarrow \phi = -2\ tan^{-1}(\infty) = -2x90° = -180° \rightarrow$ minimum phase angle

Hence, the correct option is (D).

Question Number: 7 **Question Type: MCQ**

Consider the system with following input-output relation

$$y[n] = \left(1 + (-1)^n\right) x[n]$$

Where, $x[n]$ is the input and y[n] is the output. The system is

(A) invertible and time invariant
(B) invertible and time varying
(C) non-invertible and time invariant
(D) non-invertible and time varying

Solution:

The input-output relation is

$$y[n] = [1 + (-1)^n]\, x(n)$$

Time-Invariance:

Input-delay

$$y[n_1 n_0] = [1 + (-1)^n]\, x(n - n_0)$$

output-delay

$$y[n - n_0] = [1 + (-1)^{n-n_0}]\, x(n - n_0)$$

Input delay ≠ output delay

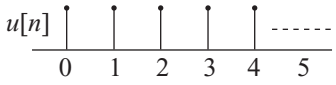
Time varying

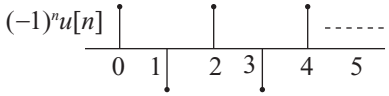
Invertibility:-

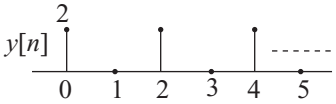
$$x[n] = u[n]$$

$$y[n] = [1 + (-1)^n]u[n]$$

$$= u[n] + (-1)^n u[n]$$





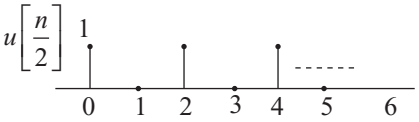


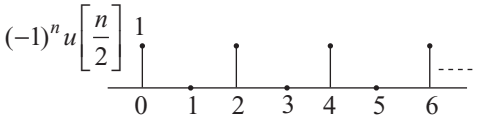
$$y[n] = \{2, 0, 2, 0, ----\}$$
$$\uparrow$$

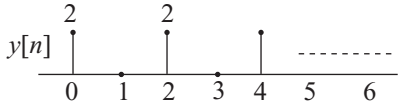
$$x[n] = u\left[\frac{n}{2}\right]$$

$$y[n] = [1 + (-1)^n]u\left[\frac{n}{2}\right]$$

$$= u\left[\frac{n}{2}\right] + (-1)^n u\left[\frac{n}{2}\right]$$







$$y[n] = \{2, 0, 2, 0, ----\}$$
$$\uparrow$$

Both the inputs are producing the same output, hence non-Invertible.

Hence, the correct option is (B).

Question Number: 8 **Question Type: MCQ**

A 4 pole induction machine is working as an induction generator. The generator supply frequency is 60 Hz. The rotor current frequency is 5 Hz. The mechanical speed of the rotor in RPM is

(A) 1350 (B) 1650
(C) 1950 (D) 2250

Solution: Given, number of poles, $P = 4$

Generator supply frequency, $f = 60Hz$

Synchronous speed, $N_S = \dfrac{120f}{P} = \dfrac{120 \times 60}{4}$

$N_S = 1800$ r.p.m.

Given Rotor current frequency, $f_2 = 5$

$Sf = 5$

Slip, $S = \dfrac{5}{60} \rightarrow (-)$ ve for induction generator

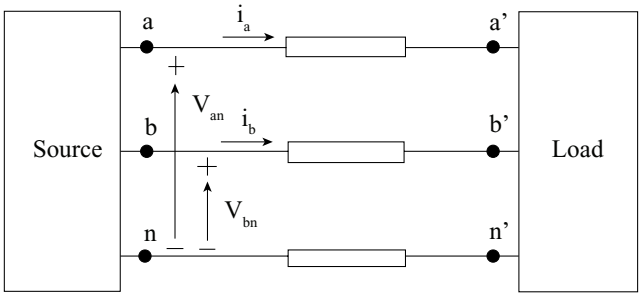
For Induction generator, $N_r = N_S(1 - S)$

$$= 1800\left(1 - \left(\frac{-5}{60}\right)\right) = 1950 \text{ r.p.m}$$

Hence, the correct option is (C).

Question Number: 9 **Question Type: MCQ**

A source is supplying a load through a 2-phase, 3-wire transmission system as shown in figure below. The instantaneous voltage and current in phase-a are $v_{an} = 220 \sin (100\pi t)$ V and $i_a = 10 \sin (100\pi t)$ A, respectively Similarly for phase-b, the instantaneous voltage and current are $v_{bn} = 220 \cos (100\pi t)$ V and $i_b = 10 \cos (100\ \pi t)$ A, respectively.



The total instantaneous power flowing from the source to the load is

(A) 2200 W
(B) 2200 $\sin^2 (100\pi t)$ W
(C) 4400 W
(D) 2200 $\sin (100\pi t) \cos (100\pi t)$ W

Solution:

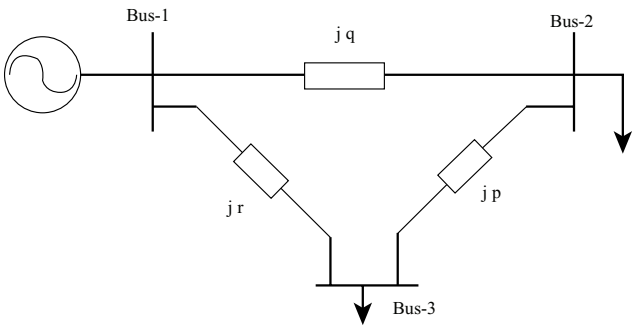
$$P_{\text{instantaneous}} = \text{Re}\left\{V_{an}I_a^* + V_{bn}I_b^*\right\}$$

$$= 2200 \text{ W}$$

Hence, the correct option is (A).

Question Number: 10 **Question Type: MCQ**

A 3-bus power system is shown in the figure below, where the diagonal elements of Y-bus matrix are: $Y_{11} = -j12$ pu, $Y_{22} = -j15$ pu and $Y_{33} = -j7$ pu.



The per unit values of the line reactances p. q and r shown in the figure are

(A) $p = -0.2$, $q = -0.1$, $r - 0.5$
(B) $p = 0.2$, $q = 0.1$, $r = 0.5$
(C) $p = -5$, $q = -10$, $r = -2$
(D) $p = 5$, $q = 10$, $r = 2$

Solution: $Y_{11} = -j12 = \dfrac{-j}{q} - \dfrac{j}{r}$

$$\frac{q+r}{qr} = 12 \rightarrow (1)$$

$$Y_{22} = -j15 = \frac{-j}{q} - \frac{j}{p} \Rightarrow 15 = \frac{q+p}{qp} \Rightarrow (2)$$

$$Y_{33} = j7 = \frac{-j}{r} - \frac{j}{p}$$

$$7 = \frac{p+r}{pr} \rightarrow (3)$$

By solving (1), (2) & (3) verify from given options.

Hence, the correct option is (B).

Question Number: 11 **Question Type: MCQ**

A closed loop system has the characteristic equation given by $s^3 + Ks^2 + (K + 2)\ s + 3 = 0$. For this system to be stable, which one of the following conditions should be satisfied?

(A) $0 < K < 0.5$ (B) $0.5 < K < 1$
(C) $0 < K < 1$ (D) $K > 1$

Solution: Given characteristic equation:

$$S^3 + kS^2 + (k + 2)S + 3 = 0$$

Form Routh Table

S^3	1	(k+2)
S^2	k	3
S^1	$\dfrac{k^2 + 2k - 3}{k}$	0
S^0	3	0

For the closed loop system to be stable, all the elements of first column should be positive.

$$K > 0;\ \frac{k^2 + 2k - 3}{k} > 0 \Rightarrow (k - 1)(k + 3) > 0$$

$$\Rightarrow k > 1;\ k > -3$$

The common range of 'K' which satisfies all condition is $k > 1$

Hence, the correct option is (D).

Question Number: 12 **Question Type: MCQ**

The slope and level detector circuit in a CRO has a delay of 100 ns. The start-stop sweep generator has a response time of 50 ns. in order to display correctly, a delay line of

(A) 150 ns has to be inserted into the *y*-channel
(B) 150 ns has to be inserted into the *x*-channel
(C) 150 ns has to be inserted into both *x* and *y* channels
(D) 100 ns has to be inserted into both *x* and *y* channels

Solution: A delay of 150ns to be inserted into the *y*-channel to get a synchronism between x-plate sweep signal and *y*-plate test signal.

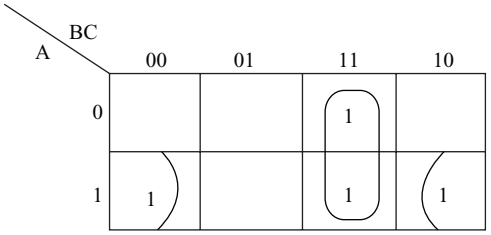
Hence, the correct option is (A).

Question Number: 13 **Question Type: MCQ**

The Boolean expression $AB + A\overline{C} + BC$ simplifies to

(A) $BC + A\overline{C}$ (B) $AB + A\overline{C} + B$
(C) $AB + A\overline{C}$ (D) $AB + BC$

Solution: Given $F = AB + A\overline{C} + BC$

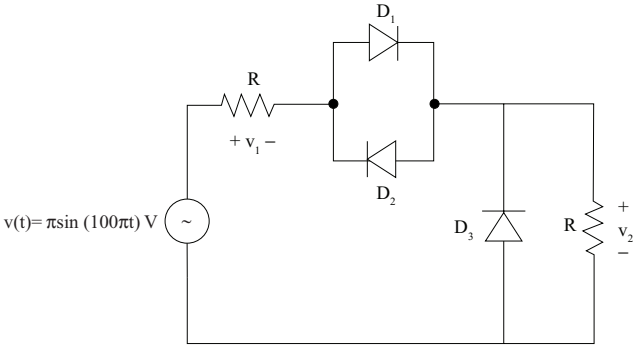


$F = I + II$
$I = BC, II = A\overline{C}$
$\therefore F = BC + A\overline{C}$

Hence, the correct option is (A).

Question Number: 14 **Question Type: MCQ**

For the circuit shown in the figure below, assume that diodes D_1, D_2 and D_3 are ideal.



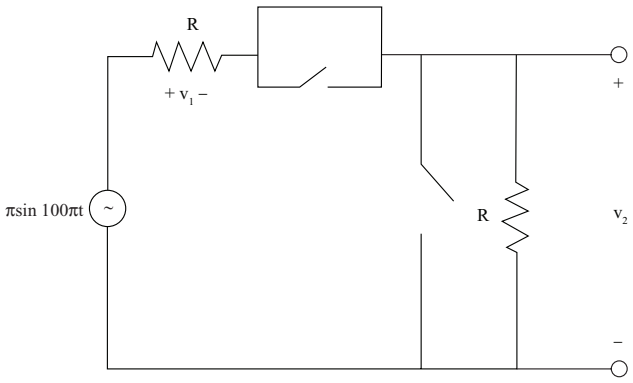
The DC components of voltages v_1 and v_2, respectively are

(A) 0 V and 1 V (B) –0.5 V and 0.5 V
(C) 1 V and 0.5 V (D) 1 V and 1 V

Solution:

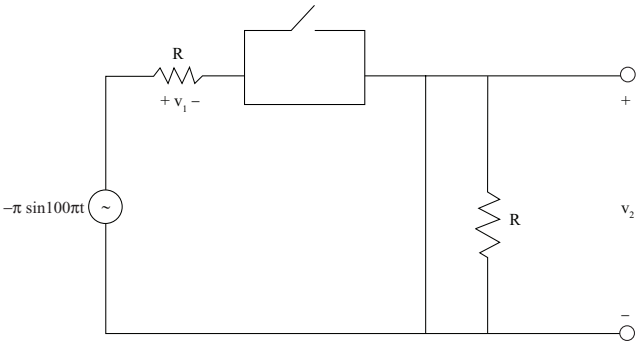
For a positive cycle of $V(t)$ D_1 is FB, D_2 & D_3 are reverse biased, so therefore D_1 is short circuited, D_2 & D_3 are open circuited

Case (i): During positive cycle of $V(t)$



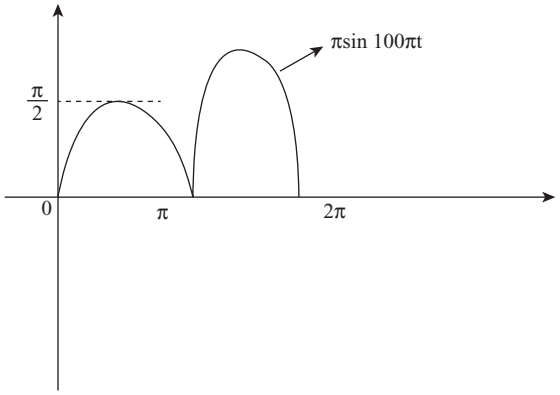
$$V_1 = \frac{\pi}{2} \sin 100\ \pi t\ ,\ V_2 = \frac{\pi}{2} \sin 100\ \pi t$$

Case (ii): During negative cycle of $V(t)$ D_1 is RB, D_2 and D_3 are *FB* so therefore D_1 is open circuit, D_2 and D_3 are short circuit



$$V_1 = -\pi \sin 100\,\pi\,t,\ V_2 = 0V$$

V_1:

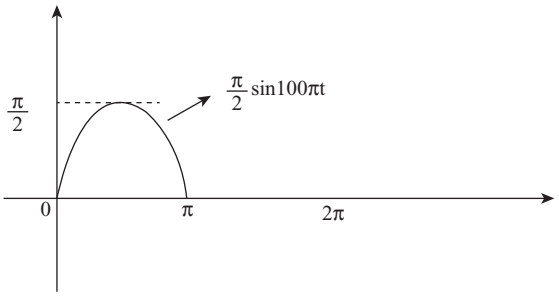


$$V_{1(\text{avg})} = \frac{1}{2\pi}\left[\int_0^{\pi} \frac{\pi}{2}\sin 100\pi t + \int_{\pi}^{2\pi} \pi \sin 100\pi t\right]$$

$$V_{1(\text{avg})} = \frac{1}{2\pi}\left[\frac{\pi}{2}\left[\frac{-Cos100\pi t}{100\pi}\right]_0^{\pi} + \pi\left[\frac{-Cos100\pi t}{100\pi}\right]_{\pi}^{2\pi}\right]$$

$$V_{1(\text{avg})} = \frac{1}{2\pi}\left[\frac{\pi}{2}[2]-\pi[+2]\right] = \frac{1}{2\pi}[-\pi] = -0.5$$

V_2:



$$V_{2(\text{avg})} = \frac{1}{2\pi}\int_0^{\pi}\frac{\pi}{2}Sin100\pi t = \left[\frac{-\pi}{2}\left[\frac{Cos100\pi t}{100\pi}\right]_0^{\pi}\right]\frac{1}{2\pi}$$

$$= \left[\frac{-\pi}{2}[-1-1]\right]\frac{1}{2\pi} = 0.5V$$

Hence, the correct option is (B).

Question Number: 15 **Question Type: MCQ**

For the power semiconductor devices IGBT, MOSFET, Diode and Thyristor, which one of the following statements is TRUE?

(A) All the four are majority carrier devices.
(B) All the four are minority carrier devices.
(C) IGBT and MOSFET are majority carrier devices, whereas Diode and Thyristor are minority carrier devices.
(D) MOSFET is majority carrier device, whereas IGBT, Diode and Thyristor are minority carrier devices.

Solution:

In *MOSFET* flow of current is due to majority carriers only.

In Diode, Thyristor and *IGBT*, current flow is due to both majority and minority carriers.

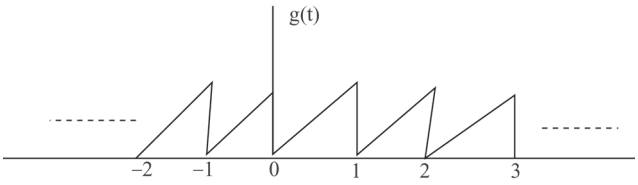
Hence, the correct option is (D).

Question Number: 16 **Question Type: NAT**

Consider $g(t) = \begin{cases} t-\lfloor t \rfloor, & t \geq 0 \\ t-\lceil t \rceil, & \text{otherwise} \end{cases}$, where $t \in \mathbb{R}$.

Here, $\lfloor t \rfloor$ represents the largest integer less than or equal to t and $\lceil t \rceil$ denotes the smallest integer greater than or equal to t The coefficient of the second harmonic component of the Fourier series representing g (t) is _____.

Solution: The given signal can be drawn as



Period $T = 1$; $W_0 = 2\pi$

Trigonometric fourier series coefficient

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} g(t) = \int_0^1 t dt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2}$$

$$a_n = \frac{2^{t_0+T}}{T}\int g(t)\cos nw_0 + dt$$

$$= 2\int_0^i t \sin n2\pi + \text{dt}$$

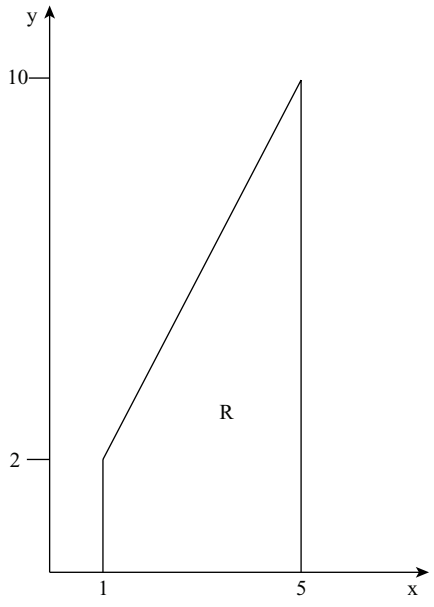
$$= \frac{-1}{n\pi}$$

Coefficient of second harmonicon is

$$b_2 = \frac{-1}{2\pi} = -0.1591 \text{ (Mark to all)}$$

Question Number: 17 **Question Type: NAT**

Let $I = c\iint_R xy^2 dxdy$ where R is the region shown in the figure and $c = 6 \times 10^{-4}$. The value of I equals _____ (Give the answer up to two decimal places.)



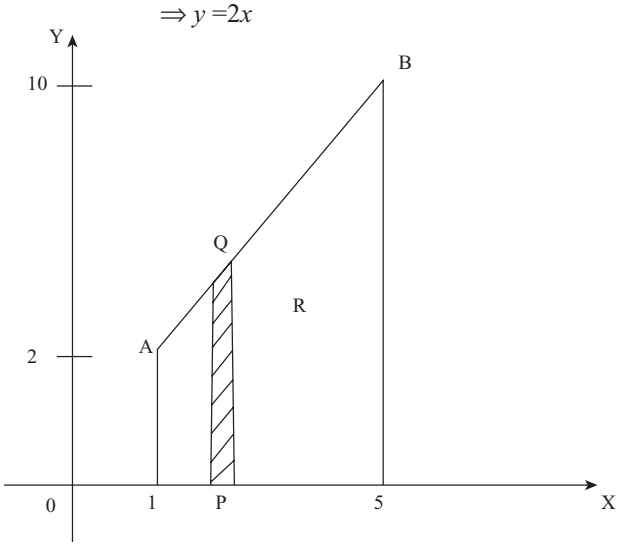
Solution: Given $I = c\iint_R xy^2 dxdy$

where $c = 6 \times 10^{-4}$

Here the equation of the straight line AB is

$$(y-2) = \frac{10-2}{5-1}(x-1)$$

$$\Rightarrow y = 2x$$



$\therefore$ First we integrate w.r.t. y along the strip PQ from $y = 0$ to $y = 2x$ followed by integration w.r.t. x from $x = 1$ to $x = 5$.

$$\therefore I = c\iint_R xy^2 dxdy = c\int_{x=1}^{5}\left(\int_{y=1}^{2x} xy^2 dy\right)dx$$

$$= c\int_{x=1}^{5}\left[x\frac{y^3}{3}\right]_{y=0}^{2x}dx$$

$$= c\int_{x=1}^{5} x\frac{(2x)^3}{3}dx$$

$$= c\int_{1}^{5}\frac{8}{3}x^4 dx$$

$$= c\left(\frac{8}{3}\times\frac{x^5}{5}\Big|_1^5\right)$$

$$= c\left(\frac{8}{3}\times\frac{(5^5-1^5)}{5}\right)$$

$$= c\left(\frac{8\times 3124}{15}\right)$$

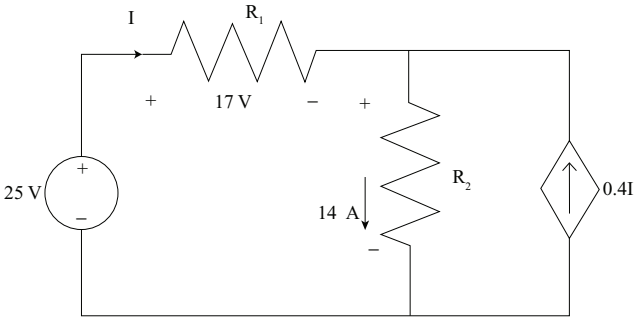
$$= 6\times 10^{-4}\times\frac{8\times 3{,}124}{15}$$

$$= 0.99968$$

Hence, the correct answer is (0.99968).

Question Number: 18 **Question Type: NAT**

The power supplied by the 25 V source in the figure shown below is _____W.



Solution: $Ve_2 = 25-17$

$= 8V$

$I + 0.4\,I = 14$

$1.4\,I = 14$

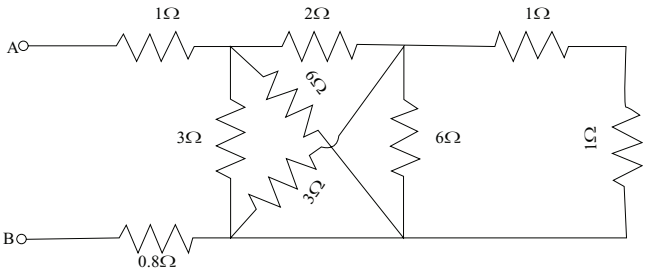
$I = 10\text{A}$

$P = V.I = 25 \times 10 = 250\text{w}$

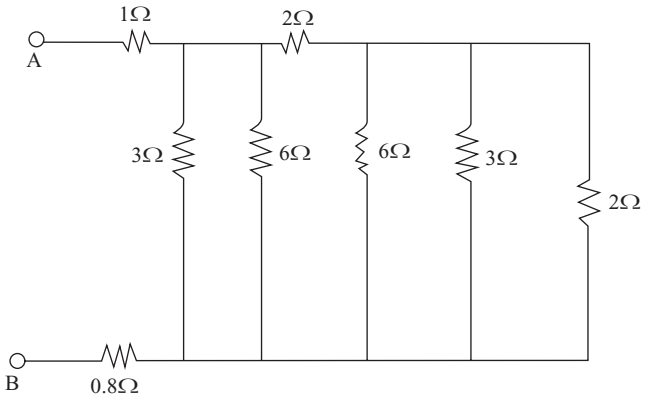
Hence, the correct answer is (248 to 252).

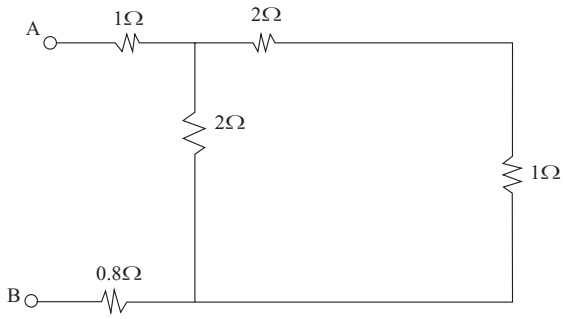
Question Number: 19 **Question Type: NAT**

The equivalent resistance between the terminals A and B is _____Ω,



Solution: Redraw the given network





$$R_{AB} = 1 + 0.8 + (3\|2)$$
$$= 1.8 + 1.2$$
$$= 3\Omega$$

Hence, the correct answer is (2.9 to 3.1).

Question Number: 20 **Question Type: NAT**

A three-phase, 50Hz. star-connected cylindrical-rotor synchronous machine is running as a motor. The machine is operated from a 6.6 kV grid and draws current at unity power factor (UPF). The synchronous reactance of the motor is 30 Ω per phase. The load angle is 30˚. The power delivered to the motor in kW is ____ (Give the answer up to one decimal place).

Solution: Power delivered, $P = \dfrac{3V_{Ph}E_{Ph}}{X_s}\sin\delta$.

Given $V_{Ph} = \dfrac{6.6\times10^3}{\sqrt{3}}\cdot V$

For synchronous motor $V_{Ph} = E_{Ph} + jI_aX_S$

$\because$ Given that the current is drawn at unity *P.F*

$\because E_{Ph}\cos\delta = V_{Ph}$

$$E_{Ph} = \frac{V_{Ph}}{cos\delta} = \frac{6.6\times10^3\times2}{\sqrt{3}\times cos30°}$$

$$\therefore P = \frac{3\times\dfrac{6.6\times10^3}{\sqrt{3}}\times\dfrac{6.6\times10^3}{\sqrt{3}\times\sqrt{3}}\times2}{30}\times\sin(30°)$$

$$= 838.312\ kW$$

Hence, the correct answer is (838.312).

Question Number: 21 **Question Type: NAT**

A 10-bus power system consists of four generator buses indexed as Gl, G2, G3, G4 and six load buses indexed as LI, L2, L3, L4, L5, L6. The generator bus G1 is considered as slack bus, and the load buses L3 and L4 are voltage controlled buses. The generator at bus G2 cannot supply the required reactive power demand, and hence it is operating at its maximum reactive power limit. The number of non-linear equations required for solving the load flow problem using Newton-Raphson method in polar form is _____

Solution: Total number of buses = $n = 10$

number of generator buses + number of voltage controlled buses = $m = 4$

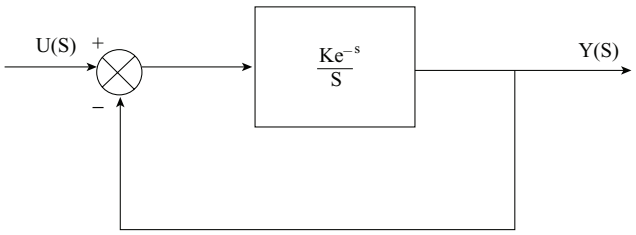
number of PQ buses = 5

number of equations = $2n - m - 2 = 20 - 4 - 2 = 14$

Hence, the correct answer is (14 to 14).

Question Number: 22 **Question Type: NAT**

Consider the unity feedback control system shown. The value of K that results in a phase margin of the system to be 30° is _____. (Give the answer up to two decimal places.)



Solution:

Loop transfer function,

$$G(S)\,H(S) = \frac{ke^{-s}}{S}$$

Given phase margin, $P.M = 30°$

$$180 + \phi_{\omega gc} = 30°$$
$$\Rightarrow \phi_{\omega gc} = -150° \quad (1)$$

$$G(j\omega) = \frac{ke^{-j\omega}}{j\omega}$$

$$M = \left|G(j\omega)H(j\omega)\right| = \frac{k}{\omega}$$

$$\phi = \angle G(j\omega)\,H(j\omega) = -\omega - \frac{\pi}{2}$$

Gain crossover frequency, ω_{gc}: $M|_{\omega gc} = 1$

$$\frac{k}{\omega_{gc}} = 1 \Rightarrow \omega_{gc} = k \quad (2)$$

From equation (1): $\phi_{\omega gc} = -\omega_{gc} - \dfrac{\pi}{2} = -150° = \dfrac{-5\pi}{6}$

$$\omega_{gc} = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{\pi}{3}$$

From equation (2): $k = \dfrac{\pi}{3} = 1.0472$

Hence, the correct answer is (1.047 to 1.05).

Question Number: 23 **Question Type: NAT**

The following measurements are obtained on a single phase load: $V = 220\ V \pm 1\ \%$, $I = 5.0\ A \pm 1\%$ and $W = 555\ W \pm 2\ \%$. If the power factor is calculated using these measurements, the worst case error in the calculated power factor in percent is _____ (Give answer up to one decimal place.)

Solution: $W = VI\cos\phi \Rightarrow \cos\phi = \dfrac{W}{VI}$

Power factor, $\cos\phi = \dfrac{555 \pm 2\%}{(220 \pm 1\%)(5 \pm 1\%)}$

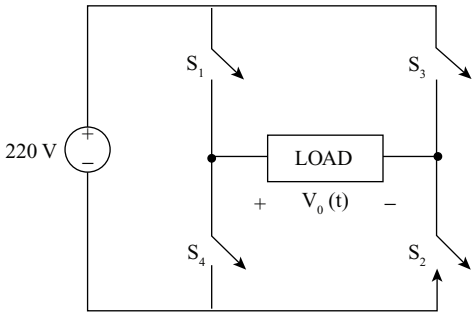
$$= \frac{555}{220 \times 5} \pm 4\%$$

$$\text{P.F} = 0.5 \pm 4\%$$

Hence, the correct answer is (4).

Question Number: 24 **Question Type: NAT**

In the converter circuit shown below, the switches are controlled such that the load voltage $v_o(t)$ is a 400 Hz square wave.



The RMS value of the fundamental component of $v_o(t)$ in volts is _____.

Solution: The given circuit is a single phase full bridge voltage source inverter.

output voltage. $V_0(t) = \dfrac{4v_{dc}}{n\pi}$ Sin nωt

$\left[\begin{array}{l}(n = 1,3,5....) \\ \because \text{square wave output}\end{array}\right]$

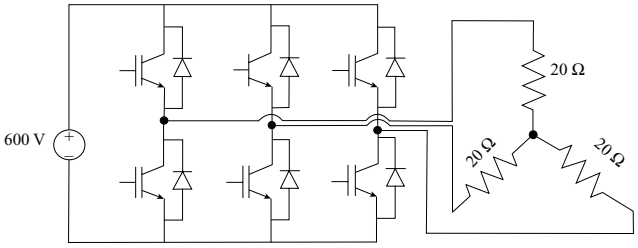
The RMS value of fundamental component of output voltage $= \dfrac{4V_{dc}}{\sqrt{2}\pi} = \dfrac{2\sqrt{2}V_{dc}}{\pi}$

$$= \frac{2\sqrt{2} \times 220}{\pi} = 198\ \text{V}$$

Hence, the correct answer is (198).

Question Number: 25 **Question Type: NAT**

A 3-phase voltage source inverter is supplied from a 600V DC source as shown in the figure below. For a star connected resistive load of 20 Ω. per phase, the load power for 120° device conduction. in kW, is _____.



Solution: For 120° conduction mode, rms value of phase

$$V_{ph} = \frac{V_{dc}}{\sqrt{6}}V = \frac{600}{\sqrt{6}}V$$

Power delivered to load $(P_L) = 3\ \dfrac{V_{ph}^2}{R} = 3 \times \left(\dfrac{600}{\sqrt{6}}\right)^2 \times \dfrac{1}{20}$

$$P_L = 9000\text{W}$$

$$P_L = 9\text{KW}$$

Hence, the correct answer is (9).

Question Number: 26 **Question Type: MCQ**

A function $f(x)$ is defined as $f(x) = \begin{cases} e^x, & x < 1 \\ \text{In}x + ax^2 + bx, & x \geq 1 \end{cases}$, where $x \in \mathbb{R}$. Which one of the following statements is TRUE?

(A) $f(x)$ is NOT differentiable at $x = 1$ for any values of a and b.

(B) $f(x)$ is differentiable at $x = 1$ for the unique values of a and b

(C) $f(x)$ is differentiable at $x = 1$ for all values of a and b such that $a + b = e$.

(D) $f(x)$ is differentiable at $x = 1$ for all values of a and b.

Solution: $f(x) = \begin{cases} e^x & ; \quad x < 1 \\ \ln x + ax^2 + bx & ; \ x \geq 1 \end{cases}$

$$\text{LHD} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{e^x - (a + b)}{x - 1}$$

LHD has to exists $\Rightarrow e^x = a + b$ at $x = 1$

$$\Rightarrow a + b = e \qquad (1)$$

$$= \lim_{x \to 1} \frac{e^x}{1} \text{ (By L' Hospital's Rule)}$$

$$\therefore \text{LHD} = e \qquad (2)$$

And RHD

$$= \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{(\ln x + ax^2 + bx)-(a+b)}{x-1}$$

$$= \lim_{x\to 1+} \frac{\frac{1}{x}+2ax+b}{1} \text{ (By L' Hospital's Rule)}$$

$$= 1 + 2a + b$$

$f(x)$ is differential at $x = 1$, if

LHD = RHD

$$\Rightarrow e = 1 + 2a + b \quad (3)$$

$$\Rightarrow a + b = 1 + 2a + b \text{ (From (1))}$$

$$a = 1$$

From (3),

$$e = 1 + 2(-1) + b$$

$$\Rightarrow b = e + 1$$

$\therefore$ $f(x)$ is differentiable at $x = 1$ for the unique values of a and b

Hence, the correct option is (B).

Question Number: 27 **Question Type: MCQ**

Consider the differential equation $(t^2-81)\frac{dy}{dt}+5t\,y=\sin(t)$ with $y(1) = 2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(A) $(-2, 2)$ (B) $(-10, 10)$

(C) $(-10, 2)$ (D) $(0, 10)$

Solution: Given differential equation is

$$(t^2-81)\frac{dy}{dt}+5ty=\sin(t) \quad (1)$$

$$\Rightarrow \frac{dy}{dt}+\frac{5t}{(t^2-81)}\quad y=\frac{\sin(t)}{t^2-81}$$

which is a linear equation of first order where $P=\frac{5t}{t^2-81}$ and $Q=\frac{\sin t}{t^2-81}$

$\therefore$ Integrating factor = I.F. = $e^{\int P dt}$

$$= e^{\int \frac{5t}{t^2-81}dt}$$

$$= e^{\frac{5}{2}\int \frac{2t}{t^2-81}dt}$$

$$= e^{\frac{5}{2}\ln(t^2-81)}$$

$$= e^{\ln(t^2-81)^{5/2}}$$

$\therefore$ I.F. $= (t^2-81)^{5/2}$

$\therefore$ The general solution of (1) is

$$y.(I.F.) = \int Q.(I.F.)dt + C$$

$$\Rightarrow y.(t^2-81)^{5/2} = \int \frac{\sin t}{(t^2-81)}\cdot(t^2-81)^{5/2}dt + C$$

$$\Rightarrow y.(t^2-81)^{5/2} = \int (t^2-81)^{3/2}\sin t dt + C$$

$$\Rightarrow y = \frac{\int (t^2-81)^{3/2}\sin t dt + C}{(t^2-81)^{5/2}}$$

Solution exists for the given differential equation only when $t^2 - 81 \neq 0$

$$\Rightarrow t \neq \pm 9$$

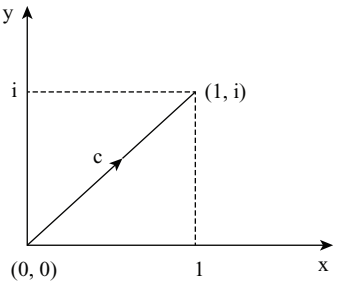
Among the intervals given in the options, only $(-2, 2)$ given in option (A) consists of neither +9 nor −9.

So, the solution is unique only when $t \in (-2, 2)$.

Hence, the correct option is (A).

Question Number: 28 **Question Type: MCQ**

Consider the line integral. $I = \int_c (x^2 + iy^2)dz$, where $z = x + iy$. The line c is shown in the figure below.



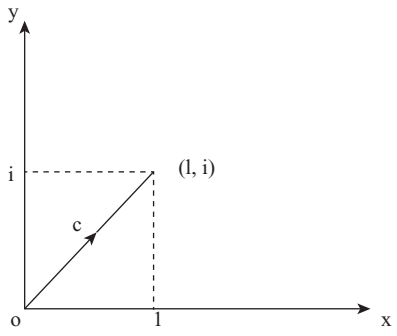
The value of I is

(A) $\frac{1}{2}i$ (B) $\frac{2}{3}i$

(C) $\frac{3}{4}i$ (D) $\frac{4}{5}i$

Solution: Given $I = \int_C (x^2 + iy^2)dz$

$$= \int_C (x^2 + iy^2)(dx + idy)$$



As C is the straight line from (0, 0) to (1, i), C is given by
$y = x$
$\Rightarrow dy = dx$

$$\therefore I = \int_C (x^2 + iy^2)(dx + idy)$$

$$= \int_{x=0}^{1} (x^2 + iy^2)(dx + idx)$$

$$= \int_{x=0}^{1} ((x^2 - y^2)dx + i(x^2 + y^2)dx)$$

$$= \int_{x=0}^{1} (0dx + 2ix^2 dx) \quad (\because y = x \text{ along } C)$$

$$= \int_0^1 2ix^2 dx$$

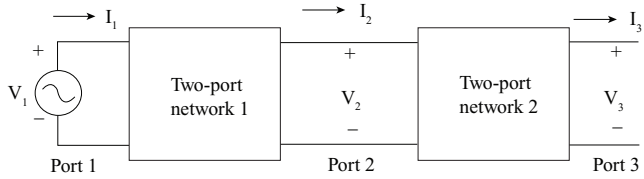
$$= \left[2i\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}i$$

Hence, the correct option is (B).

Question Number: 29 **Question Type: MCQ**

Two passive two-port networks are connected in cascade as shown in figure. A voltage source is connected at port 1.



Given

$$V_1 = A_1V_2 + B_1I_2$$
$$I_1 = C_1V_2 + D_1I_2$$
$$V_2 = A_2V_3 + B_2I_3$$
$$I_2 = C_2V_3 + D_2I_3$$

A_1, B_1, C_1, D_1, A_2, B_2, C_2. and D_2 are the generalized circuit constants. If the Thevenin equivalent circuit at port 3 consists of a voltage source V_T and an impedance Z_T connected in series, then

(A) $V_T = \frac{V_1}{A_1A_2}, Z_T = \frac{A_1B_2 + B_1D_2}{A_1A_2 + B_1C_2}$

(B) $V_T = \frac{V_1}{A_1A_2 + B_1C_2}, Z_T = \frac{A_1B_2 + B_1D_2}{A_1A_2}$

(C) $V_T = \frac{V_1}{A_1 + A_2}, Z_T = \frac{A_1B_2 + B_1D_2}{A_1 + A_2}$

(D) $V_T = \frac{V_1}{A_1A_2 + B_1C_2}, Z_T = \frac{A_1B_2 + B_1D_2}{A_1A_2 + B_1C_2}$

Solution: From the given data

$$[T] = [T_A] \cdot [T_B]$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

$$A = A_1A_2 + B_1C_2$$
$$B = A_1B_2 + B_1D_2$$

and

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Let $I_2 = 0$

$$V_2 = \frac{V_1}{A} = \frac{V_1}{A_1A_2 + B_1C_2} = V_{Th}$$

$$Z_{Th} = \frac{V_2}{I_2} \text{ at } V_1 = 0$$

$$\frac{V_2}{I_2} = \frac{B}{A} = \frac{A_1B_2 + B_1D_2}{A_1A_2 + B_1C_2}$$

Hence, the correct option is (D).

Question Number: 30 **Question Type: MCQ**

Let a causal LTI system be characterized by the following differential equation, with initial rest condition

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y(t) = 4x(t) + 5\frac{dx(t)}{dt}$$

where, $x(t)$ and $y(t)$ are the input and output respectively. The impulse response of the system is ($u(t)$ is the unit step function)

(A) $2e^{-2t}u(t) - 7e^{-5t}u(t)$ (B) $-2e^{-2t}u(t) + 7e^{-5t}u(t)$
(C) $7e^{-2t}u(t) - 2e^{-5t}u(t)$ (D) $-7e^{-2t}u(t) + 2e^{-5t}u(t)$

Solution: Given differential equation,

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y(t) = 4x(t) + 5\frac{dx(t)}{dt}$$

Apply Laplace Transform to the above *D.E.*

$$[S^2Y(s) - Sy(0) - y^1(0)] + 7[SY(s) - y(0)] + 10Y(s)$$
$$= 4X(s) + 5[SX(s) - x(0)]$$

Given, system is initially rest condition

$$\therefore y(0) = 0, y^1(0) = 0, x(0) = 0$$
$$\therefore [S^2 + 7S + 10]Y(s) = (4 + 5S)X(S)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{4 + 5S}{S^2 + 7S + 10} = \frac{4 + 5S}{(S + 2)(S + 5)}$$

$$H(s) = \frac{A}{S + 2} + \frac{B}{S + 5}$$

$$A = -2; \text{ B} = 7$$

$$\therefore \text{H(s)} = \frac{-2}{S + 2} + \frac{7}{S + 5}$$

Impulse response, $h(t) = -2e^{-2t}u(t) + 7e^{-5t}u(t)$

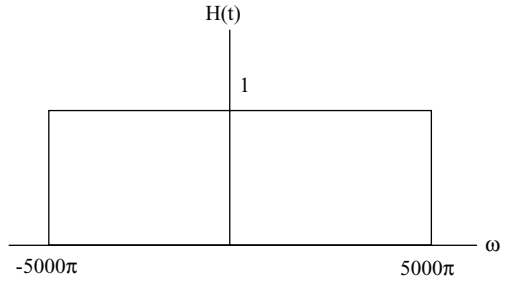
Hence, the correct option is (B).

Question Number: 31 **Question Type: MCQ**

Let the signal

$$x(t) = \sum_{k=-\infty}^{+\infty} (-1)^k \delta\left(t - \frac{k}{2000}\right)$$

be passed through an LTI system with frequency response H(ω). as given in the figure below



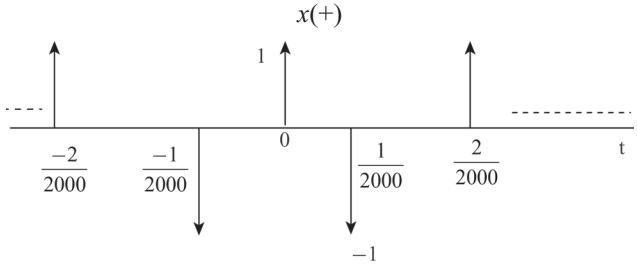
The Fourier series representation of the output is given as

(A) 4000 + 4000 cos(2000 π*t*) + 4000 cos(4000 π*t*)
(B) 2000 + 2000 cos(2000 π*t*) + 2000 cos(4000 π*t*)
(C) 4000cos(2000π*t*)
(D) 2000cos(2000π*t*)

Solution: The given signal is

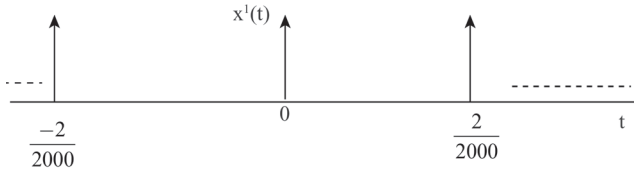
$$x(+) = \sum_{k=-d}^{\infty} (-1)^k \delta(t - \frac{k}{2000})$$

The signal can be drawn as



$x(+)$ can be broken in to two different impulse from signals, $x^1(+)$ and $x^{11}(+)$.

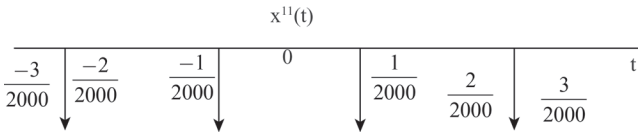
$x^1(+)$ can be shown as



$$T = \frac{1}{1000};\ W_0 = 2000\,\pi;$$

Exponential farier series Coeff is given by $C_n = \frac{1}{T} = 1000$

$X^{11}(+)$ can be shown as



$$x^{11}(+) = -x^1\left(t - \frac{1}{2000}\right)$$

By using shifting property

$$x^1(+) \leftrightarrow C_n$$

$$-x^1\left(t - \frac{1}{2000}\right) \leftrightarrow -e^{-\text{jn}\,2000\,\pi}.\frac{1}{2000}\,C_n$$

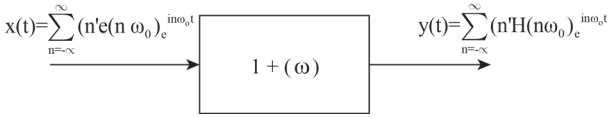
$$-x^1\left(t - \frac{1}{2000}\right) \leftrightarrow e^{-\text{jn}\,\pi}.\frac{1}{2000}\,C_n$$

$$X(t) = x^1(+) + x^{11}(t)$$

Fourier series coefficient of $x(+)$ is

$$C_n^{\,1} = (1 - e^{-\text{jn}\pi})\,C_n$$
$$= (1 - e^{-\text{jn}\pi})\,1000$$
$C_n^{\,1} = 2000$ for 'n' odd
0 for 'n' even

of a periodic input signal is to a LTI system, the o/p is given by



$$y(+) = \sum_{n=-}^{\infty} 2000H(n2000\pi)e^{jn2000\pi t}$$

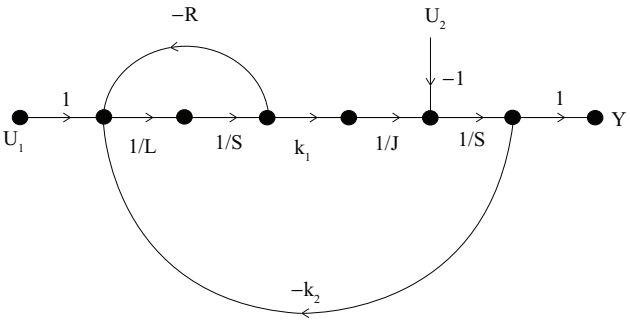
C^l_n is defined only for odd values for $n = 1$ & $n = -1$, we get the output

$$y(t) = 2000\,e^{e2000\pi t} + 2000\,e^{i2000\pi t}$$
$$y(t) = 4000\cos 2000\,\pi t$$

Hence, the correct option is (C).

Question Number: 32 **Question Type: MCQ**

In the system whose signal flow graph is shown in the figure. $U_1(s)$ and $U_2(s)$ are inputs. The transfer function $\frac{Y(s)}{U_1(s)}$ is



(A) $\dfrac{k_1}{JLs^2+JRs+k_1k_2}$

(B) $\dfrac{k_1}{JLs^2-JRs-k_1k_2}$

(C) $\dfrac{k_1-U_2(R+sL)}{JLs^2+(JR-U_2L)s+k_1k_2-U_2R}$

(D) $\dfrac{k_1-U_2(sL-R)}{JLs^2-(JR+U_2L)s-k_1k_2+U_2R}$

Solution: Apply Mason's gain formula between $Y(s)$ and $U_1(s)$:

Number of forward paths = 1

Forward path gain = $\dfrac{k_1}{LJS^2}$

Number of individual Loops = 2

Loop gains, $L_1 = \dfrac{-R}{LS}$; $L_2 = \dfrac{-K_1K_2}{LJS^2}$

$\therefore$ From Mason's Gain formula

$$\frac{Y(s)}{U_1(s)} = \frac{\dfrac{K_1}{LJS^2}}{1-\left(\dfrac{-R}{LS}-\dfrac{K_1K_2}{LJS^2}\right)}$$

$$\therefore \frac{Y(s)}{U_1(s)} = \frac{K_1}{LJS^2+RJS+K_1K_2}$$

Hence, the correct option is (A).

Question Number: 33 **Question Type: MCQ**

The transfer function of the system $Y(s)/U(s)$ whose state-space equations are given below is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 2\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}.$$

(A) $\dfrac{(s+2)}{(s^2-2s-2)}$ (B) $\dfrac{(s-2)}{(s^2+s-4)}$

(C) $\dfrac{(s-4)}{(s^2+s-4)}$ (D) $\dfrac{(s+4)}{(s^2-s-4)}$

Solution:

Given $\overline{A} = \begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}$; $\overline{B} = \begin{bmatrix}1\\ 2\end{bmatrix}$; $\overline{C} = [1\ 0]$; $\overline{D} = \overline{O}$

Transfer function, $\dfrac{Y(s)}{U(s)} = \overline{C}\,(S\overline{I}-\overline{A})^{-1}\overline{B}+\overline{D}$

$$\frac{Y(s)}{U(s)} = [1\ 0]\left\{\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}\right\}^{-1}\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = [1\ 0]\begin{bmatrix}s-1 & -2\\ -2 & s\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\end{bmatrix}$$

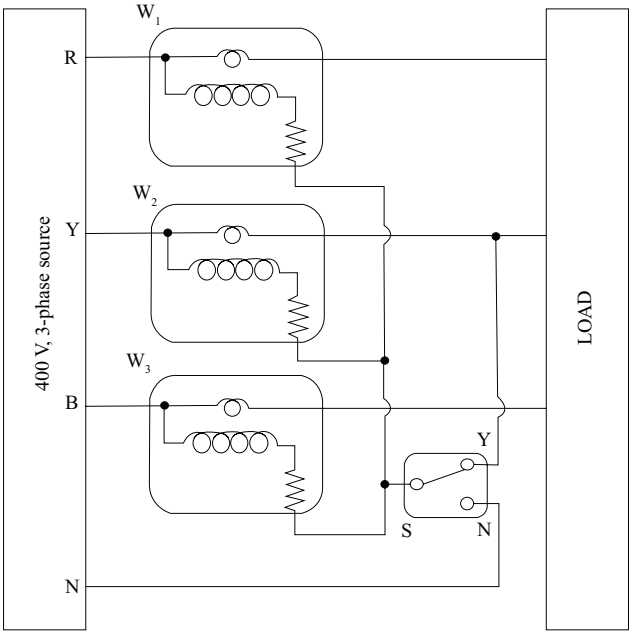
$$= [1\ 0]\ \frac{1}{s^2-s-4}\begin{bmatrix}s & 2\\ 2 & s-1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = [1\ 0]\frac{1}{s^2-s-4}\begin{bmatrix}s+4\\ 2s\end{bmatrix} = \frac{s+4}{s^2-s-4}$$

Hence, the correct option is (D).

Question Number: 34 **Question Type: MCQ**

The load shown in the figure is supplied by a 400 V (line-to-line), 3-phase source (RYB sequence). The load is balanced and inductive, drawing 3464 VA. When the switch S is in position *N*. the three watt-meters W_1, W_2 and W_3 read 577.35 *W* each. If the switch is moved to position *Y*, the readings of the watt-meters in watts will be:



(A) $W_1 = 1732$ and $W_2 = W_3 = 0$
(B) $W_1 = 0$, $W_2 = 1732$ and $W_3 = 0$
(C) $W_1 = 866$, $W_2 = 0$ and $W_3 = 866$
(D) $W_1 = W_2 = 0$ and $W_3 = 1732$

Solution: When switch *S* is in position *N*

$W = W_1 + W_2 + W_3 = 577.35 \times 3 = 1732.05W$

Given load VA = 3464 VA

$3V_{Ph}I_{Ph} = 3464 \Rightarrow V_{Ph}I_{Ph} = 1154.67$

Each watt meter reading = $V_{RN}.\ I_R \cos \underbrace{(V_{RN} \text{ and } I_R)}_{\phi}$

$577.35 = 1154.67 \cos \phi$

$\Rightarrow \phi = 60°$

When switch is moved to position Y:

$W_1 = V_{RY} I_R \cos(V_{RY} \text{ and } I_R)$

$W_1 = \sqrt{3}\ V_{Ph} I_{Ph} \cos(30° + \phi)$

$= \sqrt{3} \times 1154.67 \cos(30° + 60°) = 0$ Watt

$W_3 = V_{BY} I_R \cos(V_{BY} \text{ and } I_B)$

$W_3 = \sqrt{3}\ V_{Ph} I_{Ph} \cos(30° - \phi)$

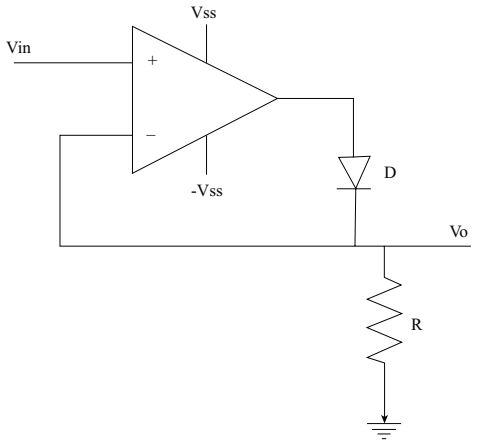
$W_3 = \sqrt{3} \times 1154.67 \times \cos(30° - 60°) = 1732$ Watt

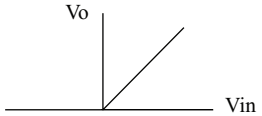
$W_2 = 0$

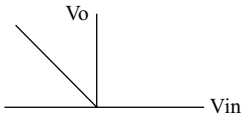
Hence, the correct option is (D).

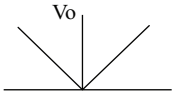
Question Number: 35 **Question Type: MCQ**

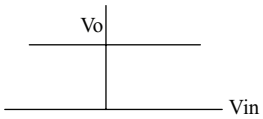
The approximate transfer characteristic for the circuit shown below with an ideal operational amplifier and diode will be



(A)


(B)


(C)


(D)


Solution:

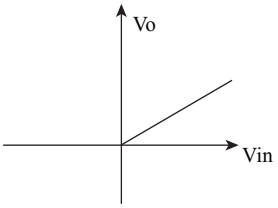
$V_{in} > 0$, $V_x = +V_{ss}$ Diode is FB

$V_0 = V_{in}$ as there is a negative feedback.

$V_{in} < 0$, $V_x = -V_{ss}$ Diode is RB

$V_0 = 0$ as there is no negative feedback

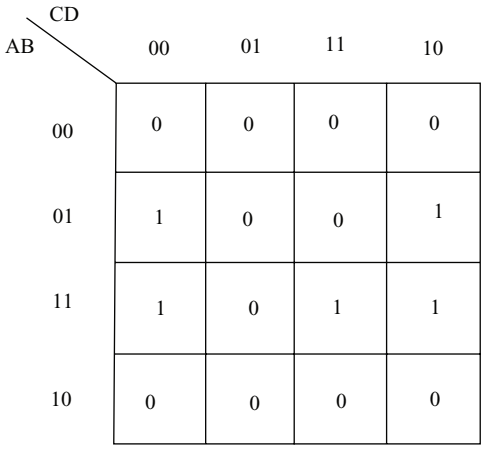
V_x is the the output of op-amp.



Hence, the correct option is (A).

Question Number: 36 **Question Type: MCQ**

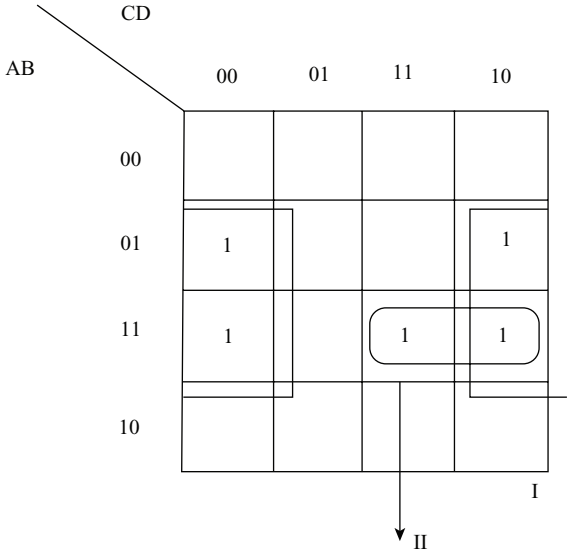
The output expression for the Karnaugh map shown below is



(A) $B\overline{D} + BCD$ (B) $B\overline{D} + AB$

(C) $\overline{B}D + ABC$ (D) $B\overline{D} + ABC$

Solution



$I = B\overline{D}$

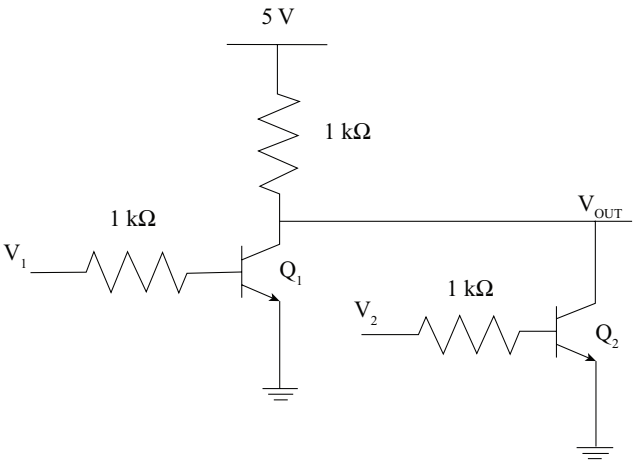
$II = ABC$

$F = I + II = B\overline{D} + ABC$

Hence, the correct option is (D).

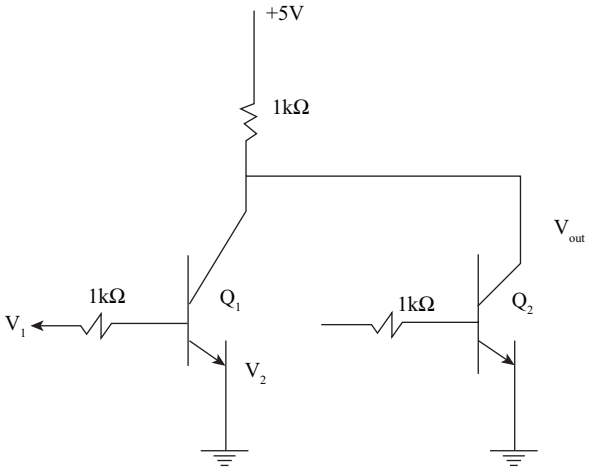
Question Number: 37 **Question Type: MCQ**

The logical gate implemented using the circuit shown below where. V_1 and V_2 are inputs (with 0 V as digital 0 and 5 V as digital 1) and V_{OUT} is the output, is



(A) NOT (B) NOR
(C) NAND (D) XOR

Solution:



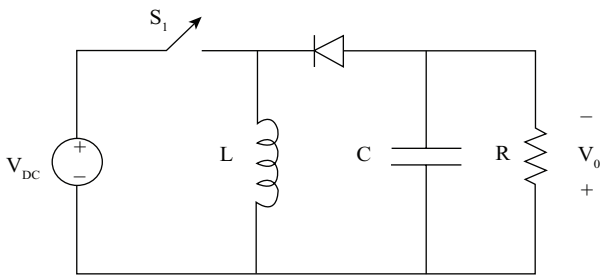
V_1	V_2	V_{oct}
0	0	1
0	1	0
1	0	0
1	1	0

Hence, it is a nor gate.

Hence, the correct option is (B).

Question Number: 38 **Question Type: MCQ**

The input voltage V_{DC} of the buck-boost converter shown below varies from 32 V to 72 V. Assume that all components are ideal, inductor current is continuous, and output voltage is ripple free. The range of duty ratio D of the converter for which the magnitude of the steady-state output voltage remains constant at 48 V is



(A) $\frac{2}{5} \le D \le \frac{3}{5}$ (B) $\frac{2}{3} \le D \le \frac{3}{4}$

(C) $0 \le D \le 1$ (D) $\frac{1}{3} \le D \le \frac{2}{3}$

Solution: In Buck-Boost converter, output voltage $V_0 = \frac{D}{1-D} V_{dc}$

$$48 = \frac{D}{1-D} V_{dc}$$

$$48 - 48D = DV_{dc}$$

$$D(V_{dc} + 48) = 48$$

$$D = \frac{48}{V_{dc} + 48}$$

When $V_{dc} = 32V$, $D = \frac{48}{32+48} = \frac{3}{5}$

When $V_{dc} = 70V$, $D = \frac{48}{72+48} = \frac{2}{5}$

The duty ratio range = $\frac{2}{5} \le D \le \frac{3}{5}$

Hence, the correct option is (A).

Question Number: 39 **Question Type: MCQ**

A load is supplied by a 230 V. 50 Hz source The active power P and the reactive power Q consumed by the load are such that 1 kW $\le P \le$ 2kW and 1kVAR $\le Q \le$ 2kVAR. A capacitor connected across the load for power factor correction generates 1 kVAR reactive power. The worst case power factor after power factor correction is

(A) 0.447 lag (B) 0.707 lag
(C) 0.894 lag (D) 1

Solution: Cos $\phi = \frac{P}{\sqrt{P^2+Q^2}}$ lag (Before correction)

Least power factor is experienced

with P = 1 kw and Q = 2 KVAR

$$\text{Cos } \phi = \frac{1}{\sqrt{1+4}} = 0.4472 \text{ lag}$$

With capacitor

$$\text{Cos } \phi \text{ (new)} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707 \text{ lag}$$

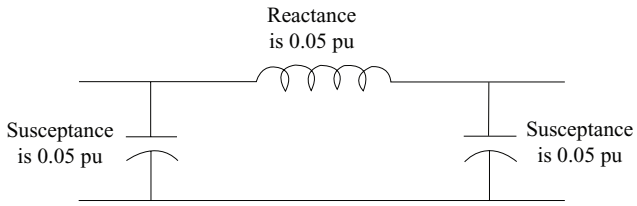
Hence, the correct option is (B).

Question Number: 40 **Question Type: MCQ**

The bus admittance matrix for a power system network is

$$\begin{bmatrix} -j39.9 & j20 & j20 \\ j20 & -j39.9 & j20 \\ j20 & j20 & -j39.9 \end{bmatrix} \text{pu.}$$

There is a transmission line, connected between buses 1 and 3. winch is represented by the circuit shown in figure.



If this transmission line is removed from service, what is the modified bus admittance matrix?

(A) $\begin{bmatrix} -j19.9 & j20 & 0 \\ j20 & -j39.9 & j20 \\ 0 & j20 & -j19.9 \end{bmatrix} \text{pu.}$

(B) $\begin{bmatrix} -j39.95 & j20 & 0 \\ j20 & -j39.9 & j20 \\ 0 & j20 & -j39.95 \end{bmatrix} \text{pu.}$

(C) $\begin{bmatrix} -j19.95 & j20 & 0 \\ j20 & -j39.9 & j20 \\ 0 & j20 & -j19.95 \end{bmatrix} \text{pu.}$

(D) $\begin{bmatrix} -j19.95 & j20 & j20 \\ j20 & -j39.9 & j20 \\ j20 & j20 & -j19.95 \end{bmatrix} \text{pu.}$

Solution:

$$Y_{13} = j20 \text{ (old value)}$$

If line between (1) & (3) is removed

$$Y_{13} = 0 \text{ (new value)}$$

$$Y_{11} = -j39.9 \text{ (old value)}$$

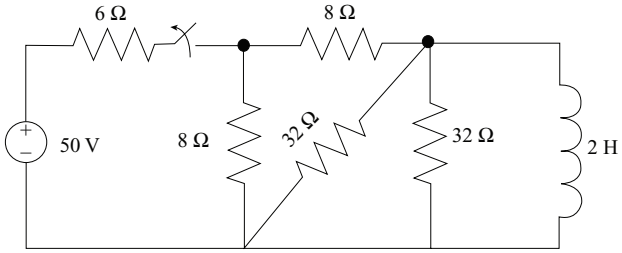
$$Y_{11(\text{new})} = -j39.9 - (-j20) - j0.05 = -j19.95$$

$$Y_{33(\text{new})} = -j39.9 - (-j20) - j0.05 = -j19.95$$

Hence, the correct option is (C).

Question Number: 41 **Question Type: MCQ**

The switch in the figure below was closed for a long time. It is opened at $t = 0$. The current in the inductor of 2 H for $t \geq 0$, is

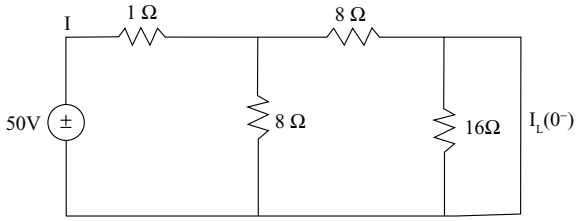


(A) $2.5e^{-4t}$ (B) $5e^{-4t}$

(C) $2.5e^{-0.25t}$ (D) $5e^{-0.25t}$

Solution: For $t < 0$

At $t = 0$: circuit is in steady state $I_L(0) = ?$



$$I = \frac{50}{6+4} = 5A$$

$$I_L(0^-) = \frac{I}{2} = 2.5A$$

<u>for $t > 0$</u>

at $t = 0^+$; the switch was opened

$$i_L(0^+) = i_L(0^-) = 2.5A$$

As $t \to \infty$, $i_L(\infty) = 0\text{A}$

$$\tau = \frac{L_{eq}}{R_{eq}}$$

$$R_{eq} = (16 \| 16) = 8\Omega$$

$$L = 2H$$

$$\tau = \frac{2}{8} = \frac{1}{4} = 0.25 \text{ sec}$$

$$\therefore i_L(t) = i(0), \ \bar{e}^{t/\tau}$$

$$i_L(t) = 2.5. \bar{e}^{4t}$$

Hence, the correct option is (A).

Question Number: 42 **Question Type: NAT**

Only one of the real roots of $f(x) = x^6 - x - 1$ lies in the interval $1 \leq .x \leq 2$ and bisection method is used to find its value. For achieving an accuracy of 0.001 - the required minimum number of iterations is _____.

Solution: Given $f(x) = x^6 - x - 1$ has a root in the interval $1 \le x \le 2$.

Here $a = 1$ and $b = 2$

As we have to achieve an accuracy of 0.001, we have $\varepsilon = 0.001 = 10^{-3}$.

We know that in bisection method, the minimum number of iterations (n) required so that error is less than ε when

$$\frac{|b-a|}{2^n} < \varepsilon$$

$$\Rightarrow \frac{|2-1|}{2^n} < 10^{-3}$$

$$\Rightarrow \frac{1}{2^n} < \frac{1}{10^3}$$

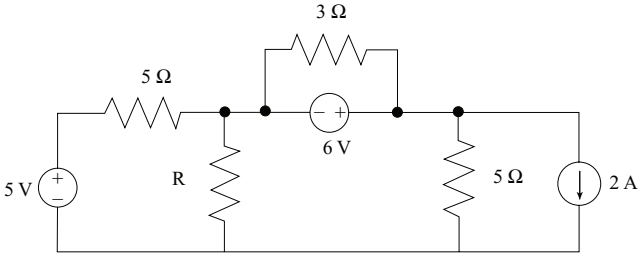
$$\Rightarrow 2^n > 10^3$$

$\Rightarrow$ The least value of n should be 10.

Hence, the correct answer is (10).

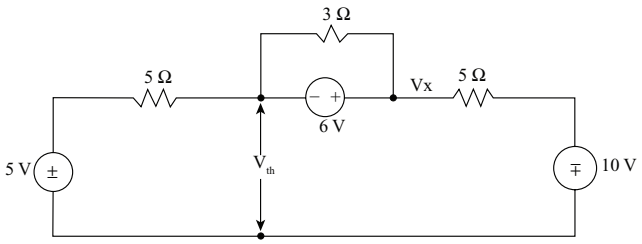
Question Number: 43 **Question Type: NAT**

In the circuit shown below, the maximum power transferred to the resistor R is _____ W.



Solution: Find V_{th} and R_{th}

V_{th}



From the above network

$$V_{th} + 6 = V_x \quad \text{(i)}$$

$$\frac{V_{th} - 5}{5} + \frac{V_x + 10}{5} = 0$$

$$V_{th} + V_x + 5 = 0 \quad \text{(ii)}$$

From (i) and (ii)

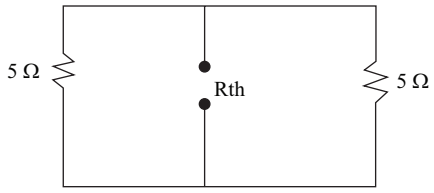
$$V_{th} + V_{th} + 6 + 5 = 0$$

$$2V_{th} = -11$$

$$V_{th} = -5.5V$$

R_{th}:

Deactivate all independent sources



$$R_{th} = (5 \| 5) = 2.5\Omega$$

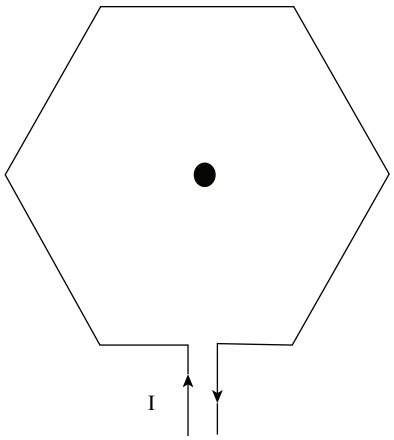
$$\therefore P_{\max} = \frac{V_{th}^2}{4R_{th}} = \frac{(5.5)^2}{4 \times 2.5}$$

$$= 3.025\text{W}$$

Hence, the correct answer is (3 to 3.1).

Question Number: 44 **Question Type: NAT**

The magnitude of magnetic flux density (B) in micro Teslas (μT), at the center of a loop of wire wound as a regular hexagon of side length 1 m carrying a current (I = 1 A), and placed in vacuum as shown in the figure is _____ (Give the answer up to two decimal places.)

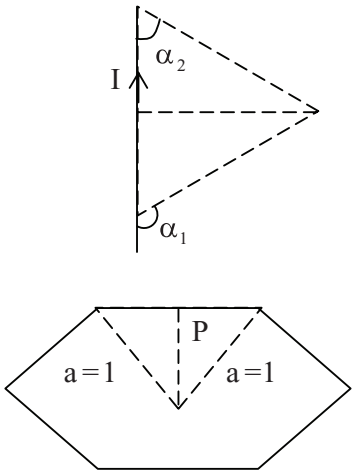


Solution: Magnetic field of finite length current carrying conductor is

$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\varphi$$

$$\alpha_2 = 60^o \quad \alpha_1 = 120^o$$

$$\rho = \sqrt{a^2 - \frac{a^2}{4}} \quad \text{where a = 1}$$



Then the magnetic field intensity at center due to one side of regular hexagon

$$H = \frac{1}{4\pi\left(\frac{\sqrt{3}}{2}\right)}\left[\cos 60^o - \cos 120^o\right]\hat{a}_\varphi$$

Due to Hexagon,

H_t = 6H = 0.551329 H/m

$$\bar{B} = \mu\bar{H}$$

$$= 4\pi \times 10^{-7} \times 0.551329$$

$$= 0.69\ \mu J$$

Hence, the correct answer is (0.65 – 0.75).

Question Number: 45 **Question Type: NAT**

A 375 W, 230 V, 50 Hz, capacitor start single-phase induction motor has the following constants for the main and auxiliary windings (at starting): Z_m = (12.50 +j 15.75) Ω (main winding), Z_a = (24.50 + j12.75) Ω (auxiliary winding). Neglecting the magnetizing branch, the value of the capacitance (in μF) to be added in series with the auxiliary winding to obtain maximum torque at starting is _____.

Ans: (98.871μF)*.

Question Number: 46 **Question Type: NAT**

Two parallel connected, three-phase, 50Hz, 11 kV. star-connected synchronous machines A and B. are operating as synchronous condensers. They together supply 50 MVAR to a 11 kV grid. Current supplied by both the machines are equal. Synchronous reactances of machine A and machine B are lΩ and 3Ω, respectively. Assuming the magnetic circuit to be linear, the ratio of excitation current of machine A to that of machine B is _____ (Give the answer up to two decimal places.)

Solution:

Given, Machine A: 11kV, 50Hz, Y-connected, X_S = 1Ω

Machine B: 11kV, 50Hz, Y-connected, X_S = 3Ω

Given both machines supplying only reactive power 50 $MVAR$ to 11kV grid equally

$$\therefore Q_A = \sqrt{3}\ V I_A \sin\phi = 25MVAR$$

$$Q_B = \sqrt{3}\ V I_B \sin\phi = 25MVAR$$

$$I_A = \frac{25\times10^6}{\sqrt{3}\times11\times10^3} = 1312.16A$$

$$\therefore I_A = I_B = 1312.16A$$

$$E_A = V - jI_AX_S = \frac{11\times10^3}{\sqrt{3}} - j\,1312.16\ \angle 90^\circ \times 1$$

$$E_B = V - jI_BX_S = \frac{11\times10^3}{\sqrt{3}} - j\,1312.16\ \angle 90^\circ \times 3$$

For linear magnetic circuit

$$E \propto I_f$$

$$\frac{E_A}{E_B} = \frac{I_{fA}}{I_{fB}}$$

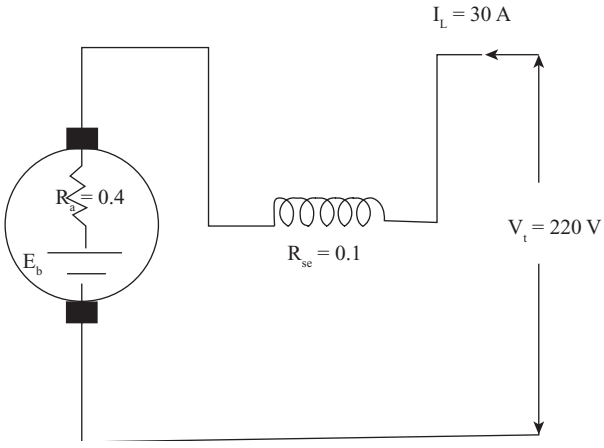
$$\therefore \frac{I_{fA}}{I_{fB}} = \frac{7662.973}{10287.333} = 0.7449 \cong 0.745$$

Hence, the correct answer is (0.745).

Question Number: 47 **Question Type: NAT**

A 220 V DC series motor runs drawing a current of 30 A from the supply. Armature and field circuit resistances are 0.4 Ωand 0.1 Ω, respectively. The load torque varies as the square of the speed. The flux in the motor may be taken as being proportional to the armature current. To reduce the speed of the motor by 50%, the resistance in ohms that should be added in series with the armature is _____ (Give the answer up to two decimal places.)

Solution:



Given: Load torque, $T_L \alpha$ (Speed)^2

$$T_{em} \alpha \phi I_a$$
$$\therefore \phi I_a \alpha N^2$$

Given $\phi \alpha I_a$

$$\therefore I^2_a \alpha N^2$$

$$\left(\frac{Ia_1}{Ia_2}\right)^2 = \left(\frac{N_1}{N_2}\right)^2 \Rightarrow \left(\frac{30}{I_{a_2}}\right)^2 = \left(\frac{N_1}{0.5N_1}\right)^2$$

$$I_{a_2} = 15A$$

$$E_{b1} = V_t - Ia_1 (R_a + R_{se}) = 220 - 30\,(0.4 + 0.1)$$
$$E_{b1} = 205V$$
$$E_{b2} = V_t - Ia_2 (R_a + R_{se} + R_{ext})$$
$$E_b \alpha \phi N$$
$$\frac{E_{b2}}{E_{b1}} = \frac{\phi_2 N_2}{\phi_1 N_1} = \frac{I_{a2} N_2}{I_{a1} N_1}$$

$$\frac{220 - 15(0.4 + 0.1 + R_{ext})}{205} = \frac{15 \times 0.5 N_1}{30 \times N_1}$$

$$R_{ext} = 10.75\Omega$$

Hence, the correct answer is (10.75Ω).

Question Number: 48 **Question Type: NAT**

A three-phase, three winding Δ/Δ/Y (1.1 kV/6.6 kV/400 V) transformer is energized from AC mains at the 1.1 kV side. It supplies 900 kVA load at 0,8 power factor lag from the 6.6 kV winding and 300 kVA load at 0,6 power factor lag from the 400 V winding. The RMS line current in ampere drawn by the 1.1 kV winding from the mains is _____ (Give the answer up to one decimal place.)

Solution: Load current on 6.6kV winding

$$I_2 = \frac{900 \times 10^3}{\sqrt{3} \times 6.6 \times 10^3} = 78.73 \angle -36.86^\circ A$$

Transformation ratio, $k = \frac{1.1}{6.6} = \frac{1}{6}$

$$I_2^1 = \frac{I_2}{k} = 472.38 \angle -36.86^\circ$$

Load current on 400V winding

$$I_3 = \frac{300 \times 10^3}{\sqrt{3} \times 400} = 433.613 \angle -53.13^\circ A$$

Transformation ratio

$$k = \frac{1.1 \times 10^3}{400/\sqrt{3}} = 4.763$$

$$I_3^1 = \frac{I_3}{k} = 90.91 \angle -53.13^\circ A$$

Line current = $\sqrt{3} \times I_3^1 = 157.46 \angle -53.13^\circ A$

$\therefore$ Current drawn by 1.1 kV winding $I_1 = I_2^1 + I_3^1$

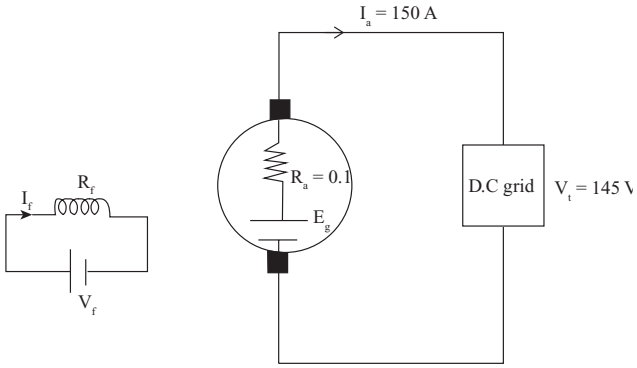
$I_1 = 472.38 \angle -36.86^\circ + 157.46 \angle -53.13^\circ = 625.093 \angle -40.91^\circ$

Hence, the correct answer is (625.09).

Question Number: 49 **Question Type: NAT**

A separately excited DC generator supplies 150 A to a 145 V DC grid. The generator is running at 800 RPM. The armature resistance of the generator is 0.1Ω. If the speed of the generator is increased to 1000 RPM. the current in amperes supplied by the generator to the DC grid is _____ (Give the answer up to one decimal place.)

Solution:



$$Eg_1 = V_t + I_{a1} R_a$$
$$Eg_1 = 145 + 150 \times 0.1$$
$$Eg_1 = 160V$$
$$Eg_2 = V_t + I_{a_2} R_a$$

$$Eg \alpha \phi \omega \Rightarrow \frac{Eg_2}{Eg_1} = \frac{N_2}{N_1} \quad (\because \phi = \text{constant})$$

$$\frac{145 + I_{a_2} \times 0.1}{160} = \frac{1000}{800}$$

$$I_{a_2} = 550A$$

Hence, the correct answer is (550A).

Question Number: 50 **Question Type: NAT**

For a system having transfer function $G(s) = \frac{-s+1}{s+1}$, a unit step input is applied at time $t = 0$. The value of the response of the system at t = 1.5 sec (rounded off to three decimal places) is _____.

Solution:

Given, transfer function, G(S) = $\frac{-S+1}{S+1}$

$$\frac{Y(S)}{U(S)} = \frac{-S+1}{S+1}$$

Given, input, $u(t)$ = unit step input

$$U(s) = \frac{1}{s}$$

$$\therefore\ Y(s) = \frac{1}{s}\left(\frac{-s+1}{s+1}\right)$$

$$Y(s) = \frac{A}{S} + \frac{B}{S+1}$$

$$A = 1;\ B = -2$$

$$Y(s) = \frac{1}{S} - \frac{2}{S+1}$$

Apply inverse Laplace transform

Response, $y(t) = 1 - 2e^{-t}$

$$y(t)|_{t=1.5} = 1 - 2\ e^{-1.5}$$

$$= 0.5537$$

Hence, the correct answer is (0.553 to 0.554).

Question Number: 51 **Question Type: NAT**

Consider a causal and stable LTI system with rational transfer function $H(z)$. whose corresponding impulse response begins at $n = 0$. Furthermore. $H(1) = \frac{5}{4}$ The poles of $H(z)$ are $p_k = \frac{1}{\sqrt{2}}\exp\left(j\frac{(2k-1)\pi}{4}\right)$ for $k = 1, 2, 3, 4$. The zeros of $H(z)$ are all at z = 0 Let $g[n] = j^n h[n]$. The value of $g[8]$ equals _____ (Give the answer up to three decimal places.)

Solution: Poles of $H(z)$ are at

$$P_K = \frac{1}{\sqrt{2}} e^{\left(\frac{(2K-1)\pi}{4}\right)}$$

$$P_1 = \frac{1}{\sqrt{2}} e^{\frac{j\pi}{4}} = 0.5 + \text{j}0.5$$

$$P_2 = \frac{1}{\sqrt{2}} e^{\frac{j3\pi}{4}} = -0.5 + j0.5$$

$$P_3 = \frac{1}{\sqrt{2}} e^{\frac{j5\pi}{4}} = -0.5 - j0.5$$

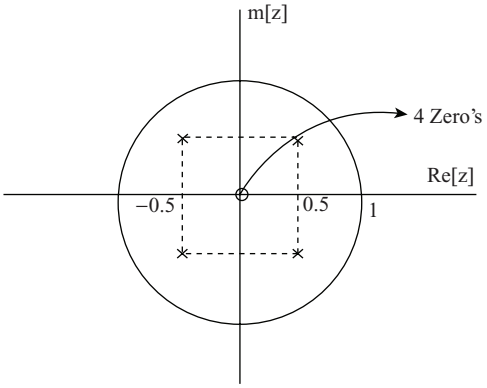
$$P_4 = \frac{1}{\sqrt{2}} e^{\frac{j7\pi}{4}} = 0.5 - j0.5$$

$$H(z) = \frac{KP(z)}{Q(z)}$$

$$H(z) = \frac{kZ^4}{(z-0.5-j0.5)(z-0.5+j0.5)(z+0.5-j0.5)(z+0.5+j0.5)}$$

$$H(z) = \frac{kz^4}{z^4 + 0.25}$$

Given that the system is causal, so, power of numerator and denominator must be same.



Given $H(1) = \frac{5}{4}$

$$\Rightarrow \frac{5}{4} = \frac{K}{1.25}$$

$$\Rightarrow K = 1.5625$$

$$H(z) = \frac{1.56252z^4}{z^4 + 0.25}$$

To get $h[n]$, we will use the scaling property of z^4-transform

$$X[n] \leftrightarrow x(z)$$

$$x\left[\frac{n}{k}\right] \leftrightarrow x(z^K)$$

$$-(0.25)^n\, u[\text{n}] \leftrightarrow \frac{\quad}{0.25}$$

$$-(0.25)^{\frac{n}{4}} u\left[\frac{n}{4}\right] \leftrightarrow \frac{z^4}{z^4 + 0.25}$$

$$g[n] = (j)^n\left[-(0.25)^{\frac{n}{4}} u\left[\frac{n}{4}\right]\right]$$

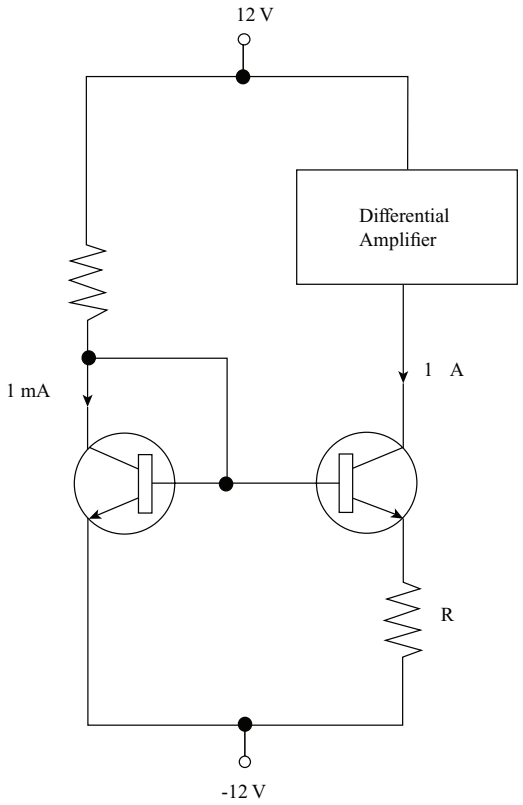
$$g[8] = 0.0625 \times 1.5625$$

$$= 0.09766$$

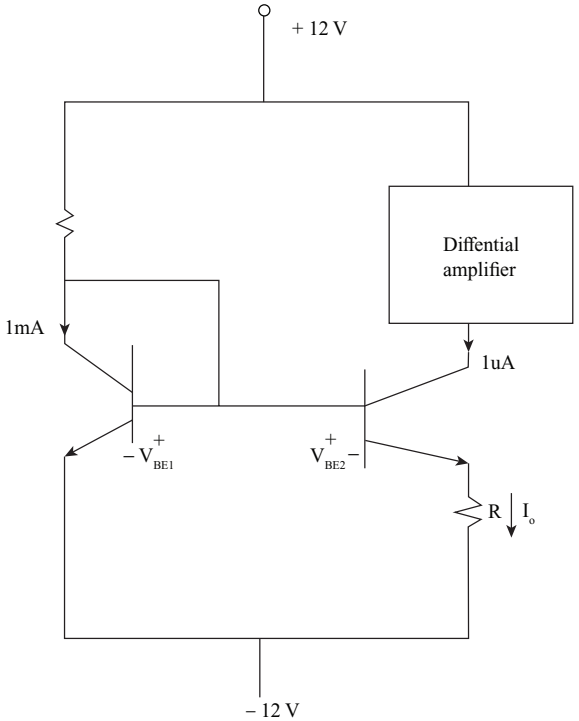
Hence, the correct answer is (0.09 to 0.1).

Question Number: 52 **Question Type: NAT**

The circuit shown in the figure uses matched transistors with a thermal voltage V_T = 25 mV. The base currents of the transistors are negligible. The value of the resistance R in $k\Omega$. that is required to provide 1 mA bias current for the differential amplifier block shown is _____ (Give the answer up to one decimal place.)



Solution:



$$V_{BE1} = V_{BE2} + I_0R$$

$$I_0R = V_{BE1} - V_{BE2}$$

$$= V_T \ln\left[\frac{I_R}{I_3}\right] - V_T \ln\left[\frac{I_0}{I_3}\right]$$

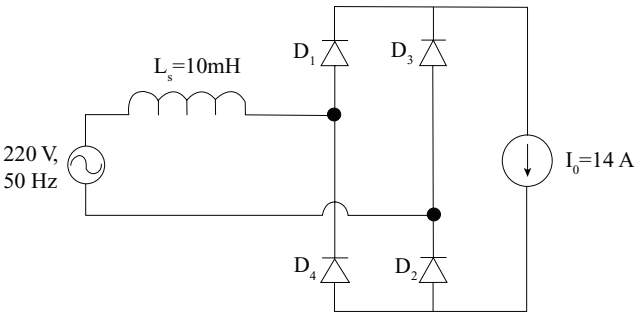
Where $I_S \rightarrow$ reverse saturation current

$$R = \frac{V_T \ln\left[\frac{I_R}{I_0}\right]}{I_0}$$

$$R = \frac{0.025\ln(10^3)}{1\mu A} = 172.69\ K\Omega$$

Question Number: 53 **Question Type: NAT**

The figure below shows an uncontrolled diode bridge rectifier supplied from a 220 V, 50 Hz. 1-phase ac source. The load draws a constant current I_0 = 14 A , The conduction angle of the diode D_1 in degrees (rounded off to two decimal places) is _____.



Solution:

As the source inductance is given, the overlap angle μ comes into picture.

For thyristor controlled bridge rectifier with source inductance,

$$\cos(\alpha + \mu) = \cos\alpha - \frac{2\omega L_S}{V_m} I_0$$

But instead of SCR, diode is given. So $\alpha = 0°$

$$\cos\mu = 1 - \frac{2\omega L_S}{V_m} I_0$$

$$\cos\mu = 1 - \frac{2\times 2\pi\times 50\times 10\times 10^{-3}}{220\sqrt{2}} \times 14\,.$$

$$= 0.7172$$

$$\mu = 44.17°$$

Due to the overlap angle, the diode D_1 conducts for (180+μ)

$\therefore$ Conduction angle of D_1 = 180 + 44.17 = 224.17°

Hence, the correct answer is (224.17°).

Question Number: 54 **Question Type: NAT**

The positive, negative, and zero sequence reactance of a wye-cormected synchronous generator are 0.2 pu. 0.2 pu. and 0.1 pu. respectively. The generator is on open circuit with a terminal voltage of 1 pu. The minimum value of the inductive reactance, in pu. required to be connected between neutral and ground so that the fault current does not exceed

3.75 pu if a single line to ground fault occurs at the terminals is _____ (assume fault impedance to be zero). (Give the answer up to one decimal place.)

Solution: $Z_1 = 0.2, Z_2 = 0.2, Z_0 = 0.1$

$V = 1\ pu \quad I_{fault} = 03.75$

$$I_f = \frac{3V}{Z_1 + Z_2 + Z_0 + 3Z_n + 3Z_f} \text{ but } Z_f = 0$$

$$I_f = \frac{3V}{Z_1 + Z_2 + Z_0 + 3Z_n}$$

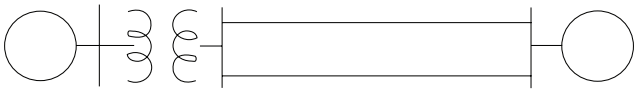
$$\frac{3 \times 1}{0.2 + 0.2 + 0.1 + 3Z_n} = 3.75$$

$$Z_n = 0.1\ pu$$

Hence, the correct answer is (0.1 to 0.1).

Question Number: 55 **Question Type: NAT**

The figure shows the single line diagram of a power system with a double circuit transmission line. The expression for electrical power is 1.5 sin δ. where δ is the rotor angle. The system is operating at the stable equilibrium point with mechanical power equal to 1 pu. If one of the transmission line circuits is removed, the maximum value of δ, as the rotor swings, is 1.221 radian. If the expression for electrical power with one transmission line circuit removed is P_{max} sin δ, the value of P_{max}. in pu is _____. (Give the answer up to three decimal places.)



Solution: $Pe_1 = Pmax_1 \sin \delta_0$

$Ps = 1Pu$

Initially $Pa = 0 = Ps = Pe_1$

$1 = 1.5 \sin\delta$

$$\delta_0 = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8103^0$$

$$Pe_2 = P_{max} \sin \delta_{max};\ \delta_{max} = 1.22^r$$

$$P_{max} = 1.22 \times \frac{180}{\pi} = 69.90085°$$

$$P_{max} = \frac{EV}{X_{2eq}}$$

$$P_{max1} = \frac{Ps(\delta_{max} - \delta_0)}{\cos\delta_0 - \cos\delta_{max}}$$

$$= \frac{(69.90085° - 41.8103^0) \times \frac{\pi}{180}}{\cos 41.8103^0 - \cos 69.90085^0}$$

$$= 1.22$$

Hence, the correct answer is (1.20 to1.24).

General Aptitude

Number of Questions: 10 **Section Marks: 15**

Q.56 to Q.60 carry 1 mark each and Q.61 to Q.65 carry 2 marks each.

Question Number: 56 **Question Type: MCQ**

After Rajendra Chola returned from his voyage to Indonesia, he _____ to visit the temple in Thanjavur.

(A) was wishing (B) is wishing
(C) wished (D) had wished

Ans: (C)

Question Number: 57 **Question Type: MCQ**

Research in the workplace reveals that people work for many reasons _____.

(A) money beside (B) beside money
(C) money besides (D) besides money

Ans: (D)

Question Number: 58 **Question Type: MCQ**

Rahul, Murali, Srinivas and Arul are seated around a square table. Rahul is sitting to the left of Murali. Srinivas is sitting to the right of Arul. Which of the following pairs are seated opposite each other'?

(A) Rahul and Murali (B) Srinivas and Arul
(C) Srinivas and Murali (D) Srinivas and Rahul

Ans: (C)

Question Number: 59 **Question Type: MCQ**

Find the smallest number y such that $y \times 162$ is a perfect cube.

(A) 24 (B) 27
(C) 32 (D) 36

Ans: (D)

Question Number: 60 **Question Type: MCQ**

The probability that a k-digit number does NOT contain the digits 0, 5, or 9 is

(A) 0.3^k (B) 0.6^k
(C) 0.7^k (D) 0.9^k

Ans: (C)

Question Number: 61 **Question Type: MCQ**

"The hold of the nationalist imagination on our colonial past is such that anything inadequately or improperly nationalist is just not history."

Which of the following statements best reflects the author's opinion?

(A) Nationalists are highly imaginative.
(B) History is viewed through the filter of nationalism.
(C) Our colonial past never happened.
(D) Nationalism has to be both adequately and properly imagined.

Ans: (B)

Question Number: 62 **Question Type: MCQ**

Six people are seated around a circular table. There are at least two men and two women. There are at least three right-handed persons. Every woman has a left-handed person to her immediate right. None of the women are right-handed. The number of women at the table is

(A) 2
(B) 3
(C) 4
(D) Cannot be determined

Ans: (A)

Question Number: 63 **Question Type: MCQ**

The expression $\frac{(x+y)-|x-y|}{2}$ is equal to

(A) the maximum of x and y
(B) the minimum of x and y
(C) 1
(D) none of the above

Ans: (B)

Question Number: 64 **Question Type: MCQ**

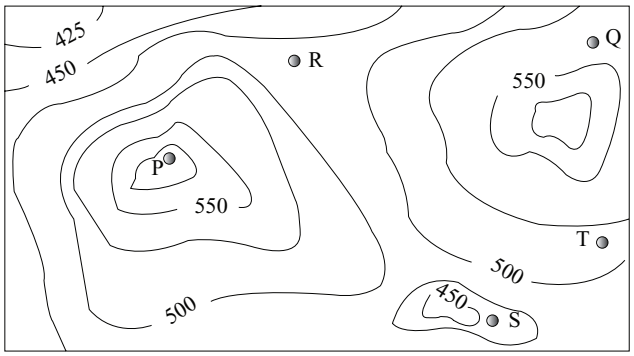
Arun, Gulab, Neel and Shweta must choose one shirt each from a pile of four shirts coloured red, pink, blue and white respectively. Arun dislikes the colour red and Shweta dislikes the colour white. Gulab and Neel like all the colours. In how many different ways can they choose the shirts so that no one has a shirt with a colour he or she dislikes?

(A) 21 (B) 18
(C) 16 (D) 14

Ans: (D)

Question Number: 65 **Question Type: MCQ**

A contour line joins locations having the same height above the mean sea level. The following is a contour plot of a geographical region. Contour lines are shown at 25 m intervals in this plot. If in a flood, the water level rises To 525 m, which of the villages P, Q, R, S, T get submerged?



(A) P, Q (B) P, Q, T
(C) R, S, T (D) Q, R, S

Ans: (C)

GATE 2017 Solved Paper
Electrical Engineering
Set – 2

Number of Questions: 65 **Total Marks:100.0**

Wrong answer for MCQ will result in negative marks, (–1/3) for 1 Mark Questions and (–2/3) for 2 Marks Question.

Electrical Engineering

Number of Questions: 55 **Section Marks: 85**

Q.1 to Q.25 carry 1 mark each and Q.26 to Q.55 carry 2 marks each.

Question Number: 1 **Question Type: MCQ**

An urn contains 5 red balls and 5 black balls. In the first draw, one ball is picked at random and discarded without noticing its colour. The probability to get a red ball in the second draw is

(A) $\frac{1}{2}$ (B) $\frac{4}{9}$

(C) $\frac{5}{9}$ (D) $\frac{6}{9}$

Solution: Number of red balls = 5

Number of black balls = 5

Case (i): First drawn ball is red.

Probability of getting a red ball in the first draw = $\frac{5}{10}=\frac{1}{2}$

Probability of getting a red ball from the remaining balls in the second draw = $\frac{4}{9}$

$\therefore$ Probability of getting a red ball in the second draw in this case = $\frac{1}{2}\times\frac{4}{9}=\frac{2}{9}$ (1)

Case (ii): First drawn ball is black.

Probability of getting a black ball in the first draw = $\frac{5}{10}=\frac{1}{2}$

Probability of getting a red ball from the remaining balls in the second draw = $\frac{5}{9}$

$\therefore$ Probability of getting a red ball in the second draw in this case = $\frac{1}{2}\times\frac{5}{9}=\frac{5}{18}$ (2)

$\therefore$ From (1) and (2),

the required probability $=\frac{2}{9}+\frac{5}{18}$

$=\frac{9}{18}=\frac{1}{2}$

Hence, the correct option is (A).

Question Number: 2 **Question Type: MCQ**

Consider a solid sphere of radius 5 cm made of a perfect electric conductor. If one million electrons are added to this sphere, these electrons will be distributed

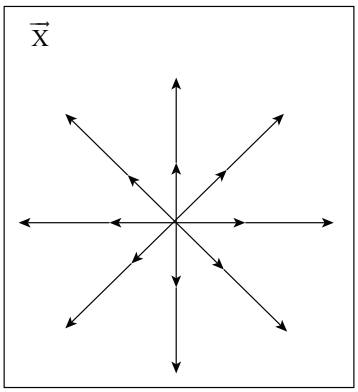
(A) uniformly over the entire volume of the sphere
(B) uniformly over the outer surface of the sphere
(C) concentrated around the centre of the sphere
(D) along a straight line passing through the centre of the sphere

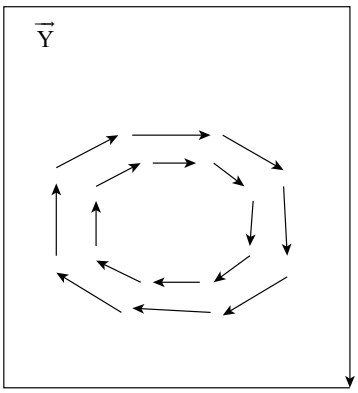
Solution: Static charge resides only on the surface of a conductor. Therefore the electrons will be uniformly over the entire surface of the sphere.

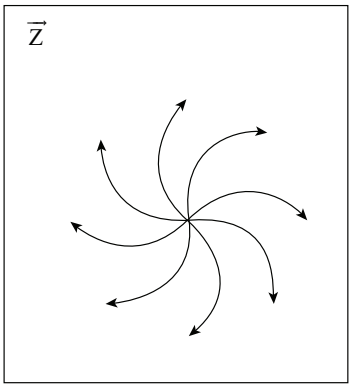
Hence, the correct option is (B).

Question Number: 3 **Question Type: MCQ**

The figures show diagrammatic representations of vector fields $\vec{X}$, $\vec{Y}$, and $\vec{Z}$, respectively. Which one of the following choices is true?







(A) $\nabla.\vec{X}=0, \nabla\times\vec{Y}\neq 0, \nabla\times\vec{Z}=0$

(B) $\nabla.\vec{X}\neq 0, \nabla\times\vec{Y}=0, \nabla\times\vec{Z}\neq 0$

(C) $\nabla.\vec{X}\neq 0, \nabla\times\vec{Y}\neq 0, \nabla\times\vec{Z}\neq 0$

(D) $\nabla.\vec{X}=0, \nabla\times\vec{Y}=0, \nabla\times\vec{Z}=0$

Solution: $\vec{x}$ is radial and irrational and hence $\nabla.\vec{x}\neq 0$

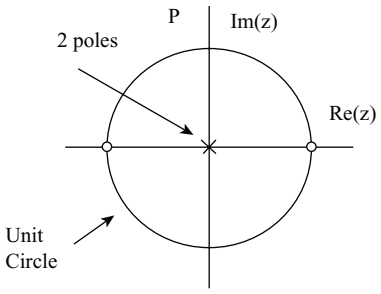
while $\vec{y}$ is rotational and its curl is non zero $(\nabla\times y\neq 0)$

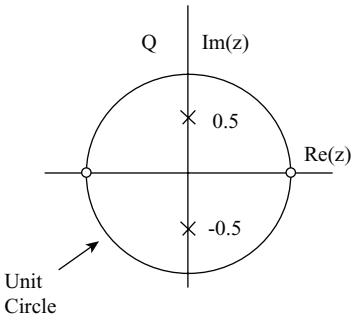
The field $\vec{z}$ is also rotational and hence $\nabla\times\bar{z}\neq 0$

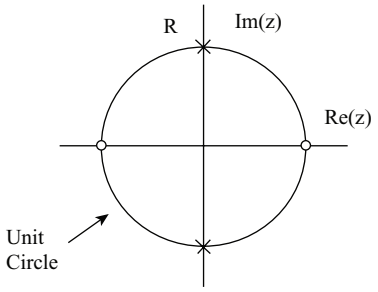
Hence, the correct option is (C).

Question Number: 4 **Question Type: MCQ**

The pole-zero plots of three discrete-time systems **P**. **Q** and **R** on the *z*-plane are shown below.







Which one of the following is TRUE about the frequency selectivity of these systems?

(A) All three are high-pass filters.
(B) All three are band-pass filters.
(C) All three are low-pass filters.
(D) **P** is a low-pass filter. **Q** is a band-pass filter and **R** is a high-pass filter.

Solution: From the pole-size plots the transfer function can be written as

$$P\rightarrow H(z)=\frac{k(z^2-1)}{z^2}$$
$$H(1)=0$$
$$H(j)=2k$$
$$H(-1)=0$$
$$Q\rightarrow H(z)=\frac{k(z^2-1)}{z^2-0.25}$$
$$H(1)=0$$
$$H(j)=1.6k$$
$$H(-1)=0$$
$$R\rightarrow H(z)=\frac{k(z^2-1)}{z^2+1}$$
$$H(1)=0$$
$$H(j)=\infty$$
$$H(-1)=0$$

The given pole-zero plots correspond to a band-pass filter.

Hence, the correct option is (B).

Question Number: 5 **Question Type: MCQ**

If a synchronous motor is running at a leading power factor, its excitation induced voltage (E_f) is

(A) equal to terminal voltage V_t
(B) higher than the terminal voltage V_t
(C) less than terminal voltage V_t
(D) dependent upon supply voltage V_t

Solution: Synchronous motor is running at a leading power *P.F* is over excited, i.e., $E_f > V_t$

Hence, the correct option is (B).

Question Number: 6

When a unit ramp input is applied to the unity feedback system having closed loop transfer function $\frac{C(s)}{R(s)}=\frac{Ks+b}{s^2+as+b}$, $(a>0, b>0, K>0)$, the steady state error will be

(A) 0 (B) $\frac{a}{b}$

(C) $\frac{a+K}{b}$ (D) $\frac{a-K}{b}$

Solution: $\frac{C(S)}{R(S)}=\frac{ks+b}{s^2+as+b}(a>0,b>0,k>0)$

$$e_{ss}\text{ (for unit ramp input)}=\left(\frac{1}{k_v}\right)$$

$$K_v=\lim_{S\to 0}\ SG\big((S)H(S)\big)$$

$$G(S)H(S)=OLTF=\frac{ks+b}{S^2+S(a-k)}$$

$$k_v=\underset{S\to 0}{Lt}\not{s}\cdot\frac{ks+b}{\not{s}\left[s+(a-k)\right]}=\left(\frac{b}{a-k}\right)$$

$$e_{ss}=\left(\frac{a-k}{b}\right)$$

Hence, the correct option is (D).

Question Number: 7 **Question Type: MCQ**

The transfer function $C(s)$ of a compensator is given below.

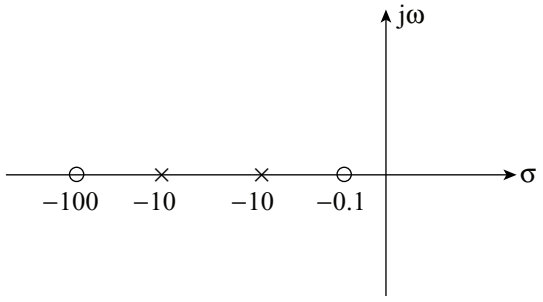
$$C(s)=\frac{\left(1+\frac{s}{0.1}\right)\left(1+\frac{s}{100}\right)}{(1+s)\left(1+\frac{s}{10}\right)}$$

The frequency range in which the phase (lead) introduced by the compensator reaches the maximum is

(A) $0.1<\omega<1$ (B) $1<\omega<10$

(C) $10<\omega<100$ (D) $\omega>100$

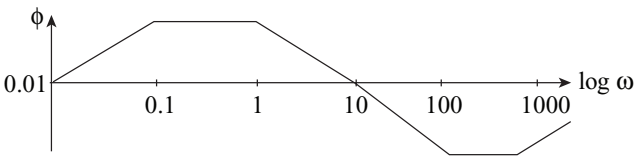
Solution: Pole-zero configuration of the given compensator is



∴ The given compensator is lead lag compensator

(or) Band Pass Filter (BPF).

The approximate bode phase plot is as shown below



∴ Maximum phase angle will occurs in between 0.1 to 1.

Hence, the correct option is (A).

Question Number: 8 **Question Type: MCQ**

Two resistors with nominal resistance values R_1 and R_2 have additive uncertainties ΔR_1 and ΔR_2, respectively. When these resistances are connected in parallel, the standard deviation of the error in the equivalent resistance R is

(A) $\pm\sqrt{\left\{\frac{\partial R}{\partial R_1}\Delta R_1\right\}^2+\left\{\frac{\partial R}{\partial R_2}\Delta R_2\right\}^2}$

(B) $\pm\sqrt{\left\{\frac{\partial R}{\partial R_2}\Delta R_1\right\}^2+\left\{\frac{\partial R}{\partial R_1}\Delta R_2\right\}^2}$

(C) $\pm\sqrt{\left\{\frac{\partial R}{\partial R_1}\right\}^2\Delta R_2+\left\{\frac{\partial R}{\partial R_2}\right\}^2\Delta R_1}$

(D) $\pm\sqrt{\left\{\frac{\partial R}{\partial R_1}\right\}^2\Delta R_1+\left\{\frac{\partial R}{\partial R_2}\right\}^2\Delta R_2}$

Solution: Standard deviation,

$$\sigma_{\text{Resistance}}=\sqrt{\left\{\frac{\partial R}{\partial R_1}\right\}^2\sigma_1^2+\left\{\frac{\partial R}{\partial R_2}\right\}^2\sigma_2^2}$$

$$=\sqrt{\left\{\frac{\partial R}{\partial R_1}\right\}^2\Delta R_1^2+\left\{\frac{\partial R}{\partial R_2}\right\}^2\Delta R_2^2}$$

Hence, the correct option is (A).

Question Number: 9 **Question Type: MCQ**

A stationary closed Lissajous pattern on an oscilloscope has 3 horizontal tangencies and 2 vertical tangencies for a horizontal input with frequency 3 kHz. The frequency of the vertical input is

(A) 1.5 kHz (B) 2 kHz

(C) 3 kHz (D) 4.5 kHz

Solution: Given horizontal input frequency $f_x=3\ KHz$

$$\frac{f_y}{f_x}=\frac{\text{Horizontal tangencies}}{\text{Vertical tangencies}}$$

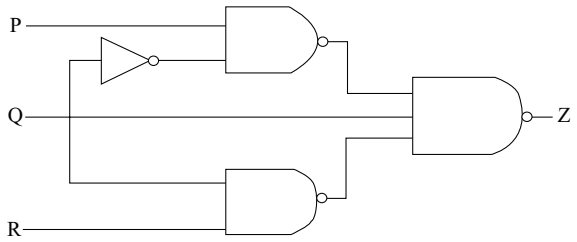
$$\frac{f_y}{3KHz}=\frac{3}{2}$$

$\Rightarrow$ Vertical input frequency, $f_y=4.5\ KHz$

Hence, the correct option is (D).

Question Number: 10 **Question Type: MCQ**

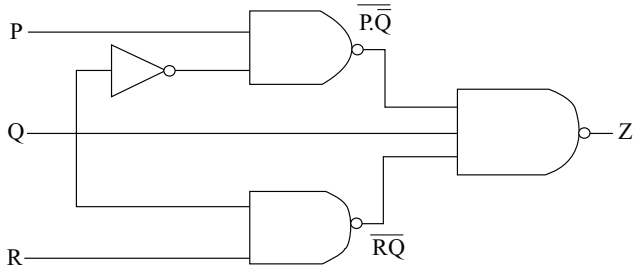
For a 3-input logic circuit shown below, the output Z can be expressed as



(A) $Q + \overline{R}$ (B) $P\overline{Q} + R$

(C) $\overline{Q} + R$ (D) $P + \overline{Q} + R$

Solution:



$$Z = \overline{P\overline{Q}} \cdot Q \cdot \overline{R\overline{Q}}$$

$$Z = P\overline{Q} + \overline{Q} + RQ$$

$$Z = \overline{Q} + R$$

Hence, the correct option is (C).

Question Number: 11 **Question Type: MCQ**

A phase-controlled, single-phase, full-bridge converter is supplying a highly inductive DC load. The converter is fed from a 230 V, 50 Hz, AC source. The fundamental frequency in Hz of the voltage ripple on the DC side is

(A) 25 (B) 50

(C) 100 (D) 300

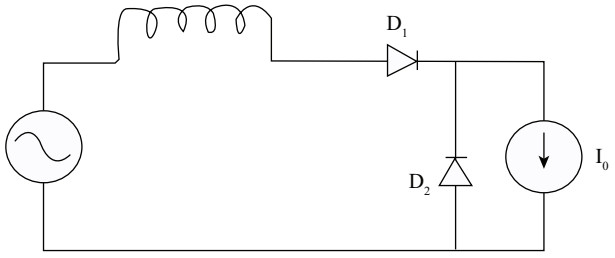
Solution: For η pulse rectifier output frequency $f_0 = \eta f_s$ full bridge rectifier is a 2 pulse converter.

So output frequency $f_0 = 2 \times 50 = 100\ Hz$

Hence, the correct option is (C).

Question Number: 12 **Question Type: MCQ**

In the circuit shown, the diodes are ideal, the inductance is small and $I_0 \neq 0$. Which one of the following statements is true?



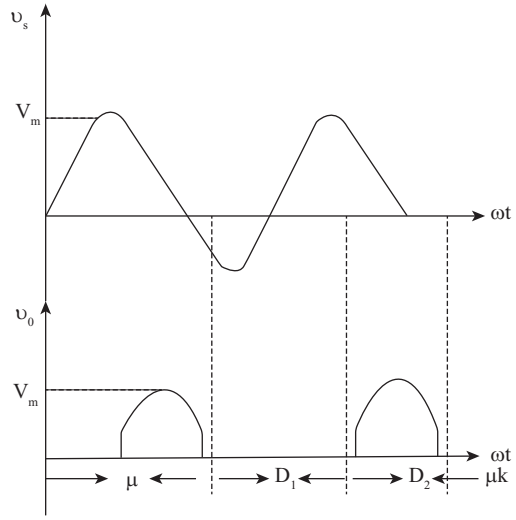
(A) D_1 conducts for greater than 180° and D_2 conducts for greater than 180°.

(B) D_2 conducts for more than 180° and D_1 conducts for 180°.

(C) D_1 conducts for 180° and D_2 conducts for 180°.

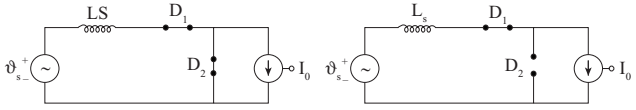
(D) D_1 conducts for more than 180° and D_2 conducts for 180°.

Solution:



$\rightarrow D_1$ conducts for 180° and D_2 conducts for 180° + μ.

During 0 to μ (or) 2π to 2π + μ during μ to π



Hence, the correct option is (A).

Question Number: 13 **Question Type: MCQ**

A three-phase voltage source inverter with ideal devices operating in 180° conduction mode is feeding a balanced star-connected resistive load. The DC voltage input is V_{dc}. The peak of the fundamental component of the phase voltage is

(A) $\frac{V_{dc}}{\pi}$ (B) $\frac{2V_{dc}}{\pi}$

(C) $\frac{3V_{dc}}{\pi}$ (D) $\frac{4V_{dc}}{\pi}$

Solution: The peak value of fundamental phase voltager $= \frac{2V_{dc}}{\pi}$.

Hence, the correct option is (B).

Question Number: 14 **Question Type: MCQ**

A 3-phase, 4-pole. 400 V. 50 Hz squirrel-cage induction motor is operating at a slip of 0.02. The speed of the rotor

flux in mechanical rad sec. sensed by a stationary observer, is closest to

(A) 1500 (B) 1470
(C) 157 (D) 154

Solution: Speed of the rotor flux w.r.t. stationary observe

$$= N_s = \frac{120f}{P}$$

$$= \frac{120 \times 50}{4} = 1500 \text{ rpm}$$

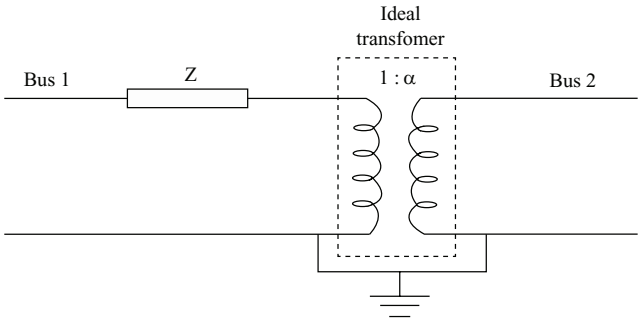
$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 1500}{60} = 157.08 \text{ rad/sec}$$

Hence, the correct option is (C).

Question Number: 15 **Question Type: MCQ**

The figure shows the per-phase representation of a phase-shifting transformer connected between buses 1 and 2, where α is a complex number with non-zero real and imaginary parts.



For the given circuit, Y_{bus} and Z_{bus} are bus admittance matrix and bus impedance matrix, respectively, each of size 2×2. Which one of the following statements is true?

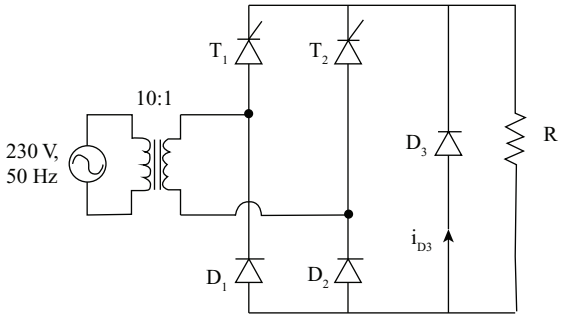
(A) Both Y_{bus} and Z_{bus} are symmetric
(B) Y_{bus} is symmetric and Z_{bus} is unsymmetric
(C) Y_{bus} is unsymmetric and Z_{bus} is symmetric
(D) Y_{bus} and Z_{bus} are unsymmetric

Solution: Both Y_{bus} and Z_{bus} are unsymmetric.

Hence, the correct option is (D).

Question Number: 16 **Question Type: NAT**

The figure below shows the circuit diagram of a controlled rectifier supplied from a 230 V, 50 Hz. 1-phase voltage sources and a 10:1 ideal transformer. Assume that all devices are ideal. The firing angles of the thyristors T_1 and T_2 are 90° and 270°, respectively.



The RMS value of the current through diode D_3 in amperes is _____

Solution: The load is purely resistive load. So the free wheeling diode D_3 never conducts. So current though the free wheeling diode D_3 is zero.

Hence, the correct answer is 0 (0 to 0).

Question Number: 17 **Question Type: NAT**

Assume that in a traffic junction the cycle of the traffic signal lights is 2 minutes of green (vehicle does not stop) and 3 minutes of red (vehicle stops). Consider that the arrival time of vehicles at the junction is uniformly distributed over 5 minute cycle. The expected waiting time (in minutes) for the vehicle at the junction is_____

Solution: Let a random variable X denotes the arrival time of a vehicle at the junction and let $f(x)$ be its probability density function.

As X is uniformly distributed over 5 minutes cycle, we have

$$f(x) = \frac{1}{5}; \quad 0 \le x \le 5$$

Also the waiting time of a vehicle at the junction depends on its arrival time.

Let $g(X)$ be the random variable denoting the waiting time of a vehicle.

$$\therefore g(X) = \begin{cases} 0, & 0 \le X \le 2 \text{ (green light)} \\ 5 - X, & 2 \le X \le 5 \text{ (red light)} \end{cases}$$

$\therefore$ Expected waiting time of a vehicle at the junction

$$= E(g(X)) = \int_0^5 g(x)\ f(x)dx$$

$$= \int_0^2 0 \times \frac{1}{5} dx + \int_2^5 (5 - x) \times \frac{1}{5} dx$$

$$= \frac{1}{5} \int_2^5 (5 - x)\, dx$$

$$= \frac{1}{5}\left[5x - \frac{x^2}{2}\right]_2^5$$

$$= \frac{1}{5} \times \frac{9}{2}$$

$$= \frac{9}{10} = 0.9$$

Hence, the correct answer is (0.9 to 0.9).

Question Number: 18 **Question Type: NAT**

Consider a function $f(x, y, z)$ given by

$$f(x, y, z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$$

The partial derivative of this function with respect to x at the point, $x = 2$, $y = 1$ and $z = 3$ is _____.

Solution: Given $f(x, y, z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$

$$\frac{\partial f}{\partial x} = zx(y^2 + z^2)$$

$$\therefore \frac{\partial f}{\partial x}\text{ at}(2,1,3) = 2 \times 2(1^2 + 3^2) = 40$$

Hence, the correct answer is (40 to 40).

Question Number: 19 **Question Type: NAT**

Let x and y be integers satisfying the following equations

$$2x^2 + y^2 = 34$$

$$x + 2y = 11$$

The value of $(x + y)$ is _____.

Solution: Given x and y are integers such that

$$2x^2 + y^2 = 34 \quad (1)$$

and

$$x + 2y = 11 \quad (2)$$

$$\Rightarrow x = 11 - 2y \quad (3)$$

From (1),

$$2(11 - 2y)^2 + y^2 = 34$$

$$\Rightarrow 2(121 - 44y + 4y^2) + y^2 = 34$$

$$\Rightarrow 9y^2 - 88y + 208 = 0$$

$$\Rightarrow (y - 4)(9y - 52) = 0$$

$$\Rightarrow y = 4; y = \frac{52}{9} \text{ (Not an integer)}$$

$$\therefore y = 4$$

From (3), $x = 11 - 2 \times 4$

$\Rightarrow x = 3$

$\therefore$ The value of $x + y$ is $x + y = 3 + 4 = 7$

Hence, the correct answer is (7 to 7).

Question Number: 20 **Question Type: NAT**

Let $y^2 - 2y + 1 = x$ and $\sqrt{x} + y = 5$. The value of $x + \sqrt{y}$ equals _____. (Give the answer up to three decimal places)

Solution: Given $y^2 - 2y + 1 = x$ (1)

and $\sqrt{x} + y = 5$ (2)

$$\Rightarrow \sqrt{y^2 - 2y + 1} + y = 5 \quad \text{(From 1)}$$

$$\Rightarrow \sqrt{(y-1)^2} + y = 5$$

$$\Rightarrow (y - 1) + y = 5$$

$$\Rightarrow y = 3$$

From (1), $3^2 - 2 \times 3 + 1 = x$

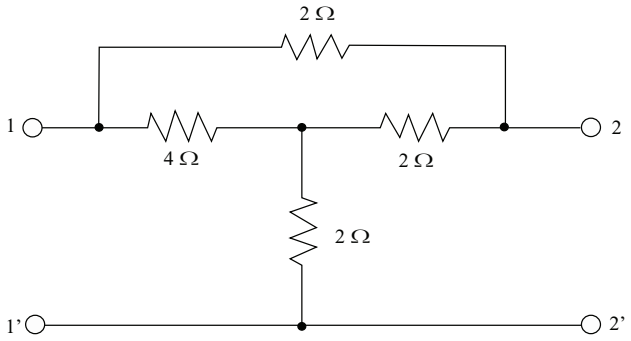
$\Rightarrow x = 4$

Now, $x + \sqrt{y} = 4 + \sqrt{3} = 4 + 1.732 = 5.732$

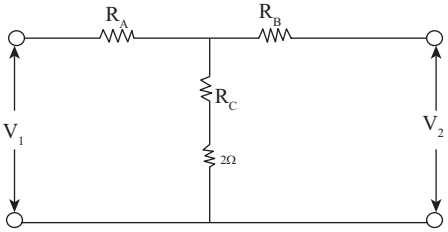
Hence, the correct answer is (5.7 to 5.8).

Question Number: 21 **Question Type: NAT**

For the given 2-port network, the value of transfer impedance Z_{21} in ohms is _____.



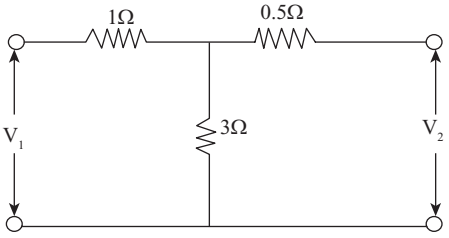
Solution: Convert $\Delta - Y$:-



$$R_A = \frac{2 \times 4}{8} = 1\Omega$$

$$R_B = \frac{2 \times 2}{8} = 0.5\Omega$$

$$R_C = \frac{2 \times 4}{8} = 1\Omega$$

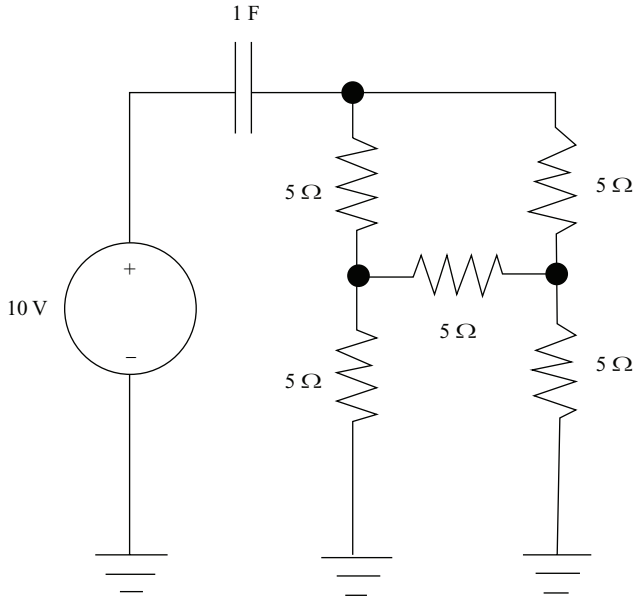


$Z_{21} = Z_c = 3\ \Omega$

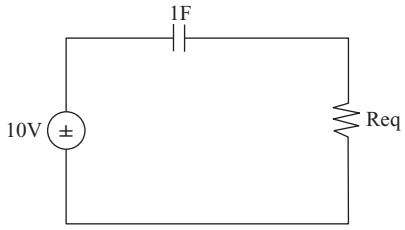
Hence, the correct answer is (3 to 3).

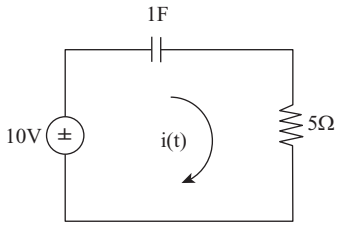
Question Number: 22 **Question Type: NAT**

The initial charge in the 1F capacitor present in the circuit shown is zero. The energy in joules transferred from the DC source until steady state condition is reached equals _____. (Give the answer up to one decimal place)



Solution: Simplify the given network





$R_{eq} = (10 \,||10) = 5\Omega$

(bridge network)

$\tau = RC = 5$ sec

$V_C(t) = 10\,[1 - e^{-0.2t}]$ volts

$V_C(\infty) = 10V$

The energy transferred from *D.C* source until steady state $= 2.Wc$

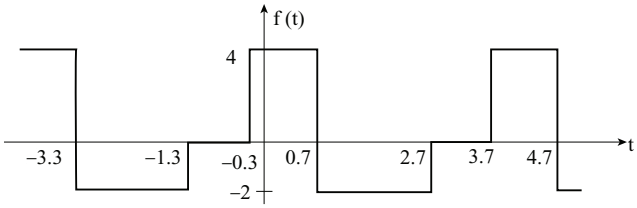
$$= 2 \times \frac{1}{2} \times CV^2$$

$$= 1 \times (10)^2 = 100V$$

Hence, the correct answer is (99 to 101).

Question Number: 23 **Question Type: NAT**

The mean square value of the given periodic waveform $f(t)$ is _____.



Solution: Mean square value is given by $\frac{1}{T}\int_{to}^{to+T} x^2(t)\,dt$

Here $T = H$

Selecting from – 0.3 to 3.7

Mean square value

$$= \frac{1}{4}\left[\int_{-0.3}^{0.7}\int 16dt + \int_{0.7}^{2.7} 4dt\right]$$

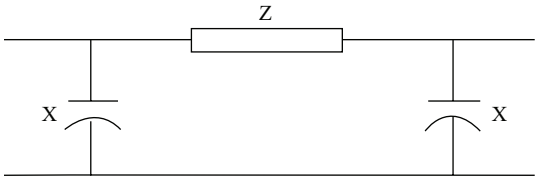
$$= \frac{1}{4}(16 + 4\,(2)) = \frac{16+8}{4}$$

$$= \frac{24}{4} = 6$$

Hence, the correct answer is (6 to 6).

Question Number: 24 **Question Type: NAT**

The nominal π– circuit of a transmission line is shown in the figure.



Impedance $Z = 100\angle 80°\ \Omega$ and reactance $X = 3300\ \Omega$. The magnitude of the characteristic impedance of the transmission line, in Ω, is _____. (Give the answer up to one decimal place)

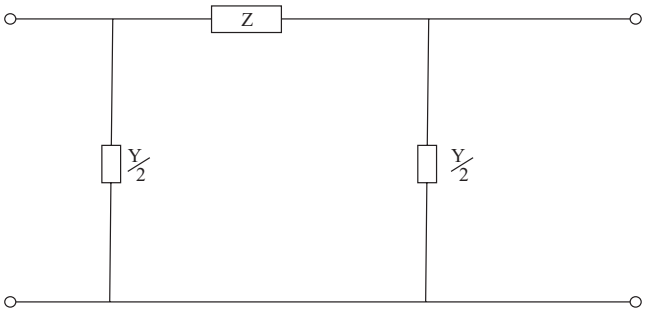
Solution:

$Z_c = \sqrt{Z_{oc} \cdot Z_{sc}} = \sqrt{A/C \cdot B/D}$ ($A = D \because$ practical transmission line is symmetrical as well as reciprocal.)

$$Z_c = \sqrt{\frac{B}{C}}$$

for nominal π model $B = Z$

$$c = Y\left[1 + \frac{ZY}{4}\right]$$



$$Y/2 = \frac{1}{X}$$

$$Y = \frac{2}{X} = \frac{1}{1650}(\mho)$$

$$C = \frac{j}{1650}\left[1 + \frac{100\angle 180^\circ}{4}\left(\frac{j}{1650}\right)\right] \quad [\because Y = jwc]$$

$$Z_c^2 = 167498.715\angle -10.153^2$$

$$|Z_c| = 409(\Omega)$$

Hence, the correct answer is (404 to 409).

Question Number: 25 **Question Type: NAT**

In a load flow problem solved by Newton-Raphson method with polar coordinates, the size of the Jacobian is 100×100, If there are 20 PV buses in addition to PQ buses and a slack bus. The total number of buses in the system is _____.

Solution:

$[J]_{(2n-m-2)\times(2n-m-2)}$

$2n - m - 2 = 100$

m = number of PV buses = 20

$2n - 20 - 2 = 100$

$n = 61$

Hence, the correct answer is (61 to 61).

Question Number: 26 **Question Type: MCQ**

Let $g(x) = \begin{cases} -x, & x \le 1 \\ x+1, & x \ge 1 \end{cases}$ and $f(x) = \begin{cases} 1-x, & x \le 0 \\ x^2, & x > 0 \end{cases}$.

Consider the composition of f and g, i.e., $(f \circ g)(x) = f(g(x))$. The number of discontinuities in $(f \circ g)(x)$ present in the interval $(-\infty, 0)$ is:

(A) 0 (B) 1
(C) 2 (D) 4

Solution: Given $g(x) = \begin{cases} -x, & x \le 1 \\ x+1, & x \ge 1 \end{cases}$

i.e., $g(x) = -x; x \in (-\infty, 0)$

and $f(x) = \begin{cases} 1-x, & x \le 0 \\ x^2, & x > 0 \end{cases}$.

i.e., $f(x) = 1 - x; x \in (-\infty, 0)$

Now $(f \circ g)(x) = f(g(x))$

$= f(-x)$

$= 1 - (-x); x \in (-\infty, 0)$

$\Rightarrow (f \circ g)(x) = 1 + x; x \in (-\infty, 0)$

And $1 + x$ is continuous everywhere in $(-\infty, 0)$.

$\therefore$ The number of discontnuities of $(f \circ g)(x)$ in $(-\infty, 0)$ is zero

Hence, the correct option is (A).

Question Number: 27 **Question Type: MCQ**

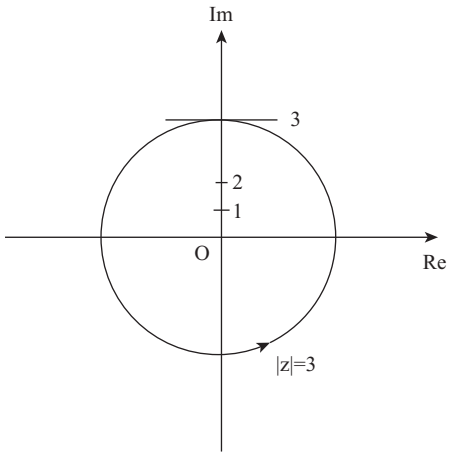
The value of the contour integral in the complex-plane

$$\oint \frac{z^3 - 2z + 3}{z - 2} dz$$

along the contour $|z| = 3$, taken counter-clockwise is

(A) $-18\pi i$ (B) 0
(C) $14\pi i$ (D) $48\pi i$

Solution: We have to evaluate the contour integral



$$\oint_C \frac{z^3 - 2z + 3}{z - 2} dz$$

where c is $|z| = 3$

$z = 2$ is the singularity of $\frac{z^3 - 2z + 3}{z - 2}$ which lies inside C.

$\therefore$ By Cauchy's integral formula, we have

$\oint_C \frac{z^3 - 2z + 3}{z - 2} dz = 2\pi i\, f(a)$, where $f(z) = z^3 - 2z + 3$ and $a = 2$

$= 2\pi i(2^3 - 2 \times 2 + 3)$

$= 14\pi i$

Hence, the correct option is (C).

Question Number: 28 **Question Type: MCQ**

The eigenvalues of the matrix given below are

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

(A) (0,–1,–3) (B) (0. –2. –3)
(C) (0. 2, 3) (D) (0, 1, 3)

Solution: Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$

The characteristic equation of A is

$\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda[-\lambda(-4-\lambda)+3] = 0$$

$$\Rightarrow \lambda[\lambda^2 + 4\lambda + 3] = 0$$

$$\Rightarrow \lambda(\lambda+1)(\lambda+3) = 0$$

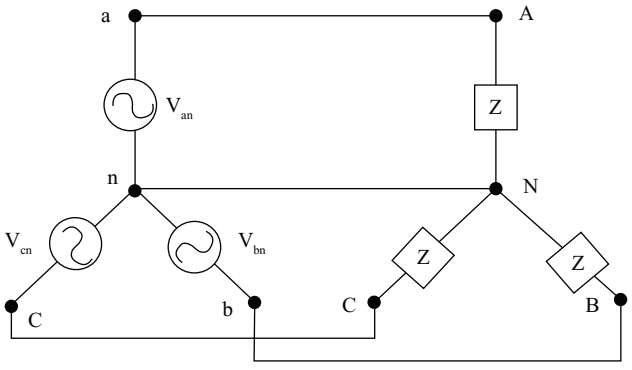
$$\Rightarrow \lambda = 0, \lambda = -1, \lambda = -3$$

$\therefore$ The eigen values of A are 0, –1 and –3

Hence, the correct option is (A).

Question Number: 29 **Question Type: MCQ**

For the balanced $Y - Y$ connected 3-phase circuit shown in the figure below, the line-line voltage is 208 V rms and the total power absorbed by the load is 432 W at a power factor of 0.6 leading.



The approximate value of the impedance Z is

(A) $33\angle -53.1^\circ\Omega$ (B) $60\angle 53.1^\circ\Omega$
(C) $60\angle -53.1^\circ\Omega$ (D) $180\angle -53.1^\circ\Omega$

Solution:

$$P = \sqrt{3}\ V_L I_L \cos\phi$$

$$432 = \sqrt{3} \times 208 \times I_L \times 0.6$$

$$432 = 1.9985 \angle 53\ 130^\circ$$

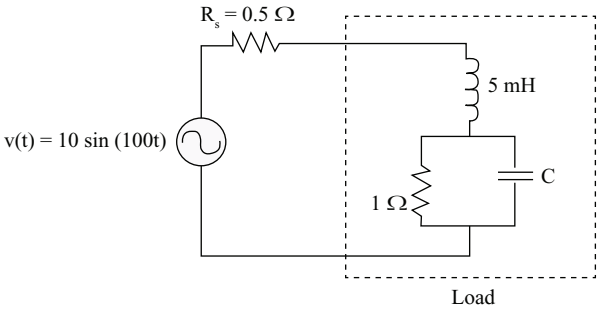
$$Z = \frac{V_{Load(ph)}}{I_{Load}} = \frac{208/\sqrt{3}\angle 0^\circ}{1.9985\angle 53.130^\circ}$$

$$Z = 60 \angle -53\ 13^\circ\ \Omega$$

Hence, the correct option is (C).

Question Number: 30 **Question Type: MCQ**

In the circuit shown below, the value of capacitor C required for maximum power to be transferred to the load is



(A) 1 nF (B) 1 μF
(C) 1 mF (D) 10 mF

Solution: We know the condition for maximum power transferred to the load is

$$Z_L = Z_s^*$$

$$Z_L = j\omega L + \left\{R \,\|\, \frac{1}{j\omega c}\right\}$$

$$Z_L = j100 \times 5 \times 10^{-3} + \left\{\frac{1 \times \frac{1}{j100C}}{1 + \frac{1}{j100C}}\right\}$$

Let $Z_L = Rs = 0.5\ \Omega$

$$0.5 = j0.5 + \frac{1}{1 + j100C}$$

$$0.5\ \{1 + j100C\} = j\ 0.5 + 1 - 50C$$

$$0.5 + j50C = j\ 0.5 + \{1 - 50C\}$$

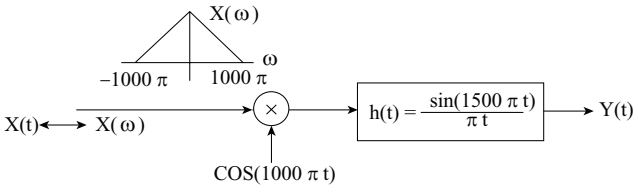
Let $50C = 0.5$

$$C = 0.01F$$

Hence, the correct option is (D).

Question Number: 31 **Question Type: MCQ**

The output $y(t)$ of the following system is to be sampled, so as to reconstruct it from its samples uniquely. The required minimum sampling rate is

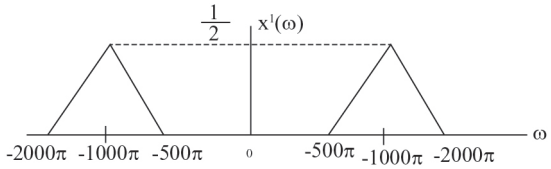


(A) 1000 samples/s (B) 1500 samples/s
(C) 2000 samples/s (D) 3000 samples/s

Solution: Let $x'(t)$ be the output of the multiplier

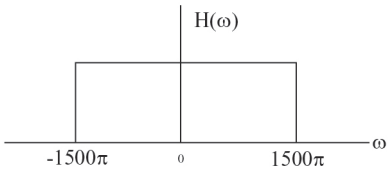
$x'(t) = x(t) \cos(1000\pi t)$

$$x'(\omega) = \frac{1}{2}[x(w - 1000\pi) + x(w + 1000\pi)]$$



$$y(t) = x'(t) * h(t)$$

$$y(\omega) = x'(\omega)\, H(\omega)$$



When $x'(w)$ and $H(w)$ one multiplied, the highest frequency component present at the output is $W_m = 1500\pi$

$\therefore$ Minimum Sampling rate

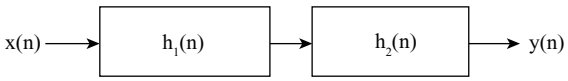
$$w_0 = 3000\ \pi \text{ rad/s}$$

$$f_s = 1500 \text{ samples/s}$$

Hence, the correct option is (B).

Question Number: 32 **Question Type: MCQ**

A cascade system having the impulse responses $h_1(n) = \{\underset{\uparrow}{1}, -1\}$ and $h_2(n) = \{\underset{\uparrow}{1}, 1\}$ is shown in the figure below, where symbol $\uparrow$ denotes the time origin.



The input sequence $x(n)$ for which the cascade system produces an output sequence $y(n) = \{\underset{\uparrow}{1}, 2, 1, -1, -2, -1\}$ is

(A) $x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$ (B) $x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$

(C) $x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$ (D) $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

Solution: If two *LTI* systems are Cascaded, then their impulse response one convolved.

Overall impulse response

$$h(n) = h_1(n) * h_2(n)$$

	1	1
1	1	1
−1	−1	−1

$$h(n) = \{1 \uparrow 0,\ -1\}$$

Output $y(n) = \{1 \uparrow, 2, 1, -1, -2, -1\}$

output samples one 6 so numbers of samples in the input must be 4

Let $x(n) = \{a \uparrow,\ b,\ c,\ d\}$

$y(n) = x(n) * h(n)$

	1	0	−1
a	a	0	$-a$
b	b	0	$-b$
c	c	0	$-c$
d	d	0	$-d$

$$y(n) = \{a, b, -a + c, -b + d, -c, -d\}$$

Comparing with given $y(n)$

$$y(n) = \{1, 2, 1, -1, -2, -1\}$$

$$a = 1, b = 2, c = 2, d = 1$$

$$x(n) = \{1 \uparrow,\ 2,\ 2,\ 1\}$$

Hence, the correct option is (D).

Question Number: 33 **Question Type: MCQ**

A 220 V, 10 kW. 900 rpm separately excited DC motor has an armature resistance $R_a = 0.02\ \Omega$. When the motor operates at rated speed and with rated terminal voltage, the electromagnetic torque developed by the motor is 70 Nm. Neglecting the rotational losses of the machine, the current drawn by the motor from the 220 V supply is

(A) 34.2 A (B) 30 A
(C) 22 A (D) 4.84 A

Solution:

$$P = E_b I_a = T_e \omega \quad (1)$$

$$E_b I_a = P \Rightarrow (V_t - I_a R_a) I_a = P$$

$$(220 - I_a \times 0.02)\, I_a = 10 \times 10^3$$

$$220\, I_a - 0.02\, I_a^2 = 10 \times 10^3$$

$$0.02\, I_a^2 - 220\, I_a + 10 \times 10^3 = 0$$

$$I_a = 45.644 \text{ A}.$$

From equation (1)

$$T_e \omega = P$$

$$T_e \omega = \frac{P}{\omega} = \frac{P}{\frac{2\pi N}{60}} = \frac{60 \times 10 \times 10^3}{2\pi \times 900}$$

$$T_e = 106.103 \text{ Nm}$$

For separately excited motor

$$T \propto I_a$$

$$\frac{T_1}{T_2} = \frac{I_{a_1}}{I_{a_2}}$$

$$I_{a_2} = \frac{T_2}{T_1} = I_{a_1}$$

$$I_{a_2} = \frac{70 \times 45.644}{106.103} = 30.113A$$

Hence, the correct option is (B).

Question Number: 34 **Question Type: MCQ**

The root locus of the feedback control system having the characteristic equation $s^2 + 6Ks + 2s + 5 = 0$ where $K > 0$, enters into the real axis at

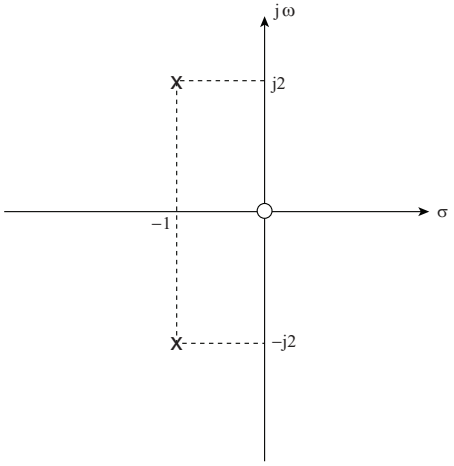
(A) $s = -1$ (B) $s = -\sqrt{5}$

(C) $s = -5$ (D) $s = \sqrt{5}$

Solution: $CE = 1 + KG(S)H(S) = S^2 + 6ks + 2S + 5 = 0$

$$C \cdot E = \left(1 + \frac{6ks}{s^2 + 2S + 5}\right) = 0$$

$$KG(S)H(S) = \left(\frac{6ks}{s^2 + 2S + 5}\right)$$



$$\left|KG((S)H(S))\right| = 1$$

$$K = \left(\frac{s^2 + 2S + 5}{6s}\right) = \left(\frac{s}{6} + \frac{1}{3} + \frac{5}{6s}\right)$$

At *BAP* (op) *BIP*, $\frac{dk}{ds} = 0$

$$\frac{dk}{dS} = \left(\frac{1}{6} + 0 - \frac{5}{6s^2}\right) = 0$$

$$\frac{1}{6} = \frac{5}{6s^2}$$

$$s^2 = 5$$

$$s = \pm\sqrt{5}$$

but $s = +\sqrt{5}$ is not valid $\because$ it does not lie on root locus.

$$\boxed{\therefore\ s = -\sqrt{5}}$$

Hence, the correct option is (B).

Question Number: 35 **Question Type: MCQ**

The range of K for which all the roots of the equation $s^3 + 3s^2 + 2s + K = 0$ are in the left half of the complex s-plane is

(A) $0 < K < 6$ (B) $0 < K < 16$

(C) $6 < K < 36$ (D) $6 < K < 16$

Solution: From *RH* criteria

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3}$	
s^0	k	

$$\left(\frac{6-k}{3}\right) > 0 \text{ and } k > 0 \Rightarrow k > 0 \text{ and } k < 6$$

Hence, the correct option is (A).

Question Number: 36 **Question Type: MCQ**

Which of the following systems has maximum peak over-shoot due to a unit step input?

(A) $\frac{100}{s^2 + 10s + 100}$ (B) $\frac{100}{s^2 + 15s + 100}$

(C) $\frac{100}{s^2 + 5s + 100}$ (D) $\frac{100}{s^2 + 20s + 100}$

Solution: Peak over shoot = $M_p = \frac{-\pi\zeta}{e\sqrt{1-\zeta^2}}$

The system with less damping ratio will have more peak over shoot. From options

(a) $\rightarrow \zeta = \frac{10}{2\sqrt{100}} = 0.5$

(b) $\rightarrow \zeta = \frac{15}{2\sqrt{100}} = 0.75$

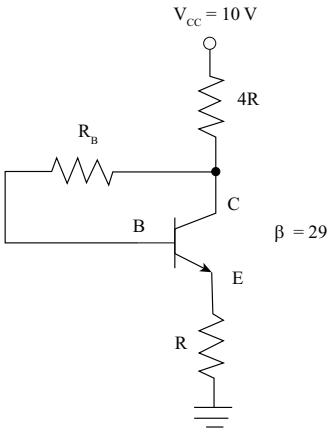
(c) $\rightarrow \zeta = \frac{5}{2\sqrt{100}} = 0.25$

(d) $\rightarrow \zeta = \frac{20}{2\sqrt{100}} = 1$

Hence, the correct option is (C).

Question Number: 37 **Question Type: MCQ**

For the circuit shown in the figure below, it is given that $V_{CE} = \frac{V_{CC}}{2}$. The transistor has $\beta = 29$ and $V_{BE} = 0.7$ V when the B-E junction is forward biased.



For this circuit, the value of $\frac{R_B}{R}$ is

(A) 43 (B) 92

(C) 121 (D) 129

Solution:

KVL at input loop

$$10 = (\beta + 1)\, I_B \times 4R + I_B \times R_B + 0.7 + (\beta + 1)\, I_B \times R$$

$$10 = 30\, I_B \times 4R + I_B \times R_B + 0.7 + 30 \times I_B \times R$$

$$9.3 = 150\, I_B \times R + I_B \times R_B \qquad (1)$$

KVL at output loop

$$10 = (1+\beta)\, I_B \times 4R + 5v + (1 + \beta)\, I_B \times R$$

$$5 = 30 I_B \times 4R + 30 \times I_B \times R$$

$$5 = 150 \times I_B \times R \quad (2)$$

Solving (1) and (2)

$$I_B R_B = 9.3 - 5 = 4.3$$

Solving above value in equation (1)

$$9.3 = I_B R\,[150 + \frac{R_B}{R}]$$

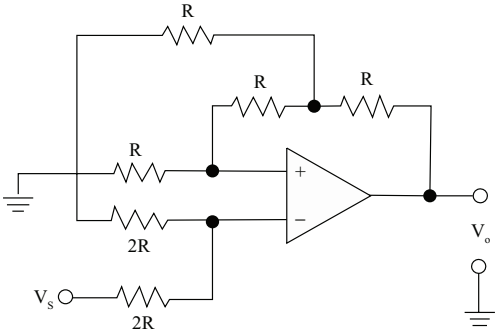
$$9.3 = \frac{5}{150}\left[150 + \frac{R_B}{R}\right]$$

$$279 = 150 + \frac{R_B}{R} \Rightarrow \frac{R_B}{R} = 129.$$

Hence, the correct option is (D).

Question Number: 38 **Question Type: MCQ**

For the circuit shown below, assume that the OPAMP is ideal.

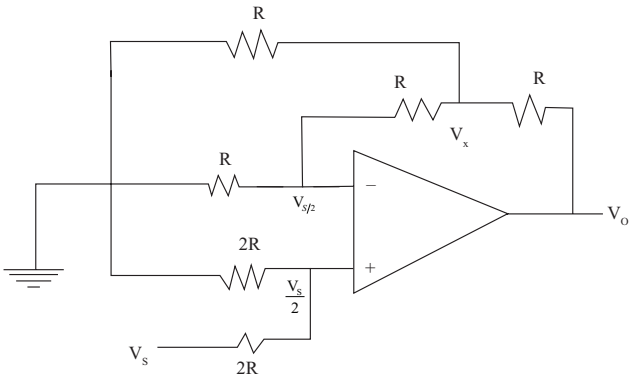


Which one of the following is TRUE?

(A) $v_o = v_s$ (B) $v_o = 1.5v_s$

(C) $v_o = 2.5v_s$ (D) $v_o = 1.5v_s$

Solution:



$$\frac{\frac{V_s}{2} - V_x}{R} + \frac{\frac{V_s}{2}}{R} = 0$$

$$V_x = V_s \qquad (1)$$

$$\frac{V_x}{R} + \frac{V_x - \frac{V_s}{2}}{R} + \frac{V_x - V_0}{R} = 0$$

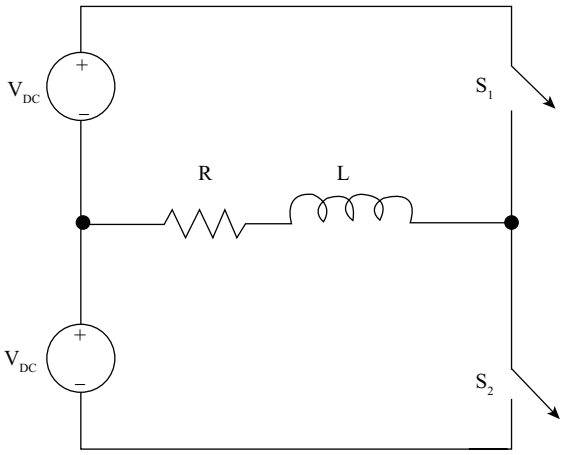
$$3V_x - \frac{V_s}{2} = V_0 \qquad (2)$$

Solving (1) and (2) $\Rightarrow V_0 = 2.5V_s$.

Hence, the correct option is (C).

Question Number: 39 **Question Type: MCQ**

The figure below shows a half-bridge voltage source inverter supplying an RL-load with R = 40 Ω and $L = \left(\frac{0.3}{\pi}\right) H$. The desired fundamental frequency of the load voltage is 50 Hz. The switch control signals of the converter are generated using sinusoidal pulse width modulation with modulation index. M = 0.6. At 50 Hz. the RL-load draws an active power of 1.44 kW. The value of DC source voltage V_{DC}. In volts is



(A) $300\sqrt{2}$ (B) 500

(C) $500\sqrt{2}$ (D) $1000\sqrt{2}$

Solution: Peak value of fundamental voltage $V_m = mV_{dc}$.

where m = modulation index.

$V_p = 0.6\ V_{dc}$.

Power $p = I_{rms}^2 \cdot R = 1.44kw$

$$I_{rms}^2 = \frac{1440}{40} = 36$$

$$I_{rms} = 6A$$

$$V_{rms} = I_{rms}|Z_L| = 6 \times \sqrt{40^2 + \left(\frac{2\pi \times 50 \times 0.3}{\pi}\right)^2}$$

$$V_{rms} = 6 \times 50 = 300V$$

peak value $V_m = \sqrt{2}V_{rms} = 300\sqrt{2}V$

$$300\sqrt{2} = 0.6V_{dc}$$

$$\Rightarrow V_{dc} = \frac{300\sqrt{2}}{0.6}$$

$$\therefore V_{dc} = 500\sqrt{2}V$$

Hence, the correct option is (C).

Question Number: 40 **Question Type: MCQ**

A person decides to toss a fair coin repeatedly until he gets a head. He will make at most 3 tosses. Let the random variable Y denote the number of heads. The value of var$\{Y\}$, where var$\{\cdot\}$ denotes the variance, equals

(A) $\frac{7}{8}$ (B) $\frac{49}{64}$

(C) $\frac{7}{64}$ (D) $\frac{105}{64}$

Solution: Given Y = Number of heads

Probability that head does not occur in any of the three tosses $= P(Y=0) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Probability that one head occurs is given in the following table

Number of tosses	01 (H)	02 (TH)	03 (TTH)
Probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

$\therefore$ Probability of getting one head

$$= P(Y=1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$\therefore$ The probability mass function of Y is

Y	0	1
$P(Y=y)$	$\frac{1}{8}$	$\frac{7}{8}$

$\therefore$ Mean of $Y = E(Y) = 0 \times \frac{1}{8} + 1 \times \frac{7}{8} = \frac{7}{8}$

Variance of $Y = Var(Y) = E(Y^2) - (E(Y))^2$

$$= 0^2 \times \frac{1}{8} + 1^2 \times \frac{7}{8} = \left(\frac{7}{8}\right)^2$$

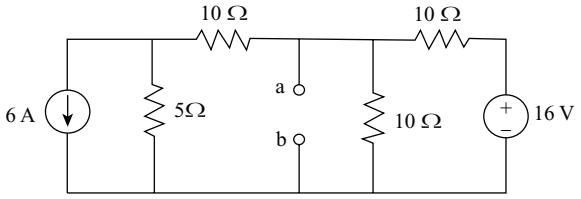
$$= \frac{7}{8} - \frac{49}{56}$$

$$= \frac{7}{64}$$

Hence, the correct option is (C).

Question Number: 41 **Question Type: MCQ**

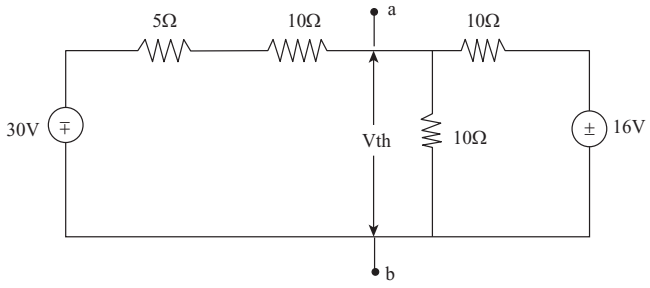
For the network given in figure below, the Thevenin's voltage V_{ab} is



(A) $-1.5\ V$ (B) $-0.5\ V$

(C) $0.5\ V$ (D) $1.5\ V$

Solution: Applying source transformation, the equivalent network becomes



$$\frac{V_{th} + 30}{15} + \frac{V_{th}}{10} + \frac{V_{th} - 16}{10} = 0.$$

$$2(V_{th} + 30) + 3V_{th} + 3(V_{th} - 16) = 0$$

$$8V_{th} + 60 - 48 = 0$$

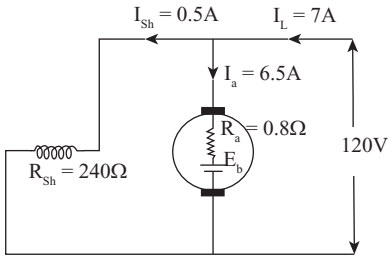
$$V_{th} = \frac{-12}{8} = -1.5V.$$

Hence, the correct option is (A).

Question Number: 42 **Question Type: NAT**

A 120 V DC shunt motor takes 2 A at no load. It takes 7 A on full load while running at 1200 rpm. The armature resistance is 0.8Ω, and the shunt field resistance is 240Ω. The no load speed, in rpm, is _____.

Solution:



Full Load:

$$I_{Sh} = \frac{V_t}{R_{Sh}} = \frac{120}{240} = 0.5A$$

$$I_a = I_L - I_{sh} = 7 - 0.5 = 6.5\ A$$

$$E_{b1} = V_t - I_a R_a$$

$$= 120 - (6.5 \times 0.8)$$

$$E_{b1} = 114.8\ V$$

$$N_1 = 1200 \text{ rpm}$$

No Load:

$$I_L = 2A,\ I_{Sh} = 0.5\ A,\ I_a = 1.5\ A$$

$$E_{b2} = 120 - (1.5 \times 0.8) = 118.8\ V$$

For shunt motor, $E_b \propto N$

$$\frac{E_{b2}}{E_{b1}} = \frac{N_2}{N_1} \Rightarrow N_2 = \frac{118.8}{114.8} \times 1200$$

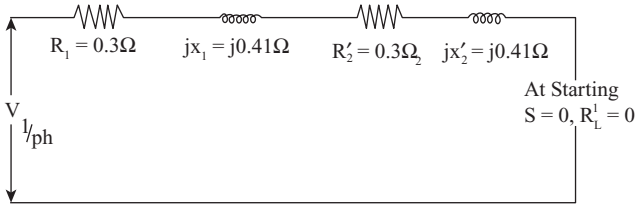
$$N_2 = 1241.81 \text{ r.p.m}$$

Hence, the correct answer is (1235 to 1250).

Question Number: 43 **Question Type: NAT**

A star-connected, 12.5k*W*, 208 *V* (line), 3-phase, 60 Hz squirrel cage induction motor has following equivalent circuit parameters per phase referred to the stator: $R_1 = 0.3\Omega$, $R_2 = 0.3\Omega$, $X_1 = 0.41\Omega$, $X_2 = 0.41\Omega$. Neglect shunt branch in the equivalent circuit. The starting current (in Ampere) for this motor when connected to an 80 V (line), 20 Hz, 3-phase AC source is _____.

Solution: The per phase equivalent circuit refer to stator side is



Total resistance referred to stator side = 0.6 Ω/ph

Total reactance referred to stator side at 60 Hz = 0.82 Ω/ph

Total reactance at 20 Hz = $0.82 \times \frac{20}{60} = 0.273\Omega$

Starting current, $I_{st} = \frac{V_{1/ph}}{R_{eq} + jX_{eq}}$

$$= \frac{80/\sqrt{3}}{0.6 + j0.273}$$

$$= 70.068\ A$$

Hence, the correct answer is (69 to 71).

Question Number: 44 **Question Type: NAT**

A 25 *kVA*. 400 *V*, Δ-connected, 3-phase, cylindrical rotor synchronous generator requires a field current of 5 *A* to maintain the rated armature current under short-circuit condition. For the same field current, the open-circuit voltage is 360 *V*. Neglecting the armature resistance and magnetic saturation, its voltage regulation (in % with respect to terminal voltage), when the generator delivers the rated load at 0.8 pf leading, at rated terminal voltage is _____.

Solution:

$$\% \text{ Voltage regulation} = \frac{E - V}{V} \times 100$$

$$V = 400;$$

$$E = \sqrt{(V\cos\phi + I_a R_a)^2 + (V\sin\phi - I_a X_s)^2}$$

$$R_a = 0;\ X_S = \frac{V_{OC}}{I_{\text{a rated(ph)}}};\ I_{\text{a rated (ph)}} = \frac{25 \times 10^3}{\sqrt{3} \times 400} \times \frac{1}{\sqrt{3}}$$

$$= 20.833\ A$$

$$X_{S(\text{Ph})} = 17.28\Omega$$

$$E = \sqrt{(400 \times 0.8 + 0)^2 + (400 \times 0.6 - 20.83 \times 17.28)^2}$$

$$E = 341.74\ V$$

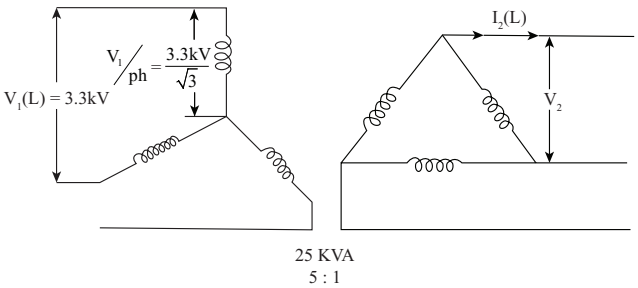
$$\% VR = \frac{(341.74 - 400)}{400} = -14.565\%$$

Hence, the correct answer is (−15 to −14).

Question Number: 45 **Question Type: NAT**

If the primary line voltage rating is 3.3 *kV* (*Y* side) of a 25 kVA. Y-Δ transformer (the per phase turns ratio is 5:1), then the line current rating of the secondary side (in Ampere) is _____.

Solution:



$$V_{2(\text{ph})} = V_{1(\text{ph})} \times \frac{1}{5} = \frac{3.3}{\sqrt{3}} \times 10^3 \times \frac{1}{5} = 381.051V$$

$$V_{2(L)} = V_{2(ph)} = 381.051V$$

$$\sqrt{3}\ V_{2(L)}\, I_{2(L)} = 25 \times 10^3$$

$$I_{2(L)} = \frac{25 \times 10^3}{\sqrt{3} V_{2(L)}} = \frac{25 \times 10^3}{\sqrt{3} \times 381.051} = 37.878A$$

Hence, the correct answer is (37 to 39).

Question Number: 46 **Question Type: NAT**

Consider the system described by the following state space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

If $u(t)$ is a unit step input and $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the value of output $y(t)$ at $t = 1$ sec (rounded off to three decimal places) is _____.

Solution: $y(t) = x_1(t)$

$$x_1^0(t) = x_2(t)$$

$$x_2^0(t) = -2x_2(t) + u(t)$$

$$s\, X_2(s) - x_2(0) = -2X_2(s) + 1/s$$

$$s\, X_2(s) + 2X_2(s) = 1/s$$

$$X_2(s) = \cdot \frac{1}{s(s+2)}$$

$$x_1^0(t) = x_2(t)$$

$$s\, X_1(s) - X_1(0) = X_2(s)$$

$$s\, X_1(s) - 1 = \frac{1}{s(s+2)}$$

$$s\, X_1(s) = 1 + \frac{1}{s(s+2)}$$

$$X_1(s) = 1/s + \frac{1}{s^2(s+2)}$$

$$X_1(s) = 1/s + \frac{0.5}{s^2} + \frac{0.25}{s+2} + \frac{-0.25}{s}$$

$y(t) = x_1(t) = u(t) + 0.5 - t\,u(t) + 0.25\, e^{-2t} - 0.25\, u(t)$

At $t = 1$

$$y(1) = 1 + 0.5 + 0.25\, e^{-2} - 0.25$$

$$y(1) = 1.2838$$

Hence, the correct answer is (1.28 to 1.287).

Question Number: 47 **Question Type: NAT**

A $10\frac{1}{2}$ digit timer counter possesses a base clock of frequency 100 MHz. When measuring a particular input, the reading obtained is the same in: (i) Frequency mode of operation with a gating time of one second and (ii) Period mode of operation (in the x 10 ns scale). The frequency of the unknown input (reading obtained) in Hz is _____.

Solution:

(i) For a $10\frac{1}{2}$ digit timer counter:

Frequency mode of operation, $f = \frac{n \rightarrow \text{base clock frequency}}{t \rightarrow \text{gating time}}$

$$f = \frac{100 \times 10^6}{1\text{sec}} = 10^8 \text{ cycles/sec}$$

on $10\frac{1}{2}$ digit display $\Rightarrow$ 100000000.00 *Hz*

(ii) Period mode of operation:

$$P = \frac{1}{f} = \frac{t}{n} = \frac{1\text{sec}}{100 \times 10^6}$$

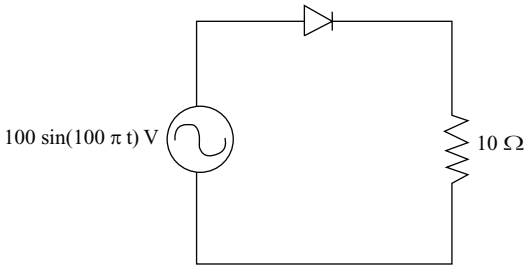
$$P = 10^{-8} = 10n\text{sec}$$

$$= 10.000000000 \text{ n sec}$$

Hence, the correct answer is (10,000 to 10,000).

Question Number: 48 **Question Type: NAT**

In the circuit shown in the figure, the diode used is ideal. The input power factor is _____. (Give the answer up to two decimal places.)



Solution:

The *rms* output voltage $V_{rms} = \frac{V_m}{2}$

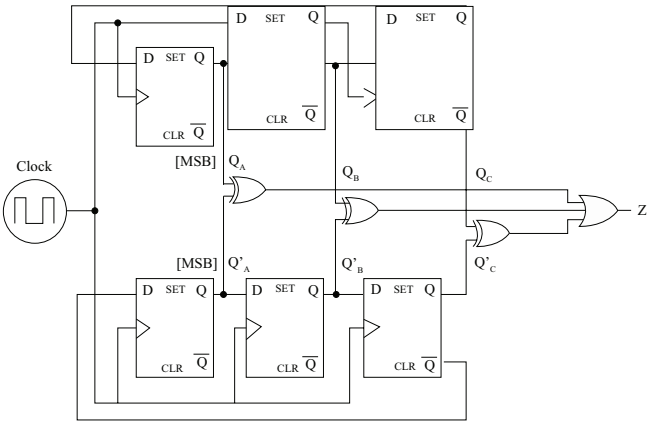
$$\text{Input power factor} = \frac{V_{rms}}{V_s} = \frac{\frac{V_m}{2}}{\frac{V_m}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = 0.707$$

Hence, the correct answer is (0.70 to 0.71).

Question Number: 49 **Question Type: NAT**

For the synchronous sequential circuit shown below, the output Z is zero for the initial conditions $Q_A Q_B Q_C = Q_A' Q_B' Q_C' = 100$.



The minimum number of clock cycles after which the output Z would again become zero is_____

Solution:

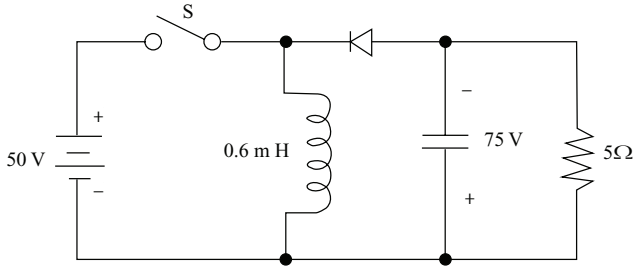
Clock	Q_A	Q_B	Q_C	Q_A'	Q_B'	Q_C'	$Q_A + Q_A'$	$Q_B + \bar{Q}_B$	$Q_C + Q_C'$	Z
0	1	0	0	1	0	0	0	0	0	0
1	0	1	0	1	1	0	1	0	0	1
2	0	0	1	1	1	1	1	1	0	1
3	1	0	0	0	1	1	1	1	1	1
4	0	1	0	0	0	1	0	1	1	1
5	0	0	1	0	0	0	0	0	1	1
6	1	0	0	1	0	0	0	0	0	0

the output Z will again become "0" after 6 clock pulses.

Hence, the correct answer is (6 to 6).

Question Number: 50 **Question Type: NAT**

In the circuit shown all elements are ideal and the switch S is operated at 10 kHz and 60% duty ratio. The capacitor is large enough so that the ripple across it is negligible and at steady state acquires a voltage as shown. The peak current in amperes drawn from the 50 V DC source is _____. (Give the answer up to one decimal place.)



Solution:

$$I_0 = \frac{V_0}{R} = \frac{75}{5} = 15$$

$$I_L = \frac{I_0}{1-D} = \frac{15}{1-0.6} = 37.5A$$

$$\Delta I_L = \frac{V_{dc}}{L} DT = \frac{50}{0.6\times10^{-3}} \times 0.6 \frac{1}{10\times10^{3}}$$

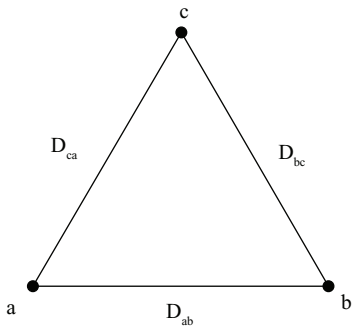
$$\Delta I_L = 5A$$

$$I_{L_{max}} = I_L + \frac{\Delta I_L}{2} = 37.5 + \frac{5}{2} = 40A$$

Hence, the correct answer is (39 to 41).

Question Number: 51 **Question Type: NAT**

Consider an overhead transmission line with 3-phase. 50 Hz balanced system with conductors located at the vertices of an equilateral triangle of length $D_{ab} = D_{bc} = D_{ca} =$ lm as shown in figure below. The resistances of the conductors are neglected. The geometric mean radius (GMR) of each conductor is 0.01 m. Neglecting the effect of ground, the magnitude of positive sequence reactance in Ω/*km* (rounded off to three decimal places) is _____.



Solution: $L/ph = 2 \times 10^{-7} \ln\left(\frac{GMD}{self\ GMD}\right)$ H/m

$$L/ph = 2 \times 10^{-7} \ln\left(\frac{1\text{m}}{0.01\text{m}}\right) \text{H/m}$$

$$L/ph = 9.21034 \times 10^{-7}$$

$$X_L = 2\pi \not{f} L = 2\pi \times 50 \times 9.21034 \times 10^{-7} \times 10^{3} \ (\Omega/\text{km})$$

$$X_L = 0.289\ \Omega/\text{km}$$

Hence, the correct answer is (0.271 to 0.301).

Question Number: 52 **Question Type: NAT**

A 3-phase, 50 Hz generator supplies power of 3MW at 17.32 *kV* to a balanced 3-phase inductive load through an overhead line. The per phase line resistance and reactance are 0.25Ω and 3 925Ω. respectively. If the voltage at the generator terminal is 17.87 kV, the power factor of the load is _____.

Solution:

$$P_1 = \sqrt{3} V_L I_L Cos\phi$$

$$3 \times 10^6 = \sqrt{3} \times 17.32 \times 10^3 \; I_L c \cos \phi$$

$$I_2 \cos \phi = 100 \qquad (1)$$

$|Vs| - |Vr| = \{I_L R \cos\phi + I_L \times \sin\phi\}$ (All are per phase values)

$$\left(\frac{17.87 - 17.32}{\sqrt{3}}\right) \times 10^3 = \{0.25 \times 100 + I_L \times 3.925 \sin\phi\}$$

$$I_L \sin \phi = 75\ 8070 \qquad (2)$$

$(2)/(1) \Rightarrow \text{Tan } \phi = 0.7580$

$\phi = \text{Tan}^{-1}(0.7580)$

$\cos \phi = \text{Load Power factors} = 0\ 7969 \text{ lag}$

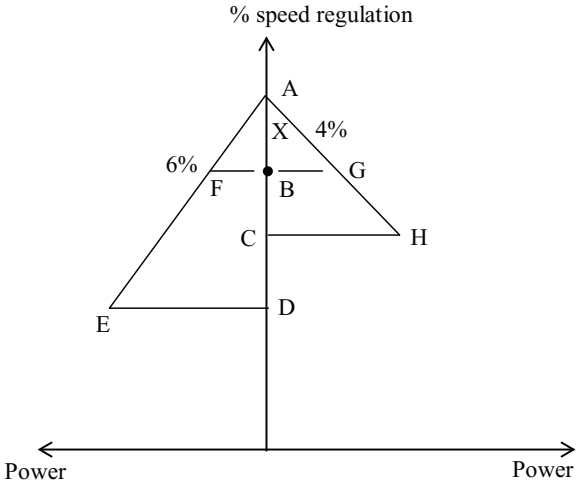
$\simeq 0\ 8 \text{ lag}$

Hence, the correct answer is (0.75 to 0.85).

Question Number: 53 **Question Type: NAT**

Two generating units rated 300 MW and 400 MW have governor speed regulation of 6% and 4% respectively from no load to full load, Both the generating units are operating in parallel to share a load of 600 MW. Assuming free governor action, the load shared by the larger unit is _____ MW.

Solution:



$$\frac{FB}{ED} = \frac{AB}{AD}$$

$$P_1 = 300 \frac{X}{6}$$

$$P_1 = 50X$$

$$\frac{BG}{CH} = \frac{AB}{AC}$$

$$P_2 = 400 \frac{X}{4}$$

$$= 100X$$

$$P_1 + P_2 = 600$$

$$X = 4$$

$$\therefore P_2 = 400 \text{ MW}$$

Hence, the correct answer is (395 to 405).

Question Number: 54 **Question Type: NAT**

A 3-phase, 2-pole, 50 Hz, synchronous generator has a rating of 250 MVA, 0.8 pf lagging. The kinetic energy of the machine at synchronous speed is 1000 MJ. The machine is running steadily at synchronous speed and delivering 60 MW power at a power angle of 10 electrical degrees. If the load is suddenly removed, assuming the acceleration is constant for 10 cycles, the value of the power angle after 5 cycles is _____ electrical degrees.

Solution:

$\cos \phi = 0.8 \text{ lag}$

$G = 250\ MVA$

$P = 2$

$$KE = \frac{1}{2} J\ \omega s^2$$

$$10^6 \times 1000 = \tfrac{1}{2} \text{ J} \times (2\pi \times N_S)^2$$

$N_s = \dfrac{120 \times 50}{2}$ rpm; $Pa = Ps - Pe = 0 \Rightarrow Ps = 60$ MW

$J = 5.62$

$$Pe = \frac{EV}{X} \sin \delta_\circ = P_{max} \sin \delta o$$

$$60 \times 10^6 = \frac{EV}{x} \sin 10^\circ$$

$$P_{max} = \frac{EV}{X} = 345.52 \times 10^6$$

$Pa = Ps - Pe$ (0 initially $Ps = Pe$)

$Pa = Ps$ ($\therefore Pe = 0$ when load is removed)

$Pa = Ps = 60$ MW

$$M = \frac{GH}{\pi f} = \frac{250 \times H}{\pi f}$$

$$H = \frac{KE}{\text{Rating of the machine}} = \frac{1000 \times 10^6}{250 \times 10^6} = 4$$

$$\delta(t) = \left(\frac{Pa}{2M}\right) t^2 + \delta_0$$

$\delta_0 = 10$ degrees.

$Pa = 60$ MW

$$M = \frac{250 \times 10^6 \times 4}{11 \times 50} = 6.366 \times 10^6$$

1 cycle $\rightarrow$ 20 m sec

5 cycles $\rightarrow$ 100 m sec

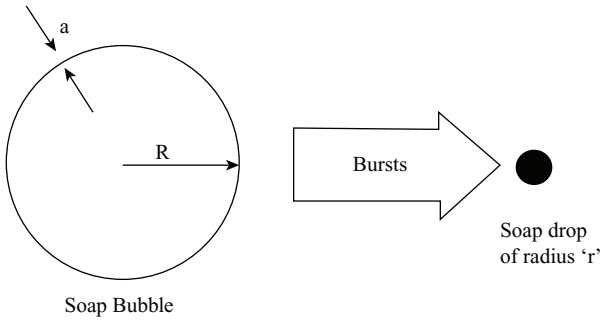
$$\delta(t)\Big|_{t=100m \sec} = \left(\frac{60 \times 10^6}{2 \times 6.366 \times 10^6}\right) \times \frac{180}{\pi} \times (100 \times 10^{-3})^2 + 10$$

$= 12.7$

Hence, the correct answer is (12.5 to 12.9).

Question Number: 55 **Question Type: NAT**

A thin soap bubble of radius $R = 1$ cm, and thickness $a = 3.3$ μm ($a \ll R$). is at a potential of 1 V with respect to a reference point at infinity. The bubble bursts and becomes a single spherical drop of soap (assuming all the soap is contained in the drop) of radius r. The volume of the soap in the thin bubble is $4\pi R^2$ a and that of the drop is $-\frac{4}{3}\pi r^3$. The potential hi volts, of the resulting single spherical drop with respect to the same reference point at infinity Is _____. (Give the answer up to two decimal places)



Charge must be same

$$(4\pi R^2)\, a = \left(\frac{4}{3}\pi r^3\right)\rho_v$$

$$r = 3\sqrt{3R^2 a}$$

$$= 0.996 \times 10^{-3}$$

The potential of thin bubble is 1 V.

$$I = \frac{Q}{4\pi\varepsilon_0 \times 1 \times 10^{-2}}$$

$$Q = 4\pi\varepsilon_0 \times 10^{-2} C$$

Potential at soap drop,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

$$= \frac{4\pi\varepsilon_0 \times 10^{-2}}{4\pi\varepsilon_0 \times 0.9966 \times 10^{-3}}$$

$$= 10.03 \text{ V}$$

Hence, the correct answer is (9.50 to 10.50).

General Aptitude

Number of Questions: 10 **Section Marks: 15**

Q.56 to Q.60 carry 1 mark each and Q.61 to Q.65 carry 2 marks each.

Question Number: 56 **Question Type: MCQ**

Choose the option with words that are not synonyms.

(A) aversion, dislike (B) luminous, radiant
(C) plunder, loot (D) yielding, resistant

Ans: (D)

Question Number: 57 **Question Type: MCQ**

Saturn is _____ to be seen on a clear night with the naked eye.

(A) enough bright (B) bright enough
(C) as enough bright (D) bright as enough

Ans: (B)

Question Number: 58 **Question Type: MCQ**

There are five buildings called V, W, X, Y and Z in a row (not necessarily in that order). V is to the West of W. Z is to the East of X and the West of V. W is to the West of Y. Which is the building in the middle?

(A) V (B) W
(C) X (D) Y

Ans: (A)

Question Number: 59 **Question Type: MCQ**

A test has twenty questions worth 100 marks in total. There are two types of questions. Multiple choice questions are worth 3 marks each and essay questions are worth 11 marks each. How many multiple choice questions does the exam have?

(A) 12 (B) 15
(C) 18 (D) 19

Ans: (B)

Question Number: 60 **Question Type: MCQ**

There are 3 red socks, 4 green socks and 3 blue socks. You choose 2 socks. The probability that they are of the same colour is

(A) 1/5 (B) 7/30
(C) 1/4 (D) 4/15

Ans: (D)

Question Number: 61 **Question Type: MCQ**

"We lived in a culture that denied any merit to literary works, considering them important only when they were

handmaidens to something seemingly more urgent - namely ideology. This was a country where all gestures, even the most private, were interpreted in political terms."

The author's belief that ideology is not as important as literature is revealed by the word:

(A) 'Culture' (B) 'seemingly'
(C) 'urgent' (D) 'political'

Ans: (B)

Question Number: 62 **Question Type: MCQ**

There are three boxes. One contains apples, another contains oranges and the last one contains both apples and oranges. All three are known to be incorrectly labelled. If you are permitted to open just one box and then pull out and inspect only one fruit, which box would you open to determine the contents of all three boxes?

(A) The box labelled 'Apples'
(B) The box labelled 'Apples and Oranges'
(C) The box labelled 'Oranges'
(D) Cannot be determined

Ans: (B)

Question Number: 63 **Question Type: MCQ**

X is a 30 digit number starting with the digit 4 followed by the digit 7. Then the number X^5 will have

(A) 90 digits (B) 91 digits
(C) 92 digits (D) 93 digits

Ans: (A)

Question Number: 64 **Question Type: MCQ**

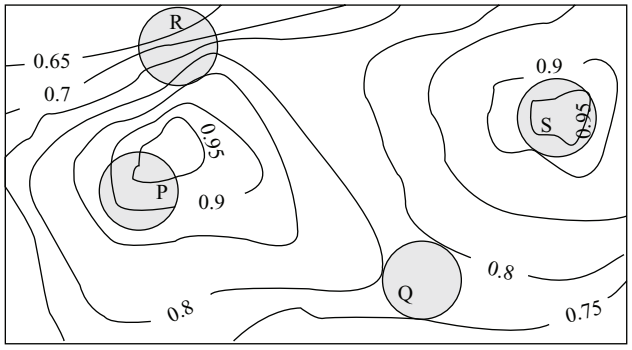
The number of roots of $e^x + 0.5x^2 - 2 = 0$ in the range $[-5, 5]$ is

(A) 0 (B) 1
(C) 2 (D) 3

Ans: (C)

Question Number: 65 **Question Type: MCQ**

An air pressure contour line joins locations in a region having the same atmospheric pressure. The following is an air pressure contour plot of a geographical region. Contour lines are shown at 0.05 bar intervals in this plot.



If the possibility of a thunderstorm is given by how fast air pressure rises or drops over a region, which of the following regions is most likely to have a thunderstorm?

(A) P (B) Q
(C) R (D) S

Ans: (C)

GATE 2018 Solved Paper
Electrical Engineering
Set – I

Number of Questions: 65 **Total Marks: 100.0**

Wrong answer for MCQ will result in negative marks, (–1/3) for 1 Mark Questions and (–2/3) for 2 Marks Question.

General Aptitude

Number of Questions: 10 **Section Marks: 15**

Q.1 to Q.5 carry one mark each.

Question Number: 1 **Question Type: MCQ**

"Since you have gone off the _____, the _____ sand is likely to damage the car." The words that best fill the blanks in the above sentence are:

(A) course, coarse (B) course, course
(C) coarse, course (D) coarse, coarse

Solution: The words that are apt for the two blanks are "course" (which means the route or direction followed) and "coarse" (rough in texture).
Hence, the correct option is (A).

Question Number: 2 **Question Type: MCQ**

"A common misconception among writers is that sentence structure mirrors though; the more _____ the structure, the more complicated the ideas."

(A) detailed (B) simple
(C) clear (D) convoluted

Solution: Options A, B, C are similar they all strengthen what the author feels about the relation between thought and sentence structure. Option D is a logical continuation of the main sentence hence it is best fits the blank.
Hence, the correct option is (D).

Question Number: 3 **Question Type: MCQ**

The three roots of the equation $f(x) = 0$ are $x = \{-2, 0, 3\}$. What are the three values of x for which $f(x - 3) = 0$?

(A) –5, –3, 0 (B) –2, 0, 3
(C) 0, 6, 8 (D) 1, 3, 6

Solution: $f(x) = 0$ for $x = -2, 0$ and 3.

$\therefore \quad f(x-3) = 0$

$\Rightarrow \quad x - 3 = -2, 0$ or 3

$\Rightarrow \quad x = 1, 3$ or 6.

Hence, the correct option is (D).

Question Number: 4 **Question Type: MCQ**

For what values of k given below is $\frac{(k+2)^2}{k-3}$ an integer?

(A) 4, 8, 18 (B) 4, 10, 16
(C) 4, 8, 28 (D) 8, 26, 28

Solution:

$$\frac{(k+2)^2}{k-3} = \frac{k^2 + 4k + 4}{k-3}$$

$$= \frac{k^2 - 3k + 7k - 21 + 25}{k-3}$$

$$= k + 7 + \frac{25}{k-3}$$

$K - 3$ has to be a factor of 25.

$\therefore \quad K - 3 = 1, 5, 25$ or $-1, -5, -25$

i.e., $K = 4, 8, 28$ or $2, -2, -22$. Among the options only 4, 8, 28 occur.

Hence, the correct option is (C).

Question Number: 5 **Question Type: MCQ**

Functions $F(a, b)$ and $G(a, b)$ are defined as follows:

$F(a, b) - (a - b)^2$ and $G(a, b) = |a - b|$, where $|x|$ represents the absolute value of x. What would be the value of $G(F(1, 3), G(1, 3))$?

(A) 2 (B) 4
(C) 6 (D) 36

Solution: $F(a, b) = (a - b)2$, $G(a, b) = |a - b|$

$F(1, 3) = 4$, $G(1, 3) = 2$

$\therefore \quad G(4, 2) = 2.$

Hence, the correct option is (A).

Q.6 to Q.10 carry two marks each.

Question Number: 6 **Question Type: MCQ**

An e-mail password must contain three characters. The password has to contain one numeral from 0 to 9, one upper case and one lower case character from the English alphabet. How many distinct passwords are possible?

(A) 6,760 (B) 13,520
(C) 40,560 (D) 1,05,456

Solution: The numeral can be selected in 10 ways, while each of other two characters can be selected in 26 ways. The 3 selected characters can be permuted in 6 ways.

$\therefore$ Number of passwords

$= 10(26)(26)(6)$

$= 40560.$

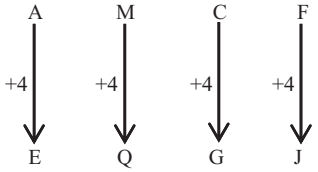
Hence, the correct option is (C).

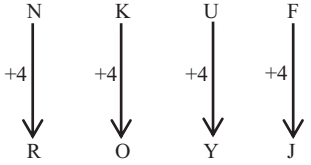
Question Number: 7 **Question Type: MCQ**

In a certain code. AMCF is written as EQGJ and NKUF is written as ROYJ. How will DHLP be written in that code?

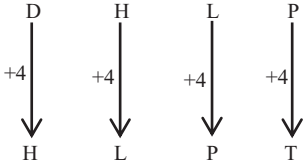
(A) RSTN (B) TLPH
(C) HLPT (D) XSVR

Solution: The code for the given words is as follows:





So, the code for the given word will be:



Hence, the correct option is (C).

Question Number: 8 **Question Type: MCQ**

A class of twelve children has two more boys than girls. A group of three children are randomly picked from this class to accompany the teacher on a field trip. What is the probability that the group accompanying the teacher contains more girls than boys?

(A) 0 (B) $\frac{325}{864}$

(C) $\frac{525}{864}$ (D) $\frac{5}{12}$

Solution: Let the number of boys and girls be b and g,

Given $b = g + 2$

and $b + g = 12$

$\therefore$ Number of boys $= 7$

Number of girls $= 5$

The probability of selecting a boy is $\frac{7}{12}$

The probability of selecting a girl is $\frac{5}{12}$

Assume that three students are selected randomly one after another with replacement.

The favorable cases that the group consists girls more than boys is

(i) all are girls
(ii) two girls and one boy

Case I: The probability that all are girls is

$$\frac{5}{12}\cdot\frac{5}{12}\cdot\frac{5}{12}=\frac{125}{1728}$$

Case II: The probability that two girls and one boy in the group is $\frac{5}{12}\times\frac{5}{12}\times\frac{7}{12}$

$\therefore$ The probability $= 3\times\frac{5}{12}\times\frac{5}{12}\times\frac{7}{12}$

$= \frac{525}{1728}$

Required probability $= \frac{125}{1728}+\frac{525}{1728}$

$= \frac{650}{1728}=\frac{325}{864}$

Hence, the correct option is (B).

Question Number: 9 **Question Type: MCQ**

A designer uses marbles of four different colours for his designs. The cost of each marble is the same irrespective of the colour. The table below shows the percentage of marbles of each colour used in the current design. The cost of each marble increased by 25%. Therefore, the designer decided to reduce equal number of marbles of each colour to keep the total cost unchanged. What is the percentage of blue marbles in the new design?

Blue	Black	Red	Yellow
40%	25%	20%	15%

(A) 35.75 (B) 40.25
(C) 43.75 (D) 46.25

Solution: Let the total number of marbles be $100n$. The number of blue, black, red, yellow are $40n$, $25n$, $20n$, $15n$.

The price of each marble increased by 25% (to $\frac{5}{4}$ its original value.) Therefore, the number of marbles has to reduce to $\frac{4}{5}$ so that the cost remains unchanged. It has to be $80n$, i.e., it has to reduce by $20n$. As the number reduced for all the colors are equal, the number in each color has to reduce by $5n$.

The number of blue, black, red, yellow marbles in the new design are $35n$, $20n$, $15n$, $10n$. The percentage of blue marbles in this new design is

35/35 + 20 + 15 + 10, i.e., 7/16, which is 43.75%

Hence, the correct option is (C).

Question Number: 10 **Question Type: MCQ**

P, *Q*, *R* and *S* crossed a lake in a boat that can hold a maximum of two persons, with only one set of oars. The following additional facts are available.

(i) The boat held two persons on each of the three forward trips across the lake and one person on each of the two trips.
(ii) P is unable to row when someone else is in the boat.
(iii) Q is unable to row with anyone else except R.
(iv) Each person rowed for at least one trip.
(v) Only one person can row during a trip.

Who rowed twice?

(A) P (B) Q
(C) R (D) S

Solution: On the first trip Q and R will travel, with Q rowing the boat. R will return alone and take P along with him. R will row the boat this time as P can not row when come one is with him. P alone will come back and take S along with him. S will row the boat this time. Only R rowed the boat twice.

Hence, the correct option is (C).

Electrical Engineering

Q.1 to Q.25 carry one mark each.

Number of Questions: 55 **Section Marks: 85**

Question Number: 1 **Question Type: MCQ**

A single-phase 100 kVA, 1000 V/100 V, 50 Hz transformer has a voltage drop of 5% across its series impedance at full load. Of this, 3% is due to resistance. The percentage regulation of the transformer at full load with 0.8 lagging power factor is

(A) 4.8 (B) 6.8
(C) 8.8 (D) 10.8

Solution: Given, 1 – ϕ, 100 kVA, 1000 V/100 V, 50 Hz

$\%Z = 5\%, \%R = 3\%$

$$\therefore \quad \%X = \sqrt{(\%Z)^2 - (\%R)^2}$$
$$= \sqrt{5^2 - 3^2} = 4\%$$

$\therefore$ % voltage regulation

$$= \%R(\cos\phi_2) + \%x(\sin\phi_2)$$

(for lagging P.F)

$$= 3 \times 0.8 + 4 \times 0.6$$
$$= 4.8\%$$

Hence, the correct option is (A).

Question Number: 2 **Question Type: MCQ**

In a salient pole synchronous motor, the developed reluctance torque attains the maximum value when the load angle in electrical degrees is:

(A) 0 (B) 45
(C) 60 (D) 90

Solution: For salient pole synchronous motor,

$$P = \frac{Ev}{X_S}\sin\delta + \frac{v^2}{2}\left[\frac{1}{x_q} - \frac{1}{x_d}\right]\sin(2\delta)$$

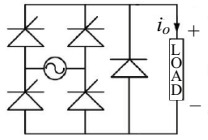
$$\therefore \quad P_{\text{reluctance}} = \frac{v^2}{2}\left[\frac{1}{x_q} - \frac{1}{x_d}\right]\sin(2\delta)$$

If $\delta = 45°$ then $P_{\text{reluctance}}$ is maximum.

Hence, the correct option is (B).

Question Number: 3 **Question Type: MCQ**

A single phase fully controlled rectifier is supplying a load with an anti-parallel diode as shown in the figure. All switches and diodes are ideal. Which one of the following is true for instantaneous load voltage and current?



(A) $v_0 \le 0$ and $i_0 < 0$ (B) $v_0 <$ and $i_0 < 0$
(C) $v_0 \ge 0$ and $i_0 \ge 0$ (D) $v_0 < 0$ and $i_0 \ge 0$

Solution: Since the freewheeling diode is connected at output section, no negative ripple appear in the output.

$$\therefore \quad V_0 \ge 0$$

The given bridge is Thyristor based bridge.

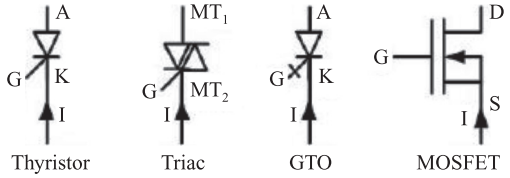
$\therefore$ Current flows only from Anode to Cathode.

$$\therefore \quad i_0 \ge 0$$

Hence, the correct option is (C).

Question Number: 4 **Question Type: MCQ**

Four power semiconductor devices are shown in the figure along with their relevant terminals. The device(s) that can carry dc current continuously in the direction shown when gated appropriately is (are):



(A) Triac only
(B) Triac and MOSFET
(C) Triac and GTO
(D) Thyristor and Triac

Solution: Among all the given devices, only TRIAC allow bidirectional current flow.

∴ TRIAC

Hence, the correct option is (A).

Question Number: 5 **Question Type: MCQ**

Two wattmeter method is used for measurement of power in a balanced three-phase load supplied from a balanced three-phase system. If one of the wattmeters reads half of the other (both positive), then the power factor of the load is:

(A) 0.532 (B) 0.632

(C) 0.707 (D) 0.866

Solution: Given, one of the wattmeter reads half of the other.

Let, $w_2 = w_1/2$

The power factor is given by,

$$\text{P.F} = \cos\phi$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{3}(w_1 - w_2)}{w_1 + w_2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}\left(w_1 - \frac{w_1}{2}\right)}{w_1 + \frac{w_1}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

$$\therefore \quad \text{P.F} = \cos 30° = 0.866$$

Hence, the correct option is (D).

Question Number: 6 **Question Type: MCQ**

Consider a lossy transmission line with V_1 and V_2 as the sending and receiving end voltages respectively. Z and X are the series impedance and reactance of the line respectively. The steady-state stability limit for the transmission line will be:

(A) greater than $\left|\frac{V_1V_2}{X}\right|$ (B) less than $\left|\frac{V_1V_2}{X}\right|$

(C) equal to $\left|\frac{V_1V_2}{X}\right|$ (D) equal to $\left|\frac{V_1V_2}{X13.5}\right|$

Solution:
$$P_r = \left\{\begin{array}{l} \frac{|V_s||V_r|}{|B|}\cos(\beta - \delta) \\ -\frac{|A|}{|B|}|V_r|^2\cos(\beta - \alpha)\end{array}\right\} \text{MW} \qquad (1)$$

If resistance of the transmission line

$= 0$

$\beta = 90°$

$$P_r = \frac{|V_s||V_r|}{X}\sin\delta,$$

$$P_{r\max} = \frac{|V_s||V_r|}{X} \qquad (2)$$

Equation (1) is always less than Equation (2).

Hence, the correct option is (B).

Question Number: 7 **Question Type: MCQ**

The graph of a network has 8 nodes and 5 independent loops. The number of branches of the graph is:

(A) 11 (B) 12

(C) 13 (D) 14

Solution: From the given data, network has

$n = 8, l = 5$

$b = ?$

$l = b - n + 1$

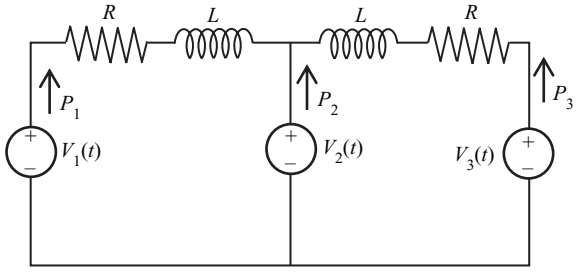
$5 = b - 8 + 1$

$b = 12$

Hence, the correct option is (B).

Question Number: 8 **Question Type: MCQ**

In the figure the voltages are $v_1(t) = 100\cos(\omega t)$, $v_2(t) = 100\cos(\omega t + \pi/18)$ and $v_3(t) = 100\cos(\omega t + \pi/36)$. The circuit is in sinusoidal steady state, and $P << \omega L$. P_1, P_2 and P_3 are the average power outputs. Which one of the following statement is true?



(A) $P_1 = P_2 = P_3 = 0$ (B) $P_1 < 0, P_2 > 0, P_3 > 0$

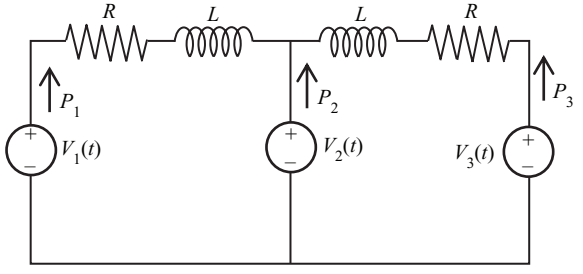
(C) $P_1 < 0, P_2 > 0, P_3 < 0$ (D) $P_1 > 0, P_2 < 0, P_3 > 0$

Solution: Given

$V_1(t) = 100\cos\omega t$

$V_2(t) = 100\cos\left(\omega t + \frac{\pi}{18}\right)$

$V_3(t) = 100\cos\left(\omega t + \frac{\pi}{36}\right)$



from the given data

$$V_1 = V_m \angle 0$$
$$V_2 = V_m \angle 10^\circ$$
$$V_3 = V_m \angle 5^\circ$$

So, V_2 leads V_1 and V_3

$$P_2 > 0, P_1 \text{ and } P_3 < 0.$$

Hence, the correct option is (C).

Question Number: 9 **Question Type: MCQ**

Match the transfer functions of the second-order systems with the nature of the systems given below:

	Transfer function		Nature of system
P.	$\frac{15}{s^2 + 5s + 15}$	I.	Over damped
Q.	$\frac{25}{s^2 + 10s + 25}$	II.	Critically damped
R.	$\frac{35}{s^2 + 18s + 35}$	III.	Under damped

(A) P-I, Q-II, R-III
(B) P-II, Q-I, R-III
(C) P-III, Q-II, R-I
(D) P-III, Q-I, R-II

Solution: P: $\frac{15}{S^2 + 5S + 15}$

$$\xi = \frac{5}{2\sqrt{15}} = 0.2581 \rightarrow \text{under damped system}$$

Q: $\frac{25}{S^2 + 5S + 25}$

$$\xi = \frac{10}{2\sqrt{15}} = 1 \rightarrow \text{critically damped system.}$$

R: $\frac{35}{S^2 + 18S + 35}$

$$= 1.521 \rightarrow \text{over damped system}$$

P-III
Q-II
R-I.

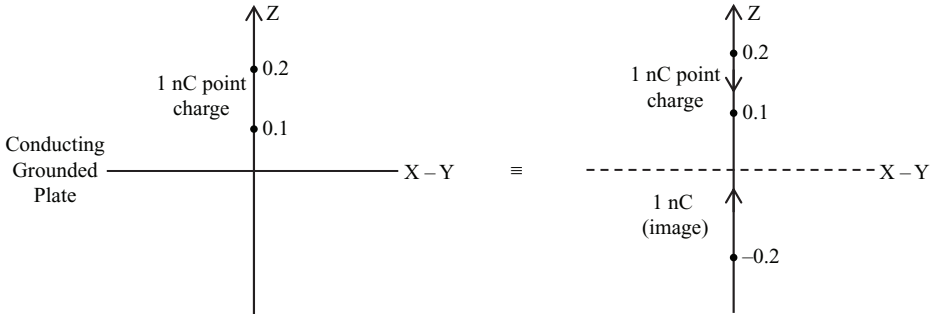
Hence, the correct option is (C).

Question Number: 10 **Question Type: MCQ**

A positive charge of 1 nC is placed at (0, 0, 0.2) where all dimensions are in metres. Consider the x–y plane to be a conducting ground plane. Take $\in_0 = 8.85 \times 10^{-12}$ F/m. The z component of the E field at (0, 0, 0.1) closest to:

(A) 899.18 V/m (B) –899.18 V/m
(C) 999.09 V/m (D) –999.09 V/m

Solution:



Electric field intensity due to point charge

$$\bar{E} = \frac{Q}{4\pi\in_0 R^2}\hat{a}_R$$

The Z–component of $\bar{E}$ at P(0, 0, 0.1) due to point charge (+1 nC) and due to its image (–1 nC) is given by

$$\bar{E} = E_z\hat{a}_Z = \frac{10^{-9}(-\hat{a}_Z)}{4\pi \times 8.85 \times 10^{-12} \times (0.1)^2} + \frac{\left(-10^{-9}\right)\hat{a}_Z}{4\pi \times 8.85 \times 10^{-12} \times (0.3)^2}$$

$$E_z \approx -999.09 \text{ v/m}$$

Hence, the correct option is (D).

Question Number: 11 **Question Type: MCQ**

Let f be a real-valued function of a real variable defined as $f(x) = x^2$ for $x \le 0$, and $f(x) = -x^2$ for $x < 0$. Which one of the following statements is true?

(A) $f(x)$ is discontinuous at $x = 0$

(B) $f(x)$ is continuous but not differentiable at $x = 0$

(C) $f(x)$ is differentiable but its first derivative is not continuous at $x = 0$

(D) $f(x)$ is differentiable but its first derivative is not differentiable at $x = 0$

Solution: Given $f(x) = \begin{cases} x^2 & \text{for } x \ge 0 \\ -x^2 & \text{for } x < 0 \end{cases}$

$f(x)$ is continuous at $x = 0$

$$f'(x) = \begin{cases} 2x & \text{for } x \ge 0 \\ -2x & \text{for } x < 0 \end{cases}$$

And at $x = 0$; LHD = RHD for $f(x)$

So, $f(x)$ is differentiable at $x = 0$ as well as

$f'(x)$ is continuous at $x = 0$

$$f''(x) = \begin{cases} 2 & \text{for } x \ge 0 \\ -2 & \text{for } x < 0 \end{cases}$$

At $x = 0$; $f''(0^-) \ne f''(0^+)$

so, $f'(x)$ is not differentiable at $x = 0$

Hence, the correct option is (D).

Question Number: 12 **Question Type: MCQ**

The value of the directional derivative of the function $\Phi(x, y, z) = xy^2 + yz^2 + zx^2$ at the point (2, –1, 1) in the direction of the vector $p = I + 2j + 2k$ is:

(A) 1 (B) 0.84

(C) 0.93 (D) 0.9

Solution: Given

$$\phi(x, y, z) = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = \left(y^2 + 2xz\right)i + \left(2xy + z^2\right)\hat{j} + (2yz + x^2)\bar{k}$$

$$\phi\nabla_{\text{at }(2,-1,1)} = 5\bar{i} - 3\bar{j} + 2\bar{k}$$

Given $\bar{P} = \bar{i} + 2\bar{j} + 2\bar{k}$

$$\therefore \quad \hat{n} = \frac{\bar{P}}{|\bar{P}|} = \frac{i + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{i + 2\bar{j} + 2\bar{k}}{3}$$

$$\therefore \quad \hat{n} = \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

So, the directional derivative of $\phi(x, y, z)$ in the direction of the vector $\bar{P}$ is

$$\nabla\phi \cdot \hat{n} = (5\bar{i} - 3\bar{j} + 2\bar{k}) \cdot \left(\frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}\right)$$

$$= \frac{5}{3} - 2 + \frac{4}{3} = 1$$

Hence, the correct option is (A).

Question Number: 13 **Question Type: MCQ**

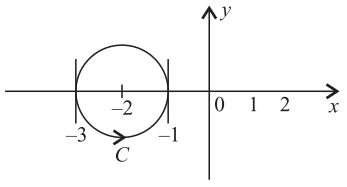
The value of the integral $\oint_C \frac{z+1}{z^2-4}dz$ in a counter clockwise direction around a circle C of radius 1 with centre at the point $z = -2$ is:

(A) $\frac{\pi i}{2}$ (B) $2\pi i$

(C) $-\frac{\pi i}{2}$ (D) $-2\pi i$

Solution: Let $I = \oint_C \frac{Z+1}{Z^2-4}dZ$

$Z = \pm 2$ are the singularities of $\frac{Z+1}{Z^2-4}$, of which $Z = -2$ lies inside C and $Z = 2$ lies outside C.



$$\therefore \quad I = \oint_C \frac{Z+1}{Z^2-4}dZ = \oint_C \frac{(Z + 1/Z - 2)}{Z+2}dZ$$

$$= 2\pi i\left(\frac{Z+1}{Z-2}\right)_{\text{at } Z=-2}$$

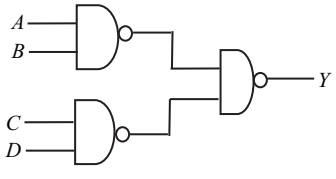
$$= 2\pi i\left(\frac{-1}{-4}\right)$$

$$= \frac{\pi i}{2}$$

Hence, the correct option is (A).

Question Number: 14 **Question Type: MCQ**

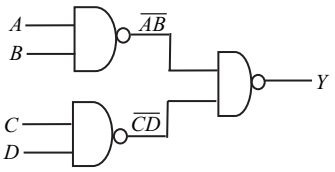
In the logic circuit shown in the figure, Y is given by



(A) $Y = ABCD$ (B) $Y = (A + B)(C + D)$

(C) $Y = A + B + C + D$ (D) $Y = AB + CD$

Solution:

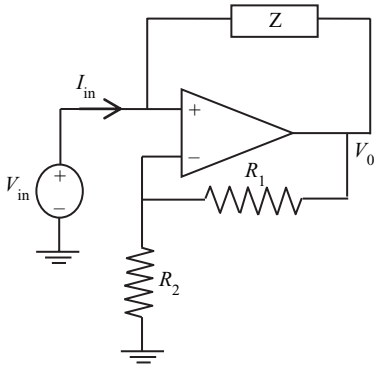


$$Y = \overline{\overline{AB} \cdot \overline{CD}} = AB + CD$$

Hence, the correct option is (D).

Question Number: 15 **Question Type: MCQ**

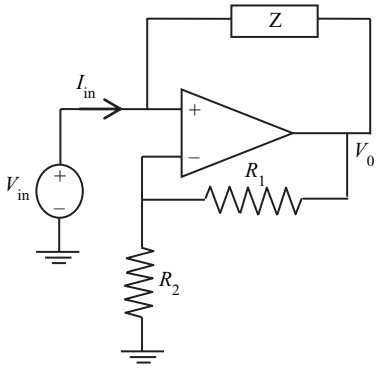
To op-amp shown in the figure is ideal. The input impedance $\frac{v_{in}}{i_{in}}$ is given by:



(A) $Z\frac{R_1}{R_2}$ (B) $-Z\frac{R_2}{R_1}$

(C) Z (D) $-Z\frac{R_1}{R_1+R_2}$

Solution:



$$I_{in} = \frac{V_{in} - V_0}{z}$$

$$\frac{V_{out}R_2}{R_1 + R_2} = V_{in}$$

$$V_{out} = \frac{V_{in}(R_1 + R_2)}{R_2}$$

$$I_{in} = \frac{1}{z}\left[V_{in} - V_{in}\left(1 + \frac{R_1}{R_2}\right)\right]$$

$$I_{in} = \frac{1}{z}\left[\frac{-V_{in}R_1}{R_2}\right]$$

$$R_{in} = \frac{V_{in}}{I_{in}} = \frac{-zR_2}{R_1}$$

Hence, the correct option is (B).

Question Number: 16 **Question Type: MCQ**

A continuous-time input signal $x(t)$ is an eigenfunction of an LTI system, if the output is:

(A) $kx(t)$, where k is an eigenvalue
(B) $ke^{j\omega t}x(t)$, where k is an eigenvalue and $e^{j\omega t}$ is a complex exponential signal
(C) $x(t)e^{j\omega t}$, where $e^{j\omega t}$ is a complex exponential signal.
(D) $kH(\omega)$, where k is an eigenvalue and $H(\omega)$ is a frequency response of the system

Solution: $x(t)$ is an eigen function of an LTI system, if the output is $k\,x(t)$, where 'k' is an eigen value.
Hence, the correct option is (A).

Question Number: 17 **Question Type: NAT**

Consider a non-singular 2 × 2 square matrix A. If trace $(A) = 4$ and trace $(A^2) = 5$, the determinant of the matrix A is _____ (up to 1 decimal place).

Solution: Given A is a 2 × 2 non-singular matrix.

Let λ_1 and λ_2 be the eigen values of A.

$\Rightarrow \lambda_1^2$ and λ_2^2 will be the eigen values of A^2.

Trace $(A) = 4 \Rightarrow \lambda_1 + \lambda_2 = 4$

Trace $(A^2) = 5 \Rightarrow \lambda_1^2 + \lambda_2^2 = 5$

Now $(\lambda_1 + \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2$

$\Rightarrow \quad 4^2 = 5 + 2\lambda_1\lambda_2$

$\Rightarrow \quad \lambda_1\lambda_2 = \frac{11}{2} = 5.5$

Hence, the correct answer is (5.5).

Question Number: 18 **Question Type: NAT**

Let f be a real-valued function of a real variable defined as $f(x) = x - [x]$, where $[x]$ denotes the largest integer less than or equal to x. The value of $\int_{0.25}^{1.25} f(x)\,dx$ is _____ (up to 2 decimal places).

Solution: Given

$$f(x) = x - [x]$$

$$\int_{0.25}^{1.25} f(x)\,dx = \int_{0.25}^{1.25} (x - [x])\,dx$$

$$= \int_{0.25}^{1.25} x dx - \int_{0.25}^{1.25} [x]\, dx$$

$$= \frac{x^2}{2}\Big]_{0.25}^{1.25} - \left(\int_{0.25}^{1} 0 dx + \int_{1}^{1.25} 1 dx \right)$$

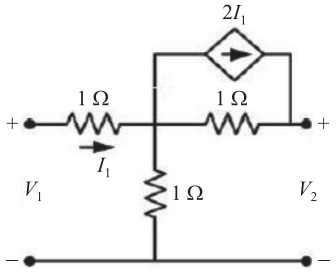
$$= \left[\frac{3}{4} - x\right]_{1}^{1.25}$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = 0.5$$

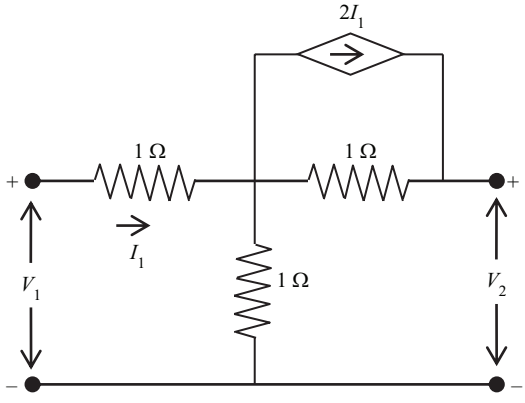
Hence, the correct answer is (0.5).

Question Number: 19 **Question Type: NAT**

In the two-port network shown, the h_{11} parameter (where, $h_{11} = \frac{V_1}{I_1}$, when $V_2 = 0$) in ohms is _____ (up to 2 decimal places).



Solution: Given

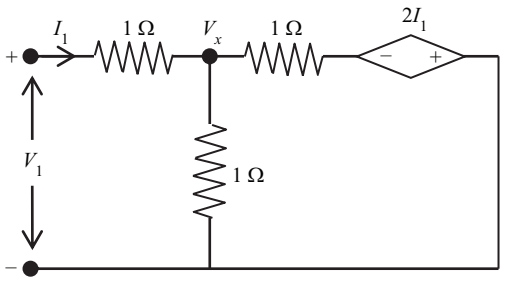


$$h_{11} = ?$$

$$h_{11} = \frac{V_1}{I_1}$$

at $V_2 = 0$

Applying source transform to given network and short the second port.



$$\frac{V_x - V_1}{1} + \frac{V_x}{1} + \frac{V_x + 2I_1}{1} = 0$$

$$-I_1 + V_x + V_x + 2I_1 = 0$$

$$2V_x + I_1 = 0$$

$$V_x = \frac{-I_1}{2}$$

But $$\frac{V_1 - V_x}{1} = I_1$$

$$V_1 + 0.5I_1 = I_1$$

$$V_1 = 0.5I_1$$

$$\frac{V_1}{I_1} = h_{11} = 0.5\,\Omega$$

Question Number: 20 **Question Type: NAT**

The series impedance matrix of a short three-phase transmission line in phase coordinates is $\begin{bmatrix} Z_s & Z_m & Z_m \\ Z_m & Z_s & Z_m \\ Z_m & Z_m & Z_s \end{bmatrix}$. If the positive sequence impedance is $(1 + j10)\,\Omega$, and the zero sequence is $(4 + j31)\,\Omega$, then the imaginary part of Z_m (in Ω) is _____ (upto 2 decimal places).

Solution:

$$Z_s - Z_m = 1 + j10$$

$$Z_s + 2Z_m = 4 + j31$$

$$2Z_s - 2Z_m = 2 + j20$$

$$Z_s + 2Z_m = 4 + j31$$

$$3Z_s = 6 + j51$$

$$Z_s = (2 + j17)\,\Omega$$

$$2 + j17 - 1 - j10 = Z_m$$

$$Z_m = j7 + 1$$

Imaginary part is 7.

Hence, the correct answer is 7 to 7.

Question Number: 21 **Question Type: NAT**

The positive, negative and zero sequence impedances of a 125 MVA, three-phase 15.5 kV, start-grounded, 50 Hz generator are $j0.1$ pu, $j0.05$ and $j0.01$ pu, respectively on the machine rating base. The machine is unloaded and working at the rated terminal voltage. If the grounding impedance of the generator is $j0.01$ pu, then the magnitude of fault

current for a b-phase to ground fault (in kA) is _____ (up to 2 decimal places).

Solution: If

$$= \frac{3Ea1}{Z_1 + Z_2 + Z_0 + 3Z_n} \text{ pu}$$

$$I_f(\text{KA}) = I_f(\text{pu}) \times I_{\text{base}}$$

$$I_{\text{base}} = \frac{125}{\sqrt{3} \times 15.5} \times 10^3$$

$$= 4656.050 \text{ (A)}$$

$$I_f = \frac{3 \times 1 \times 4656.050}{0.1 + 0.05 + 0.01 + 3(0.01)}$$

$$= 73.5236 \text{ (KA)}$$

Question Number: 22 **Question Type: NAT**

A 1000 × 1000 bus admittance matrix for an electric power system has 8000 non-zero elements. The minimum number of branches (transmission lines and transformers) in this system are _____ (up to 2 decimal points).

Solution: Number of transmission lines

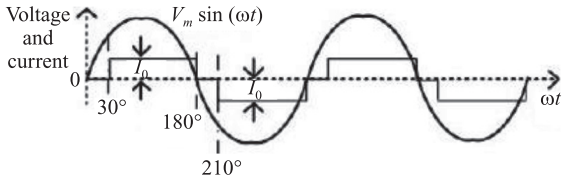
$$= \left(\frac{\text{Number of non-zero off diagonal elements}}{2} \right)$$

Number of non-zero off diagonal elements

$$= \frac{8000 - 1000}{2} = 3500.$$

Question Number: 23 **Question Type: NAT**

The waveform of the current drawn by a semi-converter from a sinusoidal AC voltage source is shown in the figure. If $I_0 = 20$ A, the rms value of fundamental component of the current is _____ A (up to 2 decimal places)



Solution:

$$I_{S1} = \frac{4I_S}{\pi} \cos\frac{\alpha}{2}$$

$$I_{S1r} = \frac{2\sqrt{2} \times 20}{\pi} \cos 15 = 17.40 \text{ A}$$

Hence, the correct answer is (17.40).

Question Number: 24 **Question Type: VAT**

A separately excited dc motor has an armature resistance R_a = 0.05 Ω. The field excitation is kept constant. At an armature voltage of 100 V, the motor produces a torque of 500 Nm at zero speed. Neglecting all mechanical losses, the no-load speed of the motor (in radian/s) for an armature voltage of 150 V is _____ (up to 2 decimal places).

Solution: Given, separately excited d.c. motor

$\Rightarrow$ ϕ = constant.

$R_a = 0.05\ \Omega,\ v_{t_1} = 100$ V.

$T_1 = 500$ Nm at speed, $N = 0$ rpm.

$\therefore$ $v_{t_1} = E_{b_1} + I_{a_1} R_a$ (1)

$E_{b_1} = k_a \phi \omega_1$

$E_{b_1} = 0$ $(\because \omega_1 = 0)$

$\therefore$ From Equation (1),

$100 = I_{a_1} \times 0.05$

$\Rightarrow$ $I_{a_1} = 2000$ A

Under no-load condition, no-load voltage drop is very small.

$\therefore$ $E_b \cong v_t \Rightarrow v_{t_2} = E_{b_2}$

$\therefore$ $E_{b_2} = 150$ V.

$T_{em_1} = k_a \phi I_{a1}$

$500 = (k_a\phi) \times 2000 \Rightarrow (k_a\phi) = \frac{1}{4}$ (2)

$E_{b_2} = k_a\phi \cdot \omega_2 \Rightarrow \omega_2 = \frac{E_{b_2}}{K_a\phi}$

$\therefore$ $\omega_2 = 150 \times 4 = 600$ rad/sec.

Hence, the correct answer is (600).

Question Number: 25 **Question Type: NAT**

Consider a unity feedback system with forward transfer function given by

$$G(s) = \frac{1}{(s+1)(s+2)}.$$

The steady-state error in the output of the system for a unit-step input is _____ (up to 2 decimal places).

Solution:

$$G(s) = \frac{1}{(S+1)(S+2)}$$

$$R(S) = \frac{1}{S}$$

$$e_{ss} = \frac{A}{1 + K_P}$$

$$K_P = \lim_{S \to 0} G(s) = \frac{1}{2}$$

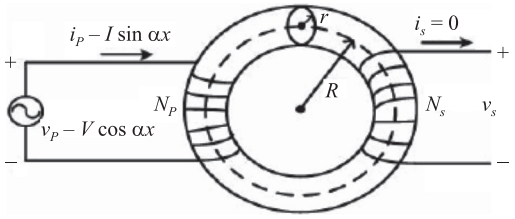
$$e_{ss} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

$$= 0.6666$$

Q.26 to Q.55 carry two marks each.

Question Number: 26 **Question Type: MCQ**

A transformer with toroidal core of permeability μ is shown in the figure. Assuming uniform flux density across the circular core cross-section of radius $r << R$, and neglecting any leakage flux, the best estimate for the mean radius R is:



(A) $\dfrac{\mu V r^2 N_P^2 \omega}{I}$ (B) $\dfrac{\mu I r^2 N_P N_s \omega}{V}$

(C) $\dfrac{\mu V r^2 N_P^2 \omega}{2I}$ (D) $\dfrac{\mu I r^2 N_P^2 \omega}{2V}$

Solution: Given Data: permeability of the core $= \mu$

Flux Density (B) is uniform,

condition: $r << R$

mean radius 'R' = ?

In General, flux (ϕ) $= \dfrac{\text{MMF}}{\text{Reluctance}}$

$\therefore$ Reluctance $= \dfrac{\text{MMF}}{\text{Flux } (\phi)}$

$\therefore$ In General, EMF induced

$$E = N \frac{d\phi}{dt}$$

$$\therefore \quad \phi = \frac{1}{N} \int_0^t E dt$$

$$\therefore \quad \phi = \frac{1}{N_p} \int_0^t E dt$$

$\because$ Lenz law states $E = -V_p$

$$\therefore \quad \phi = \frac{1}{N_p} \int_0^t -V_p dt$$

$$\therefore \quad \phi = \frac{1}{N_p} \int_0^t v \cos \omega t \, dt$$

$$\Rightarrow \quad \phi = \frac{v}{\omega N_p} \sin \omega t$$

$$\therefore \text{ Reluctance} = \frac{N_p \cdot I}{\dfrac{V}{\omega N_p}} \qquad \because (I_P = I \sin \omega t)$$

$$\Rightarrow \text{ Reluctance} = \frac{N_p^2 \cdot \omega I}{V}$$

Reluctance in terms of radius $\Rightarrow \dfrac{2R}{\mu r^2}$

$$\therefore \quad \frac{2R}{\mu r^2} = \frac{N_p^2 \cdot \omega I}{V}$$

$$\therefore \quad R = \frac{N_p^2 \cdot \omega I \mu r^2}{2V}$$

Hence, the correct option is (D).

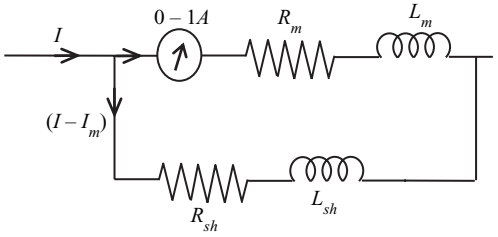
Question Number: 27 **Question Type: MCQ**

A 0–1 Ampere moving iron ammeter has an internal resistance of 50 mΩ and inductance of 0.1 mH. A shunt coil is connected to extend its range to 0–10 Ampere for all operating frequencies. The time constant in milliseconds and resistance in mΩ of the shunt coil respectively are:

(A) 2, 5.55 (B) 2, 1

(C) 2.18, 0.55 (D) 11.1, 2

Solution:



Given, $R_m = 50$ mΩ

$L_m = 0.1$ H

To make the meter independent of frequency, meter time constant should be equal to shunt coil time constant.

$$\tau_m = \tau_{sh} \Rightarrow \tau_{sh} = \frac{0.1}{50 \times 10^{-3}} = 2$$

$$\frac{L_m}{R_m} = \tau_{sh}$$

To extend meter range to 10 A,

$$I_m R_m = (I - I_m) R_{sh}$$

$$\Rightarrow \quad 1 \times 50 \times 10^{-3} = 9 \times R_{sh}$$

$$R_{sh} = 5.55$$

Hence, the correct option is (A).

Question Number: 28 **Question Type: MCQ**

The positive, negative and zero sequence impedances of a three phase generator are Z_1, Z_2 and Z_0, respectively. For a line-to-line fault with fault impedance Z_f, the fault current is $I_{f1} = kI_f$, where I_f is the fault current with zero fault impedance. The relation between Z_f and k is:

(A) $Z_f \dfrac{(z_1 + z_2)_{(1-k)}}{k}$ (B) $Z_f \dfrac{(z_1 + z_2)_{(1+k)}}{k}$

(C) $Z_f \dfrac{(z_1 + z_2)_k}{1 - k}$ (D) $Z_f \dfrac{(z_1 + z_2)_k}{1 + k}$

Solution: LL fault current without fault impedance

$$|I_{f_1}| = \frac{\sqrt{3}Ea1}{Z_1 + Z_2}$$

LL fault current with fault impedance

$$|I_{f_1}| = \frac{\sqrt{3}Ea1}{Z_1 + Z_2 + Z_f}$$

$$I_{f_1} = K_{I_f}$$

$$\frac{1}{Z_1 + Z_2 + Z_f} = \frac{K}{Z_1 + Z_2}$$

$$Z_1 + Z_2 = K(Z_1 + Z_2) + KZ_f$$

$$Z_1 + Z_2 - K(Z_1 + Z_2) = KZ_f$$

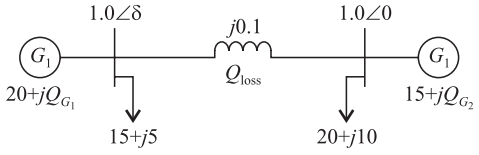
$$(Z_1 + Z_2)(1 - K) = Z_f K$$

$$Z_f = \left(\frac{1-K}{K}\right)(Z_1 + Z_2)$$

Hence, the correct option is (A).

Question Number: 29 **Question Type: MCQ**

Consider the two bus power system network with given loads as shown in the figure. All the values shown in the figure are in per unit. The reactive power supplied by generator G_1 and G_2 are Q_{G1} and Q_{G2}, respectively. The per unit values of Q_{G1} and Q_{G2}, and line reactive power loss (Q_{loss}) respectively are.



(A) 5.00, 12.68, 2.68 (B) 6.34, 10.00, 1.34
(C) 6.34, 11.34, 2.68 (D) 5.00, 11.34, 1.34

Solution:

$$P_s = \frac{|V_s||V_r|}{x}\sin\delta$$

$$5 = 10\sin\delta$$

$$\delta = 30°$$

$$Q_S = \left\{\frac{|V_S|^2}{X}\sin 90° - \frac{|V_s||V_r|}{X}\cos\delta\right\}$$

$$Q_S = 10 - 10\cos 30° = 1.33974$$

$$Q_R = -1.33974 \text{ (Receiving end supplies)}$$

$$Q_{\text{Line}} = Q_{\text{Loss}} = Q_S - Q_R$$

$$= 2.68 \text{ pu}$$

$$Q_{G1} = Q_{\text{Load}} + Q_S$$

$$= 6.34 \text{ pu}$$

$$Q_{G2} = 11.34 \text{ pu.}$$

Hence, the correct option is (C).

Question Number: 30 **Question Type: MCQ**

The pre-unit power output of a salient-pole generator which is connected to an infinite bus, is given by the expression, $P = 1.4 \sin\delta + 0.15 \sin 2\delta$, where δ is the load angle. Newton–Raphson method is used to calculate the value of δ for $P = 0.8$ pu. If the initial guess is 30°, then its value (in degree) at the end of the first iteration is:

(A) 15° (B) 27.48°
(C) 28.74° (D) 31.20°

Solution:

$$P = 1.4\sin\delta + 0.15\sin 2\delta$$

$$\delta_o = 30°$$

$$\frac{\delta P}{\partial\delta} = 1.4\cos\delta + 0.30\cos 2\delta$$

$$\begin{bmatrix}\Delta P \\ \Delta Q\end{bmatrix} = \begin{bmatrix}J_1 & J_2 \\ J_3 & J_4\end{bmatrix}\begin{bmatrix}\Delta\delta \\ \Delta|V|\end{bmatrix}$$

$$\Delta P = J_1 \Delta\delta$$

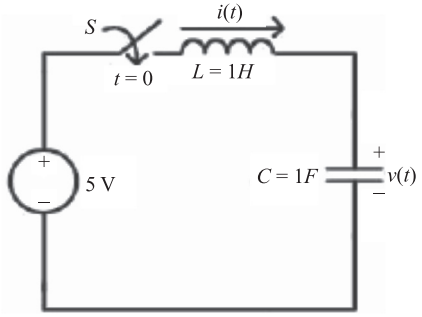
$$\Delta\delta = [J_1]^{-1}\Delta P$$

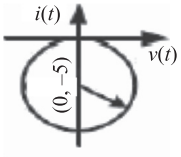
$$\delta = \delta_o - \Delta\delta$$

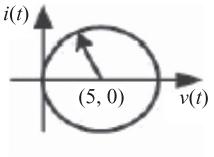
$$= 28.79°$$

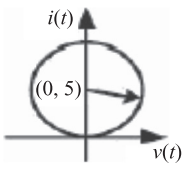
Question Number: 31 **Question Type: MCQ**

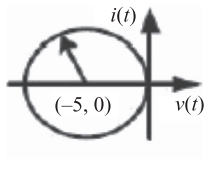
A DC voltage source is connected to a series L–C circuit by turning on the switch S at time $t = 0$ as shown in the figure. Assume $i(0) = 0$, $v(0) = 0$. Which one of the following circular loci represents the plot of $i(t)$ versus $v(t)$?



(A)


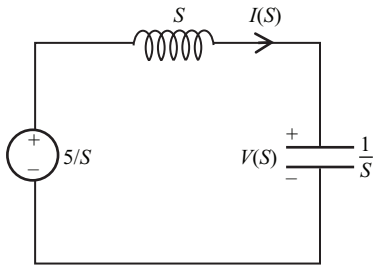
(B)


(C)


(D)


Solution: Given $i(0) = 0$,
$V(0) = 0$

Redraw the circuit in S domain



$$I(S) = \frac{\frac{5}{S}}{S + \frac{1}{S}} = \frac{5}{S^2 + 1}$$

$$i(t) = 5 \sin t \text{ Amp} \qquad \text{(i)}$$

$$V_c(t) = \frac{1}{C}\int_0^t i\,dt = \int_0^t 5 \sin t\,dt$$

$$V_c(t) = 5\left[1 - \cos t\right]V \qquad \text{(ii)}$$

From Equations (i) and (ii)

$$i^2(t) + (V_c(t) - 5)^2 = (5 \sin t)^2 + (-5 \cos t)^2$$

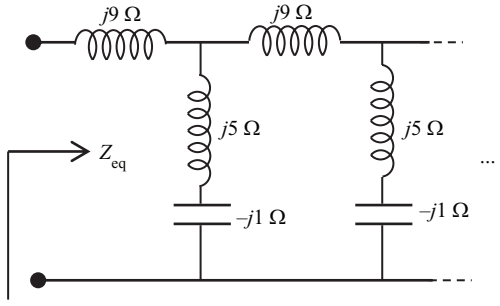
$$(V_c(t) - 5)^2 + i^2(t) = 5^2$$

It representing a circle with centre (5, 0) and $r = 5$.

Hence, the correct option is (B).

Question Number: 32 **Question Type: MCQ**

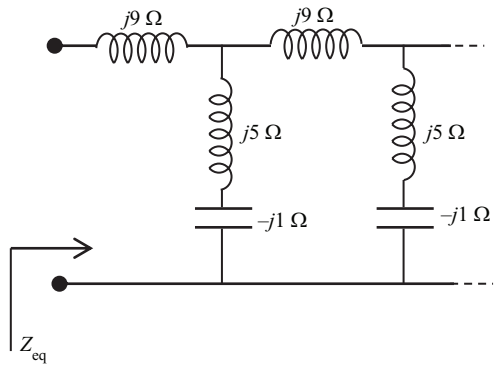
The equivalent impedance Z_{eq} for the infinite ladder circuit shown in the figure is:



(A) $j12\ \Omega$ (B) $-j12\ \Omega$

(C) $j13\ \Omega$ (D) $13\ \Omega$

Solution: Given infinite ladder circuit



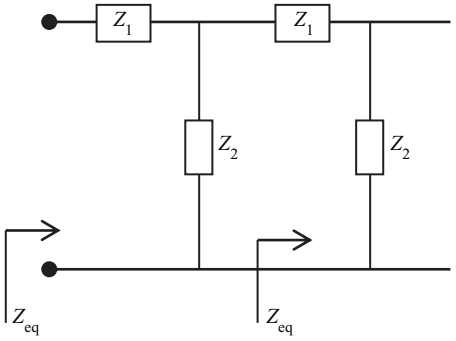
Let $Z_1 = j9\Omega$

and $Z_2 = j5 - j1$

$= j4\Omega$

It is a infinite ladder circuit redraw the approximate equivalent circuit it becomes.

From the above circuit



$$Z_{eq} = Z_1 + \{Z_2 \parallel Z_{eq}\}$$

$$Z_{eq} = Z_1 + \frac{Z_2 \cdot Z_{eq}}{Z_2 + Z_{eq}}$$

$$Z_{eq}\{Z_2 + Z_{eq}\} = Z_1Z_2 + Z_1Z_{eq} + Z_2Z_{eq}$$

sub $Z_1 = j9\ \Omega$

and $Z_2 = j4\ \Omega$

$$Z_{eq}^2 + j4Z_{eq} = -36 + j9Z_{eq} + j4Z_{eq}$$

$$Z_{eq}^2 - j9Z_{eq} + 36 = 0$$

from the given options

$$Z_{eq} = j12\Omega$$

Hence, the correct option is (A).

Question Number: 33 **Question Type: MCQ**

Consider a system governed by the following equation

$$\frac{dx_1(t)}{dt} = x_2(t) - x_1(t)$$

$$\frac{dx_2(t)}{dt} = x_1(t) - x_2(t)$$

The initial conditions are such that $x_1(0) < x_2(0) < \infty$. Let $x_{1f} = \lim_{t\to\infty} x_1(t)$ and $x_{2f} = \lim_{t\to\infty} x_2(t)$. Which one of the following is true?

(A) $x_{1f} < x_{2f} < \infty$ (B) $x_{2f} < x_{1f} < \infty$

(C) $x_{1f} = x_{2f} < \infty$ (D) $x_{1f} = x_{2f} = \infty$

Solution:

$$\frac{dx_1(t)}{dt} = x_1(t) - x_1(t) \qquad (1)$$

$$\frac{dx_2(t)}{dt} = x_1(t) - x_2(t) \qquad (2)$$

Applying laplace transform to both

$$Sx_1(s) - x_1(0) = x_2(s) - x_1(s) \qquad (3)$$

$$Sx_2(s) - x_2(0) = x_1(s) - x_2(s) \qquad (4)$$

where $x_1(0)$ and $x_2(0)$ are initial conditions

from Equation (4) $\quad x_2(s) = \dfrac{x_1(s)}{s+1} + \dfrac{x_2(0)}{s+1} \qquad (5)$

substituting Equation (5) in Equation (3)

$$x_1(s)\left[s+1-\frac{1}{s+1}\right] = \frac{x_2(0)}{s+1} + x_1(0)$$

$$x_1(s) = \left[\frac{1}{s(s+2)}\right]x_2(0) + \frac{(s+1)}{s(s+2)}x_1(0)$$

$$x_1(s) = \left[\frac{1}{2s} - \frac{1}{2(s+2)}\right]x_2(0) + \left[\frac{1}{2s} + \frac{1}{2(s+2)}\right]x_1(0)$$

$$s_1(t) = (0.5 + 0.5e^{-2t})x_1(0) + (0.5 - 0.5e^{-2t})x_2(0)$$

Similarly, $\quad x_{1f} = \underset{t\to\infty}{\text{Lt}}\, x_1(t)$

$$= 0.5x_1(0) + 0.5x_2(0)$$

$$x_{2f} = \underset{t\to\infty}{\text{Lt}}\, x_2(t)$$

$$= 0.5x_1(0) + 0.5x_2(0)$$

Here x_{1f} and x_{2f} are equal and as per the given data $x_1(0) < x_2(0) < \infty$, means

$$x_{1f} = x_{2f} < \infty$$

Hence, the correct option is (C).

Question Number: 34 **Question Type: MCQ**

The number of roots of the polynomial. $s^7 + s^6 + 7s^5 + 14s^4 + 31\,s^3 + 73s^2 + 25s + 200$, in the open left half of the complex plane is:

(A) 3 (B) 4
(C) 5 (D) 6

Solution: $C \cdot E = S^7 + 3^6 + 7S^5 + 14S^4 + 31S^3 + 73S^2 + 25S + 200 = 0$

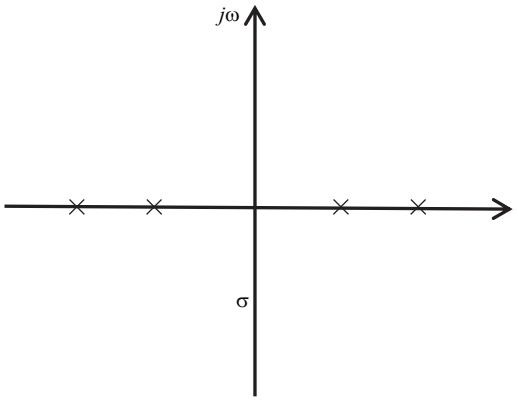
$+S^7$	1	7	31	25
$+S^6$	1	14	73	200
$-S^5$	–7	–42	–175	0
$+S^4$	8	48	200	
$+S^3$	0 (32)	0 (96)	0	
$+S^2$	24	200		
$-S^1$	$\frac{-512}{3}$			
$+S^0$	200			

$$A \cdot E = 8S^4 + 48S^2 + 200$$

$$\frac{dAE}{dS} = 32S^3 + 96S$$

Number of *s*ign change*s* below

$$A \cdot E = 2$$



And number of sign changes above auxiliary equation are 2.

Total number of RHP = 4
Total number of jω poles = 0
Total number of LHP = 3.

Hence, the correct option is (A).

Question Number: 35 **Question Type: MCQ**

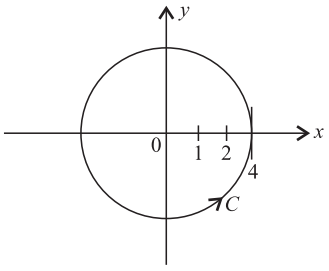
If C is a circle $|z| = 4$ and $f(z) = Z_f \dfrac{z^2}{(z^2 - 3z + 2)^2}$, then $\oint_C f(z)\,dz$ is

(A) 1 (B) 0
(C) –1 (D) –2

Solution: $\quad f(Z) = \dfrac{Z^2}{(Z^2 - 3z + 2)^2}$

$$= \frac{Z^2}{((Z-1)(Z-2))^2}$$

$\therefore \quad f(Z) = \dfrac{Z^2}{(Z-1)^2(Z-2)^2}$



$z = 1$ and $z = 2$ are the singularities of $f(z)$ and both of them lie inside C.

$\therefore$ By residue theorem,

$$\oint_C f(z)\,dz = 2\pi i\left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\,(f(z))\right] \quad (1)$$

$$\underset{z=1}{\text{Res}}\; f(z) = \underset{z\to 1}{\text{Lt}}\left[\frac{d}{dz}((z-1)^2 f(z))\right]$$

$$= \underset{z\to 1}{\text{Lt}}\left[\frac{d}{dz}\left(\frac{z^2}{(z-2)^2}\right)\right]$$

$$= \underset{z\to 1}{\text{Lt}}\left[\frac{2z(z-2)^2 - 2z^2(z-2)}{(z-2)^4}\right]$$

$$= \underset{z\to 1}{\text{Lt}}\left[\frac{2z(z^2-4z+4) - 2z^3 + 4z^2)}{(z-2)^4}\right]$$

$$= \underset{z\to 1}{\text{Lt}}\left[\frac{8z - 4z^2}{(z-2)^4}\right]$$

$$\therefore \underset{z=1}{\text{Res}}\; f(z) = 4 \quad (2)$$

$$\underset{z=2}{\text{Res}}\; f(z) = \underset{z\to 2}{\text{Lt}}\left[\frac{d}{dz}((z-2)^2 f(z))\right]$$

$$= \underset{z\to 2}{\text{Lt}}\left[\frac{d}{dz}\left(\frac{z^2}{(z-1)^2}\right)\right]$$

$$= \underset{z\to 2}{\text{Lt}}\left[\frac{2z(z-1)^2 - 2z^2(z-1)}{(z-1)^4}\right]$$

$$= \underset{z\to 2}{\text{Lt}}\left[\frac{2z(z^2-2z+1) - 2z^3 + 2z^2}{(z-1)^4}\right]$$

$$= \underset{z\to 2}{\text{Lt}}\left[\frac{2z - 2z^2}{(z-1)^4}\right]$$

$$\therefore \underset{z=2}{\text{Res}}\; f(z)$$

$$= -4 \quad (3)$$

Substituting Equation (2) and (3) in Equation (1), we have

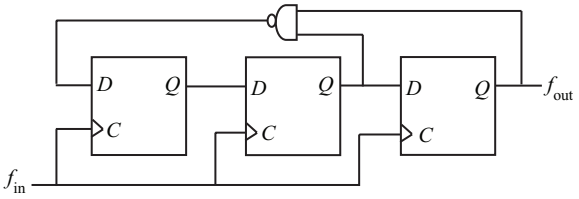
$$\oint_C f(z)\,dz = 2\pi i[4-4] = 0$$

Hence, the correct option is (B).

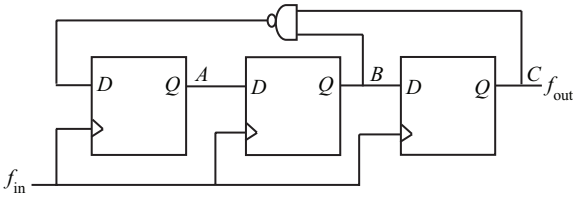
Question Number: 36 **Question Type: MCQ**

Which one of the following statements is true about the digital circuit shown in the figure given in next column.

(A) It can be sued for dividing the input frequency by *e*.
(B) It can be used for dividing the input frequency by 5.
(C) It can be used for dividing the input frequency by 7.
(D) It cannot be reliably used as a frequency divider due to disjoint internal cycles.



Solution:



CIK	A	B	C
0	0	0	0
1	1	0	0
2	1	1	0
3	1	1	1
4	0	1	1
5	0	0	1
6	1	0	0
7	1	1	0

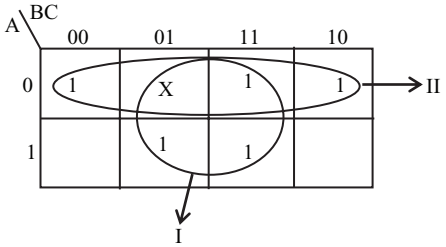
the modulus of the given counter is 5, so it is used to divide the input frequency by 5.

Question Number: 37 **Question Type: MCQ**

Digital input signals A, B, C with A as the MSB and C as the LSB are used to realized the Boolean function $F = m_0 + m_2 + m_3 + m_5 + m_7$, where m_1 denotes the i^{th} minterm. In addition, F has a don't care for m_1. The simplified expression for F is given by:

(A) $\bar{A}\bar{B} + \bar{B}C + AC$ (B) $\bar{A} + C$
(C) $\bar{C} + A$ (D) $\bar{A}C + BC + A\bar{C}$

Solution:



$$F = I + II$$

where $I = C$

$$F = \bar{A} + C$$

$$II = \bar{A}$$

Hence, the correct option is (B).

Question Number: 38 **Question Type: MCQ**

Consider the two continuous-time signals defined below:

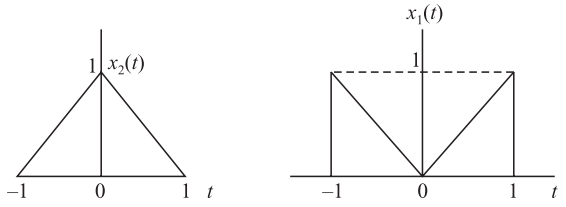
$$x_1(t) = \begin{cases} |t|, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x_2(t) = \begin{cases} 1-|t|, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

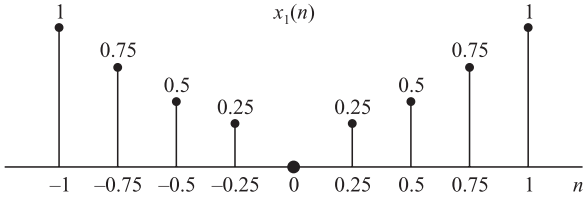
These signals are sampled with a sampling period of $T = 0.25$ seconds to obtain discretetime signals $x_1[n]$ and $x_2[n]$, respectively. Which one of the following statements is true?

(A) The energy of $x_1[n]$ is greater than the energy of $x_2[n]$.
(B) The energy of $x_2[n]$ is greater than the energy of $x_1[n]$
(C) $x_1[n]$ and $x_2[n]$ have equal energies.
(D) Neither $x_1[n]$ nor $x_2[n]$ is a finite-energy signal.

Solution: $x_1(t)$ and $x_2(t)$ can be shown as



Sampled Versions $x_1[n]$ and $x_2[n]$ can be shown as

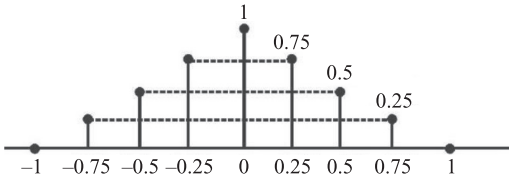


$$X_1[n] = \delta[n+1] + 0.75\delta[n+0.75] + 0.5\delta[n+0.5] + 0.25\delta[n+0.25] + 0.25\delta[n-0.25] + 0.5\delta[n-0.5] + 0.75\delta[n-0.75] + \delta[n-1]$$

Energy in $X_1[n]$ is $\sum_{n=-1}^{1} x_1^2[n]$

$$Ex_{1[n]} = 3.75$$

Sampled Version of $x_2[n]$ is



$$X_2[n] = 0.25\delta[n+0.75] + 0.5\delta[n+0.5] + 0.75\delta[n+0.25] + \delta[n] + 0.75\delta[n-0.25] + 0.5\delta[n-0.5] + 0.25\delta[n-0.75]$$

Energy in $X_2[n]$ is $\sum_{n=-1}^{1} x_2^2[n]$

$$Ex_{2[n]} = 2.75$$

Energy of $x_1[n]$ is greater than energy of $x_2[n]$

Hence, the correct option is (A).

Question Number: 39 **Question Type: MCQ**

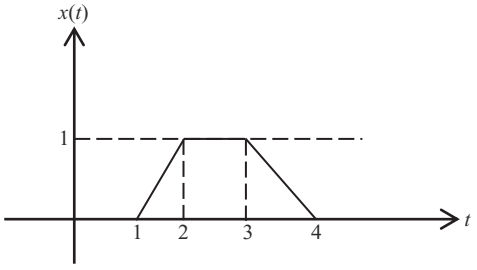
The signal energy of the continuous-time signal

$$x(t) = [(t-1)\,u(t-1)] - [(t-2)\,u(t-2)] - [(t-3)\,u(t-3)] + [(t-4)\,u(t-4)] \text{ is}$$

(A) 11/3 (B) 7/3
(C) 1/3 (D) 5/3

Solution: Given

$$x(t) = [(t-1)\,u(t-1)] - [(t-2)\,u(t-2)] - [(t-3)\,u(t-3)] + [(t-4)\,u(t-4)]$$



$1 \le t \le 2$: $x(t) = (t-1)$

$2 \le t \le 3$: $x(t) = 1$

$3 \le t \le 4$: $x(t) = (4-t)$

Energy of $x(t)$ is given by,

$$E = \int_{-\infty}^{\infty} \left(x(t)\right)^2 dt$$

$$\int_1^2 (t-1)^2\,dt + \int_2^3 1^2\,dt + \int_3^4 (4-5)^2\,dt$$

$$\int_1^2 (t^2 - 2t + 1)\,dt + \int_2^3 1dt + \int_3^4 (16 + t^2 - 8t)\,dt$$

$$1/3 + 1 + 1/3 = 5/3$$

Hence, the correct option is (D).

Question Number: 40 **Question Type: NAT**

The Fourier transform of a continuous-time single $x(t)$ is given by $X(\omega) = \dfrac{1}{(10 + j\omega)^2}, -\infty < \omega < \infty$, where $j = \sqrt{-1}$ and ω denotes frequency. Then the value of $|?\ln x(t)|$ at $t = 1$ is _____ (up to 1 decimal place). (In denotes the logarithm to base e)

Solution: Given $x(\omega) = \dfrac{1}{(10 + j\omega)^2}, -\infty < \omega < \infty$

$$x(t) \xleftrightarrow{\text{F.T}} x(\omega)$$

$$-jt\,x(t) \xleftrightarrow{\text{F.T}} \frac{dx(\omega)}{d\omega}$$

$$e^{-10t} \longleftrightarrow \frac{1}{10 + j\omega}$$

$$-jt\,e^{-10t} \longleftrightarrow \frac{-j}{(10 + j\omega)^2}$$

$$\Rightarrow \quad te^{-10t} \xleftrightarrow{\text{F.T}} \frac{1}{(10 + j\omega)^2}$$

$$\therefore \qquad X(t) = te^{-10t}$$

Now, $\quad |\text{In}\, x(t)| = |\text{In}\, te^{-10t}|$

$$= |\text{In}\, t - 10\,\text{In}\, e|$$

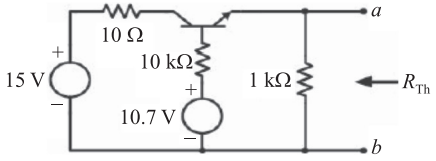
at $\quad t = 1$

$$= |\text{In}\, 1 - 10| = 10$$

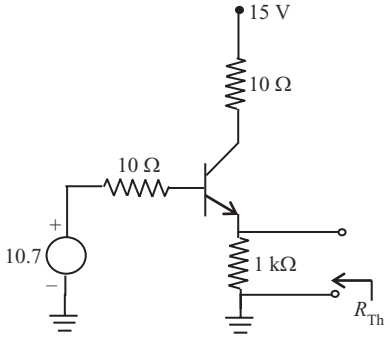
Hence, the correct answer is (10).

Question Number: 41 **Question Type: NAT**

In the circuit shown in the figure, the bipolar junction transistor (BJT) has a current gain β = 100. The base-emitter voltage drop is a constant. V_{BE} = 0.7 V. The value of the Thevenin equivalent resistance R_{Th} (in Ω) as shown in the figure is _____ (up to 2 decimal places.



Solution:



Given $\quad \beta = 100$

$$V_{BE} = 0.7$$

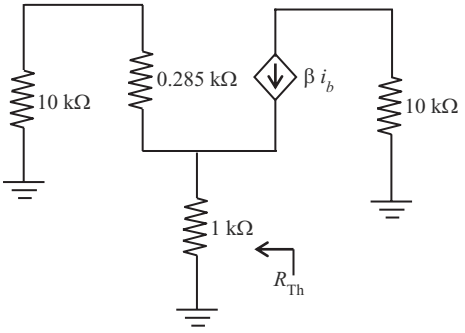
$$I_E = \frac{10.7 - 0.7}{1 + \frac{10}{101}} = 9.17\ \text{mA}$$

$$I_C = \left(\frac{\beta}{\beta + 1}\right) I_E = 9.08\ \text{mA}$$

$$\Rightarrow \qquad g_m = \frac{I_C}{V_T} = 0.35$$

$$r_\pi = \frac{\beta}{g_m} = 0.285\ \text{k}\Omega$$

a.c equivalent circuit is as shown below



$$R_{th} = 1\ \text{k}\Omega \,||\, \left(\frac{10}{101} + \frac{0.285}{101}\right)$$

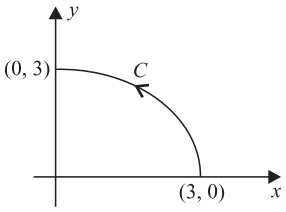
$$R_{th} = 1\ \text{k}\Omega \,||\, 0.1\ \text{k}\Omega$$

$$R_{th} = 91\ \Omega$$

Hence, the correct answer is (91).

Question Number: 42 **Question Type: NAT**

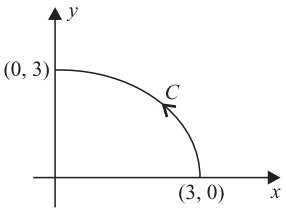
As shown in the figure, C is the arc from the point (3, 0) to the point (0, 3) on the circle $x^2 + y^2 = 9$. The value of the integral $\int_C (y^2 + 2yz)\,dx + (2xy + x^2)dy$ is _____ (up to 2 decimal places).



Solution: We have to evaluate

$$\int_C (y^2 + 2yx)\,dx + (2xy + x^2)dy$$

along the boundary of the circle $x^2 + y^2 = 9$ from (3, 0) to (0, 3)



$$\therefore \quad \int_C (y^2 + 2yx)\,dx + (2xy + x^2)dy$$

$$= \int_C [y^2dx + 2yxdx + 2xydy + x^2dy]$$

$$= \int_C [(y^2dx + 2xydy) + (2yxdx + x^2dy)]$$

$$= \int_C [d(xy^2) + d(x^2y)]$$

$$= \int_{(3,0)}^{(0,3)} [d(xy^2 + x^2y)]$$

$$= xy^2 + x^2y\Big]_{(3,0)}^{(0,3)} = 0$$

Hence, the correct answer is (0).

Question Number: 43 **Question Type: NAT**

Let $f(x) = 3x^3 - 7x^2 + 5x + 6$. The maximum value of $f(x)$ over the interval [0, 2] is _____ (up to 1 decimal place).

Solution: Given $f(x) = 3x^3 - 7x^2 + 5x + 6$

$$\Rightarrow \quad f'(x) = 9x^2 - 14x + 5$$

$$f'(x) = 0 \Rightarrow 9x^2 - 14x + 5 = 0$$

$$\Rightarrow \quad (9x - 5)(x - 1) = 0 \Rightarrow x = \frac{5}{9}; x = 1$$

$\therefore$ The maximum value of $f(x)$ in [0, 2]

$$= \max\left\{f(0), f(2), f\left(\frac{5}{9}\right), f(1)\right\}$$

$$= \max\{6, 12, 7.1317, 7\}$$

$$= 12$$

Hence, the correct answer is (12).

Question Number: 44 **Question Type: NAT**

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$, where I is the 3×3 identity matrix. The determinant of B is _____ (up to 1 decimal place)

Solution: Given $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-2 - \lambda)(2 - \lambda)(1 - \lambda) = 0$$

$$\Rightarrow \quad \lambda = 1, 2, -2$$

$\therefore$ The eigen values of A are 1, 2 and –2. Given

$$B = A^3 - A^2 - 4A + 5I$$

Eigen Values of *A*		**Eigen Values of *B***
$\lambda = 1$	$\longrightarrow$	$1^3 - 1^2 - 4 \times 1 + 5 = 1$
$\lambda = 2$	$\longrightarrow$	$2^3 - 2^2 - 4 \times 2 + 5 = 1$
$\lambda = -2$	$\longrightarrow$	$(-2)^3 - (-2)^2 - 4 \times (-2) + 5 = 1$

$\therefore$ The eigen values of B are 1, 1 and 1.

So, the determinant of

$$B = 1 \times 1 \times 1$$

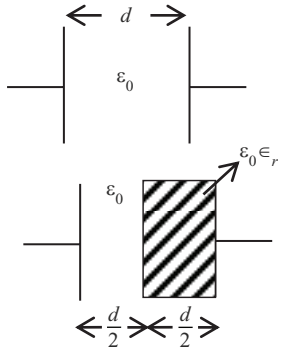
$$= 1$$

Hence, the correct answer is (1).

Question Number: 45 **Question Type: NAT**

The capacitance of an air-filled parallel-plate capacitor is 60 pF. When a dielectric slab whose thickness, is half the distance between the plates, is placed on one of the plates covering it entirely, the capacitance becomes 86 pF. Neglecting the fringing effects, the relative permittivity of the dielectric is _____ (up to 2 decimal places)

Solution:



$$C_0 = \frac{\varepsilon_0 A}{d} = 60 \text{ pF}$$

$$C_1 = 2C_0$$

$$C_2 = 2C_0 \in_r$$

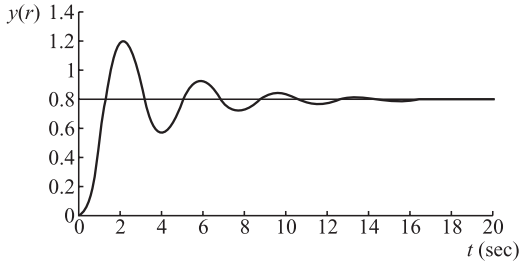
$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{120 \in_r}{1 + \in_r} = 86 \text{ pF}$$

$$\in_r = 2.53$$

Hence, the correct answer is (2.53).

Question Number: 46 **Question Type: NAT**

The unit step response $y(t)$ of a unity feedback system with open loop transfer function $G(s)\,H(s) = \dfrac{K}{(s+1)^2(s+2)}$ is shown in the figure. The value of K is _____ (up to 2 decimal places)



Solution:

$$\text{CLTF} = \frac{K}{(S+1)^2 + (S+2) + K}$$

$$0.8 = \underset{S\to 0}{\text{Lt}}\left(\frac{1}{S}\right) S\left\{\frac{K}{(S+1)^2(S+2)+K}\right\}$$

$$0.8 = \frac{K}{2+K}$$

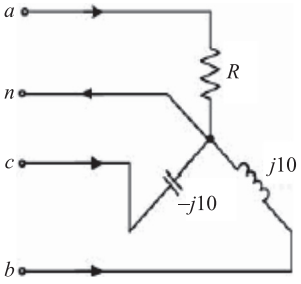
$$\Rightarrow \quad 1.6 + 0.8K = K$$

$$\Rightarrow \quad 0.2K = 1.6$$

$$\Rightarrow \quad K = 8.$$

Question Number: 47 **Question Type: NAT**

A three-phase load is connected to a three-phase balanced supply as shown in the figure. If $V_{an} = 100\ \angle 0°V$, $V_{bn} = 100\ \angle -120°V$ and $V_{cn} = 100\ \angle -240°V$ (angles are consider positive in the anti-clockwise direction), the value of R for zero current in the neutral wire is _____ Ω



Solution:

$$I_N = I_R + I_Y + I_B$$
$$= 0$$

$$I_R = \frac{K}{2+K}$$

$$I_y = \frac{V\angle -120°}{j\omega L}$$

$$I_B = \frac{V\angle -240°}{\left(\frac{1}{j\omega c}\right)}$$

$$\frac{1}{R} + \frac{1\angle -120°}{j\omega L} + j\omega c \angle -240°$$
$$= 0$$

$$\frac{1}{R}\cos 0° + \frac{1}{j\omega L}\cos 120° + j\omega c \cos 240°$$
$$= 0 \qquad (1)$$

$$\frac{1}{R}\sin 0° + \frac{1}{j\omega L}\sin(-120°) + j\omega c \sin(-240°)$$
$$= 0 \qquad (2)$$

Solve Equation (2) $\Rightarrow \omega = \frac{1}{\sqrt{LC}}$ put in Equation (1) then

$$R = \frac{1}{\sqrt{3}}\sqrt{\frac{L}{C}}$$

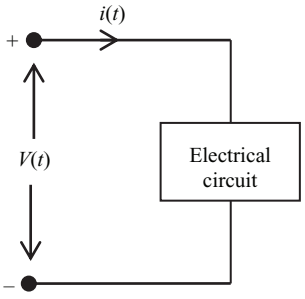
$$R = \frac{1}{\sqrt{3}}\sqrt{100} = 5.77\,(\Omega)$$

Question Number: 48 **Question Type: NAT**

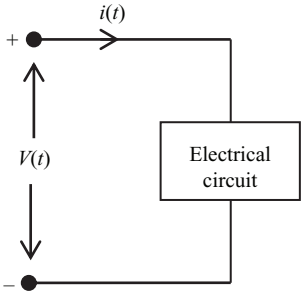
The voltage across the circuit in figure, and the current through it, are given by the following expressions:

$$V(t) = 5 - 10\cos(\omega t + 60°)V$$
$$I(t) = 5 + X\cos(\omega t)\,A$$

where ω = 100 radian/s. If the average power delivered to the circuit is zero, then the value of X (in ampere) is _____ (up to 2 decimal places).



Solution:



Given $V(t) = 5 - 10\cos(\omega t + 60°)V$

$i(t) = 5 + X\cos(\omega t)\,A$

$\omega = 100\pi$ rad/sec

$P_{del} = 0$, then $X = ?$

$P = V \cdot I$

$V(t) = 5 - 10\{\cos\omega t \cdot \cos 60° - \sin\omega t \cdot \sin 60°\}$

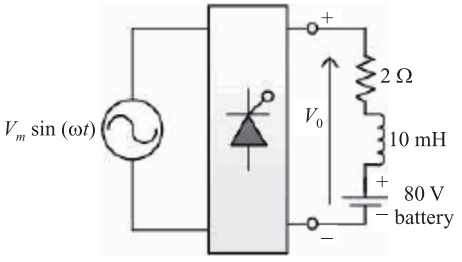
$$V(t) = 5 - 5\cos\omega t + 5\sqrt{3}\sin\omega t$$
$$i(t) = 5 + X\cos\omega t$$
$$P = V(t)\cdot i(t)$$
$$P = [5 - 5\cos\omega t + 5\sqrt{3}\sin\omega t][5 + X\cos\omega t]$$
$$= 25 + 5X\cos\omega t - 25\cos\omega t - 5X\cos^2\omega t + \ldots \text{etc}$$
$$P = 25 - \frac{5X}{2}\{1 + \cos 2\omega t\} + 5X\cos\omega t - 25\cos\omega t + \ldots \text{etc.}$$
$$P = 25 - 2.5X - 2.5X\cos\omega t + 5X\cos\omega t \ldots \text{etc}$$

given $P_{avg} = 0$

So, $25 - 2.5X = 0 \Rightarrow X = 10$

Question Number: 49 **Question Type: NAT**

A phase controlled single phase rectifier, supplied by an AC source, feeds power to an R-L-E load as shown in the figure. The rectifier output voltage has an average value given by $V_0 = \frac{V_m}{2\pi}(3 + \cos\alpha)$, where $V_m = 80\pi$ volts and α is the firing angle. If the power delivered to the lossless battery is 1600 W, α in degree is _____ (up to 2 decimal places).



Solution: Given

$$V_0 = \frac{V_m}{2\pi}(3 + \cos\alpha)$$
$$V_m = 80\pi$$
$$\Rightarrow \quad V_0 = \frac{80\pi}{2\pi}(3 + \cos\alpha) = 40(3 + \cos\alpha)$$

Current through battery,

$$I_0 = \frac{V_0 - E}{R} \qquad (1)$$

Also given,

$$P_{battery} = 80 \times I_0$$
$$= 1600$$
$$I_0 = 20\text{ A}$$

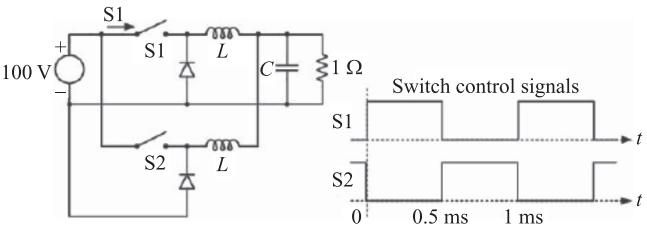
Equation (1)

$$\Rightarrow \quad 20 = \frac{40(3 + \cos\alpha) - 80}{1}\cos\alpha = 0$$
$$\alpha = 90°$$

Hence, the correct answer is $\alpha = 90°$.

Question Number: 50 **Question Type: NAT**

The figure shown two buck converters connected in parallel. The common input dc voltage for the converters has a value of 100 V. The converters have inductors of identical value. The load resistance is 1 Ω. The capacitor voltage has negligible ripple. Both converters operate in the continuous conduction mode. The switching frequency in 1 kHz and the switch control signals are as shown. The circuit operates in the steady state. Assuming that the converters share the load equally, the average value of i_{s1}, the current of switch S_1 (in Ampere), is _____ (up to 2 decimal places).



Solution:

$$V_s = 100\text{ V}$$
$$I_D = 100/1 = 100\text{ A}$$
$$P_{in} = P_{out}$$
$$V_s \cdot I_s = V_0 I_0$$
$$100 \times I_s = 100 \times 100$$
$$I_s = 100\text{ A}$$
$$I_{s1} = 50\text{ A}$$

Hence, the correct answer is 50 A.

Question Number: 51 **Question Type: NAT**

A three-phase 900 kVA, 3 kV/$\sqrt{3}$ kV (Δ/Y), 50 Hz transformer has primary (high voltage side) resistance per phase of 0.3 Ω and secondary (low voltage side) resistance per phase of 0.02 Ω Iron loss of the transformer is 10 kW. The full load % efficiency of the transformer operated at unity power factor is _____ (up to 2 decimal places).

Solution: Given, 3-ϕ, 900 kVA, 3 kV/$\sqrt{3}$ kV, Δ/λ, f
$= 50$ Hz

H.V. side: R_1/ph
$= 0.3\ \Omega$

L.V. side: R_2/ph
$= 0.02\ \Omega$

Total iron loss of transformer
$= 10$ kW.

Primary current, I_1/ph

$$= \frac{I_1\,(\text{line})}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}} \times \frac{900 \times 10^3}{\sqrt{3} \times 3 \times 10^3} = 100\text{ A}$$

Secondary current, I_2/ph
$= I_2\,(\text{line})$

$$= \frac{900 \times 10^3}{\sqrt{3} \times \sqrt{3} \times 10^3} = 300 \text{ A}$$

$\therefore$ Total Cu losses in a transformer

$$= 3I_1^2/ph\, R_1/ph + 3I_2^2/ph\, R_2/ph$$
$$= 3 \times 10^4 \times 0.3 + 3 \times 9 \times 10^4 \times 0.02$$
$$= 9000 + 5400 = 14400 \text{ W.}$$

$\therefore$ η at F.L.
(U.P.F)

$$= \frac{(1 \times 900 \times 10^3 \times 1)}{(1 \times 900 \times 10^3 \times 1) + (10 \times 10^3 + 14,400)}$$

$$= 0.9736$$

$\therefore$ % η $= 97.36\%$

Hence, the correct answer is 97.36.

Question Number: 52 **Question Type: NAT**

A 200 V DC series motor, when operating from rated voltage while driving a certain load, draws 10 A current and runs at 1000 r.p.m. The total series resistance is 1 Ω. the magnetic circuits is assumed to be linear. At the same supply voltage, the load torque is increased by 44%. The speed of the motor in r.p.m. (rounded to the nearest integer) is _____.

Solution: Given
D.C. series motor,

$V_t = 200$ V
$I_{a_1} = 10$ A, $N_1 = 1000$ r.p.m.

Total series resistance,

$R_{(\text{Total})} = 1\ \Omega$.

Given, $T_{L_2} = 1.44 T_{L_1}$

For, D.C. series motor, $\phi \alpha\, I_a$

$\therefore$ $T_{e_m} \alpha\, I_a^2$

$$\frac{T_{L_1}}{1.44 T_L} = \left[\frac{I_{a_1}}{I_{a_2}}\right]^2$$

$\therefore$ $$\frac{I_{a_1}}{I_{a_2}} = \frac{1}{1.2} \Rightarrow I_{a_2} = 12 \text{ A}$$

$$E_{b_1} = V_t - I_{a_1}$$
$$R_a = 200 - (10 \times 1) = 190 \text{ V}$$
$$E_{b2} = V_t - I_{a_2}$$
$$R_a = 200 - (12 \times 1) = 188 \text{ V}$$
$$E_b = K_a \phi \omega$$
$$E_b \alpha\, I_a N \qquad (\because \phi \alpha\, I_a)$$

$$\frac{E_{b_1}}{E_{b_2}} = \frac{I_{a_1}}{I_{a_2}} \times \frac{N_1}{N_2}$$

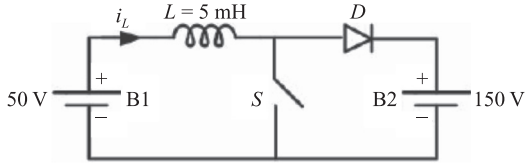
$\Rightarrow$ $$\frac{190}{188} = \frac{1}{1.2} \times \frac{1000}{N_2}$$

$\therefore$ $N_2 = 824.561$ rpm

Hence, the correct answer is 824.56.

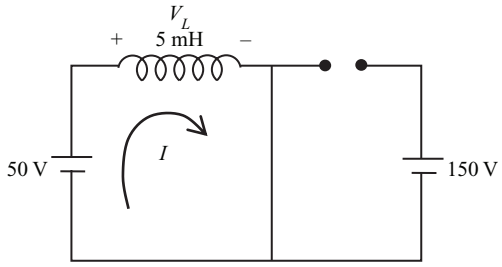
Question Number: 53 **Question Type: NAT**

A dc to dc converter shown in the figure is charging a battery bank. B_2 whose voltage is constant 150 V. B_1 is another battery bank whose voltage is constant at 50 V. The value of the inductor, L is 5 mH and the ideal switch, S is operated with a switching frequency of 5 kHz with a duty ratio of 0.4. Once the circuit has attained steady state and assuming the diode D to be ideal, the power transferred from B_1 to B_2 (in Watt) is _____ (up to 2 decimal places).



Solution: Given, $D = 0.4$
$f = 5$ kHz (see the graphs)

Switch ON:



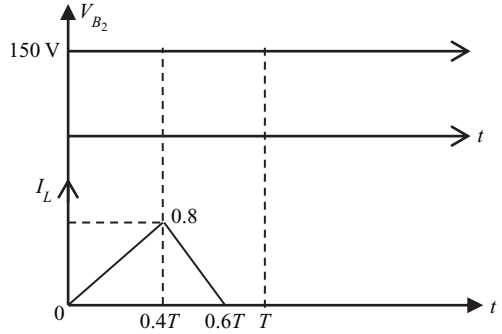
$$V_L = 50 \text{ V}$$

Inductor charges linearly.

$$V_L = \frac{LdI}{dt}$$

$$\int dI = \frac{V_L}{L} \int_0^{T_{\text{ON}}} dt$$

$$\Delta I = \frac{V_L}{L} T_{\text{ON}} \left[T_{\text{ON}} = DT\right]$$



Power transferred to B_2 is given by

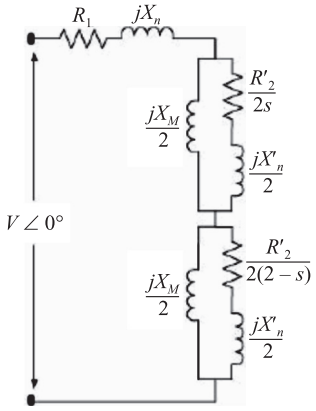
$$P = \frac{1}{T} \int_{0.4T}^{0.6T} 150 \times I_L dt$$

$$= \frac{1}{T} \times 0.2T \times \frac{0.8T}{2} \times 150 = 12 \text{ W}$$

Hence, the correct answer is (12).

Question Number: 54 **Question Type: NAT**

The equivalent circuit of a single phase induction motor is shown in the figure, where the parameters are $R_1 = R_2^1 = X_{l1} = X_{l2}^1 = 12\ \Omega$, $X_M = 240\ \Omega$. And s is the slip. At no-load, the motor speed can be approximated to be the synchronous speed. The no-load lagging power factor of the motor is _____ (up to 3 decimal places.)

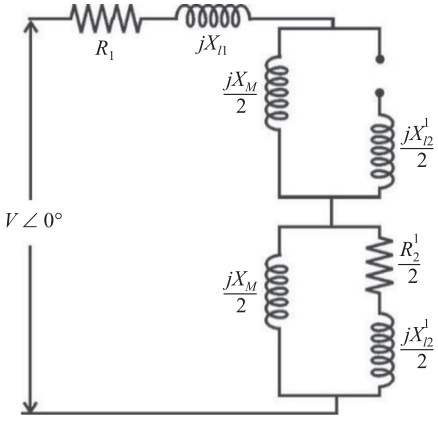


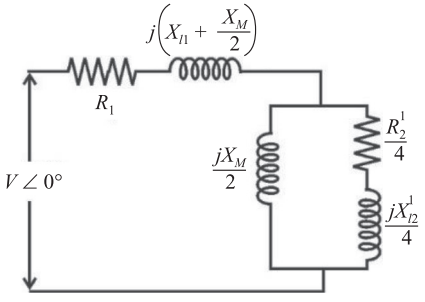
Solution: Given $R_1 = R_2^1 = X_{\lambda 1} = X_{\lambda 2}^1 = 12\ \Omega$, $X_M = 240\ \Omega$

At no-load, $N_r = N_s$

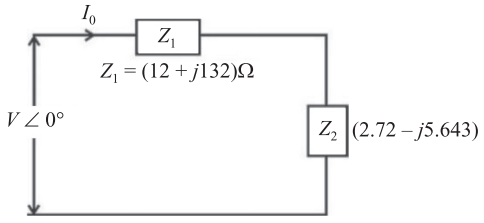
$\therefore$ $$S = \frac{N_s - N_r -}{N_s} = 0$$

$\therefore$ The equivalent circuit becomes,





$\therefore$ No load current, $$I_0 = \frac{V \angle 0°}{Z \angle \theta}$$



$$z_{eq} = 138.56 \angle 83.9\ \Omega$$

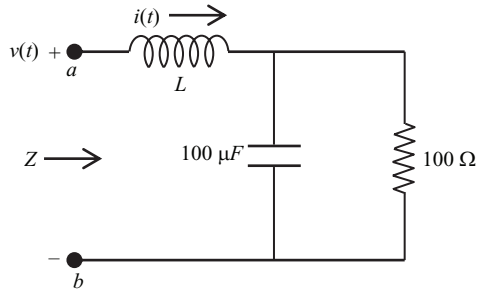
$\therefore$ No load P.F $= \cos(83.9)$

$= 0.106$ lagging

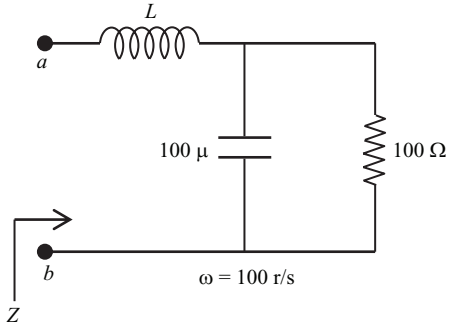
Hence, the correct answer is (0.106).

Question Number: 55 **Question Type: NAT**

The voltage $v(t)$ across the terminal a and b as shown in the figure, is a sinusoidal voltage having a frequency $\omega = 100$ radian/s. When the inductor current $i(t)$ is in phase with the voltage $v(t)$, the magnitude of the impedance Z (in Ω) seen between the terminals a and b is _____ (up to 2 decimal places).



Solution:



from the given data circuit is in resonance

So, at resonance

$$Z_{ab} = (Z_{ab})_{\text{real}}$$

$$\text{Imag}\ \{Z_{ab}\} = 0$$

$$Z_{ab} = j\omega L + \left(100 \parallel \frac{1}{SC}\right)$$

$$Z_{ab} = j\omega L + \frac{100 \times \frac{1}{SC}}{100 + \frac{1}{SC}}$$

$$Z_{ab} = j\omega L + \frac{100}{I + 100SC}$$

$$= j\omega L + \frac{100}{1 + j100 \times 100 \times 10^{-6} \times 100}$$

$$Z_{ab} = j\omega L + \frac{100}{1 + j1}$$

$$Z_{ab} = j\omega L + \frac{100\{1 - j\}}{1 + (1)^2}$$

At resonance,

$$Z_{ab} = Z_{\text{real}} = \frac{100}{2} = 50\,\Omega$$

General Aptitude

Part A: Verbal Ability 1.3

Part B: Numerical Ability 1.69

PART I

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Verbal Ability

PART A

Chapter 1: Grammar 1.5

Chapter 2: Vocabulary 1.49

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Chapter 1

Grammar

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Nouns, pronouns, articles
- Verbs, auxiliaries, modals
- Adjectives, adverbs
- Prepositions, conjunctions
- Active/passive voice, direct/indirect speech
- Verbal phrases
- Sentence completion

Introduction

The Verbal Ability section of the book includes inputs followed by Practice Exercises. It comprises inputs in following sections.

1. Grammar
2. Sentence Corrections
3. Vocabulary

Grammar

The questions under this category are based on English Grammar. These questions test the student's knowledge of grammar and ability to use English Language correctly. The grammar section includes.

1. Nouns, Pronouns, Articles
2. Verbs, Auxiliaries, Modals
3. Adjectives, Adverbs
4. Prepositions, Conjunctions
5. Active/Passive Voice, Direct/Indirect speech
6. Verbal Phrases.

Sentence Completion

This section includes sentence correction and sentence rephrasing questions. Questions in this section test a students ability to comprehend a given context and indentify grammatical errors.

Vocabulary

The verbal questions are based on the meanings of words or their opposites. These question types are designed to test the students vocabulary. This section includes.

1. Synonyms
2. Antonyms

Verbal Analogies

The word 'Analogy' means a comparison. Vocabulary plays an important part here. This question type involves identifying the logical relationship between the words. This section includes.

1. Analogy
2. Reverse Analogy

Verbal Reasoning

The ability to isolate the key issue and to identify irrelevant issues is important.

Logical reasoning is important. This section includes

1. Critical reasoning
2. Logical reasoning

Grammar—Overview

Sentence

A group of words which makes complete sense is a sentence.

Examples:

1. Birds fly.
2. It is a good college.
3. The children of that school wear blue uniform.
4. This is the theory which throws light on superconductivity.

Parts of a sentence

A sentence can be divided into two parts:
(1) Subject and (2) Predicate
Here are some sentences divided into the two parts.

	A (subject)	B (predicate)
1.	Birds	fly.
2.	Mohan	is an engineer.
3.	He	teaches Engineering Graphics.
4.	This intelligent girl	has passed her examination in the first division.
5.	The President of India	has awarded him a gold medal.

Notes:

1. The words underlined in Part A are Subject words and those underlined in Part B are Verbs.
2. The Subject is a noun or a pronoun with or without other words going with it.
3. The Predicate is the verb with or without other words going with it (other words may be object, complement, adverb, adverb phrase, etc.).
4. The subject of the sentence usually comes first but occasionally the predicate precedes.

Example:

- Down went the Royal Francis.

Sample Exercise–I

Underline the subject parts in the following sentences.

1. The train arrived late.
2. He was the captain of his team.
3. There was a large crowd on the platform to receive the team.
4. Birds of the same feather flock together.
5. The Prime Minister with all his Cabinet members has submitted his resignation to the President.

Underline the predicate parts in the following sentences.

1. Aeroplanes fly over my house frequently.
2. They are shortly moving out of their house.
3. He is suffering from a fever.
4. The college is equipped with new furniture.
5. The Secretary and the members of the club have been discussing the issue since morning.

Now study the following sentences:

Group A

1. There is a beautiful garden in front of my house.
2. It has been raining heavily since morning.
3. There is a phone call for your friend.

Note : The words underlined in these sentences are not real subjects. They are said to be grammatical or dummy subjects.
Now study the following sentences:

Group B

1. Here come my friends and their parents.
2. There go the principal and the members of the teaching staff.

Note: In these sentences you notice that the predicate part precedes the subject part. This is called S.P and P.P inversion. This construction is allowed in standard English. This is used for emphasis.

Parts of Speech: Identification

English words (according to their functions in a sentence) can be classified into Parts of Speech as follows:

1. Noun
2. Pronoun
3. Adjective
4. Verb
5. Adverb
6. Preposition
7. Conjunction
8. Interjection

Noun

A noun is a word used as the name of a person, place, animal, thing, etc.

Examples: Rajitha, Hyderabad, lion, table, etc.

Pronoun

A word used instead of a noun.

Examples: he, she, it, they, etc.

Adjective

A word that qualifies or describes a noun or a pronoun.

Examples:

1. Our team played a good (adj.) game (noun).
2. Every (adj.) sentence (noun) that he spoke was listened to with great attention.
3. All these proposals (noun) are good (adj.).

Note: Some words are used either as pronouns or as adjectives as in the following examples.

Pronoun	Adjective
a. These are good students.	1. These students are good.
b. Each of these books is worth reading.	2. Each book has some special features worth reading.
c. Some of the problems are difficult to solve	3. Some problems are difficult to solve.
d. Either of you can participate in the competition.	4. Either book is good for reference.

Verb

A verb expresses what the subject of a sentence is or does, or has, or what is done to it.

1. Dr. Rao teaches English.
   verb
2. He has taught the subject for twenty-five years.
   verb
3. He is a capable teacher.
   verb
4. She has a car.
   verb

Adverb

A word that modifies an adjective or a verb or another adverb.

1. I am deeply grateful to you for your timely help.
   adv. adj.
2. This horse runs very fast.
   verb adv. adv.
3. He read the passage quickly.
   verb adv

Preposition

A preposition is a word used with a noun or a pronoun to show how the person or thing denoted by the noun or pronoun stands in relation to something else.

1. Work in the college begins at 10 a.m.
2. He wrote the document with a pen.
3. The Professor gave a lecture on superconductivity.
4. I washed my plate after I had eaten.

In the above sentences, the words underlined are called prepositions.

Conjunction (Connector)

A Conjunction is a word that joins words, phrases or sentences.

1. Delhi and Kolkata are densely populated cities.
2. She must be either the President or the Secretary of YMCA.
3. The professor read the essay and was impressed by it.
4. He is not only intelligent but also industrious.

Interjection

A word which expresses a strong or sudden feeling such as surprise, joy, fear, sorrow, etc. It is not grammatically connected with the rest of the sentence. Usually, an exclamatory mark is put after it.

Examples: Ah!, Hurrah!, Well!, Dear!, Oh!

1. Alas! she is dead.
2. Hurrah! we won the match.

Note:
As words are divided into different parts of speech, depending on the work they do in the sentences, it is not possible to say to which part of speech a word belongs unless we see what function it performs in the sentence.

Phrases and Clauses

Phrase

A phrase is a group of words without a finite verb. There are three types of phrases.

1. Noun Phrase
   A fifty year old man is seen in the garden.
2. Adjective Phrase
   This chain is made of gold.
3. Adverb Phrase
   Please place the chemicals in the store-room.

Note: A 'finite verb' is a word that denotes tense and a 'non-finite' is a verb which does not.

Examples:

1. He has written some books. (finite verb) (present perfect tense)
2. Speaking on the occasion, the Minister said prohibition would be lifted soon. (non-finite verb)

There are three types of non-finites in English: INFINITIVE, GERUND, PARTICIPLE.
(*Please note that there is a separate chapter on non-finites.)

Clause

A Clause is a group of words which contains a 'finite verb'. Clauses are of two types:

1. Main Clause
2. Subordinate Clause

A Main Clause is also called a Principal Clause or an Independent Clause because it makes complete sense.

A Subordinate Clause, which can also be called a Dependent Clause, is divided into three types:

NOUN CLAUSE, ADJECTIVE CLAUSE, ADVERB CLAUSE

Now study the uses (functions) of these three types of clauses.

1. **THE NOUN CLAUSE** is used as

(a) the subject of a verb.
That he will be appointed to the job is certain.
subject verb

(b) the object of a verb.
We knew that the plan was impractical.
verb object

(c) the complement of a verb.
Our hope is that he will secure first rank.
v. complement

(d) the complement of an adjective.
We were certain that he would pass.
adj. complement

I am hopeful that he will agree to our proposal.
adj complement

Note: When the noun clause is an object or a complement, the conjunction that may be omitted.

Examples:
I believed he was right. (I believed that he was right)
verb object

I was sure he was right. (I was sure that he was right)
adj. compl.

(e) in apposition to the noun before it.
The suggestion, that the meeting should be adjourned, was unanimously accepted.
N N.Cl (in apposition)

(f) Wh - clauses as Noun Clauses
Wh - clauses perform all the five functions that the 'that' clauses do. In addition to these, they can function as the objects of prepositions.

Examples:
Subject: What you say is true. Where he lives is a mystery.

Object: I don't know what he would say to this proposal.

Subject Complement: The question is who will bell the cat.

In apposition: My question, who is responsible for the loss, has not been answered.

Adjectival Complement: I wasn't sure whose fault it was.

Object of a Preposition: We were not aware of what they were doing.

(g) If/whether clauses as noun clauses:
- I asked him if/whether he can speak French.
- I don't know if/whether he has been appointed or not.
- I don't care whether he attends the meeting or not.
- Whether or not he comes doesn't worry my boss.

2. **ADJECTIVE CLAUSES:** (also called relative clauses). It does the work of an Adjective. It qualifies a noun or a pronoun in the main clause.

(a) The young man who was here a little while ago is the President of the union.
antecedent adj. cl

(b) I have finished reading the novel (that) you gave me.
ant. adj.
cl

(c) This is the story (which) my friend wrote.
ant. adj. cl

Note: ( ) indicates optional use, i.e., may or may not be used.

Notes:
1. The words who, that, and which introducing the relative clauses are relative pronouns.
2. Who is used to refer to persons; That is used to refer to persons or things; and which is used to refer to things, animals, etc.
3. The noun or noun phrase, which a relative pronoun refers to is called its antecedent (ant.)

Special Note 1: The object form 'whom', can also function as a relative pronoun introducing adjective clauses.

Example: This is the person whom you want to meet.

Special Note 2: When the adjective clause defines a place, time, or reason, we use where, when, and why in place of the relative adverbs.

Examples:
1. We visited the place where Mrs. Gandhi was assassinated. (place)
2. At a time when prices are high, we must reduce our expenditure. (time)
3. Could you tell me the reason why he has not been appointed? (reason)

Note: The Possessive form whose can also introduce a relative clause. 'Whose' can be used in speaking of persons, animals and also things without life.

Example: The doctor whose car I bought wants to buy it back.

3. **ADVERBIAL CLAUSES:** Adverbial Clauses are of different kinds. They modify the verbs to which they are connected in meaning i.e., they do the work of an Adverb.

   (a) Adverbial Clauses of Time:
   (The conjunctions used are : when, as, while, till/ until, before, after, as soon as, whenever, since, as long as, etc.)
   Example: Some persons in the audience were asleep while the chief guest was speaking.

   (b) Adverbial clause of place:
   You may go wherever you like.

   (c) Adverbial clause of Reason or Cause :
   (as, because, since, now that)
   You can be punctual to school now that you have a school bus.

   (d) Adverbial clause of Purpose:
   (so that, that, in order that)
   The police officer went in plain clothes so that no one might recognize him.

   (e) Adverbial clause of Result or Effect:
   (that, so that, that)
   It rained very heavily for two days so that the rivers and lakes were flooded.

   (f) Adverbial clause of condition:
   (if/unless)
   If he apologizes to me, I shall forgive him.

   (g) Adverbial clause of concession:
   (though, although, even though)
   Though he is intelligent he cannot argue well.

   (h) Adverbial clause of manner:
   You may complete the work as you please.

Sample Exercise–2

In each of the following sentences you will find a word/ a group of words underlined. Write (P) or (C) in the bracket to denote phrase or clause respectively, as the case may be.

1. Mangoes are expensive this year. ( )
2. I found this book in the university library. ( )
3. This is the novel which Saul Bellow has written. ( )
4. Smoke from vehicle exhausts causes air pollution. ( )
5. He bought the house which belonged to his friend. ( )

Given below are a few sentences in which clauses are underlined. Identify the type of clause, (Noun clause, Adjective clause, Adverb clause) in each sentence and write its name in the brackets provided against each sentence.

1. He reached the place when the sun had set. ( )
2. Because you have done this, you must explain ( )
3. If you can do it fast, please do so. ( )
4. The table which has a broken leg is from the staff room. ( )
5. He expects that he would get through the examination. ( )
6. He is very rich, yet he is unhappy. ( )
7. Although he is an industrious student, he failed in the examination. ( )
8. The children clapped as the clown entered the ring. ( )
9. People started running out as soon as the accident took place. ( )
10. Please answer the call for me in case I go out. ( )

Conditionals

There is a great variety of conditional sentences.

The two main divisions are:

Type A

Those with clauses that contain a condition that may or may not be fulfilled.

These are called clauses of open condition or factual condition.

Example: What shall we do if it rains?

In the above example the speaker does not declare that the condition will be realized or that it will not be realized.

He leaves the question open or unanswered.

The speaker here merely puts forward the possibility that it may rain.

More examples:

- Don't come unless I tell you to come.
- We shall go provided the weather is fine.

Type B

Those with clauses in which a theoretical condition is put forward.

These are clauses in which the condition is combined with improbability or unreality.

- He would come, if he had time.
- If you were a bird, you could fly.

Conditional clauses may be introduced by if (even if, if only), so long as, suppose or supposing (that), on condition (that), provided (that), and for a negative condition, unless (which means the same as if ….not, but is more emphatic). In case (this is used to introduce a contingency or possibility against which a precaution is needed).

- If you have enough money, why don't you buy a house?
- So long as you return the book faithfully, I will lend it to you with pleasure.
- You'd better take an umbrella with you in case it rains.

If–Conditionals

Type–I

Look at the following example.

If he comes to the library, I will give him these books.
pres. fut.
(It is possible that he will come)

Note: This is an open condition i.e., this expresses a possible situation.

In the above example, the present tense is used in the if–clause (or conditional clause), and the future tense is used in the result clause.

Type–II

If she practised well, she could win the game.
(She doesn't practise well)

If she came to my house, I would give her your message.
(She doesn't come to my house)

Note: This is an unlikely condition. This is called unreal condition used for impossible or unreal situations.

The past tense is used in the 'if-clause' and would, could, or might + the simple past tense verb is used in the main clause.

Type–III

If she had come to my house, I would have given her your message. Would + have + past participle
past perf. (She didn't come to my house)

Note: This is a closed condition.

When the past perfect is used in the 'if–clause', would, could, or might + have + the past participle are used in the main clause.

We can summarize the usage of the conditionals discussed above as follows:

If–clause		Main clause
If + present	. . . . .	future (result)
If + past	. . . . .	would + verb (result)
If + past perfect	. . . . .	would + have + past participle (result)

Notes:

(A) In general, avoid using would in the if-clause.

(B) If the if–conditionals express a

(i) doubt or supposition or

(ii) a wish or desire, the form of the verb to be used is 'were'.

Examples: If he were rich, he would go to the U.S.A. for higher studies.

If I were a bird, I would fly to England without a Visa.

Examples of Errors in the Usage of Conditionals

The following examples will make clear some of the common mistakes committed in the usage of conditionals.

1. WRONG: If I will win the contest, I will buy a new house.
   RIGHT: If I win the contest, I will buy a new house.
2. WRONG: If I had been there, I would make a speech.
   RIGHT: If I were there, I would make a speech.
   If I had been there, I would have made a speech.
3. WRONG: If I would have a degree from that University, I would get a good job.
   RIGHT: If I had a degree from that University, I would get a good job.

Types of Sentences (Structural)

A sentence, for the purpose of analysis, can be examined from the point of view of its structure. Depending on the number of clauses it contains, a sentence can be called SIMPLE, COMPLEX or COMPOUND.

Let us look at each of these types of sentences with examples.

1. Simple Sentence

A simple sentence is one which contains only one subject and one predicate.

(The subject: Person or thing about which something is said).

(The predicate: What is said about the subject).

In other words a simple sentence contains only one main clause. It does not have subordinate clauses.

Examples:

1. My father is a senior manager in this organisation.
   subject (s). The predicate (p)
2. Chandigarh is a planned city.
   s p
3. Microbiology is an interesting subject for me.
   s p

Note that there is only one finite verb in each of the above sentences. That means there is only one clause. One clause sentence is a simple sentence.

2. Compound Sentence

A compound sentence is one which contains two or more main or principal clauses. It may or may not have subordinate clauses.

In other words a compound sentence consists of two or more independent sentences joined together by a coordinating conjunction.

Example: He is poor yet he is happy.

The above sentence consists of two parts.
(i) He is poor (ii) He is happy

These two are joined by the co-ordinating conjunction 'yet'.

In the above example, each part contains a subject and a predicate of its own i.e., each part is called a clause.

Each clause makes good sense by itself, and hence can stand independent of the other. Hence, each is called a principal clause or a main clause.

Study the following examples.

Examples:

1. You may either stay in the hostel or stay with your relatives.
2. The flowers were blooming, the birds were singing, and spring was in the air.

Analysis: In sentence 1, there are two main clauses.

In sentence 2, there are three main clauses.

So each of them is a compound sentence.

Some more examples of compound sentences:

1. I went to his farmhouse several times, but I was unable to meet him.
2. He is very intelligent, yet he has failed in the examination.
3. Many doctors are attending on him, still he is not out of danger.
4. He is very tired, for he has been working since morning.
5. There is no quorum, so the meeting was adjourned.

Note: In the above examples, the underlined words are called coordinating conjunctions. Coordinating conjunctions join main clauses.

The following is a list of some co-ordinating conjunctions

and, as well as, also, too, both ….. and, not only …….. but also, now, well, either ….. or, neither …….. nor, otherwise, or else, still, yet, only, however, whereas, but, nevertheless, therefore, for, so then, so then.

3. Complex Sentence

A complex sentence is one which contains only one main clause and one or more subordinate clauses. Study the following examples.

Example 1: If you work hard you will pass.

In the above example, the clause 'you will pass' makes good sense by itself, and hence can stand by itself. This is called the principal or independent clause.

The clause 'If you work hard,' cannot stand independently, it depends on the clause, 'you will pass'. It is therefore called a dependent or a subordinate clause.

Example 2: As soon as the meeting began, a member said that he wanted to raise a point of order.

Analysis: This sentence has only one main clause and two subordinate clauses. Therefore, this is a complex sentence.

Subordinate clause 1: That he wanted to raise a point of order.

Subordinate clause 2: As soon as the meeting began.

Main clause: A member said

Some more examples of Complex Sentences:

1. The children rode an elephant when they visited the zoo.
2. I think that the president will not accept this proposal.
3. They were so tired that they simply had to sit down to take rest.
4. He knew that he had to get someone to cut down the trees in the garden.
5. If you want a passport, you must consult the passport issuing authority in the city.

In the above examples, the underlined words are called subordinating conjunctions which introduce subordinate classes.

The following is a list of some subordinating conjunctions. After, before, since, as soon as, while, until, as, so long as, till, in order to, lest, so that, that, because, since, supposing, unless, whether, on condition, so ….. that, than, no less than, as much as, so far as, according as, if, though, although, however, notwithstanding, even if, whatever, whichever.

Sample Exercise–3

Identify the type of sentence—Simple, Complex or Compound.

1. As soon as the boy saw the cobra, he began to run.
2. The South Pole is too cold for human beings to live in.
3. A hundred per cent in English is a very high score and it is almost impossible for a student to achieve this.

4. On account of mismanagement, the company suffered a great loss.
5. In spite of being supplied with the best weapons, the army could not win the battle.
6. The rooms were so selected as to enable me to reach the place of my interview.
7. It is in the field of education that the seed of secularism has to be sown at the earliest stage possible so that the plant can be carefully nurtured as it grows.
8. Our hope that the clouds would disperse, was cheering.
9. To most people a long journey by motorcar is exhausting and disagreeable.
10. Although mountains are undoubtedly impressive, they have a kind of dreadful monotony that makes people like me, feel hostile to their beauty.

Types of Sentences (Semantic)

Based on the meaning (Semantic), sentences are categorised into:

1. Assertive sentences
2. Imperative sentences
3. Interrogative sentences
4. Exclamatory sentences
5. Affirmative sentences
6. Negative sentences

Assertive Sentences (or) Declarative Sentences

Sentences which state facts, describe things, or report events are called assertive or declarative sentences.

Examples:
Dr. Mohan teaches English.
Varanasi is on the banks of the Ganga.
Hyderabad is the capital of Andhra Pradesh.

Imperative Sentences

Sentences that express an order, a request, advice, a direction, a suggestion, a command, etc. are imperative sentences. These sentences start with the verb, and the subject (You) is understood.

Examples:

Take these books to the library.

Get me a cup of coffee, please.

Be neither a borrower nor a lender.

Note: There are some sentences which express 'commands' in an indirect way. Let is used at the beginning of these sentences. Subject is not omitted but expressed in these sentences.

Examples:

Let the manager talk to the director first.

Let's start at once.

Let the boys assemble in the auditorium for the seminar.

Interrogative Sentences

Sentences that are used to make enquiries and to ask questions are interrogative or question sentences.

Examples:

Can I help you?

What shall we have for lunch?

Is the bus late this morning?

Exclamatory Sentences

Sentences that express surprise, admiration, pity, sorrow and other feelings in an emphatic way are called exclamatory sentences.

Examples:

How fast time flies!

What a beautiful view we have from here!

What a wonderful piece of art this is!

What a tragedy!

Affirmative Sentences

You have seen that all the sentences given above as examples are statements, commands, questions and exclamations. They affirm something in a positive way. They are affirmative sentences.

Negative Sentences

Look at the following sentences.

Examples:

He is not a rich man.

I cannot complete this work today.

They don't understand our difficulties.

I have never watched a test match.

All these are also statements but they use words such as NOT and NEVER and are negative in meaning. They are called negative sentences.

Note: Commands can also be negative.

Example: Don't touch those glass exhibits.

Sentences of this kind are called negative commands or prohibitions. There can also be questions which are negative:

Hasn't he got a car?

Why didn't you come to the class in time?

Sentences of this kind are called negative questions.

Sentences Correction

Correct Use of Nouns

A noun is a naming word. It is the name of a person, place, thing, animal, concept, river, etc. Nouns are divided into five main kinds.

1. **Common Noun:** A name given in common to every person or thing of the same class is called a common noun.
   Examples:
   - Hyderabad is a big city.
   - She is a good girl.
2. **Proper Noun:** The name of a particular person or place is called a Proper Noun.
   Examples:
   - King Solomon was a wise king.
   - Kalidas was a great poet.
3. **Collective Noun:** It is the name of a number of persons or things taken together and spoken of as a whole.
   Examples:
   - A herd of cattle is grazing in the field.
   - A bunch of keys is on the table.
4. **Abstract Noun:** The name of a quality, a state of mind or a stage of life is called an Abstract noun.
   Examples:
   - Kindness is a virtue.
   - Childhood is the happiest stage of one's life.
5. **Material Noun:** All those nouns, which are used to refer to metals and materials, are called Material Nouns.
   Examples:
   - This chain is made of gold.
   - This house is built of brick.

In the above examples 'gold' and 'brick' are material nouns.

Nouns are further classified into two types depending on whether they can be counted or not.

1. **Countable nouns:** Countable nouns are nouns that we can count: for eg. boy - one boy, two boys, three boys, etc.
   Note: The classes of nouns known as 'common nouns' (boy, table, etc.) and 'collective nouns' (flock, regiment, crowd, troop, etc.) can be included in the larger class of countable nouns.
2. **Uncountable Nouns:** (also called non-count nouns) Uncountable nouns (NCNs) name things which we cannot count or usually don't count. The classes of nouns known as Material nouns and abstract nouns can be included in this category.

Uncountable nouns fall into the following groups:

1. Things which are considered in mass or quantity and not in numbers: sugar, sand, hair, dust etc.
2. **Materials:** wood, cloth, silver, gold, iron, etc.
3. **Liquids:** water, milk, oil, ink, honey, etc.
4. **Gases:** steam, mist, nitrogen, air, etc.
5. **Natural phenomena:** heat, cold, electricity, lightning, etc.
6. **Abstract nouns:** stages of life, states of mind, qualities, processes, actions, ideas, etc.
   courage, happiness, freedom, attention, childhood, honesty, enjoyment.

   Note: A number of words used to refer to qualities, states, actions, etc., are however countable. These may be considered exceptions.

   Examples: joys, sorrows, dreams, miseries, etc.
7. **Branches of learning:** Economics, Mathematics, Politics etc.

   Note: There are some uncountable nouns which we refer to in parts. When we do so, we use them with expressions such as a piece of, a grain of, a kilo of, etc. Here is a list of such expressions:

a piece of chalk	a piece of information
a cake of soap	a piece of furniture
a block of ice	a loaf of bread
a piece of advice	a log of wood, etc.

These expressions have their plural forms: pieces of chalk, cakes of soap, etc.

Proper Nouns are not categorised either as Countable Nouns or as Uncountable Nouns.

Now study carefully some rules related to the correct use of nouns:

Rule 1: A proper noun becomes a common noun when it is used in the plural form, or when an article is placed before it.

Examples:

There are five Gandhis in our college.

Vijay Kumar is the Milton of your college.

Kalidas is the Shakespeare of India.

Rule 2: A collective noun takes a singular verb when the whole group is considered as one unit.

Examples:

The committee consists of five members.

The family living next door has come from Chennai.

Note: If we refer to the members of the committee or family separately, the collective noun takes a plural verb, but it (the collective noun) remains singular in form, for eg.: The committee have taken their seats.

Rule 3: An abstract noun can also be used as a common noun by placing an article before it.

Example: Rajani is a beauty (a beautiful girl).

Rule 4: An abstract noun can also be used in the sense of a collective noun. When an abstract noun is used as a collective noun it takes a plural verb.

Example: Youth are the pillars of the nation. (collective noun)

Rule 5: When a material noun denotes a mass of matter, it is not used in the plural form.

Examples:
Their house is built of bricks and stones. (incorrect)

Their house is built of brick and stone. (correct)

Rule 6: Some nouns have the same form for the plural as well as for the singular. The following are some nouns that belong to this category.

Example: sheep, species, deer, aircraft, offspring, yoke, space craft, salmon, etc.

Rule 7: Some nouns are used in the singular only. The following are some nouns that belong to this category.

Example: advice, corn, grain, justice, clothing, information, poetry, scenery, machinery, hair, etc.

Note: Advice is the noun form and Advise is the verb form.

The advice of the doctor is that I should not smoke. (noun)

The doctor advised me not to smoke. (verb)

Similarly, Practice is noun and Practise is verb.

Rule 8: There are some nouns which are used only in the plural form.

Examples:

scissors	shears	spectacles
trousers	pantaloons	shorts
bowels	intestines	biceps
genitals	gymnastics	premises
proceeds	riches	thanks
nuptials	outskirts	gallows
doldrums	earnings	

Rule 9: Some nouns are apparently plural in form but are singular in use.

Example: news, economics, physics, measles, mumps, ethics.

Note: summons is singular and summonses is plural.

Rule 10: The following nouns are always used in plural.

Example: cattle, police, people, vermin, poultry, etc.

Rule 11: When a plural noun denotes a specific amount, length, weight, quantity, etc. considered as a whole, the verb must be in the singular form.

Examples:
Ten thousand rupees is a large amount.
Six kilometres is not a long distance for a runner like you.

Rule 12: Inanimate objects (lifeless objects) which are remarkable for beauty, gentleness, gracefulness, tenderness, weakness, and so on, are treated as feminine gender.

Example: the moon, the earth, spring, virtue, peace, charity, fame, nature, hope, justice, mercy, pity, fortune, truth, victory, defeat, modesty, liberty, etc.

Note: Poets refer to them as feminine gender.

Rule 13: Inanimate things which are known for strength, courage, violence, superiority, and so forth, are treated as masculine.

Example: the sun, death, summer, winter, war, anger, fear, thunder, etc.

Note: The possessive case of nouns is formed when we put 's to the noun (apostrophe s)

Examples:
- The girl's dress.
- The boy's pen

The possessive case is used with the nouns of living things but not with inanimate objects.
- Table's leg (incorrect)
- Leg of the table (correct)

The possessive is used with the names of personified objects.
- Death's icy hands

Rule 14: When the noun is plural, and ends in s; the possessive case is formed by adding only an apostrophe (after s).

Examples:
- Girls' hostel
- Birds' nests

Rule 15: When the noun is plural but does not end in s, the possessive case is formed by adding 's.

Examples:
- Men's club
- Children's park
- Women's day

Rule 16: When a noun or a title consists of several words, the possessive case is formed by adding 's to the last word.

Examples:
- The Prime Minister of India's speech.
- The king of Bhutan's visit to India…..

Rule 17: When two or more proper nouns are joined by 'and', and a common possession is meant, the possessive case will be formed by adding ('s) to the last noun.

Example: Amar and Kumar's house is really beautiful.

Note: This sentence denotes that the house belongs to both Amar and Kumar.

Rule 18: When there are two nouns in apposition, the possessive case will be formed by adding ('s) to the second noun.

Examples:
1. Penicillin, Flemming's discovery, has saved the lives of many people.
2. Mohan, the professor's son, is very intelligent.

Rule 19: The possessive case of a compound noun will be formed by adding ('s) to the last word.

Examples:
Sister-in-law's house.

Commander-in-chief's office.

Rule 20: If the last syllable of a singular noun ends in '-s' or –'ce' and the noun is followed by the word 'sake', the possessive case of the noun will be formed by adding (') only. However, modern day English accepts the use of 's' after apostrophe as the word is pronounced so–eg–goodness's sake.

Examples:
for goodness' sake, for justice' sake.

for conscience' sake, for Jesus' sake.

Sample Exercise–4

Correct the following sentences where necessary.

1. This scissor will not cut such a hard wire.
2. I have seen many cannons in the military parade ground.
3. The beggar is begging for alm.
4. He has lost all his luggages on his way home.
5. The company wants to dispose of the equipment.
6. She took great pain to write the essay properly.
7. The first innings is over just now.
8. These sceneries of Kashmir are beautiful.
9. My brother-in-laws will come to our house next week.
10. These golds have been brought from South Africa.

Correct Use of Articles

There are two types of articles:

1. The Indefinite Article
2. The Definite Article

The Indefinite Article: A and An (Please note that 'An' is a variant of 'A'.)
The Definite Article: The

Notes:
1. A or An is used before a singular countable noun.
2. The choice between A and An wholly depends on the pronunciation of the word.
3. In English the 'spelling of a word' is different from its 'pronunciation'.
4. 'An' is used before the word, the pronunciation of which starts with a vowel sound.
5. There are 44 sounds (each is given a symbol from the International Phonetic Alphabet).
6. Out of the 44 sounds, there are 20 vowel and 24 consonant sounds.
7. Please note that, strictly speaking, A, E, I, O, U are not vowels but just letters of the English Alphabet consisting of 26 letters.
8. The indefinite article 'a' is used before
   (a) a word beginning with a letter which has a consonant sound. e.g.: a book, a man, a pen.
   (b) a word that begins with a letter (like O) with the sound like 'wa' e.g.: a one-rupee note, a one-eyed man, etc.
   (c) a word beginning with 'u' or 'eu' giving the consonant sound 'yu'.
   Examples: a university, a European.
9. The indefinite article 'an' is used before:
   (a) a word beginning with a letter which has a vowel sound.
   Examples: an apple, an egg, an umbrella, etc.
   (b) a word beginning with 'h' but the pronunciation of which starts with a vowel sound.
   Examples: an heir, an hour, an honest man, etc.
   (c) an abbreviation, the first letter of which has a vowel sound
   Examples: an M.L.A., an M.P., an M.Com, an S.D.O., an F.R.C.S., an X-mas gift, etc.

Some Important Rules Regarding the Use of Articles

Rule 1: A or An is not used before:

1. plural nouns: a books, a universities, a dinners, etc. (wrong)
2. uncountable nouns: an advice, an information, etc. (wrong)
3. names of meals:

Examples:
Let us have dinner (correct)

Let us have a dinner (incorrect)

Note: An indefinite article can be used before names of meals when these are preceded by an adjective.

Example: She gave me a good breakfast at 8 a.m.

Rule 2: A or An is used in the following ways:

Before the word 'most' when it is used in the sense of very or much or exceedingly.

Examples:
Sreedhar is a most intelligent student.

This is a most unfortunate event.

Note: It must be borne in mind that 'man' or 'woman', when used in a general sense to represent 'mankind' as a whole, never takes an article.

Example: Man is mortal.

Rule 3: With a noun complement. This includes names of professions.

Examples:
He is an actor.
She is a good dancer.
It was an accident.

* The words underlined are noun complements. They are also called subject complements.

Rule 4: In certain phrases: a cold, a pain, on an average, make a noise, make an effort, make a mistake, a fever, etc.

Rule 5: With certain numbers: a hundred, a thousand, a million, etc.

Rule 6: 'A' can be used before Mr./ Miss/ Mrs. + name

Example: a Mr. Bose, a Mrs. Bose, etc.

Notes:
A Mr. Bose denotes a person who is called Bose, and implies that he is a stranger to the speaker.

Mr. Bose, without 'a', implies that the speaker knows Mr. Bose, or knows of his existence.

Rule 7: Before a proper noun to make it a common noun.

Examples:
Mohan is a Newton (Mohan is a great scientist)

Amaresh is a Shakespeare (Amaresh is a great dramatist).

Rule 8: Before certain uncountable nouns preceded by nouns + of

Example: a piece of advice, a bit of news, a drop of water, etc.

Rule 9: After the words many, rather, such, quite, etc. in certain structures.

Examples:
Many a friend of mine is attending the party.

Such a show cannot be arranged now.

He is rather a fool to take such decisions.

Rule 10: In certain expressions of quantity.

Example: a lot of, a couple, a great many, a good deal of, a good many, a great deal of, a few, a little, etc.

THE is used in the following ways:

Rule 11: Before a noun denoting a hospital, temple, school, college, prison etc, if its purpose or use is not referred to or if used not in its primary purpose.

Example: He has gone to the hospital to visit a friend.

Rule 12:
Before an adjective in the superlative degree.

Michael is the tallest boy in the class.

Rule 13: With nouns which refer to things which are unique.

Example: the sun, the moon, the sky, the earth, the equator, etc.

Rule 14: Before an abstract noun or a material if it is used with an adjunct (a qualifying clause)

(or when there is a particular reference to the material noun)

Example: The gold you have brought from the U.S.A. is of good quality.

Rule 15: Before a singular countable noun which picks out one individual, object, etc. as representative of a class.

Example: The tiger is a ferocious animal.

Notes:
All the following sentences mean the same thing.

The tiger is a fierce animal.

A tiger is a fierce animal.

Tigers are fierce animals.

Rule 16: The is used before certain adjectives to give a plural meaning.

Examples:
rich, poor, dead, sick, healthy, deaf, blind, etc.

the rich = rich people

the poor = poor people

Rule 17: With certain adjectives indicating nationality.

Example: the Dutch, the Spanish, the Chinese, the Burmese, etc.

Rule 18: Before 'only' and ordinal numbers, such as first, second, millionth, etc.

Examples:
All the students of the first year are invited.

The second ranker is my son.

Rule 19: Before a noun when special emphasis is needed.

Example: This is the novel I am talking about.

Rule 20: Before a common noun to give it the meaning of an abstract noun.

Example: At last the father in him prevailed and excused him.

Rule 21: Before an adjective in the comparative degree, when the selection of one out of only two persons, places or things is meant.

Example: Nalini is the more beautiful of the two girls in the class.

Rule 22: In special comparatives.

Examples:
The more you earn, the more you spend.

The more, the better.

The higher you go, the cooler you feel.

Rule 23: Before musical instruments.

Example: Rajani can play the piano very well.

Rule 24: Before the proper names of certain well-known or sacred books.

Example: the Mahabharat, the Ramayana, the Gita, the Bible, the Quran.

Rule 25: Before the names of certain countries each of which is a union of smaller units.

Example: the U.S.A., the U.A.E., the U.K., etc.

Rule 26: Before the words like north, south, etc. when these are used as nouns.

Example: the north of India, the Middle East, the West Asia

Rule 27: Before some proper nouns consisting of adjectives and noun or noun + of + noun.

Example: the State Bank Of India, the National Museum

Rule 28: The + singular noun + clause or phrase can be used to distinguish one person from another of the same name.

Examples:
We have two Reddys, which Reddy do you want?

I want the Reddy who signed the letter.

Rule 29: Before the names of political parties.

Example: The Congress, The B.J.P., etc.

Rule 30: When it is clear from the context that a particular person, place or thing is meant.

Examples:
I talked to the principal yesterday.

The students are playing in the garden.

Rule 31: Before the names of the historical or public buildings

Example: the Taj Mahal, the Red Fort, the Rashtrapati Bhavan, etc.

Rule 32: Before the names of rivers.

Example: the Krishna, the Ganga, the Yamuna, etc.

Rule 33: Before the names of seas.

Example: the Arabian Sea, the Red Sea, etc.

Rule 34: Before the name of Oceans.

Example: the Indian Ocean, the Atlantic Ocean, etc.

Rule 35: Before the names of certain chains of Mountains.

Example: the Himalayas, the Alps, etc.

Rule 36: Before the names of deserts

Example: the Sahara, the Thar, etc.

Rule 37: Before the names of newspapers, magazines, etc.

Example: the Hindustan Times, the Statesman, the Deccan Chronicle, etc.

Rule 38: Before groups of islands.

Example: the Andamans, the West Indies, etc.

Rule 39: Before the names of certain countries.

Example: the Yemen, the Sudan, the Hagues, etc.

Rule 40: Before the dates of months.

Example: the 23 October, 1949, the 15th August, 1947, etc.

THE is omitted in the following cases:

Rule 41: Before a common noun used in the widest sense.

Examples:
In general woman is kind-hearted.

Man is mortal.

Rule 42: Before material nouns.

Example: Gold is a precious metal.

Note: The may be used with a material noun if it is used with an adjunct which makes it definite.

Example: The gold we use in India is all imported.

Rule 43: Usually before proper nouns.

Example: Hyderabad is the capital of Andhra Pradesh.

Rule 44: Before abstract nouns used in a general sense.

Examples:

Honesty is the best policy.

Sincerity pays dividends.

Note: If an abstract noun is qualified by an adjective or an adjectival phrase or clause it may have the article 'the' before it.

Example: The wisdom of Moses is biblically significant.

Rule 45: Before the words—father, mother, aunt, uncle, etc., in general sense.

Example: Father is very angry today.

Rule 46: Before predicative nouns denoting a position that is normally held at one time by one person only.

Examples:
Mr. Kiran was elected chairman of the committee.

He became principal of our college in 1996.

Rule 47: Before plural nouns which are used to denote a class.

Examples:
Historians study old monuments for research purposes.

Camels are useful animals in deserts.

Rule 48: Before plural nouns used in general sense.

Example: Members are requested to pay their subscriptions in time.

Rule 49: Before the names of meals.

Examples:
I had lunch at 2.30 p.m.

Let's have dinner at 9.00 p.m.

Note: 'The' can be used when meals are preceded by an adjective or a clause or phrase particularising them.

Examples:
The dinner given by our colony association yesterday was not properly arranged.

What about the wedding dinner tonight?

Rule 50: Before common nouns used in pairs.

Examples:
He worked day and night to prepare for the Civil Service exam.

Both husband and wife are supposed to take part in this competition.

Rule 51: No article is used before the nouns in the following expressions:

to catch fire	to send word	to give ear
to lose heart	to leave home	to take offence
at home	in hand	in debt
by day	at sunrise	at noon
on demand	by land	by air
at night	on foot	at ease

Rule 52: 'The' is not used before the words—bed, hospital, sea, temple, prison, court, school, university, college, etc., when these places are visited or used for their primary purpose.

We go to bed to sleep.

We go to hospital as doctors.

We go to university for higher studies.

Note: When these places are visited or used for the other reasons or purposes 'the' should be used.

Example: Sometimes I go to the prison to give lectures on morality, social behaviour, etc.

The articles a, an or the can be used as follows in some cases:

Rule 53: When two or more adjectives qualify the same noun, the article is used before the first adjective only.

Example: He has a black and white coat.

Rule 54: When two or more connected nouns refer to the same person or thing, the article is placed before the first noun only.

Example: The chief accounts officer and financial adviser is supposed to certify your documents.

Rule 55: In expressing a comparison, if two nouns refer to the same person or thing, article 'a' is used before the first noun only.

Example: He is a better teacher than administrator.

Articles are omitted in the following cases:

English	⇒	the English language
the English	⇒	the English people

Examples:
We speak English at home.

The English and the French have fought various wars.

He has gone to market	⇒	He has gone to make purchases.
He has gone to the market	⇒	He has gone to the place where there is a market, not necessarily to make purchases.
To see light	⇒	to look at light
To see the light	⇒	to be born

Examples:
I see light at the end of the tunnel. (metaphoric use which means there is hope)

His proposals for a new international airport never saw the light of the day.

The article 'the' is omitted before the nouns in the phrases below:

He invited me to lunch/tea/dinner.

Note: lunch/tea/dinner are used in general sense.

Correct use of Pronouns

A pronoun is a word used in the place of a noun.

Types of pronouns:

Personal Pronouns

Person	Singular	Plural
First person	I	We
Second person	You	You
Third person	He, She, It	They

Forms of pronouns: (cases)

Nominative case (subject)	Accusative case (object)	Possessive	
I	Me	My	Mine
We	Us	Our	Ours
You	You	Your	Yours
He	Him	His	His
She	Her	Her	Hers
It	It	Its	Its
They	Them	Their	Theirs

Note: The forms my, our, your, his, her, its and their are determiners. They can also be used as possessive adjectives.

Possessive Pronouns

Example: This book is his.

Reflexive Pronouns

Examples:

1. She washed herself at a well.
2. They killed themselves for no reason.
3. I must blame myself for this.

Emphatic Pronouns

Example: I offered to drive the car myself. She herself made the mistake.

Demonstrative Pronouns

Examples:

1. That is Dr. Rao's house.
2. Those are some new buildings.
3. This is your passport.
4. These are good books.

Indefinite Pronouns

Example:

1. Some are born great.
2. Anyone can take a horse to the pond, but no one can make it drink.

Other indefinite pronouns are:
somebody, anybody, anything, something, (the) other, others, another, nobody, nothing etc.

Pronouns of Number

Examples:

1. Of the ten guests, three were men.
2. Many of them are not good books.

Pronouns of Quantity

Examples: more, much, little, etc.

Distributive Pronouns

Look at the pronouns underlined in the sentences given below:

Examples:

1. Everyone has his own problems.
2. Either of these applicants can be employed.
3. Neither of the students can get through the examination.
4. Each of those players deserves a prize.
5. Everybody will be given a chance to participate in this match.

These words, which are underlined, are distributive pronouns. They are called so because they refer to persons considered individually. The distributive pronouns are in the singular number and are therefore followed by singular verbs.

Note:

1. Neither is a negative word which is used to refer to two persons or things.
2. None is also a negative word which is used to refer to more than two persons or things.
3. The usage: Each one of the boys …… (is wrong because 'one' becomes redundant)

Reciprocal Pronouns

Look at these sentences:

Examples:

1. The two girls helped each other in every respect.
2. The political parties quarrelled with one another.

Each other and one another express a mutual or reciprocal relationship. They are considered as single units, and are called reciprocal pronouns or 'compound personal pronouns'.

Note: Each other is usually used to refer to two persons or things and one another to more than two persons or things.

Interrogative Pronouns

The words underlined in the following sentences are called interrogative pronouns because they are used in questions.

Examples:

1. Who is the president of India?
2. Whose is this calculator?
3. To whom did you give the parcel?
4. Which is the road to the hospital?
5. What have you to do now?

6. Whom did you borrow this book from?
7. Whom did she refer to in her lecture?

Relative Pronouns

Look at these sentences:

Examples:

1. The gentleman who is speaking is our principal.
2. This is the gentleman whose guest I was in Bombay.

The words underlined are pronouns standing for the nouns used before them. The nouns are called antecedents and the pronouns are called relative pronouns. They relate the adjective clauses to the main clauses.

Rules for use of Pronouns

Now let us look at some aspects related to pronouns that will be helpful in answering questions in Sentence Correction.

Rule 1: Some important uses of the pronoun 'it' are given below with examples.

1. To introduce a sentence
   Example: It is not certain that the president will come.
2. To give emphasis to the noun or pronoun that follows
   Example: It was you who began the quarrel with us.
3. As an indefinite nominative of an impersonal verb.
   Examples:
   It rains.
   It is snowing outside.
4. In sentences showing distance
   Example: It is not far to walk.
5. In sentences indicating time.
   Example: It is ten O'clock now.
6. To introduce a phrase.
   Example: It is decided to declare a holiday today.
7. In exclamatory sentences.
   Example: What a beautiful book it is!
8. To introduce a that clause.
   Example: It is said that smoking is injurious to health.
9. As a sort of object in order to avoid repetition.
   Example: Let us fight it (the issue) out.

Rule 2: While confessing a fault (or expressing a negative idea) the sequence of the personal pronouns should be as follows:

Example: I, you, and he are in the wrong and will be punished.

Sequence: First person first, second person next and third person last.

Rule 3: While expressing a positive idea or praise, the sequence of the personal pronouns should be as follows:

Example: You, he and I, will get an award for the good work we have done.

Sequence: Second person, third person and first person.

In standard English, however, the sequence of third second and first is also accepted (i.e.) He, you and I _____.

Rule 4: When two singular nouns joined by and denote the same person or thing, the pronoun used for them must be singular in number. The definite article the is placed before the first noun.

Example: The accounts officer and treasurer should be careful in his work of keeping accounts.

Rule 5: When two singular nouns are joined by 'and', and are preceded by each or every, the pronoun must be in singular number.

Example: Every student and every teacher took his or her seat.

Rule 6: When a personal pronoun is connected by a conjunction with some other word in the objective case, it must be in the objective or accusative case.

Example: These clothes are for you and me.

Rule 7: When a singular noun and a plural noun are combined by or, either ... or, neither ... nor, the singular noun usually comes first in the sentence, and the pronoun must be in the plural number, corresponding to the plural noun which is closer to it.

Example: Either the manager or his subordinates failed in their duty in sending the official message.

Rule 8: The personal pronouns—yours, ours, hers, theirs and its—are written without the apostrophe (').

Examples:

Your's sincerely (Wrong)
Yours sincerely (Right)

Note: 'It's' means 'It is' and not belonging to it.

Example: It's a mad dog which bites its tail.

Rule 9: When a personal pronoun is used as a complement to the verb to be, it (the pronoun) must be in the nominative case.

Example: It was he, who could solve the problem easily.

Rule 10: A pronoun should be used in the objective case in a sentence beginning with let.

Examples:

Let him go to his office immediately.

Let her submit the records in time.

Rule 11: One can be used to talk about people in general. the pronoun that follows one should be one's. (not his/her)*

Examples:
One should do his duty (wrong)

One should do one's duty (correct)

Note: But there is a controversy here. In American English one can be followed by his or her. Students taking TOEFL or GRE or GMAT or SAT should keep this in mind. However, usually one need not worry about this problem. If a sentence begins with one, be sure that you or they DOES NOT follow. Hence it is never correct to say:

If one takes this exam without studying, you are likely to fail.

Additional example: One should never tell his secrets to a gossip if he wishes them to remain secret. (Correct)}

Rule 12: A relative pronoun must always be placed as near its antecedent as possible. Also, it must always agree with its antecedent in number, gender and person.

Example:
This is the manager who abused the clerk.
ant. rel. pr.

Rule 13: Generally, the relative pronoun in the objective case is omitted.

Example: The student (whom is omitted) you wanted to punish is absent today.

Rule 14: The pronouns who, whom and whose are generally used for persons.

Who is used in the nominative case.

Whom is used in the objective case.

Whose is used in the possessive case.

Examples:
1. Sarita is the student who got an award.
2. They are the thieves whom the police caught.
3. This is the student whose certificates are lost.

Rule 15: When the relative pronoun is in different cases, one in the nominative and the other in the objective, it must be mentioned twice, once for each verb.

Example: The girl, who is my daughter and whom you met in the library yesterday, left for Mumbai this morning.

In the above example the subject of the sentence 'the girl' is also the object of the sentence. Hence both 'who' and 'whom' are used.

Rule 16: Uses of WHICH:

'Which' is used in the following ways:

1. For infants, small animals, and objects
   Examples:
   This is the baby which was lost in the theatre.
   This is the dog which my friend bought from the Kennel's club.
2. When selection is expressed.
   Example: Which of these television sets do you want to purchase?
3. To refer to a sentence.
   Example: He was said to be drunk, which was not true.

Rule 17: Uses of THAT:

'That' is used in the following ways:

1. For persons, lifeless things and small animals in the singular or in the plural number.
   Examples:
   This is the girl that failed in the exam.
   This is the radio that I bought yesterday.
2. As a substitute for a singular noun already mentioned. (pay attention to this use particularly in the comparative degree)
   Examples:
   The weather of Hyderabad is far better than Chennai (wrong).
   The weather of Hyderabad is far better than that of Chennai.
3. After a noun phrase used as direct object.
   I vividly remember the night that she came.
   (Here that is used in the sense of when)

Rule 18: 'Either' and 'neither' are used in speaking of two persons or places or things etc.

Examples:
Neither Mahesh nor Mohan is intelligent.
(negative meaning is implied)

Either Mahesh or Mohan is expected to get a prize.
(positive meaning is implied)

Rule 19: Uses of each other and one another

1. Each other is used for two persons or things or places etc.
   Example: These two students love each other.
2. One another is used for more than two persons or things.
   Example: Those four countries always disagree with one another.

Rule 20: Each can come in three different positions in a sentence.

1. Each of the students got a prize. (initial)
2. The students got a prize each. (end)
3. The students were each given a prize (middle)

Rule 21: ONE is used in the following ways :

1. For people in general.
   Examples:
   One must try to do one's duty.
   One must not be proud of oneself.

2. In place of a noun previously mentioned.
   Examples:
   Give me a banana which is a fresh one.
   Give me bananas which are fresh ones.

Rule 22: As regards anybody, everyone, everybody, etc. the pronoun of the masculine or feminine gender should be used according to the context.

Example: Everyone of the boys got his hall ticket.

Sample Exercise–5

Rewrite the following sentences after making the necessary corrections.

1. She availed of the opportunity.
2. Those two companies always help one another.
3. India and Pakistan should cooperate with one another in this matter.
4. Dear students, enjoy during the holidays.
5. The committee were divided in its opinion regarding this issue.
6. The team, after taking its bath, have gone for practice of cricket.
7. My opinion is the same as your.
8. We were told to let only you and she enter.
9. This is the scientist which won the first prize.
10. His problems are the same as my.

Correct Use of Adjectives

An adjective is a word which qualifies a noun or a pronoun.

Example:
She is a good teacher.
adj. noun

An adjective can be used in two ways:
(1) attributively (2) predicatively

An adjective is used attributively if it is placed immediately before the noun it qualifies.

Example:
She is a good teacher.
adj. noun

An adjective is used predicatively if it is used after the verb.

Example:
Bimala is intelligent.
v. adj.

An adjective has three forms: Positive, Comparative, and Superlative. These are the three degrees of comparison:

Look at the following examples:
Superlative degree: Mohan is the best student in the class.
Comparative degree: Mohan is better than any other student in the class.
Positive degree: No other student in the class is as good as Mohan.

- The Positive Degree of an adjective is the adjective in its simple form. It is used to refer to the mere existence of a particular quality.
- The comparative degree of an adjective denotes a higher degree of the quality than the positive, and is used, when two things are compared.
- The superlative degree denotes the highest degree of the quality.

Note:

1. The usages 'as ..... as' and 'so ..... as' appear in positive degree.
   Comparative degree has 'adjective + er ... than'.
   Superlative degree has 'the + adjective in the superlative form'.
   Note the use of 'the', before the adjective in the Superlative (superlative form).

2. There is no superlative degree if the comparison involves only two objects, persons, or places, etc.

   Examples:
   Of the two sisters she is the best. (wrong)
   Of the two sisters she is the better. (correct)
   When selection of one of the two persons or things (sets of things) of the same kind is meant, the comparative degree is preceded by 'the' and is followed by 'of'.

   Example:
   She is the better of the two sisters.
   Let us look at some important rules in the use of adjectives:

Rule 1: Most adjectives form their comparatives by the addition of -r or -er, and their superlative by the addition of -st or -est (depending on the spelling) to the positive.

Examples:

Positive	Comparative	Superlative
Great	Greater	Greatest
Brave	Braver	Bravest

Rule 2: Some adjectives (usually having more than two syllables in the spelling) form their comparative by using the adverb more with the positive, and the superlative by using the adverb most with the positive.

Examples:

Positive	Comparative	Superlative
beautiful	more beautiful	most beautiful
interesting	more interesting	most interesting

Rule 3: (very important) When two qualities in the same person or thing are compared, the comparative degree is formed by using more, instead of -r or -er with the positive.

Example:
Reddy is more wise than intelligent.

(This sentence means that Reddy is both wise and intelligent. But his wisdom is greater than his intelligence.)

Rule 4: When an object is compared with the rest of the group, the latter term of comparison must exclude the former by using 'any other'.

Examples:
Gold is more precious than any other metal. (correct)

Gold is more precious than any metal. (wrong)

Rule 5: There are some words which are used only in the positive and cannot be used in the comparitive, or in the superlative.

Example: interior, exterior, ulterior, major, minor, etc.

- This is more inferior than that. (wrong)
- This is inferior to that. (right)

Rule 6: The following are used as comparative adjectives and are followed by to and not than:

Example: superior, senior, junior, inferior, prior, anterior, posterior, prefer, etc.

Rule 7: The adjectives—empty, excellent, circular, extreme, chief, entire, complete, perfect, final, last, unique, universal, round, square, triangular, eternal, etc., are not used in the comparative or superlative degree.

Rule 8: Some adjectives can be used only in the positive and in the superlative. They are:

Positive	Superlative
top	topmost
northern	northernmost
southern	southernmost
eastern	easternmost
western	westernmost

Rule 9: The adjective 'preferable' is used as a comparative. It is followed by to. It is not used with more.

Examples:
This is more preferable than that. (wrong)

This is preferable to that. (correct)

Rule 10: The comparative adjectives ending in 'or' are followed by the preposition to.

Inferior, superior, prior, anterior, posterior, senior, junior

Examples:
He is superior to me. (correct)

He is superior than me. (incorrect)

Rule 11: Sometimes the words much, less, far, etc., are used before the comparatives to denote emphasis or excess.

Example: Ramakrishna is by far the ablest lecturer in the college.

Rule 12: Two adjectives which refer to the same noun or pronoun joined by a conjunction must be in the same degree of comparison.

Example: Gandhiji is the noblest and wisest of all national leaders.

Rule 13: Certain adjectives, when preceded by 'the' become nouns in plural and require a plural verb if they are used as subjects.

These are: rich, poor, needy, aged, blind, dead, meek, wicked, etc.

Examples: The rich (rich people) usually hate the poor (poor people).

Rule 14: Some adjectives often confused:

1. FARTHER - FURTHER
   Farther denotes more distant in space,
   Examples:
   Sheela lives at the farther end of this street.
   Further means additional; more
   Examples:
   There is nothing further to say about him.
   For further details please contact the director.
2. Many, a great many, a good many
   All these take a plural noun and a plural verb after them.
   Examples:
   1. A great many people died in the cyclone.
   2. My sister has a good many friends.
3. Use of many + a + noun (singular) + verb (singular) structure :
   Examples:
   Many a delegate is attending this seminar.
   (meaning: Many delegates are attending this seminar)
4. MODIFIERS: FEW, LITTLE, MUCH, and MANY
   Few is used with plural nouns. Little is used with singular non-count nouns.

   Without articles, few, and little usually have rather negative meanings. They often suggest 'not as much/many as one would like', or 'not as much/many as expected' or a similar idea.

   With articles 'few' and 'little' i.e., a few, and a little are more positive in meaning. Their meaning is closer to 'some'. They often suggest ideas like 'better than nothing' or 'more than expected'.

   Examples:
   (a) There are few books on 'Thermodynamics' in the library, so you have to purchase them.
   (b) You don't need to go shopping. There are a few eggs in the fridge.

(c) I've got a little bread and cheese. It'll be enough for supper.

Difference between fewer and less: In theory, fewer (the comparative of few) is used before countable nouns, and less (the comparative of little) before uncountable words.

Examples:
(a) There are fewer exploited workers than there used to be.
(b) My friend earns lesser than a postman does.

Note the following:

P.D.		C.D.		S.D.
few	-	fewer	-	fewest
little	-	less/lesser	-	least

* fewer and less (lesser) are followed by than as they are the comparative forms.

Note: 'Many' like 'few' is used before a count noun.

Example:
Many/few students.
C.N.

'Much', like 'less' is used before a non-count or a material noun.

Example:
Much/less water.
NCN

5. MODIFIERS—TOO, VERY, ENOUGH
Compare the meanings and patterns of too, very, and enough.

(a) Very means to a high degree, but does not suggest impossibility.
Mary is very intelligent.

(b) Too suggests impossibility or undesirable degree.
too + adj + to + v
She is too sick to come to class today

Note: It is better not to use 'Too' with a positive adjective.
Instead of 'too' it is better to use 'very'.

(c) Enough suggests possibility or sufficient degree.
He is tall enough to play basketball.

Notes:
1. Be careful to put enough after the adjective.
2. Be careful to put to+v (infinitive) after enough.
3. In patterns 2 and 3 above, do not use any other structure word after the adjective or adverb except to.
4. In the above patterns adverbs can be used in the same position as adjectives.
Example:
Mary sings very well.
5. Enough can come before or after a noun to express sufficiency.

Examples:
He had money enough to buy a new car.
N
OR
He had enough money to buy a new car.
N

6. MODIFIERS - NEGATION
There are some words that have negative meanings even though they do not appear to be negative, for example: hardly, scarcely, rarely, seldom, without, and only. Do not use another negative word with these words.

Examples:
(a) He had scarcely enough money to purchase books.
(b) They went to bed without dinner.

Note: Scarcely and hardly are followed by when and not than. No sooner is followed by than.
These expressions mean 'as soon as'.

Examples:
(a) As soon as I reached the station the train left.
(b) No sooner had I reached the station than the train left.
No sooner + had + Subject + Past participle + …
(c) No sooner did I reach the station than the train left.
No sooner + did + Subject + Present tense verb + …….
Similarly: Hardly/Scarcely + had + Sub + Past participle …….. when …..
OR
Scarcely/Hardly + did + Sub + Present tense verb …….. when …..

Rule 15: MODIFIERS : ADJECTIVES AFTER VERBS OF SENSATION

These verbs of sensation are generally followed by adjectives* not by adverbs: feel, look, seem, appear, smell, taste and sound.

(* These are also called predicate adjectives)

He feels bad	(correct)
He feels badly	(wrong)
The soup smells delicious	(correct)
The soup smells deliciously	(wrong)

Rule 16: MODIFIERS - HYPHENATED OR COMPOUND ADJECTIVES

Nouns are sometimes found as part of hyphenated or compound adjectives (adjectives of more than one word joined by hyphens). These nouns are never plural.

I bought a four hundred year-old painting in Germany.
HYPHENATED ADJ. N

Some more examples:
1. B.Com is a three-year degree course.
2. This is a ten-kilometre-long distance.

3. The professor has delivered a two-hour lecture today.

Rule 17: MODIFIERS - CARDINAL AND ORDINAL NUMBERS

There are two kinds of numbers, cardinal and ordinal.

Examples:

CARDINAL	ORDINAL
one	first
two	second
three	third
four	fourth
twenty-one	twenty-first

The following patterns are used to designate items in a series:

1. Ordinal numbers are used in this pattern:
   THE + ORDINAL + NOUN
   The first book of the series is about verbs.
2. Cardinal numbers are used in this pattern:
   NOUN + CARDINAL
   Book one of the series is about verbs.

Notes:

1. Use the with ordinal numbers.
2. Do not use the with cardinal numbers.
3. Be careful to use the correct word order for each pattern.

Rule 18: MODIFIERS: NOUN ADJECTIVES

The first noun (N) in the following pattern is used as an adjective.

All these are language students.
N N

Notes:

1. When nouns are used as adjectives, they do not have plural or possessive forms.
2. Exceptions: The following nouns always end in -s but are singular in number when they are used as names of courses or sciences : Physics, Mathematics, etc.

He is an Economics teacher.
N N

BUT The current economic situation is extremely uncertain.

Note:

'Economic' is an adjective here.

'Economical' means frugal, thrifty.

Sample Exercise–6

Correct the mistakes relating to the use of adjectives in the following sentences where necessary.

1. The two first chapters of this book are very important.
2. Suvarna is greater than any student in the class.
3. Shakuntala is better than any drama in Sanskrit.
4. His house is further from the station than yours.
5. The later suggestion is better than the former.
6. These flowers smell sweetly.
7. Little learning is a dangerous thing.
8. Her birthday is the most happiest event this year.
9. Usually the rich people hate poor.
10. Go and meet the concerned clerk in the university.
11. Honour is dear to me than life.
12. Little act would have saved the situation.
13. The tiger is the ferocious of all animals in the forest.
14. My knife is the sharpest of the two.

Correct use of Adverbs

An adverb is a word which modifies a verb, an adjective or another adverb in a given sentence.

Examples:

She is reading very fast.
verb adv.

She is a very intelligent girl.
adv. adj.

The horse runs very quickly
adv. adv.

(**Note:** usually the adverbs end with -ly.)

Rules Pertaining to Use of Adverbs

Rule 1: In order to make our meaning clear, an adverb must be placed as near as possible to the word it modifies.

Examples:

She has only three dollars with her.
adv. adj.

He says he often visits my place.

He often says he visits my place.

Note: Other adverbs like only are: just, nearly, hardly, almost and scarcely.

Read the following examples:

Wrong : We only have four hours to finish this paper.
Right : We have only four hours to finish this paper.
Wrong : She just wants to take one class. [not anything else]
Right : She wants to take just one class. [not even second]
Wrong : That building nearly costs sixty thousand rupees.
Right : That building costs nearly sixty thousand rupees.

Rule 2: When an adverb modifies an intransitive verb, it usually follows it.

Examples:

- She sang melodiously.
- She writes neatly.

Rule 3: When a verb consists of an auxiliary and a main verb, the adverb is placed between the auxiliary and the main verb.

Wrong : I have told him often not to come late.

Right : I have often told him not to come late

Rule 4: When an adverb modifies an adjective or another adverb, the adverb usually comes before it.

Examples:

The Cuckoo sings quite sweetly.
Adv.

Do not speak so fast.
Adv.

Note: The adverb 'enough' is always placed after the word it modifies.

- Why don't you speak loud enough to be heard.

Rule 5: The word 'only' should be placed before the word it is intended to modify.

Examples:

- Only she gave me this book. (i.e., she and nobody else)
- She only told me the truth. (and nothing else)
- I solved only two problems. (correct)
- I only solved two problems. (incorrect)

Rule 6: POSITION OF ADVERBS

The position of adverbs is often determined by shades of meaning, for which rules cannot be given, but some generalisations can be made.

Adverbs of frequency: always, often, rarely, never, ever, generally, usually, sometimes, occasionally, etc.

- If the verb is in the simple tense form, the adverb is usually placed between the subject and the verb, preferably before the verb it modifies.
  He always goes to college on foot.
  He often visits the US.
  His brother never takes alcohol.
- When the verb is some form of 'be' (is, am, was, are) the adverb follows the verb:
  They are always late.
  He is never punctual.
  If you are ever in trouble, please meet me.

If the verb is a compound one, the adverb is usually placed after the auxiliary:

I shall never forget his help.

He will always behave properly.

In negative sentences the adverb of frequency follows not.

They are not often late.

In interrogative sentences the adverb of frequency follows the subject immediately:

Does he often go fishing?

Has he ever travelled by air?

At times 'often' may be placed at the end to emphasise it. This is mainly confined to negative statements and questions.

He does not see his friend often, as he lives in a remote village,

'Never' is sometimes placed at the beginning to emphasise it. Then the verb and subject are inverted as in a question.

I never saw such an accident.

Never did I see such an accident.

(S - V becomes V - S here)

Rule 7: USE OF HARD, HARDLY, SCARCE, SCARCELY

1. Hard as an adverb means 'diligently' . It usually follows the verb.
   He works hard to make both ends meet.
2. Hardly when used as an adverb means scarcely, barely
   It conveys a negative meaning.
   Hardly (scarcely) had he reached the station, when the train left.
   Note: Hardly and scarcely are followed by when not than. No sooner is followed by than not when. (This is very important)
3. Scarce as an adjective means 'not plentiful', hard to find, not often found.
   Coal has become scarce in England.
   Scarcely as an adverb is almost synonymous with 'hardly'. I can scarcely hear you.
   They have scarcely enough money to look after their children.

Rule 8: SPLIT INFINITIVE

The infinitive is to + the simple form of the verb (V).
Do not put an adverb between to and verb.

1. He refused to do the work quickly.
   to+v Adv.
2. They have decided to repeat the experiments carefully.
   to+v adv.

Wrong: He wanted to carefully read the directions.
Right: He wanted to read the directions carefully.

Rule 9: DANGLING MODIFIER

The subject of the main clause must be the same as the understood subject of the introductory phrase. In other words, the introductory phrase modifies the subject of the main clause.

Examples:

1. Looking at his watch, Mr.Vijaykumar got up and left.
   Introductory Ph. Subject

   Who looked at his watch? Mr. Vijaykumar
   Who got up and left? Mr. Vijaykumar

2. Travelling to Bombay, Nalini injured her leg.

   Introductory Ph. Subject

Note: Both these sentences are right. In both these sentences, the subject of the introductory phrase and the subject of the main clause are same.

Wrong: When only a baby, my mother took me to the circus.

Right: When only a child, I was taken to the circus by my mother.

Sample Exercise–7

Directions for questions 1 to 4: Fill in the blanks with suitable adverbs.

1. The speaker pauses _____ in order to heighten the beauty of his speech.
(A) hastily (B) briefly
(C) lustrously (D) incandescently

2. He always speaks _____ unmindful of the seriousness of situation.
(A) facetiously (B) soberly
(C) solemnly (D) thoughtfully

3. Though they are married, they are living _____.
(A) apart (B) away
(C) differently (D) together

4. The politician was careless and _____ made promises.
(A) uncommunicatively (B) inconsiderately
(C) uncommittedly (D) unceremoniously

Correct Use of the Verb

Structure of the Verb Phrase

A verb indicates the action done by the subject or the state of being of the subject.

Examples:

1. He has completed the work (action).
2. Radha is an intelligent student (state).
   In every sentence you find a verb phrase (VP).
   A verb phrase may have a single word, two words or a group of words.

Examples:

1. The Sun rises in the east.
   1
2. He has ordered tea for us.
   1 2
3. She has been teaching English since 1965.
   1 2 3
4. She will have been typing when I meet her at 10 a.m.
   1 2 3 4

Verbs — Main Verb
Auxiliary Verb

Main Verb – Basic forms — Simple Present Tense - Sing
Simple Past Tense - Sang
Present Participle - Singing
Past Participle - Sung

Auxiliary Verbs and their Forms

	Primary Auxiliaries	Modal Auxiliaries
BE	be, is, are, am, was, were, being, been	can, might, may, must, will, needn't
HAVE	have, has, had, having	shall, daren't, should, ought
DO	do, does, did, doing, done	would, used to, could

Note: The Verb Phrase in any sentence is constituted by using only the Main Verb form or one of the basic forms of the Main Verb and the primary auxiliaries or modal auxiliaries or both the primary auxiliaries and the modal auxiliaries.

Examples:

1. I teach English.
   M.V
2. I am teaching English now.
   P.A M.V
3. I will have been teaching English.
   M.A P.A P.A M.V

Special Note: Every verb phrase has a particular structure acceptable according to standard written English.

The verb phrases in the following sentences are wrong.

1. I will the work
2. I will did the work
3. I going there
4. I can done the work

Verbs: Transitive and Intransitive

The verbs which do not require or do not have objects in the sentence in which they are used are called intransitive verbs (i.v.).

In the following sentences complements are used, not objects.

Examples:

This book costs ten rupees
i.v comp.

The play lasted an hour
i.v comp.

The birds fly in the sky
i.v adv. ph.

The verbs which require or have objects are called transitive verbs

Example:
The professor gave them some assignments
v object

Note: There are a number of verbs which can be used with or without objects, that is, 'transitively' or 'intransitively.' Here are some examples of verbs used both intransitively and transitively.

Intransitive use	Transitive use
My father is reading.	He is reading the newspaper.
The play ended at five.	Rain ended the play.

Voice

There are two voices:
1. Active voice
2. Passive voice

Notice the change in the following sentences.

1. She has done the work. (A.V)
   The work has been done by her. (P.V)
2. These engineers can draw good designs. (A.V)
   Good designs can be drawn by these engineers. (P.V)
3. The Principal read the report. (A.V)
   The report was read by the Principal. (P.V)

Note:

1. Depending on the emphasis we want to lay, we use a particular voice in a particular context.
   Examples:
   Some people dug a well to provide water to the village (A.V)
   A well was dug to provide water to the village (P.V)
   Here we would prefer to use the second rather than the first sentence, because what we want to refer to is not the action of the Panchayat but the result of the action, namely, a well, being provided in order to supply water to the village. Who dug the well is not the main idea in our minds.
2. Intransitive verbs do not have passive voice. i.e., we cannot transform a sentence from Active to Passive if the verb is intransitive.

For example the sentence 'Birds fly in the sky' cannot be transformed into passive voice.

A more detailed note on Voice is included in a later chapter.

Verb—Form and Use of the Tenses

There are 12 tense-structures in English.

Present

Simple Present : I teach English.
Present Continuous : I am teaching English.
Present Perfect : I have taught English
Present Perfect Continuous : I have been teaching English.

Past

Simple Past : I taught English.
Past Continuous : I was teaching English
Past Perfect : I had taught English.
Past Perfect Continuous : I had been teaching English

Future

Simple Future : I will teach English.
Future Continuous : I will be teaching English.
Future Perfect : I will have taught English.
Future Perfect Continuous : I will have been teaching English.

Uses of Tenses—Some Important Rules

Simple Present Tense is used to express habitual actions, permanent or verifiable truths or facts (scientific or universal).

1. He goes to church every Sunday.
2. Water boils at 100°c.
3. Calcutta stands on the banks of the Hoogli.
4. The sun rises in the east.

The Simple Present tense is used to express a planned future action, or a series of such planned actions.

1. We leave Hyderabad at 8:30 p.m. next Tuesday and arrive in New Delhi at 11 a.m. on Thursday.
   The Present Continuous tense is used to describe an action that is in progress at the time of speaking.
2. The children are playing in the garden now.
   The Present Continuous tense is also used to describe an action that is in progress and will be continued, but not necessarily going on at the moment of speaking.
3. The college authorities are building a new hostel.

The Present Continuous tense can also express an action that has been arranged to take place in the near future and one's immediate plans.

1. I am meeting the CM tomorrow morning.
2. We are going to a movie this evening.

There are a number of verbs which are not normally used in the present continuous tense. These are:

1. verbs of perception :
   see, hear, smell, notice, etc.
2. verbs used to express feelings or states of mind:
   want, desire, wish, refuse, forgive, care, hate, like, admire, love, etc.
3. verbs involving the process of thinking:
   feel, know, mean, remember, forget, recall, etc.
4. verbs denoting possession:
   have, own, belong, possess, etc.

5. verbs such as:
   contain, consist, keep, seem, cost

When some of the verbs listed above are used in the present continuous tense, their meanings change.

1. I see several mistakes in this book.
   I am seeing the principal at 3 p.m.
   (will be meeting)
2. We hear several rumours about the Minister
   The judge is hearing the case tomorrow. (conducting the trial).
3. I have a house at Malakpet
   I am having my breakfast. (act of eating)

The following sentences are wrong

1. I am loving that girl. (love ✓)
2. They are understanding the lesson. (understand ✓)
3. I am slowly understanding you. (wrong)
   I have begun to understand you. (right)
4. I am having a telephone at my residence. (have ✓)

The Present Perfect Tense is used to indicate an action that has just been completed.

Examples:
I have finished my work. (just now)

The present perfect tense is also used to represent a past action continuing to the present.

Examples:
We have lived in Hyderabad for ten years. (i.e., we are still living in Hyderabad.)

Do not use present perfect tense (has or have) when time is specified (e.g.: last year, 1994, etc.) in the sentence.

Examples:
I have seen this film last year (incorrect)

I saw this film last year (correct)

Also note that for and since are commonly used with the present perfect tense. 'For' shows length of time and 'since' shows some point of time in the past as being the starting point of the action or event.

Examples:

1. My friend has lived in Hyderabad for twenty years. (He still lives here)
2. These monuments have been here since 1650 A.D. (still they are here)
   The following are some time-expressions that go with the simple past and some that go with the present perfect.

Simple Past Tense	Present Perfect Tense
yesterday	so far
a week ago	since
recently	lately

Note the difference in meaning between the following two sentences:

Dr. Mohan lived in Delhi for fifteen years. (in the past)

Dr. Mohan has lived in Delhi for fifteen years.
(still Dr. Mohan lives in Delhi)

Simple Past Tense indicates an action completed sometime in the past.

Example: I lived in Hyderabad for ten years (in the past)

Past Progressive Tense:

1. The past progressive (continuous) tense expresses an action that was in progress at a point of time in the past, having begun before that point and probably continuing after it.
   Examples:
   (a) I was reading the newspaper at 7 a.m. this morning.
   (b) I was having my breakfast at 9 a.m.
2. The past progressive tense expresses an action continuing over a period of time in the past.
   Example: The students in the hostel were listening to the cricket commentary the whole of yesterday.
3. The past continuous tense describes two or more actions going on at the same time; often the conjunction 'while' is used to connect the clauses.
   Example: While some boys were reading in the library the others were playing.

Past Perfect Tense: This tense is used when we wish to emphasise the sequence of the two actions in the past, and when the earlier action has some relation to the later action or situation.

Examples:

1. On reaching the school I found that I had forgotten to bring my English textbook.
2. When we reached the theatre, the play had begun already.
3. I borrowed some money from a friend of mine, since I had lost my purse.

The Future Tense: There are several ways of expressing future time in English.

One of the most common ways to express this is to use shall or will with the bare forms of the verb : shall come, will go, etc. But there are other ways of expressing the future :

Example: Our cricket team leaves for Bombay this evening. They play two matches in Bombay. They play one at Poona. They return next Monday.

In the above sentences the present simple tense is used to express a series of intended or planned actions in future.

1. The P.M is visiting the city tomorrow.
2. The college team is playing a match with the city team next Sunday.

The Present Continuous tense is used in these sentences to express a future event. Usually the time is mentioned (tomorrow, next sunday, etc) and it is in the near future. Note that the verbs go and come are not usually used with going to. We don't say, for instance, He is going to go to Bombay tomorrow; we say, He is going to Bombay tomorrow.

Special expressions to indicate future:

1. The train is about to leave.
2. The President is about to speak.

This construction be about to + verb expresses events which are likely to happen in a very short while.

1. I am to be at a meeting at 5 o'clock.
2. There is to be an enquiry into the railway accident.

In these sentences the form be + to + the base form of the verb is used to express a duty or necessity or planned course of action, in the future.

Rules Regarding the Use of Verbs

Subject–Verb Agreement

General rule: The verb must agree with its subject in number and person. In other words, the verb must be of the same number and person as the subject.

Note: In the correction of sentences section of many competitive examinations the S - V concord is usually tested.

Rule 1: When two subjects are joined by 'and', the verb is plural.
My friend and his father are in India.

Rule 2: When two singular nouns joined by and refer to the same person or thing, the verb is singular.

The secretary and treasurer has been arrested.

The District Magistrate and Collector is on leave today.

Notes:

1. Article 'The' is used only once when the two nouns refer to the same person or thing.
2. If the two nouns refer to different persons or things, article 'The' is used before each noun. In such cases, the verb will be in the plural form.

Example: The secretary and the president have been given warm welcome.

Rule 3: If two different singular nouns express one idea, the verb should be in the singular form.

Bread and milk is good for breakfast.

Rice and curry is my favourite dish.

This is the long and the short of the matter.

Rule 4:
When two singular subjects are practically synonymous the verb should be in the singular form.

The law and order situation in the state is under control.

His power and influence is on the decline.

Power and position has no charm for my friend.

Peace and prosperity is the need of the day.

Rule 5: If two singular subjects (combined by and) are preceded by each or every, the verb should be in the singular.

1. Every boy and girl was present in the class yesterday.
2. Every man and every woman has the right to express his or her view.

Rule 6: When the subjects joined by 'either - or' or 'neither - nor' are of different persons, the verb will agree in person and number with the noun nearest to it. Also, the plural subject must be placed nearest to the verb. (This is very important)

Either Radha or Rajani has done this mischief.

Neither Mohini nor Ragini is beautiful.

* Either the chief minister or the cabinet ministers are responsible for this problem.

Neither you nor he is to take up this task.

Either you or I am responsible for this mistake.

Rule 7: If connectives like with, together with, as well as, accompanied by, etc. are used to combine two subjects the verb agrees with the subject mentioned first.

The President of India together with his personal secretaries is invited to this function.

The actress, along with her manager and some friends, is attending the function.

Mr. Michael, accompanied by his wife and children is arriving tonight by train.

Note: If the conjunction and is used instead, the verb would then be plural.

Compare (i) Saritha and Rajitha are our professor's daughters.

Rule 8: When 'not only ..... but also' is used to combine two subjects, the verb agrees with the subject close to it.

Not only Harish, but also his brothers were arrested.

Rule 9: None / No

None can take either a singular or a plural verb depending on the noun which follows it;

Structure: none + of the + non-count noun + singular verb

None of the counterfeit money has been found.

Structure: none + of the + plural count noun + plural verb

None of the students have finished the exam yet.

No can take either a singular or plural verb depending on the noun which follows it.

Structure: No + singular noun + singular verb.

Example: No example is relevant to this case.

Structure: No + plural noun + plural verb

Example: No examples are relevant to this case.

Rule 10: Many words indicating a number of people or animals are singular. The following nouns are usually singular. In some cases they are plural if the sentence indicates that the individual members are acting separately.

congress	family	group
committee	class	organisation
team	army	club
crowd	government	jury
minority	public	

Examples of collective nouns used with a singular verb:
The committee has met, and it has accepted the proposal.
The family was happy at the news.
The crowd was wild with excitement.
The Congress has initiated a new plan to combat inflation.
Our team is certain to win the match.
Some collective nouns are used in plural:

Examples:
The committee have arrived by different trains.
The family were fighting over inheritance.
The family living next door often quarrel among themselves

Rule 11: Majority can be singular or plural. If it is alone it is usually singular, if it is followed by a plural noun, it is usually plural.

The majority believes that the country can progress.
V.

The majority of the lecturers believe that the student has not copied in the examination.
V.

Rule 12: A number of/the number of observe the two structures:

1. A number of + plural noun + plural verb.
2. The number of + plural noun + singular verb.

Examples:
1. A number of students are going to the class picnic.
2. The number of days in a week is seven.
3. The number of residents who have been living in this colony is quite small.
4. A number of the applicants have already been interviewed.

Rule 13: Collective nouns indicating time, money, and measurements used as a whole are singular and take a singular verb.
Twenty-five rupees is not such a big amount for him.
Two miles is too much for this man to run.

Rule 14: When a lot of, a great deal of, plenty of, most of, and some of refer to number, a plural verb is used.

Examples:
A lot of people were present in the gallery.
Some of the students were absent.

Note: If these expressions refer to an amount or an uncountable noun the verb is in the singular number.

Examples:
A lot of work has to be completed before we go.
A great deal of work has been finished.

Rule 15: When the percentage or a part of something is mentioned with plural meaning the plural verb is used.

Examples:
30% of Indian women are literate.

Three-quarters of the food has been eaten.

(Here the reference is to the food as a whole.)

Rule 16: 'Barracks', 'headquarters', 'whereabouts' 'alms', etc., take a singular verb, as well as the plural verb.

The headquarters of the UNO is/are New York.

Rule 17: In sports, while referring to the players, the name of the country is followed by plural verb.

England have won the World Cup.
V

Rule 18: When the word 'enemy' is used in the sense of 'armed forces' of a nation with which one's country is at war, we have to use the plural verb:

The enemy were forced to retreat.

Sample Exercise–8

Directions for questions 1 to 4: Fill in the blanks with appropriate verbs.

1. I am sure that you will soon _____ to this environment.
(A) get used (B) be used
(C) be habituated (D) be linked

2. I _____ him from taking unnecessary loans.
(A) distinguished (B) dissuaded
(C) diverted (D) dissented

3. He _____ his studies.
(A) has left (B) is learning
(C) deserts (D) has discontinued

4. This big book _____ 30 chapters.
(A) is comprising of (B) consists
(C) comprises of (D) comprises

Non-Finites

Verbs can be classified as

1. Finites and
2. Non-finites.

A Finite Verb is a verb that denotes tense.

Example:
He goes to school.

'goes' represents the simple present tense.

Similarly 'went', 'gone' are the other forms of the verb. These have been discussed in a later chapter.

Non-finites are verb forms which do not give complete (finite) meaning. They have none of the tenses, have no number and take no modals (auxiliary verbs). There are four types of non-finites :

1. Infinitive
2. Gerund
3. Present participle
4. Past participle

Look at these sentences:

1. She wants to buy a new house. (Infinitive)
2. Buying a new house needs money. (Gerund)
3. I found him looking for a new house. (Present participle)
4. Wounded in the war the soldier was brought to the medical camp. (Past participle)

The Infinitives

1. Look at the forms of the infinitive in the following sentences:
   (a) Rajitha wanted to learn photography.
   (b) She is reported to be learning it.
   (c) She is reported to have learnt painting already.
   (d) She is known to have been learning music for the last five years.

   In these sentences you see four forms of the infinitive. The following are the structures of these four infinitives.

Sentence (a)	to + verb
Sentence (b)	to be + verb -ing (present participle)
Sentence (c)	to have + past participle
Sentence (d)	to have + past participle of be + verb -ing

   Some of these forms have the corresponding passive voice.
   to learn - to be learnt:
   There is a great deal to be learnt in photography.
   To have asked - to have been asked:
   He was happy to have been asked to deliver the convocation address.

2. The infinitives shown above have the word to before them.
   But look at these sentences:
   (a) The mother made the child drink the medicine.
   (b) Then she let the child go out and play.

'To' is very commonly used with the infinite, but it is not a necessary sign of the infinitive. The following verbs take the infinitive without 'to'.

1. The principal verbs: can, must, let, make, bid, dare, please, need, etc.
2. Verbs denoting some kind of perception: see, hear, feel, watch, know, think, etc.

Examples:

1. Make her do the work.
2. You need not go there.
3. Please go there.
4. Let him stand there.

The infinitive without 'to' is used after had better, had rather, had sooner, would rather, sooner than, rather than.

Examples:

1. You had better meet your uncle.
2. He would rather fight than give up.
3. I had rather write than play.
4. She had sooner run than walk.

Sample Exercise–9

Rewrite the following sentences using to, wherever necessary, before the verbs given in the brackets.

1. Dr. Rao asked his daughter _____ home before 9 o' clock. (come)
2. If you have planned _____ the airport before five, you had better _____ at once. (reach, start)
3. He wouldn't let anyone _____ his bicycle. (use)
4. We wanted _____ the house where Mahatma Gandhi was born. (see)
5. He would rather _____ than _____ a lie. (die, tell)

The Gerund

The gerund and the present participle have the same form, verb + ing : swimming, walking, cutting, etc. (the gerund is also called the verbal noun)

Uses of the gerund:

1. As the subject of a verb:
   Swimming is a good exercise.
   Smoking is injurious to health.
   Speaking is easier than writing.
   Looking after children needs a lot of patience.

2. As the object of a verb:
   He likes reading detective novels.
   Would you mind returning these books to him?
   I hate waiting at bus stop.
3. As the object of a preposition:
   The minister inaugurated the school by lighting a lamp.
   He was fined for being drunk.
   He is thinking of resigning his job.
   They resumed their journey after resting for an hour.
4. As the subject complement:
   Talking to him is wasting time.
   Seeing is believing.

Note: When a Noun or Pronoun is placed before a Gerund, it must be in the possessive case.

Example:
- I was pleased at Robert coming here (Incorrect)
- I was pleased at Robert's coming here (Correct)
- They insisted upon me writing the essay. (Incorrect)
- They insisted upon my writing the essay. (Correct)

But the possessive case cannot be used with the Gerund in the following cases.

1. When the Noun denotes a lifeless thing.

Examples:
There is no danger of the wall's falling down. (Incorrect)
There is no danger of the wall falling down. (Correct)

Note:
1. The following verbs can take either an infinitive or a gerund as an object.

Continue	regret	try propose
forget	neglect	remember learn

2. The following verbs can be followed by the infinitive (to + v) as the direct object.

agree	forbid	learn	offer
care	forget	plan	decide
hope	pretend	deserve	intend
mean	refuse		fail

3. The following verbs can be followed by the gerund (V + ing) as the direct object.

admit	deny	postpone	appreciate
enjoy	practise	avoid	finish
stop	cannot help	keep	suggest
consider			

4. The following verb phrases (verb + preposition) can be followed by the gerund (v + ing). Remember that gerunds, not infinitives, follow prepositions in general.

decide on	think about	keep on	think of
plan on	put off	look forward to	

The Participle

Look at the words underlined in these sentences.

1. The crying child gets milk.
2. We found a man running for the bus.
3. Inaugurating the seminar, the Education Minister spoke about the prohibition policy.

The words underlined are participles. There are two types of participles.

1. The Present Participle
2. The Past Participle

The form of the Present Participle:

Verb + ing	Example:	cry + ing

The form of the Past Participle:

V + ed	Example:	wound + ed
	(OR)	
V + en	Example:	drunk + en

Wrong use of participles:

1. Whenever a participle is used, the Noun or Pronoun to which it refers, must also be mentioned.
   Look at the following sentence:
   While taking a bath, the shampoo bottle fell.
   What is wrong with this sentence?

The sentence, as it stands means that the 'shampoo bottle' was taking a bath. In other words there is no noun or pronoun to which the participle 'taking' refers. Hence, the sentence should be re-written, as 'while I was taking bath, the shampoo bottle fell.'

Given below are a few more examples.

1. Being a rainy day, I did not go out. (Incorrect)
   It being a rainy a day, I did not go out. (Correct)
2. Crossing the road, a truck hit her. (Incorrect)
   While she was crossing the road, a truck hit her. (Correct)

Note:

1. Sometimes the spellings of the past participles are not different in the case of some verbs.
   Example: come, hit, etc.
2. Sometimes, the present participle and the past participle can function like adjectives.

Examples:
1. The wounded soldier was carried to the medical camp.
   past. part.
2. A rolling stone gathers no moss.
   pre. part.

Correct Use of Prepositions

A preposition is a word placed before a noun or a pronoun or a noun equivalent and shows some relationship between that and some other word in the sentence.

1. Mohan gave a lecture on patriotism.
   prep N
2. The prize is given to her
   prep. pr. noun
3. His objection is to what all you say.
   prep. noun eq. (clause)

The words underlined are called Prepositions because they normally take position before (pre) a noun. The preposition is said to 'govern' the noun that follows it. The noun is said to be the 'object' of the preposition.

Besides single-word prepositions, there are also phrases which do the work of prepositions and are called 'Phrase Prepositions.'

Types of Prepositions:

1. Single-word Prepositions
   in, on, after, at, with, under, above, etc. (These are Simple Prepositions)
2. Phrase Prepositions (Complex Prepositions)

Some types of Complex Preposition structures with examples are given below.

1. Adverb + Preposition
   along with, apart from, as for, as to, away from, onto, out of, together with, upto, such as, instead of
2. Verb/adjective/conjunction, etc. + Preposition
   except for, owing to, due to, but for, because of.
3. Preposition + noun + preposition
   by means of, on account of, in comparison with, in accordance with, in view of, in spite of,

Note: On certain occasions it is both necessary and correct to end a sentence with a preposition. English is a flexible language and defies pedantic rules of grammar.

Now, look at the following sentences:

This is the boy I gave the book to.

This is the house that I was born in.

In these sentences the preposition is used at the end of a sentence.

Object of the Preposition

The noun or noun-equivalent (pronoun, adverb, gerund, infinitive, adverbial clause, or any clause that can be used as the object of the preposition) before which the preposition is placed is called its object.

Examples:

The glass is on the table.	(noun)
I depend on him.	(pronoun)
Go away from here.	(adverb)
He is fond of playing.	(gerund)
She was well till a few days ago.	(adverbial phrase)
I shall see it for what it is worth.	(adverbial clause)

Rules Pertaining to the Use of Prepositions

Rule 1: A preposition is placed at the end of a sentence in the following ways:

1. When the relative pronoun is 'that':
   Example: Here is the pen that you are looking for.
2. If a preposition governs a relative pronoun:
   Example: This is the student whom I spoke about.
3. When the relative pronoun is understood:
   Example: This is the person you spoke to.
4. If a preposition governs an interrogative pronoun or an interrogative adverb:
   Example: What are you looking at?
5. When the preposition is used with the infinitive placed at the end of the sentence:
   Example: Do you have a chair to sit on?
6. When the object governed by the preposition is placed first:
   Examples: This I insist on.
   He is known all the world over.

Rule 2: A preposition can also be used at the beginning of an interrogative sentence.

Examples:

In which city do you live?

To whom are you referring?

Rule 3: There are many words which can be used as prepositions or as adverbs. The most important of these are about, above, across, along, before, below, behind, besides, by, down, in, near, off, over, past, round, through, under, up, etc.

Examples:

Preposition	Adverb
He was here before ten.	He has done this before.
He was behind us.	She is long way behind.
The shop is just round the corner.	Come round and see me this afternoon.

Rule 4: There are some words with prepositions which require gerunds after them.

Examples:

refrain from hurting	abstain from drinking
prevent from working	aid at doing
persist in disobeying	addicted to gambling
succeed in doing	bent upon doing
disqualified from doing	averse to playing
knack of getting	expert in inventing
prohibit from entering	desist from talking
tired of writing	capable of teaching
pretext for delaying	desirous of going
fond of playing	insist on going

Rule 5: The verbs, which are placed immediately after prepositions are usually in the gerund form.

Examples:
Manohar insisted on buying a television.

She left the hotel without paying the bill.

Rule 6: Some words with prepositions can take the gerund as well as the infinitive.

Examples:
He is afraid of going out alone at night.

He is afraid to go out alone at night.

Rule 7: Some intransitive verbs become transitive by placing prepositions after them.

Examples: laugh at, listen to, depend on / upon, prevail on / upon etc.

Rule 8: In and Within

1. In refers to the end of a period of time usually in the future, e.g. He will return in a month. (at the end of one month)
2. Within means before the end of a period of time (at any time before the specified period)
   He will return within a month. (he may come after two weeks also)

Rule 9: In and Into

1. 'In' indicates rest or motion inside anything.

   Examples:
   She is in the garden. (rest, inside)

   She is walking in the garden (motion inside)

2. 'Into' means motion towards the inside of anything.

   Examples:
   I walked into the garden.

   Thieves broke into my friend's house yesterday.

Rule 10:

1. On is used:
   (i) in speaking of things at rest.
   He sat on a big stone.
   (ii) before the names of days and dates.
   On Friday, on the 2nd of August, etc.
   (iii) to denote support for and concern about somebody or something.
   He lives on his maternal uncle's wealth.
   I wrote books on politics.
2. Upon is used:
   in speaking of things in motion.
   The tiger sprang upon the goat.

Rule 11: BESIDE and BESIDES

Beside means 'by the side of'

Example:
My house is beside the Kali temple.

Besides means in addition to or moreover

Examples:
- There are four professors in the department besides the head of the department.
- Besides English we are taught French.

Rule 12: BETWEEN and AMONG

Between is used for two persons or things

Distribute these sweets between the two children.

There is a good understanding between him and her.

Among is used for more than two persons or things.

The boys were fighting among themselves in the absence of their teacher.

Rule 13: BY is used to mean:

1. according to; from the evidence of
   By my watch, it is 10.30.
2. to denote the doer of an action in the passive voice.
   The thief was beaten by the policeman.

Rule 14: Prepositions from, since, for with reference to time.

From, Since indicate a point of time.

1. I have not seen her since Monday.
   I have not seen her from August.
   For indicates a length or period of time.
2. I have not seen her for six months.

SAMPLE EXERCISE–10

Directions for questions 1 to 5: Fill in the blanks with appropriate prepositions.

1. The aggrieved party demands that the CBI should investigate _____ the case.
(A) into (B) in
(C) at (D) no preposition required

2. He is persistently prevailing _____ his wife to resign her job.
(A) to (B) upon
(C) at (D) over

3. Though she searched _____ the ring everywhere, it was in vain.
(A) for (B) about
(C) out (D) no preposition required

4. The speaker has not yet entered _____ the assembly hall.
(A) into (B) in
(C) to (D) no preposition required

5. Which pen would you like to write _____?
   (A) for
   (B) with
   (C) in
   (D) no preposition required

Correct Use of Conjunctions

A Conjunction is a word which connects words, phrases, clauses or sentences. It also brings about relationship between the elements which are thus joined. There are two types of conjunctions:

1. Co-ordinate conjunctions
2. Subordinate conjunctions

A Co-ordinate Conjunction joins two clauses or sentences of equal rank. Also, it joins two words of equal grammatical rank.

And, but, for, nor, or, but, otherwise, else, also, either - or, neither - nor etc., are the chief co-ordinate conjunctions.

Examples:
He went to the hospital and met the doctor.
Dr. Rao and Dr. Reddy are best friends.
The Co-ordinate Conjunctions are of four kinds:

1. **Cumulative Conjunction:** A conjunction which adds one statement or fact to another is a cumulative conjunction.

   Example:
   The professor as well as the lecturer has accepted to conduct the examination next week.

   The following are the cumulative conjunctions:
   Not only ----- but also,
   Both ----- and,
   as well as
   too, also, moreover etc.

2. **Alternative Conjunctions:** A conjunction of this kind expresses a choice between two alternatives.
   Either ---- or, neither --- nor, otherwise, else, etc., are alternative conjunctions.

   Examples:
   She is good neither at games nor at studies.
   Work hard, otherwise you will fail.

3. **Adversative Conjunctions:** An adversative conjunction expresses a contrast between two facts or statements.
   Only, however, but, still, yet, whereas, nevertheless, etc., are adversative conjunctions.

   Examples:
   She was angry, but she kept quiet.
   She hates me, yet I love her.

4. **Illative Conjunctions:** Such a conjunction shows that a statement or fact is proved or inferred from another.
   Therefore, hence, so, consequently, for, etc., are illative conjunctions.
   Example:
   He is honest and amiable, hence he is revered.

5. The Subordinate Conjunctions are the conjunctions that connect a clause to another on which it depends for its full meaning.
   The adverbial clauses are usually connected to the main clauses by means of the subordinate conjunctions:

The Chief Subordinating Conjunctions are after, because, if, another, though, till, etc.

The following are some sentences in which important subordinate conjunctions are used.

1. The patient had died before the doctor arrived.
2. We eat so that we may live.
3. He behaved in such a manner that all disliked him.

The following compound expressions also can be used as conjunctions.

Examples: In order that, on condition that, even if, so that, provided that, as though, as well as, as if, etc.

Note: There are some words which are used both as conjunctions and as prepositions.

Conjunctions	Prepositions
We went home after he came to the office	We went home after sunset.
I went to bed early, for I was tired.	I shall do it for him.

The following conjunctions are used in pairs and hence are called correlative conjunctions.

Either …….. or
Neither …… nor
Both …… and
Whether …….. or
Not only …….. but also.

Note: When conjunctions are used as correlatives, each of the correlated words should be placed immediately before the words to be connected.
For example

- He not only visited Bombay but also Pune. (wrong)
- He visited not only Bombay but also Pune. (correct)
- He is neither good at Mathematics nor at Science. (incorrect)
- He is good at neither Mathematics nor Sceince. (correct)

Sample Exercise–11

Directions for questions 1 to 10: Fill in the blanks with appropriate conjunctions.

1. I don't know how to rate him; he is neither innocent _____ intelligent.
(A) And (B) Nor
(C) Or (D) But

2. I am ready to help him, _____ he may not accept it.
(A) Therefore (B) However
(C) Even (D) Despite

3. In functions, better be first as far as food is concerned _____ you might starve.
(A) Lest (B) Otherwise
(C) So that (D) Hence

4. He says that he is speaking the truth but I don't think _____.
(A) Still (B) Although
(C) So (D) No conjunction

5. _____his ill health, he has successfully completed the given task.
(A) Despite (B) Besides
(C) Because of (D) Owing to

6. Beyond doubt he is rich, _____ he could afford to study abroad.
(A) As well as (B) Though
(C) Still (D) Hence

7. _____ it was very hot, we could not walk further.
(A) Because (B) As such
(C) Although (D) Though

8. Many people waste power and water _____ many others are completely deprived of it.
(A) When (B) Because
(C) While (D) For

9. The weather is very hot, _____ it is middle of May.
(A) While (B) Since
(C) Supposing (D) Hence

10. He is very poor _____ he managed to get a good education.
(A) Where (B) Nevertheless
(C) Despite (D) Inspite

Notes on Phrasal Verbs

Phrasal verbs: Certain verbs when followed by certain prepositions or Adverbs, acquire a new significance.

They are said to be phrasal verbs.

Given below are a few phrasal verbs with meanings and usage. Study them carefully.

1. Back up: to support; to sustain.
   You need to back up your statement with correct data.
2. Bear down: to overthrow; to overcome.
   When people could not endure the atrocities of the government in power, they bore it down.
3. Bear with: to tolerate.
   I cannot bear with his peevish nature any further.
4. Break out: to escape from a place or a situation.
   I think Archana needs to break out of her daily routine and do something exciting.
5. Break in / into: to enter a building by force.
   Burglars broke into my neighbour's house while they were not at home.
6. Call for: to demand.
   The Reading Comprehension section calls for good reading skills from the students.
7. Call up: to recall.
   The beautiful scenery around called up memories of my childhood.
8. Come by: to obtain.
   How did you come by that beautifully embroidered skirt?
9. Come upon: to find by accident.
   I came upon some very interesting facts about our planet in this Encyclopaedia.
10. Go about: to start working on something, to tackle
    Can you just explain to me how to go about solving this problem?
11. Go by: to be guided by.
    This is a good example to go by while solving the problems.
12. Hang back: to hesitate to proceed.
    Don't hang back, you can do it as well as anyone else can do.
13. Hang on: to depend on.
    Your success hangs on how much effort you are going to put in.
14. Fall back: to recede, or to retreat.
    Having seen the armed guards, the civilians fell back.
15. Fall off: to withdraw; to drop off.
    A true friend does not fall off in adversity.
16. Live for: to devote one's life to.
    All through his life he lived for the amelioration of the poor and the destitute.
17. Live on: to depend on.
    Most people of this region live on rice and fish.
18. Lay about: to attack some one violently.
    A group of ruffians laid about him with sticks while he was returning home.
19. Lay out: to plan how something should look and arrange it in this way.
    The ancient Chinese laid out beautiful gardens with lawns, and flower beds.
20. Make away with: to squander.
    After his father's demise, he made away with the entire property.

Parallelism

1. Items in a series must be parallel : that is, they must have the same grammatical form.
   (a) NOUNS He likes music, art, and history.
   N N N
   (b) GERUNDS He likes swimming, dancing, and riding.
   V+ING V+ING V+ING
   (c) ADJECTIVES He is tall, dark, and handsome.
   ADJ ADJ ADJ
   (d) ADJECTIVES They wanted to paint the living room, to lay a new carpet, and to buy a new sofa.
   INFIN. INFIN. INFIN.
   (e) PAST TENSE The Romans conquered, colonized, and governed much of the world.
   PAST PAST PAST
   (f) PAST PERFECT TENSE

He had finished the game, had taken a shower, and had eaten lunch by the time I went to his house.

Note: The auxiliary 'had' may be omitted in the second and the third verb phrases.

2. The structure joined by and, but, as, or then, or although must have the same grammatical form.
   (a) AND
   He enjoyed the music of Spain and the sculpture of France.
   N PHRASE N PHRASE
   (b) BUT
   That verb form is not active, but passive
   ADJ ADJ
   (c) AS
   Taking the bus can be as costly as taking a plane.
   V+ING V+ING
   (d) OR
   He wanted to borrow a car or to hire one while his car was being repaired.
   INFIN. INFIN.
   (e) THAN
   Eating in a restaurant is more fun than cooking at home.
   V+ING V+ING
   (f) ALTHOUGH
   Although he liked to eat good food, he did not like to pay a high price for it.
   INFIN. INFIN. INFIN.

Question Tags

Question Tags are a feature of most languages, but English differs from many of them. Consequently the following error is sometimes found.

Examples:
You are going to the cinema this evening, isn't it? (Incorrect)
You are going to the cinema this evening, aren't you? (Correct)

The following rules pertaining to the use of question tags should be borne in mind to avoid making errors:

1. Use the same auxiliary verb as in the main clause.
2. If the sentence has no auxiliary verb use do, did or does
   Examples:
   John sings very well, doesn't he?
   John does not sing very well, does he?
3. If the given sentence is positive, the tag is negative. If the given sentence is negative, the tag is positive.
   Examples:
   Positive Statement Negative Tag
   Rajini is a music teacher, isn't she?
   Negative Statement Positive Tag
   Rajini is not a music teacher, is she?
4. Don't change the tense.
   The tense of the verb in the tag should be the same as the one in the statement.
   Examples:
   Mahesh did not accept the job, did he ? (correct)
   Mahesh did not accept the job, doesn't he? (incorrect).
5. Both the main sentence and the tag should have the same subject. The tag must contain the subject form of the pronoun.
   Example: You are teaching them grammar, aren't you?
6. If the main sentence consists of the forms like it is, there are and there is; the question tag also takes there or it.
   Example: There are only twenty boys in this class, aren't there?

Observe the following: (All of the following are correct)

1. Dr. Rao is a professor of English, isn't he?
2. My cousin cannot run fast, can he?
3. She mustn't see the film, must she?
4. We used to live in Nigeria, didn't we?
5. You couldn't swim a mile, could you?
6. Mr. Brown will be our new principal, won't he?
7. I am not a good player, am I?
8. I am a good player, aren't I?

Active & Passive Voice

In English, the active voice is more common than the passive voice, although the passive voice is acceptable and even preferred at times. It is the context that is to be taken into consideration while deciding upon the type of construction - active or passive. While both types of construction may be

grammatically correct, one of them could be more elegant. The following are some of the points of observation.

1. When the subject of the sentence is the doer of the action, denoted by the verb, then the sentence is said to be in Active Voice.*

   * **Example:** The hunter shot the tiger.
   s v o

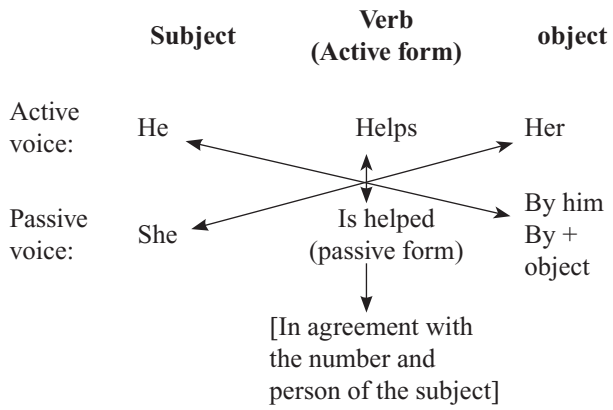
2. When the subject of the sentence is the receiver of the action denoted by the verb, then it is said to be in the passive voice.

   The tiger was shot by the hunter.
   s v o

   Note: The subject of the verb in the Active Voice is made the Object of the verb in the Passive Voice and is introduced with the preposition 'by'. Sometimes the word 'by' is omitted when the agent is not required to be mentioned.

   Example: He was elected the leader of the group.

This can be shown graphically:



3. The passive voice is preferred when the doer of the action is unknown or unimportant.

   The cure for cancer will probably be discovered by some unknown scientist in a laboratory.

   ('the cure for cancer' is the main idea in the sentence and not 'the unknown scientist')

   That church was built in the 16th century.

   (who built the church is unimportant and hence not mentioned)

4. When discussing history, the passive voice is often used.

   Example:

   The war was fought over gold.

5. The active voice is used when the subject is more important than the object.

   We watched the news.

   Archimedes discovered the laws of floating bodies.

6. Avoid using active and passive in the same sentence.

   The food was cooked and served.
   passive passive

   Susan cooked the dinner and washed the dishes.
   active active

7. Use one verb instead of two whenever possible.

   Saritha enjoys good food and music.

   (Saritha enjoys good food and music is also enjoyed by her.)

The following table will help you to note the changes from the Active Voice to the Passive Voice:

	Tense	Active	Passive
1.	*Simple Present Tense*	*He writes a letter*	*A letter is written by him*
2.	*Present continuous*	*He is writing a letter*	*A letter is being written by him.*
3.	*Present perfect tense*	*He has written a letter*	*A letter has been written by him.*
4.	*Present prefect continuous tense*	*He has been writing a letter*	*No passive form.*
5.	*Simple Past Tense*	*He wrote a letter*	*A letter was written by him.*
6.	*Past continuous tense*	*He was writing a letter*	*A letter was being written by him.*
7.	*Past perfect tense*	*He had written a letter.*	*A letter had been written by him.*
8.	*Past perfect continuous tense*	*He had been writing a letter.*	*No passive form.*
9.	*Simple future*	*He will write a letter.*	*A letter will be written by him.*
10.	*Future continuous tense*	*He will be writing a letter.*	*No passive form.*
11.	*Future Perfect tense*	*He will have written a letter.*	*A letter will have been written by him.*
12.	*Future Perfect Continuous tense*	*He will have been writing a letter.*	*No passive form.*

Note: Sentences with intransitive verbs cannot be changed to passive voice because such verbs do not take objects after them.

Example:
The sky is blue.
He goes to school. (intransitive verb)

Additional Examples

In the following examples we cannot say that the first sentence is absolutely wrong. In certain contexts it may even be preferred. However, generally speaking, the corrected sentence (the second sentence) is preferrable. We are calling the first sentence AWKWARD and the second sentence BETTER.

1. AWKWARD: Ice cream was eaten at the party by the children.
   BETTER: The children ate ice cream at the party.
2. AWKWARD: Some people painted pictures of animals on ancient cave walls.
   BETTER: Pictures of animals were painted on ancient cave walls.
3. AWKWARD: Henry likes swimming and golfing is also liked by him
   BETTER: Henry likes swimming and golfing.

Direct and Indirect Speech

Speech can be reported in two ways

1. The actual words of the speaker can be repeated. This is called DIRECT SPEECH
   Example: Meena said, 'I don't want to play now.'
2. We can report what the speaker said without quoting his exact words. This is called INDIRECT SPEECH.
   Example: Meena said that she didn't want to play then.

The verb that introduces the Reported Speech is called the Reporting Verb and the words which are put within inverted commas are called the Reported Speech.

Direct Speech is always enclosed within inverted commas and it always begins with a capital letter.

It is separated from the Reporting Verb (said) by a comma.

Rules for changing Direct Speech into Indirect Speech.

1. No inverted commas are used in Indirect Speech and no comma is used after the Reporting Verb.
2. The tense of the Reporting Verb is never changed.
3. The conjunction 'that' is used after the reporting verb in the case of statements.
4. When the Reporting Verb is in the Past Tense, the Present Tenses of the Direct Speech are changed into the corresponding Past Tenses.
   (a) Simple Present becomes Simple Past
   Examples:
   Raju said, 'I am happy.'
   Raju said that he was happy.
   (b) Present Continuous becomes Past Continuous
   Examples:
   Madhav said, 'My father is sleeping.' Madhav said that his father was sleeping.
   (c) Present Perfect becomes Past Perfect.
   Examples:
   Umesh said, 'I have failed in my duty.' Umesh said that he had failed in his duty.
   (d) 'Shall' of the Future Tense changes to 'should', 'will' changes to 'would' or 'should'.
   Example:
   He said to me, 'I will meet you tomorrow.'
   He told me that he would meet me the next day.

Exceptions

1. If the Reporting Verb is in the Present or in the Future Tense, the tense of the verb in the Reported Speech does not change.
   Examples:
   He says, 'I will never come back.'
   He says that he will never come back.
   My father will say, 'I cannot buy you a car.'
   My father will say that he cannot buy me a car.
2. If the reported speech expresses a universal truth or a habitual fact its tense remains unchanged.
   Examples:
   The teacher said, 'Honesty is the best policy.'
   The teacher said that honesty is the best policy.
   Mother said, 'Mahesh drinks milk before going to bed every night.'
   Mother said that Mahesh drinks milk before going to bed every night.
3. When the reported speech contains a time clause, and both the main verb and the verb in the time clause are in the simple past, the verbs remain unchanged.
   Examples:
   He said, 'The bus did not move till all the passengers were seated.'
   He said that the bus did not move till all the passengers were seated.
   Note: If the main verb is in the Simple Past Tense and the verb in the time clause is in the Past Continuous Tense it is usual to change the main verb to Past Perfect and leave the verb in the time clause unchanged.
   Examples:
   She said, 'Dhiren fell while he was crossing the road.'
   She said that Dhiren had fallen while he was crossing the road.

4. If the Reported Speech describes a state of affairs that still exists when the speech is reported, its tense remains unchanged.

   Examples:
   Ram said, 'My wife keeps complaining of aches and pains.'
   Ram said that his wife keeps complaining of aches and pains.

5. When the verb in the spoken sentence is in the Past Perfect Tense, there is no change in the tense of the verb in the Reported Speech.

   Examples:
   He said, 'I had waited there for half an hour before the bus arrived.'
   He said that he had waited there for half an hour before the bus arrived.

6. (a) The Simple Past in Direct Speech becomes Past Perfect in Indirect.

   Examples:
   The student said,' I wrote a letter yesterday.'
   The student said that he had written a letter the previous day.

   (b) Past Continuous Tense of Direct Speech is changed to Past Perfect Continuous.

   Examples:
   Seema said, 'We were watching TV the whole evening.'
   Seema said that they had been watching TV all evening.

7. The pronouns of the Direct speech are changed where necessary so that it is clear who said what to whom, about whom.

   Examples:
   He said, 'I don't understand you.'
   He said he didn't understand me.

   I said to him, 'I don't trust you.'
   I told him that I didn't trust him
   Note: 'Said' followed by 'to' changes to 'told'.

   They said, 'We will come again tomorrow.' They said that they would come again the next day.
   I said, 'I have done my duty.'
   I said that I had done my duty.
   The teacher said, 'You have done well.'
   The teacher said that I had done well
   He said, 'You should be regular to class.'
   He said that I should be regular to class.

8. Words expressing nearness in terms of time and place become words of distance.

Direct	Indirect
this/these	that/those
here	there
now	then
ago	before
thus	so
today	that day
tomorrow	the next day
yesterday	the day before the previous day
last night	the night before/the previous night
next week	the following week

   Examples:
   He said, 'These are the books I am looking for.'
   He said that those were the books he was looking for.

9. **Reporting different kinds of sentences**

   (a) **Statements**

   In Indirect statements the conjunction 'that' is placed after the reporting verb. However, it is often omitted. Indirect statements are introduced by such verbs as 'say', 'tell', 'reply' and 'inform'

   Examples:
   He said, 'I am feeling very weak.'
   He said that he was feeling very weak.
   She said to me, 'You must obey your parents.'
   She told me that I must obey my parents.
   Sheela said, 'No, I will not be able to come.'
   Sheela replied that she would not be able to come.
   Madhav said, 'The last train leaves at 10 o'clock.'
   Madhav informed me that the last train leaves at 10 o' clock.

   (b) **Questions**

   In reporting questions, the Indirect Speech is introduced by verbs such as 'asked', 'inquired', etc. In Indirect questions, the question form changes to the statement form. That is, the verb is placed after the subject

   Examples:
   He said, 'Where is the letter?'
   He asked where the letter was.

   If the question in Direct Speech begins with a question word (eg: who, what, which, when) this word serves as a link between the reporting verb and the reported question.

   'Why did she come?' Sekhar said.
   Sekhar asked why she had come.

   If the question has no question word and can be answered with a 'yes' or 'no', the conjunction 'whether' or 'if' is placed after the reported verb.

   Examples:
   'Do you know French?' she said.
   She asked me if I knew French.

   (c) **Commands and Requests**

   In reporting commands and requests the Indirect speech is introduced by a verb expressing

command or request and the Imperative mood is changed into the infinitive.

In indirect commands and requests, a verb such as 'tell', 'ask', 'order' 'command' and 'request', is followed by the person addressed and the 'to infinitive'

Examples:

He said to his servant, 'Bring me a glass of milk.'

He ordered his servant to bring him a glass of milk.

'Can you post this letter for me?' she said.

She requested me to post that letter for her.

(d) **Exclamations**

In reporting exclamations and wishes, the Indirect Speech is introduced by a verb which expresses an exclamation or a wish such as exclaimed, praised, blamed, applauded, called to witness, wished, desired, etc.

Examples:

'What a beautiful house you have!' my friend said.

My friend exclaimed that I had a very beautiful house.

Alice said, 'How clever you are!.'

Alice exclaimed that I was very clever.

'Bravo! You have done well,' the Principal said.

The Principal applauded the boy for doing well.

Here are some more reporting verbs for statements.

declare	inform	maintain
state	propose	announce

When the spoken sentence is by way of a reply to a question we can use 'answer', 'reply', 'respond' etc.

Some more verbs for reporting imperative sentences:

order	tell	pray
command	advise	suggest

Diction (Words with Similar pronunciation)

Diction is the choice of words. In English there are many words which are usually confused. Some words have similar meanings, but cannot be used interchangeably; that is, a choice must be made according to the grammatical situation. The following are some pairs of words which are often confused in their usage.

1.	Advice	: (Noun)	The teacher gave much advice to the students.
	Advise	: (Verb)	The doctor advised me to take nutritious food.
2.	Adapt	: (make suitable)	Novels are adapted for the stage.
	Adopt	: (take a child as one's own)	He adopted a son.
3.	Allusion	: (reference)	The allusion that I am stingy is a mistake.
	Illusion	: (false notion)	I do not have illusions about his ability.
4.	Amiable	: (pleasant and good-tempered)	Radha is quite an amiable person.
	Amicable	: (friendly)	The dispute is quite serious and therefore cannot be settled amicably.
5.	Apposite	: (Relevant)	His speech was apposite to the occasion.
	Opposite	: (contrary)	Heavy is the opposite of light.
6.	Beside	: (by the side of, close)	His mother sat beside him.
	Besides	: (in addition to)	Is anyone else coming besides you?
7.	Childish	: (silly)	I don't like his childish behaviour.
	Childlike	: (innocent)	Gandhiji always put on a childlike smile on his lips.
8.	Confidant	: (person with whom one trusts with secrets)	Nehruji was a confidant of Gandhiji in political matters.
	Confident	: (to be sure)	I am confident of success in the examination.
9.	Continual	: (very frequent)	He had continual arguments with his wife.
	Continuous	: (going on without a break)	There was continuous rain yesterday.
10.	Deny	: (ascertain that something is wrong)	The minister denied the allegation that he had taken bribe.
	Refuse	: (decline to take something that is offered or to do something that one is asked to do)	He refused the money given as bribe.

EXERCISES

Practice Problems

Grammar—Nouns/Articles/Pronouns (Basic)

Directions for questions 1 to 5: Fill in the blanks with the correct pair of words or phrases so that the sentence is grammatically correct and meaningful.

1. _____ man loves _____ money.
(A) The . . . . the
(B) A . . . .the
(C) The . . . .no article
(D) No article required . . . . no article required

2. _____ student of the class _____ to finish the work by tomorrow.
(A) All . . . .has
(B) Each . . . .have
(C) Every . . . .were
(D) Each . . . .has

3. _____ platinum is _____ any other metal.
(A) The . . . .costlier than
(B) No article required . . . . costlier than
(C) The . . . .more costlier than
(D) No article required . . . . more costlier than

4. The _____ of my professors inspired me to _____ hard and win the competition.
(A) Advise . . . .practise
(B) Advice . . . .practise
(C) Advise . . . .practice
(D) Advice. . . . practice

5. In India, _____ power to commute a death sentence is vested _____ the president.
(A) The. . . . for
(B) No article required . . . . by
(C) No article required . . . .on
(D) The. . . . in

Directions for questions 6 to 10: Fill in the blanks with suitable articles.

6. _____ British ruled India for four hundred years.
(A) No article (B) A
(C) An (D) The

7. The teacher ordered the students not to make _____ noise.
(A) A (B) An
(C) No article (D) The

8. Women in many Middle Eastern countries are deprived of _____ freedom even to this day.
(A) A (B) An
(C) The (D) No article

9. Besides being a scholar, he is also _____ eminent orator.
(A) The (B) No article
(C) An (D) A

10. _____ Rome is one of the ancient cities of the world.
(A) An
(B) No article
(C) A
(D) The

Grammar—Nouns/Articles/Pronouns (Advanced)

Directions for questions 11 to 14: Correct the following sentences where necessary.

11. The doctor has given me many informations about the disease.

12. The magistrate has issued a summon asking me to attend his court on Monday.

13. Our office has purchased new furnitures for our department.

14. Four thiefs broke into my cousin's house yesterday.

Directions for questions 15 to 20: Rewrite the following sentences after making the necessary corrections.

15. Good students like you and he should study regularly.

16. You, he and I are in the wrong.

17. Only you and him can do this work fast.

18. She helped everyone of those boys in doing their work.

19. Every teacher and every student should do their duty.

20. Neither of the boys have submitted their records.

Grammar—Verbs/Tenses/Auxiliary (Basic)

Directions for questions 21 to 25: Fill in the blanks with suitable verbs.

21. We _____ with the photograph trying to hang it, till it fell to the floor and the frame broke.
(A) Wrested (B) Wrestled
(C) Strived (D) Wrecked

22. The young boy was lying in the middle of the road, and _____ in pain but nobody picked him up.
(A) Shaking (B) Squirming
(C) Twisting (D) Writhing

23. I took some flowers and chocolates with me for I was anxious to _____ for my thoughtlessness.
(A) Atone (B) Compensate
(C) Propitiate (D) Redeem

24. The ship was scheduled to _____ a couple of days later when it hit an iceberg and sank.
(A) Port (B) Dock
(C) Wharf (D) Moor

25. He _____ on his old feet but there was nobody to extend a helping hand.
(A) Tottered (B) Drooled
(C) Doddered (D) Lurched

Directions for questions 26 to 30: Fill in the blanks with appropriate verbs.

26. Sixty miles _____ a very long distance to drive.
(A) Are (B) Were
(C) Is (D) Has been

27. We hope that they _____ good health.
(A) Kept (B) Are enjoying
(C) Keep (D) Indulge in

28. The Prime Minister _____ to the U.S. next week.
(A) Is flown (B) Fly
(C) Flied (D) Will fly

29. Did you _____ him any letter last week?
(A) Write for (B) Write to
(C) Write (D) Wrote

30. If I _____ you, I would have served them to the best of my ability.
(A) Were (B) Was
(C) Am (D) Will be like

Grammar—Verbs/Tenses/Auxiliary (Advanced)

Directions for questions 31 to 35: Fill in the blanks in the given sentences so as to make sense. Select the correct word from the answer choices and mark its number as the answer.

31. In our country Agriculture must _____ pace with Industrial development.
(A) Take (B) Make
(C) Loose (D) Keep

32. The people of South Asia _____ behind in economic development because of tensions and conflicts.
(A) Has been left (B) Had been left
(C) Left (D) Will be left

33. The motivation to _____ comes from a burning desire to achieve a purpose.
(A) Meet (B) Start
(C) Succeed (D) Idolize

34. He does not _____ the ugly aspects of human nature from his picture of life.
(A) Eliminate (B) Include
(C) Excuse (D) Extricate

35. After the shipwreck, they were _____ on the island for three days
(A) Stuck (B) Sleeping
(C) Marooned (D) Guided

Direction for questions 36 to 40: Correct the mistakes relating to the use of verbs in the sentences given below.

36. The 'Arabian Nights' are an interesting book.

37. The cost of all essential commodities have gone up.

38. Bread and milk are my only food for today.

39. Sanjay as well as his brothers have gone home.

40. The notorious dacoit with his followers have escaped.

Grammar—Adjectives/Adverbs (Basic)

Directions for questions 41 to 45: Fill in the blanks with suitable adjectives.

41. In the face of public censure, the government was compelled to take a more _____ stand.
(A) Staid (B) Acquiescent
(C) Beneficial (D) Belligerent

42. The ultra modern building looked _____ in such a quaint old village.
(A) Competent (B) Devious
(C) Incongruous (D) Decorous

43. He was already well _____ when we reached the party a little late.
(A) Advanced (B) Sloshed
(C) Incapacitated (D) Slurred

44. I keep away from _____ people who always preach about morals, values and principles to everyone.
(A) Spiritual (B) Sacrosanct
(C) Sanctimonious (D) Devout

45. The preacher's _____ remarks were well received by the audience.
(A) Sententious (B) Sensuous
(C) Sequestrable (D) Sheathed

Directions for questions 46 to 50: Fill in the blanks with suitable adverbs.

46. They struggled _____ all through the wind and rain and finally reached their destination.
(A) Madly (B) Skillfully
(C) Manfully (D) Roughly

47. After winning the match, he _____ held the trophy aloft for all to see.
(A) Urgently (B) Surreptitiously
(C) Vicariously (D) Victoriously

48. As many people repose faith in him, he is _____ a leader of the masses.
(A) Irrefutably (B) Arguably
(C) Interestingly (D) Officially

49. In spite of the big hoardings and TV commercials, there were _____ any people in the theatre for the first show of the picture.
(A) Realistically (B) Hardly
(C) Reasonably (D) Discreetly

50. He was appointed the spokesman of the party _____ to focus on the ideology of the party.
(A) Pre-eminently
(B) Conspicuously
(C) Outstandingly
(D) Uniquely

Grammar—Adjectives/Adverbs (Advanced)

Directions for questions 51 to 58: Correct the mistakes relating to the use of adjectives in the following sentences where necessary.

51. I am more wiser than that student.

52. The climate of Hyderabad is better than Madras.

53. Many a student is attending the meeting today.

54. My friend is senior than me by 4 years.

55. Death is preferable than disgrace.

56. Kolkata is a worth-seeing city.

57. Shakuntala is better than any drama in Sanskrit.

Directions for questions 58 to 60: Correct the mistakes relating to the use of adverbs in the following sentences where necessary.

58. His house is further from the station than yours. Correct the following sentences.

59. She was so quiet that hardly he noticed her.

60. No sooner did I go to the railway station when the train left.

Grammar—Conjunctions/Prepositions (Basic)

Directions for questions 61 to 65: Fill in the blanks with suitable prepositions.

61. There is no tax _____ printed books.
(A) On (B) In
(C) With (D) Towards

62. India is a noble, affluent land, teeming _____ natural wealth.
(A) At (B) About
(C) For (D) With

63. After a very long and tiring walk, we rested _____ a tree.
(A) Below (B) Under
(C) Down (D) Behind

64. He died _____ cancer at a very young age.
(A) Of (B) With
(C) By (D) Off

65. _____ all your shortcomings, I still love you.
(A) With (B) Through
(C) For (D) Besides

Directions for questions 66 to 70: Fill in the blanks with suitable conjunctions.

66. Her son hasn't come back from abroad, he hasn't written to her _____.
(A) Neither (B) Either
(C) Even (D) Though

67. He _____ plays some games in the evenings.
(A) Hardly (B) Regularly
(C) Immediately (D) Cautiously

68. You will be allowed to enter by the gate _____ you bribe the gateman.
(A) Provided
(B) Nevertheless
(C) While
(D) Whereas

69. I started early _____ I might reach in time.
(A) So that (B) Lest
(C) For (D) Since

70. He went to bed early _____ to wake up early in the morning.
(A) Such that (B) As though
(C) Such as (D) So as

Grammar—Conjunctions/Prepositions (Advanced)

Directions for questions 71 to 75: Correct the errors relating to the use of conjunctions in each of the following sentences.

71. It had been raining as he left the house.

72. Your action was either just or fair.

73. I shall not come unless I am not invited.

74. Unless you have no objection, I shall see you tomorrow.

75. God made the world so man made the town.

Directions for questions 76 to 80: Correct the error relating to the use of prepositions in each of the following sentences.

76. What is the time with your watch?

77. You must travel with a bus to reach in time.

78. He is suffering with malaria.

79. He has been working since five years.

80. It has been raining from morning.

Grammar—Active/Passive Direct-Indirect (Basic)

Directions for questions 81 to 82: Change the following sentences from direct to indirect speech.

81. Whenever you call him, he says, 'Indeed, I am thinking of calling you shortly.'
(A) Whenever you call him, he says that he was thinking of calling you just then.
(B) Whenever you call him, he said that he was thinking of calling you.
(C) Whenever you call him, he says that he is thinking of calling you just then.
(D) Whenever you call him, he says that he had been thinking of calling him just then.

82. Great people never say, 'We are busy and we don't have time.'
(A) Great people never said that they are busy and they don't have time.
(B) Great people never say that they are busy and they don't have time.

(C) Great people will never say that they are being busy and they don't have time.
(D) Great people would never say that they are busy and they don't have any time.

Directions for questions 83 to 86: Change the following sentences from indirect to direct speech.

83. My friend asked me affectionately what he could do for me when I was in trouble.
(A) When I was in troubles, my friend asked me, 'What could I do for you dear?'
(B) When I was in troubles, my friend had affectionately asked me, 'What I can do for you?'
(C) When I was in trouble, my friend asked me, 'What can I do for you dear?'
(D) My friend has asked me when I was in troubles, 'What can I do for you?'

84. The employee asked the employer whether there was possibility of a good hike in the salary.
(A) The employee asked the employer, 'Is there any possibility of a good hike in my salary?'
(B) The employee told the employer, 'Whether there is any possibility of a good hike in my salary?'
(C) The employee said to the employer, 'If there is any possibility of a good hike in my salary?'
(D) The employee said to the employer, 'Was there any possibility of a good hike in my salary?'

85. The doctor advised the patient to give up drinking.
(A) The doctor said to the patient, 'You should give up drinking.'
(B) The doctor warned the patient, 'Give up drinking.'
(C) The doctor requested the patient, 'Please give up drinking.'
(D) The doctor advised the patient, 'Give up drinking.'

Directions for questions 86 to 87: Change the following sentences from active to passive.

86. Buy this book.
(A) This book should be bought.
(B) This book shall be bought.
(C) This book is bought.
(D) Let this book be bought

87. Bring fruits immediately.
(A) Fruits may be brought immediately.
(B) Let fruits be brought immediately.
(C) Fruits might be bought immediately.
(D) All the above are correct.

Directions for questions 88 to 90: Change the following sentences from passive to active voice.

88. This law must be objected to by everyone.
(A) Everyone must object this law.
(B) Everyone had to object to this law.
(C) Everyone must object that law.
(D) Everyone must object to this law.

89. When is her household work done by her?
(A) When does she do her household work?
(B) When she does her household work?
(C) When has she done her household work?
(D) When she had done her household work?

90. Jawaharlal Nehru was said to have lived in great luxury in his childhood.
(A) People say that Jawaharlal Nehru had lived in great luxury in his childhood.
(B) People say that Jawaharlal Nehru lived in great luxury in his childhood.
(C) People say that Jawaharlal Nehru was lived in great luxury in his childhood.
(D) People had said that Jawaharlal Nehru lived in great luxury in his childhood.

Grammar—Active/Passive Direct-Indirect (Advanced)

Directions for questions 91 to 92: Select the correct passive form of the given sentences.

91. We use this room only on special occasions.
(A) Only on special occasions we use this room.
(B) This room will be used only on special occasions.
(C) This room is used by us only on special occasions.
(D) Only this room is used on special occasions by us.

92. Salesmen from all regions attended the meeting.
(A) The meeting has been attended by salesmen from all regions.
(B) Salesmen have attended the meeting from all regions.
(C) The meeting was attended by salesmen from all regions.
(D) The meeting by salesmen was attended from all regions.

Directions for questions 93 to 95: Select the correct active form of the given sentences.

93. This polish can be applied to any surface.
(A) You can apply this polish to any surface.
(B) Any surface can be applied by this polish.
(C) This polish can apply to any surface.
(D) You have applied this polish to any surface.

94. Was the window pane broken by you?
(A) Do you break the window pane?
(B) Did you break the window pane?
(C) You broke the window pane, did you?
(D) The window pane was broken by you.

95. The electric wires have been cut.
(A) They cut the electric wires.
(B) Someone had cut the electric wires.
(C) Someone has cut the electric wires.
(D) They have been cutting the electric wires.

Directions for questions 96 to 97: Select the correct reported speech of the given sentence.

96. The hotel manager asked the stranger, 'Where do you come from?'
(A) The hotel manager asked the stranger whether he knew where he came from.
(B) The hotel manager asked the stranger where he came from.
(C) The hotel manager enquired where did the stranger come from.
(D) The hotel manager asked the stranger where he had came.

97. She said, 'Sorry, I won't do this again.'
(A) She appologised and said that she wouldn't do that again.
(B) She said sorry and that she won't do it again.
(C) She said that she was sorry and that she would not do this again.
(D) She apologized and said that she shall not do it again.

Directions for questions 98 to 100: Select the correct direct speech of the given sentence.

98. The teacher asked them if they were making good progress.
(A) The teacher asked them 'Were they making good progress?'
(B) The teacher told them, 'Were you making good progress?'
(C) The teacher asked them, 'Are you making good progress?'
(D) The teacher asked them, 'If you are making good progress?'

99. He suggested that they should come early the next day and complete the work.
(A) He said, 'They should come early tomorrow and complete the work.'
(B) He said, 'Let us come early tomorrow and complete the work.'
(C) He said 'We may come early tomorrow to complete the work.'
(D) He said, 'Come early tomorrow and complete the work.'

100. The teacher asked the new boy what his name was, where he came from and which school he had attended last.
(A) The teacher asked the new boy, 'What is your name? Where do you come from? Which school did you attend last?
(B) The teacher told the new boy, 'What is your name? Where are you coming from? Which school do you attend last?'
(C) The teacher said to the new boy, 'What is his name? Where does he come from? Which school he attended last?'
(D) The teacher asked the new boy, 'What your name is? Where you come from? Which school you attended last?'

Grammar—Phrasal Verbs (Basic)

Directions for questions 101 to 110: Fill in the blanks in each sentence with the suitable word.

101. Unscrupulous builders may soon find it impossible to hide _____ terms that conceal the real floor area of the apartments they are selling.
(A) Away (B) Behind
(C) In (D) Out

102. The worker is patting _____ the bumps on the floor so that he can lay the marble properly.
(A) Up (B) On
(C) Down (D) Off

103. No sooner had the government come to know about the farmer's agitation than it sent the paramilitary forces to snuff it_____.
(A) Down (B) Out
(C) Off (D) About

104. The army of Yangon moved along the border to eliminate terrorist training camps and flush the cadres _____.
(A) In (B) Into
(C) Out of (D) Out

105. Given his special role in hounding _____ Saddam Hussein, it is Mr. Blair's coy reaction that is most interesting.
(A) To (B) At
(C) On (D) Out

106. According to Daniel Archibugi, it is impossible to deal _____ a democratic fashion with undemocratic governments.
(A) In (B) With
(C) Into (D) Out

107. South Korea is unable to shrug ____ its U.S.- oriented historical baggage of the 1950s.
(A) Away (B) Off
(C) Up (D) From

108. In Nafis' plays, we watch women writhing ____ pain in the patriarchal stranglehold.
(A) With (B) From
(C) In (D) Away

109. The media reported about victims of road accidents lying on the road and everyone hurrying by paying no attention to lives ebbing_____.
(A) Out (B) Away
(C) Off (D) In

110. The onus is clearly on the leadership of both the political parties to build on the spirit rather than to fritter it _____.
(A) On (B) Up
(C) Down (D) Away

Grammar—Phrasal Verbs (Advanced)

Directions for questions 111 to 120: Select the appropriate phrasal verb from the given options.

111. She is a very friendly girl who can _____ with everyone.
(A) Get along (B) Get off
(C) Get ahead (D) Get away

112. Though I knew she was unscrupulous, I was completely _____ by her charming face and started believing her words.
(A) Taken in (B) Taken aback
(C) Taken over (D) Taken off

113. She is a woman of many talents and I have _____ her.
(A) Taken on (B) Taken after
(C) Taken to (D) Taken up with

114. The prisoner _____ of the prison but was soon recaptured.
(A) Broke out (B) Broke through
(C) Ran out (D) Rushed out

115. They had to _____ the party because of heavy rain.
(A) Call back (B) Cancel out
(C) Call off (D) Wipe out

116. These machines are programmed to _____ automatically in case of an emergency.
(A) Shut away (B) Tune out
(C) Shut down (D) Turn down

117. Most of the water bodies in this area have _____ which has lead to serve food and water shortage.
(A) Dried out (B) Dried up
(C) Dried off (D) Dried away

118. Our manager gave us the general plan and we _____ the minor details.
(A) Worked on (B) Worked to
(C) Worked over (D) Worked in

119. She _____ in any crowd because she is very beautiful and attractive.
(A) Stands aside (B) Stands out
(C) Stamps on (D) Stamps out

120. My friend, who considers himself an excellent driver, was put off when he saw so many cars easily _____ us.
(A) Catching up with (B) Catching at
(C) Taking over (C) Taking on

Answer Keys

Exercise

Practice Problems

1. D	**2.** D	**3.** B	**4.** B	**5.** D	**6.** D	**7.** A	**8.** D	**9.** C	**10.** B
21. B	**22.** D	**23.** B	**24.** B	**25.** C	**26.** C	**27.** B	**28.** D	**29.** B	**30.** A
31. D	**32.** B	**33.** C	**34.** A	**35.** C	**41.** B	**42.** C	**43.** B	**44.** A	**45.** A
46. C	**47.** D	**48.** A	**49.** B	**50.** A	**61.** A	**62.** D	**63.** B	**64.** A	**65.** C
66. B	**67.** B	**68.** A	**69.** A	**70.** D	**81.** C	**82.** B	**83.** C	**84.** A	**85.** A
86. D	**87.** B	**88.** D	**89.** A	**90.** B	**91.** C	**92.** C	**93.** A	**94.** B	**95.** C
96. B	**97.** A	**98.** C	**99.** B	**100.** A	**101.** B	**102.** C	**103.** B	**104.** D	**105.** D
106. A	**107.** B	**108.** C	**109.** B	**110.** D	**111.** A	**112.** A	**113.** C	**114.** A	**115.** C
116. C	**117.** B	**118.** A	**119.** B	**120.** A					

Chapter 2

Vocabulary

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Synonyms
- Antonyms
- Analogy
- Reverse pnalogy
- Verbal reasoning
- Critical reasoning
- Logical reasoning

Common Roots

In this chapter, some of the most commonly used Roots are given. Corresponding to the Root in the first column, its meaning is given in the second column. Given in the third column are words (and their meanings) that contain the corresponding Root given in the first column.

Root	Meaning	Example	Meaning
ac, acr	sharp	ACRIMONIOUS ACERBITY ACIDULATE	bitter, caustic bitterness of temper to make somewhat acid or sour
aev, ev	age, era	PRIMEVAL COEVAL MEDIEVAL or MEDIAEVAL	of the first age of the same age or era of the middle ages
belli	war	BELLICOSE BELLIGERENT REBELLIOUS	inclined to fight inclined to wage war resisting authority
ben, bon	good	BENEFACTOR BENEVOLENCE BONUS	one who does good deeds charity (wishing good) something extra above regular pay
biblio	book	BIBLIOGRAPHY BIBLIOPHILE BIBLE	list of books lover of books The Book

Root	Meaning	Example	Meaning
cad, cas	to fall	DECADENT	deteriorating
		CADENCE	intonation, musical movement
		CASCADE	waterfall
carn	flesh	CARNIVOROUS	flesh-eating
		CARNAGE	destruction of life
		CARNAL	fleshly
ced, cess	to yield, to go	RECEDE	go back, withdraw
		ANTECEDENT	that which goes before
		PROCESS	go forward
chron	time	CHRONOLOGY	time-table of events
		ANACHRONISM	a thing out of time sequence
		CHRONICLE	register events in order of time
cracy	rule	THEOCRACY	that constitution of state in which god is regarded as sole sovereign
		GERONTOCRACY	government by old men.
		PLUTOCRACY	government by the wealthy
		BUREAUCRACY	a system of government by officials, responsible only to their departmental chief
		GYNAECOCRACY	government by women
		OCHLOCRACY	mob rule
		KAKISTOCRACY	government by the worst
		TIMOCRACY	a form of government in which property is the quality for office
cred, credit	to believe	INCREDULOUS	not believing, skeptical
		CREDULITY	gullibility
		CREDENCE	belief
culpa	fault, blame	EXCULPATE	free from blame
		CULPABLE	which can be punished
		CULPRIT	one who is at fault
dem	people	DEMOCRACY	rule of the people
		DEMAGOGUE	(false) leader of the people
		EPIDEMIC	widespread (among the people)
ego		EGOIST	person who is self-interested
		EGOTIST	selfish person
		EGOCENTRIC	revolving about self
err	to wander	ERROR	mistake
		ERRATIC	not reliable, wandering
		KNIGHT-ERRANT	wandering knight
eu	good, well, beautiful	EUPEPTIC	having good digestion
		EULOGIZE	praise
		EUPHEMISM	substitution of pleasant way of saying something blunt

Root	Meaning	Example	Meaning
gen	to give birth to	GENE	unit or factor of heredity
		CONGENITAL	existing from birth
		GENEALOGY	study of the family tree
		PSYCHOGENIC	born in or caused by the mind
grad, gress	go, step	DIGRESS	to astray (from the main point)
		REGRESS	go backwards
		GRADUAL	step by step, by degrees
loqu, locut	to talk	SOLILOQUY	speech by one individual
		LOQUACIOUS	talkative
		ELOCUTION	speech
luc	light	ELUCIDATE	enlighten
		LUCID	clear
		TRANSLUCENT	allowing some light to pass through
magn	great	MAGNIFY	enlarge
		MAGNANIMITY	generosity, greatness of soul
mal	bad	MALEVOLENT	wishing evil
		MALEDICTION	curse
		MALEFACTOR	evil-doer
		MALIGNANT	disposed to do harm
mob, mot, mov	move	MOBILIZE	cause to move
		MOTILITY	ability to move
		IMMOVABLE	not able to be moved
pater, patr	father	PATRIOTISM	love of one's country (fatherland)
		PATRIARCH	male ruler of a family, group or state
		PATERNITY	fatherhood
path	disease, feeling	PATHOLOGY	study of diseased tissue
		APATHETIC	lacking feeling; indifferent
		ANTIPATHY	hostile feeling
phil	to love	PHILANTHROPIST	benefactor, lover of humanity
		ANGLOPHILE	lover of everything English
		PHILANDERER	one involved in brief love affairs
		PHILOSOPHER	a lover of wisdom
		PHILATELIST	one who loves collecting stamps
phoebe	fear	ASTRAPHOBIA	fear of celestial space
		BALLISTROPHOBIA	fear of missiles
		CLAUSTROPHOBIA	fear of closed place
		ERGOPHOBIA	fear of work
		NYCTOPHOBIA	fear of night
		OCHLOPHOBIA	fear of mob
		PANOPHOBIA	a form of melancholia marked by groundless fears
		PHOTOPHOBIA	fear of light
		SITOPHOBIA	fear of food
		XENOPHOBIA	fear of strangers
		TRICHOPHOBIA	fear of hair

Root	Meaning	Example	Meaning
poten	able, powerful	OMNIPOTENT POTENTATE IMPOTENT	all-powerful powerful person powerless
psych	mind	PSYCHOLOGY PSYCHOSIS PSYCHOPATH	study of the mind mental disorder mentally ill person
sacr	holy	SACRILEGIOUS SACRAMENT	impious, violating something holy religious act
somn	sleep	INSOMNIA SOMNOLENT SOMNAMBULIST	inability to sleep sleepy sleepwalker
therm	heat	THERMOSTAT DIATHERMY	instrument that regulates heat sending heat through body tissues
tract	drag, pull	DISTRACT INTRACTABLE	pull (one's attention) away stubborn, unable to be dragged
vac	empty	VACUOUS EVACUATE	lacking content, empty-headed compel to empty an area
verb	word	VERBOSE VERBIAGE VERBATIM	wordy excessive use of words word for word
voc, vocat	to call	AVOCATION PROVOCATION INVOCATION	calling, minor occupation calling or rousing the anger of calling in prayer
vol	wish	MALEVOLENT VOLUNTARY	wishing someone ill of one's own will
volv, volut	to roll	CONVOLUTION	coiled state

EXERCISES

Practice Problems I

Sentence Completion (Basic)

Directions for questions 1 to 10: Each question gives a sentence with a part of the sentence underlined. Four alternatives for the underlined part are given. One of them is correct. Identify the correct one and mark its letter as the answer.

1. As they approaching the house, the colonel's wife calls him for tea.
(A) As they approaching the house
(B) As they will be approaching the house
(C) As they approach the house
(D) As they approach to the house

2. A writer always is having his own vision of life.
(A) Always is having (B) Every time is having
(C) Always has (D) Every time will have

3. With only two weeks before the party, the last place I would wanted to be was in the hospital recovering from surgery.
(A) Would wanted to be
(B) Wanted to be
(C) Would want myself to be
(D) Wanted me to be

4. We had been almost halfway there when the left rear tyre blew out.
(A) Had been almost
(B) Were almost

(C) Were going to be almost
(D) Were reaching

5. After I grew up and went into business, I always have had a soft spot for kids without bikes.
(A) Always have had (B) Always had had
(C) Always had (D) Have always had

6. Gayatri informed one of the directors that she has sent the material for printing.
(A) Shall be sending (B) Sends
(C) Had sent (D) Had been sending

7. Sosin suggested that not to talk about her selection in I.A.S, till she received the orders.
(A) Not to be talking
(B) That we should not talk
(C) That we would not talk
(D) That she will not talk

8. I would rather work twelve hours a day in the office than sitting and watching the idiot box at home.
(A) Than to sit and watch
(B) Instead of sitting and watching
(C) Than to sitting and watching
(D) Than sit and watch

9. He informed me that he will visit my house in a day or two.
(A) He would visit my house in a day or two.
(B) He visits my house in a day or two.
(C) He could visit my house in the coming two days.
(D) He couldn't visit my house in a day or two.

10. I shall wait here until the bus has arrived .
(A) Until the bus may arrive.
(B) Until the bus arrives.
(C) Until the bus will arrive.
(D) Until the bus has arrived.

Sentence Completion (Advanced)

Directions for questions 1 to 10: Identify the incorrect sentence or sentences:

1. a. Civilization is not inherited.
b. It had to be learned and earned by each generation anew.
c. If the transmission is interrupted for one century, civilization will die.
d. And we will be savages again.
(A) a and b (B) b and d
(C) c and d (D) b and c

2. a. Orchids can be found all over India.
b. Arunachal Pradesh, Sikkim and Meghalaya are especially blessed by these blooms.
c. More than 800 of India's 1200 species flourish in the North eastern states.
d. Infact, the state flowers of both Arunachal Pradesh and Assam is the foxtail orchid.
(A) a and c (B) b and d
(C) Only b (D) Only c

3. a. Water blessed our nation this year and also cursed it.
b. Downpours ended California's drought, and a dry summer seared South Carolina.
c. The Midwest still reels from flood.
d. Water is on everyones mind.
(A) Only b (B) b and d
(C) a, b and c (D) b, c and d

4. a. Flowers have always played a significant part in religious ceremony.
b. Their perfumes have a distinct affect on the mind and its moods.
c. The Arabians imported their perfumes from India, as they still do.
d. Kannauj, in Uttar Pradesh, is the heart of the perfume industry.
(A) Only a (B) a and b
(C) Only c (D) c and d

5. a. The country boasts about more than 400 airports.
b. But just 62 of them are in active use.
c. Mumbai and Delhi airports alone account over 40% of the passenger traffic.
d. The facilities at the airports are woefully short.
(A) Only a (B) a and c
(C) a and b (D) a, c and d

6. a. The news about coffee is surprisingly good.
b. Moderate intake of coffee does not pose a healthy threat.
c. The only serious charge against caffeine is that it may addict.
d. Its absence can result in headaches, fatigue etc.
(A) Only a (B) Only c
(C) b and c (D) a and d

7. a. The plane banked over the Amazonian rainforest.
b. Prance gazed down the dark green canopy below.
c. It was an expanse of trees, almost unbroken.
d. It had more variety of plants and animal life than any other place on earth.
(A) b and d (B) a and c
(C) a, b and c (D) b, c and d

8. a. I watched the match on T.V with my son.
b. It was about so exciting as watching a moth sleep.
c. I considered myself lucky if he grunted a monosyllable response.
d. Even eye contact was off question.
(A) a and b (B) a, b and c
(C) b, c and d (D) b and d

9. a. They had to do what no drug enforcement agents had ever done.
b. They had to infiltrate the shadowy world of the Chinese drug cartel.

c. They had to lure Johnny Kon, its crafty leader, in the open.
d. The shrewd Kon was more a match for these inexperienced investigators.
(A) a and b (B) c and d
(C) a, c and d (D) b, c and d

10. a. Shende lost the use for his arms in childhood.
b. Yet he is an expert harmonium player.
c. He is a music teacher also at Mumbai's municipal school.
d. Shende plays the harmonium deftly with his toes.
(A) a and c (B) a and b
(C) c and d (D) b and d

Practice Problems 2

Synonyms (Basic)

Directions for questions 1 to 15: Each question gives a word followed by four choices. From the choices, select the most suitable synonym (word which means the same) for the main word and mark its number as the answer.

1. ABSTRACT
(A) Peculiar (B) Summary
(C) Normal (D) Diagnostic

2. COMPLACENT
(A) Disappointed (B) Evasive
(C) Satisfied (D) Flimsy

3. EXTERMINATE
(A) Evict (B) Dissociate
(C) Accuse (D) Annihilate

4. CHAGRIN
(A) Current (B) Annoyance
(C) Accuse (D) Intoxicate

5. DUNK
(A) Douse (B) Stale
(C) Wet (D) Smell

6. DISTRAUGHT
(A) Agitated (B) Tranquil
(C) Rugged (D) Uneven

7. LISSOME
(A) Pretty (B) Petite
(C) Lithe (D) Tiny

8. PALLIATE
(A) Alleviate (B) Denude
(C) Mollify (D) Approbate

9. RESILIENT
(A) Stretchable (B) Spirited
(C) Rigid (D) Buoyant

10. IMPETUOUS
(A) Cat Like (B) Rash
(C) Impertinent (D) Jovial

11. ARRAIGN
(A) Exonerate (B) Initiate
(C) Accuse (D) Impart

12. APOCRYPHAL
(A) Huge (B) Destructive
(C) Mythological (D) Dubious

13. ABDUCT
(A) Ransack (B) Surround
(C) Destroy (D) Kidnap

14. ABODE
(A) Sanctuary (B) Asylum
(C) Reformatory (D) Dwelling

15. EPITOMIZE
(A) Disappoint (B) Distend
(C) Embody (D) Generate

Synonyms (Advanced)

Directions for questions 1 to 15: In each question given below, identify the word which is similar in meaning (synonyms) to the question word.

1. REPRIEVE
(A) Save (B) Forgive
(C) Victimize (D) Cure

2. TRIBUTE
(A) Declaration (B) Accolade
(C) Criticism (D) Anger

3. OUTFIT
(A) Costume (B) Career
(C) Behaviour (D) Employment

4. ACCREDIT
(A) Undermine (B) Legalize
(C) Enlighten (D) Recognize

5. CAUSTIC
(A) Rude (B) Sarcastic
(C) Brazen (D) Polite

6. DECREE
(A) Request (B) Worship
(C) Appeal (D) Command

7. ARBITRATE
(A) Compare (B) Attach
(C) Believe (D) Judge

8. AFFIDAVIT
(A) Document (B) Oath
(C) Deposition (D) Will

9. ASPERITY
(A) Parity (B) Ascendancy
(C) Harshness (D) Accuracy

10. VAPID
(A) Quick (B) Restless
(C) Insipid (D) Decisive

11. FUSTY
(A) Freak (B) Musty
(C) Delicate (D) Roomy

12. EXPROPRIATE
(A) Embiggle (B) Seize
(C) Wreck (D) Outsmart

13. FORLORN
(A) Forgotten (B) Lamented
(C) Courageous (D) Woebegone

14. THRIFTLESS
(A) Penurious (B) Mendacious
(C) Profligate (D) Tackless

15. COMBUSTION
(A) Ignition (B) Explain
(C) Shrinkage (D) Strength

Practice Problems 3

Antonyms (Basic)

Directions for questions 1 to 15: Each question has a word followed by four choices. From the choices, identify the one which is opposite in meaning (antonym) to the main word and mark its number as your answer.

1. IMMACULATE
(A) Unclean (B) Immobile
(C) Motherly (D) Accumulated

2. IGNOMINIOUS
(A) Crooked (B) Dishonest
(C) Ominous (D) Laudable

3. MELLOWED
(A) Ripened (B) Sensational
(C) Immature (D) Ridiculous

4. GROUCHY
(A) Cantankerous (B) Grumpy
(C) Contented (D) Surly

5. IMPUDENT
(A) Rude (B) Courteous
(C) Impertinent (D) Insolent

6. MITIGATE
(A) Soothe (B) Abate
(C) Assuage (D) Aggravate

7. EXPOSTULATION
(A) Immigration (B) Susceptibility
(C) Elegance (D) Approbation

8. DISPARAGE
(A) Evolve (B) Expose
(C) Appreciate (D) Converge

9. OSSIFICATION
(A) Liquefaction (B) Flexibility
(C) Vigour (D) Viability

10. PEDESTRIAN
(A) Mobile (B) Regular
(C) Exciting (D) Imaginative

11. TUMULTUOUS
(A) Halcyon (B) Mope
(C) Cold (D) Conniving

12. ANACHRONISM
(A) Obsession (B) Vogue
(C) Radicalism (D) Pre-emption

13. BESEECH
(A) Charge (B) Cajole
(C) Command (D) Request

14. BATTERY
(A) Slimy (B) Individual
(C) Dense (D) Clean

15. BLANDISHMENTS
(A) Smack (B) Reproof
(C) Scuttle (D) Extradition

Antonyms (Advanced)

Directions for questions 1 to 15: In each of the questions given below, identify the word which is opposite (antonym) in meaning to the question word.

1. DEMENTED
(A) Abundant (B) Opulent
(C) Sane (D) Divine

2. BRUSQUE
(A) Abrupt (B) Polite
(C) Art (D) Gruff

3. BUCOLIC
(A) Pastoral (B) Rural
(C) Rustic (D) Urban

4. CONTRITE
(A) Apologetic (B) Penitent
(C) Defiant (D) Ecstatic

5. CONTROVERT
(A) Change (B) Accept
(C) Control (D) Deny

6. COWER
(A) Come Out (B) Cringe
(C) Smother (D) Blush

7. CREDIBLE
(A) Legal (B) Unbelievable
(C) Legendary (D) Logical

8. CRESTFALLEN
(A) Abashed (B) Cheerful
(C) Broken (D) Treble

9. JEOPARDY
(A) Vivacity (B) Temerity
(C) Consummation (D) Security

10. MUSTY
(A) Desolate (B) Uncertain
(C) Fragrant (D) Clear

11. HAZY
(A) Bright (B) Misty
(C) Indolent (D) Spirited

12. PROVIDENTIAL
(A) Difficult (B) Spartan
(C) Inopportune (D) Usurping

13. ABATE
(A) Appease (B) Subside
(C) Intensify (D) Curtail

14. ABSTEMIOUS
(A) Disciplined (B) Uninteresting
(C) Unrestricted (D) Intemperate

15. SPONTANEOUS
(A) Voluntary (B) Automatic
(C) Coercive (D) Irritable

Practice Problems 4

Analogies (Basic)

Directions for questions 1 to 15: In each of the following questions a pair of words in capitals is given followed by four numbered pairs of words. Select from the choices the pair which exhibits the same relationship as the capitalised pair of words and mark the letter as your answer.

1. MONEY : MERCENARY
(A) Fame : Soldier
(B) Love : Mother
(C) Pass : Student
(D) Gold : Midas

2. POLYMER : CELL
(A) Coin : Money
(B) Food : Wheat
(C) Chain : Link
(D) Fibre : Plastic

3. POETRY : RHYME
(A) Chair : Table
(B) Mystery : Suspense
(C) Shakespeare : Drama
(D) Clothes : Paper

4. COMPUTER : RAM
(A) Book : Page
(B) Cloud : Rain
(C) Table : Tablecloth
(D) Sky : Blue

5. BELLS : JINGLE
(A) Crow : Caw
(B) Anklets : Tinkle
(C) Pig : Snuff
(D) Clock : Time

6. COGENT : CONVINCING
(A) Insane : Distinguished
(B) Laconic : Pithy
(C) Illogical : Reasonable
(D) Jovial : Abstruse

7. RETROSPECTION : PAST
(A) Syllogism : Logic
(B) Idiosyncrasy : Coherence
(C) Prognostication : Future
(D) Transience : Rigidity

8. BENEVOLENT : GRASPING
(A) Repulsive : Pushing
(B) Euphonious : Discordant
(C) Churlish : Impolite
(D) Rebellious : Disorderly

9. FRIGHTEN : PETRIFY
(A) Enamour : Protect
(B) Sneer : Appreciate
(C) Abbreviate : Interest
(D) Humiliate : Mortify

10. MUNIFICENT : STINGY
(A) Inclement : Merciless
(B) Incorrigible : Recalcitrant
(C) Articulate : Obscure
(D) Egregious : Outstanding

11. CRAVENLY : COWARD
(A) Synthetic : Savant
(B) Menial : Composer
(C) Brazen : Drunkard
(D) Duplicitous : Cheat

12. DIFFIDENT : SHY
(A) Lacklustre : Dull
(B) Covering : Encompass
(C) Infuriating : Regale
(D) Overhauling : Desert

13. RIGOUR : ONEROUS
(A) Vigour : Tentative
(B) Tedium : Boring
(C) Narration : Timely
(D) Simplicity : Rudimentary

14. ESOTERIC : SCHOLARLY
(A) Meandering : Accurate
(B) Dogmatic : Anomalous
(C) Honourable : Shameless
(D) Profuse : Plentiful

15. ADVOCATE : PROPAGATE
(A) Sap : Deliberate
(B) Exact : Demand
(C) Afflict : Ameliorate
(D) Deteriorate : Roughen

Analogies (Advanced)

Directions for questions 1 to 15: In each of the following questions a capitalized pair of words if given followed by four pairs of numbered words. Three of the numbered pairs exhibit the same relation between the words as the capitalized pair of words. Identify the numbered pair which does not exhibit the same relationship as the capitalized pair and mark its number as your answer.

1. IMPORT : EXPORT
(A) Inhale : Exhale
(B) Include : Exclude
(C) Inspire : Expire
(D) Increase : Decrease

2. MILK : MILKMAID
(A) Letters : Postman
(B) Biscuit : Baker
(C) Newspaper : Newspaper Boy
(D) Sweets : Street Vendor

3. ORNITHOLOGIST : BIRDS
(A) Entomologist : Insects
(B) Herpetologist : Reptiles
(C) Psychologist : Mind
(D) Ichthyology : Fossils

4. DISHONOUR : DEMEAN
(A) Demolish : Annihilate
(B) Glorify : Extol
(C) Consecrate : Sanctify
(D) Articulate : Obfuscate

5. LEPROSY : BACTERIA
(A) Summer : May
(B) Cloud : Evaporation
(C) Violence : Anger
(D) Day : Sun

6. MILK : CHEESE
(A) Water : Ice
(B) Cotton : Clothes
(C) Rainbow : Rain
(D) Tobacco : Cigar

7. PRINCIPAL : SCHOOL
(A) President : India
(B) Monarch : Britain
(C) God : Universe
(D) Buddha : Burma

8. FASTIDIOUS : FINICKY
(A) Parsimonious : Miserly
(B) Asinine : Unintelligent
(C) Precocious : Talented
(D) Prudent : Injudicious

9. INSIPID : ORIGINAL
(A) Ambidextrous : Unskilled
(B) Foppish : Affected
(C) Tranquil : Disturbed
(D) Winding : Succinct

10. ATTRITION : WEAR
(A) Rupture : Tear
(B) Mammon : Money
(C) Sacrilege : Honesty
(D) Melody : Rhythm

11. ANTIQUATED : ANCIENT
(A) Compassionate : Lustful
(B) Fortuitous : Coincidental
(C) Pristine : Pure
(D) Spiteful : Wicked

12. EMANCIPATE : LIBERATE
(A) Misconstrue : Misinterpret
(B) Commiserate : Sympathize
(C) Sequestrate : Align
(D) Ape : Imitate

13. COSMIC : UNIVERSE
(A) Sidereal : Stars
(B) Arboreal : Trees
(C) Aquatic : Water
(D) Despotic : Depot

14. BOUNTIFUL : PLENTITUDE
(A) Bizarre : Absurdity
(B) Consecutive : Succession
(C) Depraved : Wickedness
(D) Somnolent : Sombreness

15. STYMIE : OBSTRUCT
(A) Parry : Fight
(B) Ponder : Mull
(C) Imbue : Soak
(D) Intervene : Mediate

Practice Problems 5

Logical Reasoning (Basic)

Directions for questions 1 to 5: Select the correct alternative from the given choices.

1. If you find the next version of software products from Microsoft coming with the feminine touch, don't be surprised because the world's largest software major is planning to add more women to its development centre in Hyderabad. Microsoft India Development Centre (MSIDC), the second largest development centre of the company outside its headquarters in the U.S is planning to kick-start a diversity event, a women's recruitment drive, this Women's Day. As part of the programme, the company has invited applications from girl students who are doing final year computer science engineering across the country.

Which of the following could be the reason behind Microsoft's decision to recruit girl students?

(A) The women recruitment initiative is a pilot project which, if successful, could be replicated at the company's other centres in the country.
(B) Though, due to social conditions, women may opt for engineering colleges near their hometown, rather than IITs, their competency levels are on par with men.
(C) A diverse workforce will be able to anticipate the needs of the diverse customers spread across the world.
(D) This kind of recruitment drive will encourage more women to enrol for engineering course.

2. Its not just media agencies that are obsessed with numbers–even the creative side does not shy away from romancing with digits. Whether it is instructions (Ujala : chaar boondan wala) or just plain information (complan's 23 vital nutrients), a lot of contemporary advertising serves up a tidal wave of figures. Whether its personal care products, refrigerators, washing machine, tea or tooth paste–everyone is using numbers as a USP. While the claims work for Maggi's 2-minutes noodles in many cases it is veering towards a game of one -upmanship. Baffling concept innundate the viewers–12 times more volume mascara, natural care tea with 'paanch ayurvedic tatva', a fridge with 6th sense cooling and a tooth paste that claims to fight 10 problems.

Which of the following does not point to the advantage of using numbers?

(A) People don't believe advertising–so if you can give a quantified claim, it adds credibility to your brand.
(B) Numbers in advertisements cause disillusionment when they are not borne out
(C) When numbers are put into context, they prove to be symbolic and evocative, as in the case of Britannia cheese which says one slice in equal to two glasses of milk.
(D) The number definitely adds a competitive advantage–if one fairness cream claims results in 'six weeks' another will take a mere '30 days'.

3. Art shows in India by Indian artists settled abroad have increased over the past few years and are on the rise.

Which of the following does not account for the above occurrence?

(A) This trend is being fuelled by the booming Indian economy–people have spare funds which they can invest in art.
(B) These artists fetch higher price in India than abroad, probably because their acceptance back home is greater than it is in the oversea art scene.
(C) While acquisition of Indian art overseas continues to grow; purchases are still triggered by the Indian diaspora.
(D) A large cross section of wealthy businessmen travelling abroad drop by at galleries and museums and this develops an understanding of and love for art.

4. Andreas Gursky bestrides the art world like a colossus. For once, in the contemporary art scene, form, content and price come together like a rare planetary alignment. Yet critics conclude that A. Gursky's photographs are famous mainly because of their massive price tags.

Which one of the following prove his critics wrong?

(A) His photographs are rendered with a pin sharp focus that seems to defy the laws of physics.
(B) The hammer prices of Gursky's works simply reflect the herd instincts of billionaires for unique things.
(C) His pictures depict his ability to freeze a transient moment for ever.
(D) The photographic aesthetic embodied in his work is based on an uncanny dichotomy that makes it singular to every viewer.

5. A visit to the abattoir usually transforms a non-vegetarian into a vegetarian. Similarly a visit to the Bose Institute in Kolkatta is sure to make a tourist lead an austere life thereafter, even, perhaps, going to the extent of becoming a mineral eater.

Which of the following supports the above conclusion?

(A) The visitor can watch the graphic reaction of the plant when a leaf is plucked.
(B) One marvels at the intricacies of the composition of a plant.
(C) One realises the fact that it takes years of sunlight, water and earth to help a tree flourish.
(D) The onlooker gets a very clear picture of the intricate process of photosynthesis.

Logical Reasoning (Advanced)

Directions for questions 1 to 2: In each of the following questions a statement is followed by three assumptions. An assumption is an idea or opinion taken for granted. Find out which of the given assumptions is/are implicit considering that everything given in the statement is true.

1. Statement: We have come a long way from the barter system to cowrie currency to a complex monetary system to womb renting and sperm selling. These are innovative methods of making ends meet in hard times.

Assumptions:
I. Innovative methods are needed to make ends meet in hard times.
II. Barter system is not an innovative method.
III. Womb renting and sperm selling are innovative methods.

(A) All are implicit (B) Only I and II
(C) Only II (D) Only III

2. Statement: Why the brouhaha over dynastic politics? Have not many children followed their parents' profession? May be, genes play a part in their choice of profession.

Assumption:
I. Genes play a part in the choice of profession.
II. Choosing politics as profession because the parents are in politics is the same as choosing any other profession because the parents are in that profession.
III. There is nothing wrong with dynastic politics.

(A) Only III (B) Only II and III
(C) Only II (D) Only I and II

Directions for questions 3 to 4: Each question given below consists of a statement followed by two conclusions numbered I and II. You have to assume everything in the statement to be true and then consider the two conclusions together and decide which of them logically follows beyond a reasonable doubt from the information given in the statement. Give your answer as

(A) If only conclusion I follows.
(B) If only conclusion II follows.
(C) If neither I nor II follows.
(D) If both I and II follow.

3. Statement: 'Had I been asked to choose a portfolio, I would have probably chosen the ministry of environment and forest', Home Minister.

Conclusions:
I. It is easier to handle the environment and forest portfolio rather than the home ministry.
II. The person did not become a home minister by choice.

4. Statement: 'Getting angry makes me mentally tough'.–Mr. X.

Conclusions:
I. Mr. X does not get angry.
II. Mr. X is not mentally tough.

Directions for questions 5 to 7: In making decisions about important questions, it is desirable to be able to distinguish between 'strong' arguments and 'weak' arguments. So far as they relate to the question, 'strong' arguments are those which are both important and directly related to the question. 'Weak' arguments are those which are of minor importance and also may not be directly related to the question or may be related to a trivial aspect of the question.

Each question below is followed by two arguments numbered I and II. You have to decide which of the arguments is a 'strong' argument and which is a 'weak' argument. Mark your answer as

(A) If only argument I is strong.
(B) If only argument II is strong.
(C) If neither I nor II is strong.
(D) If both I and II are strong.

5. Statement: Should the Indian police be equipped with more sophisticated weapons?

Arguments:
I. No, they do not know how to use them.
II. Yes, the recent incidents in U.S.A. have proved that terrorists have very sophisticated weapons.

6. Statement: Is financial freedom necessary for women?

Arguments:
I. No, their husbands give them sufficient money.
II. No, women have a very important role to play at home in developing a good society.

7. Statement: Should history be removed from school curriculum?

Arguments:
I. No, it helps students to know about their past.
II. No, history provides information which helps in building the future.

Directions for questions 8 to 9: In each question below is given a statement followed by some courses of action. A course of action is a step or administrative decision to be taken for improvement follow-up or further action with regard to the problem, policy etc on the basis of the information given in the statement. Assume everything in the given statement to be true, and then decide which of the courses of action logically follow for pursuing.

8. Statement:
The competition among corporate hospitals has not done any good to the people.

Course of action:
I. Permission should be given to open more corporate hospitals.
II. The government should review the norms for corporate hospitals so that they take up some social responsibility.
III. All the corporate offices should be closed down.
(A) Only III follows
(B) Only II follows
(C) Either I or III follows
(D) Only II and III follows

9. Statement:
It is reported that 61 percent of crimes including murder are committed by those below 25 years. Unemployment and rising prices drive the youth into taking up a life of crime.

Courses of action:
I. Criminals aged below 25 should be given employment.
II. Pocket money given to the young should be increased.
III. Economics should be taught at schools, so that the young understand the mechanism of price changes and creation of employment.
(A) Only I and III follows
(B) Only III follows
(C) Only I and II follows
(D) None follows

Previous Years' Questions

1. Choose the most appropriate word from the options given below to complete the following sentence.

If we manage to _____ our natural resources, we would leave a better planet for our children. **[2010]**
(A) uphold (B) restrain
(C) cherish (D) conserve

2. Which of the following options is closest in meaning to the word: **Circuitous?** **[2010]**
(A) cyclic (B) indirect
(C) confusing (D) crooked

3. Choose the most appropriate word from the options given below to the complete the following sentence:

His rather casual remarks on politics _____ his lack of seriousness about the subject. **[2010]**
(A) masked (B) belied
(C) betrayed (D) suppressed

4. Modern warfare has changed from large scale clashes of armies to suppression of civilian populations. Chemical agents that do their work silently appear to be suited to such warfare; and regretfully, there exist people in military establishments who think that chemical agents are useful tools for their cause.

Which of the following statements best sums up the meaning of the above passage: **[2010]**
(A) Modern warfare has resulted in civil strife.
(B) Chemical agents are useful in modern warfare.
(C) Use of chemical agents in warfare would be undesirable
(D) People in military establishments like to use chemical agents in war.

5. Choose the most appropriate word from the options given below to complete the following sentence:

It was her view that the country's problems had been _____ by foreign technocrats, so that to invite them to come back would be counter-productive. **[2011]**
(A) identified (B) ascertained
(C) exacerbated (D) analysed

6. Choose the word from the options given below that is most nearly opposite in meaning to the given word:

Frequency **[2011]**
(A) periodicity (B) rarity
(C) gradualness (D) persistency

7. Choose the most appropriate word from the options given below to complete the following sentence:

Under ethical guidelines recently adopted by the Indian Medical Association, human genes are to be manipulated only to correct diseases for which _____ treatments are unsatisfactory. **[2011]**
(A) similar (B) most
(C) uncommon (D) available

8. Which one of the following options is the closest in meaning to the word given below?

Latitude **[2012]**
(A) Eligibility (B) Freedom
(C) Coercion (D) Meticulousness

9. Choose the most appropriate alternative from the options given below to complete the following sentence:

If the tired soldier wanted to lie down, he _____ the mattress out on the balcony. **[2012]**
(A) should take (B) shall take
(C) should have taken (D) will have taken

10. One of the parts (A, B, C, D) in the sentence given below contains an ERROR. Which one of the following is INCORRECT**?**

I requested that he should be given the driving test today instead of tomorrow. **[2012]**

(A) requested that
(B) should be given
(C) the driving test
(D) instead of tomorrow

11. Choose the most appropriate word from the options given below to complete the following sentence:

Given the seriousness of the situation he had to face, his _____ was impressive. **[2012]**

(A) beggary (B) nomenclature
(C) jealousy (D) nonchalance

12. One of the legacies of the Roman legions was discipline. In the legions, military law prevailed and discipline was brutal. Discipline on the battlefield kept units obedient, intact and fighting, even when the odds and conditions were against them.

Which one of the following statements best sums up the meaning of the above passage? **[2012]**

(A) Thorough regimentation was the main reason for the efficiency of the Roman legions even in adverse circumstances.
(B) The legions were treated inhumanly as if the men were animals.
(C) Discipline was the armies' inheritance from their seniors.
(D) The harsh discipline to which the legions were subjected led the odds and conditions being against them.

13. They were requested not to **quarrel** with others. Which one of the following options is the closest in meaning to the word **quarrel**? **[2013]**

(A) make out (B) call out
(C) dig out (D) fall out

14. Complete the sentence:

Dare _____ mistakes. **[2013]**

(A) commit (B) to commit
(C) committed (D) committing

15. Choose the grammatically CORRECT sentence: **[2013]**

(A) Two and two add four.
(B) Two and two become four.
(C) Two and two are four.
(D) Two and two make four.

16. Which of the following options is closest in meaning to the phrase underlined in the sentences below?

It is fascinating to see life forms cope with varied environmental conditions. **[2014]**

(A) adopt to (B) adapt to
(C) adept to (D) accept with

17. Choose the most appropriate word from the options given below to complete the following sentence.

He could not understand the judges awarding her the first prize, because he thought her performance was quite _____. **[2014]**

(A) superb (B) medium
(C) mediocre (D) exhilarating

18. In a press meet on the recent scam, the minister said, 'The buck stops here'. What did the minister convey by the statement? **[2014]**

(A) He wants all the money
(B) He will return the money
(C) He will assume final responsibility
(D) He will resist all enquiries

19. Choose the most appropriate phrase from the options given below to complete the following sentence.

India is a post-colonial country because **[2014]**

(A) it was a former British colony.
(B) Indian Information Technology professionals have colonized the world.
(C) India does not follow any colonial practices.
(D) India has helped other countries gain freedom.

20. Who —— was coming to see us this evening? **[2014]**

(A) you said (B) did you say
(C) did you say that (D) had you said

21. Match the columns. **[2014]**

	Column – I		Column – II
1	Eradicate	P	Misrepresent
2	Distort	Q	Soak completely
3	Saturate	R	Use
4	Utilize	S	Destroy utterly

(A) 1:S, 2: P, 3:Q, 4:R (B) 1:P, 2:Q, 3:R, 4:S
(C) 1:Q, 2:R, 3:S, 4:P (D) 1:S, 2:P, 3:R, 4:Q

22. While trying to collect (I) an envelope from under (II), Mr. X fell down (III) and was losing consciousness. (IV)

Which one of the above underlined parts of the sentence is NOT appropriate? **[2014]**

(A) I (B) II
(C) III (D) IV

23. If she ——— how to calibrate the instrument, she ——— done the experiment. **[2014]**
(A) knows, will have
(B) knew, had
(C) had known, could have
(D) should have known, would have

24. Choose the word that is opposite in meaning to the word 'coherent'. **[2014]**
(A) sticky (B) well-connected
(C) rambling (D) friendly

25. Didn't you buy _____ when you went shopping? **[2015]**
(A) any paper (B) much paper
(C) no paper (D) a few paper

26. Which of the following options is the closest in meaning to the sentence below?

She enjoyed herself immensely at the party. **[2015]**
(A) She had a terrible time at the party.
(B) She had a horrible time at the party.
(C) She had a terrific time at the party
(D) She had a terrifying time at the party.

27. Which one of the following combinations is incorrect? **[2015]**
(A) Acquiescence – Submission
(B) Wheedle – Roundabout
(C) Flippancy – Lightness
(D) Profligate – Extravagant

28. Select the alternative meaning of the underlined part of the sentence.

The chain snatchers took to their heels when the police party arrived. **[2015]**
(A) took shelter in a thick jungle
(B) open indiscriminate fire
(C) took to flight
(D) unconditionally surrendered

29. We _____ our friend's birthday and we _____ how to make it up to him. **[2015]**
(A) completely forgot --- don't just know
(B) forgot completely --- don't just know
(C) completely forgot --- just don't know
(D) forgot completely --- just don't know

30. Choose the statement where underlined word is used correctly. **[2015]**
(A) The industrialist has a personnel jet.
(B) I write my experience in my personnel dairy.
(C) All personnel are being given the day off.
(D) Being religious is a personnel aspect.

31. A generic term that includes various items of clothing such as a skirt, a pair of trousers and a shirt is **[2015]**
(A) fabric (B) textile
(C) fibre (D) apparel

32. Out of the following four sentences, select the most suitable sentence with respect to grammar and usage: **[2015]**
(A) Since the report lacked needed information, it was of no use to them.
(B) The report was useless to them because there were no needed information in it.
(C) Since the report did not contain the needed information, it was not real useful to them.
(D) Since the report lacked needed information, it would not had been useful to them.

33. The man who is now Municipal Commissioner worked as. **[2016]**
(A) the security guard at a university
(B) a security guard at the university
(C) a security guard at university
(D) the security guard at the university

34. Nobody knows how the Indian cricket team is going to cope with the difficult and seamer-friendly wickets in Australia.

Choose the option which is closest in meaning to the underlined phrase in the above sentence. **[2016]**
(A) put up with (B) put in with
(C) put down to (D) put up against

35. Find the odd one in the following group of words. mock, deride, praise, jeer **[2016]**
(A) mock (B) deride
(C) praise (D) jeer

36. Computers were invented for performing only high-end useful computations. However, it is no understatement that they have taken over our world today. The internet, for example, is ubiquitous. Many believe that the internet itself is an unintended consequence of the original invention. With the advent of mobile computing on our phones, a whole new dimension is now enabled. One is left wondering if all these developments are good or, more importantly, required.

Which of the statement (s) below is/are logically valid and can be inferred from the above paragraph? **[2016]**
(i) The author believes that computers are not good for us.
(ii) Mobile computers and the internet are both intended inventions
(A) (i) only (B) (ii) only
(C) both (i) and (ii) (D) neither (i) nor (ii)

37. All hills-stations have a lake. Ooty has two lakes. **[2016]**
(i) Ooty is not a hill-station.
(ii) No hill-station can have more than one lake.
(A) (i) only (B) (ii) only
(C) both (i) and (ii) (D) neither (i) nor (ii)

38. The chairman requested the aggrieved shareholders to _____ him **[2016]**
(A) bare with (B) bore with
(C) bear with (D) bare

39. Identify the correct spelling out of the given options: **[2016]**
(A) Managable (B) Manageable
(C) Mangaeble (D) Managible

40. R2D2 is a robot, R2D2 can repair aeroplanes. No other robot can repair aeroplanes. **[2016]**
(A) R2D2 is a robot which can only repair aeroplanes
(B) R2D2 is the only robot which can repair aeroplane
(C) R2D2 is a robot which can repair only aeroplane
(D) Only R2D2 is a robot

41. A poll of students appearing for masters in engineering indicated that 60% of the students believed that mechanical engineering is a profession unsuitable for women. A research study on women with masters or higher degrees in mechanical engineering found that 99% of such women were successful in their professions.

Which of the following can be logically inferred from the above paragraph? **[2016]**
(A) Many students have misconceptions regarding various engineering disciplines
(B) Men with advanced degrees in mechanical engineering believe women are well suited to be mechanical engineers
(C) Mechanical engineering is a profession well suited for women with masters or higher degrees in mechanical engineering
(D) The number of women pursuing higher degrees in mechanical engineering is small

42. Sourya committee had proposed the establishment of Sourya Institutes of Technology (SITs) in line with Indian Institutes of Technology (IITs) to cater to the technological and industrial needs of a developing country.

Which of the following can be logically inferred form the above sentence? **[2016]**

Based on the proposal,
(i) In the initial years, SIT students will get degrees from IIT
(ii) SITs will have a distinct national objective
(iii) SIT like institutions can only be established in consolation with IIT
(iv) SITs will serve technological needs of a developing country.
(A) (iii) and (iv) only (B) (i) and (iv) only
(C) (ii) and (iv) only (D) (ii) and (iii) only

Answer Keys

Exercises

Practice Problems 1

Sentence Completion (Basic)

1. C	**2.** C	**3.** C	**4.** B	**5.** C	**6.** C	**7.** B	**8.** D	**9.** A	**10.** B

Sentence Completion (Advanced)

1. D	**2.** B	**3.** B	**4.** B	**5.** B	**6.** C	**7.** A	**8.** C	**9.** C	**10.** A

Practice Problems 2

Synonyms (Basic)

1. B	**2.** C	**3.** D	**4.** B	**5.** A	**6.** A	**7.** C	**8.** A	**9.** D	**10.** D
11. C	**12.** D	**13.** D	**14.** D	**15.** C					

Synonyms (Advanced)

1. B	**2.** B	**3.** A	**4.** D	**5.** B	**6.** D	**7.** D	**8.** C	**9.** C	**10.** C
11. B	**12.** B	**13.** B	**14.** C	**15.** C					

Practice Problems 3

Antonyms (Basic)

1. A	**2.** D	**3.** C	**4.** C	**5.** B	**6.** D	**7.** D	**8.** C	**9.** B	**10.** D
11. A	**12.** B	**13.** C	**14.** B	**15.** B					

Antonyms (Advanced)

1. C	**2.** B	**3.** D	**4.** C	**5.** B	**6.** A	**7.** B	**8.** B	**9.** D	**10.** C
11. A	**12.** C	**13.** C	**14.** D	**15.** C					

Practice Problems 4

Analogies (Basic)

1. D	**2.** C	**3.** B	**4.** A	**5.** B	**6.** B	**7.** C	**8.** B	**9.** D	**10.** C
11. D	**12.** A	**13.** B	**14.** D	**15.** B					

Analogies (Advanced)

1. C	**2.** B	**3.** D	**4.** D	**5.** A	**6.** C	**7.** D	**8.** D	**9.** B	**10.** C
11. A	**12.** C	**13.** D	**14.** D	**15.** A					

Practice Problems 5

Logical Reasoning (Basic)

1. C	**2.** B	**3.** C	**4.** D	**5.** A

Logical Reasoning (Advanced)

1. D	**2.** D	**3.** B	**4.** C	**5.** C	**6.** C	**7.** B	**8.** B	**9.** D

Previous Years' Questions

1. D	**2.** B	**3.** C	**4.** C	**5.** C	**6.** B	**7.** D	**8.** B	**9.** C	**10.** B
11. D	**12.** A	**13.** D	**14.** B	**15.** D	**16.** B	**17.** C	**18.** C	**19.** A	**20.** B
21. A	**22.** D	**23.** C	**24.** C	**25.** A	**26.** C	**27.** B	**28.** C	**29.** C	**30.** C
31. D	**32.** A	**33.** B	**34.** A	**35.** C	**36.** D	**37.** D	**38.** C	**39.** B	**40.** B
41. C	**42.** C								

TEST
VERBAL ABILITY

Time: 25 min.

Directions for questions 1 to 4: In these questions you are to analyse the situation on which each question is based, and then select the answer choice that is the most appropriate response to the question.

1. The residents of a particular locality were administered a particular medicine of a company continuously for three months. The locality which had reported high incidence of Hepatitis B before the administration of the medicine reported no cases at the end of this period. The company claimed that this was because their medicine helped the residents develop immunity against Hepatitis B.

Which of the following, if true, most seriously weakens the claim of the company?

(A) A few cases of Hepatitis B were reported in the locality a month after the company started administering the medicine to the residents.
(B) The incidence of Hepatitis B is found to be very high during winter, which lasts for about four months in the region, and winter set in three months back.
(C) Research shows that any medicine capable of helping develop immunity against Hepatitis B takes about six months to show results.
(D) Usually newborn children are more affected by Hepatitis B and there were no children born in the locality in the past three months.

2. Ms. E, a serious protagonist of environmental protection, wins a mayoral election to city 'C' against Mr. D. The success of Ms. E, against Mr. D in the election could be attributed to her concern for the environment in all of the following cases EXCEPT when

(A) The citizens of 'C' are highly environment conscious.
(B) The city administration has powers to act against activities resulting in environmental degradation.
(C) Ms. E is also known to possess excellent leadership qualities.
(D) Mr. D is a champion of environmental protection.

3. The life expectancy of the population of a country has increased from forty five, (twenty five years ago), to seventy now—due to increased medical facilities and better health consciousness in people. However, an insurance company in the country has not lowered the premium substantially despite the life expectancy going up.

Which of the following, if true, would rationalize the action of the insurance company in not lowering the insurance premium?

(A) The increase in life expectancy has been significant only in the past five years.
(B) There have been fewer deaths due to cancer in recent years, whereas twenty five years back it was claiming a number of lives.
(C) The average age of the population of the country remained at around forty five only due to the increased birth rate in the past twenty five years.
(D) The country had been at war with all its neighbours during the past twenty five years, resulting in heavy casualties among its soldiers.

4. A company found that the attrition rate among its employees, especially among the skilled, has been particularly high in the last three months. As the company had introduced night shifts three months ago, it was concluded that this was the reason for the attrition. The employees working in night shifts were, therefore paid fifty percent extra wages.

Which of the following would strongly support the above conclusion?

(A) The wages of the employees compared to those of employees in other companies of comparable size in the same locality are 25% higher.
(B) The productivity of employees working in the night shift is higher.
(C) The number of night shift employees is less than that of the day shift employees.
(D) The attrition rate is higher in the case of night shift employees.

Directions for questions 5 to 10: Each question has a word followed by four choices. From the choices, identify the one which is opposite in meaning (antonym) to the main word and mark its number as your answer.

5. IGNOMINIOUS
(A) Crooked (B) Dishonest
(C) Ominous (D) Laudable

6. MELLOWED
(A) Ripened (B) Sensational
(C) Immature (D) Ridiculous

7. ROBUST
(A) Sturdy (B) Feeble
(C) Muscular (D) Ridiculous

8. VINDICTIVE
(A) Forgiving (B) Relentless
(C) Revengeful (D) Disproving

9. ALTRUISM
(A) Kindness (B) Selfishness
(C) Tenderness (D) Benevolence

10. BOISTEROUS
(A) Serene (B) Rowdy
(C) Violent (D) Tumultuous

Directions for questions 11 to 18: Fill in the blanks in the given sentences so as to make sense. Select the correct word from the answer choices and mark its number as the answer.

11. The angry officer called _____ an explanation from his subordinate for the error.
(A) on (B) for
(C) out (D) in

12. I shall keep _____ nothing from you regarding my family affairs as you are a close friend.
(A) on (B) out
(C) back (D) in

13. Mulk Raj Anand is a novelist with a/an _____.
(A) attribute (B) mission
(C) idea (D) need

14. The motivation to _____ comes from a burning desire to achieve a purpose.
(A) meet (B) start
(C) succeed (D) idolize

15. Rainfall _____ the work being done by the fire-fighters at the site of the tragedy.
(A) protected (B) cherished
(C) hindered (D) illuminated

16. The Americans appreciated the _____ made by Pervez Musharraf.
(A) posture (B) texture
(C) gesture (D) resources

17. She _____ the tray on a table next to the bed.
(A) slowed (B) kept
(C) set (D) shook

18. We have _____ some good times together, sheand I.
(A) looked (B) laughed
(C) smiled (D) seen

Directions for questions 19 to 24: Each question gives a word followed by four choices. From the choices, select the most suitable synonym (word which means the same) for the main word and mark its number as the answer.

19. ABSTRACT
(A) peculiar (B) summary
(C) normal (D) diagnostic

20. COMPLACENT
(A) disappointed (B) evasive
(C) satisfied (D) flimsy

21. EXTERMINATE
(A) evict (B) dissociate
(C) accuse (D) annihilate

22. UMPTEEN
(A) numerous (B) youthful
(C) rainy (D) convergent

23. OSTENSIBLE
(A) strong (B) desirable
(C) apparent (D) fateful

24. STALWART
(A) watchful (B) sturdy
(C) delicate (D) encomium

Directions for questions 25 to 26: In each of the following questions a pair of words in capitals is given followed by four numbered pairs of words. Select from the choices the pair which exhibits the same relationship as the capitalised pair of words and mark the number as your answer.

25. PROPERTY : MORTGAGE
(A) Money : Lend
(B) Equity : Interest
(C) Inventory : Merchandise
(D) Security : Price

26. NEGOTIABLE : CHEQUE
(A) Frozen : Asset
(B) Oil : Seed
(C) Bank : Money
(D) Security : Price

Directions for questions 27 to 30: For the word given on the top of the table, match the dictionary definitions given in the left-hand columns (A, B, C, D) with their corresponding usage given in the right-hand column (E, F, G, H). Out of the four numbered choices given in the boxes below the table, identify the one that has all definitions and usages correctly matched.

27. RAISE

A.	Levy or collect or bring together	E.	The girl has been raised in a very conservative environment.
B.	Cause to be heard or considered	F.	We decided to raise some money to help the flood victims.
C.	Increase the amount or value or strength of	G.	I will raise this issue at the right moment.
D.	Bring up; educate	H.	The doctor raised the potency of the drug before administering it to the patient.

(A)

A	G
B	H
C	F
D	E

(B)

A	F
B	G
C	H
D	E

(C)

A	G
B	E
C	H
D	F

(D)

A	F
B	H
C	G
D	E

28. SENSE

A.	One of the five powers (sight, taste, smell, touch, hearing)	E.	'When will he ever come to his senses?' she asked exasperated
B.	Show or find the meaning of	F.	I could sense that something was amiss in the function.
C.	The ability to think clearly	G.	I cannot make sense of what he says.
D.	Be vaguely aware	H.	We perceive the world through our senses.

(A)

A	H
B	G
C	F
D	E

(B)

A	G
B	E
C	F
D	H

(C)

A	E
B	F
C	H
D	G

(D)

A	H
B	G
C	E
D	F

29. MATCH

A.	Corresponding in some essential respect with what has been mentioned.	E.	The match will start at 9:00 a.m.
B.	Be as good or equal to	F.	Australia finally met its match in the finals.
C.	A contest or competition	G.	Geetanjali would be a nice match for Govind.
D.	A person viewed in regard to his or her eligibility for marriage	H.	If you buy two shirts, we will give any two matching ties free of cost.

(A)

A	H
B	F
C	E
D	G

(B)

A	G
B	E
C	H
D	F

(C)

A	H
B	G
C	E
D	F

(D)

A	G
B	H
C	F
D	E

30. MASS

A.	The Eucharist, especially in the Roman Catholic Church	E.	During peak traffic hours, the entire city seems to be just a mass of humanity.
B.	A dense aggregation of objects	F.	The Mass conducted at the Church was good for the salvation of the soul.
C.	The ordinary people; the majority	G.	Science students must know that mass and weight are entirely different.
D.	The quantity of material that something contains	H.	The masses are bothered only about the problems of day-to-day living and therefore, have no time for politics or sports.

(A)

A	H
B	G
C	F
D	E

(B)

A	G
B	H
C	E
D	F

(C)

A	F
B	E
C	H
D	G

(D)

A	E
B	F
C	H
D	G

Answer Keys

1. C	**2.** D	**3.** D	**4.** D	**5.** D	**6.** C	**7.** B	**8.** A	**9.** B	**10.** A
11. B	**12.** C	**13.** B	**14.** C	**15.** C	**16.** C	**17.** B	**18.** D	**19.** B	**20.** C
21. D	**22.** D	**23.** C	**24.** B	**25.** A	**26.** A	**27.** B	**28.** D	**29.** B	**30.** C

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Numerical Ability

PART B

UNIT I: Quantitative Aptitude 1.71

UNIT II: Reasoning 1.177

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Quantitative Aptitude

UNIT I

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Chapter 1

Simple Equations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- One equation in one unknown
- Two equations in two unknowns
- Three equations in three unknowns
- Additional cases in linear equations

Introduction

There will be linear equations of one or two unknowns invariably in every problem. A linear equation is one where each variable occurs only in its first power and not in any higher powers. Some times we get three equations in three unknowns. In general, we need as many equations as the variables we will have to solve for. So, for solving for the values of two unknowns, we need two equations (or two conditions given in the problem) and for solving for the values of three unknowns, we need three equations (and hence the problem should give three conditions from which we can frame three equations). Solving the equations by itself is not a difficult task. The most important part of the problem is framing the equation/equations. Once the equations are framed, solving them is very easy. In this chapter, we will deal with problems involving as many equations (of first degree) as the number of unknowns. Later on, we will look at equations of second degree (Quadratic Equations) and linear equations where the number of equations will be less than that of the number of variables (under the chapter Special Equations).

One Equation in One Unknown

An equation like $2x + 4 = 26$ is an equation in one unknown. We have only one variable x whose value we have to find out. The steps in solving this are:

Step I: Take all quantities added to (or subtracted from) the x term (term with the unknown) to the right side with a change of sign.
i.e., $2x = 26 - 4 = 22$.

Step II: Take the co-efficient of x from left hand side and divide right hand side with this term to get the value of x.
i.e., $x = 22/2 = 11$. Therefore, $x = 11$.

Two Equations in Two Unknowns

A set of equations like

$$2x + 3y = 8 \quad (1)$$

$$5x + 4y = 13 \quad (2)$$

is called a system of simultaneous equations in two unknowns. Here, we have two variables (or unknowns) x and y whose values we have to find out. This can be done using the two given equations. The steps for this are as follows:

Step I: Using both the equations we first eliminate one variable (so that we can then have one equation in one unknown). For this purpose, we multiply equation (1) with 5 (the co-efficient of x in the second equation) and multiply equation (2) with 2 (the co-efficient of x in the first equation) to eliminate x. Thus we have

$$(1) \times 5 \Rightarrow 10x + 15y = 40 \quad (3)$$

$$(2) \times 2 \Rightarrow 10x + 8y = 26 \quad (4)$$

Now, subtracting equation (4) from equation (3) we have

$$7y = 14 \quad (5)$$

This is one equation in one unknown.

Step II: Solve for the value of one variable from the equation (in one unknown) obtained from Step I above. Therefore, $y = 2$.

Step III: Substitute this value of the variable in one of the two equations to get the value of the second variable.

Substituting the value of y in equation (1) or equation (2), we get $x = 1$. Therefore the values of x and y that satisfy the given set of equations are $x = 1$ and $y = 2$.

Three Equations in Three Unknowns

A set of equations like

$$x + 2y + 3z = 14 \quad (6)$$

$$2x + y + 2z = 10 \quad (7)$$

$$3x + 3y + 4z = 21 \quad (8)$$

is a system of three equations in three unknowns.

Here we have three unknowns x, y and z which we have to solve for from the three given equations. The procedure for the same is as follows:

Step I: Take two out of the three equations [say, eqn. (6) and (7)] and eliminate one variable (say x) so that we get an equation in two unknowns (y and z in this case).
For this purpose, take equations (6) and (7). Multiply equation (6) by 2 and subtract equation (7) from it.
Equation (6) $\times 2 \Rightarrow 2x + 4y + 6z = 28$

$$2x + y + 2z = 10$$

$$3y + 4z = 18 \quad (9)$$

Step II: Repeat Step I for two other equations [say equations (7) and (8)] and eliminate the same variable (x in this case) so that we get one more equation in two unknowns (y and z).
For this purpose, take equations (7) and (8). Multiply equation (7) by 3 and from that subtract equation (8) multiplied by 2.
Equation (7) $\times 3 \Rightarrow 6x + 3y + 6z = 30$
Equation (8) $\times 2 \Rightarrow 6x + 6y + 8z = 42$

$$-3y - 2z = -12 \quad (10)$$

Step III: Now the equations in two unknowns that have been obtained from the above two steps have to be solved as discussed previously (in TWO EQUATIONS IN TWO UNKNOWNS) to get the values of two of the three variables (y and z in this case).
In this case, solving equations (9) and (10), we get $y = 2$ and $z = 3$.

Step IV: Substitute these values of the two variables in one of the three equations to get the value of the third variable.
Substitute the value of y and z in equation (6) to get the value of $x = 1$.
Thus the values of the three variables x, y and z that satisfy the three given equations are $x = 1$; $y = 2$ and $z = 3$

Solved Examples

Example 1: The cost of 3 tables and 4 chairs is ₹2500. The cost of 4 tables and 3 chairs is ₹2400. Find the costs of each table and each chair.

Solution: Let the cost of each table be ₹x.
Let the cost of each chair be ₹y.

$$3x + 4y = 2500 \quad (11)$$

$$4x + 3y = 2400 \quad (12)$$

Method 1:
Multiplying (11) by 3 and subtracting it from (12) multiplied by 4, we get

$$7x = 2100$$

$$x = 300$$

Substituting $x = 300$ in (1),

$$y = 400$$

Method 2:
Adding both the equations (11) and (13), we get $7(x + y) = 4900$

$$x + y = 700 \quad (13)$$

subtracting (12) from (11),

$$-x + y = 100 \quad (14)$$

Adding (13) and (14), $2y = 800$

$$y = 400$$

Substituting $y = 400$ in either (13) or (14), $x = 300$

Example 2: Raju bought 6 pens, 5 erasers and 4 sharpeners for ₹32. Had he bought 4 pens, 3 erasers and 5 sharpeners, his total expenditure would have been ₹23. Had he bought 7 pens, 2 erasers and 6 sharpeners, his total expenditure would have been ₹31. Find the cost of 1 pen, 1 eraser and 2 sharpeners.

Solution: Let the prices of each pen, each eraser and each sharpener be ₹p, ₹e and ₹s respectively.

$$6p + 5e + 4s = 32 \quad (15)$$

$$4p + 3e + 5s = 23 \quad (16)$$

$$7p + 2e + 6s = 31 \quad (17)$$

Multiplying (15) by 2 and subtracting from (16) multiplied by 3,

$$-e + 7s = 5 \quad (18)$$

Multiplying (17) by 4 and subtracting it from (16) multiplied by 7,

$$13e + 11s = 37 \quad (19)$$

Multiplying (18) by 13 and adding it to (19), $102s = 102$

$$s = 1$$

Substituting $s = 1$ in (18),

$$e = 2$$

Substituting values of e and s in (15), $p = 3$

Example 3: In a two digit number, the digits differ by 2. 10 times the number exceeds 5 times the sum of the number formed by reversing its digits and the sum of its digits by 90. Find the number.

Solution: Let the number be xy. Hence the value of the number is $10x + y$.

$$x - y = 2 \quad \text{or} \quad y - x = 2 \quad (20)$$

$$10\,(10x + y) - 5\,(10y + x + x + y) = 90$$

$$90x - 45y = 90$$

$$2x - y = 2;\ 2x - (x \pm 2) = 2$$

$$x = 4 \quad \text{or} \quad 0$$

As x cannot be 0, $x = 4$

$$\therefore \quad y = 6$$

$\therefore$ the number is 46

Example 4: The age of a man 15 years ago was 5 times his son's age. His age 10 years ago was thrice his son's age. After how may years will their combined age become 80 years?

Solution: Let the present age of the man and his son be f years and s years respectively.

$$f - 15 = 5\,(s - 15) \quad \Rightarrow \quad f = 5s - 60$$

$$f - 10 = 3\,(s - 10) \quad \Rightarrow \quad f = 3s - 20$$

$$f = 5s - 60 = 3s - 20$$

$$s = 20,\ f = 40$$

Their combined present age is 60 years. For the combined present age to become 80 years, the age of each of them must increase by 10 years.

$\therefore$ Their combined age will become 80 years after 10 years.

Example 5: If the numerator and the denominator of a fraction are both increased by 1, the fraction becomes $\frac{3}{5}$. If both are decreased by 1, it becomes $\frac{5}{9}$. Find the fraction.

Solution: Let the fraction be $\frac{x}{y}$.

$$\frac{x+1}{y+1} = \frac{3}{5}$$

$$\Rightarrow \quad 5x + 5 = 3y + 3$$

$$5x + 2 = 3y \quad (21)$$

$$\frac{x-1}{y-1} = \frac{5}{9}$$

$$\Rightarrow \quad 9x - 9 = 5y - 5$$

$$9x - 4 = 5y \quad (22)$$

Multiplying (21) by 5 and subtract it from (22) after multiplying by 3,

$$5\,(5x + 2) = 3\,(9x - 4)$$

$$\Rightarrow \quad x = 11$$

substituting $x = 11$ in (21), $y = 19$

$\therefore$ The fraction $= \frac{11}{19}$

Additional Cases in Linear Equations

1. If the number of equations is less than the number of unknowns, then we say the variables are 'indeterminate' or we have an 'indeterminate' system of equations. Here, we cannot uniquely determine the values of all the variables. There will be infinite sets of solutions that satisfy the equations.
   For example, if we take the following two equations in three unknowns,

   $$x + y + 2z = 8$$

   $$2x - y + 3z = 13$$

   this system of equations have infinite number of solutions and no unique solution is possible. For any value we take for x, we can find a corresponding set of values for y and z.
2. However, even in case of indeterminate equations, say, of three variables, it is possible that the value of one of the variables may be uniquely determined, i.e., if we have two equations and three unknowns, we may be still able to determine

the value of one variable uniquely but the other two variables will have infinite number of values. This will happen if the ratio of the coefficients of two variables in one equation is the same as the ratio of the coefficients of the same two variables in the second equation.

This depends on the equations given. Example 8 will clarify this aspect.

Example 6: Tarun bought 2 shirts, 4 trousers and 5 pairs of shoes for ₹3600. Had he bought 6 shirts, 5 trousers and 15 pairs of shoes, his total expenditure would have been ₹8700. Find the price of each trouser.

Solution: Let the prices of each shirt, each trouser and each pair of shoe be ₹x, ₹y and ₹z respectively.

$$2x + 4y + 5z = 3600 \quad (25)$$

$$6x + 5y + 15z = 8700 \quad (26)$$

Multiplying (25) by 3 and subtracting (26) from it, $7y = 2100$

$$\therefore \quad y = 300$$

3. Even in case of indeterminate equations, when some additional conditions are either implicitly built into the problem or explicitly imposed by specifying some constraints on the values of the variables, we may some times be able to determine the values of the variables uniquely or find out a finite set of values that the variables may take. Such problems are separately considered under the chapter 'SPECIAL EQUATIONS.'
4. Sometimes, even if we have equations less in number than the number of variables (i.e., indeterminate equations), while we cannot find out the values of ALL the variables uniquely, it may be possible to find out the value of some specific combination of the variables.

Example 7: The cost of 3 dosas, 5 idlis and 7 vadas is ₹154. The cost of 5 dosas, 8 idlis and 11 vadas is ₹246. Find the total cost of one idli, one dosa and one vada.

Solution: Let the cost of each dosa, each idli and each vada be ₹d, ₹i and ₹v respectively.

$$3d + 5i + 7v = 154 \quad (27)$$

$$5d + 8i + 11v = 246 \quad (28)$$

Multiplying (27) by 3 and subtracting it, from twice (28), $d + i + v = 30$

5. Sometimes, even if we have three equations in three unknown, we may not be able to uniquely determine the values of the variables if the equations are not 'INDEPENDENT,' i.e., one of the given equations can be written as a 'linear combination' of the other two equations.

For example, let us take the following system of three equations in three unknowns.

$$3x + 5y + 7z = 12 \quad (29)$$

$$x - 3y + 9z = 16 \quad (30)$$

$$9x + 8y + 31z = 54 \quad (31)$$

If we try to solve these equations, we will find that we cannot get a unique solution. That is because these equations are not independent. In this case, equation (31) can be obtained by multiplying equation (29) by 2.5 and equation (30) by 1.5 and adding them.

If there are three equations l_1, $l_2$and l_3 in three unknowns, we say that they are linearly dependent if one of the three equations can be written as a linear combination of the other two, i.e., $l_3 = l_1 + kl_2$ where k is any constant.

In such a case, the system of equations will have infinite number of solutions.

If it is not possible to write the three equations in the form above, then they are linearly independent and the system of equations will have a unique solution.

6. Sometimes, we can have 'inconsistent' equations. For example, if we know that $x + 2y = 4$, then the value of $2x + 4y$ has to be 8. The expression $(2x + 4y)$ cannot take any other value. If it is given any other value, there will be inconsistency in the data because then we will effectively be saying that $x + 2y = 4$ and at the same time $x + 2y \neq 4$.

So, if we have the system of equations $x + 2y = 4$ and $2x + 4y = k$, this system of equations will be consistent ONLY If the value of $k = 8$. For any other value of k, the system of equations will be inconsistent.

In the above system of equations, when $k = 8$, there will be infinite number of solutions (and not a unique solution).

Example 8: Find the value of k for which the following system of equations will be consistent.

$$2x - 5y = 10 \quad \text{and} \quad 6x - 15y = k$$

Solution: In the given system of equations, the ratio of the coefficients of x equals the ratio of the coefficients of y.

$\therefore$ They would be consistent only if this ratio equals the ratio of the constant terms.

$\therefore$ If $\frac{10}{k} = \frac{2}{6} = \frac{-5}{-15}$ i.e.,

if $k = 30$, the given system of equations would be consistent.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. P, Q and R are successive even natural numbers in ascending order. Five times R is eight more than seven times P. Find Q.
(A) 6 (B) 8 (C) 12 (D) 14

2. Divide 1 kg weight into two parts such that the sum of the parts is $5/4^{th}$ the difference.
(A) 550 gm, 450 gm
(B) 200 gm, 800 gm
(C) 900 gm, 100 gm
(D) 400 gm, 600 gm

3. A is greater than B by $1/3^{rd}$ the sum of A and B. If B is increased by 40, it becomes greater than twice A by 10. Find A, B.
(A) 30, 20 (B) 60, 30
(C) 20, 10 (D) 20, 40

4. Ajay was asked to find $(2/9)^{th}$ of a number. He instead multiplied the number by (9/2) and obtained an answer which was 4235 more than the correct answer. Find the number.
(A) 900 (B) 945
(C) 990 (D) 810

5. An amount of ₹5,600 is divided among A, B and C. The sum of the shares of B and C is equal to thrice the share of A. The sum of the shares of A and C is equal to nine-fifths the share of B. What is the share of C?
(A) ₹1,400 (B) ₹2,400
(C) ₹2,200 (D) ₹2,000

6. Four times the sum of the digits of a two-digit number is 18 less than the number and is also 9 less than the number formed by reversing its digits. Find the product of its digits.
(A) 12 (B) 20
(C) 30 (D) 42

7. Six years ago, Ram's age was four times Shyam's age. Six years hence, Ram's age will be thrice Shyam's age. After how many years from now will their combined age be 150 years?
(A) 21 (B) 9 (C) 36 (D) 18

8. The sum of the ages of Bharat and Sharat is twice the sum of their ages seven years ago. What is the product of their present ages, if the sum of the squares of their ages is 400?
(A) 192 (B) 180 (C) 200 (D) 164

9. Ashok has a total of 30 notes in denominations of ₹20 and ₹5. The total value of the notes with him is ₹300. Find the number of ₹20 notes with him.
(A) 5 (B) 10 (C) 8 (D) 6

10. A fraction is such that the numerator is five less than the denominator. Also four times the numerator is one more than the denominator. Find the fraction.
(A) 4/9 (B) 3/8 (C) 2/7 (D) 7/12

11. The digits of a two digit number differ by 3. Find the difference of the number and the number formed by reversing its digits.
(A) 18 (B) 27 (C) 36 (D) 45

12. Two chocolates, three milk shakes and four cakes cost ₹190. Four chocolates and eight cakes cost ₹320. Find the cost of a milkshake. (in ₹).
(A) 10 (B) 20
(C) 30 (D) Cannot be determined

13. Three consecutive even integers are such that one-third of the second number is equal to one-fourth of the third number. Find the three numbers.
(A) 4, 6, 8 (B) 8, 10, 12
(C) 12, 14, 16 (D) 2, 4, 6

14. Amar, Bhavan, Chetan and Dinesh have a total of ₹150 with them. Amar has one-fourth of the total amount with the others. Find the amount with Amar (in ₹).
(A) 20 (B) 25 (C) 30 (D) 37.5

15. Ramesh is thrice as old as Suresh. Two years hence, Ramesh will be twice as old as Suresh. Find Ramesh's present age (in years).
(A) 2 (B) 3 (C) 4 (D) 6

16. Nalini has an amount of ₹20 in coins of denominations of 50 paise and ₹1. If she has a total of 30 coins with her, how many ₹1 coins does she have?
(A) 20 (B) 10 (C) 15 (D) 30

17. A two-digit number is one more than six times the sum of its digits and also five more than forty six times the difference of its digits. Find the number.
(A) 79 (B) 97 (C) 49 (D) 94

18. Find the value of k if the equations $3x + (k/3 + 2)y = 1$ and $kx + 2ky = 4$ have infinite solutions.
(A) 9 (B) 6 (C) 18 (D) 12

19. Cost of two pens, five pencils and seven erasers is ₹37. Cost of seven pens, one eraser and two pencils is ₹49. What is the cost of nine pencils and fourtyseven pens?
(A) ₹184 (B) ₹276
(C) ₹284 (D) None of these

20. The sum of two numbers is 250. The difference of their squares is 12500. Find the larger number.
(A) 130 (B) 140 (C) 150 (D) 160

21. Five three-digit numbers including N, were to be added. While adding, the reverse of N was added by mistake instead of N. Hence, the sum increased by 11 times the

sum of the digits of N. Eight times the difference of N's units and hundreds digits is 6 more than twice its hundreds digit. Find its tens digit.
(A) 4 (B) 6 (C) 8 (D) 2

22. The cost of two pens, one eraser and three sharpeners is ₹23. The cost of six pens, three erasers and one sharpener is ₹45. The cost of fourteen pens, seven erasers and twenty one sharpeners is ₹161. Find the cost of each pen (in ₹).
(A) 3
(B) 4
(C) 5
(D) Cannot be determined

23. A child went to a shop to buy a pen, a pencil and a ruler where costs are integral values (in ₹) and are in decreasing order. Each item costs at least ₹4. The total cost is ₹15 and the cost of a pencil is ₹5. How many pencils can he purchase with the amount required to purchase ten rulers?
(A) 10 (B) 12 (C) 8 (D) 9

24. Nitya and Satya have some marbles with them. Nitya says to Satya, 'If you give one marble to me, we will have equal number of marbles'. Satya says to Nitya, 'If you give me one marble, I will have twice the number of marbles you have'. How many marbles do Nitya and Satya have respectively?
(A) 4, 6 (B) 5, 7
(C) 6, 4 (D) 7, 5

25. John covers 10 km per hour more than Peter while driving. On doubling his speed, Peter covers 15 km per hour more than John who is driving at his normal speed. What is John's speed?
(A) 40 km/hr (B) 25 km/hr
(C) 45 km/hr (D) 35 km/hr

Answer Keys

Exercise

Practice Problems

1. B	**2.** C	**3.** C	**4.** C	**5.** C	**6.** B	**7.** B	**8.** A	**9.** B	**10.** C
11. B	**12.** A	**13.** A	**14.** C	**15.** D	**16.** B	**17.** B	**18.** D	**19.** D	**20.** C
21. B	**22.** D	**23.** C	**24.** B	**25.** D					

Chapter 2

Ratio-Proportion-Variation

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Ratio
- Proportion
- Variation
- Direct variation
- Inverse variation
- Joint variation

RATIO

If the values of two quantities A and B are 4 and 6 respectively, then we say that they are in the ratio 4 : 6 (read as 'four is to six'). Ratio is the relation which one quantity bears to another of the same kind, the comparison being made by considering what multiple, part or parts, one quantity is of the other. The ratio of two quantities 'a' and 'b' is represented as $a : b$ and read as 'a is to b'. Here, 'a' is called antecedent, 'b' is the consequent. Since the ratio expresses the number of times one quantity contains the other, it's *an abstract* quantity.

Ratio of any number of quantities is expressed after removing any common factors that ALL the terms of the ratio have. For example, if there are two quantities having values of 4 and 6, their ratio is 4 : 6, i.e., 2 : 3 after taking the common factor 2 between them out. Similarly, if there are three quantities 6, 8 and 18, there is a common factor among all three of them. So, dividing each of the three terms by 2, we get the ratio as 3 : 4 : 9.

If two quantities whose values are A and B respectively are in the ratio $a : b$, since we know that some common factor $k(> 0)$ would have been removed from A and B to get the ratio $a : b$, we can write the original values of the two quantities (i.e., A and B) as ak and bk respectively. For example, if the salaries of two persons are in the ratio 7 : 5, we can write their individual salaries as $7k$ and $5k$ respectively.

A ratio $a : b$ can also be expressed as a/b. So if two items are in the ratio 2 : 3, we can say that their ratio is 2/3. If two terms are in the ratio 2, it means that they are in the ratio of 2/1, i.e., 2 : 1.

'A ratio is said to be a ratio of greater or less inequality or of equality according as antecedent is greater than, less than or equal to consequent'. In other words,

1. The ratio $a : b$ where $a > b$ is called a ratio of greater inequality (example 3 : 2)
2. The ratio $a : b$ where $a < b$ is called a ratio of less inequality (example 3 : 5)
3. The ratio $a : b$ where $a = b$ is called a ratio of equality (example 1 : 1)

From this we can find that a ratio of greater inequality is diminished and a ratio of less inequality is increased by adding same quantity to both terms, i.e., in the ratio $a : b$, when we add the same quantity x (positive) to both the terms of the ratio, we have the following results

$$\text{if } a < b \quad \text{then} \quad (a + x) : (b + x) > a : b$$
$$\text{if } a > b \quad \text{then} \quad (a + x) : (b + x) < a : b$$
$$\text{if } a = b \quad \text{then} \quad (a + x) : (b + x) = a : b$$

This idea can also be helpful in questions on Data Interpretation when we need to compare fractions to find the larger of two given fractions.

If two quantities are in the ratio $a : b$, then the first quantity will be $a/(a + b)$ times the total of the two quantities and the second quantity will be equal to $b/(a + b)$ times the total of the two quantities.

Solved Examples

Example 1: If $a : b = 3 : 4$, find $3a + 4b : 4a + 5b$.

Solution: $3a + 4b : 4a + 5b$

$$= \frac{3a+4b}{4a+5b} = \frac{\frac{3a+4b}{b}}{\frac{4a+5b}{b}}$$

$$= \frac{3\left(\frac{a}{b}\right)+4}{4\left(\frac{a}{b}\right)+5} = \frac{3\left(\frac{3}{4}\right)+4}{4\left(\frac{3}{4}\right)+5} = \frac{25}{32}$$

Example 2: The ratio of the number of marbles with Ram and Shyam is 19 : 13. If Ram gives Shyam 30 marbles, both will have equal number of marbles. Find the number of marbles with Ram.

Solution: Let the number of marbles with Ram and Shyam be $19x$ and $13x$ respectively. Total number of marbles with them = $32x$

If Ram gives Shyam 30 marbles each will have $\frac{32x}{2} = 16x$ marbles.

$\therefore$ $19x - 16x = 30$

$x = 10$

$19x = 190$

Example 3: 1400 is divided into 4 parts such that half of the first part, one third of the second part, one fourth of the third part and $\frac{1}{12}$th of the last part are all equal. Find the 4 parts.

Solution: Let the first part, second part, third part and fourth part be a, b, c and d respectively.

$$\frac{1}{2}a = \frac{1}{3}b = \frac{1}{4}c = \frac{1}{12}d$$

$$d = 6a,\ b = \frac{3}{2}a,\ c = 2a$$

Given, $a + b + c + d = 1400$

$\Rightarrow$ $a + \frac{3}{2}a + 2a + 6a = 1400$

$\Rightarrow$ $a = \frac{400}{3}$

$\therefore$ $b = 200,\ c = \frac{800}{3},\ d = 800$

Example 4: There are 2 classes A and B. If 10 students leave class A and join class B, then the ratio of the number of students in class A and class B would reverse. Find the difference in the numbers of students in class A and class B.

Solution: Let the numbers of students in class A and class B be ax and bx respectively.

Given, $\frac{ax-10}{bx+10} = \frac{b}{a}$

$a^2x - 10a = b^2x + 10b$

$\Rightarrow$ $a^2x - b^2x - 10a - 10b = 0$

$\Rightarrow$ $(ax - bx - 10)(a + b) = 0$

$\therefore$ $ax - bx = 10$

Example 5: A husband's age exceeds that of his wife by 6 years. 10 years ago, the ratio of their ages was 5 : 4. Find the present age of the husband.

Solution: Let the present age of the husband be x years.

$\Rightarrow$ Present age of the wife = $(x - 6)$ years.

10 years ago, the ages of the husband and the wife will be $(x - 10)$ years and $(x - 16)$ years respectively.

Given $x - 10 = \frac{5}{4}(x - 16)$

$\therefore$ $x = 40$

Alternative method:

Let the age of the husband 10 years ago be $5x$ years. Age of his wife at that time = $4x$ years.

The husband would then also be 6 years older than his wife.

$\therefore$ $5x = 4x + 6 \Rightarrow x = 6$

Hence the present age of the husband

$= 5x + 10$ i.e., 40 years

Proportion

When two ratios are equal, then the four quantities involved in the two ratios are said to be proportional i.e., if $a/b = c/d$, then a, b, c and d are proportional.

This is represented as $a : b :: c : d$ and is read as 'a is to b (is) as c is to d'.

When a, b, c and d are in proportion, then a and d are called the EXTREMES and b and c are called the MEANS. We also have the relationship:

Product of the MEANS = Product of the EXTREMES i.e., $b\,c = ad$

If $a : b = c : d$ then

$b : a = d : c$ (A)

$a : c = b : d$ (B)

$(a + b) : b = (c + d) : d$ (C)

(obtained by adding 1 to both sides of the given relationship)

$$(a - b) : b = (c - d) : d \quad \text{(D)}$$

(obtained by subtracting 1 from both sides of the given relationship)

$$(a + b) : (a - b) = (c + d) : (c - d) \quad \text{(E)}$$

{obtained by dividing relationship (C) above by (D)}

Relationship (A) above is called INVERTENDO
Relationship (B) is called ALTERNENDO;
Relationship (C) is called COMPONENDO;
Relationship (D) is called DIVIDENDO;
Relationship (E) is called COMPONENDO –DIVIDENDO.

The last relationship, i.e., COMPONENDO-DIVIDENDO is very helpful in simplifying problems. By this rule, whenever we know $a/b = c/d$, then we can write

$$(a + b)/(a - b) = (c + d)/(c - d).$$

The converse of this is also true – whenever we know that $(a + b)/(a - b) = (c + d)/(c - d)$, then we can conclude that $a/b = c/d$.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \ldots$, then each of these ratios is equal to $\frac{a+c+e+\ldots}{b+d+f+\ldots}$.

VARIATION

Two quantities A and B may be such that as one quantities changes in value, the other quantity also changes in value **bearing certain relationship** to the change in the value of the first quantity.

Direct Variation

One quantity A is said to vary directly as another quantity B if the two quantities depend upon each other in such a manner that if B is increased in a certain ratio, A also increases in the same ratio and if B is decreased in a certain ratio, A also decreases in the same ratio.

This is denoted as $A \propto B$ (A varies directly as B).
If $A \propto B$ then $A = kB$, where k is a constant. It is called the constant of proportionality.

For example, when the quantity of sugar purchased by a housewife doubles from the normal quantity, the total amount she spends on sugar also doubles, i.e., the quantity and the total amount increase (or decrease) in the same ratio.

From the above definition of direct variation, we can see that when two quantities A and B vary directly with each other, then $A/B = k$ or the ratio of the two quantities is a constant. Conversely, when the ratio of two quantities is a constant, we can conclude that they vary directly with each other.

If X varies directly with Y and we have two sets of values of the variables X and $Y - X_1$ corresponding to Y_1 and X_2 corresponding to Y_2, then, since $X \propto Y$, we can write down

$$\frac{X_1}{Y_1} = \frac{X_2}{Y_2} \quad \text{or} \quad \frac{X_1}{X_2} = \frac{Y_1}{Y_2}$$

Inverse Variation

A quantity A is said to vary inversely as another quantity B if the two quantities depend upon each other in such a manner that if B is increased in a certain ratio, A gets decreased in the same ratio and if B is decreased in a certain ratio, then A gets increased in the same ratio.

It is the same as saying that A varies directly with $1/B$. It is denoted as $A \propto 1/B$ i.e., $A = k/B$ where k is k the constant of proportionality.

For example, as the number of men doing a certain work increases, the time taken to do the work decreases and conversely, as the number of men decreases, the time taken to do the work increases.

From the definition of inverse variation, we can see that when two quantities A and B vary inversely with each other, then $AB = a$ constant, i.e., the product of the two quantities is a constant. Conversely, if the product of two quantities is a constant, we can conclude that they vary inversely with each other.

If X varies inversely with Y and we have two sets of values of X and $Y - X_1$ corresponding to Y_1 and X_2 corresponding to Y_2, then since X and Y are inversely related to each other, we can write down

$$X_1Y_1 = X_2Y_2 \quad \text{or} \quad \frac{X_1}{X_2} = \frac{Y_2}{Y_1}$$

Joint Variation

If there are three quantities A, B and C such that A varies with B when C is constant and varies with C when B is constant, then A is said to vary jointly with B and C when both B and C are varying. i.e., $A \propto B$ when C is constant and $A \propto C$ when B is a constant; $\Rightarrow A \propto BC$.

$A \propto BC \Rightarrow A = kBC$ where k is the constant of proportionality.

Example 6: Find the value(s) of x if $\frac{2x+5}{x+1} = \frac{x+2}{x-1}$.

Solution: $(2x + 5)(x - 1) = (x + 2)(x + 1)$

$$2x^2 + 5x - 2x - 5 = x^2 + 2x + x + 2$$

$$\Rightarrow \quad x^2 = 7$$

$$\therefore \quad x = \pm\sqrt{7}$$

Example 7: X varies directly with $Y^2 + 18$. When $Y = 18$, $X = 18$. Find Y when $X = 1$.

Solution: $\frac{X_1}{X_2} = \frac{Y_1^2+18}{Y_2^2+18}; \frac{18}{1} = \frac{18^2+18}{Y_2^2+18}$

$$Y_2^2 + 18 = 19$$

$$Y_2 = \pm 1$$

In these types of problems on variation, there are typically three parts:

1. The relationship between different variables is defined to frame an equation involving the variables and the constant of proportionality.
2. One set of values of all the values of all the variables is given to enable us find the value of the constant of proportionality.

3. The values of all but one variable of a second set are given and we are asked to find the value of the one variable whose value is not given.

Example 8: The ratio of the monthly incomes of A and B is 4 : 3. The ratio of their monthly expenditures is 5 : 4. If A saves one fourth of his monthly income, find the ratio of their monthly savings.

Solution: Let the monthly incomes of A and B be ₹$4x$ and ₹$3x$ respectively.

Let the monthly expenditures of A and B be ₹$5y$ and ₹$4y$ respectively. Monthly savings of A = ₹$(4x - 5y)$. Monthly savings of B = ₹$(3x - 4y)$.

Given that $4x - 5y = \frac{1}{4}(4x)$

$$3x = 5y.$$

$\therefore$ Monthly saving of $B = 3x - 4y = 5y - 4y$ i.e. ₹y.

Required ratio $= \frac{1}{4}(4x) : y = x : y = 5 : 3$

Example 9: If x varies directly with y, check whether $x^3 + y^3$ varies directly with $x^3 - y^3$.

Solution: Let $\frac{x}{y} = k$, where k is a constant

$$x = k\,y$$

$$x^3 + y^3 = y^3 (k^3 + 1)$$

$$x^3 - y^3 = y^3 (k^3 - 1)$$

$$\frac{x^3 + y^3}{x^3 - y^3} = \frac{k^3 + 1}{k^3 - 1} \text{ a constant}$$

$\therefore$ $(x^3 + y^3)$ varies directly with $(x^3 - y^3)$

Example 10: The monthly expenses of Raja on his car are partly constant and partly vary with the number of kilometres he travels in a month. If he travels 100 km in a month his total car expenses will be ₹3500. If he travels 200 km in a month, his total car expenses will be ₹4000. If he travels 250 km in a month, what will be his total car expenses.

Solution: Let his total car expenses be ₹T. Let the fixed expense be ₹F. Let the variable expense be ₹V.

$$T = F + V$$

If he travels D km in a month, $\frac{V}{D} = k$, where k is a constant.

$\therefore$ $$T = F + k\,D$$

Given that

$$3500 = F + 100k \quad (1)$$

$$4000 = F + 200k \quad (2)$$

Solving (1) and (2),

$$F = 3000 \text{ and } k = 5.$$

Total car expenses if he travels 250 km

$$= F + 250k = ₹4250.$$

The problems involving ratio and proportion are just different forms of the models of the basic problems we saw above. For example, the problem we just solved above might be reframed bringing in Mangoes, Bananas, Baskets, etc. Here, practice and perseverance pay you a lot. In entrance exams, there will be either direct problems on ratio, proportion and variation or indirect problems of application of these concepts just discussed to areas like Time and Work or Time and Distance.

EXERCISE

Practice Problems

Directions for questions 1 to 18: Select the correct alternative from the given choices.

1. If $p : q = 5 : 4$ and $p = a + b$ and $q = a - b$, find $a : b$.
(A) 1 : 9 (B) 9 : 1
(C) 5 : 4 (D) 4 : 5

2. The number of marbles with A and B are in the ratio of 10 : 11. Which of the following cannot be a possible number of marbles with A and B together?
(A) 189 (B) 210 (C) 231 (D) 153

3. The ratio of the number of students in three classes A, B and C is 3 : 7 : 8. If ten students, are transferred from C to B, B will have 80 students. Find the total number of students in the three classes.
(A) 150 (B) 160 (C) 180 (D) 210

4. Three positive numbers p, q and r satisfy $\frac{q+r}{p} = \frac{p+r}{q} = \frac{p+q}{r} = K$. $K =$ _____.
(A) 3/2 (B) 5/2 (C) 3 (D) 2

5. What must be subtracted from p and added to q so that the ratio of the resultants becomes 1 : 3?
(A) $\frac{p+q}{3}$ (B) $\frac{3p-q}{4}$
(C) $\frac{p-q}{p+q}$ (D) $\frac{q-3p}{4}$

6. Vipin's present age is twice the age of Kishore one year ago. What is the sum of their present ages (in years), if the ratio of the sum of their present ages to the difference of their present ages is 19 : 5?
(A) 21 (B) 19 (C) 24 (D) 34

7. Three different types of balls priced at ₹5, ₹8 and ₹13 per piece are displayed in three different boxes by a trader. Mr. Paul bought from this shop all three types of balls spending a total sum of ₹768. The numbers of the balls he bought, taken in the order in which the prices are mentioned above, are in the ratio 5 : 4 : 3. How many balls of the costliest variety did he buy?
(A) 104 (B) 64 (C) 48 (D) 24

8. If $a : b = 2 : 3$ $b : c = 4 : 3$ and $c : d = 2 : 3$, then find $a : b : c : d$.
(A) 8 : 12 : 9 : 27 (B) 16 : 24 : 18 : 27
(C) 18 : 27 : 36 : 8 (D) 12 : 18 : 15 : 20

9. The weights of Bimal and Basu are in the ratio 2 : 3 and the weights of Basu and Bali are in the ratio 4 : 3. What is Basu's weight (in kg) if the sum of the weights of Bimal, Basu and Bali is 203 kg?
(A) 84 (B) 76 (C) 49 (D) 65

10. If $3x - 4y + 2z = 0$ and $4x - 2y - z = 0$, find $x : z : y$.
(A) 8 : 10 : 11 (B) 8 : 11 : 40
(C) 11 : 40 : 8 (D) 8 : 40 : 11

11. If $a + b - c : b + c - a : a + c - b = 5 : 6 : 7$, then find $a : b : c$.
(A) 12 : 13 : 11 (B) 12 : 11 : 13
(C) 13 : 12 : 11 (D) 13 : 11 : 12

12. Which of the following represents a possible value of $p : q$ satisfying $\frac{20p^2 - 40pq}{pq + 4q^2} = 20$?
(A) 3 : 1 (B) 1 : 4 (C) 4 : 1 (D) 5 : 1

13. Ninety three is divided into two parts such that thrice the first part and twice the second part are in the ratio 25 : 4. Find the first part.
(A) 60 (B) 75 (C) 50 (D) 70

14. If three is subtracted from the numerator and five is added to the denominator of a fraction, the new fraction formed is 1/2. If two is added to the numerator of the initial fraction, the ratio of the new numerator to the denominator becomes 1 : 1. Find the original fraction.
(A) 11/13 (B) 18/23 (C) 13/15 (D) 13/11

15. The ratio of the number of students in classes A, B and C is 3 : 7 : 8. If 10 students leave C and join B, the ratio of the number of students in B and C would be reversed. Find the total number of students in the classes A, B and C.
(A) 144 (B) 162 (C) 180 (D) 198

16. A person has with him a certain number of weighing stones of 100 gm, 500 gm and 1 kg in the ratio of 3 : 5 : 1. If a maximum of 5 kg can be measured using weighing stones of 500 gm alone, then what is the number of 100 gm stones he has?
(A) 6 (B) 3 (C) 9 (D) 5

17. The ratio of the prices of tea, last year and this year is 5 : 6. The ratio of the prices of coffee last year and this year is 7 : 8. The sum of prices of a kg of tea and a kg of coffee this year is ₹48. Find the price of tea (in ₹) last year if it was $\frac{20}{21}$ of the price of coffee last year.
(A) 15 (B) 20 (C) 25 (D) 10

18. Ajay and Vijay wrote a test. The sum of Ajay's score and twice Vijay's score is 310. The sum of Vijay's score and twice Ajay's score is 290. Find the ratio of the scores of Ajay and Vijay.
(A) 9 : 11 (B) 13 : 17
(C) 11 : 19 (D) 7 : 13

Directions for questions 19 and 20: These questions are based on the data given below.

A test of 60 minutes contains questions on Mathematics and English only. The time taken to solve a Mathematics question is twice the time taken to answer an English question and the ratio of time taken to solve all Mathematics questions to time taken to answer all English questions is 8/7.

19. What is the ratio of the number of English questions to that of Mathematics?
(A) 11/7 (B) 7/4
(C) 9/4 (D) 7/5

20. If the total number of questions is 22, how many English questions can be answered in 18 minutes?
(A) 8 (B) 10
(C) 11 (D) 9

Directions for questions 21 to 25: Select the correct alternative from the given choices.

21. The pressure of a gas varies directly with the temperature when the volume is constant and varies inversely with the volume when temperature is constant. If the present temperature is 100 K what will be the increase in temperature if the pressure triples and the volume doubles?
(A) 200 K (B) 600 K
(C) 500 K (D) 100 K

22. For a body starting from rest, the distance travelled (d) is directly proportional to the square of the time elapsed from the start (t). When $t = 4$ sec, $d = 48$ m. What is the value of d (in metres) at $t = 7$ sec?
(A) 128 (B) 150
(C) 115 (D) 147

23. 'A' varies directly as the sum of two quantities 'B' and 'C'. 'B' in turn varies directly as 'x' and 'C' varies inversely as 'x'. When $x = 1$ or 2, $A = 3$. Find the value of A when $x = 4$.
(A) 5 (B) 4.5
(C) 5.5 (D) 6

24. The extension of a spring from its rest position is directly proportional to the force acting on the spring. An additional force applied on the already stretched

spring produces a further extension, which is twice that of the initial extension. What is the ratio of the additional force to the initial force?

(A) 3 : 1 (B) 2 : 1
(C) 4 : 1 (D) 1 : 3

25. The kinetic energy of a body is directly proportional to the square of its speed when the mass is kept constant and is directly proportional to mass when its speed is kept constant. A body with a mass of 2 kg and a speed of 10 m/s has a kinetic energy of 100 joules. What is the kinetic energy of a body whose mass is 20 kg and speed is 1 m/s?

(A) 100 joules (B) 1000 joules
(C) 10 joules (D) 20 joules

Answer Keys

Exercise

Practice Problems

1. B	**2.** D	**3.** C	**4.** D	**5.** B	**6.** B	**7.** D	**8.** B	**9.** A	**10.** A
11. B	**12.** C	**13.** B	**14.** C	**15.** C	**16.** A	**17.** B	**18.** A	**19.** B	**20.** D
21. C	**22.** D	**23.** B	**24.** B	**25.** C					

Chapter 3

Numbers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Rule of signs
- Classification of real numbers
- Perfect numbers
- Hierarchy of arithmetic operations
- Number of factors of a number
- Involution and evolution
- LCM and HCF models
- Successive division
- Factorial
- Number systems
- Conversions
- Binary arithmetic

Numbers

NUMBERS is one of the most important topics required for competitive entrance exams—particularly the MBA entrance exams. In this chapter, we have put together a number of models of problems—mainly based on various problems that have been appearing in different exams.

ADDITION is the process of finding out single number or fraction equal to two or more quantities taken together.

SUBTRACTION is the process of finding out the quantity left when a smaller quantity (number/fraction) is reduced from a larger one.

MULTIPLICATION signifies repeated addition. If a number has to be repeatedly added then that number is Multiplicand. The number of multiplicands considered for addition is Multiplier. The sum of repetition is Product. For example, in the multiplication $3 \times 4 = 12$, 3 is the multiplicand, 4 is the multiplier and 12 is product.

DIVISION is a reversal of multiplication. In this we find how often a given number called Divisor is contained in another given number called Dividend. The number expressing this is called the Quotient and the excess of the dividend over the product of the divisor and the quotient is called Remainder.

For example, in the division 32/5, 32 is Dividend, 5 is Divisor, 6 is Quotient and 2 is Remainder.

Rule of Signs

The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.

Example: $-1 \times -1 = +1$;

$+1 \times -1 = -1$;

$+1 \times +1 = +1$;

$-1 \times +1 = -1$;

Classification of Real Numbers

Real Numbers are classified into rational and irrational numbers.

Rational Numbers: A number which can be expressed in the form p/q where p and q are integers and $q \neq 0$ is called a rational number.

For example, 4 is a rational number since 4 can be written as 4/1 where 4 and 1 are integers and the denominator $1 \neq 0$. Similarly, the numbers 3/4, –2/5, etc., are also rational numbers.

Recurring decimals are also rational numbers. A recurring decimal is a number in which one or more digits at the end of a number after the decimal point repeats endlessly (For example, 0.333....., 0.111111...., 0.166666...., etc., are all recurring decimals). Any recurring decimal can be expressed as a fraction of the form p/q and hence it is a rational number. We will study in another section in this chapter the way to convert recurring decimals into fractions.

Between any two numbers, there can be infinite number of other rational numbers.

Irrational Numbers: Numbers which are not rational but which can be represented by points on the number line are called irrational numbers. Examples for irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\sqrt[4]{5}$, $\sqrt[3]{9}$, etc.

Numbers like π, e are also irrational numbers.

Between any two numbers, there are infinite number of irrational numbers.

Another way of looking at rational and irrational numbers is Terminating decimals and recurring decimals are both rational numbers.

Any non-terminating, non-recurring decimal is an irrational number.

Integers: All integers are rational numbers. Integers are classified into negative integers, zero and positive integers. Positive integers can be classified as Prime Numbers and Composite Numbers. In problems on Numbers, we very often use the word 'number' to mean an 'integer.'

Prime Numbers: A number other than 1 which does not have any factor apart from one and itself is called a prime number.

Examples for prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, etc.

There is no general formula that can give prime numbers. **Every prime number greater than 3 can be written in the form of $(6k + 1)$ or $(6k - 1)$ where k is an integer.** For the proof of this, refer to 4th point under 'Some important points to note' given later on in this chapter.

Composite Numbers: Any number other than 1, which is not a prime number is called a composite number. In other words, a composite number is a number which has factors other than one and itself.

Examples for composite numbers are 4, 6, 8, 9, 10, 14, 15, etc.

Note: The number 1 is neither prime nor composite.

The only prime number that is even is 2.

There are 15 prime numbers between 1 and 50 and 10 prime numbers between 50 and 100. So, there are a total of 25 prime numbers between 1 and 100.

Relative Primes: Two numbers are said to be relative primes or co-primes if they do not have any common factor other than 1. For example, the numbers 15 and 16 do not have any common factors and hence they are relative primes. Please note that none of the two numbers may individually be prime and still they can be relative primes. Unity is a relative prime to all numbers.

Multiples: If one number is divisible exactly by a second number, then the first number is said to be a multiple of the second number. For example, 15 is a multiple of 5; 24 is a multiple of 4.

Factors: If one number divides a second number exactly, then the first number is said to be a factor of the second number. For example, 5 is a factor of 15; 3 is a factor of 18. Factors are also called sub-multiples or divisors.

Even and odd numbers: Numbers divisible by 2 are called even numbers whereas numbers that are not divisible by 2 are called odd numbers.

Examples for even numbers are 2, 4, 6, 8, 10, etc. Examples for odd numbers are 1, 3, 5, 7, 9, etc.

Note: Every even number ends in 0, 2, 4, 6 or 8.

The sum of any number of even numbers is always even.

The sum of odd number of odd numbers (i.e., the sum of 3 odd numbers, the sum of 5 odd numbers, etc.) is always odd whereas the sum of even number of odd numbers (i.e., the sum of 2 odd numbers, the sum of 4 odd numbers, etc.) is always even.

The product of any number of odd numbers is always odd.

The product of any number of numbers where there is at least one even number is even.

Perfect Numbers

A number is said to be a perfect number if the sum of ALL its factors excluding itself (but including 1) is equal to the number itself.

For example, 6 is a perfect number because the factors of 6, i.e., 1, 2 and 3 add up to the number 6 itself.

Other examples of perfect numbers are 28, 496, 8128, etc.

Rules for Divisibility

In a number of situations, we will need to find the factors of a given number. Some of the factors of a given number can, in a number of situations, be found very easily either by observation or by applying simple rules. We will look at some rules for divisibility of numbers.

Divisibility by 2: A number divisible by 2 will have an even number as its last digit (For example 128, 246, 2346, etc.)

Divisibility by 3: A number is divisible by 3 if the sum of its digits is a multiple of 3.

For example, take the number 9123, the sum of the digits is $9 + 1 + 2 + 3 = 15$ which is a multiple of 3. Hence, the given number 9123 is divisible by 3. Similarly 342, 789 etc., are all divisible by 3. If we take the number 74549, the sum of the digits is 29 which is not a multiple of 3. Hence, the number 74549 is not divisible by 3.

Divisibility by 4: A number is divisible by 4 if the number formed with its last two digits is divisible by 4.

For example, if we take the number 178564, the last two digits form 64. Since this number 64 is divisible by 4, the number 178564 is divisible by 4.

If we take the number 476854, the last two digits form 54 which is not divisible by 4 and hence the number 476854 is not divisible by 4.

Divisibility by 5: A number is divisible by 5 if its last digit is 5 or zero (eg. 15, 40, etc.).

Divisibility by 6: A number is divisible by 6 if it is divisible both by 2 and 3 (18, 42, 96, etc.).

Divisibility by 7: If the difference between the number of tens in the number and twice the units digit is divisible by 7, then the given number is divisible by 7. Otherwise, it is not divisible by 7.

Take the units digit of the number, double it and subtract this figure from the remaining part of the number. If the result so obtained is divisible by 7, then the original number is divisible by 7. If that result is not divisible by 7, then the number is not divisible by 7.

For example, let us take the number 595. The units digit is 5 and when it is doubled, we get 10. The remaining part of the number is 59. If 10 (which is the units digit doubled) is subtracted from 59 we get 49. Since this result 49 is divisible by 7, the original number 595 is also divisible by 7.

Similarly, if we take 967, doubling the units digit gives 14 which when subtracted from 96 gives a result of 82. Since 82 is not divisible by 7, the number 967 is not divisible by 7.

If we take a larger number, the same rule may have to be repeatedly applied till the result comes to a number which we can make out by observation whether it is divisible by 7. For example, take 456745, We will write down the figures in various steps as shown below.

Col(1) Number	Col(2) Twice the units digit	Col(3) Remaining part of the number	Col(3) – Col(2)
456745	10	45674	45664
45664	8	4566	4558
4558	16	455	439
439	18	43	25

Since 25 in the last step is not divisible by 7, the original number 456745 is not divisible by 7.

Divisibility by 8: A number is divisible by 8, if the number formed by the last 3 digits of the number is divisible by 8.

For example, the number 3816 is divisible by 8 because the last three digits form the number 816, which is divisible by 8. Similarly, the numbers 14328, 18864 etc. are divisible by 8. If we take the number 48764, it is not divisible by 8 because the last three digits' number 764 is not divisible by 8.

Divisibility by 9: A number is divisible by 9 if the sum of its digits is a multiple of 9.

For example, if we take the number 6318, the sum of the digits of this number is 6 + 3 + 1 + 8 which is 18. Since this sum 18 is a multiple of 9, the number 6318 is divisible by 9. Similarly, the numbers 729, 981, etc., are divisible by 9. If we take the number 4763, the sum of the digits of this number is 20 which is not divisible by 9. Hence, the number 4763 is not divisible by 9.

Divisibility by 10: A number divisible by 10 should end in zero.

Divisibility by 11: A number is divisible by 11 if the sum of the alternate digits is the same or they differ by multiples of 11 - that is, the difference between the sum of digits in odd places in the number and the sum of the digits in the even places in the number should be equal to zero or a multiple of 11.

For example, if we take the number 132, the sum of the digits in odd places is 1 + 2 = 3 and the sum of the digits in even places is 3. Since these two sums are equal, the given number is divisible by 11.

If we take the number 785345, the sum of the digits in odd places is 16 and the sum of the digits in even places is also 16. Since these two sums are equal, the given number is divisible by 11.

Divisibility by numbers like 12, 14, 15 can be checked out by taking factors of the number which are relatively prime and checking the divisibility of the given number by each of the factors. For example, a number is divisible by 12 if it is divisible both by 3 and 4.

Recurring Decimals: A decimal in which a digit or a set of digits is repeated continuously is called a recurring decimal. Recurring decimals are written in a shortened form, the digits which are repeated being marked by dots placed over the first and the last of them, thus

$$\frac{8}{3} = 2.666..... = 2.\dot{6} \text{ or } 2.\overline{6};$$

$$\frac{1}{7} = 0.142857142857142857... = 0.142857$$

In case of 1/7, where the set of digits 142857 is recurring, the dot is placed on top of the first and the last digits of the set or alternatively, a bar is placed over the entire set of the digits that recur.

A recurring decimal like $0.\overline{3}$ is called a pure recurring decimal because all the digits after the decimal point are recurring.

A recurring decimal like $0.1\overline{6}$ (which is equal to 0.16666...) is called a mixed recurring because some of the digits after the decimal are not recurring (in this case, only the digit 6 is recurring and the digit 1 is not recurring).

A recurring decimal is also called a 'circulator'. The digit, or set of digits, which is repeated is called the 'period'

of the decimal. In the decimal equivalent to 8/3, the period is 6 and in 1/7 it is 142857.

As already discussed, all recurring decimals are rational numbers as they can be expressed in the form p/q, where p and q are integers. The general rule for converting recurring decimals into fractions will be considered later. Let us first consider a few examples so that we will be able to understand the rule easily.

Solved Examples

Example 1: Express $0\cdot\overline{4}$ in the form of a fraction.

Solution: Let $x = 0\cdot\overline{4} = 0.444\ldots$ (1)

$$10x = 4.444\ldots = 4.\overline{4} \quad (2)$$

Subtracting (1) from (2),

$$9x = 4$$

$$\Rightarrow \quad x = \frac{4}{9}.$$

Example 2: Express $0\cdot\overline{63}$ in the form of a fraction.

Solution: Let $x = 0\cdot\overline{63} = 0.636363\ldots$ (3)

$$100x = 63.636363\ldots = 63\cdot\overline{63} \quad (4)$$

Subtracting (3) from (4),

$$99x = 63$$

$$\Rightarrow \quad x = \frac{7}{11}.$$

We can now write down the rule for converting a pure recurring decimal into a fraction as follows:

A pure recurring decimal is equivalent to a vulgar fraction which has the number formed by the recurring digits (called the period of the decimal) for its numerator, and for its denominator the number which has for its digits as many nines as there are digits in the period.

Thus $0.\overline{37}$ can be written as equal to $\frac{37}{99}$; $0.\overline{225}$ can be written as equal to $\frac{225}{999} = \frac{25}{111}$;

$$0.\overline{63} = \frac{63}{99} = \frac{7}{11}.$$

A mixed recurring decimal becomes the sum of a whole number and a pure recurring decimal, when it is multiplied by suitable power of 10 which will bring the decimal point to the left of the first recurring figure. We can then find the equivalent vulgar fraction by the process as explained in case of a pure recurring decimal.

Now we can write the rule to express a mixed recurring decimal into a (vulgar) fraction as below:

In the numerator write the entire given number formed by the (recurring and non-recurring parts) and subtract from it the part of the decimal that is not recurring. In the denominator, write as many nines as the period (i.e., as many nines as the number of digits recurring) and then place next to it as many zeroes as there are digits without recurring in the given decimal.

i.e., $$0.1\overline{56} = \frac{156-1}{990} = \frac{155}{990} = \frac{31}{198}$$

$$0.7\overline{3} = \frac{73-7}{90} = \frac{66}{90} = \frac{11}{15}$$

Number of Factors of a Number

If N is a composite number such that $N = a^p \cdot b^q \cdot c^r \ldots$ where a, b, c are prime factors of N and p, q, r .... are positive integers, then the number of factors of N is given by the expression

$$(p+1)(q+1)(r+1)\ldots$$

For example $140 = 2^2 \times 5^1 \times 7^1$.

Hence, 140 has $(2+1)(1+1)(1+1)$, i.e., 12 factors.

Please note that the figure arrived at by using the above formula includes 1 and the given number N also as factors. So if you want to find the number of factors the given number has excluding 1 and the number itself, we find out $(p+1)(q+1)(r+1)$ and then subtract 2 from that figure.

In the above example, the number 140 has 10 factors excluding 1 and itself.

Number of Ways of Expressing a Given Number as a Product of Two Factors

The given number N (which can be written as equal to $a^p \cdot b^q \cdot c^r \ldots$ where a, b, c are prime factors of N and $p, q, r \ldots$ are positive integers) can be expressed as the product of two factors in different ways.

The number of ways in which this can be done is given by the expression $1/2\ \{(p+1)(q+1)(r+1)\ldots\}$

So, 140 can be expressed as a product of two factors in 12/2 or 6 ways {because $(p+1)(q+1)(r+1)$ in the case of 140 is equal to 12}

If p, q, r, etc. are all even, then the product $(p+1)(q+1)(r+1)\ldots$ becomes odd and the above rule will not be valid since we cannot take 1/2 of an odd number to get the number of ways. If $p, q, r, \ldots$ are all even, it means that the number N is a perfect square. This situation arises in the specific cases of perfect squares because a perfect square can also be written as {square root × square root}. So, two different cases arise in case of perfect squares depending on whether we would like to consider writing the number as {square root x square root} also as one of the ways.

Thus, to find out the number of ways in which a perfect square can be expressed as a product of 2 factors, we have the following 2 rules.

1. as a product of two DIFFERENT factors: $\frac{1}{2}\{(p+1)(q+1)(r+1)\ldots -1\}$ ways (excluding $\sqrt{N}\times\sqrt{N}$).
2. as a product of two factors (including $\sqrt{N}\times\sqrt{N}$) in $\frac{1}{2}\{(p+1)(q+1)(r+1)\ldots +1\}$ ways.

Example 3: Find the number of factors of 3025.

Solution: $3025 = (5)(605) = (5)(5)(121) = 5^2\,11^2$
Number of factors of 3025
$= (2+1)(2+1) = 9$

Example 4: In how many ways can 22500 be written as a product of two different factors?

Solution: $22500 = 150^2 = ((2)(5)(3)(5))^2 = 2^2 5^4 3^2$
Number of ways
$= \frac{1}{2}\{(2+1)(4+1)(2+1)-1\} = 22.$

Sum of all the Factors of a Number

If a number $N = a^p . b^q . c^r \ldots\ldots$ where $a, b, c\ldots..$ are prime numbers and $p, q, r \ldots$ are positive integers, then, the sum of all the factors of N (including 1 and the number itself) is:

$$\left(\frac{a^{p+1}-1}{a-1}\right).\left(\frac{b^{q+1}-1}{b-1}\right).\left(\frac{c^{r+1}-1}{c-1}\right)\ldots$$

The above can be verified by an example.

Consider the number 48, when resolved into prime factors, $48 = 2^4 \times 3^1$. Here $a = 2, b = 3, p = 4, q = 1$.

Hence, sum of all the factors

$$= \left(\frac{2^{4+1}-1}{2-1}\right)\left(\frac{3^{1+1}-1}{3-1}\right) = \frac{31}{1}\times\frac{8}{2} = 124$$

The list of factors of 48 is:

1, 2, 3, 4, 6, 8, 12, 16, 24, 48.

If these factors are added, the sum is 124 and tallies with the above result.

Product of all the Factors of a Number

We shall now consider another kind of question which has also appeared frequently in exams. These questions refer to the 'structure' of numbers, i.e., the prime factors of a number and the canonical representation of a number. We shall begin by working out the product of all the factors of a given number.

Example 5: What is the product of all the factors of 180?

Solution: $180 = 4(45) = 2^2 3^2 5^1$. There are $(2+1)(2+1)(1+1)$ or 18 factors.

If the given number is not a perfect square, at least one of the indices is odd and the number of factors is even. We can form pairs such that the product of the two numbers in each pair is the given number (180 in this example).

$\therefore$ The required product is 180^9.

In general, if $N = p^a q^b r^c$ (where at least one of a, b, c is odd), the product of all the factors of N is $N^{\frac{d}{2}}$, where d is the number of factors of N and is given by $(a+1)(b+1)(c+1)$.

Example 6: Let us see what happens when N is a perfect square, say 36.

We want the product of all the factors of 36.

Solution: $36 = 2^2 3^2$ (there are 9 factors)

$$1(36) = 2(18) = 3(12) = 4(9) = 6(6)$$

$\therefore$ The product of all the factors is $36^4\,(6)$.

In general, let $N = p^a q^b r^c$ where each of a, b, c is even.

There are $(a+1)(b+1)(c+1)$ say d factors. We can form $\frac{d-1}{2}$ pairs and we would be left with one lone factor, i.e., $\sqrt{N}$. The product of all these factors is $N^{\frac{d-1}{2}}(\sqrt{N}) = N^{\frac{d}{2}}$

$\therefore$ Whether or not N is a perfect square, the product of all its factors is $N^{\frac{d}{2}}$, where d is the number of factors of N.

Number of Ways of Writing a Number as Product of Two Co-primes

Using the same notation and convention used earlier.

If $N = a^p . b^q . c^r \ldots$, then, the number of ways of writing N as a product of 2 co-primes is 2^{n-1}, where 'n' is the number of distinct prime factors of the given number N.

Taking the example of 48, which is $2^4 \times 3^1$, the value of 'n' is 2 because only two distinct prime factors (i.e., 2 and 3 only) are involved.

Hence, the number of ways $= 2^{2-1} = 2^1 = 2$ i.e., 48 can be written as product of 2 coprimes, in two different ways. They are (1 and 48) and (3, 16).

Number of Co-primes to N, That are Less than N

If N is a number that can be written as $a^p . b^q . c^r \ldots$, then, the number of co-primes of N, which are less than N, represented by $\phi(N)$ is,

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

For example if, 48 is considered,

$$N = a^p . b^q . c^r \ldots$$

i.e., $48 = 2^4 . 3^1.$

Hence, $a = 2$, $b = 3$, $p = 4$, $q = 1$.

$$\phi(48) = 48\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$$

$$= 48\times\frac{1}{2}\times\frac{2}{3} = 16.$$

Note: If numbers less than 48 are listed, and co-primes to 48 are picked up, the count of co-primes will be 16.

Sum of Co-primes to *N* That Are Less Than *N*

The sum of the co-primes of N, that are less than N is $\frac{N}{2}\cdot\phi(N)$. If we consider the above example, already we have $\phi(48) = 16$.

Hence, sum of co-primes of 48 that are less than $48 = \frac{N}{2}\cdot\phi(N) = \frac{48}{2}\times 16 = 384$.

Note: After picking out the co-primes of 48 that are less than 48, they can be added and the sum can be verified.

Least Common Multiple (LCM) and Highest Common Factor (HCF)

Least Common Multiple (LCM) of two or more numbers is the least number which is divisible by each of these numbers (i.e., leaves no remainder; or remainder is zero). The same can be algebraically defined as 'LCM of two or more expressions is the expression of the lowest dimension which is divisible by each of them i.e., leaves no remainder; or remainder is zero.'

Highest Common Factor (HCF) is the largest factor of two or more given numbers. The same can be defined algebraically as 'HCF of two or more algebraical expressions is the expression of highest dimension which divides each of them without remainder.'

HCF is also called GCD (Greatest Common Divisor).

Product of two numbers = LCM × HCF

LCM is a multiple of HCF

For finding **LCM and HCF of fractions,** first reduce each fraction to its simplest form i.e., cancel out any common factors between the denominator and numerator and then apply appropriate formula from the following :

$$HCF\ of\ fractions = \frac{HCF\ of\ numerators}{LCM\ of\ denominators}$$

$$LCM\ of\ fractions = \frac{LCM\ of\ numerators}{HCF\ of\ denominators}$$

LCM and HCF Models

LCM - Model 1: In this model of problem, you will need to find out the smallest number (or number in a specified range like the largest five-digit number) which when divided by 2 or more other numbers (i.e., divisors) leaves the same remainder in all cases.

The basic distinguishing feature of this model of problems is that the remainder will be the **same** in all the cases (and that remainder will also be given).

The smallest such number will be the remainder itself. The next higher number that satisfies the given conditions is the LCM of the given numbers (i.e., divisors) plus the remainder given, i.e., add the remainder (which is the same in all cases) to the LCM of the given numbers (i.e., divisors).

To find any larger number that satisfies a given condition, we will first need to find out a multiple of the LCM in that range and add the remainder to this multiple of the LCM.

The general rule can be written as follows:

Any number which when divided by p, q or r leaving the same remainder s in each case will be of the form ***k (LCM of p, q and r) + s*** *where* $k = 0, 1, 2, \ldots$

If we take $k = 0$, *then we get the smallest such number.*

Example 7: Find the HCF of 1363 and 1457.

Solution:

$$\begin{array}{r} 1 \\ 1363\overline{)1457} \\ 1363 \\ \hline 94\overline{)1362} \quad 14 \\ 1316 \\ \hline 47\overline{)94} \quad 2 \\ 94 \\ \hline 0 \end{array}$$

$\therefore$ HCF (1367, 1457) = 47

Example 8: Find the smallest number which when divided by 5 or 11 leaves a remainder of 4 and is greater than the remainder.

Solution: Set of such numbers are of the form K [LCM (5, 11)] + 4 where K is a whole number.

We get the required number when $K = 1$

$\therefore$ Smallest number = LCM (5, 11) + 4

= 55 + 4 = 59.

LCM - Model 2: In this model, the remainders in the divisions given will not be the same but the difference between the divisor and the remainder (i.e., the complement of the

remainder) will be the same in each case. For example, you may be asked to find out 'the smallest number which when divided by 4 or 6 gives respective remainders of 3 and 5.' Here, the remainders are not the same as in LCM - Model 1; but the difference between the divisor and the remainder is same in each case. In the first case the difference between the divisor and the remainder is 1(= 4 – 3). In the second case also the difference between the divisor and the remainder is 1(= 6 – 5).

The smallest such number is LCM minus constant difference (the constant difference being the difference between the divisor and the corresponding remainder in all cases).

Similarly, any multiple of the LCM minus the constant remainder also will satisfy the same condition.

In the example considered above, the LCM of 4 and 6 is 12 and hence the required number is 11 (which is equal to 12 – 1).

The general rule can be written as follows:

Any number which when divided by p, q or r leaving respective remainders of s, t and u where (p – s) = (q – t) = (r – u) = v (say), will be of the form ***k (LCM of p, q and r)*** *– v*

The smallest such number will be obtained by substituting k = 1.

Example 9: Find the smallest number which when divided by 9 and 11 leaves remainders of 7 and 9 respectively.

Solution: Required number = LCM (9, 11) – 2 = 97.

Example 10: Find the largest four-digit number which when divided by 9 and 11 leaves remainders of 7 and 9 respectively.

Solution: Required number must be in the form LCM (9, 11) k – 2 i.e., $99k - 2$, where k is the largest natural number satisfying $99k - 2 < 10000$.

$$\therefore \quad k < 101\frac{1}{33}$$

$$\therefore \quad k = 101$$

$\therefore$ Largest number = 9997.

Example 11: Find the smallest number which leaves a remainder of 7 when divided by 11 and leaves a remainder of 12 when divided by 13.

Solution: Let the number be in the forms $11k_1 + 7$ and $13k_2 + 12$ where k_1 and k_2 have the least possible values.

$$11k_1 + 7 = 13k_2 + 12$$

$$k_1 = k_2 + \frac{2k_2 + 5}{11}$$

As k_1 is an integer, $2k_2 + 5$ must be divisible by 11.

Hence $k_2 = 3$.

$\therefore$ Smallest number = 51.

HCF - Model 1: In this model, we have to identify the largest number that exactly divides the given dividends (which are obtained by subtracting the respective remainders from the given numbers).

The largest number with which the numbers p, q or r are divided giving remainders of s, t and u respectively will be the ***HCF of the three numbers (p – s), (q – t) and (r – u).***

Let us understand this model with an example.

Example 12: Find the largest number which leaves remainders of 2 and 3 when it divides 89 and 148 respectively.

Solution: Largest number = HCF (89 – 2, 148 – 3) = 29

HCF - Model 2: In this model, the problem will be as follows:

'Find the largest number with which if we divide the numbers p, q and r, the remainders are the same.'

Take the difference between any two pairs out of the three given numbers. Let us say we take the two differences $(p - q)$ and $(p - r)$. The HCF of these numbers will be the required number.

Here, the required number = HCF of (p ~ q) and (p ~ r) = HCF of (p ~ q) and (q ~ r) = HCF of (q ~ r) and (p ~ r)

Let us take an example and look at this model.

Example 13: Find the largest number which divides 444, 804 and 1344 leaving the same remainder in each case.

Solution: Largest number

= HCF (804 – 444, 1344 – 804)

= HCF (360, 540) = 180.

Successive Division

If the quotient of a division is taken and this is used as the dividend in the next division, such a division is called 'successive division'. A successive division process can continue upto any number of steps – until the quotient in a division becomes zero for the first time. i.e., the quotient in the first division is taken as dividend and divided in the second division; the quotient in the second division is taken as the dividend in the third division; the quotient in the third division is taken as the dividend in the fourth division and so on.

If we say that 2479 is divided successively by 3, 5, 7 and 2, then the quotients and remainders are as follows in the successive division.

Dividend	Divisor	Quotient	Remainder
2479	3	826	1
826	5	165	1
165	7	23	4
23	2	11	1

Here we say that when 2479 is successively divided by 3, 5, 7 and 2 the respective remainders are 1, 1, 4 and 2.

Example 14: A number when divided by 6 and 4 successively leaves remainders of 5 and 2 respectively. Find the remainder when the largest such two digit number is divided by 9.

Solution: Let the quotients obtained when the number is divided by 6 and 4 successively be q_1 and q_2 respectively.

$$\text{Number} = 6q_1 + 5$$

It successive division, the quotient obtained for each division starting from the first, forms the dividend for the next division.

$\therefore \quad q_1 = 4q_2 + 2$

$\therefore \quad \text{number} = 6(4q_2 + 2) + 5 = 24q_2 + 17$

Largest two-digit number satisfying the given conditions is obtained when $24q_2 + 17 < 100$ and q_2 is maximum i.e. $q_2 < 3\frac{11}{24}$ and it is maximum i.e., $q_2 = 3$.

$\therefore$ number = 89. required remainder = 8

Alternative method:

Divisors: 6 × 4

Remainders: 5 2

The smallest number satisfying the given conditions is found using the following method. Each divisor and the remainder it leaves are written as shown above. Starting with the last remainder, each remainder is multiplied with the previous divisor and added to that divisor's remainder. This procedure is carried out until the divisor's remainder is the first remainder.

Smallest possible value of the number

$$= (6)(2) + 5 = 17$$

General form of the number $= k(6 \times 4) + 17 = 24k + 17$ where k is any whole number.

The number would be the largest two-digit number when $24k + 17 < 100$ and k is maximum i.e. $k < 3\frac{11}{24}$ and k is maximum i.e. $k = 3$.

$\therefore$ Largest two-digit number = 89

$\therefore$ Required remainder = 8

Example 15: A number when divided by 3, 5 and 6 successively leaves remainders of 1, 3 and 2 respectively. Find the number of possible values it can assume which are less than 1000.

Solution: Let the quotients obtained when the number is divided by 3, 5 and 6 successively be q_1, q_2 and q_3 respectively.

Number $= 3q_1 + 1$

$q_1 = 5q_2 + 3$

$q_3 = 6q_1 + 2$

$\therefore$ number $= 3(5q_2 + 3) = 1$

$$= 3(5(6q_1 + 2) + 3) + 1 = 90q_1 + 40$$

$$90q_1 + 40 < 1000$$

$$q_1 < 10\frac{2}{3}$$

$\therefore$ q_1 has 11 possibilities i.e., 0 to 10.

Alternative method:

Divisors: 3 × 5 × 6

Remainders: 1 3 2

Smallest possible value of the number

$$= ([(5 \times 2) + 3] \times 3) + 1 = 40$$

General form of the number $= k \times (3 \times 5 \times 6) + 40 = 90k + 40$, where k is any whole number.

If $\quad 90k + 40 < 1000, k < 10\frac{2}{3}$

$\therefore$ k has 11 possibilities (i.e., 0 to 10)

FACTORIAL

Factorial is defined for any positive integer. It is denoted by $\angle$ or !. Thus, 'Factorial n' is written as $n!$ or $\angle n$. $n!$ is defined as the product of all the integers from 1 to n.

Thus $n! = 1.2.3. \ldots (n-1)\, n$.

0! is defined to be equal to 1.

0! = 1 and, 1! is also equal to 1.

Largest Power of a Number in N!

There is a specific model of problems relating to factorial which appeared about 3 to 4 times in CAT papers. This involves finding the largest power of a number contained in the factorial of a given number. Let us understand this type of problem with the help of an example.

Example 16: Find the number of zeros that 324! ends with.

Solution: The largest power of A in $B!$ can be found using the method below when A is composite.

The largest power of each prime factor of A in $B!$ is found. The minimum of these results is the required power.

In the given problem, 10 = (2) (5). The required power is the minimum of the largest power of 2 in 324! and the largest power of 5 in 324!. Using the approach shown in the previous example, the largest power of 2 in 324! is 321. From the previous example, the largest power of 5 in 324! is 78.

$\therefore$ Required power = min (321, 781) = 78.

Alternative method:

Largest power of 10 = Largest power of (2) (5)

As 5 > 2, the largest power of 5 which can divide 324! < the largest power of 2 which can divide 324!

$\therefore$ Largest power of (2) (5) which can divide 324! = largest power of 5 which can divide 324! is 10^{78}.

$\therefore$ 324! ends with 78 zeros.

Some Important Points to Note

Please note the following points also which will be very useful in solving problems on Numbers.

1. When any two consecutive integers are taken one of them is odd and the other is even. Hence the product of any two consecutive integers is always even i.e., divisible by 2.
   Two consecutive integers can be written in the form of n and $n-1$ or n and $n+1$. Hence, any number of the form $n(n-1)$ or $n(n+1)$ will always be even.
2. Out of any 3 consecutive integers, one of them is divisible by 3 and at least one of the three is definitely even. Hence, the product of any 3 consecutive integers is always divisible by 6.
   Three consecutive integers can be of the form $(n-1)$, n and $(n+1)$. The product of 3 consecutive integers will be of the form $(n-1)n(n+1)$ or $n(n^2-1)$ or (n^3-n). Hence any number of the form $(n-1)n(n+1)$ or $n(n^2-1)$ or (n^3-n) will always be divisible by 6.
3. Out of any n consecutive integers, exactly one number will be divided by n and the product of n consecutive integers will be divisible by $n!$
4. Any prime number greater than 3 can be written in the form of $6k+1$ or $6k-1$. The explanation is:
   Let p be any prime number greater than 3. Consider the three consecutive integers $(p-1)$, p and $(p+1)$.
   Since p is a prime number greater than 3, p CANNOT be even. Since p is odd, both $(p-1)$ and $(p+1)$ will be even, i.e., both are divisible by 2.
   Also, since, out of any three consecutive integers, one number will be divisible by 3, one of the three numbers $(p-1)$, p or $(p+1)$ will be divisible by 3. But, since p is prime number – that too greater than $3-p$ cannot be divisible by 3. Hence, either $(p-1)$ or $(p+1)$, one of them – and only one of them – is definitely divisible by 3.
   If $(p-1)$ is divisible by 3, since it is also divisible by 2, it will be divisible by 6, i.e., it will be of the form $6k$. If $(p-1)$ is of the form $6k$, then p will be of the form $(6k+1)$.
   If $(p+1)$ is divisible by 3, since it is also divisible by 2, it will be divisible by 6, i.e., it will be of the form $6k$. If $(p+1)$ is of the form $6k$, then p will be of the form $(6k-1)$.
   Hence any prime number greater than 3 will be of the form $(6k+1)$ or $(6k-1)$.

Example 17: Find the HCF of $\frac{3}{5}, \frac{6}{10}$ and $\frac{9}{20}$.

Solution: To find the LCM or HCF of fractions, first empress all the fractions in their simplest term.

$$\text{HCF (fractions)} = \frac{\text{HCF(numerators)}}{\text{LCM(denominators)}} = \frac{\text{HCF}(3,3,9)}{\text{LCM}(5,5,20)} = \frac{3}{20}$$

Example 18: Find the LCM of $\frac{3}{5}, \frac{6}{10}$ and $\frac{9}{20}$.

Solution: To find the LCM or HCF of fractions, first empress all the fractions in their simplest term.

LCM (fractions)

$$= \frac{\text{LCM(numerators)}}{\text{HCF(denominators)}} = \frac{\text{LCM}(3,3,9)}{\text{HCF}(5,5,20)} = \frac{9}{5}$$

Number Systems

The numbers that are commonly used are the decimal numbers which involve ten symbols, namely 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. If we consider the number 526 in the decimal system, it means $5\times10^2+2\times10^1+6\times10^0$. Likewise, 85.67 means $8\times10^1+5\times10^0+6\times10^{-1}+7\times10^{-2}$. The role played by '10' in the decimal system is termed as the 'base' of the system. In this chapter we see the numbers expressed in various other bases.

Base: It is a number which decides the place value of a symbol or a digit in a number. Alternatively, it is the number of distinct symbols that are used in that number system.

Note:

1. The base of a number system can be any integer greater than 1.
2. Base is also termed as radix or scale of notation.

The following table lists some number systems along with their respective base and symbols.

Number System	Base	Symbols
Binary	2	0,1
Septenuary	7	0,1,2,3,4,5,6
Octal	8	0,1,2,3,4,5,6,7
Decimal	10	0,1,2,3,4,5,6,7,8,9
Duo-decimal	12	0,1,2,3,4,5,6,7,8,9,A,B
Hexa decimal	16	0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

$A=10$, $B=11$, $C=12$, $D=13$, $E=14$, $F=15$. Some books denote ten as 'E' and eleven as 'e'.

Representation: Let N be any integer, r be the base of the system and let $a_0, a_1, a_2, \ldots, a_n$ be the required digits by which N is expressed. Then $N=a_nr^n+a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+\ldots+a_1r+a_0$, where $0\le a_i<r$.

We now look into some representations and their meaning in decimal system.

Examples

1. $(100011)_2$
   $=1\times2^5+0\times2^4+0\times2^3+0\times2^2+1\times2^1+1\times2^0$
   $=32+0+0+2+1=(35)_{10}$
2. $(1741)_8$
   $=1\times8^3+7\times8^2+4\times8^1+1\times8^0$
   $=512+448+32+1=993_{10}$

3. $(A3D)_{16}$
$= A \times 16^2 + 3 \times 16^1 + D \times 16^0$
$= 10 \times 256 + 48 + 13 = 2621_{10}$

CONVERSIONS

1. **Decimal to binary:**
(a) $(253)_{10} = (11111101)_2$
Working:

2	253	
2	126	– 1
2	63	– 0
2	31	– 1
2	15	– 1
2	7	– 1
2	3	– 1
	1	– 1

Note: The remainders are written from bottom to top.

(b) $(37.3125)_{10} = (100101.0101)_2$

Working: The given decimal number has 2 parts:

(i) Integral part 37,

(ii) Fractional part 0.3125.

(i) Conversion of integral part:

2	37	
2	18	– 1
2	9	– 0
2	4	– 1
2	2	– 0
1		– 0

$\therefore (36)_{10} = (100100)_2$

(ii) Conversion of the fractional part:
Multiply the decimal part with 2 successively and take the integral part of all the products starting from the first.

	Binary digits
$0.3125 \times 2 = 0.6250$	0
$0.6250 \times 2 = 1.2500$	1
$0.2500 \times 2 = 0.500$	0
$0.5000 \times 2 = 1.0$	1

$\therefore (0.3125)_{10} = (0.0101)_2$

Note: We should stop multiplying the fractional part by 2, once we get 0 as a fraction or the fractional part is non-terminating. It can be decided depending on the number of digits in the fractional part required.

2. **Binary to decimal:**
(i) $(101011011)_2 = (347)_{10}$
Working: $(101011001)_2$
$= 1 \times 2^8 + 0 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$
$= 256 + 0 + 64 + 0 + 16 + 8 + 0 + 2 + 1$
$= (347)_{10}$

(ii) $(0.11001)_2 = (0.78125)_{10}$
Working: $(0.11001)_2$
$= 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5}$
$= 1/2 + 1/4 + 1/32 = 25/32 = (0.78125)_{10}$

3. **Decimal to octal:**
(i) $(2595)_{10} = (5043)_8$
Working:

8	2595	
8	324	– 3
8	40	– 4
	5	– 0

$\therefore (2595)_{10} = (5043)_8$

4. **Octal to decimal:**
(i) $(4721)_8 = (2513)_{10}$
Working: $(4721)_8$
$= 4 \times 8^3 + 7 \times 8^2 + 2 \times 8^1 + 1 \times 8^0$
$= 2048 + 448 + 16 + 1 = (2513)_{10}$

(ii) $(365.74)_8 = (245.9375)_{10}$
Working:

(a) Integral part:
$(365)_8 = 3 \times 8^2 + 6 \times 8^1 + 5 \times 8^0$
$= 192 + 48 + 5 = 245$
$\therefore (365)_8 = (245)_{10}$

(b) Fractional part:
$(0.74)_8 = 7 \times 8^{-1} + 4 \times 8^{-2}$
$= \frac{56+4}{64} = \frac{60}{64} = 0.9375$
$\therefore (365.74)_8 = (245.9375)_{10}$

5. **Decimal to hexa-decimal:**
(i) $(47239)_{10} = (B887)_{16}$
Working:

16	47239	
16	2952	– 7
16	184	– 8
	11	– 8

Recall: 11 is B, in hexa-decimal system.
$\therefore (47239)_{10} = (B887)_{16}$

(ii) $(30014)_{10} = (753E)_{16}$
Working:

16	30014	
16	1875	– 14 = E
16	117	– 3
	7	– 5

$\therefore (30014)_{10} = (753E)_{16}$

6. **Hexa-decimal to decimal:**
$(52B)_{16} = (1323)_{10}$
Working: $(52B)_{16}$
$= 5 \times 16^2 + 2 \times 16^1 + B \times 16^0$
$= 1280 + 32 + 11 = (1323)_{10}$
$\therefore (52B)_{16} = (1323)_{10}$

7. **Decimal to duo-decimal or duodenary (base 12):**
$(948)_{10} = (66C)_{12}$
Working:

12	948
12	78 – 12 or C
	6 – 6

$\therefore (948)_{10} = (66C)_{12}$

8. **Duo-decimal to decimal:**
$(5BC)_{12} = (864)_{10}$
Working: $(5BC)_{12}$
$= 5 \times 12^2 + B \times 12^1 + C \times 12^0$
$= 720 + 132 + 12 = (864)_{10}$

9. **Binary to octal:**
8 being the base of octal system and 2 being the base of binary system, there is a close relationship between both the systems. One can just club three digits of a binary number into a single block and write the decimal equivalent of each group (left to right).

Example:
(i) $(100101111)_2 = (100)_2\,(101)_2\,(111)_2$
$= (457)_8$
$\therefore (100101111)_2 = (457)_8$
(ii) $(11111110)_2 = (011)_2\,(111)_2\,(110)_2$
$= (376)_8$
$\therefore (11111110)_2 = (376)_8$

Note: Introduce leading zeros to form a block of 3 without changing the magnitude of the number.

10. **Binary to hexa-decimal:**
This is similar to the method disscused for octal; instead of clubbing 3, we club 4 digits.

Example:
$(10111110)_2 = (1011)_2\,(1110)_2 = (11)\,(14)$
$= (BE)_{16}$
$\therefore (10111110)_2 = (BE)_{16}$

Note: If the number of digits is not a multiple of 4, introduce leading zeros as done earlier for octal conversion.

Binary Arithmetic

Addition: Elementary Rules

$0 + 0 = 0$
$0 + 1 = 1$
$1 + 0 = 1$
$1 + 1 = 10$ (1 will be regarded as carry
$1 + 1 + 1 = 11$ as we do in decimal system)

Examples of Binary Addition

1. $(110101)_2 + (110)_2$

```
   1     →  carry
 1 1 0 1 0 1
 0 0 0 1 1 0   (Introduce leading zeros)
 1 1 1 0 1 1
```

2. $(101111)_2 + (111011)_2$

```
   1 1 1 1 1 →  carry
   1 0 1 1 1 1
   1 1 1 0 1 1
 1 1 0 1 0 1 0
```

3. $(110)_2 + (100)_2 + (010)_2$

```
   1  →  carry
   1 1 0
   1 0 0
   0 1 0
   1 1 0 0
```

Subtraction: Subtract 1101 from 11010.

1.

```
          2
     0 0 2 0 2
     1 1 0 1 0
   – 1 1 0 1
result → 1 1 0 1
```

Explanation: Say $N = 11010$,
As 1 cannot be subtracted from 0, we borrow 2 from the next place. This gives $2 - 1 = 1$, as the right most digit of the result. The penultimate digit of N would become 0. A similar calculation gives the 3rd digit of the result from the right as 1 and the 4th digit of N from the right becomes 0.

We now borrow a 2 from the 5th digit of N, this makes the 4th digit of N as 2, thereby resulting in $2 - 1 = 1$ as the 4th digit of the result.

2. Subtract 11011 from 111001

```
   2 2 1
   0 0 2 2 →  Borrow
 1 1 1 0 0 1
–1 1 0 1 1
 1 1 1 1 0
```

Example 19: If $(624)_7 = 312_k$ then find K.

Solution: $(624)_7 = (6)\,(7^2) + (2)\,(7) + (4)\,(7)^0 = 312.$

$(312)_k = 3k^2 + k + 2$

Given $(312)_k = (624)_7$

$$3k^2 + k + 2 = 312$$
$$3k^2 + k - 310 = 0$$
$$(k - 10)(3k + 31) = 0$$
$$k > 0$$
$$\therefore \quad k = 10$$

Example 20: Find the hexadecimal equivalent of the number $(234567)_8$.

Solution: $(234567)_8$
$= (10\ 011\ 100\ 101\ 110\ 111)_2$
$= (0001\ 0011\ 1001\ 0111\ 0111)_2$
$= (1\ 3\ 9\ 7\ 7)_{16}$
$= (13977)_{16}$

Example 21: A non-zero number in base 8 is such that twice the number is the number formed by reversing its digits. Find it.

Solution: Let the number be $(xy)_8$,
where $0 \le x, y < 8$.

The number formed by reversing its digits is $(yx)_8$.

$$2(xy)_8 = (yx)_8$$
$$2(8x + y) = 8y + x$$
$$\frac{x}{y} = \frac{2}{5}$$

$x = 2$ and $y = 5$ is the only possibility.

$$\therefore \quad (xy)_8 = (25)_8.$$

EXERCISE

Practice Problems

Directions for questions 1 to 50: Select the correct alternative from the given choices.

1. If x and y are irrational numbers, then $x + y - xy$ is _____.
(A) a real number (B) a complex number
(C) a rational number (D) an irrational number

2. Which of the following is a prime number?
(A) 851 (B) 589
(C) 429 (D) 307

3. Which of the following pairs of numbers are not twin primes?
(A) 131 and 133
(B) 191 and 193
(C) 157 and 159
(D) More than one of above

4. Which of the following is divisible by 11?
(A) 8787878
(B) 7777777
(C) 1234567
(D) More than one of the above

5. What is the least natural number that should be added to 52341693 so that the sum is a multiple of 8?
(A) 3 (B) 9 (C) 5 (D) 7

6. The product of 7 consecutive natural numbers is always divisible by
(A) 5040 (B) 10080
(C) 3430 (D) 6860

7. How many odd natural numbers have the same parity as their factorials?
(A) 1 (B) 2 (C) 0 (D) 3

8. N is a perfect number. What is the ratio of the sum of the factors of N and N?
(A) 1 (B) 2 (C) 3 (D) 4

9. $0.\overline{255} =$
(A) $\frac{23}{90}$ (B) $\frac{23}{99}$
(C) $\frac{253}{990}$ (D) $\frac{253}{900}$

10. $0.3\overline{21} =$
(A) $\frac{53}{165}$ (B) $\frac{106}{333}$
(C) $\frac{10}{11}$ (D) None of these

11. $0.32\bar{1} =$
(A) $\frac{289}{900}$ (B) $\frac{289}{990}$
(C) $\frac{32}{99}$ (D) $\frac{16}{45}$

12. The least natural number that must be added to 599 so that the sum is a perfect cube is
(A) 120 (B) 125 (C) 130 (D) 135

13. There are 15 consecutive odd numbers. The sum of the first ten of those odd numbers is 200. What is the sum of the last five odd numbers?
(A) 125 (B) 175 (C) 150 (D) 200

14. Find the number of prime factors of 19019.
(A) 1 (B) 2 (C) 3 (D) 4

15. If $N = 2^a \times 3^b \times 5^c$, how many numbers (in terms of N) are less than N and are co-prime to it?

(A) $\frac{2}{15}N$ (B) $\frac{4}{15}N$

(C) $\frac{8}{15}N$ (D) $\frac{2}{5}N$

16. Which of the following numbers is divisible by 40 and 72?

(A) 7560 (B) 3840 (C) 5670 (D) 3780

17. What is the least whole number that should be added to 723111 to make the resultant is a multiple of 11?

(A) 4 (B) 8 (C) 7 (D) 3

18. (a) Prime factorise: 9000

(A) $2^2 \times 3^2 \times 5^2$ (B) $2^4 \times 3 \times 5^2$

(C) $2^3 \times 3^2 \times 5^3$ (D) $2^3 \times 3 \times 5^4$

(b) Prime factorise: 1936

(A) $2^2 \times 3 \times 11^3$ (B) $2^3 \times 11^3$

(C) $2^4 \times 11^2$ (D) $2^2 \times 3^2 \times 11^2$

(c) Write 3969 as a product of prime factors.

(A) $3^5 \times 7$ (B) $3^3 \times 7^3$

(C) $3^4 \times 7^2$ (D) $3^2 \times 7^4$

(d) Write 14553 as a product of prime numbers

(A) $3 \times 7^3 \times 11$ (B) $3^2 \times 7 \times 11^3$

(C) $3^3 \times 7^2 \times 11$ (D) $3 \times 7^2 \times 11^2$

19. Simplify the following:

(a) $248 \times 555 + 148 \times 445$

(A) 203500 (B) 302500

(C) 205300 (D) 305200

(b) $4\frac{1}{2} + 3\frac{1}{5} - 2\frac{1}{10} - 4\frac{1}{20}$

(A) $1\frac{1}{10}$ (B) $1\frac{11}{20}$

(C) $1\frac{1}{5}$ (D) $1\frac{11}{40}$

(c) $\dfrac{(3.37)^3 + 10.11(6.63)^2 + 19.89(3.37)^2 + (6.63)^3}{(3.37)^2 + 2 \times (6.63)(3.37) + (6.63)^2}$

(A) 3.26 (B) 6.74 (C) 10 (D) 8

20. Find the square root of 17689

(A) 143 (B) 137 (C) 133 (D) 147

21. The number of positive integers which are co-prime to 349247 is _____.

(A) 4 (B) 5

(C) 3 (D) infinite

22. The sum of the first N natural numbers is equal to x^2 where x is an integer less than 100. What are the values that N can take?

(A) 1, 9, 27 (B) 1, 7, 26

(C) 1, 8, 48 (D) 1, 8, 49

23. What is the unit's place of $(5^n + 4^{2n} + 7^{4n})^{4n}$?

(A) 4 (B) 8 (C) 2 (D) 6

24. What is the highest power of 5 in 240!?

(A) 58 (B) 17 (C) 116 (D) 39

25. The least possible number which when successively divided by 10, 7 and 6 leaves remainders of 8, 4 and 5 respectively is

(A) 256 (B) 148 (C) 398 (D) 198

26. The LCM and HCF of a pair of numbers is 1232 and 14 respectively. How many such pairs are possible?

(A) 3 (B) 2

(C) 1 (D) None

27. Find the square root of 12345654321.

(A) 1111 (B) 11111

(C) 111111 (D) 1111111

28. There are four prime numbers written in ascending order. The product of the first three prime numbers is 2431 and that of the last three is 4199. Find the greatest of them.

(A) 17 (B) 19 (C) 23 (D) 13

29. Find the minimum number of coins required to pay three persons 69 paise, 105 paise and 85 paise, respectively using coins in the denominations of 2 paise, 5 paise, 10 paise, 25 paise and 50 paise.

(A) 9 (B) 10 (C) 14 (D) 11

30. If a, b and c are prime numbers satisfying $a = b - 2 = c - 4$. How many possible combinations exist for a, b and c?

(A) 4 (B) 3 (C) 2 (D) 1

31. Let p, q and r be distinct positive integers that are odd. Which of the following statements cannot always be true?

(A) $pq^2 r^3$ is odd.

(B) $(p + q)^2 r^3$ is even

(C) $(p - q + r)^2 (q + r)$ is even.

(D) If p, q and r are consecutive odd integers, the remainder of their product when divided by 4 is 3.

32. If $abcde$ is a five-digit number the difference of $abcde$ and $acdbe$ would always be divisible by which of the following for all values of a, b, c, d and e?

(A) 9 (B) 18

(C) 99 (D) Both (A) and (B)

33. Find the value of the expression below

$$\frac{(0.68)^3 + (0.67)^3 - (0.5)^3 + (0.68)(0.67)(1.5)}{(0.68)^2 + (0.67)^2 + (0.5)^2 - (0.68)(0.67) + (0.67)(0.5) + (0.68)(0.5)}$$

(A) 1.85 (B) 0.51 (C) 0.49 (D) 0.85

34. Find the sum of all possible distinct remainders which are obtained when squares of a prime numbers are divided by 6.
(A) 7 (B) 8 (C) 9 (D) 10

35. The least number, which when successively divided by 2, 3 and 7 leaves respective remainders of 1, 2 and 3, is
(A) 56 (B) 130
(C) 68 (D) 23

36. Find the GCD of the numbers p and q where $p = 2^3 . 3^2 . 7^2 . 11^6$ and $q = 2^2 . 3^1 . 5^4 . 11^2 . 13^2$.
(A) 776 (B) 1452
(C) 1164 (D) 2028

37. Which of the following sets of numbers are relative primes?
(a) 57,61
(b) 396,455
(c) 693,132
(d) 6561,1024
(e) 384,352
(A) (c), (e) (B) (a), (b) and (d)
(C) (a), (c), (d) (D) (b), (e)

38. (a) Find the units digit of 8^{173}.
(A) 2 (B) 4 (C) 8 (D) 6
(b) What is the last digit of $518^{163} + 142^{157}$?
(A) 2 (B) 4 (C) 6 (D) 8
(c) Find the last digit of $1567^{143} \times 1239^{197} \times 2566^{1027}$
(A) 2 (B) 3 (C) 4 (D) 6

39. If n is a positive integer, then $43^{5n} - 21^{5n}$ is always divisible by
(A) 11 (B) 18 (C) 25 (D) 64

40. Find the greatest number which when divides 6850 and 2575 leaving respective remainders of 50 and 25.
(A) 425 (B) 850 (C) 1700 (D) 1275

41. Find the least number which when divided by 12, 18 and 33 leaves a remainder of 5 in each case.
(A) 394 (B) 396 (C) 391 (D) 401

42. Find the smallest number that must be added to 1994 such that a remainder of 28 is left when the number is divided by 38 and 57.
(A) 66 (B) 68 (C) 86 (D) 98

43. Find the greatest number which divides 3300 and 3640 leaving respective remainders of 23 and 24.
(A) 13 (B) 113
(C) 339 (D) 226

44. Find the greatest number which divides 68, 140 and 248 leaving the same remainder in each case.
(A) 36 (B) 18 (C) 72 (D) 108

45. Five bells toll at intervals of 5, 6, 10, 12 and 15 seconds respectively. If they toll together at the same time, after how many seconds will they toll together again, for the first time?
(A) 300 (B) 120
(C) 60 (D) 30

46. If three numbers are in the ratio 3 : 4 : 5, and their LCM is 480, then find the sum of the three numbers.
(A) 96 (B) 72 (C) 84 (D) 108

47. If $(121)_8 = (x)_2$, then x =
(A) 101001 (B) 1010011
(C) 1010001 (D) 1011001

48. If $(ACD)_{16} = (x)_{10}$, then x =
(A) 2765 (B) 6725 (C) 5672 (D) 7625

49. Find the digit in the unit's place, in the product of $(25)^7 \times (37)^{12} \times (123)^9$.
(A) 1 (B) 5 (C) 3 (D) 9

50. What is the remainder when 3^{86} is divided by 6?
(A) 2 (B) 3 (C) 4 (D) 0

Answer Keys

Exercise

Practice Problems

1. A	**2.** D	**3.** D	**4.** A	**5.** A	**6.** A	**7.** A	**8.** B	**9.** A	**10.** A
11. A	**12.** C	**13.** B	**14.** D	**15.** B	**16.** A	**17.** C	**18.** (a) C (b) C (c) C (d) C		
19. (a) A (b) B (c) C			**20.** C	**21.** D	**22.** D	**23.** D	**24.** A	**25.** C	**26.** B
27. C	**28.** B	**29.** D	**30.** D	**31.** D	**32.** D	**33.** D	**34.** B	**35.** D	**36.** B
37. B	**38.** (a) C (b) B (c) A			**39.** A	**40.** B	**41.** D	**42.** C	**43.** B	**44.** A
45. C	**46.** A	**47.** C	**48.** A	**49.** B	**50.** B				

Chapter 4

Percentage, Profit and Loss

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Percentage
- Percentage points
- Profit and loss
- Partnerships
- Stocks and shares

PERCENTAGE

'Percent' implies 'for every hundred'. This concept is developed to make the comparison of fractions easier by equalising the denominators of all fractions to hundred.

For example, 7/11 as percentage is represented as $\frac{7}{11}=\frac{7\times100}{11\times100}=\frac{(7\times100)/11}{100}=\frac{63.63}{100}=63.63\%$

Percentages can also be represented as decimal fractions. In such a case it is effectively equivalent to the proportion of the original quantity.

For example, 20% is the same as $\frac{20}{100}$, i.e., 0.2.

Any percentage can be expressed as a decimal fraction by dividing the percentage figure by 100 and conversely, any decimal fraction can be converted to percentage by multiplying it by 100.

PERCENTAGE INCREASE or DECREASE of a quantity is the ratio expressed in percentage of the actual INCREASE or DECREASE of the quantity to the original amount of the quantity, i.e.,

$$\text{PERCENTAGE INCREASE} = \frac{\text{Actual increase}}{\text{Original quantity}}\times100$$

$$\text{PERCENTAGE DECREASE} = \frac{\text{Actual decrease}}{\text{Original quantity}}\times100$$

For example, if the production of rice went up from 225 MT in 1993 to 242 MT in 1994, then the percentage increase in rice production from 1993 to 1994 is calculated as follows:

Actual increase = 242 – 225 = 17 MT

$$\text{Percentage increase}=\frac{\text{Quality increase from 1993 to 1994}}{\text{Actual production of rice in 1993}}\times100$$

$$=\frac{17}{225}\times100=7^{5}/_{9}\%$$

Ratio of any two quantities also can be expressed as percentage.

For example, if the ratio of A and B is 3 : 2, we can say the ratio of A : B is 60% : 40%.

Whenever there is any percentage increase or decrease on a quantity, we can directly calculate the new value of the quantity instead of calculating the actual increase/decrease and then adding to/subtracting from the original quantity.

For example, if the increase on a value of 350 is 15%, the new quantity is 1.15 × 350 = 402.5 (where 1.15 = 1 + 0.15, 0.15 being the decimal equivalent of 15%).

If the production in 1994 is given as 400 MT and the increase from 1993 to 1994 is given to be 25%, then the production in 1993 will be equal to 400/1.25 = 320 MT (where 1.25 = 1 + 0.25, 0.25 being the decimal equivalent of 25%).

Similarly, if there is a decrease of 12% on a quantity of 225, then the new quantity will be equal to 225 × 0.88 (where 0.88 = 1 – 0.12, 0.12 being the decimal equivalent of 12%).

If the production in 1994 is given as 400 MT and it is a decrease of 13% from 1993, then the production in 1993 will be equal to 400/0.87 (where 0.87 = 1 – 0.13, 0.13 being the decimal equivalent of 13%).

On the basis of percentage increase, we can write down how many times the old value gives the new value. For example, if the percentage increase is 100%, then we can conclude that the new value is 2 times the old value. If the percentage increase is 300%, the new value is 4 times the old value. If the percentage increase is 450%, then the new value is 5.5 times the old value. In general, if the percentage increase is p%, then the new value is $\left(\frac{p}{100}+1\right)$ times the old value.

Conversely, if we know how many times the old value gives the new value, we can find out the percentage increase in the old value to get the new value. For example, if the new value is 3 times the old value, the percentage increase in the old value to get the new value is 200%. If the new value is 4.25 times the old value, then the percentage increase is 325%. In general, if the new value is k times the old value, then the percentage increase is $(k-1)\times 100$.

Solved Examples

Example 1: Rice production in a country increased by 25% from 2000 to 2004. It increased by 20% from 2004 to 2008. Find the percentage increase in the rice production from 2000 to 2008.

Solution: Let the rice production in 2000 be 100 tonnes.
Rice production in 2004

$$= 100+\frac{25}{100}(100) = 125 \text{ tonnes}$$

Rice production in 2008

$$= 125+\frac{20}{100}(125) = 150 \text{ tonnes}$$

∴ Percentage increase in the rice production from 2000 to 2008 is $\frac{150-100}{100}\times 100$ i.e., 50%.

Example 2: The population of a country increased by 10% from 2001 to 2002. It increased by 20% from 2002 to 2003. It increased by 30% from 2003 to 2004. Find the simple average yearly percentage increase in the population from 2001 to 2004.

Solution: Let the population of the country in 2001 be 100 million. Population in 2002 = 110 million.
Population in 2003 = 110 × 1.2 = 132 million.
Population in 2004 = 132 × 1.3

$$= 171\times 6 \text{ million}$$

Simple average percentage increase

$$= \frac{171.6-100}{\frac{100}{3}}(100) = 23\frac{13}{15}$$

Example 3: The price of a Swiss watch was ₹10000 in 2001. Due to devaluation of the rupee it becomes ₹12000 in 2002. Find the percentage increase in its price from 2001 to 2002.

Solution: Percentage increase

$$= \frac{\text{Final price} - \text{Initial price}}{\text{Initial price}}(100)$$

$$= \frac{12000-10000}{10000}(100) = 20\%$$

Example 4: 64% of a number is 416. Find 85% of that number.

Solution: Let the number be x

$$\frac{64}{100}x = 416$$

$$x = 650$$

$$\frac{85}{100}x = 552\cdot 5$$

Alternative method:

$$64\% = 416$$

$$85\% = \frac{85}{64}(416) = 552\cdot 5$$

Example 5: The ratio of the salaries of A and B is $2:2\frac{1}{7}$. By what percentage is B's salary greater than A's salary?

Solution: Let the salary of A be $2x$

$$\Rightarrow \quad \text{Salary of } B = ₹2\frac{1}{7}x$$

∴ The salary of B is more than the salary of A by $\frac{\frac{1}{7}x}{2x}(100) = 7\frac{1}{7}\%$

Example 6: The height of a triangle as well as its base are increased by 30%. Find the percentage increase in its area.

Solution: Let the original height as well as the base be 100 cm. Original area

$$= \frac{(100)(100)}{2} = \frac{100^2}{2}$$

New Height = New Base = 130 cm

$$\text{New area} = \frac{(130)(130)}{2}$$

$$= \frac{1}{2}\left((1\cdot 3)(100)^2\right) = \frac{1\cdot 69\left(100^2\right)}{2}$$

$$= 1\cdot 69 \text{ (original area)}$$

∴ the area increased by 69%.

Example 7: If the price of tea goes up by $33\frac{1}{3}\%$, what should be the percentage by which its consumption must be reduced so that the expenditure on it remains unchanged?

Solution: Let the original price be ₹300 per kg and the original consumption be 100 kg. Original expenditure = New expenditure = ₹30000.

$$\text{New price} = 300\left(1+\frac{1}{3}\right) = ₹400$$

$$\text{New consumption} = \frac{30000}{400} \text{ i.e., 75 kg.}$$

Percentage reduction in consumption

$$= \frac{25}{100}(100) = 25\%$$

In the above three examples, if the percentage given initially is x, what is asked to be found is $\frac{100x}{(100+x)}$.

We can generalize each of the three cases as below:

If the value of an item goes up/down by x%, the percentage reduction/increment to be now made to bring it back to the original level is $\frac{100x}{(100 \pm x)}\%$

If A is x% more/less than B, then B is $\frac{100x}{(100 \pm x)}\%$ *less/more than A.*

If the price of an item goes up/down by x%, then the quantity consumed should be reduced/increased by $\frac{100x}{(100 \pm x)}\%$ *so that the total expenditure remains the same.*

Percentage Points

The concept of 'percentage points' is important in the usage of percentages. Percentage points is the difference of two percentage figures.

Let us understand this with an example.

Suppose that rice forms 20% of total food grain production in Year I and 30% of total food grain production in Year II.

If we are asked to find out the percentage increase in the production of rice, calculating percentage increase from 20 to 30 as $\frac{30-20}{20}\times 100$ and saying it is 50% increase is NOT correct. With the available data, we cannot find out the percentage increase in the production of rice from Year I to Year II. We can only say that the production of rice as a percentage of total food grain production went up by 10 PERCENTAGE POINTS (the 10 being the increase from 20 to 30—both percentage figures)

We can see by taking the following figures that the percentage increase in rice production need not be 50%.

	Year I	Year II
Rice	1000	960
Total foodgrains	5000	3200
Rice as percent of Total foodgrains	20%	30%

Here, while rice is 20% of total food grains in Year I and 30% of total food grains in Year II, we find that the actual production of rice has not even increased—it decreased from 1000 in Year I to 960 in Year II.

Profit and Loss

In any business/commercial environment the most important concern is about the profit/loss of the transaction conducted.

The SELLING PRICE (S.P) and the COST PRICE (C.P) of an article determine the profit or loss made on the particular transaction.

The computation is done as follows:

$$\text{Profit} = \text{Sale Price} - \text{Cost Price} = \text{S.P.} - \text{C.P.}$$

$$\text{Percentage Profit} = \frac{\text{S.P.} - \text{C.P.}}{\text{C.P.}}\times 100$$

$$= \frac{\text{Profit}}{\text{C.P.}}\times 100$$

$$\text{Loss} = \text{C.P.} - \text{S.P.}$$

$$\text{Percentage Loss} = \frac{\text{Loss}}{\text{C.P.}}\times 100$$

It is customary to express Profit/Loss as percentage of Cost Price. However, in some problems it may specifically be given that profit/loss percentage has been calculated on the selling price or the student may be asked to calculate the profit/loss percentage on the selling price. Unless such specific directions are given, the profit/loss percentage is always to be calculated on the cost price.

Given Profit/Loss percentage along with S.P., C.P. can be found out and similarly, given Profit/Loss percentage along with C.P., S.P. can be found out by using the concepts discussed at the beginning of this chapter (where, if percentage increase or decrease is given, we can find out the new value from the old value or the old value from the new value).

The following simple rules can be remembered for this purpose.

Given the cost price (C.P.) and profit percentage p%, the selling price will be given by S.P. $= C.P.\times\frac{(100+p)}{100}$

Given the cost price (C.P.) and loss percentage p%, the selling price will be given by $S.P. = C.P. \times \frac{(100-p)}{100}$

Given the selling price (S.P.) and profit percentage p%, the cost price will be given by $C.P. = S.P. \times \frac{100}{(100+p)}$

Given the selling price (S.P.) and loss percentage p%, the cost price will be given by $C.P. = S.P. \times \frac{100}{(100-p)}$

When two articles are SOLD at the same price (i.e., their S.P. is the same) such that there is a PROFIT of $p\%$ on one article and a LOSS of $p\%$ on the other (i.e., common profit or loss percentage), then, irrespective of what the S.P. actually is, the net result of the transaction is LOSS. This percentage loss is given by

$$\text{Loss percentage} = \frac{(\text{Common profit or loss})^2}{100} = \frac{p^2}{100}$$

MARKED PRICE or LIST PRICE is the price that is indicated or marked on the product or it is the price which is given in the price list. This is the price at which the product is intended to be sold. However, there can be some DISCOUNT given on this price and consequently, the actual SELLING PRICE of the product may be less than the MARKED PRICE.

SELLING PRICE = MARKED PRICE – DISCOUNT

The amount of discount given can also be expressed as a percentage. DISCOUNT is always expressed as a percentage of the MARKED PRICE or the LIST PRICE.

$$\text{DISCOUNT percent} = \frac{\text{Marked price} - \text{Selling price}}{\text{Marked price}} \times 100$$

$$= \frac{\text{Discount}}{\text{Marked price}} \times 100$$

Certain discount is given on an article whose selling price is S.P. If further discounts are given on this discounted price, such discounts are referred to as successive discounts. If the successive discounts are $p\%$, $q\%$ and $r\%$, on a product whose selling price is S.P., then the effective price after all the discounts is given by

$$\text{Discounted price} = S.P. \times \frac{(100-p)(100-q)(100-r)}{100 \times 100 \times 100}$$

Example 8: Alok bought a watch for ₹250 and sold it for ₹300. Find his profit percentage.

Solution: Given that

$$\text{S.P} = ₹300 \quad \text{C.P} = ₹250$$

$$\Rightarrow \quad \text{Profit} = \text{S.P} - \text{C.P} = ₹300 - ₹250 = ₹50$$

$$\text{Profit \%} = \frac{50}{250}(100) = 20\%$$

Example 9: Anand gained 20% by selling a book at ₹30. Find his gain percentage if he sells it for ₹36.

Solution: Let his cost price be ₹x

$$\text{Given,} \quad \left(1 + \frac{20}{100}\right)x = 30$$

$$\Rightarrow \quad x = 25$$

Gain percent when sold at ₹36

$$= \frac{36-25}{25}(100) = 44\%$$

Example 10: Ajay calculated his profit/loss percentage on his selling prices. Find his actual profit/loss percentage if he calculated

(i) his profit percentage to be 25%
(ii) his loss percentage to be 25%

Solution: Let his selling price be ₹100

(i) Profit = ₹25 $\Rightarrow$ C.P. = S.P. – Profit
= ₹75
Actual profit percentage
$= \frac{25}{75}(100) = 33\frac{1}{3}\%$

(ii) Loss = ₹25
$\Rightarrow$ C.P. = S.P. + Loss
= ₹125
Actual loss percentage
$= \frac{25}{125}(100) = 20\%$

Example 11: The profit made by selling 5 m of a cloth equals the selling price of 2 m of that cloth. Find the profit percentage made.

Solution: S.P. (5 m) = C.P. (5 m) + Profit (5 m)
As Profit (5 m) = S.P. (2 m),

$$\text{S.P. (5 } m\text{)} = \text{C.P. (5 } m\text{)} + \text{S.P. (2 } m\text{)}$$

$$\Rightarrow \quad \text{S.P. (3 } m\text{)} = \text{C.P. (5 } m\text{)}$$

$$\Rightarrow \quad \frac{\text{S.P.}}{\text{C.P.}} = \frac{5}{3}$$

$$\therefore \quad \text{Profit \%} = \frac{5-3}{3} \times 100 = 66\frac{2}{3}\%$$

Example 12: A trader promised his customers to sell at cost price. But he cheats his customers by giving 100 gms less for every kg that he sells. Find his profit percentage.

Solution: Let the cost of each gm to the trader be ₹1. Cost price of 1000 gms = Selling price of 900 gms.
Cost of 900 gms = ₹900
Selling price of 900 gms = ₹1000

$$\text{Profit percentage} = \frac{1000-900}{900}(100)$$
$$= 11\frac{1}{9}\%.$$

Partnerships

Two or more people can get together to do business by pooling their resources. The money put in by each of the partners is called his 'INVESTMENT' or 'CAPITAL.'

All the people who have invested money in the partnership are called PARTNERS.

While two or more partners would have invested money, it is not necessary that all of them should be involved in the day-to-day running of the business. The partners involved in the day-to-day activities of the business are called 'working partners' and the others are called 'sleeping partners' or 'dormant partners.'

The profits left after paying the working partners' remuneration/commission are shared amongst all the partners.

Sometimes, the partners also take interest on their investments and only the remaining profits are shared by the partners.

Sharing of profits among the partners also depends on the understanding between the partners. However, if no special scheme of sharing the profits is specified (in a problem), then the profits are shared based on the investments of the partners. There are three different possibilities that exist here.

1. If the partners invest DIFFERENT amounts each for the SAME period of time, then the profits at the end of the year are shared in the ratio of their investments.
2. If the partners invest the SAME amounts for DIFFERENT periods of time, then the profits at the end of the year are shared in the ratio of the time periods for which their respective investments have been in business.
3. If the partners invest DIFFERENT amounts and the time periods for which their investments are in the business are also DIFFERENT, then the profits at the end of the year are shared in the ratio of the products of (investment x time period) calculated for each partner.

There CAN be problems that are modelled along the sharing of profits in partnerships. An example of this type is where a particular facility (like renting a tractor for ploughing their fields by three different people) is used by more than one party and the rent has to be shared by all the concerned parties – similar to sharing of profits in a partnership.

Example 13: Sachin started a business with ₹20000 and after 4 months Sunil joined him with ₹40000. Sachin received ₹39000 as his annual profit share which included a salary of 16% of the annual profit. Find the annual profit share of Sachin.

Solution: Let the total annual profit be ₹x
Ratio of the part of the total annual profit which is shared in the ratio of the (investments × time) = (20000) (12) : (40000) (8) = 3 : 4.

Hence the share of Sachin

$$= \frac{3}{7}\left(\frac{84}{100}x\right)+\frac{16x}{100}=39000$$
$$x = 75000$$

∴ Annual profit share of Sachin
= ₹75000 – ₹39000 = ₹36000

Example 14: A started a business with ₹20000. After 3 months, *B* joined him with ₹40000. After some more months, *C* joined them with ₹100000. *B* received ₹18000 out of the total annual profit of ₹55000. How many months after *A* started the business did *C* join?

Solution: Let us say *C* joined after x months.
Profit is shared in the ratio

$$(20000)(12) : (40000)(9) : 100000(12-x)$$
$$= 24 : 36 : 10(12-x)$$

Given $\frac{36}{180-10x} = \frac{18}{55} = \frac{36}{110}$

$$180 - 10x = 110$$
$$\therefore \quad x = 7$$

Example 15: Mohan started a business with ₹20000. After 4 months, Sachin joined him with ₹30000. At the beginning of the fifth month, Mohan added ₹10000. Find the ratio in which the annual profit will be shared.

Solution: Ratio of the profit share
= [(20000) (12) + (10000) (8)] : [30000 (8)] = 4 : 3

EXERCISE

Practice Problems

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Ganesh owns $83\frac{1}{3}\%$ of a property. Three fourths of the part of it he owns is worth ₹5 lakhs. Find the value of the property (in ₹ Lakhs)
(A) 7.2 (B) 8 (C) 6.4 (D) 8.8

2. The salaries of two persons are equal. If the salary of one of them is increased by 20% and the salary of the other is decreased by 20%, find the percentage change in the total salary of the two persons.
(A) 4% increase B) 4% decrease
(C) 0% (D) None of these

3. The price of a TV is decreased by 20%. By what percent must it be increased to bring it back to the original price?

(A) 25% (B) 20%

(C) $16\frac{2}{3}\%$ (D) 15%

4. The ratio of two numbers is 5/6 : 2/3. By what percentage is the second number more/less than the first number?
(A) 20% less (B) 25% more
(C) 25% less (D) 20% more

5. In a test, Mohan's mark was 25% more than Sohan's mark. Mohan got the minimum mark required to pass the test. The pass mark was 35. Find Sohan's mark.
(A) 21 (B) 26
(C) 27 (D) 28

6. In 2004, the price of a shampoo bottle increases by 10% with respect to that in 2003. By what percentage is its price in 2003 less than that in 2004?
(A) 10% (B) $9\frac{1}{11}\%$
(C) 11% (D) $10\frac{1}{11}\%$

7. Due to inflation the total cost of monthly household items has gone up by 20%, but the salary of the family increased by only 10%. Initially, the family used to spend 20% of the salary on household items. What percentage of the present salary should the family spend to buy the same quantities of household items?
(A) 10% (B) $20\frac{2}{11}\%$
(C) $22\frac{2}{11}\%$ (D) $21\frac{9}{11}\%$

8. If the area of a rectangle is increased by 32% and its breadth is increased by 10%, what is the percentage increase in its perimeter?
(A) 10%
(B) 12%
(C) 20%
(D) Cannot be determined

9. *A*'s salary is 20% less than *B*'s salary. If *C*'s salary is ₹10000 and it is 25% more than *B*'s salary, then what is *A*'s salary?
(A) ₹6000 (B) ₹9600
(C) ₹8000 (D) ₹6400

10. School *A* has 30% more students than school *B*. If 120 more students join school *B*, the two schools will have the same number of students. What is the sum of the number of students in school *A* and school *B* initially?
(A) 600 (B) 400 (C) 800 (D) 920

11. The price of petrol increased by 2% in a certain week and increased by 4% in the next week. Find the net percentage increase in the price of petrol over these two weeks.
(A) 6.12% (B) 6.08%
(C) 6.16% (D) 6.20%

12. Rahul got 150 marks in a test. He scored 25% more marks than the pass mark in it. Rajesh got 165 marks in it. By what percent did his mark exceed the pass mark?
(A) 40% (B) 37.5%
(C) 45% (D) 32.5%

13. In a college of total strength 1000, 30% of the students are girls. There are 600 PGs and 120 more male UGs than female UGs. What percent of the males are the female UGs?
(A) 20% (B) 15% (C) 25% (D) 10%

14. A machine costs ₹4,00,000. It depreciates by 18% in value in the first year, 16.5% in the second year, 15% in the third year and so on. Find the amount by which it depreciates in the seventh year (in ₹) (Assume all percentages apply to the original cost of the equipment).
(A) 32000 (B) 28000
(C) 36000 (D) 40000

15. The monthly income of Ram increased by 26%. His expenditure which is 70% of his monthly income increased by 20%. His savings must have increased by
(A) 40% (B) 30% (C) 50% (D) 25%

16. A shopkeeper sells an item for ₹60 at a profit of 20%. At what price (in ₹) should he sell it to gain 30%?
(A) 63 (B) 65 (C) 68 (D) 70

17. The profit made on selling 5 m of a cloth equals the cost price of 2 m of that cloth. Find the profit percentage in selling each m of the cloth.
(A) $66\frac{2}{3}\%$ (B) 50%
(C) 40% (D) $28\frac{4}{7}\%$

18. A company manufactures a product for ₹50. It sold it to a dealer for ₹60. The dealer sold it to a shopkeeper for ₹75. The shopkeeper sold it to a customer for ₹100. Find the profit percentage of the company.
(A) $16\frac{2}{3}\%$ (B) 25%
(C) 20% (D) $33\frac{1}{3}\%$

19. The cost price of 80 articles is ₹12.50 per article. Twenty of them were sold for ₹18 each. At what price should each of the remaining articles be sold so as to get an overall profit of ₹4.50 per article?
(A) ₹15 (B) ₹$16\frac{2}{3}$ (C) ₹$17\frac{1}{3}$ (D) ₹18

20. Rohit marked his goods 40% above his cost price. He sold it after a discount at 12% profit. Find his discount percentage.
(A) 20% (B) 25% (C) 15% (D) 30%

21. Two successive discounts of 30% and 10% are equal to a single discount of
(A) 33% (B) 35% (C) 37% (D) 36%

22. *P* and *Q* started a business in which *P* invested ₹10000 and *Q* invested ₹20000. They received a profit of ₹9600 at the end of a year. Find *Q*'s share in profit (in ₹).
(A) 8000 (B) 6400 (C) 4800 (D) 3200

23. Ramesh and Suresh started a business. Ramesh invested ₹9000 for ten months and Suresh invested ₹6000 for a year. If the profit at the end of a year was ₹4500, find Suresh's share.
(A) ₹3600 (B) ₹2700
(C) ₹2500 (D) ₹2000

24. Kanchan has bought 50 articles. He sells 20% of the articles and makes a profit of ₹1200, which is also equal to the cost of 5 articles. If the selling price for all 50 articles is the same, what is the value of the remaining articles at the selling price?
(A) ₹14400 (B) ₹9600
(C) ₹18000 (D) ₹15000

25. If the discount and profit percentage are both 20% by what percent is the marked price above the cost price?
(A) 40% (B) 50% (C) 60% (D) 70%

26. A car dealer sold a car at a discount of ₹100000. Even after the discount, he made a profit of 15%. What is the marked price of the car, if the marked price is 25% more than the cost price?
(A) ₹1000000 (B) ₹1115000
(C) ₹2500000 (D) ₹1250000

27. Simon gets a discount of 25% on purchasing 100 VCD's from Samuel. He sells them and makes a profit equal to the undiscounted price of 25 VCD's. What is the gain percentage?
(A) 25% (B) 30%
(C) 66.66% (D) 33.33%

28. Ashwin bought an article at ₹200 and marked it at ₹300. He offered a discount and then sold it his profit/loss percentage and discount percentage are in the ratio 3 : 2. Find his profit/loss percentage.
(A) 29% profit (B) 25% profit
(C) 20% loss (D) 25% loss

29. Feroze marks up an article by 30% and sells it at a discount of 20% to Sohail. Sohail marks up the price of the article to a certain amount which happens to be 20% more than Feroze's cost price. What is the maximum discount Sohail can offer without going into loss?
(A) 30% (B) 20%
(C) $16\frac{2}{3}\%$ (D) $13\frac{1}{3}\%$

30. Gopal, Hari and Karthik started a business with investments of ₹8000, ₹12000 and ₹16000 respectively. Hari and Karthik left the business after *x* months. Out of the annual profit share, Gopal got more than Hari but less than Karthik. If *x* is an integer, find the ratio of Gopal's, Hari's and Karthik's shares.
(A) 27 : 21 : 28 (B) 24 : 21 : 28
(C) 30 : 27 : 36 (D) 32 : 30 : 40

Answer Keys

Exercise

Practice Problems

1. B	**2.** C	**3.** A	**4.** A	**5.** D	**6.** B	**7.** D	**8.** D	**9.** D	**10.** D
11. B	**12.** B	**13.** A	**14.** C	**15.** A	**16.** B	**17.** C	**18.** C	**19.** B	**20.** A
21. C	**22.** B	**23.** D	**24.** A	**25.** B	**26.** D	**27.** D	**28.** B	**29.** D	**30.** B

Chapter 5

Simple Interest and Compound Interest

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Interest
- Simple interest
- Compound interest
- Compounding more than once a year
- Present value
- Repayment in equal instalments

Interest

Interest is money paid to the lender by the borrower for using his money for a specified period of time. Various terms and their general representation are as follows:

1. INTEREST
   Money paid by borrower for using the lender's money. Denoted by I.
2. PRINCIPAL
   The original sum borrowed. Denoted by P.
3. TIME
   Time for which money is borrowed. Denoted by n. (n is expressed in number of periods, which is normally one year)
4. RATE OF INTEREST
   Rate at which interest is calculated on the original sum. Denoted by r and is expressed as a percentage or decimal fraction.
5. AMOUNT
   Sum of Principal and Interest. Denoted by A.

Simple Interest

When interest is calculated every year (or every time period) on the original principal, i.e., the sum at the beginning of first year, such interest is called Simple Interest.

Here, year after year, even though the interest gets accumulated and is due to the lender, this accumulated interest is not taken into account for the purpose of calculating interest for latter years.

$$\text{Simple Interest} = \frac{Pnr}{100}$$

where P, n, r are as explained above.

$$\text{Total Amount } A = P + \frac{Pnr}{100} = P\left(1 + \frac{nr}{100}\right)$$

Compound Interest

Under Compound Interest, the interest is added to the principal at the end of each period to arrive at the new principal for the next period.

In other words, the amount at the end of first year (or period) will become the principal for the second year (or period); the amount at the end of second year (or period) becomes the principal for the third year (or period) and so on.

If P denotes the principal at the beginning of Period 1, then,

principal at the beginning of Period 2

$$= P\left(1 + \frac{r}{100}\right)$$

$$= PR = \text{Amount at the end of Period 1,}$$

where $R = \left\{1 + \left(\frac{r}{100}\right)\right\}$

P at the beginning of Period 3 $= P\left(1+\frac{r}{100}\right)^2$

$= PR^2$ = Amount at the end of Period 2

P at the beginning of Period $(n+1)$

$= P\left(1+\frac{r}{100}\right)^n = PR^n$

= Amount at the end of Period n

Hence the amount after n years (periods) $= PR^n = A$

$$\text{Interest} = I = A - P = P\,[R^n - 1]$$

The following table gives an example of how simple interest and compound interest operate, i.e., how the Principal is for various years under simple interest and compound interest. A principal at the beginning of 1st year, of ₹100 and a rate of 10% p.a. are considered. The details are worked out for three years and shown below.

(All figures pertaining to Principal, interest and amount are in Rupees)

	Under Simple Interest				Under Compound Interest			
Year	Principal at the beginn. of the year	Interest for the year	Interest till the end of the year	Amount at the end of the year	Principal at the beginn. of the year	Interest for the year	Interest till the end of the year	Amount at the end of the year
1	100	10	10	110	100	10	10	110
2	100	10	20	120	110	11	21	121
3	100	10	30	130	121	12.1	33.1	133.1

As can be seen from the table,

In case of Simple Interest,

1. The principal remains the same every year.
2. The interest for any year is the same as that for any other year.

In case of Compound Interest,

1. The amount at the end of an year is the Principal for the next year.
2. The interest for different years is not the same.

The compound interest for the first year (where compounding is done every year) is the same as the simple interest for one year.

Compounding More Than Once a Year

We just looked at calculating the amount and interest when the compounding is done once a year. But, compounding can also be done more frequently than once a year. For example, the interest can be added to the principal every six months or every four months and so on.

If the interest is added to the principal every six months, we say that compounding is done twice a year. If the interest is added to the principal every four months, we say that compounding is done thrice a year. If the interest is added to the principal every three months, we say that compounding is done four times a year.

The formula that we discussed above for calculating the amount will essentially be the same, i.e.,

$$\text{Amount} = P\left(1+\frac{r}{100}\right)^n$$

where r = rate % *per annum* and n = number of *years*, but the rate will *not* be for ONE YEAR but for the time period over which compounding is done and the power to which the term inside the bracket is raised (n in the above case) will *not* be the number of years but the number of years multiplied by the number of times compounding is done per year (this product is referred to as the total number of time periods).

For example, if a sum of ₹10000 is lent at the rate of 10% per annum and the compounding is done for every four months (thrice a year), then the amount will be equal to

$$10000\left(1+\frac{10}{3}\times\frac{1}{100}\right)^{2\times3}$$

Here, the dividing factor of 3 in the rate and the multiplying factor of 3 in the power (multiplying the number of years)—both shown by arrow marks—are nothing but the NUMBER OF TIMES compounding is done in a year.

If compounding is done k times a year (i.e., once every 12/k months), at the rate of r% p.a. then in n years, the principal of P will amount to $= P\left(1+\frac{r}{k\cdot100}\right)^{kn}$

When compounding is done more than once a year, the rate of interest given in the problem is called NOMINAL RATE OF INTEREST.

We can also calculate a rate of interest which will yield simple interest in one year equal to the interest obtained under the compound interest at the given nominal rate

of interest. The rate of interest so calculated is called EFFECTIVE RATE OF INTEREST.

If the number of times compounding is done in a year is increased to infinity, we say that the compounding is done EVERY MOMENT and then the amount is given by $P \cdot e^{nr/100}$, where r is the rate % p.a. and n is the number of years.

The following points should also be noted which are helpful in solving problems.

The difference between the Compound Interest and Simple Interest on a certain sum for two years is equal to the interest calculated for one year on one year's Simple Interest.

In mathematical terms, the difference between Compound Interest and Simple Interest for two years will be equal to $P(r/100)^2$, which can be written as $P(r/100)\,(r/100)$. In this $Pr/100$ is the simple interest for one year and when this is multiplied by $r/100$ again, it gives interest for one year on $Pr/100$ i.e., interest for one year on one year's simple interest.

The difference between the Compound Interest for k years and the Compound Interest for (k + 1) years is the interest for one year on the amount at the end of k^{th} year.

This can also be expressed in terms of the amount as follows:

The difference between the amount for k years and the amount for (k + 1) years under compound interest is the interest for one year on the amount at the end of the k^{th} year.

The difference between the Compound Interest for the k^{th} year and the Compound Interest for the $(k + 1)^{th}$ year is equal to the interest for one year on the compound interest for the k^{th} year.

Present Value

Consider a given sum P and a rate of interest r.

We have seen that interest is cost of using the money over a period of time. That means a sum at the beginning of a period is always higher than the same amount after a period greater than or equal to 1.

Let the sum P that is being considered at a rate of interest $r\%$ p.a., becomes Y at the end of Year 1 and Z at the end of Year 2 (i.e., Y and Z are the amounts at the end of first and second years respectively on a principal of P).

Then we can say that what is P today is equal to Y at the end of one year and equal to Z at the end of the second year. In other words, if an amount of Y were to come at the end of one year from now, its value today is equal to P. Similarly, if an amount of Z were to come at the end of two years from now, its value today is equal to P.

So, P is the PRESENT VALUE of Y coming at the end of one year and P is the PRESENT VALUE of Z coming at the end of two years.

Similarly, if we consider n years (or n periods in general), and X is the amount that P will become in n periods, then we say that P is the PRESENT VALUE of X coming at the end of n periods.

If we consider a series of payments Y_1 at the end of first year, Y_2 at the end of second year and so on, the present value of the series of payments will then be equal to the sum of the present values of each of the payments calculated separately. If Z_1 is the present value of Y_1, Z_2 is the present value of Y_2 and so on, then the present value of the series of payments $Y_1, Y_2, \ldots$ is equal to $Z_1 + Z_2 + \ldots$

Present Value can be looked at both under Simple Interest and Compound Interest.

If an amount of Y whose present value is P_1 comes at the end of Year 1, and an amount of Z whose present value is P_2 comes at the end of Year 2, then the present value of both the amounts together will be equal to $(P_1 + P_2)$, i.e., the present value of the stream of payments that come at different points of time is equal to the sum of the present values of the individual amounts coming in at various points of time.

Present Value under Simple Interest: The principal P is amounting to X in n periods. From this we know that

$$X = P\left(1 + \frac{nr}{100}\right)$$

$$\Rightarrow \quad P = \frac{X}{\left(1 + \frac{nr}{100}\right)}$$

Hence, in general, the present value P of an amount X coming (or due) after n periods is given by

$$\boxed{P = \frac{X}{\left(1 + \frac{nr}{100}\right)}}$$

where r is the rate percent per time period.

Present Value under Compound Interest: The principal P is amounting to X in n periods. From this we know that

$$X = P\left(1 + \frac{r}{100}\right)^n$$

$$\Rightarrow \quad P = \frac{X}{\left(1 + \frac{r}{100}\right)^n}$$

Hence, in general, the present value P of an amount X coming (or due) after n periods is given by

$$\boxed{P = \frac{X}{\left(1 + \frac{r}{100}\right)^n}}$$

where r is the rate percent per time period.

Repayment in Equal Instalments—compound Interest

If a sum P borrowed, is repaid in n equal instalments compound interest being calculated at r% per period of instalment, we can find out the value of each instalment. Let us consider the case of n equal ANNUAL instalments (Even if the instalments are not annual, but monthly, the approach will remain the same except that the rate of interest taken should then be the rate per month and not rate per annum).

Let each instalment (i.e., the amount paid at the end of each year) be X.

Instalment X paid after year 1 gives a present value of $\frac{X}{\left(1+\frac{r}{100}\right)}$.

Instalment X, paid at the end of year 2 gives a present value of $\frac{X}{\left(1+\frac{r}{100}\right)^2}$.

Similarly, instalment X paid for nth period (at the end of year n) gives a present value of $\frac{X}{\left(1+\frac{r}{100}\right)^n}$.

The sum of all these present values would be equal to the loan amount P (because only if the amount borrowed is equal to the amount repaid can we say that the loan is repaid).

$$\frac{X}{\left(1+\frac{r}{100}\right)}+\frac{X}{\left(1+\frac{r}{100}\right)^2}+\ldots+\frac{X}{\left(1+\frac{r}{100}\right)^n}=P$$

Call $\frac{1}{\left(1+\frac{r}{100}\right)}=k$

$$\Rightarrow \quad k=\frac{100}{100+r}$$

The above equation can then be rewritten as

$$X\{k+k^2+\ldots+k^n\}=P$$

The terms within the brackets form a G.P. with first term k and common ratio k.

The sum of this G.P. $=\frac{k\,(k^n-1)}{(k-1)}$;

Thus $\frac{X\cdot k\,(k^n-1)}{(k-1)}=P$

$$\Rightarrow \quad X=\frac{P\,(k-1)}{k\left(k^n-1\right)}$$

$$=\frac{\left[P\left\{\frac{100}{100+r}\right\}-1\right]}{\left[\left\{\frac{100}{100+r}\right\}\right]\left[\left\{\frac{100}{100+r}\right\}^n-1\right]}=\frac{P\cdot r}{100\left[1-\left\{\frac{100}{100+r}\right\}^n\right]}$$

$$\text{Each instalment}=\frac{P\cdot r}{100\left[1-\left\{\frac{100}{100+r}\right\}^n\right]}$$

Solved Examples

Example 1: Find the simple interest on a sum of ₹1000 at 10% p.a. for 4 years.

Solution: Simple interest $=\frac{Pnr}{100}$

$$\text{Interest}=\frac{(1000)(4)(10)}{100}=₹400$$

Example 2: A some of ₹4000 becomes ₹4500 in 2 years under simple interest. In how many years will ₹5000 become ₹5625 under simple interest at the same rate of interest?

Solution: Let the rate of interest be R% p.a.
Interest on ₹4000 = ₹500

$$500=(4000)\left(\frac{R}{100}\right)(2)$$

$$R=6.25$$

Interest on ₹5000 = ₹625
Let the required time be T years.

$$625=(5000)\left(\frac{6\cdot 25}{100}\right)T$$

$$\Rightarrow \quad T=2$$

Example 3: Find the value that ₹1000 would amount to under compound interest at 20% p.a., interest being compounded annually in 3 years.

Solution: Amount $=P\left(1+\frac{R}{100}\right)^N$

$$=1000\left(1+\frac{20}{100}\right)^3=₹1728$$

Example 4: Find the sum that would amount to ₹6600 under simple interest in 4 years at 8% p.a.

Solution: Let the sum be ₹P.

Given that $P\left(1+4\left(\frac{8}{100}\right)\right)=6600$

$$P=5000$$

Example 5: If a sum triples in 4 years under simple interest, find the time that it would take to become 5 times itself at the same rate of interest.

Solution: If the sum triples, the interest obtained will be twice the sum. This takes 4 years. If the sum becomes 5 times, the interest must be four times the sum.
∴ This takes a total of 8 years.

Example 6: A sum triples in 4 years under compound interest at a certain rate of interest, interest being compounded annually. Find the time it would take to become 9 times itself.

Solution: The sum triples in 4 years. If it becomes 9 times itself, it has tripled twice.

∴ This takes 8 years.

Let the sum of ₹P, triple in 4 years at R% p.a.

$$\Rightarrow \quad P\left(1+\frac{R}{100}\right)^4 = 3P$$

$$\Rightarrow \quad \left(1+\frac{R}{100}\right)^4 = 3 \qquad (1)$$

Let it take K years to become 9 times.

$$P\left(1+\frac{R}{100}\right)^K = 9P$$

$$\Rightarrow \quad \left(1+\frac{R}{100}\right)^K = 9$$

$$\Rightarrow \quad \left[\left(1+\frac{R}{100}\right)^4\right]^{\frac{K}{4}} = 3^2 \text{ from} \qquad (2),$$

$$3^{\frac{K}{4}} = 3^2$$

$$\Rightarrow \quad \frac{K}{4} = 2$$

$$\therefore \quad K = 8$$

Example 7: If ₹4000 is lent at 10% p.a, interest being compounded annually, find the interest for the fourth year.

Solution: Interest for the fourth year = Amount at the end of the first 4 years − Amount at the end of the first 3 years

$$= 4000\left(1+\frac{10}{100}\right)^4 - 4000\left(1+\frac{10}{100}\right)^3$$

$$= 4000\ (1.4641 - 1.3310)$$

$$= 4000\ (0.1331) \quad \text{i.e. ₹ } 532.40$$

Example 8: The interest on a sum is compounded every 3 months. If the rate of interest is 40% p.a., find the effective rate of interest per annum.

Solution: Let the sum be ₹100

Amount at the end of a year

$$= 100\left(1+\frac{40}{4(100)}\right)^4 = ₹146.41$$

∴ effective rate of interest = 46.41%

Example 9: The compound interest and the simple interest on a sum at certain rate of interest for 2 years are ₹2760 and ₹2400 respectively. Find the sum and the rate of interest.

Solution: Let the sum be ₹P and let the rate of interest be R% p.a.

Difference between the compound interest and the simple interest = ₹360

$$\therefore \quad P\left(\frac{R}{100}\right)^2 = 360 \qquad (5)$$

$$P(2)\left(\frac{R}{100}\right) = 2400$$

$$\Rightarrow \quad \frac{PR}{100} = 1200 \qquad (6)$$

$$\therefore \quad \frac{PR}{100}\left(\frac{R}{100}\right) = 1200\left(\frac{R}{100}\right) = 360$$

$$R = 30$$

Substituting R in (5) or (6),

$$P = 4000$$

Exercise

Practice Problems

Directions for questions 1 to 20: Select the correct alternative from the given choices.

1. Find the amount obtained by investing ₹24,000 at 18% p.a. simple interest for five years

(A) ₹21,600 (B) ₹44,000
(C) ₹45,600 (D) ₹48,000

2. The simple interest for the second year on a certain sum at a certain rate of interest is ₹1000. Find the sum of the interest accrued on it for the 6th, 7th and 8th years.

(A) ₹3200 (B) ₹3000
(C) ₹3300 (D) ₹3630

3. In how many years will a sum of money become sixteen times itself at 30% p.a. simple interest?

(A) 25 (B) 40 (C) 30 (D) 50

4. A sum of money becomes ten times itself at simple interest. If the time period (in years) is numerically equal to the rate of interest, find the annual rate of interest.

(A) 25% (B) 20% (C) 30% (D) 90%

5. An amount of ₹2400 is due after six years under simple interest at 10% p.a. Find its present value (in ₹)

(A) 2000 (B) 1600 (C) 1800 (D) 1500

6. If ₹3000 amounts to ₹3630 in two years under compound interest, interest being compounded annually, what is the annual rate of interest?
(A) 10% (B) 21% (C) 11% (D) 10.5%

7. ₹5000 is invested for two years under compound interest at 10% p.a., interest being compounded annually. Find the interest earned (in ₹)
(A) 500 (B) 1000 (C) 2100 (D) 1050

8. A sum under compound interest, interest being compounded annually amounts to ₹6000 in two years and ₹7200 in three years. Find the rate of interest.
(A) 10% p.a. (B) 20% p.a.
(C) 18% p.a (D) 15% p.a

9. The compound interest on a sum for the third year is ₹2420, interest being compounded annually. The interest on it for the fourth year is ₹2662. Find the rate of interest.
(A) 10% p.a. (B) 11% p.a.
(C) 12% p.a (C) 13% p.a

10. A sum of money becomes four times itself in eight years at compound interest. In how many years will the same sum become sixteen times itself?
(A) 64 (B) 32 (C) 44 (D) 16

11. A sum becomes 2.197 times of itself in three years at compound interest. Find the rate of interest.
(A) 30% (B) 13% (C) 39.9% (D) 235

12. Find the interest (in ₹) earned in the first year on ₹200 at 20% p.a. compound interest, interest compounded every six months.
(A) 40 (B) 42 (C) 44 (D) 48

13. Find the effective rate of interest if the rate of interest is 40% p.a., and the interest is compounded quarterly?
(A) 42% p.a (B) 40% p.a.
(C) 44% p.a (D) 46.41% p.a.

14. Ashok borrowed a total of ₹84000 from two banks at compound interest, interest being compounded annually. One of the banks charged interest at 10% p.a. while the other charged interest at 20% p.a. If Ashok paid ₹13200 as the total interest after a year, find the difference of the sums he borrowed (in ₹).
(A) 24000 (B) 48000
(C) 54000 (D) 12000

15. If the annual rate of simple interest at which a sum is lent for two years increases by 10 percentage points, the interest realized would be ₹4000 more. Find the sum (in ₹).
(A) 20000 (B) 10000 (C) 8000 (D) 16000

16. If a sum was ₹10000 more it would fetch ₹4000 extra as simple interest, if it was lent at a certain rate of interest for two years. Find the annual rate of interest.
(A) 5% (B) 10% (C) 20% (D) 25%

17. A sum was invested under compound interest, interest being compounded annually. It fetches ₹14400 as interest in the second year and ₹17280 as interest in the third year. Find the annual rate of interest.
(A) 10% (B) 15% (C) 20% (D) 25%

18. A sum takes T_1 years to double at $R_1\%$ p.a. simple interest. If it is lent at $R_2\%$ p.a. compound interest, interest being compounded annually, it would take the same time to double. Which of the following is always true if $T_1 > 1$?
(A) $R_1 > R_2$ (B) $0.5R_2 < R_1 < R_2$
(C) $R_1 = R_2$ (D) $R_2/3 < R_1 < R_2$

19. A sum takes two years to become 40% more under simple interest at a certain rate of interest. If it was lent at the same interest rate for the same time under compound interest, interest being compounded annually, it would amount to $x\%$ more than itself. Find x.
(A) 36 (B) 48 (C) 40 (D) 44

20. A sum was divided into two equal parts. One part was lent at 20% p.a. simple interest. The other part was lent at 20% p.a. compound interest, interest being compounded annually. The difference in the interests fetched by the parts in the second year is ₹400. Find the difference in the interests fetched by the parts in the fourth year (in ₹).
(A) 1414 (B) 1442 (C) 1456 (D) 1484

Answer Keys

Exercise

Practice Problems

1. C	**2.** B	**3.** D	**4.** C	**5.** D	**6.** A	**7.** D	**8.** B	**9.** A	**10.** D
11. A	**12.** B	**13.** D	**14.** D	**15.** A	**16.** C	**17.** C	**18.** A	**19.** D	**20.** C

Chapter 6

Averages, Mixtures and Alligations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Averages
- Points to remember
- An easy method to calculate averages
- Weighted average
- Mixtures
- Alligations

Averages

'Average is a very simple but effective way of representing an entire group by a single value.'

'Average' of a group is defined as:

$$\text{Average} = \frac{\text{Sum of all items in the group}}{\text{Number of items in the group}}$$

'Sum of all the items in the group' means 'sum of the values of all the items in the group'.

A batsman's performance can be expressed as the average number of runs scored per innings rather than giving the scores in individual innings. For example, let us say a cricketer scored the following runs in 9 different innings in a year: 35, 56, 124, 29, 0, 87, 98, 45 and 75. Then his average score (per innings) for the year is

$$\frac{35 + 56 + 124 + 29 + 0 + 87 + 98 + 45 + 75}{9} = 61$$

Similarly, if there are 60 students in a class, instead of talking of the height of each individual student, we can talk of 'average' height of the class. The average height of the class of students is equal to the sum of the heights of all the students of the class divided by the number of students in the class.

Average is also called the 'mean' or mean value of all the values.

Points to Remember

1. If the value of each item is increased by the same value p, then the average of the group or items will also increase by p.
2. If the value of each item is decreased by the same value p, then the average of the group or items will also decrease by p.
3. If the value of each item is multiplied by the same value p, then the average of the group or items will also get multiplied by p.
4. If the value of each item is divided by the same value p ($p \neq 0$), then the average of the group or items will also get divided by p.
5. The average of a group of items will always lie between the smallest value in the group and largest value in the group i.e., the average will be greater than the smallest value and less than the largest value in the group.

An Easy Method to Calculate Averages

As already discussed, the average of a group of items whose values are given can be found out by the rule given at the beginning of this section. However, in most of the cases, we do not need to perform such elaborate additions and divisions. The calculation of averages can be simplified greatly by taking some arbitrary number (P) as a starting point, take the deviations (differences) of the given items (Q_i) from this arbitrary number, find the average of all these

deviations ($Q_i - P$) and algebracially add it to the arbitrary number (P) to give the correct average of the given items.

If there are n items and they are denoted by Q_1, Q_2, Q_3, …, Q_n, then the average of these n items is given by

$$\text{Average} = P + \frac{1}{n}\sum_{i=1}^{n}(Q_i - P)$$

For example, the cricketer that we considered above scored the following runs in seven innings: 35, 56, 45, 43, 67, 70 and 48. Now, to find his average, we take an arbitrary figure, say 50 and first find the deviations of each of the scores from this figure. The deviations of the scores from 50 are –15, +6, –5, –7, +17, +20 and –2. The sum of these deviations is +14.

Hence, the average of the cricketer's scores is

$$50 + \frac{14}{7} = 52$$

Please note that the number P (= 50 above) can be any value. Let us work out the same example taking a different value for P. Let us take P equal to 45. The deviations of the scores from P are –10, +11, 0, –2, +22, +25 and +3. The sum of these deviations is 49. Hence, the average is 45 + 49/7 = 45 + 7 = 52.

Weighted Average

When two groups of items are combined together, then we can talk of the average of the entire group. However, if we know only the average of the two groups individually, we cannot find out the average of the combined group of items.

For example, there are two sections A and B of a class where the average height of section A is 150 cm and that of section B is 160 cm. On the basis of this information alone, we cannot find the average of the entire class (of the two sections). As discussed earlier, the average height of the entire class is

$$\frac{\text{Total height of the entire class}}{\text{Total number of students in the entire class}}$$

Since we do not have any information regarding the number of students in the two sections, we cannot find the average of the entire class. Now, suppose that we are given that there are 60 students in the section A and 40 students in section B, then we can calculate the average height of the entire class which, in this case will be equal to $\frac{60\times150+40\times160}{60+40} = 154$ cm.

This average height 154 cm of the entire class is called '**weighted average**' of the class.

The above step in calculating the weighted average of the class can be rewritten:

$$\frac{60\times150+40\times160}{60+40} = \frac{60}{100}150 + \frac{40}{100}160$$

$$= \frac{3}{5}150 + \frac{2}{5}160$$

It is clear from the above step that we would have been able to calculate the average height of the entire class even if we had not been given the number of students in the individual sections but only the **ratio** of the number of students in the two sections (which in this case is 3 : 2).

Even if there are more than two groups of items to be combined, then also the weighted average can be calculated by the same method. For example, if three sections in a class have their average marks as 75, 76 and 79 respectively and their respective strengths are 30, 35 and 35, then the average mark of the entire class is given by

$$\frac{30\times75+35\times76+35\times79}{30+35+35} = 76.75$$

The method of deviations we used for calculating averages can be applied to calculate weighted average also. Here, that method will involve finding out deviations from the arbitrarily chosen number and calculating the weighted average of these deviations. In the above example, if we take 70 as the arbitrary figure, then the deviations of the three observed values given from 70 are +5, +6 and +9. The weighted average of these deviations is

$$\frac{30\times5+35\times6+35\times9}{30+35+35} = \frac{675}{100} = 6.75.$$

Hence, the weighted average will be 70 + 6.75 = 76.75

The arbitrary figure chosen can be any figure and if it is selected, as in the previous case, between the smallest and largest observed figures, some of the deviations will be positive and some negative making the final division relatively simpler. For example, in the above case, if we take with 76 as the arbitrary figure, the deviations are –1, 0 and +3. Then, the weighted average will be

$$\frac{30-(-1)+35-0+35-(+3)}{30+35+35} = \frac{75}{100} = 0.75$$

Hence, the weighted average will be 76 + 0.75 = 76.75

'Weighted Average' can be defined or calculated for any MIXTURE.

Mixtures

Mixing of two or more qualities of things produces a mixture. When two items of different qualities are thus mixed, the quality of the resultant mixture lies in between the qualities of the original constituent items, i.e., it will be higher than the lowest quality and lower than the highest quality of the items being mixed.

In the above example that we took, the 'quality' that we looked at was the height of the students. We could also have taken their weights or the marks scored by them or any other 'quality' or 'parameter' and calculated the 'weighted average' value of that particular 'quality' for the entire group.

Similarly, if two types of a product of different prices per unit are mixed, the unit price of the resultant mixture will lie between the prices of the two types that form the mixture.

Here, the average quality is essentially the **weighted average** of the two constituent items.

If q_1 is the quantity (or number of items) of one particular item of quality p_1, and q_2 be the quantity (or number of items) of the second item of quality p_2 are mixed together to give a new mixture, then the **weighted average** value (p) of the quality of the mixture is given by

$$p = \frac{p_1 q_1 + p_2 q_2}{q_1 + q_2}$$

Even if there are more than two groups of items mixed, the weighted average rule can be applied. We will only have to take figures (as shown in the formula for the two groups) for all the groups in the numerator as well as the denominator and calculate the weighted average. For example, if there are four groups of quantities q_1, q_2, q_3 and q_4 whose respective qualities are p_1, p_2, p_3 and p_4, then the weighted average quality of the group can be written as

$$p = \frac{p_1 q_1 + p_2 q_2 + p_3 q_3 + p_4 q_4}{q_1 + q_2 + q_3 + q_4}$$

A mixture can also be a solution—that is, a liquid mixed with another liquid which is normally water. The concentration of the solution is expressed as the proportion (or percentage) of the liquid in the total solution.

For example, if 10 litres of pure alcohol is mixed with 40 litres of water, then in a total solution of 50 litres, there is 10 litres of alcohol. Hence, the concentration of this solution is 0.2 (= 10/50) or 20%.

Similarly, if 30 litres of pure milk is mixed with 10 litres of water, the concentration of this solution can be expressed as 75% (= 30/40) milk or 25% water.

We can also have two solutions mixed together to give a new solution. Such problems can also be handled in the same manner as other mixtures. In the weighted average rule, the quality of the constituents (p_1, p_2, etc.) will then be the concentrations of various solutions mixed together.

Solved Examples

Example 1: Rajiv purchased three dozen mangoes at ₹10 per dozen, two dozen mangoes at ₹15 per dozen and five dozen mangoes at ₹16 per dozen. Find the average cost per dozen of the mangoes that he purchased.

Solution: The cost of first three dozen mangoes = (3) (10) = ₹30

The cost of next two dozen mangoes = (2) (15) = ₹30

The cost of next five dozen mangoes = (5) (16) = ₹80

Total cost of the mangoes purchased = ₹140.

Average cost per dozen

$$= \frac{\text{Total cost of mangoes}}{\text{Number of dozens}} = \frac{140}{10} = ₹14.$$

Example 2: The average age of 5 men is 20 years. Their average age increased by 1 year when a new man joined them. Find the age of the new man.

Solution: Total age of 5 men = (5) (20) = 100 years

Total age of 6 men = (6) (21) = 126 years

The age of the new man = 126 – 100

i.e. 26 years

Example 3: Six kilograms of wheat costing of ₹18 per kg is mixed with nine kilograms of wheat costing of ₹12 per kg. Find the price per kg of the mixture.

Solution: Total cost of 6 kg = (6) (18) = ₹108.

Total cost of 9 kg = (9) (12) = ₹108.

$$\text{Average cost of the mixture} = \frac{\text{Total cost}}{\text{Total quantity}}$$

$$= \frac{108\,(2)}{15} = ₹\,14.40 \text{ per kg.}$$

Example 4: The average marks of three sections in the tenth class were 90, 120 and 150. If the number of students in these sections are 30, 40 and 50 respectively, find the average mark of the tenth class.

Solution: Total mark of the first section = (90) (30) = 2700

Total mark of the second section

$$= (120)\,(40) = 4800$$

Total mark of the third section = (150) (50) = 7500

$$\text{Average mark of class } X = \frac{\text{Total mark}}{\text{Number of students}}$$

$$= \frac{2700 + 4800 + 7500}{30 + 40 + 50} = \frac{15000}{120} = 125.$$

Example 5: Tarun earned an average of ₹1500 per month from January to April in a year. He earned an average of ₹1600 per month from May to October in that year. His earning in the month of December of that year was ₹300 more than his earning in the month of November of that year. His average monthly earnings for that year was ₹1675. Find his earnings in the month of November.

Solution: Total earnings of Tarun from January to April = (1500) (4) = ₹6000

Total earnings of Tarun from May to October = (1600) (6) = ₹9600.

Let his earnings in November be ₹x

His earnings in December = ₹ (x + 300)

$$\frac{6000 + 9600 + x + x + 300}{12} = 1675$$

$$x = 2100.$$

Example 6: The average age of a group of friends is 37 years. If 6 new friends whose average age is 35 years join them, the average age of the entire group becomes 36 years. How many people were there in the group initially?

Solution: Let the initial number of people in the group be n. The total age of the initial group of friends = $37n$ years

The total age of the six friends who joined the group = 35×6 i.e., 210 years.

Given that, $37n + 210 = 36\,(n + 6)$

$\Rightarrow \quad 37n - 36n = 216 - 210$

$\therefore \quad n = 6.$

ALLIGATIONS

We will take the Weighted Average rule discussed in the previous section and rewrite the formula such that the quantity terms come on one side and the price terms come on the other side. If we do this we get the rule $\frac{q_1}{q_2} = \frac{p - p_2}{p_1 - p}$

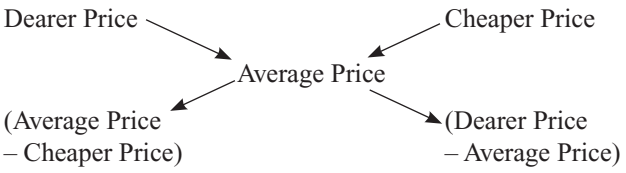
This is called the RULE OF ALLIGATION. This rule connects quantities and prices in mixtures. This can also be written as $\frac{q_1}{q_2} = \frac{p_2 - p}{p - p_1}$

In a descriptive manner, the Rule of Alligation can be written as $\frac{\text{Quantity of Cheaper}}{\text{Quantity of Dearer}}$

$$= \frac{\text{Rate of Dearer} - \text{Average Rate}}{\text{Average Rate} - \text{Rate of Cheaper}}$$

This rule is a very powerful rule and is useful in problems on weighted averages and mixtures. This rule is also useful in a number of problems which can be treated as mixtures and applied to parameters other than price also. We will take examples where alligation rule can be applied.

In actual practice, to apply alligation rule, we do not need to remember the above formula at all. It can be made very simple by representing the rule pictorially. The above formula can be represented as follows:



{We write the dearer and cheaper prices in one line with some gap in between them. Then, we write the average price in between these two but in the line below the line in which dearer and cheaper prices are written. Then, take the differences of quantities as shown along the arrows and write along the same direction of the arrows continued, i.e., in a diagonally opposite corner. The difference between the top left hand quantity (Dearer Price) and Average Price has to be written at the bottom right hand corner. Similarly, the difference between the top right hand corner (Cheaper Price) and the Average Price has to be written at the bottom left hand corner. Now the ratio of the two quantities in the bottommost line will give us the ratio of the quantities of Dearer and Cheaper varieties. Please note that since we took Dearer Price on the top left corner, the ratio of the bottom left figure to that of the bottom right figure will give the ratio of Dearer Quantity to Cheaper Quantity}

Example 7: A bag contains a total of 120 coins in the denominations of 50 p and ₹1. Find the number of 50 p coins in the bag if the total value of the coins is ₹100.

Solution: Let the number of 50 p coins be x.

Total value of the coins = $₹\left[\frac{50x}{100} + 1\,(120 - x)\right]$

$$\frac{50x}{100} + 120 - x = 100$$

$\Rightarrow \quad x = 40$

Alternative method:

Average value per coin = $\frac{(100)(100)}{120}$

$$= \frac{500}{6} \text{ paise.}$$

Using allegation rule, $\frac{x}{120 - x}$

$$= \frac{100 - \frac{500}{6}}{\frac{500}{6} - 50} = \frac{1}{2}$$

$\Rightarrow \quad 2x = 120 - x$

$\Rightarrow \quad x = 40.$

Example 8: A vessel has 60 litres of solution of milk and water having 80% milk. How much water should be added to it to make it a solution in which milk forms 60%?

Solution: Let the quantity of water to be added be x litres. Quantity of milk in the vessel

$$= \left(\frac{80}{100}\right)(60) = 48 \text{ litres}$$

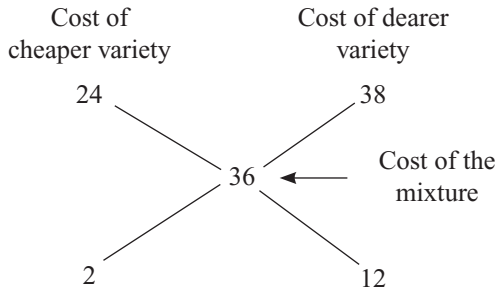
$\Rightarrow \quad 48 = 0.6\,(60 + x)$

$\therefore \quad x = 20$

Example 9: In what ratio must Anand mix two varieties of barley costing ₹24 per kg and ₹38 per kg so that by selling the mixture at ₹40 per kg he would make $11\frac{1}{9}\%$ profit?

Solution: Cost price of the mixture $= \frac{40(100)}{100+11\frac{1}{9}}$

$= ₹36$ per kg



∴ The ratio of the costs of the cheaper to the dearer variety = 2 : 12 = 1 : 6.

If there is P volume of pure liquid initially and in each operation, Q volume is taken out and replaced by Q volume of water, then at the end of n such operations, the concentration (k) of the liquid in the solution is given by $\left\{\frac{P-Q}{P}\right\}^n = k$

This gives the concentration (k) of the liquid as a PROPORTION of the total volume of the solution.

If the concentration has to be expressed as a percentage, then it will be equal to 100 k.

If the volume of the liquid is to be found out at the end of n operations, it is given by kP, i.e., the concentration k multiplied by the total volume P of the solution.

Example 10: A vessel has 400 litres of pure milk. 40 litres of milk is removed from the vessel and replaced by water. 40 litres of the mixture thus formed is replaced by water. This procedure is repeated once again. Find the percentage of milk in the resultant solution.

Solution: Let v litre be volume of milk with a concentration of $c_1\%$.

If x litres of the solution is removed and replaced with water, the new concentration is $\left(\frac{v-x}{v}\times c_1\right)\%$

Given that initial concentration is 100%. (pure milk), $v = 400$, $x = 40$ and the replacement is done thrice.

∴ Concentration of milk in the resultant solution

$$= \left(\frac{400-40}{400}\right)^3 \times 100 = 72.9\%$$

Exercise

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. Find the average of all the two digit numbers divisible by 10.
(A) 40 (B) 50 (C) 45 (D) 60

2. Find the average of all odd numbers less than 50.
(A) 26.5 (B) 25.5 (C) 26 (D) 25

3. Find the average of all the multiples of 12 less than 100.
(A) 48 (B) 54 (C) 60 (D) 66

4. The average salary per month of a man for the first four months, next four months and the last four months of a year are ₹6000, ₹8000 and ₹13000 respectively. Find his average salary per month in that year (in ₹)
(A) 7500 (B) 9000
(C) 10500 (D) 6600

5. In an office there are 20 employees. The average heights of the male employees is 180 cm. The average height of the female employees is 170 cm. Find the average height of all the employees (in cm).
(A) 172
(B) 174
(C) 176
(D) Cannot be determined

6. The average age of the boys in a class is ten years. The average age of the girls in the class is eight years. There are 50% more boys than girls in the class. Find the average age of the class (in years).
(A) 8.4 (B) 8.8 (C) 9.2 (D) 9.6

7. A vessel has 20 litres of a mixture of milk and water having 60% milk. Five litres of pure milk is added to the vessel. Find the percentage of milk in the new solution.
(A) 34% (B) 51%
(C) 68% (D) None of these

8. In what ratio must two kinds of coffee which cost ₹80 per kg and ₹108 per kg be mixed such that the resultant mixture costs ₹96 per kg?
(A) 1 : 2 (B) 2 : 3
(C) 3 : 4 (D) 2 : 1

9. Vessel A has 20 litres of a mixture of milk and water having 75% milk. Vessel B has x litres of a mixture of milk and water having 60% milk. The contents of the vessels are mixed to form a mixture having 66% milk. Find x.
(A) 25 (B) 30 (C) 20 (D) 40

10. A milkman has 15 litres of pure milk. How many litres of water have to be added to it so that he gets a 60% profit by selling at cost price?
(A) 9 (B) 10 (C) 8 (D) 12

11. From 90 litres of pure milk, 9 litres is withdrawn and replaced by water. 9 litres of the mixture is then withdrawn and replaced by water. Find the ratio of milk and water in the present mixture.
(A) 19 : 81 (B) 19 : 100
(C) 81 : 19 (D) 81 : 100

12. Just before the last match in a season, the total number of runs scored by Sachin Tendulkar added up to 2100. In his last match he scored 101 runs. As a result his average score for the season went up by one run. Find the total number of matches he played in that season if he got out in every match.
(A) 31 (B) 5
(C) 71 (D) Either 31 or 71

13. The average weight of all the students of classes I and II equals the average of the average weight of the students of the two classes. There are twice as many students in class II as in class I. The sum of twice the average weight of the students of class I and the average weight of the students of class II is 60 kg. Find the average weight of class I (in kg).
(A) 10 (B) 15 (C) 20 (D) 25

14. Two varieties of wheat are mixed in the proportion of 3 : 4 and the mixture is sold at ₹28 per kg at a profit of 40%. If the second variety of wheat costs ₹3 more than the first variety of wheat, find the cost price of the first variety of wheat.
(A) ₹128/7 per kg (B) ₹120/7 per kg
(C) ₹141/7 per kg (D) ₹149/7 per kg

15. A man buys milk at ₹4 per litre, mixes it with water and sells the mixture at the same price. If his profit is 25%, find the amount of water mixed with each litre of milk.
(A) 0.25 litres (B) 0.5 litres
(C) 0.75 litres (D) 0.6 litres

16. In what proportion can three varieties of sugar priced at ₹10 per kg, ₹12 per kg and ₹18 per kg, be mixed so that the price of the mixture is ₹14 per kg?
(A) 2 : 2 : 5 (B) 2 : 3 : 4
(C) 1 : 3 : 4 (D) 3 : 4 : 5

17. The ratio of alcohol and water in three mixtures of alcohol and water is 3 : 2, 4 : 1 and 7 : 3. If equal quantities of the mixtures are drawn and mixed, the concentration of alcohol in the resulting mixture will be_____.
(A) 65% (B) 70% (C) 75% (D) 80%

18. In what proportion should milk and water be mixed to reduce the cost of litre of milk from ₹18 per litre to ₹16?
(A) 8 : 1 (B) 6 : 1 (C) 10 : 1 (D) 7 : 1

19. A's weight equals the average weight of B, C and D. B's weight equals the average weight of A, C and D. The average weight of C and D is 30 kg. Find the average weight of A and B.
(A) 15 kg (B) 30 kg (C) 60 kg (D) 45 kg

20. Of five numbers, the first number is thrice the third, the fourth number is two less than the first, the fifth number is one-seventh of the second and the second number is three less than thrice the first. Find the fifth number, if the average of the numbers is 16.2.
(A) 3 (B) 4 (C) 5 (D) 6

21. There are nine two-digit numbers with distinct tens digits. The units digit of each number is one less than its tens digit. Find the average of the units digits.
(A) 3 (B) 4 (C) 5 (D) 6

22. A sum of ₹7.75 is made up of 100 coins, which are in the denominations of 5 paise and 10 paise. Find the number of 5 paise coins.
(A) 50 (B) 55 (C) 75 (D) 45

23. A businessman lends ₹1800 in two parts, one at 10% and the other at 12% interest. At the end of the year, the average interest he obtained worked out to be 10.5%. Find the interest earned by the businessman from the part which was lent at 10%.
(A) ₹135 (B) ₹150 (C) ₹200 (D) ₹250

24. A vessel is full of a mixture of milk and water, with 9% milk. Nine litres are withdrawn and then replaced with pure water. If the milk is now 6%, how much does the vessel hold?
(A) 27 litres (B) 18 litres
(C) 36 litres (D) 40 litres

25. Three varieties of rice, A, B and C costing ₹6/kg, ₹9/kg and ₹12/kg are mixed together in a certain ratio. The mixture is sold at $66\frac{2}{3}$% profit for ₹15 / kg. Of the total of 100 kg of the mixture, 50 kg is variety B. Find the quantity of variety A (in kgs)
(A) 15 (B) 25 (C) 20 (D) 10

Answer Keys

Exercise

Practice Problems

1. B	**2.** D	**3.** B	**4.** B	**5.** D	**6.** C	**7.** C	**8.** C	**9.** B	**10.** A
11. C	**12.** D	**13.** C	**14.** A	**15.** A	**16.** D	**17.** B	**18.** A	**19.** B	**20.** D
21. B	**22.** D	**23.** A	**24.** A	**25.** B					

Chapter 7

Time and Work

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Work and time
- Unitary method
- Pipes and cisterns

Work

Work to be done is usually considered as one unit. It may be constructing a wall or a road, filling up or emptying a tank or cistern or eating certain amount of food.

There are some basic assumptions that are made in the problems on Time and Work. These are taken for granted and are not specified in every problem.

1. If a person (or one member of the workforce) does some work in a certain number of days, then we assume (unless otherwise explicitly stated in the problem) that he does the work uniformly, i.e., he does the SAME amount of work everyday.
   For example, if a person can do some work in 15 days, he does 1/15th of the work in one day.
   If a person completes the work in 4 days, he does 1/4th of the work on each day and conversely, if a person can complete 1/4th of the work in one day, he can complete the work in 4 days.
   If a tap can fill a tank in 20 minutes, then in one minute, it can fill 1/20th part of the tank.
2. If there is more than one person (or members of 'workforce') carrying out the work, it is assumed that each person (or members of the workforce), unless otherwise specified, does the same amount of work each day. This means they share the work equally.

If two people together can do the work in 8 days it means that one man can do it in 16 days. This, in turn means, each person can do 1/16th of the work per day.

If a man works three times as fast as a boy does, the man takes one-third of the time the boy takes to complete the work. If the boy takes 12 days to complete the work, then the man takes 4 days to complete the work.

This method is known as '**UNITARY METHOD**', i.e., the time taken per '**Unit Work**' or number of persons required to complete 'Unit Work' or work completed by 'Unit Person' in 'Unit Time', etc., is what is first calculated.

We should recollect the fundamentals on variation (direct and inverse) here.

1. Time remaining constant, Work and Men are directly proportional to each other, i.e., if the work increases the number of men required to complete the work in the same number of days increases proportionately and vice-versa.
2. Work remaining constant, Men and Days are inversely proportional, i.e., if the number of men increases, the number of days required to complete the same work decreases in inverse proportion and vice-versa.
3. The number of workingmen remaining constant, Work and Days are directly proportional i.e., if the work increases, the number of days required to complete the work with the same number of working men also proportionately increases and vice-versa.

The concept of MANDAYS is very important and useful here. The number of men multiplied by the number of days that they take to complete the work will give the number of mandays required to do

the work. The total number of mandays required to complete a specific task will remain a constant. So, if we change one of the variables – men or days – the other will change accordingly so that their product will remain constant (remember from our knowledge of VARIATION, two variables whose product is a constant are said to be inversely proportional to each other). The two variables – men and days – are inversely proportional to each other, when work is constant.

Solved Examples

Example 1: If 15 men take 60 days to complete a job, find the time taken by 45 men to complete it.

Solution: Number of mandays required to complete the job = 900 mandays. Time taken by 45 men to complete it $= \frac{900}{45}$ i.e., 20 days.

Example 2: 18 men take 20 days to complete a job working 12 hours a day. Find the number of days that 15 men will take to complete it if they work 9 hours a day.

Solution: Total time for which 18 men work = 240 hours. Number of man hours required to complete the job = (18) (240) man hours.

Number of days taken by 15 men working 9 hours a day to complete it $= \frac{(18)(240)}{(15)(9)} = 32$.

Hence, in general we can say that

If M_1 men can do W_1 work in D_1 days working H_1 hours per day and M_2 men can do W_2 work in D_2 days working H_2 hours per day (where all men work at the same rate), then

$$\frac{M_1 D_1 H_1}{W_1} = \frac{M_2 D_2 H_2}{W_2}$$

Example 3: 20 men take 10 days to complete a job working 12 hours a day. Find the number of men required to complete a job, twice as large, in 30 days working 8 hours a day.

Solution: Number of man hours required to complete the job = (20) (10) (12) = 2400

Number of men required to complete a job twice as large by 240 hours $= \frac{2400 \times 2}{240}$ i.e., 20 days.

Alternative method:

$M_1 = 20, D_1 = 10, H_1 = 12$

$D_2 = 30, H_2 = 8$

$D_2 = 2W_1$

$$M_2 = \frac{M_1 D_1 H_1 W_2}{W_1 D_2 H_2} = \frac{(20)(10)(12)(2W_1)}{W_1(30)(8)} = 20.$$

If two persons A and B can individually do some work in p and q days respectively, we can find out how much work can be done by them together in one day. Since A can do $1/p$th part of the work in one day and B can do $1/q$th part of the work in one day, the two of them together do $(1/p + 1/q)$ th part of the work in one day.

From this we can find out the number of days that they take to complete the work.

If A can do a piece of work in p days and B can do it in q days then A and B together can complete the same in $\frac{pq}{p+q}$ days.

Example 4: A and B can complete a job in 10 days and 12 days respectively. Find the time taken to complete it, if both A and B work together.

Solution: Time taken by them to complete it

$$= \frac{(10)(12)}{10+12} = \frac{60}{11} \text{ days.}$$

Example 5: A and B together can complete a job in 12 days. A alone can complete it in 24 days. Find the time taken by B to complete it.

Solution: Part of the job that A and B can complete in a day $= \frac{₹1320}{6}$

Part of the job that A can complete in a day $= \frac{1}{24}$

Part of the job that B can complete in a day

$$= \frac{1}{12} - \frac{1}{24} = \frac{1}{24}$$

$\therefore$ B can complete it in 24 days.

Example 6: Ajay and Bala working together can complete a job in 16 days. Ajay alone can complete it in 18 days. Both work together for 4 days and then Bala leaves. Find the time taken by Ajay to complete the remaining work.

Solution: Part of the job that can be done by both in a day $= \frac{1}{16}$

Part of the job that can be done by them in 4 days $= 4 \times \frac{1}{16} = \frac{1}{4}$

Remaining part of the job $= \frac{3}{4}$

Time taken by Ajay to complete it

$$= \frac{3}{4}(18) = 13.5 \text{ days.}$$

Example 7: A can complete a job in 16 days. He started the work and after 4 days, B joined him. They completed the job in 4 more days. Find the number of days in which B alone can complete it.

Solution: Part of the job done by A in a day $= \frac{1}{16}$
A worked for a total of 8 days

$\therefore$ A completed $(8)\left(\frac{1}{16}\right) = \frac{1}{2}$ of the job.

Hence, B can complete the remaining $\frac{1}{2}$ of the job in 4 days.

$\therefore$ B alone can complete the entire job in 8 days.

Example 8: P and Q together can complete a job in $14\frac{2}{5}$ days. Q and R together can complete it in $20\frac{4}{7}$ days. P and R together can complete it in 16 days. Find the time taken by each of them to complete the job.

Solution: Part of the job that P and Q can do in a day $= \frac{5}{72}$
Part of the job that Q and R can do in a day $= \frac{7}{144}$

Part of the job P and R can do in a day $= \frac{1}{16}$

Let the time taken by P, Q and R to complete the job be p days, q days and r days respectively.

$$\frac{1}{p}+\frac{1}{q}=\frac{5}{72} \quad (5)$$

$$\frac{1}{q}+\frac{1}{r}=\frac{7}{144} \quad (6)$$

$$\frac{1}{p}+\frac{1}{r}=\frac{1}{16} \quad (7)$$

Adding (5) and (6) and subtracting (7),

$$\Rightarrow \quad \frac{2}{q}=\frac{5}{72}+\frac{7}{144}-\frac{1}{16}=\frac{8}{144}$$

$$q = 36$$

substitute $q = 36$, in (5),
we get $p = 24$
substi $q = 36$ in (6),
we get $r = 48$.

Example 9: A contractor decided to complete a job in 30 days for which he employed 20 men in the beginning. After 10 days he released that the job could not be completed on time. Hence, he employed 15 more men and thus completed the job on time. Find the number of extra days it would have taken to complete the job if the additional men were not employed.

Solution: Number of mandays required to complete the job $= (20)(10) + (20 + 15)(20) = 900$ mandays.

If the additional men were not employed, number of extra days $= \frac{900}{20} - 30 = 15$ days.

Example 10: A and B together can complete a job in 18 days and 36 days respectively. They work on alternate days with A starting the job. In how many days will the job be completed?

Solution: Part of the job completed in the first 2 days

$$= \frac{1}{18}+\frac{1}{36}=\frac{1}{12}$$

$\therefore$ To complete the job, 12 cycles of 2 days i.e., a total of 24 days will be required.

Example 11: P and Q together can complete a job in 8 days and 16 days respectively. They work on alternate days with Q starting the job. In how many days will the job be completed?

Solution: Part of the job completed by P and Q in the first 2 days $= \frac{1}{8}+\frac{1}{16} = \frac{3}{16}$

After 5 cycles of 2 days, i.e., after 10 days, $\frac{15}{16}$th of the job will be completed.

Remaining part $= \frac{1}{16}$th. Q will work on the 11th day and he takes exactly one day to complete the remaining part.

$\therefore$ The job will be completed in 11 days.

In general, money earned should be shared by people doing the work together in the ratio of the **SHARE OF WORK** done by each of them.

For example, if A does two-fifth of the work, then he should get two-fifth of the total earnings for the work. If the remaining three-fifth of the work is done by B and C in the ratio of 1 : 2, then the remaining three-fifth of the earnings (after paying A) should be shared by B and C in the ratio of 1 : 2. Suppose ₹500 is paid to A, B and C together for doing the work, then A will get ₹200 (which is 2/5 of ₹500), B will get ₹100 and C, ₹200 (because the remaining ₹300 after paying A is to be divided in the ratio 1 : 2 between B and C).

When people work for the same number of days each, then the ratio of the total work done will be the same as the work done by each of them PER DAY. Hence, if all the people involved work for the same number of days, then the earnings can directly be divided in the ratio of work done per day by each of them.

Example 12: P, Q and R can together earn ₹3100 in 10 days. Q and R together can earn ₹1320 in 6 days. P and R together can earn ₹1050 in 5 days. Find R's daily earning.

Solution: Total daily wage of P, Q and $R = \frac{₹3100}{10} = ₹310$

Total daily wage of Q and $R = \frac{₹1320}{6} = ₹220$

Total daily wage of P and $R = \frac{₹1050}{5} = ₹210$

Total daily wage of P, Q and $2R = ₹430$

$\therefore$ R's daily wage = ₹120.

Example 13: Two men under take a job for ₹960. They can complete it in 16 days and 24 days. They work along with a third man and take 8 days to complete it. Find the share that the third man should get.

Solution: The amount payable should be proportional to the fraction of work done.

Part of the job done by the third man

$$= 1-\left(\frac{8}{16}+\frac{8}{24}\right)=\frac{1}{6}.$$

$\therefore$ The third man should get $\frac{₹960}{6}=₹160.$

Pipes and Cisterns

There can be pipes (or taps) filling (or emptying) tanks with water. The time taken by different taps (to fill or empty the tank) may be different. Problems related to these can also be dealt with in the same manner as the foregoing problems on Work have been dealt with.

There is only one difference between the problems on regular Work (of the type seen earlier on in the chapter) and those in Pipes and Cisterns. In Pipes and Cisterns, a filling pipe or tap does positive work and an emptying pipe or a leak does negative work.

Example 14: Pipes P and Q take 24 minutes and 36 minutes respectively to fill an empty tank. If both take 18 minutes to fill a tank along with an outlet pipe R, find the time R would take to empty the full tank.

Solution: Let the time taken by R to empty the tank be r minutes.

$$\frac{1}{24}+\frac{1}{36}-\frac{1}{r}=\frac{1}{18};\ r=72.$$

Example 15: Pipes X and Y can fill a tank in 30 minutes and 60 minutes respectively. Both pipes are opened simultaneously. After how much time should X be closed so that the tank is filled in 30 minutes?

Solution: Let us say pipe X should be closed after n minutes.

i.e. pipe X is in operation for n minutes and pipe Y for all the 30 minutes.

So, $$\frac{n}{30}+\frac{30}{60}=1$$

$\Rightarrow$ $$n=15.$$

Example 16: Pipes P, Q and R together can empty a full tank in 6 hours. All the three pipes are opened simultaneously and after 2 hours, P is closed. The tank is emptied in another 6 hours. Find the time in which P can empty the tank.

Solution: Part of the tank that can be emptied by P, Q and R per hour $=\frac{1}{6}$.

Part of the tank that was emptied by P, Q and R in 2 hours $=\frac{1}{3}$

Part of the tank which was emptied by Q and R per hour

$$=\frac{1-\frac{1}{3}}{6}=\frac{1}{9}$$

Time in which P can empty the tank

$$=\frac{1}{\frac{1}{6}-\frac{1}{9}}$$ i.e. 18 hours.

Example 17: A tank has a leak at its bottom which empties it at 6 litres/minutes. It also has a filling tap which can fill the tank in 6 hours. The tank takes 18 hours to become full. Find the capacity of the tank.

Solution: Let the time that would be taken by the leak to empty the full tank be x hours.

$\therefore$ $$\frac{1}{6}-\frac{1}{x}=\frac{1}{18};\ x=9$$

$\therefore$ Capacity of the tank = (6) (9) (60)

= 3240 litres.

Exercise

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. X men can complete a work in 120 days. If there were 10 men more, the work would be completed in 20 days less. Find the value of X.

(A) 75 (B) 50 (C) 90 (D) 60

2. Nine men can complete a job in 15 days. If a man works thrice as fast as a woman, find the number of days taken by 15 women to complete the job.

(A) 20 (B) 24 (C) 27 (D) 36

3. The ratio of the time taken by A, B and C to complete a job is 3 : 4 : 6. Find the ratio of the work they can complete in an hour.

(A) 6 : 4 : 3 (B) 4 : 3 : 2
(C) 2 : 3 : 4 (D) 3 : 4 : 6

4. Amar, Bharat and Charu can complete a job in 12, 24 and 24 days respectively. If they all work together, how long will they take to complete the same work?

(A) 18 days
(B) 6 days
(C) 20 days
(D) 16 days

5. Adam can complete a job in 25 days. Adam and Chris together can complete it in $9\frac{3}{8}$ days. In how many days can Chris alone complete the job?
(A) $12^5/_8$ (B) 10 (C) 25 (D) 15

6. P and Q can complete a job in 10 days. Q and R can complete it in 12 days. P and R can complete it in 20 days. Who is the slowest of the three workers?
(A) P
(B) Q
(C) R
(D) Cannot be determined

7. Ten men can do a piece of work in 15 days. How many men are needed to complete a work which is five times as large as the first one, in 10 days?
(A) 60 (B) 75 (C) 70 (D) 85

8. Tap X can fill a tank in 10 hours. Tap Y can fill it in 15 hours. If the two taps fill the tank together, what fraction of the tank is filled by X?
(A) 1/10 (B) 1/6 (C) 2/3 (D) 3/5

9. Pipe A can fill an empty tank in 9 hours. Pipe B can empty a full thank in 18 hours. If both pipes are opened simultaneously when the tank is empty, find the time taken to fill the tank. (in hours)
(A) 24 (B) 27 (C) 18 (D) 36

10. Raj can build a wall in 18 days and Kiran can do the same in 30 days. After Raj had built half the wall, Kiran joins him. What is the total number of days taken to build the wall?
(A) 24 (B) $14\frac{5}{8}$ (C) $15\frac{1}{2}$ (D) $16\frac{1}{2}$

11. Kaushik is one and a half times more efficient than Ravi. Kaushik can do a piece of work in 20 days. What portion of the total work can both of them together complete in 10 days?
(A) 3/10 (B) 4/5 (C) 9/10 (D) 7/10

12. Had then been one men less, then the number of days required to do a piece of work would have been one more. If the number of mandays required to complete the work is 56, how many workers were there?
(A) 6 (B) 8 (C) 9 (D) 14

13. In 8 days, Peter can do as much work as Pan can do in 12 days. To do a certain job both together take 36 days. In how many days can Pan, working alone, complete the job?
(A) 60 days (B) 80 days
(C) 108 days (D) 90 days

14. X can complete a job in 36 days and Y can complete it in 45 days. Z can complete the job in z days. Z started the job. After 28 days, X and Y joined. The job was completed in 4 more days. Find z.
(A) 40 (B) 35 (C) 30 (D) 50

15. Working in pairs, PQ, QR and RP can complete a job in 24 days, 20 days and 30 days respectively. Find the respective times taken by P, Q and R individually to complete the same job (in days).
(A) $48, 80, \frac{240}{7}$ (B) $80, 48, \frac{240}{7}$
(C) $80, \frac{240}{7}, 48$ (D) $48, \frac{240}{7}, 80$

16. A frog was at the bottom of a 80 m deep well. It attempted to come out of it by jumping. In each jump it covered 1.15 m but slipped down by 0.75 m. Find the number of jumps after which it would out of the well.
(A) 198 (B) 201 (C) 200 (D) 199

17. A man, a woman and a boy can do a piece of work in 2, 4 and 8 days respectively. How many boys must work together with one man and one woman to complete the work in one day?
(A) 5 (B) 4 (C) 2 (D) 1

18. A machine of type A which has to produce a set of 1500 bolts, can do so in 30 days. The machine breaks down after 10 days. A machine of type B completes the remaining work in 10 days. In 30 days how many bolts can both of them together produce?
(A) 3,000 (B) 4,500 (C) 6,000 (D) 2,500

19. In a farm, each cow eats twice as much grass as each sheep. The cost of grass for 10 cows and 40 sheep for 20 days is ₹ 900. Find the cost of grass for 20 cows and 10 sheep for 18 days (in ₹).
(A) 600 (B) 675 (C) 750 (D) 800

20. The cost of grass for 20 cows and 30 sheep for 30 days is ₹720. If the 30 sheep eat double the grass eaten by the 20 cows, then what is the cost of grass eaten by 20 sheep in 15 days?
(A) ₹200 (B) ₹160 (C) ₹240 (D) ₹100

21. George and Gagan together repair a bridge in 45 days and receive ₹13,500. If Gagan is three times as efficient as George, what is the amount of money he earns in 10 days?
(A) ₹2,000 (B) ₹2,250 (C) ₹2,500 (D) ₹2,750

22. Two pipes A and B which can fill a tank in 20 and 30 hours respectively were opened simultaneously. But there was a leak and it took 3 hours more to fill the tank. In how many hours can the leak empty the tank?
(A) 60 (B) 50 (C) 30 (D) 40

23. Gokul, Govardhan and Ganesh can do a piece of work in 10, 20 and 30 days respectively. They begin a new job of similar nature and each of them works on it for one third of the total period of work. If they get ₹6,600 for the new job, how much should Govardhan get, given that the amounts distributed are in proportion to the work done by them?
(A) ₹1,800 (B) ₹2,200 (C) ₹3,300 (D) ₹2,400

24. Rakesh and Ramesh take 30 days and 60 days respectively to complete a job. They work on alternate days to complete it with Rakesh starting the job. Find the time in which the job is completed (in days).
(A) 60 (B) 80 (C) 40 (D) 90

25. If Rakesh and Ramesh had instead taken 10 days and 12 days respectively to complete the job, find the time in which the job would have been completed (in days).

(A) $10\frac{1}{3}$ (B) $10\frac{5}{6}$

(C) 11 (D) $10\frac{1}{2}$

Answer Keys

Exercise

Practice Problems

1. B	**2.** C	**3.** B	**4.** B	**5.** D	**6.** C	**7.** B	**8.** D	**9.** C	**10.** B
11. D	**12.** B	**13.** D	**14.** A	**15.** C	**16.** D	**17.** C	**18.** B	**19.** B	**20.** B
21. B	**22.** A	**23.** A	**24.** C	**25.** B					

Chapter 8

Time and Distance

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- General problems on time, speed and distance
- Speed
- Relative speed
- Average speed
- Boats and streams
- Races and circular tracks

Introduction

In this chapter, we will look at problems in the following different areas:

1. General problems on Time, Speed and Distance
2. Relative Speed
3. Boats and Streams
4. Races and Circular Tracks

Before we look at problems in various areas, let us first look at some basic concepts pertaining to speed, time and distance.

Speed

Distance covered per unit time is called speed.

i.e., Speed = Distance/time

The above relationship between the three variables distance, speed and time can also be expressed as follows:

Distance = Speed × Time or Time = Distance/Speed

1. If two bodies travel with the same speed, distance covered ∝ time (Direct Variation).
2. If two bodies travel for the same period of time, distance covered ∝ speed (Direct Variation).
3. If two bodies travel the same distance,

$$\text{Time} \propto \frac{1}{\text{Speed}} \quad \text{(Inverse Variation)}$$

Distance is normally measured in kilometres, metres or miles; time in hours or seconds and speed in km/hr (also denoted by kmph), miles/hr (also denoted by mph) or metres/second (denoted by m/s).

To convert speed in kmph to m/sec, multiply it with 5/18.

To convert speed in m/sec to kmph, multiply it with 18/5.

In the case of moving trains, three different situations need to be considered.

When a train passes a stationary point, the distance covered (in the passing) is the length of the train. If the train is crossing a platform (or a bridge), the distance covered by the train (in the crossing) is equal to the length of the train plus the length of the platform (or bridge). If two trains pass each other (travelling in the same direction or in opposite directions), the total distance covered (in the crossing or the overtaking, as the case may be) is equal to the sum of the lengths of the two trains.

Average Speed

Average speed of a body travelling at different speeds is defined as follows:

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

Please note that the **Average speed** of a moving body is **Not equal** to the **Average of the speeds.**

If a body travels from point A to point B with a speed of p and back to point A (from point B) with a speed of q, then the average

speed of the body can be calculated as $\mathbf{2pq/(p+q)}$. Please note that this does not depend on the distance between A and B.

If a body covers part of the journey at speed p and the remaining part of the journey at speed q and the distances of the two parts of the journey are in the ratio $m : n$, then the average speed for the entire journey is $\mathbf{(m+n)pq/(mq+np)}$.

Solved Examples

Example 1: Ashok covered a certain distance at a certain speed. If his speed was 20% more, he would take 10 minutes less to cover the same distance. Find the time he takes to cover the distance.

Solution: If his speed was 20% more, it would be 120% i.e., $\frac{6}{5}$ times his actual speed.

$\therefore$ Time taken would be $\frac{5}{6}$ times his actual time.

$$\text{Reduction in time} = \frac{1}{6}(\text{actual time}) = 10 \text{ minutes}$$

$\therefore$ Actual time = 60 minutes.

Example 2: A car covered a certain distance at 90 kmph and returned back at 60 kmph. Find its average speed for the entire journey.

Solution: Let x km be the distance to be covered, each way.
Total time of travel (in hours)

$$= \frac{x}{90} + \frac{x}{60} = \frac{5x}{180} = \frac{x}{36}$$

Average speed (in km/hr)

$$= \frac{\text{Total disance travelled}}{\text{Total time taken}} = \frac{2x}{\frac{x}{36}} = 72$$

Example 3: Find the time taken by a train, 100 m long, travelling at a speed of 63 kmph to cross a platform of length 250 m.

Solution: Speed of the train $= (63)\left(\frac{5}{18}\right) = \frac{35}{2}$ m/sec

Distance to be travelled by the train to cross the platform = length of the train + length of the platform.

Time taken to cross the platform

$$= \frac{100+250}{\frac{35}{2}} = 20 \text{ seconds}$$

Example 4: Ashok would reach his office 15 minutes early if he walked at 4 kmph from his house. He would reach it 45 minutes late if he walked at 3 kmph from his house. Find the distance between his house and office.

Solution: Let the distance be x km. Time taken by Ashok if he walked at 4 kmph = $\frac{x}{4}$ hours.

Time taken by Ashok if he walked at 3 kmph = $\frac{x}{3}$ hours.

In this case he would take one hour more to reach his office compared to the time taken if he had walked at 4 kmph.

$$\therefore \quad \frac{x}{3} - \frac{x}{4} = 1$$

$$\Rightarrow \quad x = 12.$$

In general, if a person travelling between two points reaches p hours late travelling at a speed of u kmph and reaches q hours early travelling at v kmph, the distance between the two points is given by $\frac{vu}{v-u}(p+q)$.

Relative Speed

The speed of one (moving) body in relation to another moving body is called the relative speed of these two bodies, i.e., it is the speed of one moving body as observed, from the second moving body.

If two bodies are moving in the same direction, the relative speed is equal to the difference of the speeds of the two bodies.

If two bodies are moving in opposite directions, the relative speed is equal to the sum of the speeds of the two bodies.

Example 5: Find the time taken by a train 175 m long running at a speed of 54 kmph to overtake another train 75 m long running at a speed of 36 kmph.

Solution: Relative speed = 18 kmph = 5 m/sec
Time taken for the faster train to overtake the slower train

$$= \frac{(\text{Length of the faster train}) + (\text{Length of the slower train})}{\text{Their relative speed}}$$

$$= \frac{175+75}{5} = 50 \text{ seconds}$$

Example 6: A train overtakes two persons, cyling at 9 kmph and 18 kmph in 40 seconds and 48 seconds respectively. Find its length and speed.

Solution: Let the length and the speed of the train be l m and s kmph respectively.

$$\ell = 40(s-9)\frac{5}{18} = 48(s-18)\frac{5}{18}$$

$$\Rightarrow \quad \frac{s-9}{s-18} = \frac{48}{40}$$

$$\Rightarrow \quad s = 63$$

$$\therefore \quad \ell = 40\,(63-9) \times \frac{5}{18} = 600\ m$$

Boats and Streams

Problems related to boats and streams are different in the computation of relative speed from those of trains/cars.

When a boat is moving in the same direction as the stream or water current, the boat is said to be moving **WITH THE STREAM OR CURRENT.**

When a boat is moving in a direction opposite to that of the stream or water current, it is said to be moving **AGAINST THE STREAM OR CURRENT.**

If the boat is moving with a certain speed in water that is not moving, the speed of the boat is then called the **SPEED OF THE BOAT IN STILL WATER.**

When the boat is moving upstream, the speed of the water opposes (and hence reduces) the speed of the boat.

When the boat is moving downstream, the speed of the water aids (and thus adds to) the speed of the boat. Thus, we have

Speed of the boat against stream = Speed of the boat in still water – Speed of the stream

Speed of the boat with the stream = Speed of the boat in still water Speed of the stream

These two speeds, the speed of the boat against the stream and the speed of the boat with the stream, are RELATIVE SPEEDS.

If u is the speed of the boat down the stream and v is the speed of the boat up the stream, then we have the following two relationships.

$$\text{Speed of the boat in still water} = (u + v)/2$$

$$\text{Speed of the water current} = (u - v)/2$$

In problems, instead of a boat, it may be a swimmer but the approach is exactly the same. Instead of boats/swimmers in water, it could also be a cyclist cycling against or along the wind. The approach to solving the problems still remains the same.

Example 7: A boat travels 30 km upstream in 5 hours and 100 km downstream in 10 hours. Find the speed of the boat in still water and the speed of the stream.

Solution: Upstream speed $= \frac{30}{5} = 6$ kmph

Downstream speed $= \frac{100}{10} = 10$ kmph

Speed in still water $= \frac{6+10}{2} = 8$ kmph

Speed of the stream $= \frac{10-6}{2} = 2$ kmph

Example 8: Anand can row 20 km in 2 hours in still water. If the speed of the stream is 6 kmph, he would take 3.75 hours to cover a round trip journey. Find the distance that he would then cover each way.

Solution: Speed of the boat in still water $= \frac{20}{2} = 10$ kmph
Let the total distance covered be $2x$ km.

Given that, $\frac{x}{10+6} + \frac{x}{10-6} = 3 \cdot 75$

$$x = 12$$

Races and Circular Tracks

When two persons P and Q are running a race, they can start the race at the same time or one of them may start a little later than the other. In the second case, suppose P starts the race and after 5 seconds, Q starts. Then we say P has a 'start' of 5 seconds. Alternatively, in a race between P and Q, P starts first and then when P has covered a distance of 10 metres, Q starts. Then we say that P has a 'start' of 10 metres.

In a race between P and Q where Q is the winner, by the time Q reaches the winning post, if P still has another 15 metres to reach the winning post, then we say that Q has won the race by 15 metres. Similarly, if P reaches the winning post 10 seconds after Q reaches it, then we say that Q has won the race by 10 seconds.

In problems on RACES, we normally consider a 100 m race or a 1 km race. The length of the track.

NEED NOT necessarily be one of the two figures mentioned above but can be as given in the problem.

When two or more persons running around a circular track (starting at the same point and at the same time), then we will be interested in two main issues:

1. When they will meet for the first time and
2. When they will meet for the first time at the starting point

To solve the problems on circular tracks, you should keep the following points in mind.

When two persons are running around a circular track in **OPPOSITE** directions

1. The relative speed is equal to the sum of the speeds of the two individuals and
2. From one meeting point to the next meeting point, the two of them TOGETHER cover a distance equal to the length of the track.

When two persons are running around a circular track in the **SAME** direction

1. The relative speed is equal to the difference of the speeds of the two individuals and
2. From one meeting point to the next meeting point, the faster person covers one COMPLETE ROUND more than the slower person.

We can now tabulate the time taken by the persons to meet for the first time ever or for the first time at the starting point in various cases.

When TWO people are running around a circular track

Let the two people A and B with respective speeds of a and $b(a > b)$ be running around a circular track (of length L) starting at the same point and at the same time. Then,

	When the two persons are running in the SAME direction	When the two persons are running in OPPOSITE directions
Time taken to meet for the FIRST TIME EVER	$\frac{L}{(a-b)}$	$\frac{L}{(a+b)}$
Time taken to meet for the first time at the STARTING POINT	LCM of $\left\{\frac{L}{a}, \frac{L}{b}\right\}$	LCM of $\left\{\frac{L}{a}, \frac{L}{b}\right\}$

Please note that when we have to find out the time taken by the two persons to meet for the first time at the starting point, what we have to do is to find out the time taken by each of them to complete one full round and then take the LCM of these two timings (L/a and L/b are the timings taken by the two of them respectively to complete on full round).

Example 9: In a 200 m race, A beats B by 10 m or 2 seconds. Find B's speed and A's speed.

Solution: A beat B by 10 m or 2 seconds.
$\Rightarrow$ When A reached the finishing line B was 10 m behind the finishing line and took 2 seconds to cover it.

$$\therefore \quad B\text{'s speed} = \frac{10}{2} = 5 \text{ m/sec}$$

Time taken by B to complete the race

$$= \frac{200 \text{ m}}{5 \text{ m/s}} = 40 \text{ seconds}$$

$\therefore$ Time taken by A to complete the race

$= 38$ seconds

$$A\text{'s speed} = \frac{200}{38} = \frac{100}{19} \text{ m/sec}$$

Example 10: Ramu is 50% faster than Somu. In a race, Ramu gave Somu a head start of 200 m. Both finished the race simultaneously. Find the length of the race.

Solution: Let the length of the race be x m.

$$\frac{x}{x-200} = \frac{150}{100}$$

$$\Rightarrow \quad x = 600.$$

Example 11: In a 1200 m race, Ram beats Shyam by 300 m. In the same race, Shyam beats Tarun by 400 m. Find the distance by which Ram beats Tarun.

Solution: Let the speeds of Ram, Shyam and Tarun be r m/sec, s m/sec, and t m/sec respectively

$$\frac{r}{s} = \frac{1200}{1200-300} = \frac{4}{3}$$

$$\frac{s}{t} = \frac{1200}{1200-400} = \frac{3}{2}$$

$$\frac{r}{t} = \left(\frac{r}{s}\right)\left(\frac{s}{t}\right) = 2$$

$\Rightarrow$ By the time Ram covers 1200 m, Tarun covers 600 m.
$\therefore$ Ram beats Tarun by (1200 – 600) i.e. by 600 m.

EXERCISE

Practice Problems

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Convert the following speeds into meters per second

(a) 36 km/hr

(A) 10 (B) 12 (C) 15 (D) 20

(b) 12.6 km/hr

(A) 3.5 (B) 4 (C) 0.35 (D) 6

(c) 252/35 km/hr

(A) 2.2 (B) 2.4 (C) 2 (D) 2.6

2. If a man runs at 6 metres per second, what distance (in km) will he cover in 3 hours and 45 minutes?

(A) 81 (B) 96 (C) 91 (D) 27

3. Travelling at 5/6th of his usual speed a man is 10 minutes late. What is the usual time he takes to cover the same distance?

(A) 50 minutes (B) 70 minutes
(C) 1 hour (D) 75 minutes

4. X and Y are 270 km apart. At 9:00 a.m, buses A and B left X and Y for Y and X respectively. If the speeds of A and B are 50 kmph and 40 kmph respectively, find their meeting time.

(A) 11:00 a.m (B) 12:00 p.m
(C) 1: 00 p.m (D) 2:00 p.m

5. Car A left X for Y at 9:00 a.m. Car B left Y for X at 10:00 a.m. $XY = 180$ km. Speeds of A and B are 30 kmph and 20 kmph respectively. Find their meeting time.

(A) 12:36 p.m. (B) 1:36 p.m.
(C) 1:00 p.m (D) 2:00 p.m

6. Ashok left X and reached Y in 4 hours. His average speed for the journey was 90 kmph. Find the distance between X and Y (in km).
(A) 180 (B) 360 (C) 720 (D) 900

7. Alok travelled from Hyderabad to Tirupati at 60 kmph and returned at 90 kmph. Find his average speed for the journey (in kmph)
(A) 72 (B) 75 (C) 66 (D) 78

8. What is the time taken by a train 650 m long travelling at 72 km/hr to cross a 750 m long platform?
(A) 60 sec (B) 65 sec
(C) 70 sec (D) 75 sec

9. What is the time taken by a 750 m long train travelling at 99 km/hr to cross a boy running at 9 km/hr towards the train?
(A) 30 sec (B) 33 sec
(C) 36 sec (D) 25 sec

10. In a 200 m race, Eswar gives Girish a start of 10 m and beats him by 10 m. Find the ratio of their speeds
(A) 1 : 1 (B) 9 : 10 (C) 10 : 9 (D) 19 : 20

11. In a 100 m race, Ganesh beats Harish by 10 m or 2 seconds. Find Harish's speed (in m/sec).
(A) 5 (B) $5\frac{5}{9}$
(C) $4\frac{1}{2}$ (D) 6

12. In a 100 m race, Akbar gives Birbal a start of 2 seconds. Birlbal covers 10 m by the time Akbar starts. If both of them finish together, find Akbar's speed. (in m/sec)
(A) 5 (B) $5\frac{5}{9}$
(C) $4\frac{1}{2}$ (D) 4

13. In a race, P beats Q by 20 seconds. Q beats R by 30 seconds. By how many seconds did P beat R?
(A) 44 (B) 25 (C) 50 (D) 36

14. In a 100 m race, A beats B by 10 m and B beats C by 20 m. Find the distance by which A beats C. (in m)
(A) 30 (B) 28 (C) 32 (D) 36

15. Anand can row a boat in still water at a speed of 5 kmph. The speed of the stream is 3 kmph. Find the time taken by him to row 40 km downstream (in hours).
(A) 5 (B) 20 (C) 8 (D) 10

16. Ram, Shyam and Tarun started cycling from a point on a circular track 600 m long with speeds of 10 m/sec, 15 m/sec and 20 m/sec respectively. Find the time taken by them to meet at the starting point for the first time (in seconds).
(A) 120 (B) 60 (C) 240 (D) 600

17. Ashwin and Bhaskar started running simultaneously from a point on a 300 m long circular track. They ran in opposite directions with speeds of 6 m/sec and 4 m/sec respectively. After meeting for the first time, they exchange their speeds. Who will reach the starting point first?
(A) Ashwin
(B) Bhaskar
(C) Both reach simultaneously
(D) Cannot be determined

18. A man reaches his destination which is 16 km away, 9 min late, if he travels at 8 kmph. What should his speed be if he wishes to reach 15 minutes ahead of the right time?
(A) 10 kmph (B) 3 m/sec
(C) 20/9 m/sec (D) 12 kmph

19. The distance between two points P and Q is 84 km. Two persons start at the same time but one travelling from P towards Q and the other travelling from Q towards P. If their respective speeds are 36 kmph and 27 kmph, where do they meet each other?
(A) 48 km from Q (B) 24 km from P
(C) 36 km from P (D) 48 km from P

20. Towns P and Q are 80 km apart. Cars A and B are stationed at towns P and Q respectively. If they start simultaneously towards each other, they would meet in an hour. If both start simultaneously in the same direction, the faster car would overtake the slower car in 4 hours. Find the speed of the faster car (in kmph).
(A) 50 (B) 55 (C) 60 (D) 65

21. A cat on seeing a dog 100 m away turns around and starts running away at 24 kmph. The dog spots him one minute later and starts chasing the cat at a speed of 33 kmph. After how much time, from the start of the cat's run, will the chase end?
(A) 160 s (B) 220 s
(C) 260 s (D) 280 s

22. Train A starts at 6 a.m. from city P towards city Q at a speed of 54 kmph. Another train 'B' starts at 9 a.m. from P towards Q at 72 kmph. If the distance between P and Q is 1440 km, find at what distance from Q would the two trains meet each other?
(A) 648 km (B) 792 km
(C) 486 km (D) 954 km

23. Mahesh travelled from Hyderabad to Tirupati at a certain speed and returned at a certain speed. His average speed for the entire trip was the average of his onward and return speeds. He travelled a total distance of 1200 km in 12 hours. Find his onward speed (in kmph).
(A) 100 (B) 80 (C) 60 (D) 40

24. Two cars left simultaneously from two places P and Q, and headed for Q and P respectively. They crossed each other after x hours. After that, one of the cars took y hours to reach its destination while the other took z hours to reach its destination. Which of the following always holds true?

(A) $x = \frac{y+z}{2}$ (B) $x = \frac{2yz}{y+z}$

(C) $x = \sqrt{yz}$ (D) $x = \frac{y^2+z^2}{y+z}$

25. A boat travels 30 km upstream in 5 hours and 24 km downstream in 3 hours. Find the speed of the boat in still water and the speed of the water current

(A) 7 kmph, 2 kmph (B) 14 kmph, 1 kmph
(C) 7 kmph, 1 kmph (D) 8 kmph, 2 kmph

26. Amar, Akbar and Anthony start running in the same direction and from the same point, around a circular track with speeds 7 m/sec, 11 m/sec and 22 m/sec respectively. If Akbar can complete 5 revolutions around the track in 40 sec, when will they meet for the first time after they start?

(A) 56 s (B) 88 s (C) 118 s (D) 79 s

27. If Ashok travelled at 4/5th of his usual speed, he would reach his destination 15 minutes late. By how many minutes would he be early if he travelled at 6/5th of his usual speed?

(A) 12 (B) 10 (C) 15 (D) 20

28. In a 500 ft race, Habib beats Akram by 60 ft. If Habib takes 5 paces for every 4 paces taken by Akram, what is the ratio of the length of Habib's pace to that of Akram?

(A) 10 : 11 (B) 11 : 10
(C) 25 : 22 (D) 22 : 25

29. Girish takes 1 minute to complete a round around a circular track. Harish is twice as fast as Girish, Suresh is thrice as fast as Harish. All three start at the same point. Find the time taken by them to meet at the starting point for the first time (in minutes).

(A) 1 (B) 2
(C) 6 (D) 12

30. Two cars *C* and *D* start from a junction along two perpendicular roads at 8:00 a.m. and 9:00 a.m. respectively. If at 12 noon, the cars, which travel at the same speed, are 150 km apart, then, find the speed of each car.

(A) 15 kmph (B) 45 kmph
(C) 60 kmph (D) 30 kmph

Answer Keys

Exercise

Practice Problems

1. **(a)** A	**(b)** A	**(c)** C	**2.** A	**3.** A	**4.** B	**5.** C	**6.** B	**7.** A	**8.** C
9. D	**10.** C	**11.** A	**12.** B	**13.** C	**14.** B	**15.** A	**16.** A	**17.** C	**18.** A
19. D	**20.** A	**21.** C	**22.** B	**23.** A	**24.** C	**25.** C	**26.** B	**27.** B	**28.** A
29. A	**30.** D								

Chapter 9

Indices, Surds, Logarithms

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Indices
- Surds
- Rationalisation of a surd
- Square root of a surd
- Comparison of surds
- Logarithms

INDICES

If a number 'a' is added three times to itself, then we write it as $3a$. Instead of adding, if we multiply 'a' three times with itself, we write it as a^3.

We say that 'a' is expressed as an exponent. Here, 'a' is called the 'base' and 3 is called the 'power' or 'index' or 'exponent'.

Similarly, 'a' can be expressed to any exponent 'n' and accordingly written as a^n. This is read as 'a to the power n' or 'a raised to the power n.'

$$a^n = a \times a \times a \times a \times \ldots \; n \text{ times}$$

For example,

$$2^3 = 2 \times 2 \times 2 = 8 \text{ and } 3^4 = 3 \times 3 \times 3 \times 3 = 81$$

While the example taken is for a positive integer value of n, the powers can also be negative integers or positive or negative fractions. In the sections that follow, we will also see how to deal with numbers where the powers are fractions or negative integers.

If a number raised to a certain power is inside brackets and quantity is then raised to a power again, {i.e., a number of the type $(a^m)^n$—read as 'a raised to the power m whole raised to the power n' or 'a raised to power m whole to the power n'}, then the number inside the brackets is evaluated first and then this number is raised to the power which is outside the brackets.

For example, to evaluate $(2^3)^2$, we first find out the value of the number inside the bracket (2^3) as 8 and now raise this to the power 2. This gives 8^2 which is equal to 64. Thus $(2^3)^2$ is equal to 64.

If we have powers in the manner of 'steps', then such a number is evaluated by starting at the topmost of the 'steps' and coming down one 'step' in each operation.

For example, 2^{4^3} is evaluated by starting at the topmost level '3'. Thus we first calculate 4^3 as equal to 64. Since 2 is raised to the power 4^3, we now have 2^{64}.

Similarly, 2^{3^2} is equal to '2 raised to the power 3^2' or '2 raised to the power 9' or 2^9 which is equal to 512.

There are certain basic rules/formulae for dealing with numbers having powers. These are called Laws of Indices. The important ones are listed below but you are not required to learn the proof for any of these formulae/rules. The students have to know these rules and be able to apply any of them in solving problems. Most of the problems in indices will require one or more of these formulae. These formulae should be internalised by the students to the extent that after some practice, application of these rules should come naturally and the student should not feel that he is applying some specific formula.

Table 1 Table of Rules/Laws of Indices

	Rule/Law	Example
(1)	$a^m \times a^n = a^{m+n}$	$5^2 \times 5^7 = 5^9$
(2)	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{7^5}{7^3} = 7^2 = 49$
(3)	$(a^m)^n = a^{mn}$	$(4^2)^3 = 4^6$
(4)	$a^{-m} = \frac{1}{a^m}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$
(5)	$\sqrt[m]{a} = a^{1/m}$	$\sqrt[3]{64} = 64^{1/3} = 4$
(6)	$(ab)^m = a^m \cdot b^m$	$(2 \times 3)^4 = 2^4 \cdot 3^4$
(7)	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$
(8)	$a^0 = 1$ (where $a \neq 0$)	$3^0 = 1$
(9)	$a^1 = a$	$4^1 = 4$

These rules/laws will help you in solving a number of problems. In addition to the above, the student should also remember the following rules:

Rule 1: When the bases of two EQUAL numbers are equal, then their powers also will be equal. (If the bases are neither zero nor ±1.)

For example: If $2^n = 2^3$, then it means $n = 3$

Rule 2: When the powers of two equal numbers are equal (and not equal to zero), two cases arise:

1. If the power is an odd number, then the bases are equal. For example, if $a^3 = 4^3$ then $a = 4$.
2. If the powers are even numbers, then the bases are numerically equal but can have different signs. For example, if $a^4 = 3^4$ then $a = +3$ or -3.

The problems associated with indices are normally of THREE types:

Simplification: Here, the problem involves terms with different bases and powers which have to be simplified using the rules/formulae discussed in the table above.

Solving for the value of an unknown: Here, the problem will have an equation where an unknown (like x or y) will appear in the base or in the power and using Rule 1 and Rule 2 discussed above, values of unknown are to be determined.

Comparison of numbers: Here two or more quantities will be given—each being a number raised to a certain power. These numbers have to be compared in magnitude—either to find the largest or smallest of the quantities or to arrange the given quantities in ascending or descending order.

The following examples will make clear the different types of problems that you may be asked.

Solved Examples

Example 1: Simplify: $\left(\frac{729}{1728}\right)^{\frac{-2}{3}} \times \left(\frac{1024}{9}\right)^{\frac{1}{2}} \div \left(\frac{24}{324}\right)$

Solution: $\left(\frac{729}{1728}\right)^{\frac{-2}{3}} \times \left(\frac{1024}{9}\right)^{\frac{1}{2}} \div \left(\frac{24}{324}\right)$

$$= \left(\frac{9^2}{12^2}\right)^{-1} \times \left(\frac{32^2}{3^2}\right)^{\frac{1}{2}} \times \left(\frac{324}{24}\right)$$

$$\left(\frac{9^3}{12^3}\right)^{\frac{-2}{3}} \times \frac{32}{3} \times \frac{324}{24} = \frac{144}{81} \times \frac{32}{3} \times \frac{324}{24} = 256$$

Example 2: In the equation given below, solve for x

$$\sqrt[3]{\left(\frac{5}{7}\right)^{x+1}} = \frac{125}{343}$$

Solution: Given, $\left(\frac{5}{7}\right)^{x+1} = \left[\left(\frac{5}{7}\right)^3\right]^3 = \left(\frac{5}{7}\right)^9$.

By equating their indices.

$$x + 1 = 9$$

$$x = 8.$$

Example 3: If $\left(\frac{49}{2401}\right)^{4-x} = 49^{2x-6}$, find x.

Solution: $\left(\frac{49}{2401}\right)^{4-x} = \left(49^{-1}\right)^{4-x} = 49^{x-4}$

Given, $49^{x-4} = 49^{2x-6}$

$$x - 4 = 2x - 6$$

$$x = 2$$

Example 4: Arrange the following in ascending order 625^6, 125^7 and 25^{10}

Solution: $625^6 = (5^4)^6 = 5^{24}$

$125^7 = (5^3)^7 = 5^{21}$

$25^{10} = (5^2)^{10} = 5^{20}$

$25^{10} < 125^7 < 625^6$

SURDS

Any number of the form p/q, where p and q are integers and $q \neq 0$ is called a rational number. Any real number which is not a rational number is an irrational number. Amongst irrational numbers, of particular interest to us are SURDS. Amongst surds, we will specifically be looking at 'quadratic

surds'—surds of the type $a+\sqrt{b}$ and $a+\sqrt{b}+\sqrt{c}$, where the terms involve only square roots and not any higher roots. We do not need to go very deep into the area of surds—what is required is a basic understanding of some of the operations on surds.

If there is a surd of the form $(a+\sqrt{b})$, then a surd of the form $\pm(a-\sqrt{b})$ is called the conjugate of the surd $(a+\sqrt{b})$. The product of a surd and its conjugate will always be a rational number.

Rationalisation of a Surd

When there is a surd of the form $\frac{1}{a+\sqrt{b}}$, it is difficult to perform arithmetic operations on it. Hence, the denominator is converted into a rational number thereby facilitating ease of handling the surd. This process of converting the denominator into a rational number without changing the value of the surd is called rationalisation.

To convert the denominator of a surd into a rational number, multiply the denominator and the numerator simultaneously with the conjugate of the surd in the denominator so that the denominator gets converted to a rational number without changing the value of the fraction. That is, if there is a surd of the type $a+\sqrt{b}$ in the denominator, then both the numerator and the denominator have to be multiplied with a surd of the form $a-\sqrt{b}$ or a surd of the type $(-a+\sqrt{b})$ to convert the denominator into a rational number.

If there is a surd of the form $(a+\sqrt{b}+\sqrt{c})$ in the denominator, then the process of multiplying the denominator with its conjugate surd has to be carried out TWICE to rationalise the denominator.

Square Root of a Surd

If there exists a square root of a surd of the type $a+\sqrt{b}$, then it will be of the form $\sqrt{x}+\sqrt{y}$. We can equate the square of $\sqrt{x}+\sqrt{y}$ to $a+\sqrt{b}$ and thus solve for x and y. Here, one point should be noted—when there is an equation with rational and irrational terms, the rational part on the left hand side is equal to the rational part on the right hand side and, the irrational part on the left hand side is equal to the irrational part on the right hand side of the equation.

However, for the problems which are expected in the entrance exams, there is no need of solving for the square root in such an elaborate manner. We will look at finding the square root of the surd in a much simpler manner. Here, first the given surd is written in the form of $(\sqrt{x}+\sqrt{y})^2$ or $(\sqrt{x}-\sqrt{y})^2$. Then the square root of the surd will be $(\sqrt{x}+\sqrt{y})$ or $(\sqrt{x}-\sqrt{y})$ respectively.

Comparison of Surds

Sometimes we will need to compare two or more surds either to identify the largest one or to arrange the given surds in ascending/descending order. The surds given in such cases will be such that they will be close to each other and hence we will not be able to identify the largest one by taking the approximate square root of each of the terms. In such a case, the surds can both be squared and the common rational part be subtracted. At this stage, normally one will be able to make out the order of the surds. If even at this stage, it is not possible to identify the larger of the two, then the numbers should be squared once more.

Example 5: Rationalize the denominator: $\frac{1}{1+\sqrt{6}-\sqrt{7}}$

Solution: The rationalizing factor of

$$1+\sqrt{6}-\sqrt{7} \text{ is } 1+\sqrt{6}+\sqrt{7}$$

$$\frac{1}{1+\sqrt{6}-\sqrt{7}}=\frac{(1+\sqrt{6}+\sqrt{7})}{(1+\sqrt{6}-\sqrt{7})(1+\sqrt{6}+\sqrt{7})}$$

$$=\frac{1+\sqrt{6}+\sqrt{7}}{(1+\sqrt{6})^2-(\sqrt{7})^2}=\frac{1+\sqrt{6}+\sqrt{7}}{2\sqrt{6}}$$

The rationalizing factor of $\sqrt{6}$ is $\sqrt{6}$

$$=\frac{\sqrt{6}+6+\sqrt{42}}{12}$$

Example 6: Find the value of $\sqrt{62+\sqrt{480}}$.

Solution: Let $\sqrt{62+\sqrt{480}}=\sqrt{a}+\sqrt{b}$

Squaring both sides, $62+\sqrt{480}=a+b+2\sqrt{ab}$

$$62+\sqrt{480}=a+b+\sqrt{4ab}$$

Equating the corresponding rational and irrational parts on both sides, $a+b=62$

$$4ab=480 \quad \Rightarrow \quad ab=120$$

As $a+b=60+2$ and $ab=(60)(B)$ it follows that $a=60$ and $b=2$ or vice versa.

$$\therefore \quad \sqrt{a}+\sqrt{b}=\sqrt{60}+\sqrt{2}$$

Example 7: Which of the surds given below is greater?

$$\sqrt{3}+\sqrt{23} \text{ and } \sqrt{6}+\sqrt{19}$$

Solution: $(\sqrt{3}+\sqrt{23})^2=26+2\sqrt{69}$

$\sqrt{69}$ lies between $\sqrt{64}$ and $\sqrt{81}$

$\therefore 26+2\sqrt{69}$ lies between $26+2(8)$ and $26+2(9)$ i.e., 42 and 44.

Similarly $(\sqrt{6}+\sqrt{19})^2$ lies between 45 and 47.

$$\therefore \quad (\sqrt{3}+\sqrt{23})^2<(\sqrt{6}+\sqrt{19})^2$$

$$\therefore \quad \sqrt{6}+\sqrt{19}>\sqrt{3}+\sqrt{23}$$

LOGARITHMS

In the equation $a^x = N$, we are expressing N in terms of a and x. The same equation can be re-written as, $a = N^{1/x}$. Here we are expressing a in terms of N and x. But, among a, x and N, by normal algebraic methods known to us, we cannot express x in terms of the other two parameters a and N. This is where logarithms come into the picture. When $a^x = N$, then we say x = logarithm of N to the base a, and write it as $x = \log_a N$. The definition of logarithm is given as: 'the logarithm of any number to a given base is the index or the power to which the base must be raised in order to equal the given number.'

Thus,

$$\text{if } a^x = N \text{ then } x = \log_a N$$

This is read as 'log N to the base a'.

In the above equation, N is a **POSITIVE NUMBER** and a is a **POSITIVE NUMBER OTHER THAN 1**.

This basic definition of logarithm is very useful in solving a number of problems on logarithms.

Example of a logarithm: $216 = 6^3$ can be expressed as $\log_6 216 = 3$.

Since logarithm of a number is a value, it will have an 'integral' part and a 'decimal' part. The integral part of the logarithm of a number is called the CHARACTERISTIC and the decimal part of the logarithm is called the MANTISSA.

Logarithms can be expressed to any base (positive number other than 1.) Logarithms from one base can be converted to logarithms to any other base. (One of the formulae given below will help do this conversion). However, there are two types of logarithms that are commonly used.

(i) **Natural Logarithms or Napierian Logarithms:** These are logarithms expressed to the base of a number called 'e.'

(ii) **Common Logarithms:** These are logarithms expressed to the base 10. For most of the problems under LOGARITHMS, it is common logarithms that we deal with. In examinations also, if logarithms are given without mentioning any base, it can normally be taken to be logarithms to the base 10.

Given below are some **important rules/ formulae** in logarithms:

(i) $\log_a a = 1$ (logarithm of any number to the same base is 1)

(ii) $\log_a 1 = 0$ (log of 1 to any base other than 1 is 0)

(iii) $\log_a (mn) = \log_a m + \log_a n$

(iv) $\log_a (m/n) = \log_a m - \log_a n$

(v) $\log_a m^p = p \times \log_a m$

(vi) $\log_a b = \dfrac{1}{\log_b a}$

(vii) $\log_a m = \dfrac{\log_b m}{\log_b a}$

(viii) $\log_{a^q} m^p = \dfrac{p}{q} \log_a m$

(ix) $a^{\log_a N} = N$

(x) $a^{\log b} = b^{\log a}$

You should memorize these rules/formulae because they are very helpful in solving problems.

Like in the chapter on INDICES, in LOGARITHMS also there will be problems on

(i) Simplification using the formulae/rules listed above and

(ii) Solving for the value of an unknown given in an equation

In solving problems of the second type above, in most of the cases we take recourse to the basic definition of logarithms (which is very important and should be memorized).

The following examples will give problems of both the above types and also some problems on common logarithms.

The following rules also should be remembered while solving problems on logarithms:

Given an equation $\log_a M = \log_b N$,

(i) if $M = N$, then a will be equal to b; if $M \neq 1$ and $N \neq 1$.

(ii) if $a = b$, then M will be equal to N.

The examples that follow will explain all the above types of problems. Please note that unless otherwise specified, all the logarithms are taken to the base 10).

Example 8: Solve for x : $\log_{10} 20x = 4$

Solution: Given that $\log_{10} 20x = 4$

$\Rightarrow \quad 20x = 10^4 = 10000$

$\therefore \quad x = 500$

Example 9: Solve for x : $\log (x + 3) + \log (x - 3) = \log 72$

Solution: $\log (x + 3) + \log (x - 3) = \log 72$

$\log (x + 3)(x - 3) = \log 72$

$(x + 3)(x - 3) = 72$

$x^2 = 81$

$x = 9$ (If $x = -9$, $\log (x - 3)$ would be undefined)

Example 10: If $\log 2 = 0.301$, find the value of $\log 1250$, $\log 0.001250$ and $\log 125000$.

Solution: $\log 1250 = \log \dfrac{10000}{8}$

$= 4 \log 10 - 3 \log 2 = 4 - 3 (0.3010) = 3.097$

$$\log 0.001250 = \log \frac{1250}{10^6}$$
$$= 3.097 - 6 = -2.903$$
$$\log 125000 = \log (1250)(100)$$
$$= \log 1250 + 2 = 5.097$$

Example 11: Find the number of digits in 294^{20} given that log 6 = 0.778 and log 7 = 0.845

Solution: $\log 294^{20} = 20 \log (7^2\ 6)$

$= 20 (2 \log 7 + \log 6)$

$= 20 (2 (0.845) + 0.778) = 20 (1.69 + 0.778)\ 49.36$

Characteristic = 49.

$\therefore$ 294^{20} has 50 digits.

Example 12: Find the value of $\log_{\sqrt[3]{2}} 32 \sqrt[3]{16}$.

Solution: $\log_{\sqrt[3]{2}} 32 \sqrt[3]{16} = \log_{2^{1/3}} 2^5 \left(2^4\right)^{\frac{1}{3}}$

$$= \log_{2^{1/3}} \left(2^{\frac{1}{3}}\right)^{19} = 19$$

Example 13: Find the number of zeros after the decimal point in $\left(\frac{3}{4}\right)^{500}$, given that log 3 = 0.4771 and log 2 = 0.3010.

Solution: $\log \left(\frac{3}{4}\right)^{500} = 500 \left(\log \frac{3}{4}\right)$

$= 500 (\log 3 - 2 \log 2)$

$= 500 (0.4771 - 2 (0.3010) = -62.4500.$

$\therefore$ Number of zeros after the decimal point is 62.

EXERCISE

Practice Problems

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Simplify the following:

$$\left(\frac{243}{1024}\right)^{-2/5} \times \left(\frac{144}{49}\right)^{-1/2} \div \left(\frac{8}{343}\right)^{-2/3}$$

(A) $2^5 \times 3^{-1} \times 7$ (B) $2^5 \times 3^{-3} \times 7^{-2}$

(C) $2^4 \times 3^{-3} \times 7^{-1}$ (D) $2^4 \times 3^{-1} \times 7^{-2}$

2. Simplify the following:

$$\left(\frac{x^2 \cdot y^{-3}}{z^4}\right)^{-2} \times \left(\frac{x^2 \cdot y}{z^{-2}}\right)^{3} \div \left(\frac{x^{-12} \cdot y^7}{z^{-8}}\right)^{-1}$$

(A) $x^{10} . y^{16} . z^{-22}$ (B) $x^7 . y^{-16} . z^{-22}$

(C) $x^{-7} . y^{16} . z^{-22}$ (D) $x^{-10} . y^{16} . z^{22}$

3. Simplify the following: $\frac{1-[1-\{1-(1+y)^{-1}\}]}{(1-y)}$

(A) $\frac{y}{(1-y^2)}$ (B) $\frac{y}{(1-y)^2}$

(C) $\frac{1+y}{(1-y)^2}$ (D) $\frac{1+y^2}{1-y^2}$

4. $(x^{a-b})^{(a^2+ab+b^2)} \times (x^{b-c})^{(b^2+bcc^2)} \times (x^{c-a})^{(c^2+ac+a^2)}$

(A) 0

(B) 1

(C) $x^{a^3+b^3+c^3}$

(D) $x^{3\left(a^2+b^2+c^2+ac+bc+ca\right)}$

5. $343^{0.12} \times 2401^{0.08} \times 49^{0.01} \times 7^{0.1} =$

(A) 7 (B) $7^{4/5}$ (C) 7^8 (D) $7^{3/5}$

6. Solve for x: $9^{2x+1} = 27^{5x-3}$

(A) 1 (B) 2 (C) –1 (D) –2

7. If $\frac{p}{q} = \frac{r}{s}$ and $p^a = q^b = r^c = s^d$, then $\frac{1}{a} - \frac{1}{b} =$

(A) $\frac{1}{d} - \frac{1}{c}$ (B) $\frac{1}{c} + \frac{1}{d}$

(C) $\frac{1}{c} - \frac{1}{d}$ (D) $-\left(\frac{1}{c} + \frac{1}{d}\right)$

8. Which of the following is the largest in value?

(A) $6^{1/2}$ (B) $7^{1/3}$ (C) $8^{1/4}$ (D) $9^{1/5}$

9. $2\sqrt{\frac{5}{2}} - 5\sqrt{\frac{2}{5}} + \sqrt{10} + \sqrt{1000} =$

(A) $9\sqrt{10}$ (B) $8\sqrt{10}$

(C) $8\sqrt{10}$ (D) $11\sqrt{10}$

10. $\left(\frac{\sqrt{p} - \sqrt[4]{pq}}{\sqrt[4]{pq} - \sqrt{q}}\right)^{-4} =$

(A) $\frac{p}{q}$ (B) $\sqrt{\frac{p}{q}}$

(C) $\sqrt{\frac{q}{p}}$ (D) $\frac{q}{p}$

11. If $y = 12 + 2\sqrt{35}$, then $\sqrt{y} - \frac{1}{\sqrt{y}} =$

(A) $\frac{\sqrt{7}+\sqrt{5}}{2}$ (B) $\frac{3\sqrt{5}-\sqrt{7}}{2}$

(C) $\frac{2\sqrt{5}+\sqrt{7}}{2}$ (D) $\frac{3\sqrt{5}+\sqrt{7}}{2}$

12. Arrange the following in ascending order.
$a = \sqrt{2} + \sqrt{11}$, $b = \sqrt{6} + \sqrt{7}$,
$c = \sqrt{3} + \sqrt{10}$ and $d = \sqrt{5} + \sqrt{8}$
(A) *abcd* (B) *abdc*
(C) *acdb* (D) *acbd*

13. Arrange the following in descending order.
$a = \sqrt{13} + \sqrt{11}$, $b = \sqrt{15} + \sqrt{9}$, $c = \sqrt{18} + \sqrt{6}$,
$d = \sqrt{7} + \sqrt{17}$.
(A) *abdc* (B) *dcab*
(C) *adcb* (D) *acdb*

14. Solve for x and y:
$3.5^x + 2^{y+2} = 107$, $5^{x+1} + 8.2^y = 189$
(A) 3, 2 (B) 5, 7 (C) 7, 5 (D) 2, 3

15. Solve for x, if $\left(5\sqrt{7}\right)^{5x-4} = (35)^3 (25)^{3/2}$.
(A) 2 (B) 5/4 (C) 7/2 (D) 3

16. If $5^{x+3} - 5^{x-3} = 78120$, find x.
(A) 4 (B) 3 (C) 5 (D) 6

17. If $a^a \cdot b^b \cdot c^c = a^b \cdot b^c \cdot c^a = a^c \cdot b^a \cdot c^b$ and a, b, c are positive integers greater than 1, then which of the following can NOT be true for any of the possible values of a, b, c?
(A) $abc = 8$ (B) $a + b + c = 8$
(C) $abc = 27$ (D) $a + b + c = 27$

18. The ascending order of $16^{\frac{7}{12}}$, $81^{\frac{3}{8}}$, $625^{\frac{2}{3}}$ is _____.
(A) $16^{\frac{7}{12}}$, $81^{\frac{3}{8}}$, $625^{\frac{2}{3}}$ (B) $16^{\frac{7}{12}}$, $625^{\frac{2}{3}}$, $81^{\frac{3}{8}}$,
(C) $625^{\frac{2}{3}}$, $16^{\frac{7}{12}}$, $81^{\frac{3}{8}}$ (D) $81^{\frac{3}{8}}$, $16^{\frac{7}{12}}$, $625^{\frac{2}{3}}$

19. If $2\sqrt{2} + \sqrt{3} = x$, what is the value of $\frac{11 + 4\sqrt{6}}{2\sqrt{2} - \sqrt{3}}$ in terms of x?
(A) $\frac{x^2}{\sqrt{2}}$ (B) x^3
(C) $\frac{x^3}{8}$ (D) $\frac{x^3}{5}$

20. Simplify: $\sqrt{(a+b+c) + 2\sqrt{ac + bc}}$.
(A) $\sqrt{a} + \sqrt{b} + \sqrt{c}$ (B) $\sqrt{a+b} + \sqrt{c}$
(C) $\sqrt{ab + bc}$ (D) $\sqrt{abc}$

21. Find the value of $x^2 - y^2$, if $\log_y(x-1) + \log_y(x+1) = 2$.
(A) 2 (B) $2y$ (C) 1 (D) $2xy$

22. If $a > 1$, $\log_a a + \log_{\frac{1}{a^2}} a + \log_{\frac{1}{a^3}} a + \ldots + \log_{\frac{1}{a^{20}}} a =$
(A) 420 (B) 210 (C) 380 (D) 190

23. If $\log_7(x-7) + \log_7(x^2 + 7x + 49) = 4$, then $x =$
(A) 196 (B) 7 (C) 49 (D) 14

24. If $\frac{\log a}{5} = \frac{\log b}{6} = \frac{\log c}{7}$, then $b^2 =$
(A) ac (B) a^2
(C) bc (D) ab

25. What is the value of $\log_{(1/5)} 0.0000128$?
(A) –7 (B) –5 (C) 5 (D) 7

26. If $(\log\tan5°)(\log\tan10°)(\log\tan15°)\ldots(\log\tan60°) = x$, what is the value of x?
(A) $\log(\sin5°)^{12}$
(B) 1
(C) 0
(D) $\log(\cos60°)$

27. Solve for x, if $\log_x[\log_5(\sqrt{x+5} + \sqrt{x})] = 0$.
(A) 1 (B) 9 (C) 12 (D) 4

28. If a, b, c are distinct values, what is the value of abc if $(\log_b a)(\log_c a) + (\log_a b)(\log_c b) + (\log_a c)(\log_b c) - 3 = 0$?
(A) 2
(B) 1
(C) $1 - \log a - \log b - \log c$
(D) 0

29. If $\log_6 161 = a$, $\log_6 23 = b$, what is the value of $\log_7 6$ in terms of a and b?
(A) a/b (B) $a + b$
(C) $1/(a - b)$ (D) b/a

30. $x = y^2 = z^3 = w^4 = u^5$, then find the value of $\log_x xyzwu$.
(A) $1\frac{47}{60}$ (B) $\frac{111}{120}$
(C) $2\frac{17}{60}$ (D) $2\frac{13}{60}$

Answer Keys

Exercise

Practice Problems

1. C	**2.** D	**3.** A	**4.** B	**5.** B	**6.** A	**7.** C	**8.** A	**9.** D	**10.** D
11. D	**12.** C	**13.** A	**14.** D	**15.** A	**16.** A	**17.** B	**18.** A	**19.** D	**20.** B
21. C	**22.** B	**23.** D	**24.** A	**25.** D	**26.** C	**27.** D	**28.** B	**29.** C	**30.** C

Chapter 10

Quadratic Equations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Quadratic equations
- Finding the roots by factorisation
- Finding the roots by using the formula
- Sum and product of roots of a quadratic equation
- Nature of the roots
- Signs of the roots
- Constructing a quadratic equation
- Maximum or minimum value of a quadratic expression

Quadratic Equations

'If a variable occurs in an equation with all positive integer powers and the highest power is two, then it is called a Quadratic Equation (in that variable).'

In other words, a second degree polynomial in x equated to zero will be a quadratic equation. For such an equation to be a quadratic equation, the co-efficient of x^2 should not be zero.

The most general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real)

Some examples of quadratic equations are

$$x^2 - 5x + 6 = 0 \quad (1)$$

$$x^2 - x - 6 = 0 \quad (2)$$

$$2x^2 + 3x - 2 = 0 \quad (3)$$

$$2x^2 + x - 3 = 0 \quad (4)$$

Like a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x will have TWO values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or imaginary.

For the four quadratic equations given above, the roots are as given below:

Equation (1) : $x = 2$ and $x = 3$
Equation (2) : $x = -2$ and $x = 3$
Equation (3) : $x = 1/2$ and $x = -2$
Equation (4) : $x = 1$ and $x = -3/2$

In general, the roots of a quadratic equation can be found out in two ways.

1. By factorising the expression on the left-hand side of the quadratic equation.
2. By using the standard formula.

All the expressions may not be easy to factorise whereas applying the formula is simple and straightforward.

Finding the Roots by Factorisation

If the quadratic equation $ax^2 + bx + c = 0$ can be written in the form $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first write it in the form of $(x - \alpha)(x - \beta) = 0$, i.e., the left hand side $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorised into two factors.

For this purpose, we should go through the following steps. We will understand these steps with the help of the equation $x^2 - 5x + 6 = 0$ which is the first of the four quadratic equations we looked at as examples above.

1. First write down b (the co-efficient of x) as the sum of two quantities whose product is equal to ac.

In this case –5 has to be written as the sum of two quantities whose product is 6. We can write –5 as (–3) + (–2) so that the product of (–3) and (–2) is equal to 6.

2. Now rewrite the equation with the 'bx' term split in the above manner.
   In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$
3. Take the first two terms and rewrite them together after taking out the common factor between the two of them. Similarly, the third and fourth terms should be rewritten after taking out the common factor between the two of them. In other words, you should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and the fourth term (after removing their common factor).
   In this case, the equation can be rewritten as $x(x-3) - 2(x-3) = 0$; Between the first and second terms as well as the third and fourth terms, we are left with $(x-3)$ is a common factor.
4. Rewrite the entire left-hand side to get the form $(x-\alpha)(x-\beta)$.
   In this case, if we take out $(x-3)$ as the common factor, we can rewrite the given equation as $(x-3)(x-2) = 0$
5. Now, α and β are the roots of the given quadratic equation.
   $\therefore$ For $x^2 - 5x + 6 = 0$, the roots of the equation are 3 and 2.

For the other three quadratic equations given above as examples, let us see how to factorise the expression and get the roots.

For equation (2), i.e., $x^2 - x - 6 = 0$, the co-efficient of x which is –1 can be rewritten as (–3) + (+2) so that their product is –6 which is equal to ac (1 multiplied by –6). Then we can rewrite the equation as $(x-3)(x+2) = 0$ giving us the roots as 3 and –2.

For equation (3), i.e., $2x^2 + 3x - 2 = 0$, the co-efficient of x which is 3 can be rewritten as (+4) + (–1) so that their product is –4 which is the value of ac (–2 multiplied by 2). Then we can rewrite the equation as $(2x-1)(x+2) = 0$ giving the roots as 1/2 and –2. For equation (4), i.e., $2x^2 + x - 3 = 0$, the co-efficient of x which is 1 can be rewritten as (+3) + (–2) so that their product is –6 which is equal to ac (2 multiplied by –3). Then we can rewrite the given equation as $(x-1)(2x+3) = 0$ giving us the roots as 1 and –3/2.

Finding the Roots by Using the Formula

If the quadratic equation is $ax^2 + bx + c = 0$, then we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots of the four quadratic equations we took as examples above can be taken and their roots found out by using the above formula. The student is advised to check it out for himself/herself that the roots can be obtained by using this formula also.

Sum and Product of Roots of a Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots and the product of the roots can be given by the following:

Sum of the roots = $-b/a$

Product of the roots = c/a

These two rules will be very helpful in solving problems on quadratic equation.

Nature of the Roots

We mentioned already that the roots of a quadratic equation with real co-efficients can be real or complex. When the roots are real, they can be equal or unequal. All this will depend on the expression $b^2 - 4ac$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the 'DISCRIMINANT' of the quadratic equation.

If $b^2 - 4ac > 0$, then the roots of the quadratic equation will be real and distinct.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation will be complex conjugates.

Thus we can write down the following about the nature of the roots of a quadratic equation when a, b and c are all *rational*.

when $b^2 - 4ac < 0$	the roots are complex and unequal
when $b^2 - 4ac = 0$	the roots are rational and equal
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal
when $b^2 - 4ac > 0$ but not a perfect square	the roots are irrational and unequal

Whenever the roots of the quadratic equation are irrational, (a, b, c being rational) they will be of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e. whenever $a + \sqrt{b}$ is one root of a quadratic equation, then $a - \sqrt{b}$ will be the second root of the quadratic equation and vice versa.

Signs of the Roots

We can comment on the signs of the roots, i.e., whether the roots are positive or negative, based on the sign of the sum of the roots and the product of the roots of the quadratic equation. The following table will make clear the relationship between the sum and the product of the roots and the signs of the roots themselves.

Sign of product of the roots	Sign of sum of the roots	Sign of the roots
+ ve	+ ve	Both the roots are positive.
+ ve	– ve	Both the roots are negative.
– ve	+ ve	The numerically larger root is positive and the other root is negative.
– ve	– ve	The numerically larger root is negative and the other root is positive.

Constructing a Quadratic Equation

We can build a quadratic equation in the following three cases:

1. When the roots of the quadratic equation are given.
2. When the sum of the roots and the product of the roots of the quadratic equation are given.
3. When the relation between the roots of the equation to be framed and the roots of another equation is given.

If the roots of the quadratic equation are given as α and β, the equation can be written as

$$(x-\alpha)(x-\beta)=0 \quad \text{i.e., } x^2-x(\alpha+\beta)+\alpha\beta=0$$

If p is the sum of the roots of the quadratic equation and q is the product of the roots of the quadratic equation, then the equation can be written as $x^2 - px + q = 0$.

Maximum or Minimum Value of a Quadratic Expression

An equation of the type $ax^2 + bx + c = 0$ is called a quadratic equation. An expression of the type $ax^2 + bx + c$ is called a 'quadratic expression'. The quadratic expression $ax^2 + bx + c$ takes different values as x takes different values.

As x varies from $-\infty$ to $+\infty$, (i.e., when x is real) the quadratic expression $ax^2 + bx + c$

1. Has a minimum value whenever $a > 0$ (i.e., a is positive). The minimum value of the quadratic expression is $(4ac - b^2)/4a$ and it occurs at $x = -b/2a$.
2. Has a maximum value whenever a < 0 (i.e. a is negative). The maximum value of the quadratic expression is $(4ac - b^2)/4a$ and it occurs at $x = -b/2a$.

Solved Examples

Example 1: Find the roots of the equation $11x^2 - 37x + 30 = 0$.

Solution: We have to write –37 as the sum of two parts whose product should be equal to $(11) \times (30)$

$$(-22) + (-15) = -37 \text{ and } (-22)(-15) = 11 \times 30$$

Therefore, $11x^2 - 37x + 30 = 0$

$$\Rightarrow \quad 11x^2 - 22x - 15x + 30 = 0$$

$$\Rightarrow \quad 11x(x-2) - 15(x-2) = 0$$

$$\Rightarrow \quad (11x-15)(x-2) = 0 \; x = \frac{15}{11} \text{ or } 2.$$

Example 2: Discuss the nature of the roots of the equation $8x^2 - 2x - 4 = 0$.

Solution: For the quadratic equation $ax^2 + bx + c = 0$ the nature of the roots is given by the discriminant $b^2 - 4ac$.

Discriminant of $8x^2 - 2x - 4 = 0$ is

$$(-2)^2 - 4(8)(-4) = 132.$$

Since the discriminant is positive but not a perfect square, the roots of the equation are irrational and unequal.

Example 3: If the sum of the roots of the equation $Rx^2 + 5x - 24 = 0$ is 5/11, then find the product of the roots of that equation.

Solution: For a quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is $(-b/a)$ and the product of the roots is (c/a).

The sum of the roots of the equation

$$Rx^2 + 5x - 24 = 0 \text{ is } \left(\frac{-5}{R}\right) \text{ which is given as } \frac{5}{11}$$

$\therefore R = -11$

In the given equation, product of the roots $= \dfrac{-24}{R} = \dfrac{-24}{-11} = +\dfrac{24}{11}$.

Example 4: Find the value of k, so that the roots of $6x^2 - 12x - k = 0$ are reciprocals of each other.

Solution: If the roots of the equation are reciprocals of each other, then the product of the roots should be equal to 1.

$$\Rightarrow \quad \frac{-k}{6} = 1.$$

Therefore $k = -6$.

Example 5: If $4 + \sqrt{7}$ is one root of a quadratic equation with rational co-efficients, then find the other root of the equation.

Solution: When the coefficients of a quadratic equation are rational and the roots are irrational, they occur only in pairs like $p \pm \sqrt{q}$ i.e., if $p + \sqrt{q}$ is one root, then the other root of the equation will be $p - \sqrt{q}$. So, in this case, the other root of the equation will be $4 - \sqrt{7}$.

Example 6: Form a quadratic equation with rational coefficients, one of whose roots is $5 + \sqrt{6}$.

Solution: If $5 + \sqrt{6}$ is one root, then the other root is $5 - \sqrt{6}$ (because the coefficients are rational).

The sum of the roots $= 5 + \sqrt{6} + 5 - \sqrt{6} = 10$.

The product of the roots $= (5+\sqrt{6})(5-\sqrt{6}) = 25-6 = 19$.

Thus the required equation is $x^2 - 10x + 19 = 0$.

Example 7: If the price of each book goes up by ₹5, then Atul can buy 20 books less for ₹1200. Find the original price and the number of books Atul could buy at the original price.

Solution: Let the original price of each book be x.
Then the new price of each book will be $x + 5$.

The number of books that can be bought at the original price $= \frac{1200}{x}$

The number of books that can be bought at the new price $= \frac{1200}{x+5}$

Given that Atul gets 20 books less at new price i.e.,

$$\frac{1200}{x} - \frac{1200}{x+5} = 20$$

$$\Rightarrow \quad \frac{60}{x} - \frac{60}{x+5} = 1$$

$$\Rightarrow \quad \frac{60(x+5-x)}{x^2+5x} = 1$$

$$\Rightarrow \quad 300 = x^2 + 5x$$

$$\Rightarrow \quad x^2 + 5x - 300 = 0$$

$$\Rightarrow \quad (x+20)(x-15) = 0$$

$$\Rightarrow \quad x = -20 \quad \text{or} \quad 15$$

As the price cannot be negative, the original price is ₹15.

Example 8: If α and β are the roots of the equation $x^2 - 3x - 180 = 0$ such that $\alpha < \beta$, then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iii) $\alpha - \beta$

Solution: From the given equation, we get $\alpha + \beta = 3$ and $\alpha\beta = -180$

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= (3)^2 - 2(-180) = 369$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{+3}{-180} = \frac{-1}{60}$

(iii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$$\Rightarrow \quad \alpha - \beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

$$= \pm\sqrt{(+3)^2 - 4(-180)} = \pm\sqrt{9+720}$$

$$= \pm\sqrt{729} = \pm 27; \text{ as } \alpha < \beta,\ \alpha - \beta = -27$$

Example 9: If $\sqrt{x+4} + \sqrt{x+8} = 7$, then find the value of x.

Solution: Given $\sqrt{x+4} + \sqrt{x+8} = 7$
Squaring on both sides, we get

$$x + 4 + x + 8 + 2\left(\sqrt{x+4}\sqrt{x+8}\right) = 49$$

$$\Rightarrow \quad 2x + 12 + 2\sqrt{x^2+12x+32} = 49$$

$$\Rightarrow \quad 2x - 37 = -2\sqrt{x^2+12x+32}$$

Squaring again on both sides, we have

$$(2x-37)^2 = 4(x^2 + 12x + 32)$$

$$\Rightarrow \quad 4x^2 - 148x + 1369 = 4x^2 + 48x + 128$$

$$\Rightarrow \quad 1241 = 196x$$

$$\Rightarrow \quad x = \frac{1241}{196}$$

Example 10: If $4^{2x+1} + 4^{x+1} = 80$, then find the value of x.

Solution: Given $4^{2x+1} + 4^{x+1} = 80$

$$\Rightarrow \quad 4^{2x} \times 4 + 4^x \times 4 = 80$$

$$4^{2x} + 4^x = 20$$

Substituting $4^x = a$,

we get $a^2 + a = 20$

$$\Rightarrow \quad a^2 + a - 20 = 0$$

$$\Rightarrow \quad (a+5)(a-4) = 0$$

$$\Rightarrow \quad a = -5 \quad \text{or} \quad 4$$

If $4^x = -5$, there is no possible value for x as no power of 4 gives negative value.

If $4^x = 4$, then $x = 1$.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The roots of the quadratic equation $2x^2 - 7x + 2 = 0$ are
(A) Rational and unequal (B) Real and equal
(C) Imaginary (D) Irrational

2. Find the nature of the roots of the quadratic equation $2x^2 + 6x - 5 = 0$.
(A) Complex conjugates
(B) Real and equal
(C) Conjugate surds
(D) Unequal and rational

3. Construct a quadratic equation whose roots are one third of the roots of $x^2 + 6x + 10 = 0$.
(A) $x^2 + 18x + 90 = 0$ (B) $x^2 + 16x + 80 = 0$
(C) $9x^2 + 18x + 10 = 0$ (D) $x^2 + 17x + 90 = 0$

4. A quadratic equation in x has its roots as reciprocals of each other. The coefficient of x is twice the coefficient of x^2. Find the sum of the squares of its roots.
(A) 5 (B) 4 (C) 3 (D) 2

5. If one root of the quadratic equation $4x^2 - 8x + k = 0$, is three times the other root, find the value of k.
(A) 3 (B) 9 (C) -3 (D) -6

6. The roots of the quadratic equation $(m - k + \ell)\, x^2 - 2mx + (m - \ell + k) = 0$ are
(A) $1, \dfrac{\ell + m - k}{k + m - \ell}$ (B) $1, \dfrac{2m}{\ell + m - k}$
(C) $1, \dfrac{k + m - \ell}{\ell + m - k}$ (D) $1, \dfrac{2k}{k - m + \ell}$

7. The expression $\dfrac{4ac - b^2}{4a}$ represents the maximum/minimum value of the quadratic expression $ax^2 + bx + c$. Which of the following is true?
(A) It represents the maximum value when $a > 0$.
(B) It represents the minimum value when $a < 0$.
(C) Both (A) and (B)
(D) Neither (A) nor (B)

8. Find the signs of the roots of the equation $x^2 + x - 420 = 0$.
(A) Both are positive.
(B) Both are negative.
(C) The roots are of opposite signs with the numerically larger root being positive.
(D) The roots are of opposite signs with the numerically larger root being negative.

9. If k is a natural number and $(k^2 - 3k + 2)\,(k^2 - 7k + 12) = 120$, find k.
(A) 7 (B) 6 (C) 5 (D) 9

10. Both A and B were trying to solve a quadratic equation. A copied the coefficient of x wrongly and got the roots of the equation as 12 and 6. B copied the constant term wrongly and got the roots as 1 and 26. Find the roots of the correct equation.
(A) 6, 16 (B) $-6, -16$
(C) 24, 3 (D) $-3, -24$

11. If the roots of the equation $(x - k_1)\,(x - k_2) + 1 = 0$, k_1 and k_2 are integers, then which of the following must be true?
(A) k_1, k_2 are two consecutive integers.
(B) $k_2 - k_1 = 2$
(C) $k_1 - k_2 = 2$
(D) Either (B) or (C)

12. The roots of the equation $ax^2 + bx + c = 0$ are k less than those of the equation $px^2 + qx + r = 0$. Find the equation whose roots are k more than those of $px^2 + qx + r = 0$.
(A) $ax^2 + bx + c = 0$
(B) $a(x - 2k)^2 + b(x - 2k) + c = 0$
(C) $a(x + 2k)^2 + b(x + 2k) + c = 0$
(D) $a(x - k)^2 + b(x - k) + c = 0$

13. If one root of the equation $x^2 - 10x + 16 = 0$ is half of one of the roots of $x^2 - 4Rx + 8 = 0$. Find R such that both the equations have integral roots.
(A) 1 (B) 2/3 (C) 3/2 (D) 4

14. If $x + y = 4$, find the maximum/minimum possible value of $x^2 + y^2$.
(A) Minimum, 8 (B) Maximum, 8
(C) Maximum, 16 (D) Minimum, 16

15. Find positive integral value(s) of p such that the equation $2x^2 + 8x + p = 0$ has rational roots.
(A) 8 (B) 4
(C) 6 (D) (A) or (C)

16. Two equations have a common root which is positive. The other roots of the equations satisfy $x^2 - 9x + 18 = 0$. The product of the sums of the roots of the two equations is 40. Find the common root.
(A) 1 (B) 2 (C) 3 (D) 4

17. If one root of the equation $x^3 - 11x^2 + 37x - 35 = 0$ is $3 - \sqrt{2}$, then find the other two roots.
(A) $5, 3 - \sqrt{2}$ (B) $-5, 3 + \sqrt{2}$
(C) $5, 3 + \sqrt{2}$ (D) $-5, 3 - \sqrt{2}$

18. The roots [the values of x (and not $|x|$)] of the equation $|x|^2 + 6|x| - 55 = 0$ are α and β. One of the roots of $py^2 + qy + r = 0$ is $\alpha\beta$ times the other root. Which of the following can be concluded?
(A) $25q^2 = -576pr$ (B) $25pr = -576q^2$
(C) $25q^2 = 576pr$ (D) $25pr = 576q^2$

19. The sides of a right-angled triangle are such that the sum of the lengths of the longest and that of the shortest side is twice the length of the remaining side. Find the longest side of the triangle if the longer of the sides containing the right angle is 9 cm more than half the hypotenuse.
(A) 30 cm (B) 25 cm (C) 20 cm (D) 15 cm

20. Solve for x: $2\{3^{2(1+x)}\} - 4(3^{2+x}) + 10 = 0$
(A) $-1, \log_3\left(\dfrac{5}{3}\right)$ (B) $-1, \log_3 2$
(C) $-1, \dfrac{5}{3}$ (D) $-1, \log_3\left(\dfrac{3}{5}\right)$

21. If $\sqrt{x^2 - 2x - 3} + \sqrt{x^2 + 5x - 24} = \sqrt{x^2 + 7x - 30}$, then find x.
(A) 2 (B) 3 (C) 4 (D) 6

22. Two software professionals Ranjan and Raman had 108 floppies between them. They sell them at different prices, but each receives the same sum. If Raman had sold his at Ranjan's price, he would have received ₹722 and if Ranjan had sold his at Raman's price, he would have received ₹578. How many floppies did Ranjan have?
(A) 51 (B) 57 (C) 68 (D) 40

23. The sum and product of the roots of a quadratic equation E are a and b respectively. Find the equation whose roots are the product of first root of E and the square of the second root of E, and the product of the second root of E and the square of the first root of E.
(A) $x^2 - abx + b^3 = 0$
(B) $x^2 + abx + b^3 = 0$
(C) $x^2 + abx - b^3 = 0$
(D) $x^2 - abx - b^3 = 0$

24. Which of the following options represent(s) a condition for the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ to have exactly one common root, given that the roots of both the equations are real?
(A) $a - b = 1$ (B) $b - a = 1$
(C) $1 + a + b = 0$ (D) Either (A) or (B)

25. If the roots of $2x^2 + (4m + 1)x + 2(2m - 1) = 0$ are reciprocals of each other, find m.
(A) –1 (B) 0 (C) 1 (D) 3/4

Answer Keys

Exercise

Practice Problems

1. D	**2.** C	**3.** C	**4.** D	**5.** A	**6.** C	**7.** D	**8.** D	**9.** B	**10.** C
11. D	**12.** B	**13.** C	**14.** A	**15.** D	**16.** B	**17.** C	**18.** A	**19.** A	**20.** A
21. B	**22.** A	**23.** A	**24.** C	**25.** C					

Chapter 11

Inequalities

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Inequalities and modulus
- Symbols and notations
- Absolute value
- Properties of modulus

Inequalities and Modulus

If 'a' is any real number, then 'a' is either positive or negative or zero. When 'a' is positive, we write $a > 0$ which is read 'a is greater than zero'. When 'a' is negative, we write $a < 0$ which is read 'a is less than zero'. If 'a' is zero, we write $a = 0$ and in this case, 'a' is neither positive nor negative.

Symbols and Notations

'$>$' means 'greater than'
'$<$' means 'less than'
'$\geq$' means 'greater than or equal to'
'$\leq$' means 'less than or equal to'

For any two non-zero real numbers a and b,

1. a is said to be greater than b when $a - b$ is positive.
2. a is said to be less than b when $a - b$ is negative.

These two statements are written as

1. $a > b$ when $a - b > 0$ and
2. $a < b$ when $a - b < 0$.

For example,
3 is greater than 2 because $3 - 2 = 1$ and 1 is greater than zero. -3 is less than -2 because $-3 - (-2) = -1$ and -1 is less than zero.

Certain properties and useful results pertaining to inequalities are given below. A thorough understanding of these properties/results is very essential for being able to solve the problems pertaining to inequalities.

[In the following list of properties and results, numbers like a, b, c, d, etc. are real numbers]

1. For any two real numbers a and b, either $a > b$ or $a < b$ or $a = b$.
2. If $a > b$, then $b < a$.
3. If $a \not< b$, then $a \geq b$ and if $a > b$, then $a \leq b$.
4. If $a > b$ and $b > c$, then $a > c$.
5. If $a < b$ and $b < c$, then $a < c$.
6. If $a > b$, then $a \pm c > b \pm$ c.
7. If $a > b$ and $c > 0$, then $ac > bc$.
8. If $a < b$ and $c > 0$, then $ac < bc$.
9. If $a > b$ and $c < 0$, then $ac < bc$.
10. If $a < b$ and $c < 0$, then $ac > bc$.
11. If $a > b$ and $c > d$, then $a + c > b + d$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. Let A, G and H be the Arithmetic mean, Geometric mean and Harmonic mean of n positive real numbers. Then $A \geq G \geq H$, the equality occurring only when the numbers are all equal.

14. If the sum of two positive quantities is given, their product is the greatest when they are equal; and if the product of two positive quantities is given, their sum is the least when they are equal.
15. If $a > b$ and $c > d$, then we cannot say anything conclusively about the relationship between $(a - b)$ and $(c - d)$; depending on the values of a, b, c and d, it is possible to have
$(a - b) > (c - d)$, $(a - b) = (c - d)$ or $(a - b) < (c - d)$

Absolute Value

(written as $|x|$ and read as 'modulus of x')

For any real number x, the absolute value is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \text{ and} \\ -x, & \text{if } x < 0 \end{cases}$$

Properties of Modulus

For any real number x and y,

1. $x = 0 \Leftrightarrow |x| = 0$
2. $|x| \geq 0$ and $-|x| \leq 0$
3. $|x + y| \leq |x| + |y|$
4. $||x| - |y|| \leq |x - y|$
5. $-|x| \leq x \leq |x|$
6. $|x \cdot y| = |x| \cdot |y|$
7. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$; $(y \neq 0)$
8. $|x|^2 = x^2$

In inequalities, the variables generally take a range of values unlike in the case of equations where the variables in general, take one value or a discrete set of values. (In some specific cases, the variables may take only one value).

Solved Examples

Example 1: If $13x - 19 \leq 4x + 26$, find the range of x.

Solution: $13x - 4x \leq 26 + 19$

$$x \leq 5.$$

$(-\infty, 5]$ in the interval notation.

Example 2: Solve the following inequalities:

$$5x + 21 < 46 \text{ and } 4x + 18 < 54.$$

Solution: $5x + 21 < 46 \Rightarrow x < 5$ (1)

$$4x + 18 < 54 \Rightarrow x < 9 \quad (2)$$

The common inequality satisfying (1) and (2) is $x < 5$ or $(-\infty, 5)$ in the interval notation.

Example 3: Which of the numbers 50^{51} and 51^{50} is greater?

Solution: Let $a = 50^{51}$ and $b = 51^{50}$.

$$\frac{b}{a} = \frac{51^{50}}{50^{51}} = \left(\frac{51}{50}\right)^{50}\left(\frac{1}{50}\right) = \left(1+\frac{1}{50}\right)^{50}\left(\frac{1}{50}\right)$$

$\left(1+\frac{1}{x}\right)^x$ where $x > 0$ always lies between 2 and 2.8.

$\therefore$ $\frac{b}{a}$ lies between

$$\frac{2}{50} = 0.04 \text{ and } \frac{2.8}{50} = 0.056$$

$\therefore$ $\frac{b}{a} < 1$

$\therefore$ $a > b$.

Example 4: Solve for x if $4x^2 - 21x + 20 > 0$

Solution: $4x^2 - 21x + 20 > 0 \Rightarrow (4x - 5)(x - 4) > 0$
Both factors are positive (i.e., the smaller is positive) or both are negative (i.e. the greater is negative) i.e., $x > 4$ or $x < \frac{5}{4}$ or it can be expressed in the interval notation as $(4, \infty) \cup \left(-\infty, \frac{5}{4}\right)$

Example 5: Solve for x, if $\frac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0$

Solution: $x^2 + 5x - 24 = (x + 8)(x - 3)$
Similarly $2x^2 - 5x - 3 = (2x + 1)(x - 3)$

Given: $\frac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0$

$\Rightarrow$ $\frac{(x+8)(x-3)}{(2x+1)(x-3)} < 0$

$\Rightarrow$ $\frac{x+8}{2x+1} < 0$

$\frac{(x+8)(2x+1)}{(2x+1)^2} < 0$

$\Rightarrow$ $(x + 8)\left(x + \frac{1}{2}\right) < 0$

$\therefore$ $-8 < x < -\frac{1}{2}$.

Example 6: Solve the inequality $|3x + 6| > -12$.

Solution: The modulus of any number is always non-negative.

$\therefore$ $|3x + 6| \geq 0$.

$\therefore$ The given inequality is always satisfied.

$\therefore \quad -\infty < x < \infty$

Example 7: Solve the inequality $|2x + 4| < -6$.

Solution: The modulus of any number is always non negative.

$\therefore \quad |2x + 4| \geq 0$

$\therefore$ The given inequality will never satisfy.
The solution is null set.

Example 8: Solve for x: $|2x - 3| = 5$

Solution: $2x - 3 = 5$ or $2x - 3 = -5$

$$(\text{If } |y| = a, y = \pm a) \quad \Rightarrow \quad x = 4 \text{ or } x = -1.$$

Example 9: Find the maximum value of

$$g(x) = 16 - |-x - 6|; x \in R.$$

Solution: $g(x)$ is maximum when $|-x - 6|$ is minimum.
The minimum value of the modulus of all numbers is 0.
$\therefore$ The maximum value of $g(x) = 16 - 0 = 16$.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. If $a < b$ and $c < 0$ then which of the following is true?

(A) $ac < bc$ (B) $\frac{a}{c} < \frac{b}{c}$

(C) $ac > bc$ (D) None of these

2. If p and q are two real numbers, then which of the following statements is always true?

(A) $\frac{p}{q} < 1 \Rightarrow p < q$

(B) $p > 0, q > 0$ and $\frac{p}{q} > 1 \Rightarrow p > q$

(C) $\frac{p}{q} > 1 \Rightarrow p > q$

(D) All the above

3. If $5x - 8 < 2x + 9$ and $4x + 7 > 7x - 8$, then the range of the values of x that satisfies the inequalities is

(A) $(5, \infty)$ (B) $(-\infty, 5)$

(C) $\left(5, \frac{17}{3}\right)$ (D) $\left(-\infty, \frac{17}{3}\right)$

4. Solve for real values of x; $5x^2 - 3x - 2 \geq 0$.

(A) $\left[\frac{-2}{5}, 1\right]$ (B) $R - \left(\frac{-2}{5}, 1\right)$

(C) $[1, \infty)$ (D) $R - (0, 1)$

5. If $x^2 - 9x - 36$ is negative, then find the range of x.

(A) $(-3, 12)$ (B) $[-3, 12]$

(C) $(-12, 3)$ (D) $[-12, 3]$

6. Which of the following is true?

(A) $|x + y| \leq |x| + |y|$ (B) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$

(C) $|x - y| \geq ||x| - |y||$ (D) All the above

7. If $6x + 8 > 7x - 9$ and $4x - 7 < 6x - 3$, then the values of x is

(A) $(-17, 2)$ (B) $(2, 17)$

(C) $(-2, 17)$ (D) $(-\infty, 17)$

8. The solution set of the inequality $|x - 5| < 9$ is

(A) $(0, 14)$ (B) $(-4, 14)$

(C) $(-4, 0)$ (D) $(9, 14)$

9. The number of integral values of x, that do not satisfy the inequation $\frac{x+5}{x-2} \geq 0$ is

(A) 7 (B) 5 (C) 6 (D) 4

10. If $(x + 5)(x + 9)(x + 3)^2 < 0$, then the solution set for the inequality is

(A) $(-9, -3)$ (B) $(-9, -5)$

(C) $(-3, \infty)$ (D) $(-9, \infty)$

11. Find the range of the real values of x satisfying $8 - 3x \leq 5$ and $4x + 5 \leq -7$.

(A) $[-3, 1]$ (B) $(-\infty, -3] \cup [1, \infty)$

(C) $(-3, 1)$ (D) ϕ

12. Which of the following is true?

(A) $30^{31} < 31^{30}$ (B) $71^{69} > 70^{70}$

(C) $(155)^{29} < (150)^{30}$ (D) Both (B) and (C)

13. At what value of x is $-|x - 3| + \frac{21}{2}$ maximum?

(A) -3 (B) $\frac{21}{2}$ (C) 0 (D) 3

14. Find the range of all real values of x if $|3x + 5| < 5x - 11$.

(A) $(8, \infty)$ (B) $(-\infty, -5/3) \cup (8, \infty)$

(C) $(-5/3, 8)$ (D) $(-5/3, \infty)$

15. If $ac = bd = 2$, then the minimum value of $a^2 + b^2 + c^2 + d^2$ is

(A) 4 (B) 6 (C) 8 (D) 16

16. If $x, y > 0$ and $x + y = 3$ then

(A) $xy \leq 0.72$ (B) $xy \leq 1.8$

(C) $xy \leq 2.25$ (D) $xy \leq 1.25$

17. Find the complete range of values of x that satisfies $|x - 16| > x^2 - 7x + 24$.

(A) $(0, 2)$ (B) $\left(\frac{3}{2}, \frac{5}{2}\right)$

(C) $(1, 3)$ (D) $(2, 4)$

18. For which of the following range of values of x is $x^2 + x$ less than $x^3 + 1$?
(A) $(-\infty, -1)$ (B) $(1, \infty)$
(C) $(-1, 1) \cup (1, \infty)$ (D) $[-1, 1]$

19. If x, y, z are positive, then the value of
$A = \frac{(4x^2 + x + 4)(5y^2 + y + 5)(7z^2 + z + 7)}{xyz}$ can be
(A) 400 (B) 500 (C) 1000 (D) 1500

20. The range of x for which $2x^2 - 5x - 8 \le |2x^2 + x|$ is
(A) $\left[-\frac{4}{3}, \infty\right)$ (B) $\left(-\frac{4}{3}, -1\right)$
(C) $[-1, \infty)$ (D) $[-1, 2]$

21. For how many integral values of x, is the inequation $\frac{x-5}{x+7} > 4$ satisfied?
(A) 5 (B) 4 (C) 2 (D) 3

22. If $1 \le x \le 3$ and $2 \le y \le 5$, then the minimum value of $\frac{x+y}{y}$ is
(A) $\frac{3}{5}$ (B) $\frac{1}{5}$ (C) $\frac{6}{5}$ (D) $\frac{5}{6}$

23. If $|b| \ge 5$ and $x = |a|\, b$, which of the following is true?
(A) $a - xb > 0$ (B) $a + xb < 0$
(C) $a + xb > 0$ (D) $a - xb \le 0$

24. Find the number of solutions of the equation $\big|x - |x - 2|\big| = 6$.
(A) 2 (B) 1 (C) 3 (D) 4

25. If x, y and z are positive real numbers, then the minimum value of
$\frac{x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2}{xyz}$ is
(A) 6 (B) 9 (C) 12 (D) 14

Answer Keys

Exercise

Practice Problems

1. C	**2.** B	**3.** B	**4.** B	**5.** A	**6.** D	**7.** C	**8.** B	**9.** A	**10.** B
11. D	**12.** C	**13.** D	**14.** A	**15.** C	**16.** C	**17.** D	**18.** C	**19.** D	**20.** A
21. D	**22.** C	**23.** D	**24.** B	**25.** A					

Chapter 12

Progressions

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Progressions
- Arithmetic progression (A.P.)
- Geometric progression (G.P.)
- Infinite geometric progression
- Some important results

Progressions

In this chapter, we will look at the problems on sequences or progressions of numbers, where the terms of the sequence follow a particular pattern either addition of a constant (Arithmetic Sequence or Arithmetic Progression) or multiplication by a constant (Geometric Sequence or Geometric Progression). A third type of progression–Harmonic Progression–has also been defined later.

Arithmetic Progression (A.P.)

An arithmetic progression is a sequence of numbers in which any number (other than the first) is more (or less) than the immediately preceding number by a constant value. This constant value is called the common difference. In other words, any term of an arithmetic progression can be obtained by adding the common difference to the preceding term.

Let a be the first term of an arithmetic progression, d the common difference and n the number of terms in the progression.

The n^{th} term is normally represented by T_n and the sum to n terms of an arithmetic progression is denoted by S_n

$$T_n = n^{th} \text{ term} = a + (n-1)\,d$$

S_n = Sum of n terms $= \frac{n}{2} \times [2a + (n-1)\,d]$, then the progression can be represented as a, $a + d$, $a + 2d$, …, $[a + (n-1)d]$. Here, quantity d is to be added to any chosen term to get the next term of the progression.

The sum to n terms of an arithmetic progression can also be written in a different manner.

$$\text{Sum of first } n \text{ terms} = \frac{n}{2} \times [2a + (n-1)d]$$

$$= \frac{n}{2} \times [a + \{a + (n-1)d\}]$$

But, when there are n terms in an arithmetic progression, a is the first term and $\{a + (n-1)\,d\}$ is the last term. Hence, $S_n = \frac{n}{2} \times$ [First Term + Last Term]

The average of all the terms in an arithmetic progression is called their Arithmetic Mean (A.M.). Since average is equal to {sum of all the quantities/number of quantities}, arithmetic progression must be equal to the Sum of the terms of the arithmetic progression divided by the number of terms in the arithmetic progression.

Arithmetic Mean of n terms in arithmetic progression

$$= \frac{S_n}{n} = \frac{1}{2}\,\{2a + (n-1)d\}$$

$$= \frac{1}{2} \times (\text{First Term} + \text{Last Term})$$

$$= \frac{(\text{First Term} + \text{Last Term})}{2}$$

i.e., A.M. is the average of the first and the last terms of the A.P.

Arithmetic Mean can also be obtained by taking the average of any two terms which are EQUIDISTANT from the two ends of the A.P. i.e.,

1. The average of the second term from the beginning and the second term from the end will be equal to the A.M.
2. The average of the third term from the beginning and the third term from the end will also be equal to the A.M. and so on.

In general, the average of the k^{th} term from the beginning and the k^{th} term from the end will be equal to the A.M.

Conversely, if the A.M. of an A.P. is known, the sum to n terms of the series (S_n) can be expressed as

$$S_n = n \times \text{A.M.}$$

If three numbers are in arithmetical progression the middle number is called the Arithmetic Mean, i.e., if a, b, c are in A.P., then b is the A.M. of the three terms and $b = \frac{a+c}{2}$.

If a and b are in Arithmetic Progression (A.P.), then their A.M. $= \frac{(a+b)}{2}$.

If three numbers are in A.P., we can represent the three numbers as $(a - d)$, a and $(a + d)$.

If four numbers are in A.P., we can represent the four numbers as $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$; (in this case, $2d$ is the common difference).

If five numbers are in A.P., we can represent the five numbers as $(a - 2d)$, $(a - d)$, a, $(a + d)$ and $(a + 2d)$.

Solved Examples

Example 1: The sixth and the tenth terms of an arithmetic progression are 22 and 38 respectively. Find the first term and the common difference.

Solution: Let the first term and the common difference be a and d respectively.

$$a + 5d = 22 \quad (1)$$

$$a + 9d = 38 \quad (2)$$

Subtracting (1) from (2),

$$4d = 16,\ d = 4$$

Substituting d in (1) or (2),

we get $a = 2$

Example 2: The 12^{th} term, the 14^{th} term and the last term of an arithmetic progression are 25, 31 and 37 respectively. Find the first term, common difference and the number of terms.

Solution: Let the first term, the common difference and the number of terms be a, d and n respectively.
Given that

$$a + 11d = 25 \quad (3)$$

$$a + 13d = 31 \quad (4)$$

Subtracting (3) from (4),

$$2d = 6$$

$$d = 3$$

Substituting $d = 3$ in (3) or (4),

$$a = -8$$

given, $t_n = -8 + (n - 1)3 = 37$

$$n = 16$$

Example 3: Three terms in arithmetic progression have a sum of 45 and a product of 3240. Find them.

Solution: Let the terms be $a - d$, a and $a + d$.

$$a - d + a + a + d = 45$$

$$a = 15$$

$$(a - d)\, a\, (a + d) = 3240$$

$$15^2 - d^2 = 216$$

$$d = \pm 3$$

If $d = 3$ the terms are 12, 15 and 18. If $d = -3$, the terms are same but in the descending order.

Example 4: The first term and the last term of an arithmetic progression are 9 and 69 respectively. If the sum of all the terms is 468, find the number of terms and the common difference.

Solution: Let the number of terms and the common difference be n and d respectively,

$$S_n = \frac{n}{2}[9 + 69] = 468$$

$$\Rightarrow \quad 39n = 468$$

$$n = 12$$

$$t_n = 9 + 11d$$

$$\Rightarrow \quad 11d = 60$$

$$\Rightarrow \quad d = \frac{60}{11}$$

Example 5: The sum of three numbers which are in arithmetic progression is 24. The sum of their square is 200. Find the numbers.

Solution: Let the numbers be $a - d$, a and $a + d$.

Given, $a - d + a + a + d = 24$

$\therefore \quad a = 8$

$$(a - d)^2 + a^2 + (a + d)^2 = 200$$

$$3a^2 + 2d^2 = 200$$

$$\Rightarrow \quad d^2 = 4$$

$$\therefore \quad d = \pm 2$$

If $d = 2$, the numbers are 6, 8 and 10. If $d = -2$, the numbers are same, but in the descending order.

Geometric Progression (G.P.)

Numbers taken in a certain order, are said to be in Geometrical Progression, if the ratio of any (other than the first number) to the preceding one is the same. This ratio is called the Common Ratio. In other words, any term of a geometric progression can be obtained by multiplying the preceding number by the common ratio.

The common ratio is normally represented by r. The first term of a geometric progression is denoted by a.

A geometric progression can be represented as a, ar, $1ar^2$, … where a is the first term and r is the common ratio of the geometric progression.

n^{th} term of the geometric progression is ar^{n-1}.

Sum to n terms:

$$\frac{a\left(1-r^n\right)}{1-r} \text{ or } \frac{a\left(r^n-1\right)}{r-1}$$

$$= \frac{xar^{n-1}-a}{r-1} = \frac{r\times\text{Last term}-\text{First term}}{r-1}$$

Thus the sum to n terms of a geometric progression can also be written as

$$S_n = \frac{r\times\text{Last term}-\text{First term}}{r-1}$$

If n terms $a_1, a_2, a_3, \ldots, a_n$ are in G.P., then the Geometric Mean (G.M.) of these n terms is given by $= \sqrt[n]{a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n}$

If three terms are in geometric progression then the middle term is a Geometric Mean of the other two terms, i.e., if a, b and c are in G.P., then b is the geometric mean of the three terms and $b^2 = ac$.

If there are two terms a and b, their geometric mean (G.M.) is given by $\sqrt{ab}$.

For any two unequal positive numbers a and b, their Arithmetic Mean is always greater than their Geometric Mean, i.e.

For any two unequal positive numbers a and b. $\frac{a+b}{2} > \sqrt{ab}$; $(a+b) > 2\sqrt{ab}$ $b) >$

When there are three terms in geometric progression, we can represent the three terms to be a/r, a and ar

When there are four terms in geometric progression, we can represent the four terms as $\frac{a}{r^3}$, $\frac{a}{r}$, ar and ar^3.

(In this case r^2 is the common ratio)

Infinite Geometric Progression

If $-1 < r < +1$ or $|r| < 1$, then the sum of a geometric progression does not increase infinitely; it 'converges' to a particular value. Such a G.P. is referred to as an infinite geometric progression. The sum of an infinite geometric progression is represented by S_∞ and is given by the formula $S_\infty = \frac{a}{1-r}$.

Harmonic progression: If the reciprocals of the terms of a sequence are in arithmetic progression, the sequence is said to be a harmonic progression, For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a harmonic progression. In general, the sequence $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ is a harmonic progression.

If a, b, c are in harmonic progression, b is said to be the harmonic mean of a and c. In general, if $x_1, x_2, \ldots, x_n$ are in harmonic progression $x_2, x_3, \ldots, x_{n-1}$ are the $n-2$ harmonic means between x_1 and x_n.

Some Important Results

The results of the sums to n terms of the following series are quite useful and hence should be remembered by students.

Sum of the first n natural numbers $= \sum n = \frac{n(n+1)}{2}$

Sum of squares of the first n natural numbers

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of cubes of first n natural numbers

$$\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \left[\sum n\right]^2$$

Example 6: Find the 7^{th} term of the geometric progression whose first term is 6 and common ratio is 2.

Solution: n^{th} term of a G.P. $= ar^{n-1}$

7^{th} term $= 6\ (2^6) = 384$

Example 7: A geometric progression has its first term as 64 and its common ratio as $\frac{1}{2}$. Find the sum of its first five terms.

Solution: Sum of the first n terms of a G.P. $= \frac{a(1-r^n)}{1-r}$

$$\text{Sum of its first five terms} = \frac{64\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}} = 124$$

Example 8: Find the common ratio of the geometric progression whose first and last terms are 5 and $\frac{1}{25}$ respectively and the sum of its terms is $\frac{624}{100}$.

Solution: Sum of the terms of a geometric progression whose common ratio is r, is given by $\frac{r(\text{last term}) - (\text{first term})}{r-1}$

$$\frac{r\left(\frac{1}{25}\right)-5}{r-1} = \frac{624}{100}.$$

$$\Rightarrow \quad 4r - 500 = 624r - 624$$

$$\Rightarrow \quad r = \frac{1}{5}.$$

Example 9: Three numbers in geometric progression have a sum of 42 and a product of 512. Find the numbers.

Solution: Let the numbers be $\frac{a}{r}, a$ and ar.

$$\frac{a}{r} + a + ar = 42$$

$$\left(\frac{a}{r}\right)(a)(ar) = 512$$

$$a = 8$$

$$\frac{8}{r} + 8 + 8r = 42$$

$$8r^2 - 34r + 8 = 0$$

$$8r^2 - 32r - 2r + 8 = 0$$

$$(r-4)(4r-1) = 0$$

$$r = 4 \quad \text{or} \quad \frac{1}{4}$$

If $r = 4$, the numbers are 2, 8 and 32. If $r = \frac{1}{4}$, the numbers are same, but in the descending order.

Example 10: The sum of the terms of an infinite geometric progression is 27. The sum of their squares is 364.5. Find the common ratio.

Solution: Let the first term and the common ratio be a and r respectively.

Given that $\frac{a}{1-r} = 27 \Rightarrow \left(\frac{a}{1-r}\right)^2 = 729$

And $\frac{a^2}{1-r^2} = 364.5$

$$\Rightarrow \quad a^2 = 729(1-r)^2$$

$$= 364.5(1-r^2)$$

$$729(1-r)^2 - \frac{729}{2}(1-r)(1+r) = 0$$

$$\frac{729}{2}(1-r)[2(1-r) - (1+r)] = 0$$

$$\Rightarrow \quad (1-r)(1-3r) = 0$$

$$r \neq 1 \quad (\because |r| < 1)$$

$$\therefore \quad r = \frac{1}{3}$$

Example 11: If $|x| < 1$, find the value of $3 + 6x + 9x^2 + 12x^3 + \ldots$

Solution: Let $S = 3 + 6x + 9x^2 + 12x^3 + \ldots$ (7)

$$xS = 3x + 6x^2 + 9x^3 + \quad (8)$$

Subtracting (8) from (7)

$$S(1-x) = 3(1 + x + x^2 + \ldots)$$

As $|x| < 1$, $S = \frac{3\left(\frac{1}{1-x}\right)}{1-x} = \frac{3}{(1-x)^2}$

Example 12: Evaluate $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{99(100)}$.

Solution: $\frac{1}{1(2)} = \frac{1}{1} - \frac{1}{2}$

$$\frac{1}{2(3)} = \frac{1}{2} - \frac{1}{3}$$

Finally $\frac{1}{99(100)} = \frac{1}{99} - \frac{1}{100}$

The given expression is $1 - \frac{1}{100} = \frac{99}{100}$

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The sixth term and the eleventh term of an arithmetic progression are 30 and 55 respectively. Find the twenty-first term of the series.

(A) $88\frac{1}{3}$ (B) 105 (C) 110 (D) $92\frac{1}{2}$

2. What is the 15^{th} term of an arithmetic progression whose first term is equal to its common difference and whose 3^{rd} term is 9?

(A) 15 (B) 30 (C) 45 (D) 60

3. If $x + 4$, $6x - 2$ and $9x - 4$ are three consecutive terms of an arithmetic progression, then find x.

(A) 2 (B) 4 (C) 6 (D) 8

4. Find the number of terms and the sum of the terms of the arithmetic progression 32, 28, … 4.

(A) 8; 144 (B) 7; 126
(C) 14; 252 (D) 15; 270

5. Find the sum of the first 31 terms of the arithmetic progression whose first term is 6 and whose common difference is $\frac{8}{3}$.

(A) 1410 (B) 1418 (C) 1426 (D) 1434

6. The sum of five terms of an arithmetic progression is 70. The product of the extreme terms is 132. Find the five terms.
(A) 4, 8, 12, 16, 20 (B) 10, 12, 14, 16, 18
(C) 6, 10, 14, 18, 22 (D) 8, 12, 16, 20, 24

7. The sum to n terms of an arithmetic progression is $5n^2 + 2n$. Find the n^{th} term of the series.
(A) $10n + 5$ (B) $10n - 3$
(C) $5n - 1$ (D) $5n - 2$

8. Which term of the geometric progression 4, $4\sqrt{2}$, 8 … is $64\sqrt{2}$?
(A) 8 (B) 9 (C) 10 (D) 12

9. Find the sixth term of the geometric progression whose first term is 2 and common ratio is 3.
(A) 96 (B) 486 (C) 1458 (D) 162

10. Find the sum of the first 4 terms of a geometric progression whose first term is 6 and whose common ratio is 2.
(A) 90 (B) 84 (C) 96 (D) 102

11. What is the sum of the first 7 terms of a geometric progression whose first term is 1 and 4^{th} term is 8?
(A) 129 (B) 128
(C) 127 (D) None of these

12. If the sum to 37 terms of an arithmetic progression is 703, then find the middle term of the arithmetic progression.
(A) 34 (B) 17 (C) 38 (D) 19

13. Find the sum of the 20 terms of the series 1, (1 + 2), (1 + 2 + 3), (1 + 2 + 3 + 4), (1 + 2 + 3 + 4 + 5), …
(A) 1540 (B) 1435 (C) 1450 (D) 1345

14. If the real numbers a, c and b as well as $a^2 + b^2$, $a^2 + c^2$ and $b^2 + c^2$ are in geometric progression, then which of the following is necessary true?
(A) $a = b$ (B) $b = c$
(C) $a = c$ (D) $a = b = c$

15. How many numbers between 450 and 950 are divisible by both 3 and 7?
(A) 20 (B) 24 (C) 30 (D) 35

16. $S = 2 + 4x + 6x^2 + 8x^3$ … where $|x| < 1$. Which of the following is the value of S?
(A) $\frac{4}{(1-x)^2}$ (B) $\frac{3}{(1-x)^2}$
(C) $\frac{2}{(1-x)^2}$ (D) $\frac{1}{(1-x)^2}$

17. The sum of the first eight terms of a geometric progression. is 510 and the sum of the first four terms of the geometric progression. is 30. Find the first term of the geometric progression., given that it is positive.
(A) 2 (B) 4 (C) 6 (D) 8

18. Find the integer value of y, if $-x$, $2y$ and $2(y + 3)$ are in arithmetic progression and $(x + 2)$, $2(y + 1)$ and $(5y - 1)$ are in geometric progression.
(A) 2 (B) 3 (C) 4 (D) 5

19. Find the number of terms common to the progressions 2, 8, 14, 20, …, 98 and 6, 10, 14, 18, …, 102.
(A) 7 (B) 6 (C) 8 (D) 9

20. Find the sum of the series $2 + 3x + 4x^2 + 5x^3 + \ldots$ to infinity, if $|x| < 1$.
(A) $\frac{2-x}{(1-x)^2}$ (B) $\frac{2+x}{(1+x)^2}$
(C) $\frac{2-x}{(1+x)^2}$ (D) $\frac{2+x}{(1-x)^2}$

21. The mean of the sequence 3, 8, 17, 30, …, 1227 is _____.
(A) 531 (B) 431 (C) 314 (D) 315

22. Find the value of $-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 + \ldots - 19^2 + 20^2$
(A) 210 (B) 420 (C) 630 (D) 720

23. Find the sum of the given terms in the following series: $\frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \ldots + \frac{1}{\sqrt{119}+\sqrt{121}}$
(A) $2\sqrt{3}+1$ (B) 5
(C) $11 - 2\sqrt{3}$ (D) 10

24. If $\log_3 x + \log_{\sqrt[3]{3}} x + \log_{\sqrt[9]{3}} x + \ldots + \log_{\sqrt[2^n]{3}} x = 432$, then find x.
(A) 9 (B) 27 (C) $3\sqrt{3}$ (D) 81

25. The sum of the first n terms of two arithmetic progressions S_1 and S_2 are in the ratio $11n - 17 : 5n - 21$. Find the ratio of the 16^{th} terms of S_1 and S_2.
(A) 3 : 2 (B) 162 : 67
(C) 9 : 4 (D) 27 : 8

Answer Keys

Exercise

Practice Problems

1. B	**2.** C	**3.** A	**4.** A	**5.** C	**6.** C	**7.** B	**8.** C	**9.** B	**10.** A
11. C	**12.** D	**13.** A	**14.** A	**15.** B	**16.** C	**17.** A	**18.** A	**19.** C	**20.** A
21. B	**22.** A	**23.** B	**24.** B	**25.** B					

Chapter 13

Permutations and Combinations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Permutations
- Combinations
- Number of linear permutations of 'n' dissimilar items taken 'r' at a time without repetition
- Number of arrangements of 'n' items of which 'p' are of one type, 'q' are of a second type and the rest are distinct
- Number of arrangements of 'n' distinct items where each item can be used any number of times
- Total number of combinations
- Dividing given items into groups
- Circular Permutations
- Rank of a word
- Arrangements

Introduction

Permutations and Combinations is one of the important areas in many exams because of two reasons. The first is that solving questions in this area is a measure of students' reasoning ability. Secondly, solving problems in areas like Probability requires thorough knowledge of Permutations and Combinations.

Before discussing Permutations and Combinations, let us look at what is called as the 'fundamental rule'.

'If one operation can be performed in 'm' ways and (when, it has been performed in any one of these ways), a second operation then can be performed in 'n' ways, the number of ways of performing the two operations will be $m \times n$'.

This can be extended to any number of operations.

If there are three cities A, B and C such that there are 3 roads connecting A and B and 4 roads connecting B and C, then the number of ways one can travel from A to C is 3×4, i.e., 12.

This is a very important principle and we will be using it extensively in Permutations and Combinations. Because we use it very extensively, we do not explicitly state every time that the result is obtained by the fundamental rule but directly write down the result.

Permutations

Each of the arrangements which can be made by taking some or all of a number of items is called a Permutation. Permutation implies 'arrangement' or that 'order of the items' is important.

The permutations of three items a, b and c taken two at a time are ab, ba, ac, ca, cb and bc. Since the order in which the items are taken is important, ab and ba are counted as two different permutations. The words 'permutation' and 'arrangement' are synonymous and can be used interchangeably.

The number of permutations of n things taking r at time is denoted by ${}^nP_r$ (and read as '${}^nP_r$').

Combinations

Each of the groups or selections which can be made by taking some or all of a number of items is called a Combination. In combinations, the order in which the items are taken is not considered as long as the specific things are included.

The combination of three items a, b and c taken two at a time are ab, bc and ca. Here, ab and ba are not considered separately because the order in which a and b are taken is not important but it is only required that a combination including a and b is what is to be counted. The words 'combination' and 'selection' are synonymous.

The number of combinations of n things taking r at time is denoted by ${}^nC_r$ (and read as '${}^nC_r$').

Number of linear permutations of 'n' dissimilar items taken 'r' at a time without repetition (${}^nP_r$)

Consider r boxes each of which can hold one item. When all the r boxes are filled, what we have is an arrangement of r items

taken from the given n items. So, each time we fill up the r boxes with items taken from the given n items, we have an arrangement of r items taken from the given n items without repetition. Hence the number of ways in which we can fill up the r boxes by taking things from the given n things is equal to the number of permutations of n things taking r at a time.

Boxes □ □ □ □ ..... □
1 2 3 4 r

The first box can be filled in n ways (because any one of the n items can be used to fill this box). Having filled the first box, to fill the second box we now have only $(n-1)$ items; any one of these items can be used to fill the second box and hence the second box can be filled in $(n-1)$ ways; similarly, the third box in $(n-2)$ ways and so on the r^{th} box can be filled in $\{n-(r-1)\}$ ways, i.e. $[n-r+1]$ ways. Hence, from the Fundamental Rule, all the r boxes together be filled up in

$$n \times (n-1) \times (n-2) \ldots (n-r+1) \text{ ways}$$

So, $${}^nP_r = n \times (n-1) \times (n-2) \ldots (n-r+1)$$

This can be simplified by multiplying and dividing the right hand side by $(n-r)\,(n-r-1) \ldots 3.2.1$ giving us ${}^nP_r = n(n-1)\,(n-2) \ldots [n-(r-1)]$

$$= \frac{(n-1)(n-2)\ldots[n-(r-1).(n-r)\ldots 3.2.1]}{(n-r)\ldots\ldots\ldots 3.2.1} = \frac{n!}{(n-r)!}$$

The number of permutations of n distinct items taking r items at a time is

$$\boxed{{}^n p_r = \frac{n!}{(n-r)!}}$$

If we take n items at a time, then we get ${}^nP_n$. From a discussion similar to that we had for filling the r boxes above, we can find that ${}^nP_n$ is equal to $n!$

The first box can be filled in n ways, the second one in $(n-1)$ ways, the third one in $(n-2)$ ways and so on, then the n^{th} box in 1 way; hence, all the n boxes can be filled in $n(n-1)\,(n-2) \ldots 3.2.1$ ways, i.e., $n!$ ways. Hence,

$$\boxed{{}^n\boldsymbol{P}_n = \boldsymbol{n!}}$$

But if we substitute $r = n$ in the formula for ${}^nP_r$, then we get ${}^nP_n = \frac{n!}{0!}$; since we already found that ${}^nP_n = n!$, we can conclude that $0! = 1$

Number of combinations of n dissimilar things taken r at a time.

Let the number of combinations ${}^nC_r$ be x. Consider one of these x combinations. Since this is a combination, the order of the r items is not important. If we now impose the condition that order is required for these r items, we can get $r!$ arrangements from this one combination. So each combination can give rise to $r!$ permutations. x combinations will thus give rise to $x \,.\, r!$ permutations. But since these are all permutations of n things taken r at a time, this must be equal to ${}^nP_r$. So,

$$x.r! = {}^nP_r = \frac{n!}{(n-r)!}$$

$$\Rightarrow \qquad {}^nC_r = \frac{n!}{r!.(n-r)!}$$

The number of combinations of n dissimilar things taken all at a time is 1.

Out of n things lying on a table, if we select r things and remove them from the table, we are left with $(n-r)$ things on the table – that is, whenever r things are selected out of n things, we automatically have another selection of the $(n-r)$ things. Hence, the number of ways of making combinations taking r out of n things is the same as selecting $(n-r)$ things out of n given things, i.e.,

$$\boldsymbol{{}^nC_r = {}^nC_{n-r}}$$

When we looked at ${}^nP_r$, we imposed two constraints which we will now release one by one and see how to find out the number of permutations.

Number of arrangements of *n* items of which *p* are of one type, *q* are of a second type and the rest are distinct

When the items are all not distinct, then we **cannot** talk of a general formula for $\boldsymbol{{}^nP_r}$ for any r but we can talk of only ${}^nP_n$ (which is given below). If we want to find out ${}^nP_r$ for a specific value of r in a given problem, we have to work on a case to case basis (this has been explained in one of the solved examples).

The number of ways in which n things may be arranged taking them all at a time, when p of the things are exactly alike of one kind, q of them exactly alike of another kind, r of them exactly alike of a third kind, and the rest all distinct is $\frac{n!}{p!\,q!\,r!}$.

Number of arrangements of n distinct items where each item can be used any number of times (i.e., repetition allowed)

You are advised to apply the basic reasoning given while deriving the formula for ${}^nP_r$ to arrive at this result also. The first box can be filled up in n ways; the second box can be filled in again inn ways (even though the first box is filled with one item, the same item can be used for filling the second box also because repetition is allowed); the third box can also be filled in n ways and so on … the r^{th} box can be filled in n ways. Now all the r boxes together can be filled in $\{n.n.n.n \ldots r \text{ times}\}$ ways, i.e., $\boldsymbol{n^r}$ ways.

The number of permutations of n things, taken r at a time when each item may be repeated once, twice, … up to r times in any arrangement is $\boldsymbol{n^r}$.

What is important is not this formula by itself but the reasoning involved. So, even while solving problems of this type, you will be better off if you go from the basic reasoning and not just apply this formula.

Total number of combinations: Out of n given things, the number of ways of selecting **one or more** things is where we can select 1 or 2 or 3 … and so on n things at a time; hence the number of ways is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$

This is called 'the total number of combinations' and is equal to $2^n - 1$ where n is the number of things.

The same can be reasoned out in the following manner also.

There are n items to select from. Let each of these be represented by a box.

	1	2	3	4		n
No. of ways of dealing	☐	☐	☐	☐	.....	☐
with the boxes	2	2	2	2		2

The first box can be dealt with in two ways. In any combination that we consider, this box is **either** included **or** not included. These are the two ways of dealing with the first box. Similarly, the second box can be dealt with in two ways, the third one in two ways and so on, the n^{th} box in two ways. By the Fundamental Rule, the number of ways of dealing with all the boxes together in $2 \times 2 \times 2 \times \ldots$ n times ways, i.e., in 2^n ways. But out of these, there is one combination where we 'do not include the first box, do not include the second box, do not include the third box and so on, do not include the n th box.' That means, no box is included. But this is not allowed because we have to select **one or more** of the items (i.e., at least one item). Hence this combination of no box being included is to be subtracted from the 2^n ways to give the result of

Number of ways of selecting one or more items from n given items is $2^n - 1$

Dividing given items into groups: Dividing $(p + q)$ items into two groups of p and q items respectively.

Out of $(p + q)$ items, if we select p items (which can be done in ${}^{p+q}C_p$ ways), then we will be left with q items, i.e., we have two groups of p and q items respectively. So, the number of ways of dividing $(p + q)$ items into two groups of p and q items respectively is equal to ${}^{p+q}C_p$ which is equal to $\frac{(p+q)!}{p!\,q!}$

The number of ways of dividing $(p + q)$ items into two groups of p and q items respectively is $\frac{(p+q)!}{p!\cdot q!}$

If $p = q$, i.e., if we have to divide the given items into two EQUAL groups, then two cases arise

1. When the two groups have distinct identity and
2. When the two groups do not have distinct identity.

In the first case, we just have to substitute $p = q$ in the above formula which then becomes

The number of ways of dividing 2p items into two equal groups of p each is $\frac{(2p)!}{(p!)^2}$ where the two groups have distinct identity.

In the second case, where the two groups do not have distinct identity, we have to divide the above result by 2!, i.e., it then becomes

The number of ways of dividing 2p items into two equal groups of p each is $\frac{(2p)!}{2!(p!)^2}$ where the two groups do not have distinct identity.

Dividing $(p + q + r)$ items into three groups consisting of p, q and r items respectively

The number of ways in which $(p + q + r)$ things can be divided into three groups containing p, q and r things respectively is $\frac{(p+q+r)!}{p!q!r!}$

If $p = q = r$, i.e., if we have to divide the given items into three EQUAL groups, then we have two cases where the three groups are distinct and where the groups are not distinct.

When the three groups are distinct, the number of ways is $\frac{(3p)!}{(p!)^3}$

When the three groups are not distinct, then the number of ways is $\frac{(3p)!}{3!(p!)^3}$

Circular Permutations

When n distinct things are arranged in a straight line taking all the n items, we get $n!$ permutations. However, if these n items are arranged in a circular manner, then the number of arrangements will not be $n!$ but it will be less than that. This is because in a straight line manner, if we have an arrangement ABCDE and if we move every item one place to the right (in cyclic order), the new arrangement that we get EABCD is not the same as ABCDE and this also is counted in the $n!$ permutations that we talked of. However, if we have an arrangement ABCDE in a circular fashion, by shifting every item by one place in the clockwise direction, we still get the same arrangement ABCDE. So, if we now take $n!$ as the number of permutations, we will be counting the same arrangement more than once.

The number of arrangements in circular fashion can be found out by first fixing the position of one item. Then the remaining $(n - 1)$ items can be arranged in $(n - 1)!$ ways. Now even if we move these $(n - 1)$ items by one place in the clockwise direction, then the arrangement that we get will not be the same as the initial arrangement because one item is fixed and it does not move.

Hence, the number of ways in which n distinct things can be arranged in a circular arrangement is $(n-1)!$

*The number of **circular arrangements** of **n distinct items** is **(n – 1)!** if there is **DIFFERENCE** between clockwise and anticlockwise arrangements and **(n – 1)!/2** if there is **NO DIFFERENCE** between clockwise and anticlockwise arrangements.*

The number of diagonals in an *n*-sided regular polygon

An n-sided regular polygon has n vertices. Joining any two vertices we get a line of the polygon which are ${}^nC_2$ in number. Of these ${}^nC_2$ lines, n of them are sides. Hence diagonals are ${}^nC_2 - n = \dfrac{n(n-3)}{2}$

Number of integral solution of the equation

$$x_1 + x_2 + \ldots + x_n = s$$

Consider the equation $x_1 + x_2 + x_3 = 10$

If we consider all possible integral solutions of this equation, there are infinitely many. But the number of positive (or non-negative) integral solutions is finite.

We would like the number of positive integral solutions of this equation, i.e., values of (x_1, x_2, x_3) such that each $x_i > 0$.

We imagine 10 identical objects arranged on a line. There are 9 gaps between these 10 objects. If we choose any two of these gaps, we are effectively splitting the 10 identical objects into 3 parts of distinct identity. Conversely, every split of these 10 objects corresponds to a selection of 2 gaps out of the 9 gaps.

Therefore, the number of positive integral solutions is ${}^9C_2$. In general, if $x_1 + x_2 + \ldots + x_n = s$ where $s \geq n$, the number of positive integral solutions is ${}^{s-1}C_{n-1}$.

If we need the number of non negative integral solutions, we proceed as follows. Let $a_1, a_2, \ldots$ be a non-negative integral solution. Than $a_1 + 1, a_2 + 1, \ldots, a_n + 1$ is a positive integral solution of the equation $x_1 + x_2 + \ldots + x_n = s + n$. Therefore, the number of non-negative integral solutions of the given equation is equal to the number of positive integral solutions of $x_1 + x_2 + \ldots + x_n = s + n$, which is ${}^{s+n-1}C_{n-1}$.

*For $x_1 + x_2 + x_3 + \ldots + x_n = s$ where $s \geq 0$, the number of **positive integral solutions** (when $s \geq n$) is ${}^{s-1}\mathbf{C}_{n-1}$ and the number of **non-negative integral solutions** is ${}^{n+s-1}\mathbf{C}_{n-1}$*

Some additional points

1. Suppose there are n letters and n corresponding addressed envelopes. The numbers of ways of placing these letters into the envelopes such that no letter is placed in its corresponding envelope is often referred as derangements. The number of derangements of n objects is given by

$$D(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right]$$

For example, when $n = 3$, the number of derangements is

$D(3) = 3!\left[1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!}\right] = 2$ and when $n = 4$,

$D(4) = 4!\left[1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!}\right] = 9$

2. The total number of ways in which a selection can be made by taking some or all out of $p + q + r + \ldots$ things where p are alike of one kind, q alike of a second kind, r alike of a third kind and so on is

$$[\{(p+1)(q+1)(r+1)\ldots\} - 1].$$

3. ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ and ${}^nP_r = r.{}^{n-1}P_{r-1} + {}^{n-1}P_r$

Solved Examples

Example 1: Consider the word PRECIPITATION. Find the number of ways in which

(i) a selection
(ii) an arrangement

of 4 letters can be made from the letters of this word.

Solution: The word PRECIPITATION has 13 letters I, I, I, P, P, T, T, E, R, C, A, O, N of 9 different sorts.

In taking 4 letters, the following are the possibilities to be considered.

(a) all 4 distinct.
(b) 3 alike, 1 distinct.
(c) 2 alike of one kind, 2 alike of other kind.
(d) 2 alike, 2 other distinct.

Selections

(a) 4 distinct letters can be selected from 9 distinct letters (I, P, T, E, R, C, A, O, N) in ${}^9C_4 = 126$ ways.
(b) As 3 letters have to be alike, the only possibility is selecting all the *I*'s. Now the 4th letter can be selected from any of the remaining 8 distinct letters in ${}^8C_1 = 8$ ways.
(c) Two pairs of two alike letters can be selected from I's, *Q*'s and *T*'s in ${}^3C_2 = 3$ ways.
(d) The two alike letters can be selected in ${}^3C_1 = 3$ ways and the two distinct letters can now be selected from the 8 distinct letters in ${}^8C_2 = 28$ ways. Hence required number of ways are $3 \times 28 = 84$.

Hence, the total selections are $126 + 8 + 3 + 84 = 221$.

Arrangements: For arrangements, we find the arrangements for each of the above selections and add them up.

(a) As the 4 letters are distinct, there are 4! arrangements for each selection. Hence required arrangements are $126 \times 4! = 3024$.
(b) Since 3 of the 4 letters are alike, there are $\dfrac{4!}{3!}$ arrangements for each of the selection. Hence required arrangements are $8 \times \dfrac{4!}{3!} = 32$.

(c) The required arrangements here are $3 \times \frac{4!}{2!\,2!} = 18$

(d) The required arrangements are $84 \times \frac{4!}{2!} = 1008$.

Total number of arrangements are 3024 + 32 + 18 + 1008 = 4082.

Example 2: How many four letter words can be formed using the letters of the word 'ROAMING'?

Solution: None of the letters in the word are repeated.
∴ The number of four letter words that can be formed = ${}^{7}P_{4}$
$= \frac{7!}{3!} = (7)\,(6)\,(5)\,(4) = 840.$

Example 3: In a party, each person shook hands with every other person present. The total number of hand shakes was 28. Find the number of people present in the party.

Solution: Let the number of people present in the party be n.

Method 1: The first people shakes hands with a total of $(n-1)$ persons, the second with $(n-2)$ other people and so on.

The total number of hand shakes is

$$(n-1) + (n-2) + \ldots + 2 + 1$$

$$\frac{n(n-1)}{2} = 28 \quad \text{(given)}$$

$$\Rightarrow \quad n = 8$$

Method 2: Number of hand shakes = Number of ways of selecting 2 people out of $n = {}^{n}C_{2}$.

$${}^{n}C_{2} = 28$$

$$\frac{n(n-1)}{2!} = 28$$

$$\Rightarrow \quad n = 8$$

Directions for Examples 4 to 7: The following examples are based on the data below.

The letters of NESTLE are permuted in all possible ways.

Example 4: How many of these words begin with *T*?

Solution: NESTLE has 6 letters of which the letter *E* occurs two times. There fore the required number of words = Number of ways of filling *N, E, S, E* and *L* in the second to sixth positions $= \frac{5!}{2!} = 60.$

Example 5: How many of these words begin and end with *E*?

Solution: The required number of words = The number of ways of filling *N, S, T* and *L* in the second to fifth positions = 4! = 24.

Example 6: How many of these words begin with *S* and end with *L*?

Solution: The required number of words = The number of ways of filling *N, E, T* and *E* in the second to fifth positions $= \frac{4!}{2!} = 12.$

Example 7: How many of these words neither begin with S nor end with *L*?

Solution: The required number of words = The total number of words which can be formed using the letters *N, E, S, T* and *E* – (Number of words which begin with *S* or end with L) $= \frac{6!}{2!}$ – (Number of words beginning with *S* + Number of words ending with *L* – Number of words beginning with *S* and ending with *L*)

$$= \frac{6!}{2!} - \left(\frac{5!}{2!} + \frac{5!}{2!} - \frac{4!}{2!}\right)$$

$$= 360 - (60 + 60 - 12) = 252.$$

Example 8: How many of these words begin with *T* and do not end with *N*?

Solution: The required number of words = The number of words beginning with *T* – The number of words beginning with *T* and ending with $N = \frac{5!}{2!} - \frac{4!}{2!} = 48.$

Directions for Examples 8 to 11: The following examples are based on the data below.

The letters of FAMINE are permuted in all possible ways.

Example 9: How many of these words have all the vowels occupying odd places?

Solution: FAMINE has 3 vowels and 3 consonants.

The vowels can be arranged in the odd places in 3! or 6 ways.

The consonants would have to be arranged in even places. This is possible in 3! or 6 ways as well.

∴ The required number of words = $6^2 = 36$.

Example 10: How many of these words have all the vowels together?

Solution: If all the vowels are together, the vowels can be arranged in 3! ways among themselves.

Considering the vowels as separate a unit and each of the other letters as a unit, we have a total of 4 units which can be arranged in 4! ways.

∴ The required number of words = 4! 3! = 144

Example 11: How many of these words have at least two of the vowels separated?

Solution: The required number of words = The total number of words which can be formed using the letters *F, A, M, I, N* and *E* – The number of words with all the vowels together = 6! – 4! 3! = 576.

Example 12: How many of these words have no two vowels next to each other?

Solution: To ensure that no two vowels are together, we first arrange the 3 consonants say $-c_1 - c_2 - c_3 -$ and place

the vowels in the gaps between the consonants or the initial or final position. For each arrangement of the consonants, there are 4 places where the vowels can go. The vowels can be dealt with in 4 (3) (2) ways.

∴ The total number of words is 3! 4! = 144.

Direction for Examples 13 and 14: The following examples are based on the data below.

A committee of 5 is to be formed from 4 women and 6 men.

Example 13: In how many ways can it be formed if it consists of exactly 2 women?

Solution: The committee must have 2 women and 3 men.
∴ The required number of ways = ${}^4C_2\ {}^6C_3 = 120$.

Example 14: In how many ways can it be formed if it consists of more women than men?

Solution: The committee must have either 4 women and 1 man or 3 women and 2 men.
∴ The required number of ways

$$= {}^4C_4\ {}^6C_1 + {}^4C_3\ {}^6C_2 = 6 + 60 = 66.$$

Example 15: Find the number of four-digit numbers which can be formed using four of the digits 0, 1, 2, 3 and 4 without repetition.

Solution: The first digit has 4 possibilities (1, 2, 3 and 4).
The second digit has 4 possibilities (0 and any of the three digits not used as the first digit).
The third digit has 3 possibilities.
The last digit has 2 possibilities.
∴ The required number of numbers = (4) (4) 3 (2) = 96.

Example 16: The number of diagonals of a regular polygon is four times the number of its sides. How many sides does it have?

Solution: Let the number of sides in the polygon be n.

$$\frac{n(n-3)}{2} = 4n$$

$$n(n-11) = 0;\ n > 0$$

$$\therefore \quad n - 11 = 0;$$

$$\therefore \quad n = 11$$

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. A man has 12 blazers, 10 shirts and 5 ties. Find the number of different possible combinations in which he can wear the blazers, shirts and ties.
(A) 27 (B) 300 (C) 240 (D) 600

2. How many different words can be formed by using all the letters of the word INSTITUTE?
(A) $\frac{9!}{2!}$ (B) 9 ! (C) $\frac{9!}{3!}$ (D) $\frac{9!}{3!2!}$

3. In how many ways can a cricket team of 11 members be selected from 15 players, so that a particular player is included and another particular player is left out?
(A) 216 (B) 826 (C) 286 (D) 386

4. A group contains n persons. If the number of ways of selecting 6 persons is equal to the number of ways of selecting 9 persons, then the number of ways of selecting four persons from the group is
(A) 1365 (B) 273 (C) 455 (D) 285

5. The number of ways of arranging 10 books on a shelf such that two particular books are always together is
(A) 9! 2! (B) 9! (C) 10! (D) 8

6. Find the number of ways of inviting at least one among 6 people to a party.
(A) 2^6 (B) $2^6 - 1$ (C) 6^2 (D) $6^2 - 1$

7. An eight-letter word is formed by using all the letters of the word 'EQUATION'. How many of these words begin with a consonant and end with a vowel?
(A) 3600 (B) 10800
(C) 2160 (D) 720

8. A committee of 5 members is to be formed from a group of 6 men and 4 women. In how many ways can the committee be formed such that it contains more men than women?
(A) 180 (B) 186 (C) 126 (D) 66

9. In how many ways can 10 boys and 10 girls be arranged in a row so that all the girls sit together?
(A) 10! (B) 11!
(C) 20! (D) 10! 11!

10. In how many ways can 6 boys and 5 girls be arranged in a row so that boys and girls sit alternately?
(A) $(6!)^2$ (B) $(5!)^2$
(C) 6! 5! (D) 2.5! 6!

11. There are seven letters and corresponding seven addressed envelopes. All the letters are placed randomly into the envelopes – one in each envelope. In how many ways can exactly two letters be placed into their corresponding envelopes?
(A) 44 (B) 924 (C) 308 (D) 189

12. We are given 3 different green dyes, 4 different red dyes and 2 different yellow dyes. The number of ways

in which the dyes can be chosen so that at least one green dye and one yellow dye is selected is
(A) 336 (B) 335 (C) 60 (D) 59

13. Prahaas attempts a question paper that has 3 sections with 6 questions in each section. If Prahaas has to attempt any 8 questions, choosing at least two questions from each section, then in how many ways can he attempt the paper?
(A) 18000 (B) 10125
(C) 28125 (D) 9375

14. Find the number of selections that can be made by taking 4 letters from the word INKLING.
(A) 48 (B) 38 (C) 28 (D) 18

15. A man has $(2n + 1)$ friends. The number of ways in which he can invite at least $n + 1$ friends for a dinner is 4096. Find the number of friends of the man.
(A) 11 (B) 15 (C) 17 (D) 13

16. How many four-digit numbers are there between 3200 and 7300, in which 6, 8 and 9 together or separately do not appear?
(A) 1421 (B) 1420
(C) 1422 (D) 1077

17. Raju has forgotten his six-digit ID number. He remembers the following: the first two digits are either 1, 5 or 2, 6, the number is even and 6 appears twice. If Raju uses a trial and error process to find his ID number at the most, how many trials does he need to succeed?
(A) 972 (B) 2052
(C) 729 (D) 2051

18. A matrix with four rows and three columns is to be formed with entries 0, 1 or 2. How many such distinct matrices are possible?
(A) 12 (B) 36 (C) 3^{12} (D) 2^{12}

19. In how many ways can 4 postcards be dropped into 8 letter boxes?
(A) ${}^8P_4$ (B) 4^8 (C) 8^4 (D) 24

20. In how many ways can 12 distinct pens be divided equally among 3 children?
(A) $\frac{12!}{(3!)^4}$ (B) $\frac{12!}{(4!)^3\,3!}$
(C) $\frac{12!}{3!4!}$ (D) $\frac{12!}{(4!)^3}$

21. If all possible five-digit numbers that can be formed using the digits 4, 3, 8, 6 and 9 without repetition are arranged in the ascending order, then the position of the number 89634 is
(A) 91 (B) 93 (C) 95 (D) 98

22. Manavseva, a voluntary organisation has 50 members who plan to visit 3 slums in an area. They decide to divide themselves into 3 groups of 25, 15 and 10. In how many ways can the group division be made?
(A) 25! 15! 10! (B) $\frac{50!}{25!\,15!\,10!}$
(C) 50! (D) 25! + 15! + 10!

23. In how many ways is it possible to choose two white squares so that they lie in the same row or same column on an 8×8 chessboard?
(A) 12 (B) 48 (C) 96 (D) 60

24. The number of four digit telephone numbers that have at least one of their digits repeated is
(A) 9000 (B) 4464 (C) 4000 (D) 3986

25. There are 4 identical oranges, 3 identical mangoes and 2 identical apples in the basket. The number of ways in which we can select one or more fruits from the basket is
(A) 60 (B) 59 (C) 57 (D) 55

Answer Keys

Exercise

Practice Problems

1. D	**2.** D	**3.** C	**4.** A	**5.** A	**6.** B	**7.** B	**8.** B	**9.** D	**10.** C
11. B	**12.** A	**13.** C	**14.** D	**15.** D	**16.** D	**17.** B	**18.** C	**19.** C	**20.** D
21. C	**22.** B	**23.** C	**24.** B	**25.** B					

Chapter 14

Data Interpretation

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Methods of presenting data
- Data table
- Pie-charts
- Two-dimensional graphs

Introduction

Not a day passes without our coming across figures and statistics. Study and manipulation of such data leads us to an important area – Data Interpretation. Data can be organised in a number of ways so that larger volume of data can be presented in a more compact and precise form. Data thus presented has to be deciphered correctly by the user of the data. This process of deciphering the data from its compactly presented form is called Data Interpretation.

Methods of Presenting Data

Numerical data can be presented in one or more of the following ways

1. Data Tables
2. Pie Charts
3. 2-Dimensional Graphs
4. Bar Charts
5. 3-Dimensional Graphs
6. Venn Diagrams
7. Geometrical Diagrams
8. Pert Charts
9. Others

The 'Others' category covers miscellaneous forms like descriptive case format, etc. customised for the situation. Data can also be presented by using a combination of two or more of the above forms.

While some data can be presented in many different forms, some other data may be amenable to be presented only in a few ways. In real life situations, the style of data presentation is based on the end-objective. In certain situations data has to be presented as a combination of two or more forms of data presentation.

Let us understand each of the above forms of data presentation with an example.

Data Table

Here data is presented in the form of simple table. While any type of data can be presented in tabular form, that too in a very accurate manner, interpreting the data in tabular form is more difficult and time consuming than the other modes, all of which are basically pictorial or graphical in presentation.

Data tables can be of a number of types. They can be of a single-table variety or combination of tables. Some examples of tables are given below.

Table 1 Movement of goods by different modes of transport

(in 000's of metric-ton-kms)

Year	Road	Rail	Air	Water	Total
1985	1000	1500	120	20	2640
1986	1600	2000	129	24	3753
1987	2907	3090	139	28	6164
1988	4625	5200	152	27	10004
1989	6346	7540	174	33	14093
1990	7920	10250	212	40	18422
1991	9540	13780	266	50	23636

From the table we can deduce the following:

1. Rate of growth by each mode of transport in successive years as well as cumulative annual growth.
2. Rate of growth of total haulage by all modes of transport together in any year.
3. Contribution by each mode of transport to the total haulage in any given year.
4. Trends of growth over time for various modes of transport.
5. Given the cost of transportation for each mode, we can calculate the total annual cost of transportation over the years for various modes of transport as well make a cost comparison.
6. Finding out the mode of transportation in any given year that forms the largest percentage of total haulage.
7. For a given mode of transport, finding out the year in which the percentage increase in haulage over the previous year was the highest.

Pie-Charts

This is probably the simplest of all pictorial forms of data presentation. Here, total quantity to be shown is distributed over one complete circle or 360 degrees. In pie-charts, data is essentially presented with respect to only one parameter (unlike in two and 3-dimensional graphs described later). This form essentially presents shares of various elements as proportion or percentage of the total quantity. Each element or group in the pie-chart is represented in terms of quantity (or value, as the case may be) or as the angle made by the sector representing the elements or as a proportion of the total or as a percentage of the total.

Figure 1 gives distribution of the population in different geographical zones.

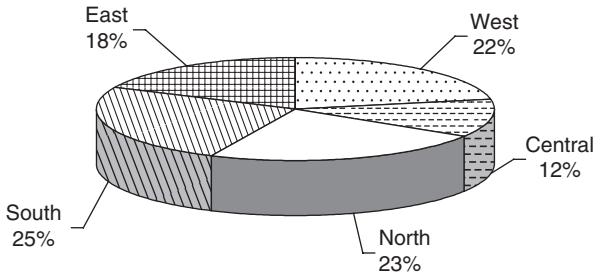


Figure 1 Distribution of population in geographical zones

From the above pie chart, we can calculate the following:

1. Population in any zone given the total population.
2. Population of any zone as a percentage of that of another zone.
3. Percentage increase in the total population given the percentage increase in the population of one or more zones.

Pie Charts are also very frequently used in combination with other forms of data or along with other Pie-Charts.

Two-Dimensional Graphs

This is essentially used for continuous data but can also be used for depicting discrete data provided we understand the limitation. Also known as Cartesian Graphs, they represent variation of one parameter with respect to another parameter each shown on a different axis. These types of graphs are useful in studying the rate of change or understanding the trends through extrapolations.

These graphs can be of various types and a few of them are shown below (Figure 2 and 3):

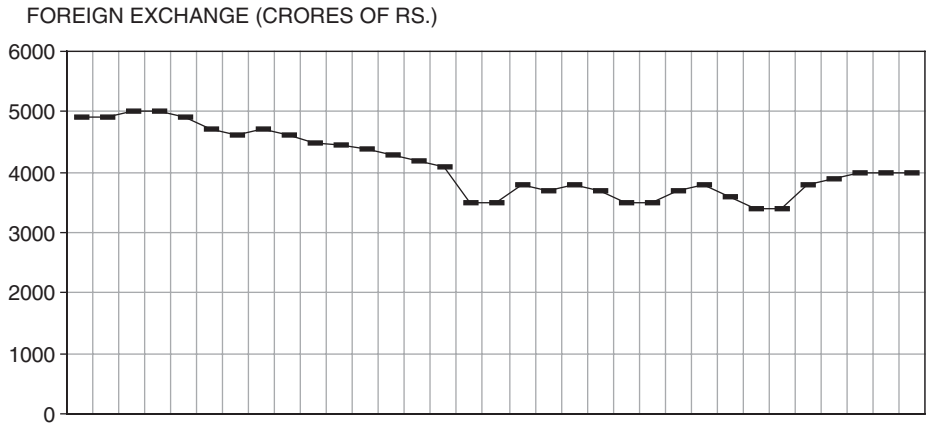


Figure 2 Foreign exchange reserves of India

The graph in Figure 2 shows the changes in the foreign exchange reserves of our country during a period of time. One can find out trends and the growth rates of foreign exchange reserves.

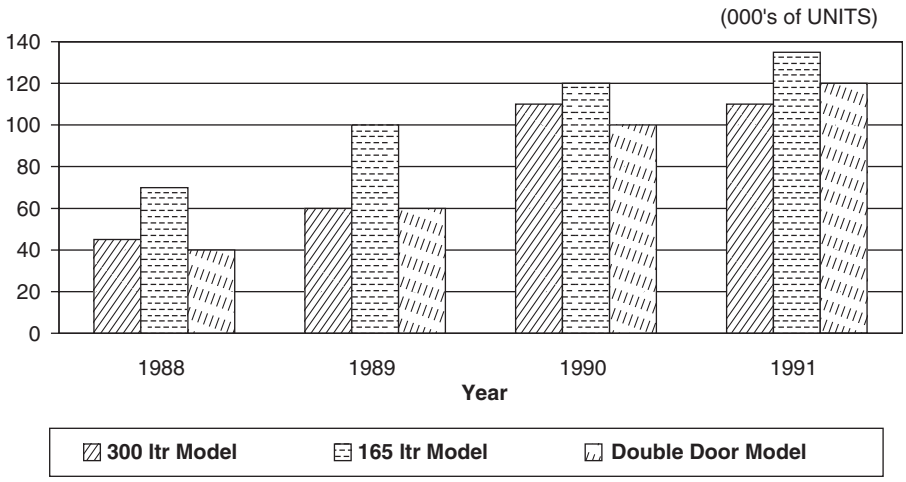


Figure 3 Refrigerator sales of company ABC

Chart 3 shows model wise sales of refrigerators during four years. From this graph we can obtain the following:

1. Percentage contribution of each model to the company's total sales for four years.
2. Relative increase or decrease in the share of each model.
3. Sales trend of various models.

Using this bar chart one can carry out a detailed performance evaluation of the company with respect to the sales of the four year period 1988 to 1991 for any given model. These bar charts can also be depicted horizontally. Another variation could be showing each product at one place (rather than each year at one place).

Example: These questions are based on the following table, which gives the details of the sports which students in all the classes of a school like.

The table gives the number of students in each class and the percentage of students in it who like Cricket, Volleyball, Basketball and Football.

Class	Number of students	Cricket	Volleyball	Basketball	Football
6	120	60%	70%	50%	60%
7	140	50%	60%	60%	50%
8	160	40%	65%	55%	45%
9	180	65%	75%	65%	55%
10	240	70%	80%	75%	45%

Solved Examples

Example 1: How many students in the school like cricket?

(A) 436 (B) 432
(C) 491 (D) 511

Solution: (C)

Number of students who like Cricket

$$= \frac{60}{100}(120) + \frac{50}{100}(140) + \frac{40}{100}(160) + \frac{65}{100}(180) + \frac{70}{100}(240) = 491$$

Example 2: By what percentage is the number of students who like Volleyball in class 6 more/less than those who like Basketball in class 10?

(A) 40% (B) 50%
(C) 53.33% (D) 56.67%

Solution: (C)

Number of students who like Volleyball in class 6

$$= \frac{70}{100}(120) = 84$$

Number of students who like Basketball in class 10

$$= \frac{75}{100}(240) = 180$$

84 is less than 180 by $\frac{180-84}{180}(100)$

$= 53.33\%$.

Example 3: The number of students who like Cricket in class 7 is what percentage of the number of students who like Football in class 8?

(A) 88% (B) 93.5%
(C) 95.6% (D) 97.2%

Solution: (D)
Number of students who like Cricket in class 7 = $\frac{50}{100}(140) = 70$

Number of students who like Football in class 8 = $\frac{45}{100}(160) = 72$

Required percentage = $\frac{70}{72} \times 100$
$= 97.2\%$

Example 4: (A)
In how many of the given classes can more than 90 students like all the three games?
(A) 2 (B) 3 (C) 1 (D) 0

Solution: In any class, the maximum value of the number of students who like all the three games would be the number of students who like the game liked by the least number of students.

In class 6, the percentage of students who like a game is the least for Basketball. Number of those who like basketball = $\frac{50}{100}(120) = 60 < 90$

In class 7, the percentage of students who like a game is the least for cricket and football. Number of students who like cricket

$$= \frac{50}{100}(140) = 70 < 90$$

In class 8, the number of students who like a game is the least for cricket. Number of students who like cricket = $\frac{40}{100}(160) = 64 < 90$

In class 9, the percentage of students who like a game is the least for Football. Number of those who like Football

$$= \frac{55}{100}(180) = 99 > 90$$

In class 10, the percentage of students who like a game is the least for Football. Number of students who like Football

$$= \frac{45}{100}(240) = 108 > 90$$

$\therefore$ In two classes, more than 90 students can like all the games.

Example 5: What can be the maximum percentage of students in class 6 who do not like any of the given games?
(A) 40% (B) 10% (C) 50% (D) 30%

Solution: (D)
In class 6, the maximum percentage of students who like a game = Percentage of students who like Volleyball i.e., 70%. Percentage of students who like at least one game would be minimum when all students who like other games are the ones who like Volleyball.

$\therefore$ Maximum percentage required
$= 100 - 70 = 30\%$.

EXERCISES

Practice Problems I

Directions for question 1: Select the correct alternative from the given choices.

1. The table shows the total marks of four students P, Q, R and S in all their subjects for the two years 2012 and 2013.

Students	2012	2013
P	997	1295
Q	664	876
R	585	732
S	480	689

How many students had a percentage Increase in their total marks of more than 35% from 2012 to 2013?
(A) 1 (B) 2 (C) 3 (D) 4

Directions for questions 2 to 4: These questions are based on the following data which gives some details of new states joining the United States of America across time.

State	Capital	Joined the union	Union rank	Population	Number of representatives in the house of representatives
Washington	Olympia	Nov, 11, 1889	42	62,87,759	9
Texas	Austin	Dec, 29, 1845	28	2,28,59,968	32
Delaware	Dover	Dec, 7, 1781	1	8,43,524	1
Virginia	Raleigh	Nov, 21, 1789	12	86,83,242	13
Minnesota	St. Paul	May, 11, 1832	32	51,32,799	8
Kansas	Topeka	Jan, 29, 1861	34	27,44,687	4
Illinois	Springfield	Dec, 3, 1818	21	1,27,63,371	19
New Hampshire	Concord	June 21, 1788	9	13,09,940	2

(*Continued*)

State	Capital	Joined the union	Union rank	Population	Number of representatives in the house of representatives
Arizona	Phoenix	Feb, 14, 1912	48	59,39,292	8
Hawai	Honolulu	Aug 21, 1959	50	12,75,194	2
Indiana	Indianapolis	Dec 11, 1816	19	62,71,973	9
Vermont	Montpelier	March 14, 1791	14	6,23,050	1
Nebraska	Lincoln	March 1, 1867	37	17,58,787	3
Georgia	Atlanta	Jan 2, 1788	4	9,07,256	13

Union rank is the chronological order in which the states joined the Union.

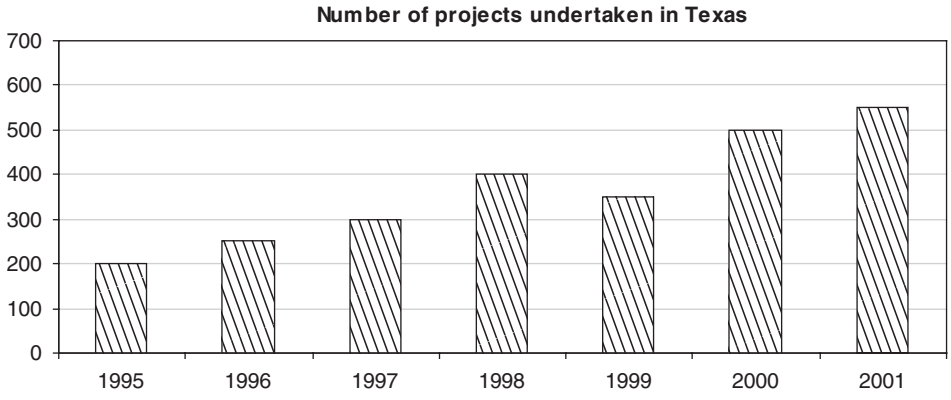
2. How many states joined the Union from March 1, 1867 to Feb 14, 1912?
(A) 11 (B) 12 (C) 13 (D) 14

3. If it is known that the House of Representatives of USA has a strength of 535 members, then the number of representatives in the House of Representatives of the given states will form what approximate percentage of the total strength of the House of Representatives?
(A) 16 (B) 19 (C) 21 (D) 23

4. In how many of the given states is the population less than 15 million but the number of representatives is not less than six?
(A) 4 (B) 5 (C) 6 (D) 7

Directions questions 5 to 7: These questions are based on the following graphs.





5. In how many years was the number of projects undertaken in Mexico greater than that in Texas?
(A) 3 (B) 6 (C) 5 (D) 4

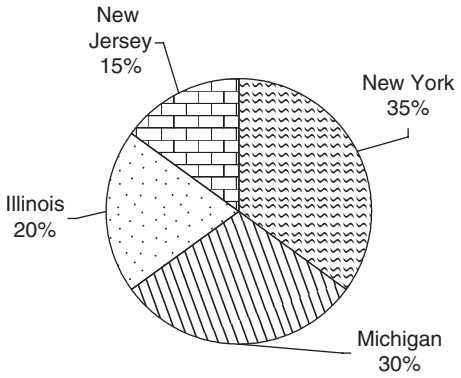
6. How many projects were undertaken in the year 1998 in both places together?
(A) 1000 (B) 900 (C) 750 (D) 500

7. In which of the following years was the average (arithmetic mean) number of projects undertaken in both places the highest?
(A) 2000 (B) 1996 (C) 1998 (D) 1999

Directions for questions 8 to 10: The table below shows the percentages of colleges offering the courses mentioned in medicine in four states New York, New Jersey, Illinois and Michigan in a certain year.

Sl. No.	Course	New York	New Jersey	Illinois	Michigan
1.	Biochemistry	86	80	74	68
2.	Biophysics	74	92	88	64
3.	Biomechanics	59	82	84	68
4.	Biostatistics	56	84	86	70

The total number of colleges offering courses in medicine in the four states is 2000. The percentage wise distribution of the number of colleges in the four states is as shown below.



8. The number of colleges offering the Biochemistry course is more than 230 in
(A) All the four states.
(B) Exactly three states.
(C) Exactly two states.
(D) Exactly one state.

9. What percentage of the colleges in the four states do not offer Biophysics as well as Biochemistry?
(A) 41%
(B) 36%
(C) 34%
(D) Cannot the determined

10. What is the total number of colleges offering Biostatistics in all the four states?
(A) 1392 (B) 1408
(C) 1432 (D) 1476

Directions for questions 11 and 12: Study the given table and answer the questions that follow.

Percentage break-up of the workers working in six different factories *J, K, L, M, N* and *P*.

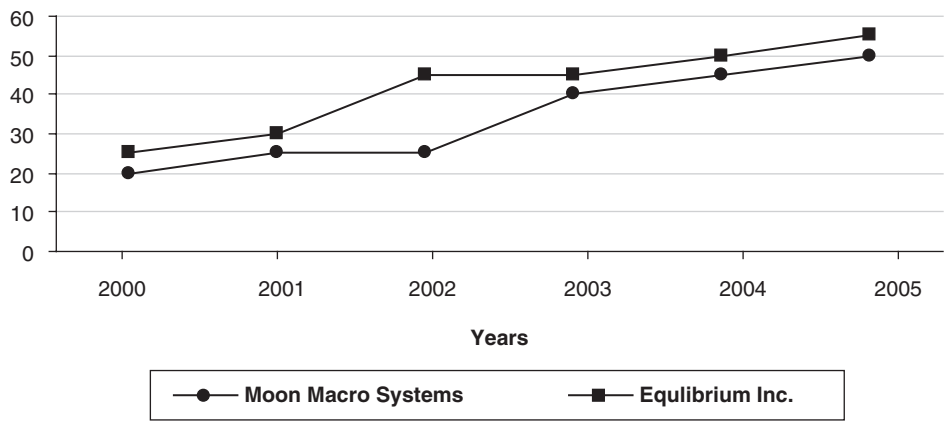
Factory	Total no. of workers	Percentage		
		Men	Women	Boys
J	4800	50	37.5	12.5
K	8750	40	36	24
L	5250	24	56	20
M	12000	35	25	40
N	8500	38	30	32
P	2700	45	40	15

11. By what percent is the number of women working in factory *P* is more than the number of boys working in factory *J*?
(A) 20% (B) $44\frac{4}{9}\%$
(C) 80% (D) 180%

12. What is the ratio of the number of men working in factory *M* to the number of women working in factory *L*?
(A) 7 : 10 (B) 10 : 7 (C) 7 : 5 (D) 5 : 7

Directions for questions 13 to 15: These questions are based on the following line graph.

Numbers of employees of two companies each years over the period 2000 to 2005 (in thousands)

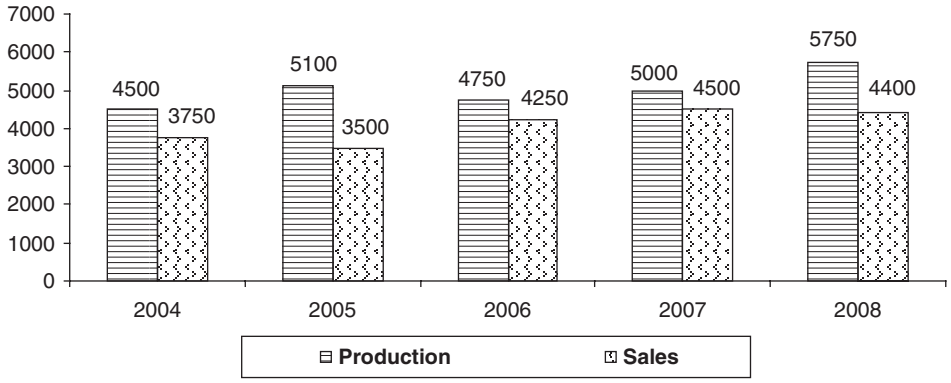


13. What is the ratio of the average number of employees of company Moon Macro Systems for the period 2001, 2002 and 2003 and the average number of employees of company Equilibrium Inc. for the same period?
(A) 2 : 5 (B) 3 : 5 (C) 3 : 4 (D) 4 : 3

14. During which of the following years was the percentage increase in the number of employees of company Equilibrium Inc. over that in the previous year the highest?
(A) 2000 (B) 2001 (C) 2002 (D) 2004

15. What is the approximate percentage increase in the total number of employees of the two companies from 2004 to 2005?
(A) 8.5% (B) 9% (C) 9.5% (D) 10.5%

Directions for questions 16 and 17: These questions are based on the following bar graph which gives the production and sales of a company across five years from 2004 to 2008.



16. In the given period what percentage was the average production more than the average sales?
(A) 20% (B) 23% (C) 25% (D) 28%

17. The percentage increase/decrease in the total sales of the company in a given year with respect to that in the previous year was highest in which of the following years?
(A) 2005 (B) 2006 (C) 2007 (D) 2008

Practice Problems 2

Directions for questions 1 to 3: These questions are based on the following table which shows the Btu content of common types of energy.

Type of energy	Btu content
1 barrel of Crude oil (42 gallons)	5,838,000
1 gallon of gasoline	124,000
1 gallon of heating oil	139,000
1 barrel of residual fuel oil	6,287,000
1 cubic foot of natural gas	1,026
1 gallon of propane	91,000
1 short ton of coal	20,681,000
1 kilowatt hour of electricity	3,412

Note 1: British Thermal Unit (Btu) represents the amount of energy required to raise the temperature of 1 pound of water by 1°F. Also 1 calorie represents the amount of energy required to raise the temperature of 2.2 pounds (1 kg) of water by 1.8°F (1°C).

Note 2: 1 Joule = 0.2390 calories and

1 Btu = 1.06 kilojoules

1. The Btu content in 42 gallons of heating oil is equal to the Btu content in which of the following?
(A) 60 gallons of propane
(B) 42 gallons of crude oil
(C) 48 gallons of gasoline
(D) 57 cubic feet of natural gas

2. The Btu content in 60 gallons of propane is less than that in one short ton of coal by
(A) 15,221,000 units (B) 16,290,000 units
(C) 17,820,000 units (D) 18,630,000 units

3. 1 Btu is equal to _____ calories
(A) 228.72 (B) 244.68
(C) 253.34 (D) 262.44

Directions for questions 4 to 6: These questions are based on the following data which shows the number of representatives of different political parties in the Parliament of UK which has three wings – The Northern Ireland Assembly, The Scottish Parliament and the National Assembly of Wales.

Political parties	Northern ireland assembly		Political parties	Scottish parliament		Political parties	National assembly of wales	
	Males	Females		Males	Females		Males	Females
DUP	36	12	Labour	48	8	Labour	12	4
UUP	10	3	Democrats	16	4	SDLP	15	2
SDLP	24	2	Conservative	24	6	Conservative	6	5
Others	18	3	Republicans	14	9	Others	9	7

4. Only among parties with at least 25 members in any wing, the number of young parliamentarians is more than eight. What is the maximum number of parliamentarians in the two wings – Northern Ireland Assembly and Scottish Parliament who are not young?
(A) 201 (B) 144
(C) 165 (D) 176

5. If there are six Catholic parliamentarians for each party in the Scottish Parliament, then how many Scottish parliamentarians are not members of the Conservative party nor are Catholics?
(A) 27 (B) 42
(C) 60 (D) 81

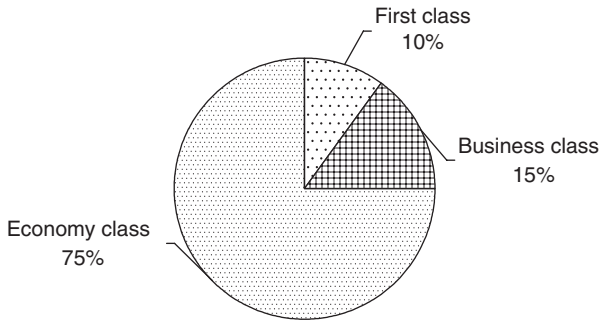
6. The number of parliamentarians who are graduates in the National Assembly of Wales is 45. How many parliamentarians in the National Assembly of Wales are either females or graduates?
(A) 27
(B) 30
(C) 32
(D) Cannot be determined

Directions for questions 7 to 9: These questions are based on the table and the pie chart given below.

Details of Airports in USA

Name of the Airport	Flights per day		Number of passengers per day (in hundreds)
	Arrivals	Departures	
Atlanta	1420	1380	560
JFK	1380	1420	700
Florida	1300	1500	360
Chicago	1440	1560	720
National	1200	1300	360
Los Angeles	920	880	468
Dallas	800	700	405
Detroit	690	710	392
Seattle	1100	1000	441

Classification of passengers in each of of the flights

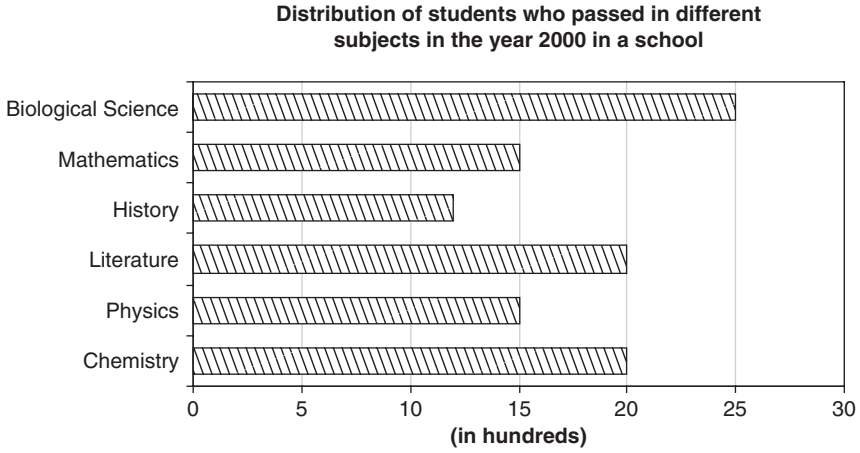


7. How many airports handle more than 1500 flights per day?
(A) 5 (B) 6 (C) 7 (D) 8

8. Which of the following airports carries the maximum number of passengers per flight per day?
(A) Chicago (B) JFK
(C) Detroit (D) Atlanta

9. How many passengers (in hundreds) travel in the Economy class at JFK Airport?
(A) 310 (B) 490 (C) 525 (D) 595

Directions for questions 10 to 12: These questions are based on the graph and the table given below.



Percentage change in the number of students who passed over the previous year in the school

	2000	2001
Biological Science	10%	20%
Mathematics	15%	20%
History	–25%	–10%
Literature	30%	–20%
Physics	20%	20%
Chemistry	15%	25%

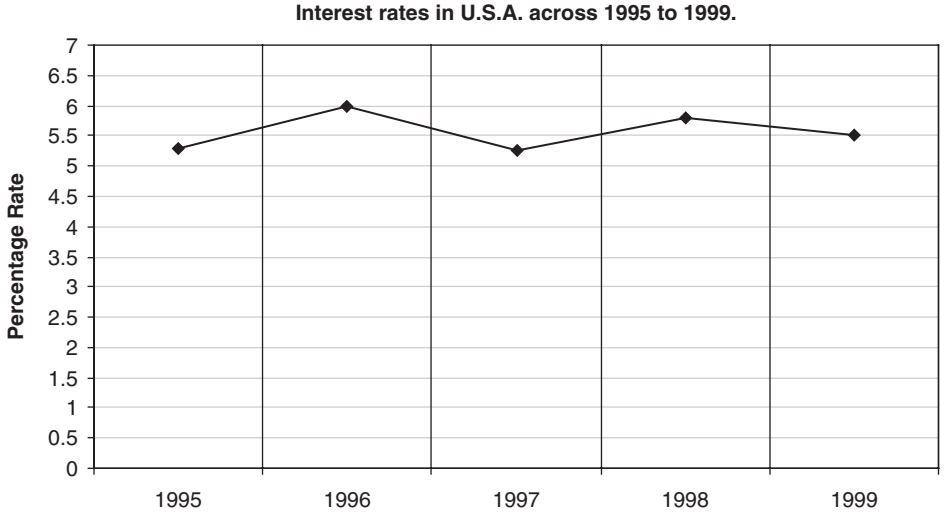
Note: The negative sign indicates a decrease.

10. What is the ratio of the number of students who passed in History in 1999 to that in 2001?
(A) 9 : 4 (B) 8 : 5 (C) 40 : 27 (D) 6 : 5

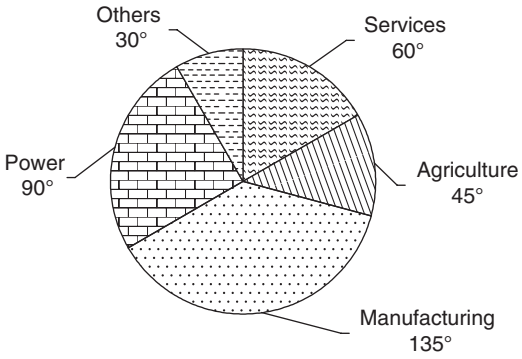
11. By what percent did the number of students who passed in Chemistry increase from 1999 to 2001?
(A) 35% (B) 37.5% (C) 40% (D) 43.75%

12. In which of the following pairs of subjects did an equal number of students pass in the year 2001?
(A) Mathematics and Literature
(B) Literature and Chemistry
(C) Mathematics and Physics
(D) History and Physics

Directions for questions 13 to 15: These questions are based on the graph, pie chart and the table given below.



Distribution of loans availed of by various sectors of the industry ofor each of the years from 1995 to 1999



Total loan amount availed of by the industry (In Billions of US$)	
1995	36
1996	48
1997	72
1998	108
1999	144

13. In the years 1998 and 1999 combined the total interest paid by exactly one sector as a percentage of the total interest paid by all the sectors of the industry cannot be which of the following?

(A) $8\frac{1}{3}\%$ (B) $16\frac{2}{3}\%$

(C) 25% (D) 40%

14. ENRON Inc. has availed 25% of the loan pertaining to one of the sectors in 1997. What is the minimum possible amount of interest that they could have paid in 1997 in millions of dollars? [Consider others to be a single sector]

(A) 60.75 (B) 66.75

(C) 72.75 (D) 78.75

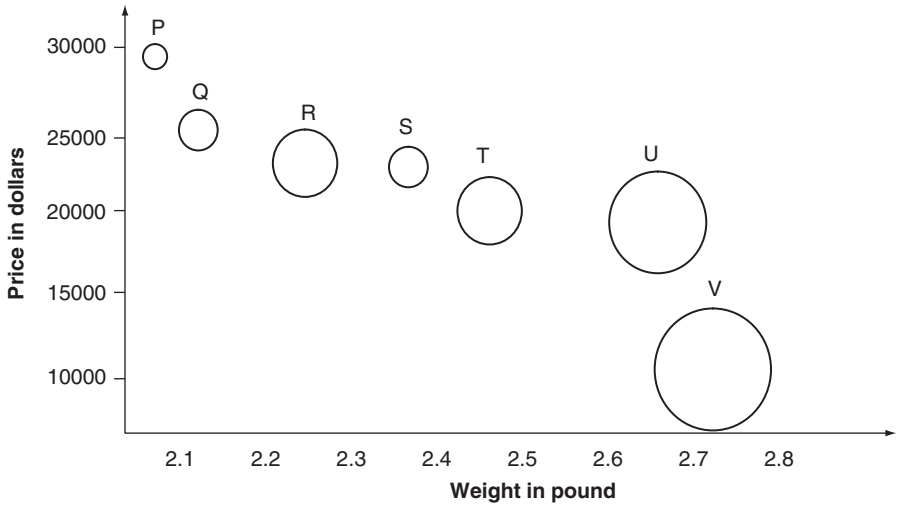
15. The total interest paid in 1995 and in 1996 was more than \$390 million for

(A) at least three sectors (B) exactly one sector

(C) exactly two sectors (D) exactly three sectors

Directions for questions 16 to 18: Answer the questions on the basis of the information given below.

In the bubble graph below, the horizontal axis shows the weight of the product in pounds and the vertical axis shows the price in dollars. The relative size of the circles indicates how many of the products were sold. The largest circle indicates 4000 products and there is a decrease of 10% in the number of products for an incremental decrease in the size of the circles.



16. A shop sells one unit each of all products having a weight of 2.2 to 2.5 pound. The sales generated in dollars is _____

(A) 42500 (B) 70000
(C) 47500 (D) 72500

17. The number of the heaviest product sold was _____ that of any product shown and was the _____

(A) greater than; most expensive
(B) greater than; least expensive
(C) lesser than; most expensive
(D) lesser than; most expensive

18. The total numbers of products of type *U* and *T* sold were _____.

(A) 5400 (B) 6000
(C) 6840 (D) 8000

Directions for questions 19 to 21: These questions are based on the information given below.

Six activities – *A, B, C, D, E* and *F* – need to be performed. Time required for each activity is given in the table below, along with the list of activities that need to be completed before that activity is started.

Activity	Time to complete (in min.)	Activities to be performed before
A	50	–
B	60	*A, C*
C	45	–
D	40	*B, C*
E	30	*D*
F	20	*B*

Assume that two or more activities can be performed simultaneously.

19. Find the minimum time in which all the activities can be completed.

(A) 180 minutes (B) 175 minutes
(C) 195 minutes (D) 225 minutes

20. If the work is to be completed in the least possible time, then what is the maximum possible time gap between starting the activities *D* and *F*?

(A) 20 minutes (B) 40 minutes
(C) 50 minutes (D) 80 minutes

21. Now, two new activities – *Y*, taking 10 minutes, and *Z*, taking 30 minutes – are included. If *Y* and *Z* are to be completed before *A* and *D* respectively, then what is the least possible time in which the entire work is completed?

(A) 205 minutes (B) 210 minutes
(C) 180 minutes (D) 190 minutes

Previous Years' Questions

1. 5 skilled workers can build a wall in 20 days, 8 semi-skilled workers can build the wall in 25 days, and 10 unskilled workers can build the wall in 30 days. If a team has 2 skilled, 6 semi-skilled and 5 unskilled workers, how long will it take to build the wall? **[2010]**

(A) 20 (B) 18
(C) 16 (D) 15

2. From the digits 2, 2, 3, 3, 3, 4, 4, 4, 4, how many distinct 4–digit numbers greater than 3000 can be formed? **[2010]**

(A) 50 (B) 51
(C) 52 (D) 54

3. If 137 + 276 = 435, how much is 731 + 672? **[2010]**

(A) 534 (B) 1403
(C) 1623 (D) 1513

4. There are two candidates P and Q in an election. During the campaign, 40% of the voters promised to vote for P, and rest for Q. However, on the day of election 15% of the voters went back on their promise to vote for P and instead voted for Q. 25% of the voters went back on their promise to vote for Q and instead voted for P. Suppose, P lost by 2 votes, then what was the total number of voters? **[2011]**

(A) 100 (B) 110
(C) 90 (D) 95

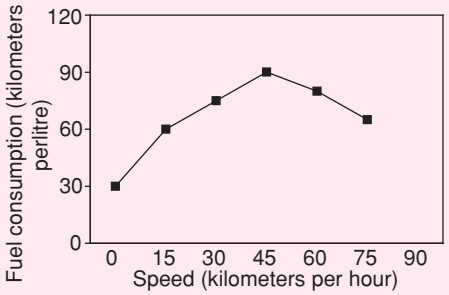
5. Three friends R, S and T shared toffee from a bowl. R took $\frac{1}{3}$ of the toffees, but returned four to the bowl. S took $\frac{1}{4}$ of what was left but returned three toffees to the bowl. T took half of the remainder but returned two back into the bowl. If the bowl had 17 toffees left, how many toffees were originally there in the bowl? **[2011]**

(A) 38 (B) 31
(C) 48 (D) 41

6. Given that $f(y) = \frac{|y|}{y}$, and q is any non-zero real number, the value of $|f(q) - f(-q)|$ is **[2011]**

(A) 0 (B) −1
(C) 1 (D) 2

7. The fuel consumed by a motorcycle during a journey while travelling at various speeds is indicating in the graph below: **[2011]**



The distances covered during four laps of the journey are listed in the table below.

Lap	Distance (Kilometres)	Average Speed (Kilometres per Hour)
P	15	15
Q	75	45
R	40	75
S	10	10

From the given data, we can conclude that the fuel consumed per kilometre was least during the lap

(A) P (B) Q
(C) R (D) S

8. The sum of n terms of the series $4 + 44 + 444 + ...$ is **[2011]**

(A) $\left(\frac{4}{81}\right)[10^{n-1} - 9n - 1]$

(B) $\left(\frac{4}{81}\right)[10^{n-1} - 9n - 1]$

(C) $\left(\frac{4}{81}\right)[10^{n-1} - 9n - 10]$

(D) $\left(\frac{4}{81}\right)[10^{n} - 9n - 10]$

9. If $(1.001)^{1259} = 3.52$ and $(1.001)^{2062} = 7.85$, then $(1.001)^{3321} =$ **[2012]**

(A) 2.23 (B) 4.33
(C) 11.37 (D) 27.64

10. The data given in the following table summarizes the monthly budget of an average household. **[2012]**

Category	Amount (₹)
Food	4000
Clothing	1200
Rent	2000
Savings	1500
Other expenses	1800

The approximate percentage of the monthly budget NOT spent on savings is

(A) 10% (B) 14%
(C) 81% (D) 86%

11. Raju has 14 currency notes in his pocket consisting of only ₹20 notes and ₹10 notes. The total money value of the notes is ₹230. The number of ₹10 notes that Raju has is **[2012]**

(A) 5 (B) 6
(C) 9 (D) 10

12. A and B are friends. They decide to meet between 1 p.m. and 2 p.m. on a given day. There is a condition that whoever arrives first will not wait for the other for more than 15 minutes. The probability that they will meet on that day is **[2012]**

(A) $\frac{1}{4}$ (B) $\frac{1}{16}$
(C) $\frac{7}{16}$ (D) $\frac{9}{16}$

13. In the summer of 2012, in New Delhi, the mean temperature of Monday to Wednesday was 41°C and of Tuesday to Thursday was 43°C. If the temperature on Thursday was 15% higher than that of Monday, then the temperature in °C on Thursday was **[2013]**

14. What is the chance that a leap year, selected at random, will contain 53 Saturdays? **[2013]**

(A) $\frac{2}{7}$ (B) $\frac{3}{7}$
(C) $\frac{1}{7}$ (D) $\frac{5}{7}$

15. The set of values p for which the roots of equation $3x^2 + 2x + p(p - 1) = 0$ are opposite sign is **[2013]**

(A) $(-\infty, 0)$ (B) $(0, 1)$
(C) $(1, \infty)$ (D) $(0, \infty)$

16. A car travels 8 km in the first quarter of an hour, 6 km in the second quarter and 16 km in the third quarter. The average speed of the car in km per hour over the entire journey is **[2013]**

(A) 30 (B) 36
(C) 40 (D) 24

17. Find the sum to n terms of the series $10 + 84 + 734 + ...$ **[2013]**

(A) $\frac{9(9^n + 1)}{10} + 1$ (B) $\frac{9(9^n - 1)}{8} + 1$

(C) $\frac{9(9^n - 1)}{8} + n$ (D) $\frac{9(9^n - 1)}{8} + n^2$

18. If $\left(z + \frac{1}{z}\right)^2 = 98$, compu3te $\left(z^2 + \frac{1}{z^2}\right)$. **[2014]**

19. The roots of $ax^2 + bx + c = 0$ are real and positive. If a, b and c are real, then $ax^2 + b|x| + c = 0$ has **[2014]**

(A) no roots (B) 2 real roots
(C) 3 real roots (D) 4 real roots

20. Round-trip tickets to a tourist destination are eligible for a discount of 10% on the total fare. In addition, groups of 4 or more get a discount of 5% on the total fare. If the one way single person fare is ₹100, a group of 5 tourists purchasing round-trip tickets will be charged ₹_____. **[2014]**

21. In a survey, 300 respondents were asked whether they own a vehicle or not. If yes, they were further asked to mention whether they own a car or scooter or both. Their responses are tabulated below. What percent of respondents do not own a scooter? **[2014]**

		Men	Women
Own vehicle	Car	40	34
	Scooter	30	20
	Both	60	46
Do not own vehicle		20	50

22. When a point inside of a tetrahedron (a solid with four triangular surfaces) in connected by straight lines to its corners, how many (new) internal planes are created with these lines? ———. **[2014]**

23. What is the average of all multiples of 10 from 2 to 198? **[2014]**

(A) 90 (B) 100
(C) 110 (D) 120

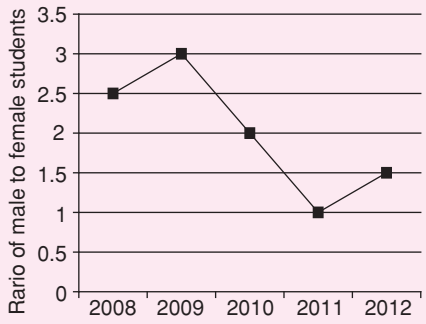
24. The value of $\sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$ is **[2014]**

(A) 3.464 (B) 3.932
(C) 4.000 (D) 4.444

25. If x is real and $|x^2-2x+3|=11$, then possible values of $|-x^2+2x-x|$ include. **[2014]**

(A) 2, 4 (B) 2, 14
(C) 4, 52 (D) 14, 52

26. The ratio of male to female students in a college for five years is plotted in the following line graph. If the number of female students doubled in 2009, by what percent did the number of male students increase in 2009? **[2014]**



27. The table below has question-wise data on the performance of students in an examination. The marks for each question are also listed. There is no negative or partial marking in the examination. **[2014]**

Q. No.	Marks	Answered Correctly	Answered Wrongly	Not Attempted
1	2	21	17	6
2	3	15	27	2
3	2	23	18	3

What is the average of the marks obtained by the class in the examination?

(A) 1.34 (B) 1.74
(C) 3.02 (D) 3.91

28. The Gross Domestic Product (GDP) in Rupees grew at 7% during 2012–2013. For international comparison, the GDP is compared in US Dollars (USD) after conversion based on the market exchange rate. During the period 2012–2013 the exchange rate for the USD increased from ₹50 per $ to ₹60 per $. India's GDP in USD during the period 2012–2013 **[2014]**

(A) increased by 5% (B) decreased by 13%
(C) decreased by 20% (D) decreased by 11%

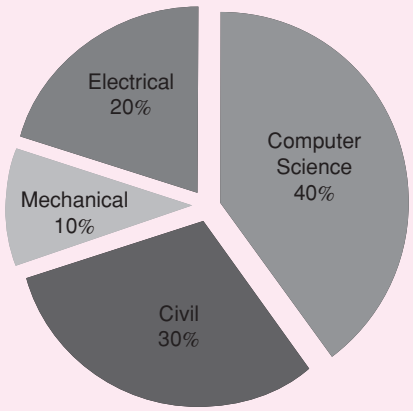
29. Consider the equation: $(7256)_8-(Y)_8=(4364)_8$, where $(X)_N$ stands for X to the base N. Find Y. **[2014]**

(A) 1634 (B) 1737
(C) 3142 (D) 3162

30. Given Set A = {2, 3, 4, 5} and Set B = {11, 12, 13, 14, 15}, two numbers are randomly selected, one from each set. What is the probability that the sum of the two numbers equals 16? **[2015]**

(A) 0.20 (B) 0.25
(C) 0.30 (D) 0.33

31. The pie chart below has the breakup of the number of students from different departments in an engineering college for the year 2012. The proportion of male to female students in each department is 5:4. There are 40 males in Electrical Engineering. What is the difference between the numbers of female students in the Civil department and the female students in the Mechanical department? **[2015]**



32. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, the student has 75% chance of passing

in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Following relations are drawn in m, p, c: **[2015]**

(I) $p + m + c = 27/20$

(II) $p + m + c = 13/20$

(III) $(p) \times (m) \times (c) = 1/10$

(A) Only relation I is true.

(B) Only relation II is true.

(C) Relations II and III are true.

(D) Relations I and III are true.

33. The number of students in a class who have answered correctly, wrongly, or not attempted each question in an exam, are listed in the table below. The marks for each question are also listed. There is no negative or partial marking.

Q.No.	Marks	Answered Correctly	Answered Wrongly	Not Attempted
1	2	21	17	6
2	3	15	27	2
3	1	11	29	4
4	2	23	18	3
5	5	31	12	1

What is the average of the marks obtained by the class in the examination? **[2015]**

(A) 2.290 (B) 2.970

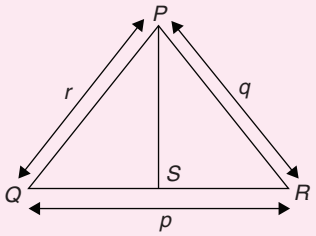
(C) 6.795 (D) 8.795

34. Consider a function $f(x) = 1 - |x|$ on $-1 \le x \le 1$. The value of x at which the function attains a maximum and the maximum value of the function are: **[2015]**

(A) 0, –1 (B) –1, 0

(C) 0, 1 (D) –1, 2

35.



In a triangle PQR, PS is the angle bisector of $\angle QPR$ and $\angle QPS = 60°$. What is the length of PS? **[2015]**

(A) $\frac{(q+r)}{qr}$ (B) $\frac{qr}{(q+r)}$

(C) $\sqrt{(q^2 + r^2)}$ (D) $\frac{(q+r)^2}{qr}$

36. If p, q, r, s are distinct integers such that:

$f(p, q, r, s) = \max(p, q, r, s)$

$g(p, q, r, s) = \min(p, q, r, s)$

$h(p, q, r, s)$ = remainder of $(p \times q)/(r \times s)$ if $(p \times q) > (r \times s)$ or remainder of $(r \times s)/(p \times q)$ if $(r \times s) > (p \times q)$

Also a function $fgh(p, q, r, s) = f(p, q, r, s) \times g(p, q, r, s) \times h(p, q, r, s)$

Also the same operations are valid with two variable functions of the form $f(p, q)$

What is the value of $fg(h(2,5,7,3), 4,6,8)$? **[2015]**

37. If the list of letters, P, R, S, T U is an arithmetic sequence, which of the following are also in arithmetic sequence? **[2015]**

I. $2P, 2R, 2S, 2T, 2U$

II. $P–3, R–3, S–3, T–3, U–3$

III. P^2, R^2, S^2, T^2, U^2

(A) I only

(B) I and II

(C) II and III

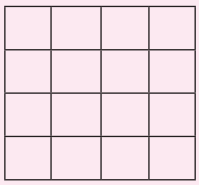
(D) I and III

38. In a quadratic function, the value of the product of the roots (α, β) is 4. Find the value of $\frac{\alpha^n + \beta^n}{\alpha^{-n} + \beta^{-n}}$ **[2016]**

(A) n^4 (B) 4^n

(C) 2^{2n-1} (D) 4^{n-1}

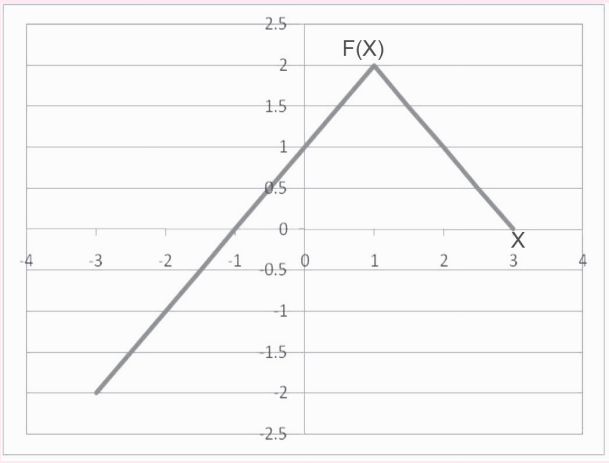
39. In a 2 × 4 rectangle grid shown below, each cell is a rectangle. How many rectangles can be observed in the grid? **[2016]**



(A) 21 (B) 27

(C) 30 (D) 36

40.

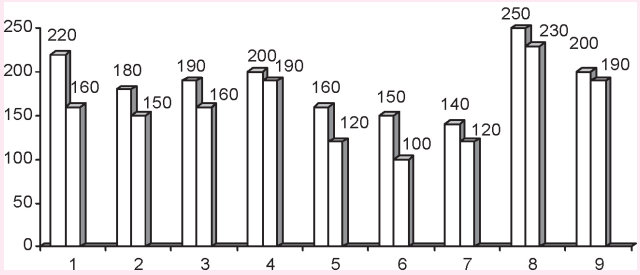


Choose the correct expression for $f(x)$ given in the graph. **[2016]**

(A) $f(x) = 1 - |x - 1|$ (B) $f(x) = 1 + |x + 1|$
(C) $f(x) = 2 - |x - 1|$ (D) $f(x) = 2 + |x - 1|$

41. If $|9y - 6| = 3$, then $y^2 - 4y/3$ is _____ **[2016]**
(A) 0 (B) +1/3
(C) –1/3 (D) Undefined

42.



The graph represents the installed capacity for cement production (in tonnes) and the actual production (in tonnes) of nine cement plants of a cement company. Capacity utilization of a plant is defined as ratio of actual production of cement to installed capacity. A plant with installed capacity of at least 200 tonnes is called a large plant and a plant with lesser capacity is called a small plant. The difference between total production of large plants and small plants in tonnes is _____ **[2016]**

43. Shaquille O' Neal is a 60% career free throw shooter, meaning that he successfully makes 60 free throws out of 100 attempts on average. What is the probability that he will successfully make exactly 6 free throws in 10 attempts? **[2016]**
(A) 0.2508 (B) 0.2816
(C) 0.2934 (D) 0.6000

44. The numeral in the units position of $211^{870} + 146^{127} \times 3^{424}$ is _____ **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** B	**3.** D	**4.** D	**5.** A	**6.** C	**7.** A	**8.** A	**9.** D	**10.** B
11. C	**12.** B	**13.** C	**14.** C	**15.** D	**16.** B	**17.** B			

Practice Problems 2

1. B	**2.** A	**3.** C	**4.** A	**5.** D	**6.** D	**7.** C	**8.** C	**9.** C	**10.** C
11. D	**12.** C	**13.** D	**14.** D	**15.** A	**16.** B	**17.** B	**18.** C	**19.** A	**20.** C
21. D									

Previous Years' Questions

1. D	**2.** B	**3.** C	**4.** A	**5.** C	**6.** D	**7.** B	**8.** C	**9.** D	**10.** D
11. A	**12.** C	**13.** C	**14.** A	**15.** B	**16.** C	**17.** D	**18.** 96	**19.** D	**20.** 850
21. 48	**22.** 6	**23.** B	**24.** C	**25.** D	**26.** 140	**27.** C	**28.** D	**29.** C	**30.** A
31. 32	**32.** D	**33.** C	**34.** C	**35.** B	**36.** 8	**37.** B	**38.** B	**39.** C	**40.** C
41. C	**42.** 120	**43.** A	**44.** 7						

TEST
QUANTITATIVE ABILITY

Time: 30 min.

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. P, Q, R and S have a total amount of ₹220 with them. P has ₹30 more than Q. S has half the amount with Q. R has ₹10 more than S. Find the amount with S (in ₹).
(A) 20 (B) 40
(C) 30 (D) 50

2. In a room, there are some girls and some benches. If 5 girls sit on each bench, three girls will have no bench to sit on. If there is one bench less, 6 girls can sit on each bench. Find the number of benches.
(A) 6 (B) 8
(C) 7 (D) 9

3. A test has 60 questions. Each correct answer fetches 1 mark. For each wrong answer and each unanswered question 1 mark is deducted. A candidate who wrote this test scored 20 marks. Find the number of questions he correctly answered.
(A) 50 (B) 45
(C) 35 (D) 40

4. If $\frac{a+b-c}{x}=\frac{a-b+c}{y}=\frac{c+b-a}{z}$ then $x(b-a)+y(a-c)+z(c-b)=$
(A) 0 (B) 2
(C) 3 (D) 1

5. The value of a diamond varies directly with the cube of its weight. It broke into two pieces whose weights are in the ratio 3 : 4. The loss due to breakage is ₹504000. Find its initial value (in ₹).
(A) 1029000 (B) 686000
(C) 1372000 (D) 1715000

6. The average of 25 observations is 120. By mistake one of the observations, 144, is taken as 169. Find the average of the 25 observations, after the mistake is corrected.
(A) 120 (B) 119
(C) 125 (D) 132

7. The average of 13 observations is 50. The average of first seven observations is 45 and the average of last seven observations is 52. Find the value of seventh observation.
A) 41 (B) 30
(C) 29 (D) 62

8. A container contains 100 litres of milk. 10 litres of milk is replaced by 10 litres of water. From the solution formed, 10 litres of solution is replaced by 10 litres of water and this process is repeated one more time. Find the percentage of water in the resulting solution.
(A) 33.3% (B) 67%
(C) 36.5% (D) 27.1%

9. In a 729 ml of solution, the ratio of acid to water is 7 : 2. How much more water should be mixed so that the resulting mixture contains acid and water in the ratio 7 : 3 (in ml)?
(A) 100 (B) 40
(C) 37 (D) 81

10. A merchant buys sulphuric acid at a certain rate per gallon and after mixing it with water, sells it at the same rate. If the merchant makes a profit of 20%, how many gallons of water are there per gallon of acid?
(A) 0.2 (B) 0.5
(C) 0.7 (D) 0.25

11. If A travelled a certain distance at 6 kmph, he would have reached his destination 10 minutes early. If he travelled it at 4 kmph, he would have reached his destination 10 minutes late. Find the speed at which he must travel to reach his destination on time (in kmph).
(A) 5 (B) 5.4
(C) 4.8 (D) 4.5

12. A car travelled the first hour of its journey at 30 kmph, the next 5 hours of its journey at 50 kmph and the remaining 4 hours of its journey at 75 kmph. Find its average speed for its journey (in kmph).
(A) 56 (B) 60
(C) 58 (D) 62

13. Without stoppages, a train can cover 54 km in an hour. With stoppages it can cover 36 km in an hour. Find its stoppage time per hour in a journey it covers with stoppages (in minutes).
(A) 15 (B) 18
(C) 20 (D) 12

14. A and B can complete a job in 40 days. B and C can complete it in 30 days. A and C can complete it in 20 days. Find the time taken by A to complete it (in days).
(A) $\frac{180}{7}$ (B) 30
(C) 48 (D) $\frac{240}{7}$

15. 3 men and 4 women can complete a job in 10 days. 24 men and 2 women can complete it in 2 days. Find the time taken by five men and 10 women to complete it (in days).
(A) 4 (B) 5
(C) 3 (D) 6

16. Abhilash spends 25% of his income towards rent, 20% of the remaining income towards food, 8% of the remaining towards medical expenses, and 25% of the remaining towards miscellaneous expenses. If he saves ₹82,800, what is his income?
(A) ₹2,00,000 (B) ₹2,25,000
(C) ₹2,40,000 (D) ₹2,50,000

17. By selling 30 articles, a shopkeeper gained the selling price of 10 articles. Find the profit percent.
(A) 20% (B) 30%
(C) 50% (D) 40%

18. When 1036 is divided by N, the remainder is 12 and when 1545 is divided by N, the remainder is 9. Find the greatest possible value of N.
(A) 128 (B) 512
(C) 250 (D) 64

19. Five bells toll at regular intervals of 10, 15, 20, 25 and 30 seconds respectively. If they toll together at 8:00 a.m., then at what time will they toll together for the first time after 8:00 a.m.?
(A) 8:04 a.m. (B) 8:06 a.m.
(C) 8:05 a.m. (D) 8:07 a.m.

20. Find the value of $\frac{3}{\sqrt{2}+\sqrt{11}}+\frac{3}{\sqrt{5}+\sqrt{8}}+\frac{3}{\sqrt{8}+\sqrt{11}}+\cdots+\frac{3}{\sqrt{26}+\sqrt{29}}$.
(A) $\sqrt{29}-\sqrt{2}$ (B) $\sqrt{26}+\sqrt{5}$
(C) $\sqrt{29}+\sqrt{26}$ (D) $\sqrt{26}-\sqrt{8}$

21. If $a+b+c=0$, find the value of $(3^{a^8})^{\frac{1}{a^6bc}}(3^{b^8})^{\frac{1}{ab^6c}}(3c^8)^{\frac{1}{abc^6}}$.
(A) 2 (B) 6
(C) 27 (D) 81

22. If $3^{x+3}-3^{x+2}=486$, then find x.
(A) 3 (B) 5
(C) 6 (D) 2

23. $\log_p q=\frac{5}{4}, \log_r q=\frac{5}{6}$ and $\log_r p=3x$, then find x.
(A) $\frac{1}{9}$ (B) $\frac{2}{3}$
(C) $\frac{1}{18}$ (D) $\frac{2}{9}$

24. $\frac{5}{1+\log_p qr}+\frac{5}{1+\log_q pr}+\frac{5}{1+\log_r pq}=$
(A) 0 (B) 1
(C) 5 (D) 10

25. If $\log_{10}2 = 0.3010$ then find the number of digits in 2^{55}.
(A) 17 (B) 11
(C) 18 (D) 16

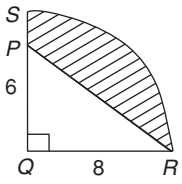
26. The maximum sum of the arithmetic progression 45, 41, 37, ... is
(A) 256 (B) 274
(C) 276 (D) 264

27. The greatest value of n such that $1 + 3 + 3^2 + 3^3 + \cdots + 3^n$, which is less than 3000 will be:
(A) 6 (B) 8
(C) 7 (D) 9

28. $3 + 33 + 333 + \cdots +$ upto n terms =
(A) $\frac{(10^n-1)}{3}-n$ (B) $\frac{10(10^n-1)}{27}-n$
(C) $\frac{(10^n-1)}{3}-\frac{n}{9}$ (D) $\frac{10(10^n-1)}{27}-\frac{n}{3}$

29. What is the minimum value of the function $f(x) = x^2 - 15x + 9$?
(A) $\frac{289}{4}$ (B) $\frac{-200}{9}$
(C) $\frac{295}{2}$ (D) $\frac{-189}{4}$

30. Find the area of the shaded region, where PQR is a triangle and QRS is a quadrant. PQ = 6 cm and QR = 8 cm.



(A) $4\pi - 8$ sq. cm. (B) $2\pi - 3$ sq. cm.
(C) $8\pi - 16$ sq. cm. (D) $16\pi - 24$ sq. cm.

Answer Keys

1. C	**2.** D	**3.** D	**4.** A	**5.** B	**6.** B	**7.** C	**8.** D	**9.** D	**10.** A
11. C	**12.** C	**13.** C	**14.** C	**15.** B	**16.** A	**17.** C	**18.** B	**19.** C	**20.** A
21. C	**22.** A	**23.** D	**24.** C	**25.** A	**26.** C	**27.** C	**28.** D	**29.** D	**30.** D

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Reasoning

UNIT II

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Chapter 1

Number and Letter Series

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Number series
- Difference series
- Product series
- Squares/cubes series
- Miscellaneous series
- Combination series
- Letter series

Introduction

Number and Letter Series form an important part of the Reasoning section in various competitive examinations. There are two or three broad categories of questions that appear in various exams from this particular chapter.

In the first category of questions, a series of numbers/letters is given with one number/letter (or two numbers/letters) missing, represented by a blank or a question mark. The given series of numbers/letters will be such that each one follows its predecessor in a certain way, i.e., according to a definite pattern. Students are required to find out the way in which the series is formed and hence work out the missing number/numbers or letter/letters to complete the series. For the purpose of our discussion, we will refer to this category of questions as Number Series Type I or Letter Series Type I questions. Under Type I questions, there are a large variety of patterns that are possible and the student requires a proper understanding of various patterns to be able to do well in these types of questions.

In the second category of questions, a series of numbers/letters is given and the student is required to count how many numbers/letters in that series satisfy a given condition and mark that as the answer. For the purpose of our understanding, we will refer to this category of questions as Number Series Type II or Letter Series Type II questions. These questions will mainly involve counting of numbers/letters satisfying a given condition.

Number Series

For better understanding, we will classify this into the following broad categories.

1. Difference series
2. Product series
3. Squares/Cubes series
4. Miscellaneous series
5. Combination series

Difference Series

The difference series can be further classified as follows.

1. Number series with a constant difference.
2. Number series with an increasing or decreasing difference.

In the number series with a constant difference, there is always a constant difference between two consecutive numbers. For example, the numbers of the series 1, 4, 7, 10, 13, … are such that any number is obtained by adding a constant figure of 3 to the preceding term of the series.

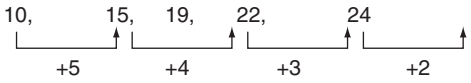
If we have to find the next number in the above series, we need to add a 3 to the last term 13. Thus, 16 is the next term of the series.

Under the series with constant difference, we can have series of odd numbers or series of even numbers also.

In the series with increasing/decreasing difference, the difference between consecutive terms keeps increasing (or decreasing, as the case may be). For example, let us try to find out the next number in the series 2, 3, 5, 8, 12, 17, 23, …

Here, the difference between the first two terms of the series is 1; the difference between the second and third terms is 2; the difference between the third and the fourth terms is 3 and so on. That is, the difference between any pair of consecutive terms is one more than the difference between the first number of this pair and the number immediately preceding this number. Here, since the difference between 17 and 23 is 6, the next difference should be 7. So, the number that comes after 23 should be (23 + 7) = 30.

We can also have a number series where the difference is in decreasing order (unlike in the previous example where the difference is increasing). For example, let us find out the next term of the series 10, 15, 19, 22, 24, …

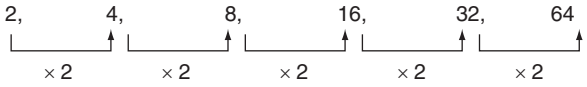


Here the differences between 1st and 2nd, 2nd and 3rd, 3rd and 4th numbers, etc., are 5, 4, 3, 2, and so on. Since the difference between 22 and 24 is 2, the next difference should be 1. So, the number that comes after 24 should be 25.

Product Series

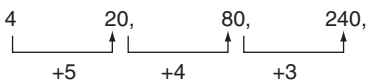
A product series is usually a number series where the terms are obtained by a process of multiplication. Here also, there can be different types of series. We will look at these through examples.

Consider the series 2, 4, 8, 16, 32, 64, …



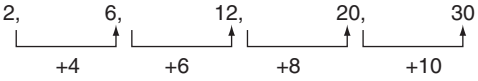
Here, each number in the series is multiplied by 2 to get the next term. So, the term that comes after 64 is 128. So, each term is multiplied by a fixed number to get the next term. Similarly we can have a series where we have numbers obtained by **dividing** the previous term with a constant number. For example, in the series 64, 32, 16, 8, …, each number is obtained by dividing the previous number by 2 (or in other words, by multiplying the previous term by $\frac{1}{2}$). So, here, the next term will be 4 (obtained by dividing 8 with 2).

Consider the series 4, 20, 80, 240, …

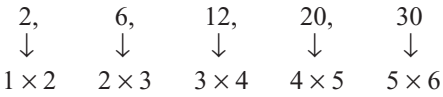


Here, the first term is multiplied by 5 to get the second term; the second term is multiplied by 4 to get the third term; the third term is multiplied by 3 to get the fourth term. Hence, to get the fifth term, we have to multiply the fourth term by 2, i.e., the fifth term is 480. So, each term is multiplied by a decreasing factor (or it could also be an increasing factor) to get the next term. That is, with whatever number a particular term is multiplied to get the next term, this latest term is multiplied by a number different from the previous multiplying factor to get the next term of the series. All the multiplying factors follow a certain pattern (normally of increasing or decreasing order).

Consider the series 2, 6, 12, 20, 30, …



This can be looked at a series of increasing differences. The differences of consecutive pairs of terms are 4 (between 2 and 6), 6 (between 6 and 12), 8 (between 12 and 20), 10 (between 20 and 30) and so on. Hence, the difference between 30 and the next term should be 12 and so the next term will be 42. But this series can also be looked at as a product series.

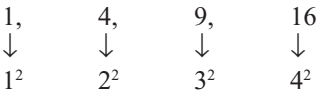


The first term is the product of 1 and 2; the second term is the product of 2 and 3; the third term is the product of 3 and 4; the fourth term is the product of 4 and 5; the fifth term is the product of 5 and 6. Hence, the next term will be the product of 6 and 7, that is 42.

Squares/Cubes Series

There can be series where all the terms are related to the squares of numbers or cubes of numbers. With squares/cubes of numbers as the basis, there can be many variations in the pattern of the series. Let us look at various possibilities of series based on squares/cubes.

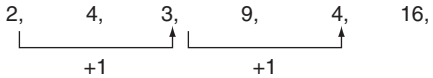
Each term of the series may be the square of a natural number, such as 1, 4, 9, 16, …



The numbers are squares of 1, 2, 3, 4 … respectively. The number which follows 16 (which is the square of 4) will be 25 (which is the square of 5).

The terms of the series may be the squares of odd numbers (for example, 1, 9, 25, 49, …) or even numbers (for example, 4, 16, 36, 64, …).

The terms of the series could be such that a number and its square are both given one after the other and such pairs are given in some specific pattern. For example, take the series 2, 4, 3, 9, 4, 16, …



Here, 2 is followed by its square 4; then comes the number 3 (which is one more than 2) followed by its square 9 and so on. Hence, the next number in the series is 5 and the one after that is its square i.e., 25.

Similarly, each term could be the square root of its predecessor. For example, in the series 81, 9, 64, 8, 49, 7, 36, …, 81 is the square of 9, 64 the square of 8, and so on. Therefore the next number which follows in the series should be the square root of 36, i.e., 6.

The terms of the series could be the squares of natural numbers increased or reduced by certain number. For example, in the series 3, 8, 15, 24, …

3,	8,	15,	24
↓	↓	↓	↓
$2^2 - 1$	$3^2 - 1$	$4^2 - 1$	$5^2 - 1$

We have {Squares of natural numbers – 1} as the terms. The first term is $2^2 - 1$; the second term is $3^2 - 1$; the third term is $4^2 - 1$ and so on. Hence, the next term will be $6^2 - 1$, i.e., 35 [Please note that the above series can also be looked at as a series with increasing differences. The differences between the 1st and 2nd terms, the 2nd and 3rd terms, and so on are 5, 7, 9, and so on. Hence, the next difference should be 11 giving us the next term as 35]. There could also be a series with {squares of natural numbers + some constant}.

Like we have seen series with squares of numbers, we can have similar series with cubes of numbers. For example, take the series 1, 8, 27, 64, …

1,	8,	27,	64
↓	↓	↓	↓
1^3	2^3	3^3	4^3

Here, all the terms are cubes of natural numbers. So, the next term will be 5^3, i.e., 125.

Consider the series 2, 9, 28, 65, …

2,	9,	28,	65
↓	↓	↓	↓
$1^3 + 1$	$2^3 + 1$	$3^3 + 1$	$4^3 + 1$

Here, the terms are {Cubes of natural numbers + 1}. The first term is $1^3 + 1$; the second term is $2^3 + 1$; the third term is $3^3 + 1$ and so on. Hence the next term will be $5^3 + 1$, i.e., 126.

Miscellaneous Series

There are series that do not come under the other patterns and are of general nature but are important and are fairly common. Even here, some times, there can be a specific pattern in some cases.

Take the series 3, 5, 7, 11, 13, … . This is a series of consecutive PRIME NUMBERS. It is an important series and the student should look out for this as one of the patterns. The next term in this series is 17.

There can also be variations using prime numbers. Take the series 9, 25, 49, 121, … . In this series, the terms are squares of prime numbers. Hence, the next term is 13^2, i.e., 169.

Take the series 15, 35, 77, … . The first term is 3×5; the second term is 5×7; the third term is 7×11; here the terms are product of two consecutive prime numbers. So, the next term will be the product of 11 and 13, i.e., 143.

Take the series 8, 24, 48, 120, 168, … Here, the 2nd term is 3 times the first term and the 3rd term is 2 times the 2nd term, but after that it does not follow this pattern any more. If you look at the terms carefully, you will find that the terms are {one less than squares of prime numbers}. Hence, the next term will be $17^2 - 1$, i.e., 288.

Consider the series 1, 4, 9, 1, 6, 2, 5, 3, …

At first sight there is nothing we can say about the series. This is actually a series formed by squares of natural numbers. However, if any of the squares is in two or more digits, each of the digits is written as a separate term of the series. Thus, the first terms are 1, 4 and 9, the squares of 1, 2 and 3 respectively. After this, we should get 16 (which is the square of 4). Since this has two digits 1 and 6, these two digits are written as two different terms 1 and 6 in the series. Similarly, the next square 25 is written as two different terms 2 and 5 in the series. So, the next square 36 should be written as two terms 3 and 6. Of these, 3 is already given. So, the next term of the series is 6.

Consider the series 1, 1, 2, 3, 5, 8, …

1,	1,	2,	3,	5,	8
		↓	↓	↓	↓
		1 + 1	1 + 2	2 + 3	3 + 5

Here, each term, starting with the third number, is the sum of the two preceding terms. After taking the first two terms as given (1 and 1), then onwards, to get any term, we need to add the two terms that come immediately before that position. Hence, to get the next term of the series, we should take the two preceding terms 5 and 8 and add them up to get 13. So, the next term of the series is 13. The term after this will be 21 (= 8 + 13).

Combination Series

A number series which has more than one type of (arithmetic) operation performed or more than one series combined together is a combination series. The series that are combined can be two series of the same type or could be different types of series as described above. Let us look at some examples.

First let us look at those series which are formed by more than one arithmetic operation performed on the terms to get the subsequent terms.

Consider the series: 2, 6, 10, 3, 9, 13, 4, 12, … Here, the first term 2 is multiplied by 3 to get the second term, and 4 is added to get the third term. The next term is 3 (one more than the first term 2) and it is multiplied by 3 to get 9 (which is the next term) and then 4 is added to get the next term 13. The next term 4 (which is one more than 3) which is multiplied with 3 to get 12. Then 4 is added to this to get the next number 16.

Consider the series: 1, 2, 6, 21, 88, … Here, we can observe that 88 is close to 4 times 21. It is in fact $21 \times 4 + 4$. So, if we now look at the previous term 21, it is related to the previous term 6 as $6 \times 3 + 3$. Now we get the general pattern: to get any term, multiply the previous term with k and then add k where k is a natural number with values in increasing order from 1. So, to get the second term, the first term has to be multiplied with 1 and then 1 is added. To get the third term, the second term is multiplied with 2 and then 2 is added and so on. Hence, after 88, the next term is $88 \times 5 + 5$, i.e., 445.

Now, let us look at a series that is formed by combining two (or more) different series. The two (or more) series can be of the same type or of different types described above.

Consider the series: 8, 12, 9, 13, 10, 14, … Here the 1st, 3rd, 5th, … terms which are 8, 9, 10, … form one series whereas the 2nd, 4th, 6th, etc. terms which are 12, 13, 14 form another series. Here, both series that are being combined are two simple constant difference series. Therefore the missing number will be the next term of the first series 8, 9, 10, … which is equal to 11.

Consider the series: 0, 7, 2, 17, 6, 31, 12, 49, 20, … Here, the series consisting of 1st, 3rd, 5th, … terms (i.e., the series consisting of the odd terms) which is 0, 2, 6, 12, 20, … is combined with another series consisting of 2nd, 4th, 6th, … terms (i.e., the series consisting of the even terms) which is 7, 17, 31, 49, … . The first series has the differences in increasing order 2, 4, 6, 8, 10 and so on. The second series also has the difference in increasing order 10, 14, 18, … Since, the last term 20 belongs to the first series, a number from the second series should follow next. The next term of the second series will be obtained by adding 22 to 49, that is 71.

Consider the series: 1, 1, 2, 4, 3, 9, 4, 16, … Here, one series consisting of odd terms, which is 1, 2, 3, 4, …, is combined with the series of even terms which is 1, 4, 9, 16, … The first series is a series of natural numbers. The second series is the squares of natural numbers. Hence, the next term is 5.

Consider the series: 1, 1, 4, 8, 9, 27, … Here, the series of squares of natural numbers is combined with the series of cubes of natural numbers. The next term in the series will be 4.

Consider the series: 2, 4, 5, 9, 9, 16, 14, ? , 20, … Here, we have to find out the term that should come in place of the question mark. The odd terms form one series 2, 5, 9, 14, 20, … where the difference is increasing. The differences are 3, 4, 5, 6, … This series is combined with the series of even terms 4, 9, 16, … where the terms are squares of numbers 2, 3, 4, … . Hence, the term that should come in place of the question mark is the next term of the second series which is 5^2, i.e., 25.

A general approach to Number Series: The best way of approaching the number series questions is to first observe the difference between terms. If the difference is constant, it is a constant difference series. If the difference is increasing or decreasing by a constant number, then it is a series with a constant increasing or decreasing difference. If there is no constant increasing or decreasing difference, then try out the product series approach. For this, first divide the second term with the first term, third with the second, and so on. If the numbers obtained are the same, then it is a product series. Alternatively, try writing each term of the series as a product of two factors and see if there is any pattern that can be observed. If still there is no inference, but the difference is increasing or decreasing in a rapid manner, then check out the square series. If the increase is very high, and it is not a square series, then try out the cube series.

If the difference is alternately decreasing and increasing (or increasing for some time and alternately decreasing), then it should most probably be a mixed series. Therefore test out the series with alternate numbers. If still the series is not solved, try out the general series.

Letter Series

The questions here are similar to the questions in Number Series Type I. Instead of numbers we have letters of the alphabet given here. We have to first identify the pattern that the series of letters follow. Then, we have to find the missing letter based on the pattern already identified. In Number Series, we saw different patterns that the numbers in the series can follow – like squares, cubes. In letter series, obviously, patterns like squares, cubes will not be possible. In Letter Series, in general, we have a series with constant or increasing or decreasing differences. The position of the letters in the English alphabet is considered to be the value of the alphabet in questions on Letter Series. Also, when we are counting, after we count from *A* to *Z*, we again start with *A*, i.e., we treat the letters as being cyclic in nature. Like in Number Series, in this type of Letter Series also, we can have a 'combination' of series, i.e., two series are combined and given. We need to identify the pattern in the two series to find out the missing letter. Sometimes, there will be some special types of series also. Let us look at a few examples to understand questions on Letter Series.

Solved Examples

Example 1: Find the next letter in the series

D, G, J, M, P, _____.

(A) Q (B) R (C) S (D) T

Solution: (C)

Three letters are added to each letter to get the next letter in the series.

i.e., $D^{+3}, G^{+3}, J^{+3}, M^{+3}, P^{+3}, \underline{S}$

P + 3 and P = 16 and 16 + 3 = 19 and the 19th letter in the alphabet is S.

Example 2: Find the next letter in the series

A, B, D, H, _____.

(A) L (B) N (C) R (D) P

Solution: (D)

Each letter in the given series is multiplied with 2 to get the next letter in the series.

$A \times 2 \Rightarrow 1 \times 2 = 2$ and the 2nd letter is B, $B \times 2 \Rightarrow 2 \times 2 = 4$ and the 4th letter is D.

Similarly, $H \times 2 \Rightarrow 8 \times 2 = 16$ and the 16th letter is P.

Example 3: What is the next letter in the series?

B, D, G, K, P, _____

(A) S (B) V (C) W (D) X

Solution: (B)

B^{+2}, D^{+3}, G^{+4}, K^{+5}, P^{+6}, _____

P + 6 = 16 + 6 = 22 and the 22nd letter is V.

Example 4: I, X, J, W, K, V, L, _____.

(A) M (B) U (C) S (D) T

Solution: (B)

The given series is an alternate series.

I^{+1}, J^{+1}, K^{+1}, L is one series and X^{-1}, W^{-1}, V^{-1}, _____ is the other series.

X – 1 = W, W – 1 = V and V – 1 = 22 – 1 = 21 and the 21st letter is U.

Example 5: 97, 83, 73, 67, 59, _____

(A) 53 (B) 49

(C) 47 (D) 51

Solution: (C)

The given numbers are alternate prime numbers in decreasing order, starting with 97. Hence, the next number in the series is 47.

Example 6: 75, 291, 416, 480, 507, _____

(A) 515 (B) 532

(C) 511 (D) 521

Solution: (A)

75^{+216}, 291^{+125}, 416^{+64}, 480^{+27}, 507, _____

The differences are cubes of consecutive natural numbers in decreasing order. Hence, the next number in the series in $507 + (B)^3 = 515$.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Complete the following series.

1. 17, 19, 23, 29, 31, 37, _____

(A) 41 (B) 43 (C) 40 (D) 42

2. 225, 196, 169, _____, 121, 100, 81

(A) 156 (B) 144 (C) 136 (D) 125

3. 64, 125, 216, 343, _____

(A) 64 (B) 424 (C) 317 (D) 512

4. 54, 66, 82, 102, 126, _____

(A) 146 (B) 130 (C) 154 (D) 144

5. 7, 11, 20, 36, 61, _____, 146

(A) 25 (B) 91 (C) 97 (D) 92

6. 8, 16, 48, 96, 288, 576, _____

(A) 1152 (B) 1728

(C) 1052 (D) 1428

7. 125, 375, 377, 1131, 1133, _____

(A) 3399 (B) 1136

(C) 1135 (D) 1234

8. 12, 35, 106, 317, 952, _____

(A) 2851 (B) 2855

(C) 1851 (D) 1849

9. 2, 4, 7, 35, 42, 462, _____

(A) 5016 (B) 470

(C) 4712 (D) 475

10. $13\frac{1}{3}$, 15, $\frac{120}{7}$, 20, 24, _____

(A) 30 (B) 36 (C) 40 (D) $37^{1}/_{3}$

11. 6, 15, 35, 77, 143, 221, _____

(A) 357 (B) 437 (C) 323 (D) 383

12. 29, 29, 27, 23, 25, 19, 23, 17, _____, _____

(A) 19, 13 (B) 19, 15 (C) 21, 13 (D) 19, 13

13. 24, 625, 26, 729, 28, 841, _____

(A) 30 (B) 29 (C) 900 (D) 961

14. 3731, 2923, 1917, 1311, _____

(A) 117 (B) 119 (C) 917 (D) 75

15. 11, 28, 327, 464, _____

(A) 525 (B) 5625 (C) 5125 (D) 5250

16. 6, 24, 60, 120, 210, _____

(A) 336 (B) 343 (C) 368 (D) 322

17. 132, 182, 306, 380, 552, 870, _____

(A) 930 (B) 1010 (C) 992 (D) 1142

18. KPD, LOE, MNF, NMG, _____

(A) ONF (B) OLH (C) MLH (D) MNH

19. BEP, CIQ, DOR, FUS, GAT, _____

(A) HEV (B) HIT (C) IET (D) IEU

20. GKF, IPC, LTY, PWT, UYN, _____

(A) ABZ (B) XBX (C) XAH (D) AZG

21. QLR, JPD, RNU, GNC, SPX, DLB, _____
(A) TRA (B) AJA (C) BTU (D) KJE

22. GTB, CYV, YDP, _____, QND
(A) DIV (B) UIJ (C) DDV (D) UVV

23. ABDH, BDHP, CFLX, DHPF, _____
(A) EKNT (B) TNEK (C) EJTN (D) JNTE

24. TCFK, RADI, OXAF, JSVA, _____
(A) DMPU (B) DMOT
(C) CMOT (D) CLOT

25. KJAM, GGWJ, _____, YAOD, UXKA
(A) CDUI (B) DFTC
(C) DCTF (D) CDSG

Answer Keys

Exercise

Practice Problems

1. A	**2.** B	**3.** D	**4.** C	**5.** C	**6.** B	**7.** A	**8.** B	**9.** D	**10.** A
11. C	**12.** C	**13.** A	**14.** D	**15.** C	**16.** A	**17.** C	**18.** B	**19.** A	**20.** D
21. A	**22.** B	**23.** C	**24.** D	**25.** D					

Chapter 2

Analogies

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Analogy
- Number analogies
- Letter analogies
- Verbal analogies

Analogy

Analogy means 'similarity' or 'similar relationship'. In questions on number or letter analogies, a pair, that has a certain relationship between them, is given. This number/letter pair is followed by a third number/letter. The student is expected to identify the relationship between the pair given and find out a FOURTH number such that the relationship between the third and the fourth is similar to the relationship that exists between the first and the second. (In some cases, it may not be the fourth one that has to be found out. The fourth one will be given and the student has to find out one of the other three, whichever is not given).

Number Analogies

Typical relationships between the numbers in a given pair can be any of the following:

- One number is a multiple of the other.
- One number is the square or square root of the other.
- One number is the cube or cube root of the other.
- The two numbers are squares of two other numbers which themselves are related. For example, the two numbers are squares of two consecutive integers or squares of two consecutive even integers or squares of two consecutive odd integers.
- The two numbers are such that they are obtained by subtracting a certain number from the squares or cubes of the two related numbers.
- The two numbers are such that they are obtained by adding a certain number to the squares or cubes of the two related numbers.
- The two numbers can be consecutive, even, odd or prime numbers.

There can be many more combinations that one can think of but the student has to note an important point in solving questions on Number Analogies. In Number Series related questions, since a series of numbers (more than two numbers) will be given, the relationship or pattern can be identified uniquely. In Number Analogies, since only two numbers are given, it may be possible to think of more than one relationship existing between the two numbers in the given pair. But, it should be kept in mind that generally, simple addition of one number or subtraction of one number is not what is given in Number Analogies. The questions try to test the insight that the student has got into the relationship between the numbers.

Let us take a few examples and understand the questions on Number Analogies.

Solved Examples

Example 1: Find the missing number 25 : 36 : : 49 : _____.
(A) 61 (B) 63 (C) 65 (D) 60

Solution: (D)
When the numbers in the question are considered the students tend to consider 25 and 36 as squares of two consecutive natural numbers. But the answer choices does not consist of an answer suitable to the above logic. Hence, it is important that, the student keeps the answer choices in view in arriving at the logic.

$$25 + 11 = 36$$

Similarly, $49 + 11 = 60$

Example 2: Find the missing number 27 : 51 : : 83 : _____.
(A) 102 (B) 117 (C) 123 (D) 138

Solution: (C)
The given analogy can be written as

$$5^2 + 2 : 7^2 + 2 : : 9^2 + 2 : _____.$$

5 and 7 are successive odd numbers.

Similarly, next odd number to 9 is 11 and $11^2 + 2 = 121 + 2 = 123$.

Example 3: Find the missing number.

11 : 25 : : 17 : _____.

(A) 33 (B) 28 (C) 41 (D) 37

Solution: (D)

$$11 \times 2 + 3 = 22 + 3 = 25$$

Similarly, $17 \times 2 + 3 = 34 + 3 = 37$.

Letter Analogies

The questions in this area are similar to Verbal Analogies. Here, the questions are based on the relationship between two groups of letters (instead of two words as in Verbal Analogies). Typically, three sets of letters are given followed by a question mark (where a fourth set of letters is supposed to inserted). The student has to find the relation or order in which the letters have been grouped together in the first two sets of letters on the left hand side of the symbol : : and then find a set of letters to fit in place of the question mark so that the third and the fourth set of letters will also have the same relationship as the first and the second. The sequence or order in which the letters are grouped can be illustrated by the following examples.

Example 4: BDEG : DFGI : : HKMO : _____.
(A) ILNP (B) JMOP
(C) JMOQ (D) JNOQ

Solution: (C)
Two letters are added to each letter to get the next letters in the analogy.

B	D	E	G;	Similarly,	H	K	M	O
+2	+2	+2	+2		+2	+2	+2	+2
D	F	G	I		J	M	O	Q

Example 5: ACDF : CGJN : : BEHI : _____.
(A) DJNQ (B) DINQ
(C) DINR (D) DHNQ

Solution: (B)

A	C	D	F;	Similarly,	B	E	H	I
+2	+4	+6	+8		+2	+4	+6	+8
C	G	J	N		D	I	N	Q

Example 6: SUWY : LPTX : : PRTV : _____.
(A) INRU (B) INQU
(C) IMRU (D) IMQU

Solution: (D)

S	U	W	Y;	Similarly,	P	R	T	V
–7	–5	–3	–1		–7	–5	–3	–1
L	P	T	X		I	M	Q	U

Example 7: BCDE : DFHH : : FGHI : _____.
(A) LJPL (B) LKPL
(C) JKPJ (D) IKPL

Solution: (A)

B	C	D	E;	Similarly,	F	G	H	I
×2	+3	×2	+3		×2	+3	×2	+3
D	G	H	H		L	J	P	L

Verbal Analogies

Here, the questions are based on relationship between two words. In these kind of questions three words are followed by a blank space, which the student has to fill up in such a way that the third and the fourth words have the same relationship between them as the first and the second words have. The following examples help in understanding the concepts.

Example 8: Gum : Stick : : Needle : _____
(A) Cloth (B) Prick
(C) Taylor (D) Stitch

Solution: (D)
Gum is used to Stick and Needle is used to Stitch.

Example 9: Socks : Feet : _____ : Hands
(A) Arms (B) Shirt
(C) Gloves (D) Fingers

Solution: (C)
Socks are worn on Feet, similarly Gloves are worn on Hands.

Example 10: Soft : Hard : : Cold : _____
(A) Hot
(B) Shirt
(C) Gloves
(D) Fingers

Solution: (A)
Soft and Hard are antonyms similarly the antonym of Cold is Hot.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Find the missing term.

1. 97 : 89 : : 43 : _____
(A) 37 (B) 31 (C) 39 (D) 41

2. 196 : 256 : : 324 : _____
(A) 361 (B) 400 (C) 411 (D) 484

3. 121 : 169 : : 361 : _____
(A) 529 (B) 400 (C) 484 (D) 576

4. 125 : 343 : : 343 : _____
(A) 512 (B) 1331 (C) 1728 (D) 81

5. 4 : 256 : : 5 : _____
(A) 625 (B) 1025 (C) 525 (D) 875

6. 12 : 144 : : 18 : _____
(A) 160 (B) 180 (C) 190 (D) 150

7. 25 : 21 : : 59 : _____
(A) 42 (B) 46 (C) 76 (D) 56

8. 8 : 72 : : 10 : _____
(A) 95 (B) 106 (C) 99 (D) 90

9. 8 : 0.125 : : 4 : _____
(A) 0.5 (B) 0.4 (C) 0.35 (D) 0.25

10. 11 : 143 : : 19 : _____
(A) 443 (B) 450 (C) 420 (D) 437

11. 568 : 352 : : 732 : _____
(A) 516 (B) 496 (C) 526 (D) 536

12. 6 : 222 : : 9 : _____
(A) 738 (B) 720 (C) 729 (D) 744

13. 5 : 120 : : 8 : _____
(A) 520 (B) 504 (C) 448 (D) 512

14. 16 : 68 : : 36 : _____
(A) 216 (B) 210 (C) 222 (D) 226

15. 10 : 95 : : 16 : _____
(A) 218 (B) 318 (C) 248 (D) 102

16. 3829 : 3851 : : 2987 : _____
(A) 301 (B) 3007
(C) 3017 (D) 3023

17. 47 : 121 : : 89 : _____
(A) 183 (B) 187 (C) 193 (D) 195

18. NATURE : PEVASI : : ISOMERS : _____
(A) OTUNJTV (B) OTUNIST
(C) PUVNJST (D) OVTNJST

19. BAD : BBL : : JDFE : _____
(A) JHRI (B) JHPX
(C) JFTV (D) JHRT

20. FIELD : LRJXH : : CRICKET : _____
(A) FHRDXLJ (B) FJPDTLN
(C) FJRDXAL (D) FJRFVJN

21. TAP : SUZBOQ : : RED : _____
(A) QTDGDE (B) PSDEDF
(C) QSDFCE (D) QRDGBE

22. Train : Track : : Bus : _____
(A) Driver (B) Road
(C) Petrol (D) Passengers

23. Earth : Planet : : Carrot : _____
(A) Vegetable (B) Plant
(C) Cooking (D) Nut

24. Wood : Carpenter : : Iron : _____
(A) Goldsmith (B) Instrument
(C) Melting (D) Blacksmith

25. Pen : Write : : Knife : _____
(A) Vegetable (B) Cut
(C) Sharp (D) Shoot

ANSWER KEYS

EXERCISE

Practice Problems

1. D	**2.** B	**3.** A	**4.** B	**5.** A	**6.** D	**7.** D	**8.** D	**9.** D	**10.** D
11. A	**12.** A	**13.** B	**14.** C	**15.** C	**16.** A	**17.** B	**18.** B	**19.** D	**20.** D
21. C	**22.** B	**23.** A	**24.** D	**25.** B					

Chapter 3

Odd Man Out (Classification)

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Odd man out
- Alphabet classification
- Word classification
- Number classification

Introduction

Finding the odd man out from the given alternatives is a very common type of questions that one comes across in different competitive examinations. In the questions on odd man out, all the items – except one – follow a certain pattern (in their formation) or belong to a group. The item that does not follow the pattern or does not belong to the group has to be marked as the answer choice.

The problems of this variety often fall under the category of CLASSIFICATION. When a given set of elements is classified under a single head, one of the items will not fall into that group to which the rest belong, i.e. it will not have the common property, which the others will have. Hence it becomes the odd man out.

Questions on classification can be asked in any form. Some of the commonly asked ones are given below.

Alphabet Classification

In this type, a group of jumbled letters typically consisting of three letters, (but can be four or two or just a single letter) are put together. The pattern or order in which they are grouped is to be studied and we need to find out which groups have the same pattern or relationship between the letters. There will be one choice, which will have a pattern different from the rest and that is our answer.

Solved Examples

Example 1: Find the odd one among the following.

(A) ZW (B) TQ
(C) SP (D) NL

Solution: (D)

$$Z^{-3}W, T^{-3}Q, S^{-3}P, N^{-2}L$$

Hence, NL is the odd one.

Example 2: Find the odd one among the following.

(A) CFD (B) GJH
(C) KNM (D) JMK

Solution: (C)

$$C^{+3}F^{-2}D, G^{+3}J^{-2}H, K^{+3}N^{-1}M, J^{+3}M^{-2}K$$

Hence, KNM is the odd one.

Word Classification

Here, different items are classified based on common properties like names, places, professions, parts of speech, etc. A few examples are illustrated below.

Example 3: Find the odd one among the following.
(A) Mercury (B) Moon
(C) Jupiter (D) Saturn

Solution: (B)
All others except Moon are planets whereas Moon is a satellite.

Example 4: Find the odd one among the following.
(A) SORE (B) SOTLU
(C) NORGAE (D) MEJNIAS

Solution: (C)
The words are jumbled. The actual words are ROSE, LOTUS, ORANGE and JASMINE. All, except ORANGE, are flowers whereas ORANGE is a fruit.

Example 5:
(A) Cow (B) Goat
(C) Horse (D) Dog

Solution: (D)
All except Dog are herbivorous animals.

Example 6:
(A) Shoe (B) Spectacle
(C) Scissor (D) Shirt

Solution: (D)
All except shirt are in prises.

Number Classification

In this case, we need to choose the odd number from the given alternatives. The numbers may belong to a particular set, i.e., they may be odd, even, prime, rational, squares, cubes, and they may also be coded into binary digits (involving 0's and 1's) etc. and only one of the choices will not follow the rule which others do and that is our answer. A few illustrations are given below.

Example 7: Find the odd one among the following.
(A) 17 (B) 27 (C) 37 (D) 47

Solution: (B)
All the given numbers except 27 are prime numbers whereas 27 is a composite number.

Example 8:
(A) 441 (B) 289
(C) 361 (D) 343

Solution: (D)
The given numbers can be written as $(21)^2$, $(17)^2$, $(19)^2$, $(7)^3$. All except 343 are the squares whereas 343 is a cube.

Example 9:
(A) 10 (B) 50
(C) 120 (D) 290

Solution: (D)
All expect 120 can be expressed as $n^2 + 1$

$$10 = 3^2 + 1,\ 50 = 7^2 + 1$$

$290 = 17^2 + 1$ but $120 = 11^2 - 1$.

Example 10:
(A) 235 (B) 352 (C) 523 (D) 253

Solution: (B)
All expect 352 are odd numbers but whereas 352 is an even number.

Exercise

Practice Problems

Directions for questions 1 to 25: Find the odd man out.

1. (A) 16 (B) 28
(C) 36 (D) 64

2. (A) 27 (B) 37
(C) 47 (D) 67

3. (A) 8 (B) 27
(C) 64 (D) 125

4. (A) 42624 (B) 37573
(C) 84284 (D) 93339

5. (A) 30 (B) 630
(C) 10 (D) 520

6. (A) 8 : 9 (B) 25 : 25
(C) 64 : 81 (D) 16 : 16

7. (A) $\frac{3}{\sqrt{4+25}}$ (B) $\frac{7}{\sqrt{36+64}}$
(C) $\frac{11}{\sqrt{49+169}}$ (D) $\frac{5}{\sqrt{9+49}}$

8. (A) $13\frac{17}{23}$ (B) $41\frac{45}{49}$
(C) $71\frac{73}{79}$ (D) $83\frac{89}{97}$

9. (A) 4422 (B) 2442
(C) 4242 (D) 2244

10. (A) 350 (B) 70
(C) 30 (D) 520

11. (A) N (B) O
(C) B (D) K

12. (A) $E\frac{V}{R}$ (B) $O\frac{L}{B}$
(C) $I\frac{R}{V}$ (D) $U\frac{B}{L}$

13. (A) ABD (B) BDH
(C) CEJ (D) DFL

14. (A) BCDE (B) FGHI
(C) RSTU (D) WXYZ

15. (A) DFRTH (B) ABEJM (C) NBEJM (D) DHKVY

16. (A) Cat (B) Dog (C) Tiger (D) Elephant

17. (A) Chameleon (B) Crocodile (C) Turtle (D) Allegator

18. (A) Trivandrum (B) Hyderabad (C) Calicut (D) Bangalore

19. (A) Part (B) Trap (C) Cart (D) Dart

20. (A) Rocket (B) Star (C) Planet (D) Comet

21. (A) Skin (B) Tongue (C) Leg (D) Nose

22. (A) Baseball (B) Boxing (C) Chess (D) Wrestling

23. (A) Walk (B) Talk (C) Drink (D) Plank

24. (A) Ganga (B) Nagarjuna sagar (C) Yamuna (D) Sutlez

25. (A) HEWAT (B) CERI (C) ROWAJ (D) EECRALS

Answer Keys

Exercise

Practice Problems

1. B	**2.** A	**3.** C	**4.** C	**5.** B	**6.** B	**7.** B	**8.** B	**9.** C	**10.** B
11. B	**12.** D	**13.** A	**14.** D	**15.** A	**16.** D	**17.** A	**18.** C	**19.** B	**20.** A
21. C	**22.** A	**23.** D	**24.** B	**25.** D					

Chapter 4

Coding and Decoding

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Coding
- Decoding

Coding and Decoding

Before looking at the different types of questions and some of the codes that can be used with the help of examples, let us first understand what we mean by ***coding*** and ***decoding***. When we say ***coding***, a particular code or pattern is used to express a word in English language as a different word or in a different form. The coded word itself does not make any sense unless we know the pattern or code that has been followed. ***Decoding*** refers to the process of arriving at the equivalent English word from the code word given.

In the questions, a particular code is given and on the basis of this given code, we have to find out how another word (in English language) can be coded. The correct code for the given word has to be selected from the answer choices on the basis of the code given in the question.

Solved Examples

Example 1: In a certain code language, if the word 'PARTNER' is coded as OZQSMDQ, then what is the code for the word 'SEGMENT' in that language?

(A) TFHNFOU (B) RDFLDMS
(C) RDELDMS (D) RDFEDNS

Solution: (B)

Word : P A R T N E R
Logic : –1 –1 –1 –1 –1 –1 –1
Code : O Z Q S M D Q

Similarly the code for SEGMENT is

Word : S E G M E N T
Logic : –1 –1 –1 –1 –1 –1 –1
Code : R D F L D M S

Example 2: In a certain code language, if the word RECTANGLE' is coded as TGEVCPING, then how is the word 'RHOMBUS' coded in that language?

(A) TJOQDWV (B) TJQNDWU
(C) TJQODWU (D) TJQOEWU

Solution: (C)

Word : R E C T A N G L E
Logic : +2 +2 +2 +2 +2 +2 +2 +2 +2
Code : T G E V C P I N G

Similarly, the code for RHOMBUS is

Word : R H O M B U S
Logic : +2 +2 +2 +2 +2 +2 +2
Code: T J Q O D W U

Example 3: In a certain code language, if the word 'SPHERE' is coded as EREHPS, then how is the word 'EXHIBITION' coded in that language?

(A) NOTITBIHXE (B) NOITIDIHXE
(C) NOITIBIHWE (D) NOITIBIHXE

Solution: (D)

Word: S P H E R E
Logic: The letters in the given word are reversed.
Code: E R E H P S

Similarly, the code for EXHIBITION, is

Word: E X H I B I T I O N
Logic: The letters in the given word are reversed.
Code: N O I T I B I H X E

Example 4: In a certain code language, if the word 'REJECTION' is coded as SGMIHZPWW, then how is the word 'MECHANIC' coded in that language?

(A) NGFLFTPK (B) NGPLFTPK
(C) NGFKFTPK (D) NGPTPKIL

Solution: (A)

Word:	R	E	J	E	C	T	I	O	N
Logic :	+1	+2	+3	+4	+5	+6	+7	+8	+9
Code :	S	G	M	I	H	Z	P	W	W

Similarly, the code for MECHANIC is

Word :	M	E	C	H	A	N	I	C
Logic :	+1	+2	+3	+4	+5	+6	+7	+8
Code :	N	G	F	L	F	T	P	K

Example 5: In a certain code language, if the word 'PLAYER' is coded as AELPRY, then how is the word 'MANAGER' coded in that language?

(A) AEAGMNR (B) AAGEMNR
(C) AAEGMNR (D) AAEGNMR

Solution: (C)

Word: P L A Y E R
Logic: The letters in the word are arranged in the increasing order of their value as in the alphabet.
Code: A E L P R Y

Similarly, the code for MANAGER is AAEGMNR.

Example 6: In a certain code language, if the number 1 is assigned to all the letters in odd numbered places in the alphabet and the remaining letters are assigned the number 2, then what is the code for the word 'INDIAN'?

(A) 121212 (B) 111222
(C) 112212 (D) 122112

Solution: (D)

The code for the word INDIAN is 122112.

Example 7: In a certain code language, if CRICKET is coded as 3923564, ROCKET is coded as 913564 and KETTLE is coded as 564406, then how is LITTLE coded in that language?

(A) 244060 (B) 024406
(C) 020446 (D) 200446

Solution: (B)

As we observe that the letters and their corresponding codes are given in order i.e., the code for C is 3, R is 9, I is 2 and so on. Hence, the code for LITTLE is 024406.

Directions for questions 8 to 10: In a certain code language, the codes for some words are as follow.

NATION	-	agvnab
REMOTE	-	rzgrbe
STAIR	-	efgnv
FORMAL	-	bensyz
COMMON	-	zabzpb
FOR	-	ebs

Based on the above coding pattern answer the following questions.

8. What is the code for 'SCREEN'?
(A) fepcra
(B) fpersa
(C) fpreba
(D) fperra

9. What is the code for 'RATION'?
(A) ensvba
(B) engvba
(C) engrba
(D) engvca

10. What is the code for 'CREATOR'?
(A) prengbc
(B) persbgc
(C) perngbe
(D) pebryc

Solutions for questions 8 to 10:

The given words and their codes are as follow

(A)	NATION	-	agvnab
(B)	REMOTE	-	rzgrbi
(C)	STAIR	-	efgnv
(D)	FORMAL	-	bensyz
(E)	COMMON	-	zabzpb
(6)	FOR	-	ebs

In the 1st word the letter N is repeated and so is the code *a*. Hence, for N, the code is a. Similarly, from the 2nd word, the code for E is '*r*'. In 1st and 6th words the letter *o* is common and so is the code *b*. Hence, the code for *o* is *b*. In the 5th word the letter *m* is repeated and so is the code *z*. Hence, the code for *m* is *z*. Similarly the codes for the remaining letters can be determined.

The letters and their respective codes are as follows

Letter	A	C	E	F	I	L	M	N	O	R	S	T
Code letter	n	p	R	s	v	y	z	a	b	e	f	G

8. The code for 'SCREEN' is fperra.
Hence, the correct option is (D).

9. The code for 'RATION' is engvba.
Hence, the correct option is (B).

10. The code for 'CREATOR' is perngbe.
Hence, the correct option is (C).

EXERCISE

Practice Problems

Directions for questions 1 to 12: Select the correct alternative from the given choices.

1. In a certain code language, if the word CIRCUMSTANCE is coded as CRUSACICMTNE, then how is the word HAPPINESS coded in that language?
(A) HPEISAPNS (B) HPISEAPNS
(C) HPIESPANS (D) HPIESAPNS

2. In a certain code language, if the word REGISTRATION is coded as TSIGERNOITAR, then how is the word ACCURATE coded in that language?
(A) UCCAETAR (B) UACCETAR
(C) UCACETAR (D) UCCATEAR

3. In a certain code language, if the word LIBERAL is coded as MJCFSBM, then how is the word REDUCTION coded in that language?
(A) EDCTBSHNM (B) SFEVDUJPO
(C) SFEVCTJPO (D) SFDUCTJPO

4. In a certain code language, if the word STRUCTURE is coded as TVUYHZBZN, then how is the word REMEDY coded in that language?
(A) SGPIJE (B) SGPEJD
(C) SGPIHE (D) SGPIIE

5. In a certain code language, if the word SEARCH is coded as IDSBFT, then how is the word FURNISH coded in that language?
(A) ITKNSVG (B) ITJORWG
(C) ITJOSVG (D) ITHNRVG

6. In a certain code language 'two' is called 'three', 'three' is called 'four', 'four' is called 'one', 'one' is called 'five', 'five', is called 'six' and 'six' is called 'nine', then what in the code language is the sum of one and three?
(A) six (B) two (C) nine (D) one

7. In a certain code language if 'pink' means 'black', 'black' means 'white', 'white' means 'yellow', 'yellow' means 'orange', 'orange' means 'red' and 'red' means 'green', then which colour stands for peace in that code?
(A) Red (B) Black
(C) Orange (D) Green

8. In a certain code language, if MENTION = 49 and NEUROTIC = 64, then MARVELLOUS = ?
(A) 81 (B) 88 (C) 64 (D) 100

9. In a certain code language, if CABINET = 70 and BEAUTY = 60, then PRODUCTION = ?
(A) 90 (B) 100 (C) 110 (D) 120

10. In a certain code language, if IMPEND = 61 and DISH = 40, then FRUIT = ?
(A) 86 (B) 68 (C) 74 (D) 76

11. In a certain code language, if BUG = 90 and ALMS = 180, then CADET = ?
(A) 153 (B) 165 (C) 175 (D) 148

12. In a certain code language, if INFER = 25 and JERSEY = 28, then CHOICE =?
(A) 34 (B) 39 (C) 41 (D) 47

Directions for questions 13 to 15: These questions are based on the following data.

In a certain code language, if the word ROUTINE is coded as JMPRRLJ and the word FIDELITY is coded as LGHCXGNW, then how will you code the following words in that language?

13. PREVAIL
(A) FPLRDGX (B) FPJTBGX
(C) FTJBNKX (D) FPJVBIX

14. LANGUAGE
(A) XYBDPXNC (B) XYBDPXMC
(C) XYCEPXNC (D) XYBEPYNC

15. TOBACCO
(A) NMDXEAF (B) NMDYEBF
(C) NMCYFBD (D) NMDYFAD

Directions for questions 16 to 20: For the following groups of letters given in column I, the codes are given in column II. Answer the following questions by finding the codes for the groups from the given columns.

	Column I	Column II
(A)	lit kit bit dit	b r p d
(B)	fit git mit kit	t d s v
(C)	rit bit git tit	x p v w
(D)	nit dit fit rit	r s x j

16. What is the code for lit?
(A) v (B) r (C) p (D) b

17. What is the code for tit?
(A) w (B) x (C) p (D) v

18. What is the code for rit?
(A) j (B) s (C) r (D) x

19. What is the code for nit?
(A) x (B) s (C) j (D) r

20. What is the code for kit?
(A) r (B) p (C) x (D) d

Directions for questions 21 to 25: For the words given in column I, the codes are given in column II. Answer the following questions by finding the codes for the letters from the words and their codes given in the columns.

	Column I	Column II
(A)	PRETEND	4396408
(B)	COMMON	615715
(C)	HOUSE	4*2&1
(D)	SUPPORT	3*21839
(E)	DRUM	5*08

21. What is the code for the word PROTECT?
(A) 3895479 (B) 3846978
(C) 3819479 (D) 3814978

22. What is the code for the word HORMONE?
(A) &385364 (B) &176561
(C) &175184 (D) &185164

23. What is the code for the word EMPEROR?
(A) 5495717 (B) 4534818
(C) 3453919 (D) 4537178

24. What is the code for the word DETHRONE?
(A) 049&7264
(B) 049&8164
(C) 059&7164
(D) 059&8164

25. What is the code for the word COMPOUND?
(A) 71531*60 (B) 72532*80
(C) 91531*70 (D) 72542*60

Answer Keys

Exercise

Practice Problems

1. D	**2.** A	**3.** B	**4.** D	**5.** C	**6.** D	**7.** B	**8.** D	**9.** B	**10.** C
11. B	**12.** A	**13.** B	**14.** D	**15.** D	**16.** D	**17.** A	**18.** D	**19.** C	**20.** D
21. C	**22.** D	**23.** B	**24.** B	**25.** A					

Chapter 5

Blood Relations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

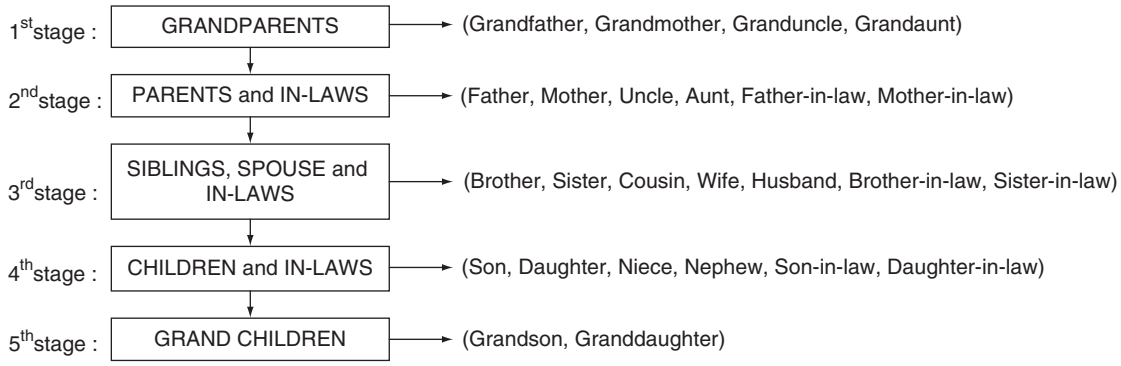
- Blood relations
- Grandparents
- Parents and in-laws
- Siblings, spouse and in-laws
- Children and in-laws
- Grand children

Blood Relations

There are two types of questions based on blood relations that are given in different competitive examinations. For the sake of convenience we will refer to the two types of questions as Type I and Type II. (Please note that the questions on blood relationships are not categorised as above in the actual exam papers. It is being done purely from the point of better understanding).

In the exams, the success of a candidate in the questions on blood relations depends upon his knowledge about various blood relations. Some of the relationships given below help in solving the problems.

The easiest and non-confusing way to solve these types of problems would be to draw a family tree diagram and increase the levels in the hierarchy as shown below:



Mother's or Father's son	:	Brother
Mother's or Father's daughter	:	Sister
Mother's or Father's brother	:	Uncle
Mother's or Father's sister	:	Aunt
Mother's or Father's mother	:	Grandmother
Mother's or Father's father	:	Grandfather
Grandmother's brother	:	Granduncle
Grandmother's sister	:	Grandaunt
Grandfather's brother	:	Granduncle
Grandfather's sister	:	Grandaunt

Sister's or Brother's son	:	Nephew
Sister's or Brother's daughter	:	Niece
Uncle or Aunt's son or daughter	:	Cousin
Son's wife	:	Daughter-in-law
Daughter's husband	:	Son-in-law
Husband's or Wife's sister	:	Sister-in-law
Husband's or Wife's brother	:	Brother-in-law
Sister's husband	:	Brother-in-law
Brother's wife	:	Sister-in-law
Children of same parents	:	Siblings (could be all brothers, all sisters or some brothers and some sisters)
Children	:	Son, Daughter
Children's Children	:	Grandchildren (Grandson, Granddaughter)

In addition, remember the word spouse which means either husband or wife.

Grandfather and grandmother will come in the first stage; mother, father, uncle and aunt will come in the second stage; sister, brother and cousin will come at the third stage; son, daughter, niece and nephew will come in the fourth stage and finally, granddaughters and grandsons will come. The above stages are made from the point of view of an individual.

In Type-I questions, the relationship between two people is given through a roundabout way of relating them through other people. We have to go through the series of relationships and finally determine the relationship between the two people given in the question. The relationship can be given as a simple statement or as a statement made by a person. In the first example given below, a person is involved in making a statement whereas in the second question, there is no person involved in making a statement.

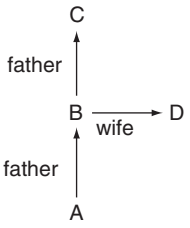
Solved Examples

Example 1: A's father's mother-in-law's only daughter's son is B. How is A related to B?
(A) Brother
(B) Sister
(C) Nephew
(D) Cannot be determined

Solution: (D)
A's father's mother-in-law's only daughter is A's mother. A's mother's son is A's brother.
But A can be either brother or sister to B.

Example 2: If A's father is B, C is the father of B and D is A's mother, then How is C related to D?
(A) Father (B) Grandfather
(C) Father-in-law (D) Uncle

Solution: (C)
A's father is B and mother is D. Therefore D is B's wife and C is the father of B. Hence C is D's father-in-law.



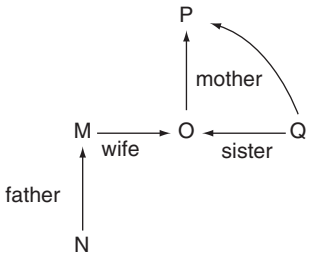
Example 3: A + B means A is the son of B.
A – B means A is the daughter of B.
A × B means A is the father of B.
A ÷ B means A is the mother of B.
If M × N + O – P ÷ Q, then how is M related to Q?
(A) Husband (B) Cousin
(C) Brother-in-law (D) Uncle

Solution: (C)
M × N + O – P ÷ Q means M is the father of N, N is the son of O, O is the daughter of P, P is the mother of Q.

M is the father of N and N is the son of O means M is the husband of O.

O is the daughter of P and P is the mother of Q means O is the sister of Q.

M is the husband of O and O is the sister of Q means M is the brother-in-law of Q.



Example 4: A + B means A is the son of B.
A – B means A is the daughter of B.
A × B means A is the father of B.
A ÷ B means A is the mother of B.
Which of the following means S is the son-in-law of P?
(A) P + Q ÷ R × S – T
(B) P × Q ÷ R – S + T
(C) P + Q × R – S ÷ T
(D) P × Q – R ÷ S × T

Solution: (B)
P + Q ÷ R × S – T means P is the son of Q. Q is the mother of R, R is the father of S and S is the daughter of T. Hence S is the nephew of P.

P × Q ÷ R – S + T means P is the father of Q, Q is the mother of R, R is the daughter of S and S is the son of T. Hence S is the son-in-law of P.

P + Q × R – S ÷ T means P is the son of Q, Q is the father of R, R is the daughter of S and S is the mother of T. Hence S is the mother of P.

P × Q – R ÷ S × T means P is the father of Q, Q is the daughter of R, R is the mother of S and S is the father of T. Hence S is the son of P.

Example 5: Pointing to a person, Raju said, 'He is the only brother of my father's mother's daughter.' How is the person related to Raju?
(A) Brother
(B) Father
(C) Uncle
(D) Nephew

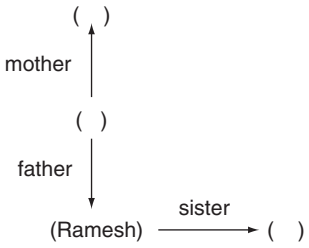
Solution: (B)
Raju's father's mother's daughter is Raju's father's sister. Raju's father's sister's only brother is Raju's father. Hence the person is Raju's father.

Example 6: A's mother's father is the husband of B's mother. How is A related to B, if A and B are both male.
(A) Uncle (B) Father
(C) Nephew (D) Son

Solution: (C)
A's mother's father is the husband of B's mother. That means A's mother is the sister of B. Hence all the nephew of B.

Example 7: Pointing to a photograph Ramesh said, 'she is the sister of my father's mother's only child's son.' How is the person in the photograph related to Ramesh?
(A) Sister (B) Aunt
(C) Mother (D) Cousin

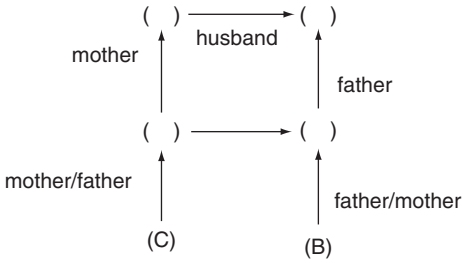
Solution: (A)
My father's mother's only child is my father. My father's son's sister is in the photograph. Hence she is Ramesh's sister.



Example 8: B's father's father is the husband of C's mother's mother. How is B related to C?
(A) Brother
(B) Sister
(C) Cousin
(D) Cannot be determined

Solution: (D)
B's father's father is B's grandfather.

B's grandfather is the husband of C's mother's mother i.e., grandmother. It is possible that B and C are sibling's and the persons mentioned are their paternal/maternal grand parents. It is also possible that B and C are cousins. Hence the relationship cannot be determined.



Directions for questions 9 and 10:
A × B means A is the daughter of B
A * B means A is the son of B
A + B means A is the mother of B
A – B means A is the brother of B
A ÷ B means A is the sister of B
A = B means A is the father of B

Example 9: Which of the following means S is the nephew of P?
(A) P – Q ÷ R = S
(B) P – Q × R * S
(C) S * R – Q ÷ P
(D) P + Q – R × S

Solution: (C)
P – Q ÷ R = S means P is the brother of Q, Q is the sister of R and R is the father of S. As we do not know whether S is the son or daughter of R, we cannot determine that S is the nephew.

P – Q × R * S means P is the brother of Q, Q is the daughter of R and R is the son of S. Here S is the grandfather of P.

S * R – Q ÷ P means S is the son of R. R is the brother of Q and Q is the sister of P. Hence, S is the nephew of P.

Example 10: Which of the following means T is the husband of V?
(A) T = S – R * V
(B) T + S – R × V
(C) R × V – T * S
(D) R + S – T × V

Solution: (A)
T = S – R * V means T is the father of S, S is the brother of R and R is the son of V. Therefore, T is the husband of V.

EXERCISE

Practice Problems

***Directions for questions 1 to 15*:** Select the correct alternative from the given choices.

1. A person who is the husband of my son's sister is my
(A) Nephew (B) Son-in-law
(C) Son (D) Brother

2. Y is the daughter of X's brother's wife's father-in-law. Y is X's _____.
(A) Niece (B) Daughter
(C) Sister (D) Sister-in-law

3. Showing a photograph P said,' She is my mother's mother's son's daughter'. How is the person in the photograph related to P?
(A) Sister (B) Cousin
(C) Niece (D) Mother

4. How is my father's mother's only daughter-in-law's sister related to me?
(A) Aunt (B) Sister
(C) Cousin (D) Niece

5. How is my grandmother's only child's husband's mother related to me?
(A) Mother (B) Grandmother
(C) Aunt (D) Sister

6. How is Ramu's mother-in-law's only daughter's son related to Ramu?
(A) Nephew (B) Brother
(C) Son (D) Uncle

7. How is my son's mother's daughter related to me?
(A) Niece (B) Granddaughter
(C) Daughter (D) Aunt

8. How is my father's brother's only sibling's mother related to me?
(A) Mother (B) Cousin
(C) Daughter (D) Grandmother

9. A is the father of B. C is the son of D. E is the brother of C while D is the sister of B. How is B related to E?
(A) Uncle (B) Aunt
(C) Mother (D) Either (A) or (B)

10. My mother's sister's son's father's mother-in-law is related to me as
(A) Mother (B) Grandmother
(C) Mother-in-law (D) Aunt

11. How is David's father's only daughter-in-law's son's wife related to David?
(A) Daughter (B) Daughter-in-law
(C) Niece (D) Granddaughter

12. How is Ravi's mother's father's son related to Ravi's father?
(A) Cousin (B) Uncle
(C) Brother-in-law (D) Son-in-law

13. Divya's father, pointing towards a person, said, 'He is the brother of my father's only sibling'. How is the person related to Divya?
(A) Father (B) Uncle
(C) Brother (D) Grandfather

14. Tinku, introducing a person to Rinku, said 'He is the father of your sister's son and he is also my mother's husband'. How is Tinku's father related to Rinku's mother?
(A) Nephew (B) Uncle
(C) Son-in-law (D) Father

15. A is B's father, B is C's daughter, E is D's only sibling. C is D's only daughter. How is B related to E's niece?
(A) Niece
(B) Granddaughter
(C) Daughter
(D) Mother

Directions for questions 16 to 20: Use the relations defined below and answer the following questions.
A + B means A is the mother of B.
A – B means A is the sister of B.
A × B means A is the father of B.
A ÷ B means A is the son of B.
A = B means A is the brother of B.
A ≠ B means A is the daughter of B.

16. Which of the following means P is the aunt of Q?
(A) P – R ÷ Q (B) P + R × Q
(C) P ≠ R × Q (D) P – R + Q

17. Which of the following means, S is the son of T's daughter?
(A) T × M + S + N (B) T × M + S = N
(C) T + M × S – N (D) S ÷ M ÷ T – N

18. Which of the following means W is the uncle of Z?
(A) W × A – B + Z (B) W = A + B – Z
(C) W = A + B + Z (D) W × A × B = Z

19. Which of the following means C is the grandfather of both D and E?
(A) C × A ÷ D – E (B) C + A + D ≠ E
(C) C ÷ A ≠ D = E (D) C × A × D – E

20. Which of the following means I is the mother of L?
(A) I + B – C ≠ D × L (B) I ≠ B + C × L
(C) I + B × C ≠ D – L (D) I + B – C × L

Directions for questions 21 to 25: These questions are based on the information given below.

A, B, C, D, E and F are six members of a family. A is the mother of B, who is the husband of D. F is the brother of one of the parents of C. D is the daughter-in-law of E and has no siblings. C is the son of D.

21. How is C related to A?
(A) Nephew (B) Son-in-law
(C) Grandson (D) Father

22. How is F related to D?
(A) Cousin (B) Brother-in-law
(C) Brother (D) Father

23. How is E related to F?
(A) Mother
(B) Son
(C) Father
(D) Father-in-law

24. If F is married to G, then how is G related to B?
(A) Sister (B) Sister-in-law
(C) Cousin (D) Mother

25. How many male members are there in the family?
(A) Two
(B) Three
(C) Four
(D) Cannot be determined

Answer Keys

Exercise

Practice Problems

1. B	**2.** C	**3.** B	**4.** A	**5.** B	**6.** C	**7.** B	**8.** D	**9.** D	**10.** B
11. B	**12.** C	**13.** D	**14.** C	**15.** C	**16.** D	**17.** B	**18.** B	**19.** D	**20.** A
21. C	**22.** B	**23.** C	**24.** B	**25.** C					

Chapter 6

Venn Diagrams

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

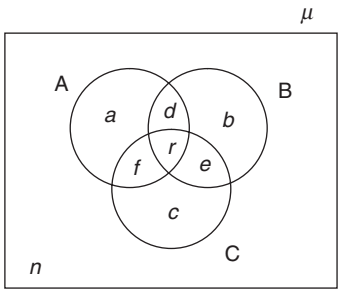
- Venn diagrams
- Venn diagram Type I
- Venn diagram Type II

Venn Diagrams

Venn Diagrams are diagrammatic representation of sets, using geometrical figures like circles, triangles, rectangles, etc. Each geometrical figure represents a group as shown in the examples. The area common to two or more figures represent those elements which are common to two or more groups. There are various models in Venn Diagrams which we will discuss with examples.

Venn Diagram Type I

In this type, two, three or four different groups could be given with some elements common to two or more groups. Let us observe the diagram given below.



Here, A, B and C are three different groups, and the various regions can be explained as given below.

Only A = a
Only B = b
Only C = c
A and B only = d
B and C only = e
C and A only = f
All the three (A, B and C) = r
Both A and B = $d + r$
Both B and C = $e + r$
Both C and D = $f + r$
Neither A, nor B, nor C = n
A, B or C and none = μ
Also, $\mu = (A \cup B \cup C) + n$

Here, the rectangle represents the sample space, which consists of three groups A, B and C, and also n, which is the number of people belonging to neither A, nor B, nor C.

Some more formulae are as given under:

1. $A' = (b + e + c) + n$; where A′ = A complement (not in A)
2. $B' = (a + f + c) + n$; where B′ = B complement (not in B)
3. $C' = (a + d + b) + n$; where C′ = C complement (not in C)
4. $A - B = A - (A \cap B)$
5. $A \Delta B = (A - B) \cup (B - A)$
6. Number of people (or things) belonging to at least one out of the three groups = $A \cup B \cup C$
   $= \underset{\text{exactly one}}{(a + b + c)} + \underset{\text{exactly two}}{(d + e + f)} + \underset{\text{exactly three}}{r}$
7. $A + B + C = (A \cup B \cup C) + (d + e + f) + 2r$
   $= (a + b + c) + 2(d + e + f) + 3r$

Solved Examples

Directions for questions 1 to 3: These questions are based on the data given below.

In a class of 165 students, 45 students are passed in Maths a well as in English, whereas 60 students are failed in Maths and 65 students are failed in English.

Example 1: How many students are passed in exactly one subject?

(A) 160 (B) 100
(C) 115 (D) 165

Example 2: How many students are failed in both the subjects?

(A) 25 (B) 20 (C) 45 (D) 5

Example 3: How many students are failed only in Maths?

(A) 55 (B) 60 (C) 65 (D) 70

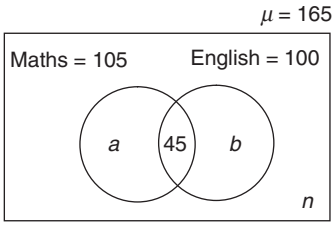
Solutions for questions 1 to 3:

Since, 60 students are failed in Maths.

$\therefore$ 165 – 60 = 105 students passed in Maths similarly, 65 students failed in English.

$\therefore$ 165 – 65 = 100 students passed in English.

The respective Venn – diagrams is as follows which shows the number of students who passed the subject.



45 students passed in Maths as well as English.

$\therefore$ a = 105 – 45 = 60 students passed only in Maths and b = 100 – 45 = 55 students passed only English.

Number of students passed in atleast one subject = 60 + 55 + 45 = 160

Hence, n = 165 – 160 = 5 students failed in both.

Solution 1: $a + b$ = 60 + 55 = 115 students passed exactly in one subject.

Hence, the correct option is (C).

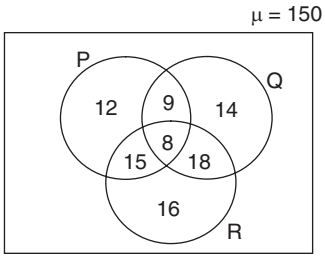
Solution 2: 5 students failed in both the subjects.

Hence, the correct option is (D).

Solution 3: As 55 students passed only in English which implies that 55 students failed only in Maths.

Hence, the correct option is (A).

Directions for questions 4 to 7: These questions are based on the given diagram.



Example 4: How many elements are there in Q′ (complement of Q)?

(A) 100 (B) 49 (C) 101 (D) 50

Example 5: How many elements are there in P′ ∩ Q′ ∩ R′?

(A) 35 (B) 8 (C) 58 (D) 48

Example 6: How many elements are there in R?

(A) 16 (B) 57 (C) 41 (D) 8

Example 7: How many elements are there in P ∩ (Q ∪ R)?

(A) 32 (B) 48 (C) 54 (D) 44

Solutions for questions 4 to 7:

Solution 4: Number of elements in Q′ = (μ) – (number of elements in Q)

= 150 – (14 + 18 + 8 + 9)

$\Rightarrow$ 150 – 49 = 101.

Hence, the correct option is (C).

Solution 5: Number of elements in P′ ∩ Q′ ∩ R′

$= \mu - (P \cup Q \cup R)$

= 150 – (12 + 15 + 9 + 8 + 18 + 16 + 14)

= 150 – 92 = 58.

Hence, the correct option is (C).

Solution 6: Number of elements in R = 16 + 15 + 8 + 18 = 57

Hence, the correct option is (B).

Solution 7: Number of elements in Q ∪ R =

(14 + 9 + 8 + 15 + 18 + 16)

Number of elements in P = (12 + 9 + 15 + 8)

P ∩ (Q ∪ R) is the region common to P and Q ∪ R

Number of elements in P ∩ (Q ∪ R)

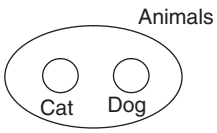
= 9 + 8 +15 = 32.

Hence, the correct option is (A).

Venn Diagram Type II

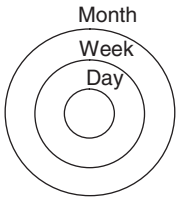
In this type, Venn diagrams are used to establish relationship between the given groups. In other words, two or more groups are given and the Venn diagram, which most correctly establishes a relation between them, has to be chosen out of the various Venn diagrams given in the choices. Let us look at some of the examples given below.

Example: Animals, Cat, Dog



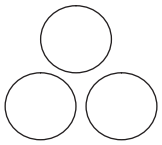
Here, in animals we have many species of which cat and dog are two different kinds of species, having nothing in common. So the above diagram is the most appropriate representation of the given groups.

Example: Month, week, day



We know that day is a part of the week and week is a part of the month. So the above diagram is the most appropriate representation of the given groups.

Example: Mars, Earth, Jupiter

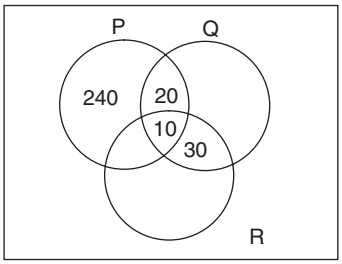


We know that Mars, Earth and Jupiter are three independent entities having nothing in common. So the above diagram is the most appropriate representation of the given groups.

EXERCISE

Practice Problems

Directions for questions 1 to 5: There are 1500 students in a college. Each student can be a member of three student communities namely P, Q, and R. Now using the data mentioned and the diagram given below answer the questions that follow.



- Total members in community P is 300.
- Total members in community Q is 420.
- Total members in community R is 490.

1. How many students are part of only community R?
(A) 360 (B) 420 (C) 210 (D) 350

2. How many students is not part of any community?
(A) 390 (B) 420 (C) 410 (D) 490

3. How many students are part of at least two communities?
(A) 10 (B) 30
(C) 80 (D) 90

4. How many students are part of at least one community?
(A) 1000 (B) 1090
(C) 1110 (D) 1100

5. How many students are part of exactly two communities?
(A) 90 (B) 80
(C) 100 (D) 120

Directions for questions 6 to 10: These questions are based on the data given below.

In a class of 95 students, 40 play cricket, 50 play football and 10 play both cricket and football.

6. How many students play only football?
(A) 45 (B) 30 (C) 40 (D) 28

7. How many students play at least one game?
(A) 80 (B) 70 (C) 60 (D) 50

8. How many students play only cricket?
(A) 30 (B) 35 (C) 40 (D) 25

9. How many students play exactly one game?
(A) 85 (B) 80 (C) 70 (D) 75

10. How may students play neither cricket nor football?
(A) 12 (B) 15 (C) 18 (D) 20

Directions for questions 11 to 15: Study the following data and answer the questions given below.

In a certain college, 37% of the students write EAMCET exam, 47% of the students write IIT-JEE exam and 50% of the students write AIEEE exam. Also known that, 11% of the students write both EAMCET and IIT-JEE, 11% of the students write both EAMCET and AIEEE, 15% of the students write both IIT-JEE and AIEEE, while 15 students write all the three exams. Each student in the college writes at least one of the three exams.

11. How many students appear for the exams from the college?
(A) 400 (B) 200
(C) 500 (D) 600

12. How many students write exactly two exams?
(A) 120 (B) 110
(C) 140 (D) 150

13. The number of students who write only EAMCET as a percentage of the number of students who write only AIEEE is

(A) $33\frac{1}{3}\%$ (B) $66\frac{2}{3}\%$

(C) $33\frac{2}{3}\%$ (D) $66\frac{1}{3}\%$

14. How many students write exactly one exam?

(A) 345 (B) 395 (C) 198 (D) 398

15. What is the ratio of the number of students who write only AIEEE to that of those who write only IIT JEE?

(A) 3 : 2 (B) 2 : 3 (C) 8 : 9 (D) 9 : 8

Directions for questions 16 to 20: These questions are based on the data given below.

In a library maintained by a student, there are books on different subjects. It was found that 35 books are on sports, 45 books are on business and 15 books are on current affairs. 14 books are on at least two subjects among sports, business and current affairs. 3 books have sports, business as well as current affairs in them. Every book in the library is assumed to contain at least one of sports, business or current affairs in them.

16. How many books are there which contain information regarding only one subject?

(A) 58 (B) 64 (C) 60 (D) 62

17. What are the total number of books in his library?

(A) 78 (B) 72 (C) 68 (D) 80

18. How many books contained information regarding exactly two subjects?

(A) 11 (B) 10 (C) 9 (D) 14

19. How many books are there which contain information regarding at most two subjects?

(A) 11 (B) 64 (C) 72 (D) 75

20. If the number of books on only sports is equal to 26, then how many books are there in the library, which are on both business and current affairs but not sports?

(A) 5 (B) 3 (C) 2 (D) 8

Directions for questions 21 to 25: These questions are based on the data given below.

In a colony, it is known that three brands of mobile phones are used, namely Nokia, Sony Ericsson and Motorola. 70 families use only one brand, 47 families use exactly two brands and 8 use all the three brands. It is assumed that each family uses at least one of these three brands.

21. How many families are there in the colony?

(A) 75 (B) 100 (C) 105 (D) 125

22. How many families use at least two brands?

(A) 117 (B) 55 (C) 47 (D) 125

23. If 10 families stop using Nokia and start using Motorola, then what is the maximum number of families who use exactly two brands?

(A) 57 (B) 37 (C) 47 (D) 67

24. What is the ratio of the number of families which use exactly one brand to that which use at least one brand?

(A) 14 : 25 (B) 14 : 11

(C) 11 : 25 (D) 11 : 14

25. How many families do not use all the three brands?

(A) 125 (B) 117 (C) 0 (D) 8

Answer Keys

Exercise

Practice Problems

1. B	**2.** A	**3.** D	**4.** C	**5.** B	**6.** C	**7.** A	**8.** A	**9.** C	**10.** B
11. C	**12.** C	**13.** B	**14.** A	**15.** D	**16.** B	**17.** A	**18.** A	**19.** D	**20.** A
21. D	**22.** B	**23.** A	**24.** A	**25.** B					

Chapter 7

Seating Arrangements

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Seating arrangement
- Linear sequencing
- Circular arrangement

Linear Sequencing

Linear sequencing is essentially arranging the items given in a sequence (in a single line). The questions of this type are also referred to as 'Seating Arrangement'. The word 'seating arrangement' should not be misconstrued – it should not be treated as consisting of questions involving only persons sitting as per specified conditions. Essentially, these questions involve arranging subjects (people or things) satisfying the given conditions. The arrangement is done only on one 'axis' and, hence, the position of the subjects assumes importance here in terms of order like first position, second position, etc.

Let us look at the examples:

Directions for questions 1 to 5: Read the data given below carefully and answer the questions that follow.

Seven persons Paul, Queen, Rax, Sam, Tom, Unif and Vali are sitting in a row facing us. Rax and Sam sit next to each other. There must be exactly four persons between Queen and Vali. Sam sits to the immediate right of Queen.

Solved Examples

Example 1: If Paul and Tom are separated exactly by two persons, then who sits to the immediate left of Vali?
(A) Paul (B) Tom (C) Unif (D) Rax

Example 2: If Queen is not sitting at either extreme of the row, then who among the following has as many persons on his left as on his right?
(A) Sam (B) Unif (C) Rax (D) Vali

Example 3: If Queen sits at one extreme, then who is at the other extreme?
(A) Paul (B) Tom
(C) Vali (D) Cannot be determined

Example 4: Tom sits to the right of Queen, and Paul is separated from Tom by exactly three persons. Then, who is sitting to the immediate left of Vali?
(A) Unif (B) Paul (C) Tom (D) Rax

Example 5: In how many different ways can the seven persons sit in a row?
(A) 3 (B) 2 (C) 10 (D) 12

Solutions for questions 1 to 5:

Let us write down the conditions given in short form and then represent them pictorially. Also, let us treat the left of the persons sitting as 'left' and their right as 'right' for interpreting the conditions.

Rax and Sam sit next to each other → RS or SR.

There are exactly 4 persons between Queen and Vali → Q — — — — V or V — — — — Q.

Sam sits to the immediate right of Queen → SQ.

Now let us analyse the data/conditions that we are given and then put the three conditions together. Let us number the seats from *our* left to right as Seat 1 to Seat 7.

Since S is to the right of Q and since R and S have to be next to each other, R can come only to the immediate right of S. Thus, R, S and Q, will be in the order RSQ. Since there are four persons between Q and V, Q can be placed in seats 1, 2, 6 or 7. But if Q is in Seat 1or 2, then there are no seats for R and S. Hence, there are only two seats available for Q. Let us fix the positions of R, S and V in each of these two positions of Q and write them down. The directions Left and Right are as shown below.

◀ R ▶ L

Arrangement I:

1	2	3	4	5	6	7
	V			R	S	Q

Arrangement II:

1	2	3	4	5	6	7
	V			R	S	Q

These are the only two arrangements possible for the four persons V, R, S and Q. The other three persons Paul, Tom and Unif can sit in the three vacant seats in any order, as no information is given about them. Now let us look at each of the questions.

Example 1: Paul and Tom are separated by exactly two persons.

Arrangement I is the only one possible as in Arrangement II, Paul and Tom cannot have exactly two persons between them. So, we have the arrangement as follows:

T/P, V, U, P/T, R, S, Q

So, Unif must be sitting to the immediate left of Vali.

Hence, the correct option is (C).

Example 2: If Queen is not at the extreme right, then only Arrangement II above is possible. The person who has as many persons on his left as on his right can only be the person who is sitting in the middle seat, i.e., seat 4. In this arrangement, Rax is sitting in seat 4.

Hence, the correct option is (C).

Example 3: 'Queen sits at one extreme' means that we should look at arrangement I. In this arrangement, any one out of the three persons Paul, Tom and Unif can be in seat 1, i.e., extreme right.

Hence, the correct option is (D).

Example 4: If Tom and Paul are separated by exactly three persons, then only Arrangement II is possible. So, Tom and Paul have to be in seats 3 and 7, Since, we are also given that Tom is to the right of Queen, Tom has to be in seat 3 and Paul, in seat 7. So, the arrangement must be as follows:

V, U, T, R, S, Q, P

The person sitting to the immediate left of Vali is Unif.

Hence, the correct option is (A).

Example 5: We have two possible arrangements—Arrangement I and Arrangement II that we looked at already. In each arrangement, the remaining three people can sit in the remaining three seats in 6 ways. Thus, a total of 12 ways of seating the seven persons is possible.

Hence, the correct option is (D).

Directions for questions 6 to10: Read the data given below carefully and answer the questions that follow.

Seven boys – Rajan, Shyam, Vardhan, Mithra, Vimal, Raj and Kishan – are sitting in a row. Shyam sits to the immediate left of Vardhan and third to the right of Rajan, whereas Mithra, who sits at the left extreme, is next to Kishan.

Example 6: Who is sitting to the immediate right of Shyam?

(A) Mithra (B) Kishan
(C) Vimal (D) Vardhan

Example 7: If Vardhan and Kishan exchange places with each other without changing the rest of the arrangement that is already done, who will be sitting to the immediate left of Rajan?

(A) Kishan (B) Raj
(C) Vimal (D) Vardhan

Example 8: If only Shyam sits between Raj and Vardhan, who is exactly in the middle of the row?

(A) Raj (B) Vardhan
(C) Vimal (D) Rajan

Example 9: Which of the following cannot confirm the seating arrangement of all the boys?

(A) Raj is to the immediate right of Rajan, whereas Vimal is to the left of Shyam.
(B) Mithra and Raj have two persons between them.
(C) Raj and Kishan have two persons between them.
(D) Rajan and Shyam have two persons in between them.

Example 10: After arranging all the boys as per the conditions given in the data, if Rajan now exchanges his place with Mithra, and Vardhan exchanges his place with Vimal, then how many persons will be there between Vimal and Rajan?

(A) Three (B) Two
(C) Five (D) Cannot be determined

Solutions for question 6 to10:

Let us denote Left and Right as shown below:

◀ L ▶ R

Now, let us represent the data given in pictorial form (We use R for Raj and Rn for Rajan; Va for Vardhan; Vi for Vimal; S for Shyam; M for Mithra and K for Kishan).

Mithra sits at the left extreme – next to Kishan → M K — — — — —.

Shyam sits to the immediate left of Vardhan and third to the right of Rajan → Rn — — S Va.

Putting both the above together, Va can go only to extreme right position. Thus, we have the arrangement as M K Rn — — S Va.

Raj and Vimal occupy the two vacant seats between Rajan and Shyam.

Example 6: From the seating arrangement figure above, Vardhan is to the immediate right of Shyam.

Hence, the correct option is (D).

Example 7: If Kishan and Vardhan exchange places, as can be seen from the arrangement, the person to the immediate left of Rajan will be Vardhan.

Hence, the correct option is (D).

Example 8: If Shyam sits between Raj and Vardhan, then the seating arrangement is as follows: Mithra, Kishan, Rajan, Vimal, Raj, Shyam, Vardhan. Then, Vimal will be exactly in the middle of the row.

Hence, the correct option is (D).

Example 9: Statement (A) makes the arrangement as: Mithra, Kishan, Rajan, Raj, Vimal, Shyam, Vardhan

Statement (B) gives the seating arrangement as: Mithra, Kishan, Rajan, Raj, Vimal, Shyam, Vardhan.

Statement (C) makes the seating arrangement as: Mithra, Kishan, Rajan, Vimal, Raj, Shyam, Vardhan.

So, only statement (D) cannot make the seating arrangement unique while others can.

Hence, the correct option is (D).

Important point to note is that on the basis of the given data, we know that the places of only Raj and Vimal have not been fixed. Hence, if there is an additional statement that we are considering to determine the arrangement uniquely, it SHOULD have at least one of the two people Raj and Vimal. In this case, choice (D) does not have either one of the two names and hence, this statement cannot help us determine the arrangement uniquely. So, this becomes the answer choice.

Hence, the correct option is (D).

Example 10: The arrangement is

M K Rn R/Vi Vi/R S Va

Rajan exchanges his place with Mithra, and Vimal with Vardhan, then we have the following arrangement:

Rn K M R/Va Va/R S Vi.

While we still do not know the exact position of Vardhan (or which place Vimal sits), we can see that there are five persons between Rajan and Vimal.

Hence, the correct option is (C).

In addition to the questions that we saw above, where a set of questions are based on the data given, there are also 'stand-alone' questions. In these questions, on the basis of the data given, only one question is asked. Given below is an example of this type.

Directions for question 11: Select the correct alternative from the given choices.

Example 11: Four persons A, B, C and D arrive to attend a meeting. D arrives 10 minutes after B and twenty minutes before A, who arrives 10 minutes before C. Who is the first person to arrive at the meeting?

(A) A (B) B (C) C (D) D

Solution for question 11:

Example 11: C arrived after A. A arrived after D. D arrived after B. This implies that B arrived first.

Hence, the correct option is (B).

Circular Arrangement

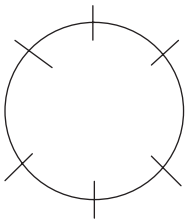
Questions on circular arrangement involve seating of people around a table or arrangement of things in a circular manner (for example, different coloured beads strung to form a necklace).

In case of people sitting around a table, the table could be of any shape – rectangular, square, circular or any other.

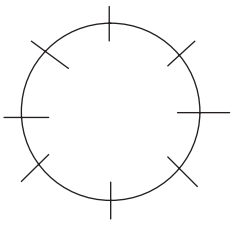
The data given in such sets of questions specify the positions of some or all of the individuals (or things) in the arrangement. The positions are specified through conditions involving specified persons sitting (or not sitting) opposite each other or a particular person sitting to the right or left of another person, etc.

Once you read the data, first draw the shape specified in the data and then draw the slots in the seating arrangement.

Six people around a circular table



Eight people around a circular table



Statements like 'A and B are sitting farthest from each other' or 'A and B sit across the table' imply that A and B sit opposite each other.

On the other hand, you should remember that, unlike in straight-line arrangement, the words 'immediately' and 'directly' do not play any role in circular arrangement. In general, there is no left side or right side (unless we are talking of 'immediate right' or 'immediate left').

So if it is given that C sits to right of B, then it is clear that C must be to the immediate right of B. Go 'anti-clockwise' if anybody's right has to be located, and go 'clockwise' if somebody's left has to be located.

Let us take some examples.

Directions for questions 12 to 16: Read the following information and answer the questions that follow.

P, Q, R, S and T sit around a table.

P sits two seats to the left of R and Q sits two seats to the right of R.

12. If S sits in between Q and R, who sits to the immediate right of P?
(A) T (B) S (C) Q (D) R

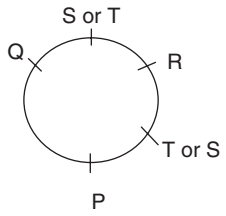
13. Which of the following cannot be the correct seating arrangement of the five persons in either the clockwise direction or the anti-clockwise direction?
(A) P, Q, R, S, T (B) P, S, R, T, Q
(C) P, Q, S, R, T (D) P, T, R, S, Q

14. If S is not sitting next to Q, who is sitting between Q and S?
(A) R (B) P
(C) T (D) Both (R) and (P)

15. If a new person U joins the group such that the initial conditions for the seating arrangement should be observed and also a new condition that U does not sit next to R be satisfied, then which of the following statements is true?
(A) U sits to the immediate right of S.
(B) U sits to the immediate left of T.
(C) U sits to the immediate left of P.
(D) Either (A) or (B) above

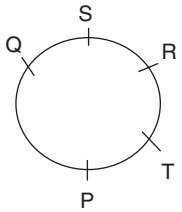
16. If a new person U joins the group such that the initial conditions for the seating arrangement should be observed and also a new condition that U does not sit next to P, S or T be satisfied, then who will be the neighbours of P (one on either side)?
(A) S and T (B) S and Q
(C) T and R (D) R and Q

Solutions for questions 12 to 16:



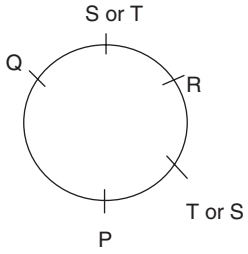
P sits two seats to the left of R, and Q sits two seats to the right of R. We can represent this information in the diagram below.

12. If S sits between Q and R, then the arrangement is as follows.



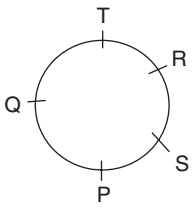
As can be seen from the diagram, T is to the immediate right of P.
Hence, the correct option is (A).

13. We will take each choice and see whether it fits in the arrangement that we represented through a diagram in the analysis of the data (the same diagram is reproduced below).



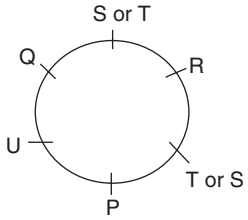
We can see that the arrangement given in choice (A) is not possible and hence the answer choice is (A).
Hence, the correct option is (A).

14. If S is not next to Q, then the seating arrangement is fixed as follows.



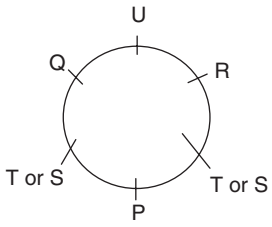
Now P is between Q and S.
Hence, the correct option is (B).

15. On the basis of the diagram that we drew, we find that to accommodate U we have to create a new slot between P and Q.



Hence, choice (C) is the correct answer.

16. We create a new slot for the sixth person. But since U will not sit next to P, S or T, he will have to sit between R and Q. The arrangement will then look as follows:



As we can see from the diagram, the neighbours of P will be T and S.
Hence, the correct option is (A).

Directions for questions 17 to 21: Read the following information and answer the questions that follow.

There are 10 persons at a round table conference, consisting of a Professor, a Lawyer, a Doctor, a Scientist, an Accountant, a Grocer, two Computer Specialists and two Marketing Executives. The Professor sits opposite to the Lawyer. The Scientist and the Doctor sit opposite each other. The two Marketing Executives sit opposite each other with one of them sitting to the immediate left of the scientist. The Professor sits to the immediate right of the Scientist.

17. If the two Computer Specialists sit opposite each other but neither of them is immediately next to any Marketing Executive, who sits to the immediate right of the professor?
(A) Computer Specialist
(B) Marketing Executive
(C) Grocer
(D) Accountant

18. If the Grocer and Accountant do not sit opposite each other, then which of the following must be TRUE?
(A) The Computer Specialist cannot sit beside the Lawyer.
(B) One of the Computer Specialists is next to a Marketing Executive.
(C) The Professor cannot have the Scientist and a Computer Specialist on his either side.
(D) The Computer Specialists must sit next to one another.

19. If the Grocer and the Accountant each have a Marketing Executive as his immediate neighbour, then which of the following is definitely FALSE?
(A) The two Computer Specialists are opposite each other.
(B) A Computer Specialist is an immediate neighbour of the Scientist.
(C) The Grocer is next to a Computer Specialist.
(D) A Computer Specialist is an immediate neighbour of the Lawyer.

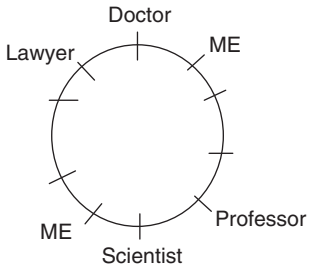
20. If a Computer Specialist is the immediate neighbour of a Marketing Executive and the Grocer is the immediate neighbour of the Lawyer, how many different kinds of seating arrangements are possible? (Assume that the two Computer Specialists are indistinguishable from each other and the two Marketing Executives are indistinguishable from each other.)
(A) 3 (B) 6 (C) 16 (D) 8

21. The maximum number of persons you can count if you start counting with the Scientist and end with a Marketing Executive (excluding both) is
(A) 0 (B) 8 (C) 5 (D) 6

Solutions for questions 17 to 21:

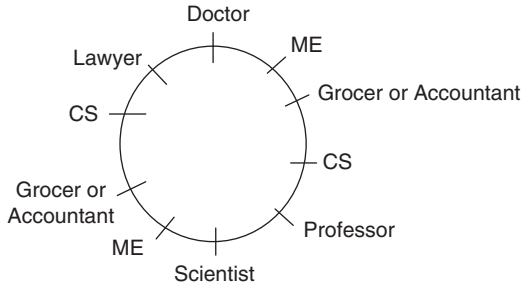
The Professor sits to the immediate right of the Scientist and opposite to the Lawyer. The Scientist sits opposite to the Doctor and one Marketing Executive is to the immediate left of the Scientist.

Choosing to place the Scientist in one of the 10 seats, we have the arrangement as follows.



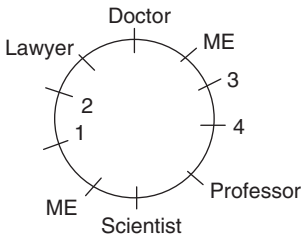
The vacant seats are one each for the two Computer Specialists, one for the Grocer and one for the Accountant.

17. The two Computer Specialists sit opposite each other. Neither of them is next to any Marketing Executive. So, the arrangement must be as follows



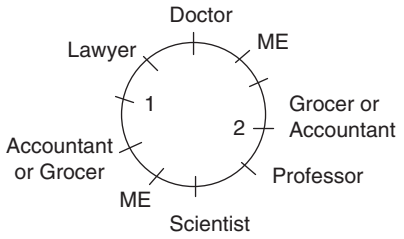
So, a Computer Specialist sits to the immediate right of the professor.
Hence, the correct option is (A).

18. The Grocer and the Accountant do not sit opposite each other. Then the arrangements can be as follows:



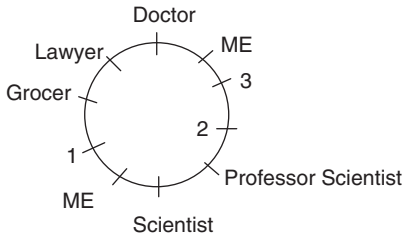
The Grocer and the Accountant can occupy the following pairs of seats: 3 and 4, 1 and 4, 1 and 2 or 2 and 3. Then, the two computer specialists may occupy one of the pairs of seats 1 and 2, 2 and 3, 3 and 4 or 4 and 1. We check for the choices given in the question, one by one, and find that whichever combination is taken, there is a Computer Specialist in seat 1 or seat 3, both of, which are next to the Marketing Executives seats. So, choice (B), which states that one of the Computer Specialists is next to a Marketing Executive, is true.
Hence, the correct option is (B).

19. The Grocer and the Accountant have one each of the Marketing Executives on their immediate side. So, the arrangement must be as follows.



Now the Computer Specialists must sit in chairs 1 and 2 only. But no Computer Specialist can be the immediate neighbour of the Scientist. Choice (B) is definitely FALSE. (Note that choices (A), (C) and (D) are true). Hence, choice (B) is the correct answer.
Hence, the correct option is (B).

20. Given that the Grocer is the immediate neighbour of the Lawyer, we have the three slots numbered 1, 2 and 3 (in the following diagram) free for the two Computer Specialists and the Accountant. Since a Computer Specialist has to be next to a Marketing Executive, he should be in slot 1 or 3. By fixing the Accountant in any one of the three slots 1, 2 or 3, we can ensure that there is a Computer Specialist next to a Marketing Executive. Hence, there are three possible seating arrangements.



Hence, the correct option is (A).

21. Based on the seating arrangement that we discussed, the number of persons between the Scientist and a Marketing Executive can be 3 or 8 (counted clockwise) or 0 or 5 (counted anti-clockwise). Maximum number that can be counted is 8.
Hence, the correct option is (B).

EXERCISE

Practice Problems

Directions for questions 1 to 3: These questions are based on the following information.

Five boys – Anil, Charan, David, John and Kamal sit in a row facing north, not necessarily in the same order.

I. John sits exactly in between Anil and David.
II. John sits exactly in between Charan and Kamal.

1. Who sits exactly at the middle of the row?
(A) John
(B) Kamal
(C) David
(D) Cannot be determined

2. In how many different ways these five boys can sit?
(A) 2 (B) 4 (C) 8 (D) 16

3. If Anil sits to the immediate left of John and if a boy sits to the immediate right of Kamal then who is that boy?
(A) David
(B) Anil
(C) Charan
(D) None of these

Directions for questions 4 to 6: These questions are based on the following information.

Seven girls – A, B, C, D, E, F and G sit in a row facing north, not necessarily in the same order. It is also known that,

I. Two girls sit in between B and F.
II. Three girls sit in between C and G.
III. Four girls sit in between A and D.

4. Who sits exactly at the middle of the row?
(A) B
(B) D
(C) E
(D) Cannot be determined

5. If B sits to the immediate right of D then who sits in between A and E?
(A) F
(B) C
(C) G
(D) Cannot be determined

6. If F and G sit on either sides of E then who sits at the right end of the row?
(A) A
(B) C
(C) D
(D) Cannot be determined

Directions for questions 7 to 9: These questions are based on the following information.

Five persons P, Q, R, S and T sit in a row facing North not necessarily in the same order. The following information is known about them:

I. Either P or S sits at the one end of the row.
II. Either Q and T or S and T sit on either sides of P.
III. R sits to the left of S and to the immediate left of Q.

7. In how many different ways can these five people sit?
(A) 2 (B) 3
(C) 1 (D) 4

8. If Q sits to the immediate left T then who sits exactly at the middle of the row?
(A) P (B) R
(C) T (D) Cannot be determined

9. If P is not sitting adjacent to S, then who sits to the immediate right of Q?
(A) Q (B) P
(C) R (D) Cannot be determined

Directions for questions 10 to 12: These questions are based on the following information.

Each of six persons – Pavan, Raman, Kiran, Charan, Shravan and Rajan stay in a different floor of a six storied (1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, and 6^{th} from bottom to top respectively) building.

I. Raman stays above Kiran but below Charan.
II. Pavan stays below Rajan but above Shravan.
III. Kiran stays above Pavan but below Raman who stays above Rajan.

10. Who stays in the 2^{nd} floor?
(A) Pavan
(B) Shravan
(C) Rajan
(D) Cannot be determined

11. Who stays in the 4^{th} floor?
(A) Raman
(B) Rajan
(C) Kiran
(D) Cannot be determined

12. If one person stays in between Pavan and Kiran then who stays in the 3^{rd} floor?
(A) Shravan (B) Pavan
(C) Rajan (D) Charan

Directions for questions 13 to 15: These questions are based on the following information.

There are five buildings of different heights in a row. These houses are painted with a different colour among – Red, Blue, White, Green and Yellow such that each house is painted with exactly one colour.

The following information is know about them:

I. Yellow and green coloured buildings are on either sides of the white coloured building.
II. The shortest building is painted in red colour but it is neither at any end of the row nor adjacent to the tallest building.
III. The white coloured building is exactly in between the tallest and the second tallest buildings.

13. Which among the following buildings is definitely at one end of the row?
(A) Yellow coloured building
(B) Green coloured building
(C) The tallest building
(D) The third tallest building

14. Which among the following is definitely false?
(A) The White coloured building is the third tallest
(B) The third tallest and the shortest buildings are together
(C) Blue and Yellow and coloured buildings are at either ends of the row.
(D) Yellow and Green coloured buildings are at either ends of the row.

15. If the yellow coloured building is to the immediate left of the third tallest building then what could be the order of these buildings in the descending order of their heights?
(A) Blue, Yellow, Red, White, Green
(B) Blue, Green, White, Yellow, Red
(C) Green, White, Blue, Yellow, Red
(D) Green, Yellow, White, Blue, Red

Directions for questions 16 to 18: These questions are based on the following information given below.

Each of the six persons – John, Ted, Humpty, Dumpty, Jack and Jill is from one different country among India, Japan, China, Australia, America and England and are sitting around a circular table, may not be in the same order. John, who is from China is sitting adjacent to American, who is not Humpty. Ted is not an Indian and Chinese is not sitting adjacent to Indian. The person from England is sitting one place away to the left of the Australian. Humpty is sitting opposite the Indian, who is adjacent to the Japanese. Australian and Dumpty are sitting opposite each other. Jack is not from India and Ted is not from Japan but both are not adjacent to each other.

16. Who among them is from India?
(A) Jill (B) Dumpty
(C) Humpty (D) None of these

17. If Jack is the Japanese, then who is sitting opposite the American?
(A) Jill (B) Ted
(C) Jack (D) Dumpty

18. Which country does Humpty belong to?
(A) Japan (B) Australia
(C) America (D) England

Directions for questions 19 to 22: These questions are based on the following information given below.

Eight persons – Ram, Ramesh, Mohan, Sohan, Seema, Saroj, Sakshi and Saloni are sitting around a circular table. Each of them is one among Doctor, Engineer, Dancer, Singer, Teacher, Lawyer, Accountant and Pilot, not necessarily in the given order. Further it is known that

I. Pilot is sitting opposite Ramesh, who is adjacent to the Accountant.
II. Dancer is sitting opposite the Lawyer and is not adjacent to Sakshi who is not sitting adjacent to the Lawyer.
III. Saloni is sitting opposite the Engineer, Ramesh is not a Lawyer or Doctor or Engineer.

IV. Sakshi, the Singer, is sitting one place away to the right of Saroj.
V. Seema is sitting opposite the Lawyer and Ram is sitting opposite the Dancer.
VI. Ramesh is sitting three places to the right of Singer. Mohan is neither the Accountant nor adjacent to the Dancer.

19. Who among the following is the Doctor?
(A) Ramesh
(B) Saloni
(C) Saroj
(D) Cannot be determined

20. What is the profession of Mohan?
(A) Accountant
(B) Pilot
(C) Engineer
(D) Cannot be determined

21. Who is sitting opposite Ramesh?
(A) Seema (B) Sakshi
(C) Saroj (D) None of these

22. Who is sitting opposite the Accountant?
(A) Sakshi (B) Mohan
(C) Seema (D) Saroj

Directions for questions 23 to 25: These questions are based on the following information given below.

Eight persons – Arun, Pankaj, Rohan, Veda, Suman, Shanu, Dimple and Pinky are sitting around a circular table for a group discussion. Suman is not sitting opposite Pinky and Shanu is sitting three places away to the right of Pankaj. Dimple is sitting in between Pankaj and Suman. Rohan is sitting adjacent to Pankaj who is sitting opposite Arun.

23. Who is sitting opposite Dimple?
(A) Piniky
(B) Shanu
(C) Rohan
(D) Cannot be determined

24. Who is sitting opposite Veda?
(A) Suman
(B) Pinky
(C) Shanu
(D) Cannot be determined

25. If Rohan is sitting to the left of Veda, then who is sitting opposite Shanu?
(A) Rohan
(B) Dimple
(C) Suman
(D) Cannot be determined

Answer Keys

Exercise

Practice Problems

1. A	**2.** C	**3.** B	**4.** C	**5.** D	**6.** D	**7.** A	**8.** C	**9.** B	**10.** A
11. D	**12.** C	**13.** C	**14.** D	**15.** D	**16.** B	**17.** A	**18.** B	**19.** B	**20.** C
21. C	**22.** A	**23.** D	**24.** A	**25.** B					

Chapter 8

Puzzles

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Distribution
- Order sequence
- Selections
- Questions on routes/networks

Puzzles

In this type of problems, you have to match two or more 'variables' (Variable means a 'subject' as used in the discussion of Linear Arrangement). In double line-up, the data given may talk of four people living in four houses each of a different colour. What we need to find out is the colour of the house of each of the four persons. There is no first position or second position of the houses.

Sometimes, Double line-up is also called as 'Distribution.' An example of data given for this variety of questions is:

'Each of the four persons A, B, C and D wears a different coloured shirt – Red, Pink, Blue and White. A has a Red shirt and D does not have a Pink shirt.'

From the above statement, it becomes clear that no person among A, B, C and D can have shirts of two different colours among Red, Pink, Blue and White.

As discussed in the questions on Single Line-up, questions can be solved easily by representing the given data pictorially. In case of Double Line-up, it will help us if we represent the data in the form of a matrix or a table.

Let us see how to draw a matrix for the data given above.

Names	Colours			
	Red	Pink	Blue	White
A	✓			
B				
C				
D		×		

As it is given that A has red colour shirt, it is clear that he does not have any other colour shirt. Similarly B, C, D do not Red colour shirt. So, in all the other cells in the row belonging to A, we put a cross ('×'). Then, the table will look as follows:

Names	Colours			
	Red	Pink	Blue	White
A	✓	×	×	×
B	×			
C	×			
D	×	×		

In this manner, we can fill up the cells on the basis of the data given to us. Once, we use up all the data, we will draw any conclusions that can be drawn and then answer the questions given in the set.

Let us Take a Few Examples

Directions for questions 1 to 5: These questions are based on the following information.

P, Q, R, S, T, U, V and W are eight employees of a concern. Each is allotted a different locker, out of eight lockers numbered 1 to 8 in a cupboard. The lockers are arranged in four rows with two lockers in each row.

Lockers 1 and 2 are in the top row from left to right respectively while lockers 7 and 8 are in the bottom row – arranged from left to right respectively. Lockers 3 and 4 are in the second row from the top – arranged from right to left respectively. So are lockers 5 and

6 – arranged from right to left respectively – in the second row from the bottom. P has been allotted locker 1 while V has been allotted locker 8. T's locker is just above that of Q which is just above that of R, whereas W's locker is in the bottom row.

Solved Examples

Example 1: Which of the following cannot be the correct locker number–occupant pair?
(A) 3-Q (B) 7-W (C) 4-U (D) 6-R

Example 2: If U's locker is not beside Q's locker, whose locker is just above that of W?
(A) U (B) S (C) R (D) Q

Example 3: Which of these pairs cannot have lockers that are diagonally placed?
(A) P-Q (B) S-R
(C) U-R (D) Either (B) or (C)

Example 4: Which of the following groups consists only occupants of odd numbered lockers?
(A) Q, R, W (B) R, V, W
(C) T, R, Q (D) P, T, Q

Example 5: If U's locker is in the same row as that of R, and S exchanges his locker with V, then who is the new neighbour of V in the same row? (Assume that nothing else is disturbed from the original arrangement)
(A) P (B) Q (C) R (D) U

Solutions for questions 1 to 5:
Let us first try to locate the lockers in the cupboard as per the conditions given. Then, we will do the allotment to the persons.

Lockers 1 and 2 are in the top row and lockers 7 and 8 are in the bottommost row. In these two rows, the lockers are numbered from left to right. In the other two rows, the lockers are numbered from right to left.

L	R	
1	2	Top Row
4	3	
6	5	
7	8	Bottom Row

Now let us look at the conditions given for the allotment of the lockers.

P has locker 1. V has locker 8.

1-P	2
4	3
6	5
7	8-V

Locker of W is in the bottom row → W's locker must be 7.

1-P	2
4	3
6	5
7-W	8-V

T's locker is just above that of Q, which is just above that of R → The lockers of T, Q and R must be 2, 3 and 5 respectively (there are no other group of lockers which satisfy this condition).

1-P	2-T
4	3-Q
6	5-R
7-W	8-V

S and U have lockers 4 and 6 left for them.

Thus, on the basis of the data given to us, we can show the final arrangement of lockers as below:

1-P	2-T
4-S/U	3-Q
6-U/S	5-R
7-W	8-V

Now we can answer the questions easily on the basis of the above.

Example 1: By looking at the final arrangement of lockers above, we find that choice (D) does not represent the correct combination of locker number-occupant pair.

Hence, the correct option is (D).

Example 2: If U's locker is not beside Q's locker, then U's locker must be locker 6. So, it is U's locker that will be immediately above W's.

Hence, the correct option is (A).

Example 3: R's locker is in the same row as that of exactly one of S or U and diagonally placed to the other one. Hence, 'either S-R or U-R' is the answer.

Hence, the correct option is (D).

Example 4: The odd-numbered lockers 1, 3, 5 and 7 which belong to P, Q, R and W respectively. Of the choices, we find that Q, R, W appear in choice (A). Hence, this is the correct choice.

Hence, the correct option is (A).

Example 5: U's locker is in the same row as that of R which means that locker 6 belongs to U. So locker 4 belongs to S. Now V and S exchange lockers. Then the new neighbour of V is Q.

Hence, the correct option is (B).

Directions for questions 6 to 10: These questions are based on the following information.

There are four trees – Lemon, Coconut, Mango and Neem – each at a different corner of a rectangular plot. A well is located at one corner and a cabin at another

corner. Lemon and Coconut trees are on either side of the Gate which is located at the centre of the side opposite to the side at whose extremes, the well and the cabin are located. The mango tree is not at the corner where the cabin is located.

Example 6: Which of the following pairs can be diagonally opposite to each other in the plot?
(A) Neem tree and Lemon tree
(B) Cabin and Neem tree
(C) Mango tree and Well
(D) Coconut tree and Lemon tree

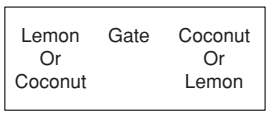
Example 7: If the Lemon Tree is diagonally opposite to the well, then the Coconut tree is diagonally opposite to the
(A) Mango tree (B) Well
(C) Cabin (D) Gate

Example 8: If the Coconut tree and the Neem tree cannot be at adjacent corners of the plot, then which of the following will necessarily have to be at diagonally opposite corners of the plot?
(A) Coconut tree and Well
(B) Lemon tree and Cabin
(C) Lemon tree and Coconut tree
(D) Lemon tree and Well

Example 9: Which of the following must be TRUE?
(A) Cabin and Well are not at adjacent corners.
(B) Cabin and Coconut tree cannot be at the adjacent corners.
(C) Neem tree and Well are at adjacent corners.
(D) Neem tree and Well are not at adjacent corners.

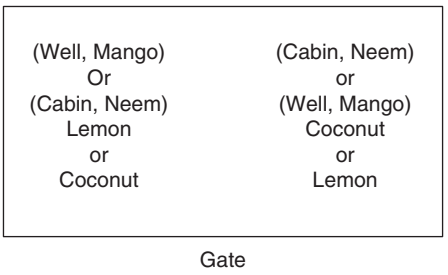
Example 10: Which of the following is definitely FALSE?
(A) Mango Tree is adjacent to the well at one corner.
(B) Neem Tree is adjacent to the Cabin at one corner.
(C) Coconut Tree is at the corner adjacent to the Well.
(D) Lemon Tree is not on the same side of the plot as the gate.

Solutions for questions 6 to 10:
Lemon and Coconut are on either sides of the gate.



The Well and the Cabin are at either end of the Wall opposite to the Gate.

Mango tree and Cabin are not at the same corner. So, Neem tree and Well are not at the same corner. This means that Mango tree and the Well are at the same corner and Neem tree and the Cabin are at the same corner.



Example 6: Let us take each choice and check with the above diagram to see if it is possible or not.

Neem and Lemon can be diagonally opposite each other. Hence, this is the correct answer choice. (In an exam, you do not need to check the other choices since the first choice is correct. But, for the sake of clarity and proper understanding, we will check all the choices).

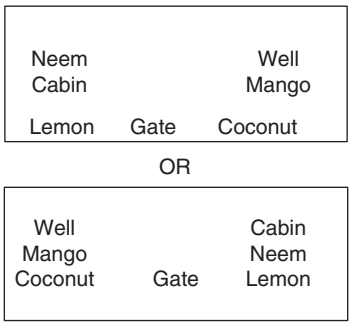
From the diagram given above, we can see that Cabin and Neem cannot be located diagonally opposite each other.

Mango and Well cannot be located diagonally opposite to each other.

Coconut and Lemon cannot be located diagonally opposite each other.

Hence, the correct option is (A).

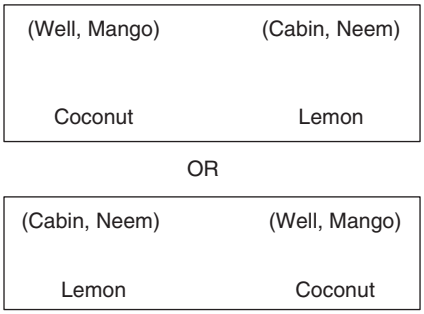
Example 7: If Lemon tree is diagonally opposite to the Well, then we can have the following two possible arrangements.



The Coconut tree is diagonally opposite the Cabin and Neem.

Hence, the correct option is (C).

Example 8: Since Coconut and Neem trees cannot be at adjacent corners, the following arrangements are possible.



From the above diagrams, we find that choice (D) is the correct answer.

Hence, the correct option is (D).

Example 9: We can check each statement with the diagram that we drew initially. We find that the statement in Choice (C) which says that Neem tree and the Well are at the adjacent corners is true. Hence, choice (C) is the correct answer. Hence, the correct option is (C).

Example 10: We check each statement with the diagram that we drew initially to find out which of the statements has to be false.

We find that choice (D) has to be false.

Hence, the correct option is (D).

Directions for question 11: Select the correct answer from the given choices.

Example 11: A, B, C and D play four different games among Baseball, Cricket, Kabaddi and Volley ball. A does not play Baseball or Cricket. B does not play Kabaddi or Volleyball. C plays Volleyball and D plays either Baseball or Volleyball. Who plays Cricket?

(A) A (B) B (C) C (D) D

Solution for question 11:

Example 11: C plays Volleyball. A does not play Cricket and D does not play Cricket as he plays either Baseball or Volleyball.

∴ B should play Cricket.

Hence, the correct option is (B).

Order Sequence

The term 'Order Sequence' is self-explanatory. In questions for this category, you will be asked to deal with relative positions of subjects. The absolute values of the subjects is not what you should be interested in. It is comparison between different subjects that you have to deal with. The data also specifies the relationships like 'A is greater than B' or 'C is not less than D' and so on. You have to decide the positions of the subjects in ascending or descending order on the parameters given. The subjects of comparison can be people or things.

In short, data will be given to compare the quality or quantity. The parameters on which the subjects are compared can be heights or weights of people, the money with them, complexion, sizes of things, etc.

In such questions, you will come across typical statements like 'A is taller than B,' 'B is not shorter than C', and so on. You may use the following symbols to symbolically represent the conditions given and then later, represent all the subjects pictorially.

Greater than	>
Less than	<
Greater than or equal	≥
Less than or equal	≤.

'Not greater than' is the same as 'less than or equal to.' Similarly, 'not less than' is the same as 'greater than or equal to'.

Words like 'Who, And, Which, But' used in the data play a significant role in analysing the data. 'AND' and 'BUT' play the same role whereas 'Who' and 'Which' play the same role.

Let us take one statement.

'A is taller than B, who is shorter than C and taller than D but shorter than E, who is taller than F and G but shorter than H'.

By using appropriate symbols, the above statement can be represented as follows.

$$A > B;\ B < C;\ B > D;\ B < E;\ E > F;\ E > G;\ E < H$$

Questions on the above data can be as follows.

1. Who is the tallest?
2. Who is the shortest?
3. Who is the second tallest in the group?, etc.

Let us take some examples.

Directions for questions 12 to 16: Read the information given below and answer the questions that follow.

A, B, C, D and E are five cars while P, Q and R are three motorcycles. A is the fastest of the cars and R is the slowest of the motorcycles. C is costlier than D and Q but cheaper than B. Among cars, A is not the costliest. D is cheaper than E and there is no car whose cost lies between the cost of these two. E is faster than three of the cars and all the motorcycles. Q is costlier than R but cheaper than P, who is faster than Q.

Example 12: Which of the following cars cannot stand exactly in the middle position among cars as far as their cost is concerned?

(A) A (B) C

(C) E (D) D

Example 13: Which of the following statements is true about the motorcycles?

(A) P is the costliest as well as the fastest motorcycle.

(B) The fastest motorcycle is not the costliest motorcycle.

(C) The slowest motorcycle is also the cheapest motorcycle.

(D) Both (A) and (C)

Example 14: If P is costlier than E, how many cars are cheaper than P?

(A) 1

(B) 2

(C) 3

(D) Cannot be determined

Example 15: If P is cheaper than A which is not costlier than E, which of these is the cheapest of all the cars and motorcycles put together?

(A) R

(B) Q

(C) E

(D) Cannot be determined

Example 16: Which of these is the slowest of the cars, if B and C are faster than D?
(A) B (B) D (C) E (D) A

Solutions for questions 12 to 16:
Let us first write down all the comparisons given for costs and speeds. Then we will tabulate them.

Speed
A → fastest car
E → Faster than three of the cars ➔ E is the second fastest car
R → slowest motorcycle
P > Q
Cost
C > D
C > Q
B > C
A → Not the costliest among cars
E > D → No other car lies between these two
Q > R
P > Q

Now let us tabulate this data.
Speed
Cars

Fastest	A	E				Slowest

Motorcycles

Fastest	P	Q	R	Slowest

Cost
Cars

Costliest	B C E D	Cheapest

Here, we know that A is not the costliest car but we do not know where it will fit in. It can come anywhere after B except between E and D.

Motorcycles

Costliest	P	Q	R	Cheapest

In addition to the above, we have to also keep in mind that C > Q in cost. (From this we can conclude that B > Q, B > R, C > R in cost).

Example 12: In terms of cost of the cars, A can come between B and C or between C and E or to the right of E. In each of the above cases, the middle car will be C, A and E respectively. Hence, among the cars given, D cannot be in the middle.
Hence, the correct option is (D).

Example 13: By looking at the tables above, we can make out that choices (A) and (C) are both correct and hence, the correct answer is choice (D).
Hence, the correct option is (D).

Example 14: If P is costlier than E, we can also conclude that it is costlier than D but we cannot conclude anything about the relationship between the cost of P and that of B, C and A.
Hence, the correct option is (D).

Example 15: Since A is not costlier than E, it means that A is at the same level of E or cheaper than E. We cannot conclude which of these two positions A is in. Hence, we cannot conclude which the cheapest of all the vehicles is. {Please note that if A is the cheapest car, then R will be the cheapest of all the vehicles. However, if A is at the same level as E in cost, then there is a possibility of R or D being the cheapest of all the vehicles.}
Hence, the correct option is (D).

Example 16: If B and C are faster than D, then the order will be as follows:

1	2	3	4	5
A	E	B/C	C/B	D

Hence, D is the slowest of all the cars.
Hence, the correct option is (B).

Directions for questions 17 to 21: Read the information given below and answer the questions that follow.
J, K, L, M and N are five boys in a class. They are ranked in the order of heights – from the tallest to the shortest – and in order of cleverness – from the cleverest to the dullest. K is taller than N, but not as clever as J and L, whereas M is the cleverest of all but shorter than J. While L is shorter than M but taller than K, L is not as clever as J. No two persons got the same ranks in any of these parameters.

Example 17: Who is the third in the order of heights?
(A) J (B) N (C) K (D) L

Example 18: If N is not the last in at least one of the two comparisons, which of the following is the dullest of all the five?
(A) K (B) L (C) M (D) J

Example 19: If L is the third in order of cleverness, who is the dullest of all?
(A) M
(B) N
(C) L
(D) Cannot be determined

Example 20: Who among the following is cleverer as well as taller than K?
(A) L and J only (B) N
(C) L and N (D) J, L and M

Example 21: How many people are definitely shorter than K?
(A) 1 (B) 2
(C) 3 (D) None of these

Solutions for questions 17 to 21:
Let us first write down all the conditions given and then tabulate the data.

Clever ness
J > K
L > K
M is the cleverest.
J > L

Height
K > N
J > M
M > L
L > K

Now let us put together all the information we have.

Cleverness

Cleverest	M J L K	Dullest

We do not know where N will come in the order of cleverness but he will definitely be after M.
Height

Tallest	J M L K N	Shortest

Example 17: From the table above, we can clearly see that L is ranked third in order of heights.
Hence, the correct option is (D).

Example 18: N is the last in terms of height. Since we are given that he is not the last in at least one of the lists, he CANNOT be the last in cleverness. So, K is the dullest of all.
Hence, the correct option is (A).

Example 19: If L is the third in the order of cleverness, as can be seen from the table above, either N or K can be the dullest.
Hence, the correct option is (D).

Example 20: By looking at the tables we made above and from the answer choices, we find that L, J and M are taller as well as cleverer than K.
Hence, the correct option is (D).

Example 21: Only N is shorter than K.
Hence, the correct option is (A).

Directions for question 22: Select the correct alternative from the given choices.

Example 22: P, Q, R, S, and T are five girls competing in a running race. R and P have at least two girls ahead of each of them. T and P do not have more than one girl behind each of them. Who arrives at the finishing line after two girls as well as before two other girls, if no two girls finish the race at the same time?
(A) Q (B) S (C) T (D) R

Solution for question 22:

Example 22: R and P have at least two girls before them
➔ R and P have to be in two out of 3rd, 4th and 5th positions.
T and P have not more than one girl behind each of them
➔ T and P have to be in the 4th or 5th positions.
The above two statements together mean that R will have to be in the third position.
Hence, the correct option is (D).

SELECTIONS

In this category of questions, a small group of items or persons has to be selected from a larger group satisfying the given conditions. The conditions will specify as to when a particular item or person can be included or cannot be included in the subgroup. For example, the condition may specify that two particular persons should always be together or that two particular persons should not be together.

Sometimes, the conditions given for selection or non-selection of items or persons may be based on logical connectives like if-then, either-or, unless, etc. You should be careful in interpreting the logical connectives used in the conditions.

Directions for questions 23 to 27: These questions are based on the following information.

Amit, Bittu, Chintu, Dumpy, Falgun, Hitesh, Ronit, Purav and Saurav are nine players from among whom three teams consisting respectively of 4 members, 3 members and 2 members must be formed subject to the following conditions.

Chintu must have three more players with him while Dumpy must have only two more with him.
Chintu and Saurav cannot be in the same team.
Purav and Bittu cannot be in the same team.
Ronit and Hitesh must be in the same team.

Example 23: If Dumpy, Falgun, Purav form the team of 3 members, then which of the following must be TRUE?
(A) Hitesh must be in a team with Bittu.
(B) Saurav must form a two-member team with Amit or Chintu.
(C) Saurav must form a two-member team with Bittu or Amit.
(D) Chintu should form a team of 4 members with Hitesh, Ronit and Amit.

Example 24: If Dumpy takes Amit as a part of his three-member team, which of the following must go into Chintu's team?
(A) Bittu and Hitesh (B) Hitesh and Ronit
(C) Purav and Ronit (D) Purav and Falgun

Example 25: If Chintu and Falgun are together and Saurav is in the team of two members, then how many sets of different teams are possible?
(A) 4 (B) 3 (C) 2 (D) 1

Example 26: If Chintu does not have Purav in his team and the two member team consists of Saurav and Amit, then Chintu should take
(A) Hitesh, Bittu and Ronit.
(B) Bittu but not Ronit.
(C) Bittu and Falgun.
(D) Hitesh and Ronit.

Example 27: If Purav is in the same team as Chintu and Falgun, then Saurav must be in the same team as
(A) Bittu (B) Bittu and Amit.
(C) Amit (D) Bittu and Dumpy.

Solutions for questions 23 to 27:
It is given that:
Chintu must form a team of 4 members only
Dumpy must form a team of 3 members only.
Since Chintu and Dumpy are in two different teams, let us, for convenience, denote the two teams as the respective teams of these two persons. Let us call the team with four members as the first team and the team with three members as the second team. The third team should have two persons.

Number of members		
4	3	2
Chintu	Dumpy	Saurav
	Saurav	

Now let us take the other conditions and fill them up in the table above.
Chintu and Saurav cannot be in the same team.
➔ Saurav will be in the second or the third team.

Purav and Bittu cannot be in the same team.
Hitesh and Ronit must be in the same team.
We cannot represent these two conditions right now in the table above but we will use them as we go along.

Example 23: If Dumpy, Falgun, Purav form the team of 3 members, then Saurav should be in the third team.
Since Hitesh and Ronit must be in the same team, they have to be in the first team. That leaves only Amit or Bittu to be with Saurav in the third team.
Hence, the correct option is (C).
(Also, note that we can eliminate choice (B) easily.)

Example 24: Dumpy takes Amit as a member of his team.
If we take Hitesh and Ronit as the two members of the third team, then Saurav has to be in the second team, in which case we will have both Purav and Bittu coming into the same team – the first team – which is not possible.
Since Saurav cannot be in Chintu's team and Purav and Bittu cannot be in the same team, the three people required for Chintu's team will **have to be** Hitesh and Ronit check-font Falgun or Purav or Bittu.
Hence, the correct option is (B).

Example 25: Let us analyse the conditions. It is given that Chintu and Falgun are together, whereas Saurav is in the team of two members. Let us fill up these details in the box that we made above and then see in how many ways we can fill up the remaining cells in the box.

Chintu	Dumpy	Saurav
Falgun		

First let us look at Hitesh and Ronit who must be in the same team.
They can go into the first team or the second team. Let us consider these two cases.

Case 1: Hitesh and Ronit go into the first team.
Then, one out of Bittu and Purav will to go into the third team and the other into the second team. This gives rise to two ways of forming the teams – one with Bittu in the second team and the other with Bittu in the third team.

Case 2: Hitesh and Ronit go into the second team.
In this case too, one out of Bittu and Purav will go into the third team and the other into the second team. Hence, this will also give rise to two ways of forming the teams.
Hence, there are total four ways of forming the teams.
Hence, the correct option is (A).

Example 26: Let us use the table that we built in the initial analysis and fill up the details that we have in this problem. Since the two member team is already formed and Chintu does not take Purav, hence Purav will have to go into the second team.

Chintu	Dumpy	Saurav
	Purav	Amit

Since Ronit and Hitesh have to be in the same team, they should go into the first team. Since Bittu cannot go with Purav, he should also be in the first team. This leaves Falgun for the second team. Thus, we can fill up the table as follows:

Chintu	Dumpy	Saurav
Ronit	Purav	Amit
Hitesh	Falgun	
Bittu		

Hence, the correct option is (A).

Example 27: If Purav is with Chintu and Falgun, then Bittu cannot be with them. Since Ronit and Hitesh should be together, the only other person left is Amit. These four members form the first team.

If Hitesh and Ronit together form the two member team, then Bittu and Saurav will be part of the three member team.

Instead, if Hitesh and Ronit are in the three-member team, then Saurav and Bittu will form the two-member team.

In either case, Saurav and Bittu are together in one team. Hence, the correct option is (A).

Directions for questions 28 to 31: These questions are based on the following information.

A, B, C, D, E, F and G are seven players. They form two teams of two players each and one team of three players. A and B cannot be in the same team. B and C cannot be in the same team whereas E and F must be in the same team. G and D cannot be in the same team.

Example 28: If C, D and A form a team of three players, which of the following can be the members of one of the other teams?

(A) A and E (B) G and B
(C) E and F (D) Both (B) and (C)

Example 29: If E, F and G form a team of three members, then in how many ways can the remaining two teams of two players each be formed?

(A) 2 (B) 4 (C) 3 (D) 1

Example 30: If D and A are not in the same team, then altogether in how many ways can the teams of two members be formed?

(A) 4 (B) 7 (C) 8 (D) 5

Example 31: If B, E and F form a team of three members, which of the following cannot be the two teams of two members each?

(A) AC, GD (B) AD, CG
(C) AG, CD (D) Both (A) and (B)

Solutions for questions 28 to 31:

Let Team I be of 3 players, Team II be of 2 players and Team III be of 2 players.

It is given that A and B cannot be together. We will represent it as A × B.

Similarly, we have B × C and G × D.

E and F must be in the same team. So E and F can form a team of 2 members on their own or can form a team of 3 members with another person.

Let us now take up the questions and work them out.

Example 28: Given that C, D, A form a team of 3 members, one of the other teams **has** to have E and F together. Hence, B and G should form one team.

Hence, the correct option is (D).

Example 29: Given that E, F, G form a team of 3 players. Since A and B or B and C cannot be in the same team, we must necessarily have A and C together in one team and B and D in the other team. So the teams can be formed only in one way.

Hence, the correct option is (D).

Example 30: Given that A and D are not in the same team. Hence A × B, B × C, G × D and A × D.

We already know that E and F must be in the same team. They may form a team of 3 members or they themselves be a team of 2 members. Let us consider the above two possibilities and then fill up the other teams. They can be formed as follows:

	Team I	Team II	Team III
1.	A E F	B D	C G
2.	A E F	C D	B G
3.	B E F	A G	C D
4.	C E F	A G	B D
5.	D E F	A C	B G
6.	G E F	A C	B D
7.	A C G	B D	E F

Thus the teams can be formed in 7 ways.

Hence, the correct option is (B).

Example 31: If B, E, F form a team of 3 members, then the two members teams must be formed from A, C, D, G. The teams can be AD and CG or AG and CD. As D and G cannot form a team, AC and GD cannot be formed. Choice (A). {Please note that we can answer this question from the answer choices – from choice (A), we find that G and D are together in one team which is not possible. Thus, choice (A) is the answer.

Hence, the correct option is (A).

Directions for question 32: Select the correct alternative from the given choices.

Example 32: At least two boys out of A, B, C and D and at least two girls out of P, Q, R and S have to be chosen to form a group of 5 members.

Neither A nor C can go with Q.

Neither P nor S can go with B.

Q and R cannot be together.

Which of the following is an acceptable team?

(A) ARCQP
(B) ASQPD
(C) ASQRP
(D) PSRAD

Solution for question 32:

Example 32: The required group of 5 members must be formed with at least two boys from A, B, C, D and at least 2 girls from P, Q, R, S.

Answers 1, 2 and 3 can be ruled out as A and Q cannot be together.

In choice (D), P, S, R, A, D can be together without violating any of the given conditions.

Hence, the correct option is (D).

Questions on Routes/Networks involve different points or locations between which there is some movement or communication. The way the movement or communication is effected is described in the data/conditions. Sometimes, these are also referred to as 'Maps' because the routes given resemble a map.

The data given in these types of questions may not always have the word 'route' or 'network' in them but a network is indicated by some sort of connectivity between two 'points.' The way the statements are worded is important. The wording includes statements like

1. Some poles are connected through wires.
2. Some towers send signals to one another.
3. Some cabins, market, cities, etc. are connected via. passages or roads and so on.

The connectivity between the two 'points' can be only one-way or two-way. In one-way connectivity, the flow will be in only one direction whereas in two-way connectivity, the flow will be in both directions between the points.

Read the data carefully and then draw the diagram or network. The words 'from' and 'to' play an important role in these questions and hence, care should be taken while interpreting the data. While drawing a diagram, arrow marks can be used very effectively to indicate the direction of connectivity as explained below.

1. If the statement mentions that there is a one-way route from city A to B then it can be represented as follows.

   A ⟶ B

2. If the statement mentions that cities X and Y have roads on which you can travel in either direction, it means that it is a two-way connectivity. Then it can be represented as follows.

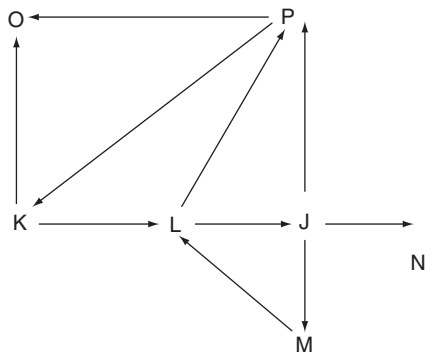
   X ⟷ Y

3. If the statement mentions that all the projected roads are one-ways and there is a route from point K to P and then from P to K, then it should be represented as follows.

   P ⇄ K

   i.e., from P to K is one route and from K to P is another route.

   Now, consider the following network.



In this network, let us say that a person starts from J and he wants to reach K. We want to find out the number of distinct routes he can take without touching any point twice. Starting from J, the possible directions of movement are from J to N, J to M and J to P, but if he goes from J to N then coming back or travelling in some other direction is not possible. Hence, the person has only two options for movement from J (J to M and J to P). Now, if he goes to M, he has to go to L from M. At L, it appears that he has two options – he can go to P or to J. But, since he started from J and as he cannot touch any point twice, he cannot go to J. So, there is only one option at L – that is going to P. So, to reach P from J, there are two options – one directly to P from J and the other via. M and L. Once he reaches P, he has only one way of reaching K – along the diagonal PK. If he goes to O from P, then he cannot travel to K from O (the route is one way in the O to K). Thus the total number of ways from J to K is two (JPK and JMLPK).

Thus one has to look at all the possible routes carefully in the manner discussed above.

In the above example, if the route between J and L is two-way and then we have to find out the number of ways to reach O stating from J, the routes we have will be as follows:

JPO, JPKO, JMLPO, JMLPKO, JLPO, JLPKO

This gives us a total of six distinct ways of reaching O from J.

Directions for questions 33 to 37: Read the following information and answer the questions given below.

P, Q, R, S, T, U, V are seven places on a map. The following places are connected by two-way roads: P and Q; P and U; R and U; R and S; U and V; S and T; Q and R; T and V. No other road exists.

Example 33: The shortest route (the route with the least number of intermediate places) from P to V is

(A) P-R-V (B) P-T-V
(C) P-Q-R-U-V (D) P-U-V

Example 34: How many distinct routes exist from S to U (without touching any place more than once)?

(A) 3 (B) 2 (C) 1 (D) 4

Example 35: The route covering the maximum number of places and going from P to R does not pass through

(A) U (B) T (C) S (D) Q

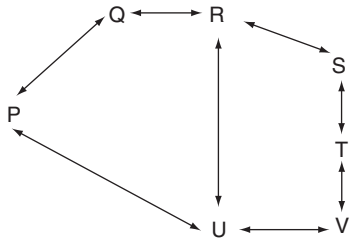
Example 36: If U to V and S to R are only one-way routes, then which of the following places lose contact with P?

(A) V
(B) T
(C) S
(D) No place loses contact with P

Example 37: If the number of places to which a place is connected directly considered as the measure of importance, then which of the following places is of the highest importance?

(A) Q (B) P (C) R (D) S

Solutions for questions 33 to 37:
The route map of the places is as follows:



Example 33: As is seen, P-U-V is the shortest way (with only one intermediate point) from P to V.

Hence, the correct option is (D).

Example 34: To travel from S to U, the routes available are: S-R-U; S-T-V-U and S-R-Q-P-U – a total of 3 routes.

Hence, the correct option is (A).

Example 35: First let us write down the route from P to R with the maximum number of intermediate points. By observation, we find that it is P-U-V-T-S-R. It does not touch Q.

Hence, the correct option is (D).

Example 36: If U to V and S to R are only one-way routes, from the figure, we find that all places can still be reached from P. Hence, none of the places loses contact with P.

Hence, the correct option is (D).

Example 37: For each of the places given in the choices, Q, P, R and S, let us see how many places are directly connected.

Q is directly connected to 2 places.
P is directly connected to 2 places.
R is directly connected to 3 places.
S is directly connected to 2 places.
Hence, the correct option is (C).

Directions for questions 38 to 41: Read the following information and answer the questions given below.

Five cities P, Q, R, S and T are connected by different modes of transport as follows:

P and Q are connected by boat as well as by rail.
S and R are connected by bus and by boat.
Q and T are connected only by air.
P and R are connected only by boat.
T and R are connected by rail and by bus.

Example 38: Which of the following pair of cities are connected by any of the routes directly [without going through any other city]?

(A) P and T (B) T and S
(C) Q and R (D) None of these

Example 39: Which mode of transport would help one to reach R starting from Q but without changing the mode of transport?

(A) Boat (B) Rail (C) Bus (D) Air

Example 40: If a person visits each of the places starting from P and gets back to P, which of the following places must he visit twice?

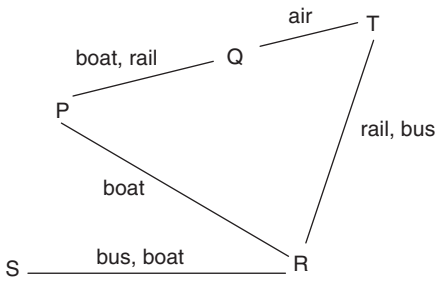
(A) Q (B) R
(C) S (D) T

Example 41: Between which two cities among the pairs of cities given below are there maximum travel options available? (If there is more than one route possible between two cities, consider the route with least number of cities enroute.)

(A) Q and S (B) P and R
(C) P and T (D) Q and R

Solutions for questions 8.38 to 8.41:

Example 38:



P and T are connected through Q.
T and S are connected through R
Q and R are connected through T or P. Q and S are connected through R and P or T. So none of the pairs in the choices are directly connected.

Hence, the correct option is (D).

Example 39: From Q to P, he can reach by boat.
From P to R also he can travel by boat.

So, a person should travel by boat to reach R from Q without changing the mode of transport.

Hence, the correct option is (A).

Example 40: If a person wants to visit all the places and again return to P, then he can go in the order of P→ Q → T → R → S and then
S → R → P (OR)
P → R → S and then
S → R → T → Q → P
He must visit R twice.

Hence, the correct option is (B).

Example 41: One has to travel between any of the two cities with a restriction that if there is more than one possible route, he has to go by the least number of cities enroute. It is better to take the pair of cities given in each of the choices.

Choice (A) Q and S: A person can go from Q to T to R to S (or) Q to P to R to S or vice-versa. In both the routes, there are two cities enroute. We can calculate the number of options in the entire route by multiplying the options available in each segment of the route.

For the route Q-T-R-S, the number of options = $1 \times 2 \times 2 = 4$
For the route Q-P-R-S, the number of options = $2 \times 1 \times 2 = 4$

Choice (B) P and R: The route between P and R has only one mode of travel, that is boat.

Choice (C) P and T: A person can go by PQT in either way or by PRT in either way. If it is by PQT, then the options are boat-air or rail-air i.e. two ways. If it is by PRT, then the options are boat-rail or boat-bus in either way. Hence, they are only two options.

Choice (D) Q and R: If a person travels between Q and R, then he can go by QTR or by QPR. If he goes by QTR, then the options are air-rail or air-bus, that gives two options or if he goes by QPR, then the options are rail-boat or boat-boat, that will again give us two options.

So, it is very clear that Q and S have maximum number of travel options available between them.

Hence, the correct option is (A).

***Directions for question* 42:** Select the correct alternative from the given choices.

Example 42: Four computers P, Q, A and B are interconnected for the transmission of data. A and B each can send data to both P and Q but B cannot receive data from A. P and Q can have data flow in both directions between them but they cannot transmit the data so received to B but can otherwise send the data directly to B. Which of the following routes can be followed if B has to receive data from A?

I. AQPB II. APB
III. AQB IV. APQB
(A) I and III (B) II and III
(C) III and IV (D) All four

Solution for question 8.42:

Example 42: AQPB and APQB can be eliminated because P and Q cannot send the data to B.
The paths AQB and APB do not violate any conditions and hence can be possible routes to send data from A to B.

Hence, the correct option is (B).

EXERCISE

Practice Problems

Directions for questions 1 to 3: These questions are based on the following information.

Each of the seven delegates A through G came to India to attend a conference from seven different countries – China, Japan, Malaysia, England, Australia, Germany and Poland.

(i) China, Japan, and Malaysia are the only Asian countries.
(ii) A and B are from Asian countries whereas D is neither from England nor from Australia.
(iii) E and F are from non Asian countries but neither of them came from either Australia or England.
(iv) C is not from England and the person from Poland is not F.
(v) A is from China.

1. Who is from Germany?
(A) E (B) C (C) F (D) G

2. Who is from Malaysia?
(A) B (B) D
(C) A (D) Either (A) or (B)

3. Which country did G come?
(A) England
(B) Australia
(C) Poland
(D) Cannot be determined

Directions for questions 4 to 6: These questions are based on the following information.

Bingo, Pingo, Tingo, Hingo and Mingo are five friends, each of whom is working in a different company among – C_1, C_2, C_3, C_4 and C_5 and they belongs to the same city but a different locality – l_1, l_2, l_3 and l_4 and l_5.

(i) The persons who are working with C_1 and C_2 are from l_3 and l_4
(ii) Bingo is from l_5 but does not work for C_5.
(iii) Tingo is not from l_4 but works for C_2.
(iv) Pingo works neither for C_5 nor in C_3 and is not form l_2.
(v) The person working for C_3 is from l_1.
(vi) Mingo does not work for C_3.

4. For which company does Hingo work?
(A) C_3 (B) C_4 (C) C_5 (D) C_2

5. Who is from l_4?
(A) Mingo (B) Hingo
(C) Tingo (D) Pingo

6. Who works for C_4?
(A) Bingo (B) Mingo
(C) Pingo (D) Hingo

Directions for questions 7 to 9: These questions are based on the following information.

A team of three is to be selected from six persons – Amar, Bhavan, Chetan, Dawan, Ekta and Farheen under the following constraints.

(i) If Amar or Bhavan is selected, then Chetan must not be selected.
(ii) If Chetan or Dawan is selected, then at least one of Ekta and Farheen must be selected.

7. If Dawan is selected, then who must not be selected?
(A) Amar
(B) Bhavan
(C) Chetan
(D) None of these

8. If Amar is selected, then in how many ways the team can be selected?
(A) 5 (B) 6 (C) 4 (D) 7

9. If Bhavan is selected, then who must be selected?
(A) Dawan
(B) Ekta
(C) Farheen
(D) Either (B) or (C)

Directions for questions 10 to 12: These questions are based on the following information.

Three girls – Anjali, Bharathi and Chandrika and four boys – Kiran, Lala, Manoj and Naveen are to be divided into two teams under the following constraints.

(i) Each team must have at least one girl and at least one boy and at least three persons in total.
(ii) If Anjali and Bharathi are selected in a team, then the team must have only one boy.
(iii) Kiran and Lala cannot be in the same team.
(iv) Chandrika and Naveen can be in the same team, only if Bharathi is selected in that team.

10. If Kiran and Chandrika are in the same team, then in how many ways can the other team be selected?
(A) 6 (B) 3 (C) 4 (D) 5

11. If Manoj is not in the same team as Bharathi, then in how many ways can the teams be selected?
(A) 3 (B) 4 (C) 5 (D) 6

12. If three boys are selected into one team, then in how many ways can the teams be selected?
(A) 4 (B) 5 (C) 3 (D) 6

Directions for questions 13 to 15: These questions are based on the following information.

Seven persons – P, Q, R, S, T, U and V, who are of different ages, are comparing their ages. We know the following information.

(i) P is younger than R, who is not older than S.
(ii) S is younger than only two persons.
(iii) Q is not the oldest but older than fourth youngest person.
(iv) T is older than only U.

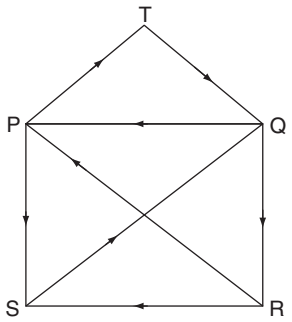
13. Who is oldest?
(A) S (B) T (C) U (D) V

14. Who is the third youngest?
(A) V (B) P (C) R (D) S

15. Who is the fourth eldest?
(A) R (B) P (C) S (D) V

Directions for questions 16 to 19: These questions are based on the diagram given below:

Five cities P, Q, R, S and T are connected by one way rail routes as shown below. One takes one hour duration to travel between any two directly connected cities.



At station S, for every 2 hours one train departs and the departure time of the first train is 6:00 a.m. Similarly at station R, for every 3 hours one train departs in each route and the departure time of the first train is 4 a.m.

At station Q, one train departs for every 2 hour in each route and the departure time of the first train is 7 a.m.

At station P, train departs for every 1 hour in each route and the departure time of the first train is 8 a.m.

At station T, for every 3 hours one train departs and the departure time of the first train is 5:30 a.m.

16. What is the least time will it take to reach P from R, if one takes the longest route without visiting any station more than once?
(A) 6 hours (B) 3 hours
(C) 5 hours (D) 4 hours

17. If a person reaches Q at 1:00 p.m. from R, which of the following can be the time at what time he must have started from R if that person takes the shortest route?
(A) 10:00 a.m. (B) 9:00 a.m.
(C) 11:00 a.m. (D) None of these

18. A person wants to travel from R to T and he takes the longest route without visiting any station more than once. If he starts at 4:00 a.m. then for how much time he has to wait for the trains altogether in all stations before reaching T.
(A) 2 hours (B) 3 hours
(C) 2 hours (D) 1 hour

19. If a person starts from P at 10:00 a.m. to reach S and he takes the longest route without visiting any station more than once then at what time will he be reachings?
(A) 1:00 p.m. (B) 4:00 p.m.
(C) 5:00 p.m. (D) 2:00 p.m.

Directions for question 20: This question is based on the information given below.

Five cities - Ahmedabad, Bangalore, Calicut, Delhi and Indore are connected by one way routes from Ahmedabad to Bangalore, Delhi to Ahmedabad, Indore to Delhi, Delhi to Calicut, Ahmedabad to Calicut, Bangalore to Calicut, Calicut to Indore, Indore to Bangalore and Ahmedabad to Indore.

20. In how many ways a person can travel from Delhi to Indore without visiting any city more than once?
(A) 5 (B) 3 (C) 6 (D) 4

Directions for questions 21 and 22: Select the correct alternative from the given choices.

21. There are 15 identical coins out of which fourteen are of equal weights and one coin lighter than each of the other coins. What is the minimum number of weighings required using a common balance to definitely identify the counterfeit coin?
(A) 3 (B) 4
(C) 5 (D) None of these

22. Beside a lake, there are three temples and a flower garden. Whenever some flowers are dipped into the lake, the flowers gets triplet. A person brought some flowers from the garden and dipped then into the lake. He placed x flowers in front of the first temple and dipped the remaining flowers into the lake. He placed x flowers in front of the second temple and dipped the remaining flowers into the lake. Now, he placed x flowers in front of the third temple and has no flowers. Which of the following numbers can be the value of x?
(A) 9 (B) 18 (C) 27 (D) 36

Directions for questions 23: These questions are based on the following letter – multiplication in which each letter is represents a unique non-zero digit.

A	B	C		
×	C	B	A	
C	D	E	F	C

Also, it is known that D = 3C and F = 4B

23. What is the value of D?
(A) 3
(B) 6
(C) 9
(D) Cannot be determined

Directions for questions 24 and 25: These questions are based on the following data.

Each individual of a city called 'Josh' belongs exactly to one of the two types, viz., Yes-type or No-type. Yes-type people always give the true reply while the No-type always lies. Answer the following questions based on the above information.

24. You met three residents A, B and C, of the city and asked them, 'who among you are married?' and got the following replies.
A: I am married to B.
B: I am married to C.
C: I am not married to A.
If it is further known that A is married to one of B and C and there is exactly one married couple among the three, then which of the following is definitely true?
(A) C is married to A.
(B) B is married to A.
(C) A is of Yes-type.
(D) B is of No-type.
(E) A is of No-type.

25. You approached three inhabitants A, B and C of the city and asked them, 'Who is of No-type among you?', and got the following replies.
A: B is of No-type.
B: C is of No-type.
C: A is of No-type.
It can be concluded that:
(A) A is a No-type.
(B) B is a No-type.
(C) C is a No-type.
(D) Either A or B is of No-type.
(E) Data inconsistent.

Answer Keys

Exercise

Practice Problems

1. C	**2.** D	**3.** A	**4.** A	**5.** D	**6.** A	**7.** D	**8.** A	**9.** D	**10.** D
11. D	**12.** A	**13.** D	**14.** B	**15.** A	**16.** B	**17.** A	**18.** D	**19.** C	**20.** D
21. A	**22.** C	**23.** C	**24.** D	**25.** C					

Chapter 9

Clocks and Calendars

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Clocks
- Calendar
- Leap and non-leap year
- Counting the number of odd days
- Counting of number of odd days, when only one date is given
- Counting number of odd days, when two dates are given

Clocks

The hour hand and the minute hand of a clock move in relation to each other continuously and at any given point of time, they make an angle between 0° and 180° with each other.

If the time shown by the clock is known, the angle between the hands can be calculated. Similarly, if the angle between two hands is known, the time shown by the clock can be found out.

When we say angle between the hands, we normally refer to the acute/obtuse angles (upto 180°) between the two hands and not the reflex angle (> 180°).

For solving the problems on clocks, the following points will be helpful.

- Minute hand covers 360° in 1 hour, i.e., in 60 minutes. Hence, MINUTE HAND COVERS 6° PER MINUTE.
- Hour hand covers 360° in 12 hours. Hence, hour hand covers 30° per hour. Hence, HOUR HAND COVERS 1/2° PER MINUTE.

The following additional points also should be remembered. In a period of 12 hours, the hands make an angle of

- 0° with each other (i.e., they coincide with each other), 11 times.
- 180° with each other (i.e., they point exactly in opposite directions), 11 times.
- 90° or any other angle with each other, 22 times.

Note: We can also solve the problems on clocks using the method of 'Relative Velocity'

In 1 minute, Minute Hand covers 6° and Hour hand covers 1/2°.

Therefore, Relative Velocity = 6 – 1/2 = $5\frac{1}{2}°$ per minute.

Alternately, in 1 hour, the minute hand covers 60 minute divisions whereas the hour hand covers 5 minute divisions.

$\therefore$ Relative Speed = 60 – 5 = 55 minutes per hour.

However, adopting the approach of actual angles covered is by far the simplest and does not create any confusion.

Points to Note

- Any angle other than (0° and 180°) is made 22 times in a period of 12 hours.
- In a period of 12 hours, there are 11 coincidences of the two hands, when the two hands are in a straight line facing opposite directions.
- The time gap between any two coincidences is 12/11 hours or $65\frac{5}{11}$ minutes.
- If the hands of a clock (which do not show the correct time) coincide every p minutes, then

If $p > 65\frac{5}{11}$, then the watch is going slow or losing time. If $p < 65\frac{5}{11}$, then the watch is going fast or gaining time.

To calculate the angle 'θ' between the hands of a clock, we use the following formula (where m = minutes and h = hours)

1. $\theta = \frac{11}{2}m - 30h$ $\left(\text{when } \frac{11}{2}m > 30h\right)$
2. $\theta = 30h - \frac{11}{2}m$ $\left(\text{when } 30h > \frac{11}{2}m\right)$

Solved Examples

Example 1: What is the angle between the minute hand and the hour hand of a clock at 3 hours 40 minutes?
(A) 20° (B) 70° (C) 90° (D) 130°

Solution: (D)
The angle between the hands can be calculated by $\theta = \left|\frac{11}{2}m - 30h\right|$, where m is minutes and h is hours. Here, $m = 40$ and $h = 3$

$$\therefore \quad \theta = \left|\frac{11}{2} \times 40 - 30 \times 3\right| = |220 - 90| = 130°$$

The angle between the two hands is 130°.

Example 2: Find the time between 2 and 3 O'clock at which the minute hand and the hour hand make an angle of 60° with each other.

Solution: In the formula $\theta = \left|\frac{11}{2}m - 30h\right|$,

$$\theta = 60° \text{ and } h = 2$$

$$\therefore \quad 60 = \frac{11}{2}m - 30 \times 2$$

$$\frac{11}{2}m = 120$$

$$m = \frac{240}{11} = 21\frac{9}{11}m \text{ past } 2$$

(or) $$60 = 30 \times 2 - \frac{11}{2}m$$

$$\therefore \quad \frac{11}{2}m = 0$$

$$m = 0$$

Therefore, the angle between the hour hand and the minute hand is 60° at 2 O'clock and at $21\frac{9}{11}$ minutes past 2 O'clock.

Example 3: Find the time between 2 and 3 O'clock at which the minute hand and the hour hand overlap.

Solution: When the two hands overlap, the angle between them is 0°.

$$\theta = \left|\frac{11}{2}m - 30h\right|$$

$$\therefore \quad \theta = 0° \quad \text{and} \quad h = 2$$

$$\frac{11}{2}m = 30 \times 2$$

$$m = \frac{120}{11} = 10\frac{10}{11} \text{ min past } 2.$$

Example 4: Find the time between 2 and 3 O'clock at which the minute hand and the hour hand are perpendicular to each other.

Solution: When two hands are perpendicular, $\theta = 90°$ and $h = 2$

$$\therefore \quad \theta = \left(\frac{11}{2}m - 30h\right) \quad \text{or} \quad \left(30h - \frac{11}{2}m\right)$$

$$90 = \frac{11}{2}m - 30 \times 2; \quad \frac{11}{2}m = 150$$

$$m = \frac{300}{11} = 27\frac{3}{11} \text{ minutes past } 2$$

(or) $$90 = 30 \times 2 - \frac{11}{2}m$$

$$\Rightarrow \quad \frac{11}{2}m = -30$$

As m cannot be negative, this case is not possible.
So, the hands are perpendicular to each other only once i.e. at $27\frac{3}{11}$ minutes past 2 O'clock.

Example 5: Find the time between 2 and 3 O'clock at which the minute hand and the hour hand are on the same straight line but are pointing in opposite directions.

Solution: When two hands are pointing opposite directions and are on a straight line the angle between them would be 180°. i.e. $\theta = 180°$ and $h = 2$.

$$180° = \frac{11}{2}m - 30h; \quad \frac{11}{2}m = 180 + 60 = 240$$

$$m = \frac{480}{11} = 43\frac{7}{11}$$

So, at $43\frac{7}{11}$ minutes past 2 O'clock the hands will be at 180°.

Calendar

Suppose you are asked to find the day of the week on 30th June, 1974, it would be a tough job to find it if you do not know the method. The method of finding the day of the week lies in the number of 'odd days'.

Note: Every 7th day will be the same day count wise, i.e. if today is Monday, then the 7th day counting from Tuesday onwards will once again be Monday. Odd days is the days remaining after completion of an exact number of weeks. Odd days is the reminder obtained on dividing the total number of days with seven.

Example: 52 days ÷ 7 = 3 odd days.

Leap and Non-leap Year: A Non-leap year has 365 days whereas a leap year has one extra day because of 29 days in the month of February. Every year which is divisible by 4 is called a leap year. Leap year consists of 366 days, (52 complete weeks + 2 days), the extra two days are the odd days. So, a leap year has two odd days.

An non-leap year consists of 365 days (52 complete weeks + 1 day). The extra one day is the odd day.

Note: Every century, year which is a multiple of 400, is a leap year. A century year which is not divisible by 400 is a non-leap year.

Example: 400, 800, 1200, 1600 … are leap years.
500, 700, 900, 1900 … are non-leap years.

Counting the number of Odd Days: 100 years consist of 24 leap years + 76 ordinary years. (100 years when divided by 4, we get 25. But at the 100th year is not a leap year, hence only 24 leap years).

$= 2 \times 24$ odd days $+ 1 \times 76$ odd days $= 124$ days

$= 17$ weeks $+ 5$ days

The extra 5 days are the odd days.

So, 100 years contain 5 odd days.

Similarly, for 200 years we have 10 extra days (1 week + 3 days).

$\therefore$ 200 years contains 3 odd days.

Similarly, 300 years contain 1 odd day and 400 years contain 0 odd days.

Counting of number of odd days, when only one date is given: Here we take January 1st 1 AD as the earlier date and we assume that this day is a Monday. We take its previous day, i.e., Sunday as the reference day. After this the above mentioned method is applied to count the number of odd days and find the day of the week for the given date.

Counting number of odd days, when two dates are given: Any month which has 31 days has 3 odd days.

($\because$ 31 ÷ 7 leaves 3 as remainder) and any month which has 30 days has 2 odd days (30 ÷ 7 leaves 2 as remainder).

Then, the total number of odd days are calculated by adding the odd days for each month. The value so obtained is again divided by 7 to get the final number of odd days. The day of the week of the second date is obtained by adding the odd days to the day of the week of the earlier date.

Example 6: If you were born on 14th April, 1992, which was a Sunday, then on which day of the week does your birthday fall in 1993?
(A) Monday
(B) Tuesday
(C) Wednesday
(D) Cannot be determined

Solution: (A)
14th April 1992 to 14th April 1993 is a complete year, which has 365 days. Hence, the number of odd days from 14th April 1992 to 14th April 1993 is 1.

Hence, 14th April 1993 is one day after Sunday, i.e., Monday.

Example 7: If 1st Jan, 1992 is a Tuesday then on which day of the week will 1st Jan, 1993 fall?
(A) Wednesday (B) Thursday
(C) Friday (D) Saturday

Solution: (B)
Since 1992 is a leap year there are 2 odd days.

Hence, 1st January 1992 is two days after Tuesday, i.e., Thursday.

Example 8: If 1st April, 2003 was Monday, then which day of the week will 25th December of the same year be?
(A) Tuesday (B) Wednesday
(C) Thursday (D) Friday

Solution: (B)
The number of days from 1st April to 25th December

(29 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 25) days = 268 days

$$= \frac{268}{7} = 38 + 2 \text{ odd days.}$$

Hence, 25th December is two days after Monday, i.e., Wednesday.

Example 9: Which year will have the same calendar as that of 2005?
(A) 2006 (B) 2007
(C) 2008 (D) 2011

Solution: (D)
Year: 2005 + 2006 + 2007 + 2008 + 2009 + 2010
Odd days : 1+ 1+ 1+ 2+ 1 + 1

Total number of odd days from 2005 to 2010 are $7 \cong 0$ odd days.

Hence, 2011 will have the same calendar as that of 2005.

Example 10: What day of the week was 18th April 1901?
(A) Monday
(B) Tuesday
(C) Wednesday
(D) Thursday

Solution: (D)
18th April 1901 $\Rightarrow$ (1600 + 300) years + 1st January to 18th April 1901.

1600 years have – 0 odd days

300 years have – 1 odd day

The number of days from 1st January, 1901 to 18th April 1901 is (31 + 28 + 31 + 18) days

108 days $\cong$ 3 odd days

$\therefore$ Total number of odd days = 3 + 1 = 4

Hence, 18th April 1901 is Thursday.

EXERCISE

Practice Problems

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. What is the angle covered by the minute hand in 22 minutes?
(A) 66° (B) 110° (C) 132° (D) 220°

2. By how many degrees will the minute hand move, in the same time, in which the hour hand moves 6°?
(A) 54° (B) 84° (C) 72° (D) 60°

3. What is the angle between the hands of the clock, when it shows 40 minutes past 6?
(A) 40° (B) 70° (C) 80° (D) 90°

4. What is the angle between the two hands of a clock when the time is 25 minutes past 7 O'clock?
(A) $62\frac{1}{2}°$ (B) $66\frac{1}{2}°$
(C) $72\frac{1}{2}°$ (D) $69\frac{1}{2}°$

5. At what time between 9 and 10 O'clock, will the two hands of the clock coincide?
(A) $43\frac{3}{11}$ minutes past 9 O'clock
(B) $45\frac{6}{11}$ minutes past 9 O'clock
(C) $49\frac{1}{11}$ minutes past 9 O'clock
(D) $49\frac{6}{11}$ minutes past 9 O'clock

6. At what time between 4 and 5 O'clock are the two hands of a clock in the opposite directions?
(A) $52\frac{3}{11}$ minutes past 4 O'clock
(B) $54\frac{6}{11}$ minutes past 4 O'clock
(C) $51\frac{7}{11}$ minutes past 4 O'clock
(D) $53\frac{9}{11}$ minutes past 4 O'clock

7. The angle between the two hands of a clock is 20° and the hour hand is in between 2 and 3. What is the time shown by the clock?
(A) $7\frac{3}{11}$ minutes past 2
(B) $14\frac{6}{11}$ minutes past 2
(C) $15\frac{5}{11}$ minutes past 2
(D) Both (A) and (B)

8. Which of the following can be the time shown by the clock, when the hour hand is in between 4 and 5 and the angle between the two hands of the clock is 60°?
(A) $16\frac{4}{11}$ min past 4 (B) $18\frac{9}{11}$ min past 4
(C) $32\frac{8}{11}$ min past 4 (D) $36\frac{5}{11}$ min past 4

9. How many times, the two hands of a clock will be at 30° with each other in a day?
(A) 36 (B) 40 (C) 44 (D) 48

10. If the time in a clock is 10 hours 40 minutes, then what time does its mirror image show?
(A) 2 hours 20 minutes (B) 1 hour 15 minutes
(C) 1 hour 10 minutes (D) 1 hour 20 minutes

11. There are two clocks on a wall, both set to show the correct time at 5:00 p.m. The clocks lose 2 minutes and 3 minutes respectively in an hour. When the clock which loses 2 minutes in one hour shows 9:50 p.m. on the same day, then what time does the other clock show?
(A) 9:30 p.m. (B) 9:40 p.m.
(C) 9:45 p.m. (D) 10:15 p.m.

12. A watch which gains uniformly was observed to be 1 minute slow at 8:00 a.m. on a day. At 6:00 p.m. on the same day it was 1 minute fast. At what time did the watch show the correct time?
(A) 12:00 noon (B) 1:00 p.m.
(C) 2:00 p.m. (D) 3:00 p.m.

13. A watch, which gains uniformly was observed to be 6 minutes slow at 9:00 a.m. on a Tuesday and 3 minutes fast at 12:00 noon on the subsequent Wednesday. When did the watch show the correct time?
(A) 9:00 p.m. on Tuesday
(B) 12:00 a.m. on Wednesday
(C) 3:00 a.m. on Wednesday
(D) 6:00 a.m. on Wednesday

14. The number of odd days in a non-leap year is
(A) 0 (B) 1 (C) 2 (D) 3

15. What will be next leap year after 2096?
(A) 2100 (B) 2101 (C) 2104 (D) 2108

16. If 21st March 2000 was a Monday, what day of the week will 21st March 2003 be?
(A) Tuesday (B) Friday
(C) Thursday (D) Wednesday

17. If 5th January 2001 was a Friday then what day of the weak will 25th December 2001 be?
(A) Monday (B) Tuesday
(C) Wednesday (D) Thursday

18. If 14th February 2001 was a Wednesday, then what day of the week will 14th February 2101 be (i.e., after a century)?

(A) Friday (B) Saturday
(C) Sunday (D) Monday

19. If 8th February 1995 was a Wednesday, then what day of the week will 8th February 1994 be?
(A) Wednesday (B) Thursday
(C) Tuesday (D) Monday

20. If holidays are declared only on Sundays and in a particular year 12th March is a Sunday, is 23rd September in that year a holiday?
(A) Yes
(B) No
(C) Yes, if it is a leap year.
(D) No, if it is a leap year.

21. Which day of the week was 1601, Jan 15?
(A) Monday (B) Tuesday
(C) Wednesday (D) Thursday

22. In a year, if 23rd November is a Friday then what day of the week will 14th March in that year be?
(A) Monday
(B) Wednesday
(C) Sunday
(D) Cannot be determined

23. The calendar of which of the following years is the same as that of the year 2001?
(A) 2005 (B) 2006 (C) 2007 (D) 2008

24. Pankaj met his friend three days ago. He told his friend that he has his last exam five days later. He met his friend again, three days after the last exam. Six days after he met his friend after the last exam, they left for a vacation. The day on which they left for a vacation is a Saturday. What is today?
(A) Saturday
(B) Tuesday
(C) Sunday
(D) Cannot be determined

25. Five days ago Shweta lost her phone. Two days after loosing the phone she lodged a complaint with the police. Six days after lodging the complaint she bought a new phone. Four days after buying a new phone i.e., on a Thursday she found her old phone. On which day did she loose her phone?
(A) Friday (B) Saturday
(C) Thursday (D) None of these

Previous Years' Questions

1. 25 persons are in a room. 15 of them play hockey, 17 of them play football and 10 of them play both hockey and football. The number of persons playing neither hockey nor football is _____. **[2010]**
(A) 2 (B) 17
(C) 13 (D) 3

2. The question below consists of a pair of related words followed by four pairs of words. Select the pair that best expresses the relation in the original pair.

Unemployed: Worker **[2010]**
(A) fallow: land (B) unaware: sleeper
(C) wit: jester (D) renovated: house

3. Hari (H), Gita (G), Irfan (I) and Saira (S) are siblings (i.e., brothers and sisters). All were born on 1st January, in different years. The age difference between any two successive siblings (that is born one after another) is less than 3 years. Given the following facts:

(i) Hari's age + Gita's age > Irfan's age + Saira's age

(ii) The age difference between Gita and Saira is 1 year. However Gita is not the oldest and Saira is not the youngest.

(iii) There are no twins.

Which of the following in a possible order in which they were born? **[2010]**
(A) HSIG (B) SGHI
(C) IGSH (D) IHSG

4. The question below consists of a pair of related words followed by four pairs of words. Select the pair that best expresses the relation in the original pair: **[2011]**
(A) Dancer : stage (B) Commuter : train
(C) Teacher : classroom (D) Lawyer : courtroom

5. The horse has played a little known but very important role in the field of medicine. Horse were injected with toxins of diseases until their blood built up immunities. Then a serum was made from their blood. Serums to fight with diphtheria and tetanus were developed this way.

It can be inferred from the passage, that horses were **[2011]**
(A) given immunity to diseases
(B) generally quite immune to diseases
(C) given medicines to fight toxins
(D) given diphtheria and tetanus serums

6. There are eight bags of rice looking alike, seven of which have equal weight and one is slightly heavier. The weighing balance is of unlimited capacity. Using this balance, the minimum number of weighing required to identify the heavier bag is **[2012]**
(A) 2 (B) 3
(C) 4 (D) 8

7. Statement: You can always give me a ring whenever you need.

Which one of the following is the best inference from the above statement? **[2013]**

(A) Because I have a nice caller tune.
(B) Because I have a better telephone facility.
(C) Because a friend in need is a friend indeed.
(D) Because you need not pay towards the telephone bills when you give me a ring.

8. Statement: There were different steams of freedom movements in colonial India carried out by the moderates, liberals, radicals, socialists, and so on

Which one of the following is the best inference from the above statement? **[2013]**

(A) The emergence of nationalism in colonial India led to our Independence.
(B) Nationalism in India emerged in the context of colonialism.
(C) Nationalism in India is homogeneous.
(D) Nationalism in India is heterogeneous.

9. The Palghat Gap (or Palakkad Gap), a region about 30 km wide in the southern part of the Western Ghats in India, is lower than the hilly terrain to its north and south. The exact reasons for the formation of this gap are not clear. It results in the neighboring regions of Tamil Nadu getting more rainfall from the South West monsoon and the neighboring regions of Kerala having higher summer temperatures.

What can be inferred from this passage? **[2014]**

(A) The Palghat gap is caused by high rainfall and high temperatures in Southern Tamil Nadu and Kerala.
(B) The regions in Tamil Nadu and Kerala that are near the Palghat Gap are near the low- lying.
(C) The low terrain of the Palghat Gap has a significant impact on weather patterns in neighboring parts of Tamil Nadu and Kerala.
(D) Higher summer temperatures result in higher rainfall near the Palghat Gap area.

10. Geneticists say that they are very close to confirming the genetic roots of psychiatric illnesses such as depression and Schizophrenia, and consequently, that doctors will be able to eradicate these diseases through early identification and gene therapy.

On which of the following assumptions does the statement above rely? **[2014]**

(A) Strategies are now available for eliminating psychiatric illnesses.
(B) Certain psychiatric illnesses have a genetic basis.
(C) All human diseases can be traced back to genes and how they are expressed.
(D) In the future, genetics will become the only relevant field for identifying psychiatric illness.

11. The old city of Koenigsberg, which had a German majority population before World War 2, is now called Kaliningrad. After the events of the war, Kaliningrad is now a Russian territory and has a predominantly Russian population. It is bordered by the Baltic Sea on the north and the countries of Poland to the South and West Lithuania to the east respectively. Which of the statements below can be inferred from this passage? **[2014]**

(A) Kaliningrad was historically Russian in its ethnic make up
(B) Kaliningrad is part of Russia despite it not being contiguous with the rest of Russia.
(C) Koenigsberg was renamed Kaliningrad, as that was its original Russian name
(D) Poland and Lithuania are on the route from Kaliningrad to the rest of Russia.

12. The number of people diagnosed with dengue fever (contracted from the bite of a mosquito) in north India is twice the number diagnosed last year. Municipal authorities have concluded that measures to control the mosquito population have failed in this region.

Which one of the following statements, if true, does not contradict this conclusion? **[2014]**

(A) A high proportion of the affected population has returned from neighbouring countries where dengue is prevalent.
(B) More cases of dengue are now reported because of an increase in the Municipal Office's administrative efficiency.
(C) Many more cases of dengue are being diagnosed this year since the introduction of a new and effective diagnostic test
(D) The number of people with malarial fever (also contracted from mosquito bites) has increased this year.

13. At what time between 6 am and 7 am, will the minute hand and hour hand of a clock make an angle closest to 60°? **[2014]**

(A) 6:22 am (B) 6:27 am
(C) 6:38 am (D) 6:45 am

14. Which number does not belong in the series below?

2, 5, 10, 17, 26, 37, 50, 64 **[2014]**

(A) 17 (B) 37
(C) 64 (D) 26

15. A dance programme is scheduled for 10:00 am. Some students are participating in the programme and they need to come an hour earlier than the start of the event. These students should be accompanied by a parent. Other students and parents should come in time for the programme. The instruction you think that is appropriate for this is **[2014]**

(A) Students should come at 9:00 am and parents should come at 10:00 am.
(B) Participating students should come at 9:00 am accompanied by a parent, and other parents and students should come by 10:00 am.

(C) Students who are not participating should come by 10:00 am and they should not bring their parents. Participating students should come at 9:00 am.

(D) Participating students should come before 9:00 am. Parents who accompany them should come at 9:00 am All others should come at 10:00 am.

16. By the beginning of the 20th century, several hypotheses were being proposed, suggesting a paradigm shift in our understanding, of the universe. However, the clinching evidence was provided by experimental measurements of the position of a star which was directly behind our sun. Which of the following inference(s) may be drawn from the above passage? **[2014]**

(i) Our understanding of the universe changes based on the positions of stars.

(ii) Paradigm shifts usually occur at the beginning of centuries.

(iii) Stars are important objects in the universe.

(iv) Experimental evidence was important in confirming this paradigm shift.

(A) (i), (ii) and (iv) (B) (iii) only
(C) (i) and (iv) (D) (iv) only

17. Based on the given statements, select the most appropriate option to solve the given question.

If two floors in a certain building are 9 feet apart, how many steps are there in a set of stairs that extends from the first floor to the second floor of the building? **[2015]**

Statements:

(I) Each step is $\frac{3}{4}$ foot high.

(II) Each step is 1 foot wide.

(A) Statement I alone is sufficient but statement II alone is not sufficient.

(B) Statement II alone is sufficient, but statement I alone is not sufficient.

(C) Both statements together are sufficient, but neither statement alone is sufficient.

(D) Statements I and II together are not satisfied.

18. The given statement is followed by some courses of action. Assuming the statement to be true, decide the correct option.

Statement:

There has been significant drop in the water level in the lakes supplying water to the city.

Course of action: **[2015]**

(I) The water supply authority should impose a partial cut in supply to tackle the situation.

(II) The government should appeal to all the residents through mass media for minimal use of water.

(III) The government should ban the water supply in lower areas.

(A) Statements I and II follow.
(B) Statements I and III follow.
(C) Statements II and III follow.
(D) All statements follow.

19. Based on the given statements, select the most appropriate option to solve the given question. What will be the total weight of 10 poles each of same weight? **[2015]**

Statements:

(I) One fourth of the weight of a pole is 5 Kg.

(II) The total weight of these poles is 160 kg more than the total weight of two poles.

(A) Statement I alone is not sufficient.
(B) Statement II alone is not sufficient.
(C) Either I or II alone is sufficient.
(D) Both statements I and II together are not sufficient.

20. Four branches of a company are located at M, N, O and P. M is north of N at a distance of 4 km; P is south of O at a distance of 2 km; N is southeast of O by 1 km. What is the distance between M and P in km? **[2015]**

(A) 5.34 (B) 6.74
(C) 28.5 (D) 45.49

21. Pick the odd one from the following options. **[2016]**

(A) CADBE (B) JHKIL
(C) XVYWZ (D) ONPMQ

22. Among 150 faculty members in an institute, 55 are connected with each other through Facebook® and 85 are connected through WhatsApp®. 30 faculty members do not have Facebook® or WhatsApp® accounts. The number of faculty members connected only through Facebook® accounts is _____ **[2016]**

(A) 35 (B) 45
(C) 65 (D) 90

23. Pick the odd one out in the following **[2016]**

13, 23, 33, 43, 53

(A) 23 (B) 33
(C) 43 (D) 53

Answer Keys

Exercise

Practice Problems

1. C	**2.** C	**3.** A	**4.** C	**5.** C	**6.** B	**7.** D	**8.** C	**9.** C	**10.** D
11. C	**12.** B	**13.** C	**14.** B	**15.** C	**16.** C	**17.** B	**18.** D	**19.** C	**20.** B
21. A	**22.** B	**23.** C	**24.** B	**25.** B					

Previous Years' Questions

1. D	**2.** A	**3.** B	**4.** A	**5.** B	**6.** A	**7.** C	**8.** D	**9.** C	**10.** B
11. B	**12.** D	**13.** A	**14.** C	**15.** B	**16.** D	**17.** A	**18.** A	**19.** C	**20.** A
21. D	**22.** A	23. B							

TEST
LOGICAL ABILITY

Time: 25 min.

Directions for questions 1 to 7: Complete the following series.

1. 11, 26, 51, 76, ____
(A) 101 (B) 115
(C) 125 (D) 133

2. 23, 57, 1113, 1719, ____
(A) 2329 (B) 2931
(C) 3137 (D) 3743

3. VIQ, TAC, WJR, VCE, XKS, XEG, _____
(A) YGL (B) ZFH
(C) YLT (D) YNR

4. 25 : 343 : : 49 : _____
(A) 121 (B) 343
(C) 512 (D) 1331

5. BIDM : DLPR : : HSBC : _____
(A) PVEH (B) PXDH
(C) PVHH (D) RVHD

6. 2Y5 : 4W9 : : 3J6 : _____
(A) 4W9 (B) 6L4
(C) 8C1 (D) 6N4

7. Aeroplane : Pilot : : Ship : _____
(A) Driver (B) Chef
(C) Captain (D) Marshal

Directions for questions 8 to 11: Find the odd man out.

8. (A) 38 – 121 (B) 48 – 144
(C) 68 – 196 (D) 98 – 361

9. (A) BDGC (B) DHKR
(C) FLOH (D) EJMZ

10. (A) 6V12 (B) 2H4
(C) 9F18 (D) 3R6

11. (A) Mercury (B) Mars
(C) Moon (D) Venus

Directions for questions 12 to 14: Choose the correct alternative from the given choices.

12. In a certain code language the word PRIVATE is coded as AEIPRTV then how is the word PRESENT coded in that language?
(A) EEPNRST (B) EENPRST
(C) EPSNERT (D) EENRPST

13. In a certain code, if the word CHLORATE is written as DFOKWUAW then how is the word PHOSPHATE written in that code?
(A) QFRUOBHLN (B) QFROBUHLN
(C) QFHROUBLN (D) QFROUBHLN

14. In a certain code, if the word PRESSURE is written as KIVHHFIV then how is the word SOLUTION written in that code?
(A) HLOUTRLM
(B) HLPGFRLM
(C) HLOFGRLM
(D) HLOGTROM

Directions for questions 15 and 16: These questions are based on the following information.

Six persons – P through U – are standing in a queue in the increasing order of their heights so that the shortest is at the front of the queue and the tallest is at the back. Further,

(i) U is the shortest.
(ii) Exactly two persons are taller than T.
(iii) P is taller than S and exactly two persons stand between P and S.
(iv) Q is taller than P.

15. Who is the second tallest person?
(A) T (B) R
(C) S (D) P

16. Who is/are the persons in between P and R?
(A) Only T
(B) Q and S
(C) T and U
(D) Only S

Directions for questions 17 to 19: These questions are based on the following data.

Eight persons – A, B, C, D, E, F, G and H attended a conference and are sitting around a circular table. Among them, there are C.E.Os of 4 companies who came along with one assistant each. Each C.E.O. has his assistant sitting to his right.

(i) Assistants of C and A are sitting opposite each other.
(ii) E, who is the assistant of B, is sitting opposite F.
(iii) E was not sitting adjacent to A.
(iv) G is neither adjacent nor opposite to D.

17. Who is to the left of A?
(A) D (B) G
(C) F (D) H

18. Who is the assistant of C?
(A) D (B) G
(C) E (D) H

19. If H is opposite to G, then D is to the right of ____.
(A) A (B) B
(C) F (D) H

Directions for questions 20 to 24: These questions are based on the following data.

In a class, 50 students failed in Maths.

40 students failed in Physics.

30 students failed in Chemistry.

10 students failed in Physics and Chemistry.

10 students failed in Maths and Physics.

No student failed in both Maths and Chemistry.

None of the students failed in all the three subjects.

20. How many students failed atleast in one subject?
(A) 50 (B) 100
(C) 75 (D) 125

21. What is the ratio of the number of students who failed in Maths and Physics to that who failed in Physics and Chemistry?
(A) 1 : 2 (B) 2 : 1
(C) 1 : 1 (D) 4 : 3

22. How many students failed in exactly two subjects?
(A) 10 (B) 20
(C) 30 (D) 40

23. The number of students who failed in only Maths, in only Physics and in only Chemistry respectively is
(A) 40, 20, 20
(B) 20, 40, 20
(C) 20, 20, 40
(D) 50, 40, 30

24. Which of the following statements is true?
(A) The number of students who failed in only Maths equals to that of the students who failed in only Physics.
(B) The number of students who failed in only Maths equals to that of the students who failed in only Physics or in only Chemistry.
(C) The number of students who failed in all the three subjects is 10.
(D) None of these.

Directions for question 25 to 30: select the correct alternative from the given choices.

25. In the following addition each latter represents a different digit from of O to 9. which of the following is a possible number represented by FAN ?

$$\begin{array}{cccc} & H & A & N \\ F & A & F & N \\ \hline 5 & 5 & 8 & 8 \end{array}$$

(A) 434 (B) 534
(C) 345 (D) 135

26. At what time between 10 O' clock and 11 O' clock are the hands of the clock together?
(A) $54\frac{6}{11}$ past 10 (B) $27\frac{8}{11}$ past 10
(C) Both A and B (D) None of these

27. How many times the hands of a clock are at right angles in a day?
(A) 24 (B) 22
(C) 44 (D) 48

28. On a particular day if it is found that a clock is showing 10 minutes less at 1:00 pm and 5 minutes more at 6:00 pm on the same day. At what time did the clock show the correct time?
(A) 3 hr 20 min (B) 4 hr 20 min
(C) 5 hr 40 min (D) 6 hr 40 min

29. If 21st March 2000 was a Monday, which day of the week will be 21st March 2003?
(A) Tuesday (B) Friday
(C) Thursday (D) Wednesday

30. The movie of my favorite hero is going to be released on Wednesday. To watch the movie on the first day of release, I booked my ticket the day before yesterday. If I watch the movie on the fourth day from today, on which day of the week did I book my ticket?
(A) Wednesday (B) Thursday
(C) Friday (D) Saturday

Answer Keys

1. B	**2.** A	**3.** C	**4.** D	**5.** C	**6.** B	**7.** C	**8.** D	**9.** A	**10.** A
11. C	**12.** B	**13.** D	**14.** C	**15.** D	**16.** A	**17.** C	**18.** D	**19.** D	**20.** B
21. C	**22.** B	**23.** A	**24.** B	**25.** B	**26.** A	**27.** C	**28.** B	**29.** C	**30.** B

Engineering Mathematics

PART II

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Chapter 1

Linear Algebra

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Matrix
- Type of matrices
- Determinants
- Minor and cofactor of a matrix
- Singular and non-singular matrices
- Adjoint of a matrix
- Evaluating inverse of a square matrix
- Elementary operations (or) elementary transformations
- Systems of linear equations
- Cramers method
- Inverse method
- Gauss–Jordan method
- Gauss elimination method
- The characteristic equation of a matrix
- Caley–Hemilton theorem

Introduction

A set of 'mn' elements arranged in the form of rectangular array having 'm' rows and 'n' columns is called an $m \times n$ matrix (read as "m by n matrix") and is denoted by $A = [a_{ij}]$ where $1 \le i \le m$; $1 \le j \le n$

or
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{pmatrix}$$

The element a_{ij} lies in the i-th row and j-th column.

Type of Matrices

Square matrix A matrix $A = [a_{ij}]_{m\times n}$ is said to be a square matrix, if $m = n$ (i.e., No. of rows of A = No. of columns of A)

The elements a_{11}, a_{22}, a_{33}, …, a_{nn} are called 'DIAGONAL ELEMENTS'.

The line containing the diagonal elements is the 'PRINCIPAL DIAGONAL'.

The sum of the diagonal elements of 'A' is the 'TRACE' of A.

Row matrix A matrix $A = [a_{ij}]_{m\times n}$ is said to be row matrix, if $m = 1$ (i.e., the matrix has only one row)

General form is $A = [a_1, a_2, \ldots, a_n]$ or $[a_{ij}]_{1\times n}$

Column matrix A matrix which has only one column is called column matrix.

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ or } [a_{ij}]_{n\times 1}$$

Diagonal matrix A square matrix is said to be a diagonal matrix if all its elements except those in the principal diagonal are zeros. That is, if

1. $m = n$ (A is a square matrix) and
2. $a_{ij} = 0$ if $i \ne j$ (The non-diagonal elements are zeros)

A diagonal matrix of order 'n' with diagonal elements $d_1, d_2, \ldots, d_n$ is denoted by Diag $[d_1\ d_2 \ldots d_n]$.

Scalar matrix A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. That is, if

1. $m = n$
2. $a_{ij} = 0$ if $i \ne j$
3. $a_{ij} = k$ if $i = j$ for some constant 'k'.

Unit or identity matrix A scalar matrix of order 'n' in which each diagonal elements is '1' (unity) is called a unit matrix or identity matrix of order 'n' and is denoted by I_n. That is,

1. $m = n$
2. $a_{ij} = 0$ if $i \ne j$
3. $a_{ij} = 1$ if $i = j$

Null matrix or zero matrix A matrix is a 'null matrix' or zero matrix if all its elements are zeros.

Upper triangular matrix A square matrix is said to be an upper triangular matrix, if each element below the principal diagonal is zero. That is,

1. $m = n$
2. $a_{ij} = 0$ if $i > j$

Lower triangular matrix A square matrix is said to be a lower triangular matrix, if each element above the principal diagonal is zero, i.e., if

1. $m = n$
2. $a_{ij} = 0$ if $i < j$

Horizontal matrix If the number of rows of a matrix is less than the number of columns, i.e., $m < n$, then the matrix is called horizontal matrix.

Vertical matrix If the number of columns in a matrix is less than the number of rows, i.e., if $m > n$, then the matrix is called a vertical matrix.

Comparable matrices Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are said to be comparable, if they are of same order, i.e., $m = p$; $n = q$.

Equal matrices Two comparable matrices are said be 'equal', if the corresponding elements are equal, i.e., $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are equal if

1. $m = p$; $n = q$ (i.e., they are of the same order)
2. $a_{ij} = b_{ij}\ \forall\ i, j$ (i.e., the corresponding elements are equal)

Transpose of a matrix

The matrix obtained by interchanging the rows and the columns of a given matrix 'A' is called the 'transpose' of A and is denoted by A^T or A'. If A is an $(m \times n)$ matrix, A^T will be an $(n \times m)$ matrix. Thus if $A = [a_{ij}]_{m\times n}$ then $A^T = [u_{ij}]_{n\times m}$, where $u_{ij} = a_{ji}$.

Properties of transpose

$T - 1$: $(A')' = A$, for any matrix A
$T - 2$: $(A + B)' = A' + B'$, for any two matrices A, B of same order
$T - 3$: $(KA)' = KA'$, for any matrix A
$T - 4$: $(AB)' = B'A'$, for any matrices A, B such that number of columns of A = number of rows of B (REVERSAL LAW)
$T - 5$: $(A^n)' = (A')^n$, for any square matrix A

Trace of a matrix

Let 'A' be a square matrix. The trace of A is defined as the sum of elements of 'A' lying in the principal diagonal.

Thus if $A = [a_{ij}]_{n\times n}$ then trace of 'A' denoted by $t_r A = a_{11} + a_{22} + \cdots + a_{nn}$.

Properties of trace of a matrix Let A and B be any two square matrices and K any scalar then,

1. $t_r(A + B) = t_r A + t_r B$
2. $t_r(KA) = Kt_r A$
3. $t_r(AB) = t_r(BA)$

Conjugate of a matrix

A matrix obtained by replacing each element of a matrix 'A' by its complex conjugate is called the 'conjugate matrix' of A and is denoted by $\bar{A}$. If $A = [a_{ij}]_{m\times n,}$ then $\bar{A} = \left[\overline{a_{ij}}\right]$ where $\bar{a}_{ij}$ is the conjugate of 'a_{ij}'.

Properties of conjugate of a matrix

$C - 1\overline{(\overline{(A)})} = A$ for any matrix 'A'
$C - 2\ (\overline{A + B}) = \bar{A} + \bar{B}$ for any matrices A, B of same order.
$C - 3\ (\overline{KA}) = \bar{K}\ \bar{A}$ for any matrix 'A' and any Scalar K.
$C - 4\,(\overline{AB}) = (\bar{A})\cdot\bar{B}$ for any matrices A and B with the condition that number of columns of A = number of rows of B.
$C - 5(\bar{A})^n = \left(\overline{A^n}\right)$ for any square matrix 'A'.

Tranjugate or transposed conjugate of a matrix

Tranjugate of a matrix 'A' is obtained by transposing the conjugate of A and is denoted by A^0. Thus $A^0 = (\bar{A})^T$.

Properties of tranjugate of a matrix

TC – 1 $(A^\theta)^\theta = A$ for any matrix A
TC – 2 $(A + B)^\theta = A^\theta + B^\theta$ for any matrices A, B of the same order.
TC – 3 $(KA)^\theta = KA^\theta$ for any matrix A and any scalar K.
TC – 4 $(BA)^\theta = B^\theta A^\theta$ for any matrix A, B with the condition that number of columns of A = number of rows of B.
TC – 5 $(A^n)^\theta = (A^\theta)^v$ for any square matrix 'A'.

Symmetric matrix A matrix A is said to be symmetric, if $A^T = A$ (i.e., transpose of $A = A$).

Note: A symmetric matrix must be a square matrix.

Skew-symmetric matrix A matrix 'A' is said to be skew-symmetric matrix, if $A^T = (-A)$, i.e., $A = [a_{ij}]_{m\times n}$ is skew symmetric if

1. $m = n$
2. $a_{jI} = -a_{ij}\ \forall\ i, j$

Note: In a skew-symmetric matrix, all the elements of the principal diagonal are zero.

Orthogonal matrix A square matrix 'A' of order $n \times n$ is said to be an orthogonal matrix, if $AA^T = A^TA = I_n$.

Involutory matrix A square matrix 'A' is said to be involutory matrix, if $A^2 = I$ (where I is identity matrix).

Idempotent matrix A square matrix 'A' is said to be an idempotent matrix, if $A^2 = A$.

Nilpotent matrix A square matrix 'A' is said to be nilpotent matrix, if there exists a natural number 'n' such that $A^n = O$. If 'n' is the least natural number such that $A^n = O$, then 'n' is called the index of the nilpotent matrix 'A'. (Where O is the null matrix).

Unitary matrix A square matrix 'A' is said to be a unitary matrix if, $AA^\theta = A^\theta A = I$. (Where A^θ is the transposed conjugate of A.)

Hermitian matrix A matrix 'A' is said to be a hermitian matrix, if $A^\theta = A$, i.e., $A = [a_{ij}]_{m\times n}$ is hermitian if

1. $m = n$
2. $a_{ij} = \overline{a}_{ij} \forall i, j$

Note: The diagonal elements in a hermitian matrix are real numbers.

Skew-hermitian matrix A matrix 'A' is said to be a skew-hermitian matrix, if $A^\theta = -A$.

Operations on Matrices

Scalar multiplication of matrices

If A is a matrix of order $m \times n$ and 'K' be any scalar (a real or complex number), then KA is defined to be a $m \times n$ matrix whose elements are obtained by multiplying each element of 'A' by K, i.e., if $A = [a_{ij}]_{m\times n}$ then $KA = [Ka_{ij}]_{m\times n}$ in particular if $K = -1$; then $KA = -A$ is called the negative of A and is such that,

$A + (-A) = [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] = [0] = O$ (zero matrix)

$(-A) + A = [-a_{ij}] + [a_{ij}] = [-a_{ij} + a_{ij}] = [0] = O$

That is, $A + (-A) = (-A) + A = O$.

Properties of scalar multiplication

Let A, B are two matrices of same order and α, β are any scalars, then

S – 1: $\alpha(A + B) = \alpha A + \alpha B$

S – 2: $(\alpha + \beta)A = \alpha A + \beta A$

S – 3: $\alpha(\beta A) = (\alpha\beta)A$

S – 4: $1A = A$

Addition of matrices

If A and B are two matrices of the same order, then they are 'conformable' for addition and their sum '$A + B$' is obtained by adding the corresponding elements of A and B, i.e., if $A = [a_{ij}]_{m\times n}$; $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

Properties of addition Let A, B and C be three matrices of same order say $m \times n$, then

$A - 1$: $A + B$ is also a $m \times n$ matrix (CLOSURE)

$A - 2$: $(A + B) + C = A + (B + C)$ (ASSOCIATIVITY)

$A - 3$: If 'O' is the $m \times n$ zero (null) matrix, then $A + O = O + A = A$ ('O' is the ADDITIVE IDENTITY)

$A - 4$: $A + (-A) = (-A) + A = O$ ($-A$ is the ADDITIVE INVERSE)

$A - 5$: $A + B = B + A$ (COMMUTATIVITY)

Note: The set of matrices of same order form an 'Abelian Group' under addition.

Multiplication of matrices

Let A and B be two matrices. A and B are conformable for multiplication, only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix, $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product 'AB' is defined as the matrix $C = [c_{ik}]$ of order $m \times p$ where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$

c_{ij} calculated for $i = 1,2,\ldots m$ and $k = 1,2,\ldots m$ will give all the elements of the matrix C.

Properties of multiplication

$M - 1$: If A, B, C be $m \times n$, $n \times p$, $p \times q$ matrices respectively, then $(AB)C = A(BC)$ (ASSOCIATIVITY).

$M - 2$: If A is a $m \times n$ matrix, then $A\,I_n = A$ and $I_m\,A = A$ and if A is a square matrix, i.e., $m = n$, then $AI = IA = A$ (I is the MULTIPLICATIVE IDENTITY).

$M - 3$: If A, B, C be $m \times n$, $n \times p$, $p \times q$ matrices respectively, then $A(B + C) = AB + AC$ (DISTRIBUTIVE LAW).

$M - 4$: Matrix multiplication is NOT COMMUTATIVE in general.

$M - 5$: The INVERSE of a given matrix may not always exist.

Determinants

Let $A = [a_{ij}]$ be a square matrix of order 'n'. Then the determinant of order 'n' associated with 'A' is denoted by $|A|$ or $|a_{ij}|$ or Det(A) or Δ.

Notes:

1. Determinant of a matrix exists, only if it is a square matrix.
2. The value of a determinant is a single number.

Determinant of order 1 (first order determinant)

If 'A' is a square matrix of order 2 given by $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ then Det $(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is determinant of order 2 and its value is $\Delta = a_1b_2 - a_2b_1$

Minor and cofactor of a matrix

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be a 3×3 matrix

Then the minor of an element a_{ij} of 'A' is the determinant of the 2×2 matrix obtained after deleting the i-th row and j-th column of A and is denoted by M_{ij}.

The cofactor of a_{ij} small size is denoted by A_{ij} and is defined as $(-1)^{i+j} M_{ij}$, i.e., $A_{ij} = (-1)^{i+j} M_{ij}$

Determinant of order 2 (second order determinant)

If A is a square matrix of order '3', given by $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Then the determinant of 'A' is given by

$\Delta = \text{Det } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a determinant of order 3 and the value is obtained by taking the sum of the products of the elements of any row (or column) by their corresponding cofactors.

Thus for A, $\Delta = a_1A_1 + b_1B_1 + c_1C_1$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

or also $\Delta = a_1A_1 + a_2A_2 + a_3A_3$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

(This is by expanding by C_1) and so on.

Properties of determinant

1. If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by (−1).
2. If the rows and columns of a determinant are interchanged, the value of the determinant remains unchanged, i.e., Det(A) = Det(A^T).
3. If all the elements of a row (or column) of a determinant are multiplied by a scalar (say 'K'), the value of the new determinant is equal to 'K' times the value of the original determinant.
4. If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
5. If the elements of a row (or a column) in a determinant are proportional to the elements of any other row (or column), then the determinant is '0'.
6. If every element of any row (or column) is zero, then determinant is '0'.
7. If each element in a row (or column) of a determinant is the sum of two terms, then its determinant can be expressed as the sum of two determinants of the same order.
8. (The theorem of 'false cofactor') The sum of products of elements of a row (or column) with the cofactors of any other row (or column) is zero.

   Thus in $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

   $a_1A_2 + b_1B_2 + c_1C_2 = 0$

   $a_2A_1 + b_2B_1 + c_2C_1 = 0$ and so on in general

   $a_rA_s + b_rB_s + c_rC_s = 0$ if $r \neq s$
9. If the elements of a determinant are polynomials in x and the determinant vanishes for $x = a$, then $x - a$ is a factor of the determinant.

Singular and non-singular matrices

A square matrix 'A' is said to be singular, if Det(A) = 0 and is non-singular, if Det(A) ≠ 0.

Notes:

1. A unit matrix is non-singular (since its Det = 1)
2. If A and B are non-singular matrices of the same 'type', then AB is non-singular of the same 'type'.

Inverse of a matrix

Let 'A' be a square matrix. A matrix 'B' is said to be an inverse of 'A', if $AB = BA = I$.

Note: If B is the inverse of 'A', then 'A' is the inverse of 'B'.

Some results of inverse

1. Inverse of a square matrix, when it exists, is unique.
2. The inverse of a square matrix exists, if and only if it is non-singular.
3. If 'A' and 'B' are square matrices of the same order, then 'AB' is invertible (i.e., inverse of AB exists) if 'A' and 'B' are both invertible.
4. If 'A' and 'B' are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$.
5. If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.
6. If A is invertible, then so is A^θ and $(A^\theta)^{-1} = (A^{-1})^\theta$.

Adjoint of a matrix

The adjoint of a square matrix 'A' is the transpose of the matrix obtained by replacing the elements of 'A' by their corresponding cofactors.

Note: The adjoint is defined only for square matrices and the adjoint of a matrix 'A' is denoted by Adj(A). If $A = \begin{pmatrix} a_1 & a_2 \cdots & a_n \\ b_1 & b_2 \cdots & b_n \\ \cdots & \cdots & \cdots \\ \ell_1 & \ell_2 & \ell_n \end{pmatrix}$

$$\text{Adj } A = \begin{pmatrix} A_1 & A_2 \cdots A_n \\ B_1 & B_2 \cdots B_n \\ \cdots\cdots & \\ \cdots\cdots & \\ L_1 & L_2 \cdots L_n \end{pmatrix}^T = \begin{pmatrix} A_1 & B_1 \cdots L_1 \\ A_2 & B_2 \cdots L_2 \\ \cdots\cdots & \\ A_n & B_n \cdots L_n \end{pmatrix}$$

Results

1. If 'A' is of order 3×3 and K is any number, then $\text{Adj}(KA) = K^2(\text{Adj } A)$.
2. $A(\text{Adj } A) = (\text{Adj } A)A = |A|\, I$ for any square matrix 'A'.
3. Adj $I = I$; Adj $O = O$ where I is the identity matrix and O is the null matrix.
4. Adj(AB) = (Adj B) (Adj A) if A, B are non-singular and are of same type.
5. If $A = A_{n\times n}$, then
   $\det(\text{Adj } A) = (\det A)^{n-1}$.
   $\text{Adj}(\text{Adj } A) = (\det A)^{n-2}(A)$.
   $|\text{Adj (Adj } A)| = (\det A)^{(n-1)^2}$

Evaluating inverse of a square matrix

If A is a square matrix, then $A^{-1} = \dfrac{1}{|A|}(\text{Adj } A)$

Notes:

1. The inverse of an identity matrix is itself.
2. $(\text{Adj } A)^{-1} = \dfrac{1}{|A|} A$
3. If A is a non-singular square matrix (say of order 3) and K is any non-zero number, then
$$(KA)^{-1} = \frac{1}{K} A^{-1}$$

Rank and nullity of a matrix

Rank of a matrix The Matrix 'A' is said to be of rank 'r', if and only if it has at least one non-singular square sub-matrix of order 'r' and all square sub-matrices of order $(r + 1)$ and higher orders are singular. The rank of a matrix A is denoted by rank (A) or $\rho(A)$.

Nullity of a matrix If A is a square matrix of order 'n', then $n - \rho(A)$, i.e., n – rank (A) is defined as nullity of matrix 'A' and is denoted by $N(A)$.

Remark 1: If there is a non-singular square sub-matrix of order 'K', then $\rho(A) \geq K$.

Remark 2: If there is no non-singular square sub-matrix of order 'K', then $\rho(A) < K$.

Remark 3: If A^I is the transpose of A, then $\rho(A) = \rho(A^1)$.

Remark 4: The rank of a null matrix is '0'.

Remark 5: The rank of a non-singular square matrix of order 'n' is 'n' and its nullity is '0'.

Remark 6: Elementary operations do not change the rank of a matrix.

Remark 7: If the product of two matrices A and B is defined, then $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$. That is, the rank of product of two matrices cannot exceed the rank of either of them.

Elementary Operations or Elementary Transformations

(a) Elementary row operations
   (i) $R_i \leftrightarrow R_j$: Interchanging of ith and jth rows
   (ii) $R_i \to KR_i$: Multiplication of every element of ith row with a non-zero scalar K
   (iii) $R_i \to T_i + KR_j$: Addition of k times the elements of jth row to the corresponding elements of ith row.

(b) Elementary column operations
   (i) $C_i \leftrightarrow C_j$: Interchanging of ith and jth columns
   (ii) $C_i \to KC_i$: Multiplication of every element of ith column with a non-zero scalar K.
   (iii) $C_i \to C_i + KC_j$: Addition of K times the elements of jth column to the corresponding elemnt of ith column.

Consider the matrix $A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ 3 & 0 & 1 & 5 \\ 4 & 7 & 1 & 2 \end{bmatrix}$

$$R_2 \to 2R_2 \sim \begin{bmatrix} 2 & 3 & -4 & 1 \\ 6 & 0 & 2 & 10 \\ 4 & 7 & 1 & 2 \end{bmatrix}$$

$$C_2 \leftrightarrow C_3 \sim \begin{bmatrix} 2 & -4 & 3 & 1 \\ 3 & 1 & 0 & 5 \\ 4 & 1 & 7 & 2 \end{bmatrix}$$

$$C_1 \to C_1 - 2C_4 \sim \begin{bmatrix} 0 & -4 & 3 & 1 \\ -7 & 1 & 0 & 5 \\ 0 & 1 & 7 & 2 \end{bmatrix}$$

Note: The rank of a matrix is invariant under elementary operations

Row and column equivalence matrices

Row equivalence matrix If B is a matrix obtained by applying a finite number of elementary row operations successively on matrix A, then matrix B is said to be row equivalent to A (or a row equivalent matrix of A).

Column equivalence matrix If B is obtained by applying a finite number of elementary column operations successively on matrix A, then matrix B is said to be column equivalent to A (or a column equivalent matrix of A).

Example: $A=\begin{bmatrix}1 & 3 & 4\\ 2 & 5 & -2\\ 1 & 4 & -3\end{bmatrix}$

$$R_2-2R_1\begin{bmatrix}1 & 3 & 4\\ 0 & -1 & -10\\ 0 & 1 & -7\end{bmatrix}=B(\text{say})$$

B is a row equivalent matrix of A.

Example: $B=\begin{bmatrix}1 & 3 & 2\\ 3 & 4 & -4\\ 1 & 1 & 6\end{bmatrix}C_2-3C_1,$

$$\frac{C_3}{2}\to\begin{bmatrix}1 & 0 & 1\\ 3 & -5 & -2\\ 1 & -2 & 6\end{bmatrix}=C(\text{say})$$

C is a column equivalent to B.

Row reduced matrix A matrix A of order $m \times n$ is said to be row reduced if,

(a) The first non-zero element of a non-zero row is 1.
(b) Every other element in the column in which such 1's occur is 0.

$$A=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ 0 & 0 & 0\end{pmatrix}\text{ is a row reduced matrix}$$

$$B=\begin{pmatrix}1 & 0 & 4\\ 0 & 5 & 0\\ 0 & 0 & 0\end{pmatrix}\text{ is not a row reduced matrix.}$$

Row reduced echelon matrix A matrix 'X' is said to be row reduced echelon matrix if,

(a) X is row reduced.
(b) There exists integer $\underline{P}(0 \le p \le m)$ such that first 'p' rows of X are non-zero and all the remaining rows are zero rows.
(c) For the ith non-zero row, if the first non-zero element of the row (i.e., 1)occurs in the jth column then, $j_1 < j_2 < j_3 < \cdots < j_p$.

Example: $P=\begin{pmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\end{pmatrix};\ Q=\begin{pmatrix}0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$

are echelon matrices. The number of non-zero rows (i.e., value of P and Q) are 3 and 2 respectively. The value of i and j are tabulated below

P:	i	1	2	3
	j	1	2	3

Q:	i	1	2
	j	1	4

Normal form of a matrix

By means of elementary transformations, every matrix 'A' of order $m \times n$ and rank r (> 0) can be reduced to one of the following forms.

(i) $\begin{pmatrix}I_r & 0\\ 0 & 0\end{pmatrix}$
(ii) $[I_r/0]$
(iii) $[\underline{I_r}]$

and these are called the normal forms. I_r is the unit matrix of order 'r'.

Note: If a $m \times n$ matrix 'A' has been reduced to the normal form say $\begin{pmatrix}I_r & 0\\ 0 & 0\end{pmatrix}$ then 'r' is the rank of A.

Systems of Linear Equations

Let
$$\left.\begin{aligned}&a_{11}x_1+a_{12}x_2+\cdots a_{1n}X_n=b_1\\ &a_{12}x_1+a_{22}x_2+\cdots a_{2n}X_n=b_2\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\\ &a_{n1}x_1+a_{n2}x_2+\cdots a_{nn}x_n=b_n\end{aligned}\right\}\qquad(1)$$

be a system of 'n' linear equations in 'n' variables $x_1, x_2, \ldots, x_n$. The above system of equations can be written as

$$\begin{pmatrix}a_{11} & a_{12}\cdots a_{1n}\\ a_{21} & a_{22}\cdots a_{2n}\\ \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots\\ a_{n1} & a_{2n}\cdots a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \cdots\\ \cdots\\ x_n\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\\ \cdots\\ \cdots\\ b_n\end{pmatrix}\quad\text{or}\quad AX=B$$

where

$$A=\begin{pmatrix}a_{11} & a_{12}\cdots a_{1n}\\ a_{21} & a_{22}\cdots a_{2n}\\ \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots\\ a_{m1} & a_{m2}\cdots a_{mn}\end{pmatrix},\ X=\begin{pmatrix}x_1\\ x_2\\ \cdots\\ \cdots\\ x_n\end{pmatrix},\ B=\begin{pmatrix}b_1\\ b_2\\ \cdots\\ \cdots\\ b_m\end{pmatrix}$$

A is called the co-efficient matrix.

Any set of values of $x_1, x_2, x_3, \ldots$ which simultaneously satisfy these equations is called a solution of the system. When the system of equations has one or more solutions, the equations are said to be CONSISTENT and the system of equations are said to be INCONSISTENT if it does not admit any solution. The system of equation (1) is said to be

HOMOGENEOUS, if B = 0
NON-HOMOGENEOUS, if $B \neq 0$

The matrix $\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} \cdots & a_{2n} & b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \cdots a_{mn} & b_m \end{pmatrix}$ is called the augmented matrix of the system of equations and is denoted by $[A : B]$.

Let $AX = B$ represents 'm' linear equations with 'n' variables. Let rank of $A = r$ and rank $(A, B) = r_1$ [where (A, B) is an augmented matrix]. If $r_1 \neq r$, then the system of equations are inconsistent.

If $r_1 = r$, the table follows:

	$m = n$		$m > n$		$m < n$	
	$r = n$	$r < n$	$r = n$	$r < n$	$r = m$	$r < m$
Homo geneous	Only trivial solution	Infinite solutions	Only trivial solution	Infinite solutions	Infinite solutions	Infinite solutions
Non-homo geneous	Unique solution	Infinite solutions	Unique solution	Infinite solutions	Infinite solutions	Infinite solutions

The characteristic equation of a matrix

Characteristic matrix If A is any square matrix, the matrix $A - \lambda I$ where λ is a scalar, is called the characteristic matrix of A.

Characteristic polynomial If A is any square matrix of order n, then the determinant $|A - \lambda I|$ yields a polynomial $\phi(\lambda)$ of degree n in λ which is known as the characteristic polynomial of the matrix A.

Characteristic equation If $\phi(\lambda)$ is the characteristic polynomial of a matrix A, then $\phi(\lambda) = 0$, is called the characteristic equation of A.

And the roots of this equation, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the characteristic roots or latent roots or eigen values. If λ is a characteristic root of ordert, then t is called the algebraic multiplicity of λ.

Characteristic vectors Corresponding to each characteristic root λ, there is a non-zero vector which satisfies the characteristic equation $|A - \lambda I| = 0$. These non-zero vectors are called the characteristic vectors or eigen vectors or latent vectors.

Notes:

1. The characteristic roots of a matrix and its transpose are the same.
2. 0 is the characteristic roots of a matrix, if the matrix is singular.
3. The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.
4. If K is any scalar, the characteristic roots of matrix KA are K times the characteristic roots of matrix A.
5. If $a_1, a_2, a_3, \ldots, a_n$ are characteristic roots of matrix A and K is a scalar, then the characteristic roots of matrix $A - KI$ are $a_1 - K, a_2 - K, \ldots, a_n - K$.
6. If λ is a characteristic root of a non-singular matrix, then λ^{-1} is a characteristic root of A^{-1}.
7. If the eigen values of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$.

Caley–Hemilton Theorem

Every square matrix satisfies its characteristic equation.

Solved Examples

Example 1: Find the value of

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

Solution: $c_1 \to c_1 + c_2 + c_3$

$$\begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$R_2 \to R_2 - R_1$$

$$R_3 \to R_3 - R_1$$

$$2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)^3 \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(a+b+c)^3.$$

Example 2: How many solutions are there for the system of linear equations $x + 2y + z = 0$, $3x + 2y - z = 0$ and $4x + y - 3z = 0$?

Solution: Determinant of the co-efficient matrix of the given equation is $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 4 & 1 & -3 \end{vmatrix}$

$$= 1(-6 + 1) - 2(-9 + 4) + 1(3 - 8) = 0$$

$\therefore$ The system has infinite number of solutions.

Example 3: Solve: $x + y + z = 6$, $3x - 2y - z = -4$ and $2x + 3y - 2z = 2$.

Solution:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 3 & -2 \end{vmatrix} = 1(7) - 1(-4) + 1(9+4) \neq 0$$

$\therefore$ The set of given equations are non-homogeneous and the number of equations is equal to the number of variables.

$\therefore$ The given system of equations is consistent and has a unique solution.

Augmented matrix

$$[AB] \text{ is } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & -1 & -4 \\ 2 & 3 & -2 & 2 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, \text{ and } R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & -4 & -22 \\ 0 & 1 & -4 & -10 \end{bmatrix}$$

$$R_1 \to R_1 + \frac{1}{5}R_2 \text{ and } R_3 \to R_3 + \frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{8}{5} \\ 0 & -5 & -4 & -22 \\ 0 & 0 & -\frac{24}{5} & -\frac{72}{5} \end{bmatrix}$$

$$R_2 \to \frac{-5}{6}R_3 + R_2;\ R_3 \to \frac{-5}{24}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{8}{5} \\ 0 & -5 & 0 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$R_1 \to -\frac{1}{5}R_3 \text{ and } R_2 \to \frac{-1}{5}R_2$$

$$R_2 \to -\frac{1}{5}R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ Solution is $x = 1$, $y = 2$ and $z = 3$.

Example 4: Determine the eigen values and eigen vectors of $A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$.

Solution: Characteristic equation of the given matrix is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$$
$$(\lambda - 6)(\lambda + 1) = 0$$

$\Rightarrow$ $\lambda = -1$ and $\lambda = 6$ are the eigen values. Eigen vector corresponding to $\lambda = -1$ is obtained as follows:

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 3x_1 + 4x_2 = 0$$

$$3x_1 + 4x_2 = 0 \Rightarrow x_1 = -4/3x_2$$

$\therefore$ Eigen vector corresponding to $\lambda = -1$ is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{-4}{3}x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{-4}{3} \\ 1 \end{pmatrix}$$

Similarly eigen vector corresponding to $\lambda = 6$ is obtained as follows:

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -4x_1 + 4x_2 = 0 \text{ and } 3x_1 - 3x_2 = 0 >$$

$$\Rightarrow \quad x_1 = x_2$$

Eigen vector corresponding to $\lambda = 6$ is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 5: Find the rank of the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -1 \\ 1 & 4 & 1 \end{bmatrix}$.

Solution: Given

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -1 \\ 1 & 4 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 4 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 \to R_1$ and $R_3 \to \mathrm{P}_{3-3}R_1$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & -8 & -3 \\ 0 & -11 & -5 \end{bmatrix} R_3 \to R_3 \frac{-11}{8} R_2 \begin{bmatrix} 1 & 4 & 1 \\ 0 & -8 & -3 \\ 0 & 0 & \frac{18}{5} \end{bmatrix}$$

which is a row echelon form. The number of non zero rows = 3.

The rank of the matrix = The number of non-zero rows in it = 3

$\therefore$ Rank of the matrix = 3.

Example 6: Find whether the vectors given below are linearly dependent or independent $\{(1, 3, 2), (1, -4, 1), (-1, 2, 5)\}$.

Solution: Let $x, y, z \in R$ such that $x(1, 3, 2) + y(1, -4, 1) + z(-1, 2, 5) = (0, 0, 0)$

$$\left.\begin{aligned} & x + y - z = 0 \\ \Rightarrow \quad & 3x - 4y + 2z = 0 \\ & 2x + y + 5z = 0 \end{aligned}\right\} \qquad (2)$$

The above system of equations when expressed in determinant form, we have

$$\begin{vmatrix} 1 & 1 & -1 \\ 3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} \xrightarrow{R_2 - 3R_1,\ R_3 - 2R_1} \begin{vmatrix} 1 & 1 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 7 \end{vmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{7}R_2} \begin{vmatrix} 1 & 1 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{44}{7} \end{vmatrix}$$

$\therefore$ Rank = 3 = number of unknowns

$\therefore$ There exists a unique solution $x = 0, y = 0$ and $z = 0$

$\Rightarrow \quad x(1, 3, 2) + y(1, -4, 1) + z(-1, 2, 5)$

$= (0, 0, 0)$ and $x = 0, y = 0, z = 0$

$\therefore$ The set of vectors are linearly independent.

Example 7: If $A = \begin{bmatrix} 4 & 2 \\ -7 & -4 \end{bmatrix}$, then find A^{16} by using Cayley–Hamilton theorem.

Solution: The characteristic equation of $A = \begin{bmatrix} 4 & 2 \\ -7 & -4 \end{bmatrix}$ is

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{bmatrix} 4 & -\lambda & 2 \\ -7 & -4 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (4 - \lambda)(-4 - \lambda) + 14 = 0$$

$$\Rightarrow \quad -16 - 4\lambda + 4\lambda + \lambda^2 - 14 = 0$$

$$\Rightarrow \quad \lambda^2 - 2 = 0 \qquad (3)$$

By Cayley–Hamilton theorem, the matrix A satisfies its characteristic equation (3)

$\therefore A^2 - 2I = O$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad A^2 = 2I \qquad (4)$$

Now $A^{16} = (A^2)^8 = (2I)^8$ (From (4))

$$= 2^8 I^8 \quad \Rightarrow \quad 256I \quad \Rightarrow \quad 256\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \qquad A^{16} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$$

EXERCISE

Practice Problem

Directions for questions 1 to 47: Select the correct alternative from the given choices.

1. If the trace of the matrix $\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 2 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & n \end{pmatrix}$ is 55 then the value of n is

(A) 10 (B) 11
(C) 9 (D) Cannot be determined

2. Which of the following statement is/are false?

(A) $A^T \cdot B^{Tx}$ always defined for square matrices of same order.
(B) $A^T . B$ is defined for matrices of the same order.
(C) $t_r(A^T) + t_r(B^T)$ is always defined for matrices A, B of same order.
(D) $A^T + B^T$ is always defined for matrices A, B of same order.

3. Consider the following statements about two square matrices A and B of the same order:

P: $(A + B)^2 = A^2 + 2AB + B^2$

Q: $(A + B)(A - B) = A^2 - B^2$ then,

(A) both P and Q are true.
(B) both P and Q are false
(C) both P and Q are true if A and B commute.
(D) P is true but Q is false.

4. If $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} -2x & 3 & x \\ x & -2 & 0 \\ 2 & -2x & -x \end{pmatrix} = I_{3\times 3}$, then $x =$

(A) -1 (B) 1
(C) 1/2 (D) 2

5. The determinant value of $\begin{pmatrix} 2 & 3 & -3 \\ 1 & -2 & 2 \\ 7 & 4 & -4 \end{pmatrix}$ is

(A) 0 (B) 10
(C) -10 (D) 15

6. If $f(x) = \begin{vmatrix} {}^xC_0 & {}^xC_1 & {}^{x+1}C_1 \\ 2^xC_1 & 2^xC_2 & 2^{(x+1)}C_2 \\ 6^xC_2 & 6^xC_3 & 6^{(x+1)}C_3 \end{vmatrix}$, then $f(200)$ is

(A) 200 (B) -200
(C) 0 (D) -2001

7. The determinant $\begin{vmatrix} 2 & 3+i & -1 \\ 3-i & 0 & -1+i \\ -1 & -1-i & 1 \end{vmatrix}$ is

(A) purely imaginary (B) zero
(C) real (D) 10

8. If $A = \begin{pmatrix} x & y & z \\ 2x & y & 3z \\ \frac{x}{2} & \frac{y}{2} & \frac{z}{2} \end{pmatrix}$, then $|A| =$ then $|A| =$

(A) $10xyz$
(B) 1
(C) 0
(D) $1/2(x^3 + y^3 + z^3 - 3xyz)$

9. If A is a square matrix of order k and $\det(kA) = 27 \det(A)$, then $k =$

(A) 9 (B) 1
(C) 2 (D) 3

10. If A and B are two square matrices of order 4 such that $|A| = -2$ and $|B| = 5$, then $|4AB|$ is

(A) -80 (B) -160
(C) -2560 (D) -256

11. A lower triangular matrix $A = (a_{ij})_{n\times n}$ is singular if and only if

(A) $a_{ii} = 0$ for all $i = 1, 2, \ldots n$
(B) $a_{ii} = 0$ for atleast one $i = 1, 2, \ldots n$
(C) $a_{ii} \neq 0$ for all $i = 1, 2, \ldots n$
(D) $a_{ii} \neq 0$ for atleast one i, $i = 1, 2, \ldots n$

12. Inverse of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 1 & -1 \end{pmatrix}$ is

(A) $\begin{pmatrix} -5 & -11 & 9 \\ -1 & -2 & 2 \\ -3 & -6 & 5 \end{pmatrix}$ (B) $\begin{pmatrix} -5 & -1 & -3 \\ -11 & -2 & -6 \\ 9 & 2 & 5 \end{pmatrix}$

(C) $\begin{pmatrix} 5 & 11 & 9 \\ 1 & 2 & -2 \\ 3 & 6 & -5 \end{pmatrix}$ (D) $\begin{pmatrix} 5 & 1 & 3 \\ 11 & 2 & 6 \\ 9 & -2 & -5 \end{pmatrix}$

13. (i) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$, then A^{-1} is symmetric.

(ii) If a non-singular matrix A is symmetric, then A^{-1} is also symmetric.

Which of the following is correct?

(A) Both (i) and (ii) true. (B) Both (i) and (ii) false.
(C) (i) is true, (ii) is false. (D) (i) is false, (ii) is true.

14. A is a third order matrix. If the value of the square of the determinant of the matrix of co-factors of A is 28561, then $|A|$ equals

(A) 25 (B) ±13
(C) 120 (D) ±169

15. If A is a square matrix of order 3, then the product of A and its transpose is

(A) unit matrix (B) zero matrix
(C) identity matrix (D) symmetric matrix

16. If A and B are two skew symmetric matrices of the same order then AB is skew symmetric if and only if

(A) $AB + BA = O$ (B) $AB - BA = O$
(C) $AB + BA = I$ (D) $AB - BA = I$

17. The matrix $\begin{pmatrix} 2 & 1+i & 3 \\ 1-i & 6 & i \\ 3 & -i & 4 \end{pmatrix}$ is

(A) symmetric.
(B) skew-symmetric.
(C) hermition matrix.
(D) skew-hermition matrix.

18. Which of the following matrices do not satisfy the relation $A^2 = I$?

(A) Involutory matrix
(B) Identity matrix
(C) Idempotent matrix
(D) None of these

19. The matrix $\begin{pmatrix} 1 & -2 & -3 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{pmatrix}$ is

(A) idempotent. (B) nilpotent.
(C) involuntary. (D) orthogonal.

20. Rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is

(A) 1 (B) 2
(C) 3 (D) 4

21. The rank of the matrix $\begin{pmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \\ -10 & 5 & 15 \end{pmatrix}$ is

(A) 0 (B) 1
(C) 2 (D) 3

22. If $A = (a_{ij})_{mxn}$ and $a_{ij} = 3$ for all i, j then $\rho(A)$ is

(A) 0 (B) 1
(C) 2 (D) 3

23. If $A = (1 \quad 2 \quad 3)$ and $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $\rho(AB)$ is

(A) 0 (B) 1
(C) 2 (D) 4

24. Which of the following matrix is row echelon form?

(A) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Which of the following set of vectors are linearly dependent?

(A) (2, 3, 3), (3, –1, 3), (4, –2, 5)
(B) (3, 4, –1), (–1, 3, 1), (–2, –7, –2)
(C) (2, 1, 4), (1, –2, 2), (–3, 1, –6)
(D) (1, 3, –5), (–5, –1, 3), (4, –2, –2)

26. The system of equations $6x + 7y + 8z = 1$, $13x + 14y + 15z = 2$ and $x + 2y + 3z = 2$ is

(A) consistent with unique solution.
(B) consistent with infinite solutions.
(C) inconsistent.
(D) None of these

27. If the system of equations $2y + z = 2$, $x + ky + 3z = 1$ and $x - z = -1$ has a unique solution, then the value of K is
(A) equal to 8 (B) not equal to 8
(C) not equal to 7 (D) equal to 7

28. If the system of equations $4x + 2y - 3z = 7$, $x + 3y + kz = 4$ and $x + y - z = 5$ have no solution, then the value of k is
(A) -2 (B) -3
(C) 2 (D) 3

29. The value of λ for which the following system of equations does not have a solution is
$x + y + z = 6$
$4x + ly - lz = 0$
$3x + 2y - 4z = -8$
(A) 3 (B) -3
(C) 0 (D) 1

30. If the system $2x - 3y + z = p$, $-x + 2y + 3z = q$, $5x - 8y - z = r$ has a solution, then
(A) p, q, r are in AP
(B) p, q, r are in GP
(C) p, q, r are in HP
(D) q, p, r are in AP

31. If the equations $2x - y - z = 0$, $kx - 3y + 2z = 0$ and $-31 + 2y + kz = 0$ have a non-zero solution, then the value of k is
(A) 2 (B) 1
(C) 7 (D) Both 1 and 7

32. The system of equations $x - ky - z = 0$, $kx - y - z = 0$ and $x + y - z = 0$ has non-trivial solutions. Then the possible values of k are
(A) ± 1 (B) ± 2
(C) 0 (D) ± 4

33. The system of equations $\alpha + 3y + 5z = 0$, $2x - 4\alpha y + \alpha z = 0$, $-4x + 18y + 7z = 0$ has only trivial solution if α is
(A) -1 or -3
(B) 1 or -3
(C) not equal to 1, -3
(D) not equal to -1 and 3

34. Solve the given system of equations by Gauss elimination method.

$$2x + y - z = 4;$$
$$3x - 2y + 4z = 8$$
$$x - 3y + 2z = 1,$$

(A) (2, –1, 2) (B) (–1, 2, –2)
(C) (2, 1, 1) (D) (–1, 1, 2)

35. The eigen values of $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ is
(A) 0, 0, 0 (B) 0, 1, 0
(C) 2, 1, 3 (D) –2, –1, –3

36. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then characteristic root of A is
(A) 3 (B) 2
(C) 1 (D) 0

37. The minimum eigen value of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & -6 \\ 1 & 1 & 14 \end{bmatrix}$ is
(A) 0 (B) 1
(C) 2 (D) 3

38. The characteristic roots of the inverse of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are
(A) –1, –1, 5 (B) 1, 1, 5
(C) 1, 1, 1/5 (D) –1, –1, 1/5

39. The sum and product of the eigen values of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & -2 \\ 1 & 3 & -5 \end{bmatrix}$ is respectively
(A) 0, 24 (B) 1, –24
(C) 2, 20 (D) 4, –24

40. The eigen values of a matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & p \\ 1 & 0 & q \end{bmatrix}$ are 1, 2, and 3. Then the values of p and q are _____
(A) $p = 0, q = 0$
(B) p = any real number, $q = 2$
(C) $p = 2, q = 0$
(D) $p = 2, q = 2$

41. The eigen values of the matrix $\begin{bmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & 4 & 6 \\ -2 & -4 & 0 & 5 \\ 3 & -6 & -5 & 0 \end{bmatrix}$ is
(A) real only (B) imaginary
(C) zero only (D) imaginary or zero

42. The number of linearly independent eigen vectors of $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$ is _____
(A) 0 (B) 1
(C) 2 (D) infinite

43. Which of the following is an eigen vector for the matrix $\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$?

(A) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(C) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$

44. For a matrix $A = \begin{bmatrix} 6 & -6 & 2 \\ -6 & 5 & -4 \\ 2 & -4 & 1 \end{bmatrix}$, $x = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector. The corresponding eigen value is _____.

(A) –2 (B) 1

(C) 2 (D) 13

45. Let A be a 2×2 square matrix with $\lambda_1 = -2$ and $\lambda_2 = -3$ as its eigen values and $x_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$, $x_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ as its eigen vectors then A is given by

(A) $\begin{bmatrix} 0 & 2 \\ 4 & -5 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & -6 \\ 7 & -9 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & 6 \\ 7 & -3 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 6 \\ -4 & 3 \end{bmatrix}$

46. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, then $A^{15} =$

(A) $8^{14}A$ (B) $8^{15}A$

(C) $8^{16}A$ (D) $15A$

Previous Years' Questions

1. In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x: **[2005]**

(A) Augmented matrix $[Pq]$ must have the same rank as matrix P

(B) Vector q must have only non-zero elements

(C) Matrix P must be singular

(D) Matrix P must be square

2. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to –2. Which of the following is an eigen vector **[2005]**

(A) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (B) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (D) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

3. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, the top row of R^{-1} is: **[2005]**

(A) [5 6 4] (B) [5 –3 1]

(C) [2 0 –1] (D) $\left[2 \; -1 \; \frac{1}{2}\right]$

4. $x = [x_1 \; x_2 \ldots x_n]^T$ is an n-tuple non-zero vector. The $n \times n$ matrix $V = xx^T$ **[2007]**

(A) has rank zero (B) has rank 1

(C) is orthogonal (D) has rank N

5. Let x and y be two vectors in a 3 dimensional space and $\langle x, y \rangle$ denote their dot product. Then the determinant $\det \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{bmatrix}$ **[2007]**

(A) is zero when x and y are linearly independent

(B) is positive when x and y are linearly independent

(C) is non-zero for all non-zero x and y.

(D) is zero only when either x or y is zero.

6. The linear operation $L(x)$ is defined by the cross product $L(x) = b \times x$, where $b = [0 \; 1 \; 0]^T$ and $x = [x_1 \; x_2 \; x_3]^T$ are three dimensional vectors. The 3×3 matrix M of this operation satisfies

$$L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then the eigen values of M are **[2007]**

(A) 0, +1, –1 (B) 1, –1, 1

(C) i, $-i$, 1 (D) i, $-i$, 0

Common Data for Questions 7 and 8: Cayley–Hamilton theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

7. A satisfies the relation **[2007]**

(A) $A + 3I + 2A^{-1} = 0$ (B) $A^2 + 2A + 2I = 0$

(C) $(A + I)(A + 2I) = 0$ (D) $\exp(A) = 0$

8. A^9 equals **[2007]**

(A) $511\,A + 510I$ (B) $309\,A + 104I$

(C) $154\,A + 155I$ (D) $\exp(9\,A)$

9. The characteristic equation of a (3×3) matrix P is defined as $\propto(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$. If I denotes identity matrix, then the inverse of matrix P will be **[2008]**

(A) $(P^2 + P + 2I)$ (B) $(P^2 + P + I)$

(C) $-(P^2 + P + I)$ (D) $-(P^2 + P + 2I)$

10. If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct? **[2008]**

(A) Q will have four linearly independent rows and four linearly independent columns

(B) Q will have four linearly independent rows and five linearly independent columns

(C) QQ^T will be invertible

(D) Q^TQ will be invertible

11. A is a $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A^+ = (A^TA)^{-1}A^T$. Then, which one of the following statements is FALSE? **[2008]**

(A) $AA^+A = A$ (B) $(AA^+)^2 = AA^+$

(C) $A^+A = I$ (D) $AA^+A = A^+$

12. Let P be a 2×2 real orthogonal matrix and $\vec{x}$ is a real vector $[x_1, x_2]^T$ with length $\|\vec{x}\| = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$ Then, which one of the following statements is correct? **[2008]**

(A) $\|P\vec{x}\| \le \|\vec{x}\|$ where at least one vector satisfies $\|P\vec{x}\| < \|\vec{x}\|$

(B) $\|P\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$

(C) $\|P\vec{x}\| \ge \|\vec{x}\|$ where at least one vector satisfies $\|P\vec{x}\| > \|\vec{x}\|$

(D) No relationship can be established between $\|\vec{x}\|$ and $\|P\vec{x}\|$

13. An eigen vector of $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is **[2010]**

(A) $[-1\ 1\ 1]^T$ (B) $[1\ 2\ 1]^T$

(C) $[1\ -1\ 2]^T$ (D) $[2\ 1\ -1]^T$

14. For the set of equations, $x_1 + 2x_2 + x_3 + 4x_4 = 2$ and $3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$. Which of the following statement is true **[2010]**

(A) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists

(B) There are no solutions

(C) A unique non-trivial solution exists

(D) Multiple non-trivial solutions exist

15. The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of a lower triangular matrix $[L]$ and an upper triangular matrix $[U]$. The properly decomposed $[L]$ and $[U]$ matrices respectively are **[2011]**

(A) $\begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

16. Given that $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the value of A^3 is **[2012]**

(A) $15A + 12I$ (B) $19A + 30I$

(C) $17A + 15I$ (D) $17A + 21I$

17. The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has **[2013]**

(A) no solution

(B) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(C) non-zero unique solution

(D) multiple solutions

18. A matrix has eigen values -1 and -2. The corresponding eigen vectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is **[2013]**

(A) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

19. Given a system of equations:

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

Which of the following is true regarding its solutions **[2014]**

(A) The system has a unique solution for any given b_1 and b_2.

(B) The system will have infinitely many solutions for any given b_1 and b_2.

(C) Whether or not a solution exists depends on the given b_1 and b_2.

(D) The system would have no solution for any values of b_1 and b_2.

20. A system matrix is given as follows.

$$A=\begin{bmatrix}0 & 1 & -1\\ -6 & -11 & 6\\ -6 & -11 & 5\end{bmatrix}$$

The absolute value of the ratio of the maximum eigen value to the minimum eigen value is _____. **[2014]**

21. Which one of the following statements is true for all symmetric matrices? **[2014]**

(A) All the eigenvalues are real.
(B) All the eigenvalues are positive.
(C) All the eigenvalues are distinct.
(D) Sum of all the eigenvalues is zero.

22. Two matrices A and B are given below:

$$A=\begin{bmatrix}p & q\\ r & s\end{bmatrix};\ B\begin{bmatrix}p^2+q^2 & pr+qs\\ pr+qs & r^2+s^2\end{bmatrix}$$

If the rank of matrix A is N, then the rank of matrix B is **[2014]**

(A) $N/2$ (B) $N-1$
(C) N (D) 2N

23. If a continuous function $f(x)$ does not have a root in the interval $[a, b]$, then which one of the following statements is TRUE? **[2015]**

(A) $f(a)\cdot f(b)=0$ (B) $f(a)\cdot f(b)<0$
(C) $f(a)\cdot f(b)>0$ (D) $f(a)/f(b)\le 0$

24. If the sum of the diagonal elements of a 2×2 matrix is -6, then the maximum possible value of determinant of the matrix is _____. **[2015]**

25. The maximum value of "a" such that the matrix $\begin{pmatrix}-3 & 0 & -2\\ 1 & -1 & 0\\ 0 & a & -2\end{pmatrix}$ has three linearly independent real eigenvectors is **[2015]**

(A) $\dfrac{2}{3\sqrt{3}}$ (B) $\dfrac{1}{3\sqrt{3}}$
(C) $\dfrac{1+2\sqrt{3}}{3\sqrt{3}}$ (D) $\dfrac{1+\sqrt{3}}{3\sqrt{3}}$

26. We have a set of 3 linear equations in 3 unknowns. '$X \equiv Y$' means X and Y are equivalent statements and '$X \not\equiv Y$' means X and Y are not equivalent statements.

P: There is a unique solution.

Q: The equations are linearly independent.

R: All eigen values of the coefficient matrix are non zero.

S: The determinant of the coefficient matrix is nonzero.

Which one of the following is TRUE? **[2015]**

(A) $P \equiv Q \equiv R \equiv S$
(B) $P \equiv R \not\equiv Q \equiv S$
(C) $P \equiv Q \not\equiv R \equiv S$
(D) $P \not\equiv Q \not\equiv R \not\equiv S$

27. Consider a 3 × 3 matrix with every element being equal to 1. Its only non-zero eigenvalue is _____. **[2016]**

28. Let the eigenvalues of a 2 x 2 matrix A be 1, –2 with eigenvalues and eigenvectors x_1 and x_2 respectively. Then the eigenvectors of the matrix $A^2 - 3A + 4I$ would respectively be **[2016]**

(A) 2, 14; x_1, x_2
(B) 2, 14; x_1+x_2, x_1-x_2
(C) 2, 0; x_1, x_2
(D) 2, 0; x_1+x_2, x_1-x_2

29. Let A be a 4 × 3 real matrix with rank 2. Which one of the following statement is TRUE? **[2016]**

(A) Rank of $A^T A$ is less than 2.
(B) Rank of $A^T A$ is equal to 2.
(C) Rank of $A^T A$ is greater than 2.
(D) Rank of $A^T A$ can be any number between 1 and 3

30. A 3×3 matrix P is such that, $P^3 = P$. Then the eigenvalues of P are **[2016]**

(A) 1, 1, –1
(B) 1, 0.5 + j0.866, 0.5 – j0.866
(C) 1, –0.5 + j0.866, –0.5, –j0.866
(D) 0, 1, –1

31. Let $P=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$. Consider the set S of all vectors $\begin{pmatrix}x\\ y\end{pmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{pmatrix}a\\ b\end{pmatrix} = P\begin{pmatrix}x\\ y\end{pmatrix}$ Then S is _____. **[2016]**

(A) a circle of radius $\sqrt{10}$
(B) A circle of radius $\dfrac{1}{\sqrt{10}}$
(C) an ellipse with major axis along $\begin{pmatrix}1\\ 1\end{pmatrix}$
(D) an ellipse with minor axis along $\begin{pmatrix}1\\ 1\end{pmatrix}$

32. Consider a linear time invariant system $x = Ax$, with initial condition $x(0)$ at $t = 0$. Suppose α and β are eigenvectors of (2×2) matrix A corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then the response $x(t)$ of the system due to initial condition $x(0) = \alpha$ is **[2016]**

(A) $e^{\lambda_1 t}\alpha$ (B) $e^{\lambda_2 t}\beta$
(C) $e^{\lambda_2 t}\alpha$ (D) $e^{\lambda_1 t}\alpha + e^{\lambda_2 t}\beta$

Answer Keys

Exercise

Practice Problem

1. A	**2.** C	**3.** C	**4.** B	**5.** A	**6.** C	**7.** C	**8.** C	**9.** D	**10.** C
11. B	**12.** B	**13.** D	**14.** B	**15.** D	**16.** A	**17.** D	**18.** C	**19.** B	**20.** B
21. B	**22.** B	**23.** B	**24.** A	**25.** C	**26.** C	**27.** B	**28.** A	**29.** A	**30.** D
31. D	**32.** A	**33.** C	**34.** C	**35.** C	**36.** D	**37.** A	**38.** C	**39.** B	**40.** B
41. B	**42.** B	**43.** B	**44.** D	**45.** B	**46.** A				

Previous Years' Questions

1. A	**2.** D	**3.** B	**4.** B	**5.** D	**6.** D	**7.** AC	**8.** A	**9.** D	**10.** A
11. D	**12.** B	**13.** B	**14.** D	**15.** D	**16.** B	**17.** D	**18.** D	**19.** B	
20. 0.333	**21.** A	**22.** C	**23.** C	**24.** 9	**25.** B	**26.** A	**27.** 3	**28.** A	**29.** B
30. D	**31.** D	**32.** A							

Chapter 2

Calculus

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Limit of a function
- Continuous functions
- Algebra of continuous functions
- Intermediate-value theorem
- Uniform continuity
- Types of discontinuity
- Derivatives
- Product rule
- Quotient rule
- Second derivative
- Application of derivatives
- Cauchy's mean value theorem
- Taylor's theorem
- Maclaurin's theorem
- Partial differentiation
- Indefinite integrals
- Definite integrals
- Applications of integration
- Rectification
- Improper integrals
- Triple integrals
- Change of variables

Limit of a Function

Let $y = f(x)$ be a function of x and let 'a' be any real number.

We must first understand what a 'limit' is. A limit is the value, function approaches, as the variable within that function (usually 'x') gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, what is $f(x)$ very close to?

Meaning of '$x \to a$'

Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a', then we say that 'x tends to a' or 'x approaches a' and is written as '$x \to a$'. By $x \to a$, we mean that $x \neq a$ and x may approach 'a' from left or right, which is explained in the example given below.

Let us look at an example of a limit: What is the limit of the function $f(x) = x^3$ as x approaches 2? The expression 'the limit as x approaches to 2' is written as : $\lim_{x \to 2}$ Let us check out some values of $\lim_{x \to 2}$ as x increases and gets closer to 2, without even exactly getting there.

When $x = 1.9, f(x) = 6.859$
When $x = 1.99, f(x) = 7.88$
When $x = 1.999, f(x) = 7.988$
When $x = 1.9999, f(x) = 7.9988$

As x increases and approaches 2, $f(x)$ gets closer and closer to 8 and since x tends to 2 from left this is called 'left-hand limit' and is written as $\lim_{x \to 2^-}$.

Now, let us see what happens when x is greater than 2.

When $x = 2.1, f(x) = 9.261$
When $x = 2.01, f(x) = 8.12$
When $x = 2.001, f(x) = 8.01$
When $x = 2.0001, f(x) = 8.001$

As x decreases and approaches 2, $f(x)$ still approaches 8. This is called 'right-hand limit' and is written as $\lim_{x \to 2^+}$

$$\underset{x}{\overset{\rightarrow}{}} \quad 2 \qquad 2 \quad \underset{x}{\overset{\leftarrow}{}}$$

We get the same answer while finding both, left and right hand limits. Hence we write that $\lim_{x \to 2} x^3 = 8$.

Meaning of the Symbol: $\lim_{x \to a} f(x) = \ell$

Let $f(x)$ be a function of x where x takes values closer and closer to 'a' ($\neq a$), then $f(x)$ will assume values nearer and nearer to ℓ. Hence we say, $f(x)$ tends to the limit 'ℓ' as x tends to a.

The following are some of the simple algebraic rules of limits.

1. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$
2. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{x \to a}{\lim_{x \to a} g(x)} (\lim_{x \to a} g(x) \neq 0)$

Notes:

1. If the left hand limit of a function is not equal to the right hand limit of the function, then the limit does not exist.
2. A limit equal to infinity is not the same as a limit that does not exist.

Continuous Functions

Let $f: A \to B$ be any given function and let $c \in A$. We say f is continuous at c, if given $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \in$ whenever $|x - c| < \delta$

In words, this means that, if x is very close to c in domain, then $f(x)$ is very close to $f(c)$ in range.

Equivalently f is continuous at c. If $\lim_{x \to c} f(x) = f(c)$

We observe

1. $c \in A$, i.e., $f(c)$ must exist
2. $\lim_{x \to c} f(x)$ exists
3. $f(c)$ and $\lim_{x \to c} f(x)$ are equal.

If any of these three conditions fail, then f is discontinuous at $x = c$.

Algebra of continuous functions

If f, g be two continuous functions at c, then $f + g, f - g, fg$ are also continuous at $x = c$.

To solve a problem of continuous functions at a point a, you can take the following approach.

1. Find the value $f(x)$ at $x = a$. If a is in the domain of f, $f(a)$ must exist. If a is not in the domain, then $f(a)$ does not exist. In such a case, f is not continuous at $x = a$.
2. Find $\lim_{x \to a} f(x)$. For this you have to first find $\lim_{x \to a^-} f(x) = \ell_1$(say) and $\lim_{x \to a^+} f(x) = \ell_2$(say). If $\ell_1 \neq \ell_2$ then $\lim_{x \to a} f(x)$ does not exist and so f is not continuous at $x = a$. If $\ell_1 = \ell_2$, then $\lim_{x \to a} f(x)$ exists.
3. If $\lim_{x \to a} f(x)$ exists and also $f(a)$ exists.

Then verify whether $\lim_{x \to a} f(x) = f(a)$.

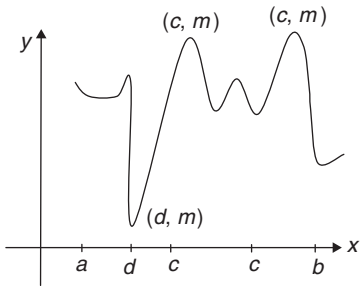
If $\lim_{x \to a} f(x) = f(a)$. Then f is continuous, otherwise it is not continuous at $x = a$.

Problems on continuous functions can be grouped into the following categories.

1. Using $\in, \delta$ notation.
2. Using existence of right and left hand limits.
3. To find the value of the unknown in $f(x)$ when f is given to be continuous at a point.
4. To find $f(a)$ when f is given to be continuous at $x = a$.

For functions that are continuous on (a, b) the following holds:

f is bounded and attains its bounds at least once on $[a, b]$, i.e., for some $c, d \in [a, b]$,
M = supremum of $f = f(c)$ and m = Infimum of $f = f(d)$



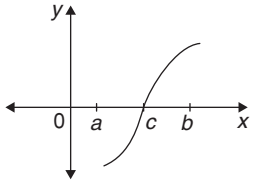
Note: The converse may not be true as $f(x) = \begin{cases} 1; & 0 < x \le 1 \\ -1; & 1 < x \le 2 \end{cases}$ is bounded on [1, 2] but it is not continuous at $x = 1$.

Intermediate-value Theorem

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$ then f takes every value between $f(a)$ and $f(b)$.

Equivalently, if f is continuous on $[a, b]$ and $f(a) < k < f(b)$ or $f(b) < k < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

Equivalently, If $f(a)$ and $f(b)$ are of opposite signs then there exists $c \in (a, b)$ such that $f(c) = 0$.



$f(a) < 0$ and $f(b) > 0$, clearly $f(c) = 0$.

Notes:

1. If $f(x)$ is continuous in $[a, b]$ then f takes all values between m and M at least once as x moves from a to b, where M = Supremum of f on $[a, b]$ and m = infimum of f on $[a, b]$.
2. If $f(x)$ is continous in $[a, b]$, then $|f|$ is also continuous on $[a, b]$, where $|f|(x) = |f(x)| \; x \in [a, b]$.
3. Converse may not be true

For instance, $f(x) = \begin{cases} 1; & 0 < x \le 3 \\ -1; & 3 < x \le 5 \end{cases}$ is not continuous at $x = 3$, but $|f|(x) = 1 x \in [0, 5]$, being a constant function is continuous [0, 5].

Inverse-function theorem

If f is a continuous one-to-one function on $[a, b]$ then f^{-1} is also continuous on $[a, b]$.

Uniform continuity A function f defined on an interval I is said to be uniformly continuous on I if given $\in > 0$ there exists a $\delta > 0$ such that if x, y are in I and $|x - y| < \delta$ then $|f(x) - f(y)| < \in$.

Note: Continuity on $[a, b]$ implies uniform continuity whereas continuity on (a, b) does not mean uniform continuity.

Types of discontinuity If f is a function defined on an interval I, it is said to have

(TD1) **a removable discontinuity** at $p \in I$, if $\lim_{x\to p} f(x)$ exists, but is not equal to $f(p)$.

(TD2) **a discontinuity of first kind from the left** at p if $\lim_{x\to p^-} f(x)$ exists but is not equal to $f(p)$.

(TD3) **a discontinuity of first kind from the right** at p if $\lim_{x\to p^+} f(x)$ exists but is not equal to $f(p)$.

(TD4) **a discontinuity of first kind** at p if $\lim_{x\to p^-} f(x)$ and $\lim_{x\to p^+} f(x)$ exists but they are unequal.

(TD5) **a discontinuity of second kind from the left** at p if $\lim_{x\to p-} f(x)$ does not exist as x approaches p from left.

(TD6) **a discontinuity of second kind from the right** at p if $\lim_{x\to p^+} f(x)$ does not exist as x approaches p from right.

(TD7) **a discontinuity of second kind** at p if neither $\lim_{x\to p^-} f(x)$ nor $\lim_{x\to p^+} f(x)$ exist.

Examples for each type are presented in the following table:

Type	Example	Point of Discontinuity
TD1	$f(x)=\frac{x^2-1}{x-1}$, $x\neq$ $f(1)=3$	$x=1$
TD2	$f(x)=x+3$ for $0<x<1$ $f(x)=5$ for $x\geq 1$	$x=1$
TD3	$f(x)=x+3$, for $x>2$ $f(x)=8$ for $x\leq 2$	$x=2$
TD4	$f(x)=\begin{cases} x+3 & x>2 \\ 7; & x=2 \\ x-3; & x<2 \end{cases}$	$x=2$
TD5	$f(x)=\tan x$ for $x<\pi/2$ $f(x)=1$, for $x\geq\pi/2$	$x=\frac{\pi}{2}$
TD6	$f(x)=1$, for $x\leq\pi/2$ $f(x)=\tan x$ for $x>\pi/2$	$x=\frac{\pi}{2}$
TD7	$f(x)=1/x$ at $x\neq 0$ $f(0)=3$ at $x=0$	$x=0$

Notes:

1. Every differentiable function is continuous, but the converse is not true.
   The example of a function which is continuous but not differentiable at a point $f(x)=|x-3|$ for $x\in R$ is continuous at $x=3$, but it is not differentiable at $x=3$.
2. The function may have a derivative at a point, but the derivative may not be continuous.
   For example the function

$$f(x)=\begin{cases} x^3\sin\frac{1}{x}; & x\neq 0 \\ 0; & x=0 \end{cases}$$ has the derivative function as

$$f'(x)=\begin{cases} 3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}; & x\neq 0 \\ 0; & x=0 \end{cases}$$

However $\lim_{x\to 0} f'(x)$ doesn't exist.

Solved Examples

Example 1: Discuss the continuity of the function at $x=1$ where $f(x)$ is defined by

$$f(x)=\frac{3x-2}{x} \text{ for } 0<x\leq 1$$

$$=\frac{\sin(x-1)}{(x-1)} \text{ for } x>1$$

Solution: Consider the left and right handed limits

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}\frac{3x-2}{x}=1$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}\frac{\sin(x-1)}{x-1}$$

$$=\lim_{(x-1)\to 0}\frac{\sin(x-1)}{(x-1)}=1 \text{ and } f(1)$$

$$=\frac{3(1)-2}{1}=1$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$\therefore f$ is continuous at $x=1$.

Example 2: If $f(x)=\frac{(2^x-1)^2}{\sin 2x\log(1+x)}$ for $x\neq 0$ and $f(x)=$ log 2 for $x=0$, discuss the continuity at $x=0$.

Solution:

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{(2^x-1)^2}{\sin 2x\log(1+x)}$$

$$=\lim_{x\to 0}\frac{\left(\frac{2^x-1}{x}\right)^2}{2\left(\frac{\sin 2x}{2x}\right)\log(1+x)\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\left(\frac{2^x-1}{x}\right)^2}{2\left(\frac{\sin 2x}{2x}\right)\log(1+x)\frac{1}{x}}$$

$$= \frac{1}{2}\frac{\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)^2}{\lim_{2x\to 0}\frac{\sin 2x}{2x}\log\lim_{x\to 0}(1+x)^{\frac{1}{x}}}$$

$$= \frac{1}{2}(\log 2)^2.$$

But given $f(x) = 2 \log 2$ at $x = 0$

$\therefore \lim_{x\to 0} f(x) \neq f(0)$

$\therefore f(x)$ is not continuous at $x = 0$.

Example 3: Find the value of k if

$$f(x) = \frac{2x^3 - 5x^2 + 4x + 11}{x+1}, \text{ for } x \neq -1$$

And $f(-1) = k$ is continuous at $x = -1$.

Solution: Given $f(x)$ is continuous at $x = -1$

$$\Rightarrow \lim_{x\to -1} f(x) = f(-1) = k.$$

$$\Rightarrow \lim_{x\to -1} f(x) = \lim_{x\to -1}\left[\frac{2x^3 - 5x^2 + 4x + 11}{x+1}\right]$$

$$= \lim_{x\to -1}\frac{(x+1)(2x^2 - 7x + 11)}{x+1}$$

$$= 2(-1)^2 - 7(-1) + 11$$

$$= 2 + 7 + 11 = 20$$

$$\therefore k = 20$$

Example 4: If $f(x) = \frac{x-4}{|x-4|} + a$, for $x < 4, = a + b$ at $x = 4, = \frac{x-4}{|x-4|} + b$, for $x > 4$

And $f(x)$ is continuous at $x = 4$, then find the values of a and b.

Solution: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\frac{x-4}{|x-4|} + a$

$$= \lim_{x\to 4^-}\frac{x-4+a}{-(x-4)} = -1 + a$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\frac{x-4}{|x-4|} + b$$

$$= \lim_{x\to 4^+}\frac{x-4}{(x-4)} + b = 1 + b$$

Since given $f(x)$ is continuous at $x = 4$

$$\lim_{x\to 4^-} f(x) = f(4) = \lim_{x\to 4^-} f(x)$$

$\Rightarrow -1 + a = a + b = 1 + b \Rightarrow a = 1, b = -1$

Example 5: Examine the continuity of the given function at origin where,

$$f(x) = \begin{cases} \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}; & x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = 0$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0^+}\frac{x}{e^{-1/x}_{+1}} = 0$$

Then,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 0$$

Thus the function is continuous at the origin.

Derivatives

In this section we will look at the simplistic form of the definition of a derivative, the derivatives of certain standard functions and application of derivatives.

For a function $f(x)$, the ratio $\frac{[f(a+h) - f(a)]}{h}$ is the rate of change of $f(x)$ in the interval $[a, (a + h)]$.

The limit of this ratio as h tends to zero is called the derivative of $f(x)$. This is represented as $f'(x)$, i.e.,

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = f'(x)$$

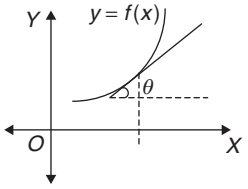
The derivative $f'(x)$ is also represented as $\frac{d\{f(x)\}}{dx}$ or $\frac{d}{dx}\{f(x)\}$

Hence, if $y = f(x)$, i.e., y is a function of x, then $\frac{dy}{dx}$ is the derivative of y with respect to x.

Notes:

1. $\frac{dy}{dx}$ is the rate of change of y with respect to x.
2. If the function y can be represented as a general curve, and a tangent is drawn at any point where the tangent makes an angle θ with the horizontal (as shown in the figure), then $\frac{dy}{dx} = \tan\theta$, In other words, derivative of a function at a given point is the slope of the curve at

that point, i.e., tans of the angle, the tangent drawn to the curve at that point, makes with the horizontal.



Standard Results

If $f(x)$ and $g(x)$ are two functions of x and k is a constant, then

1. $\frac{d}{dx}(c) = 0$ (c is a constant)
2. $\frac{d}{dx} k \cdot f(x) = k \frac{d}{dx} f(x)$ (k is a constant)
3. $\frac{d}{dx}(f(x) \pm g(x))$

   $= \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

Product rule

4. $\frac{d}{dx}\{f(x) \cdot g(x)\} = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

Chain rule

6. If $y = f(u)$ and $u = g(x)$ be two functions, then $\frac{dy}{dx} = \left(\frac{dy}{du}\right) \times \left(\frac{du}{dx}\right)$

Derivatives of Some Important Functions

1. (i) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

   (ii) $\frac{d}{dx}\left[\frac{1}{x^n}\right] = \frac{n}{x^{n+1}}$

   (iii) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$; $x \neq 0$
2. $\frac{d}{dx}[ax^n + b] = an \cdot x^{n-1}$
3. $\frac{d}{dx}[ax + b]^n = n\, a\, (ax + b)^{n-1}$
4. $\frac{d}{dx}[e^{ax}] = a \cdot e^{ax}$
5. $\frac{d}{dx}[\log x] = \frac{1}{x}$; $x > 0$
6. $\frac{d}{dx}[a^x] = a^x \log a$; $a > 0$
7. (i) $\frac{d}{dx}[\sin x] = \cos x$

   (ii) $\frac{d}{dx}[\cos x] = -\sin x$

   (iii) $\frac{d}{dx}[\tan x] = \sec^2 x$

   (iv) $\frac{d}{dx}[\cot x] = -\text{cosec}^2 x$

   (v) $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$

   (vi) $\frac{d}{dx}[\text{cosec } x] = -\text{cosec } x \cdot \cot x$

Inverse rule

If $y = f(x)$ and its inverse $x = f^{-1}(y)$ is also defined, then $\frac{dy}{dx} = \frac{1}{dx/dy}$.

Second derivative

If $y = f(x)$, then the derivative of derivative of y is called as second derivative of y and is represented by $\frac{d^2y}{dx^2}$.

$\frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ where $\frac{dy}{dx}$ is the first derivative of y.

Successive Differentiation

If f is differentiable function of x and the derivative f' is also a differentiable function of x, then f'' is called the second derivative of f. Similarly 3rd, 4th ... nth derivative of f may be defined and are denoted by f''', f'''', ... f^n or $y_3, y_4 \ldots y_n$.

Application of Derivatives

Errors in measurement

Problems relating to errors in measurement can be solved using the concept of derivatives. For example, if we know the error in measurement of the radius of a sphere, we can find out the consequent error in the measurement of the volume of the sphere. Without going into further details of theory, we can say dx = error in measurement of x and dy = consequent error in measurement of y, Where $y = f(x)$. Hence, we can rewrite $\frac{dy}{dx} = f'(x)$ as $dy = f'(x) \cdot dx$. Thus, if we know the function $y = f(x)$ and dx, error in measurement of x, we can find out dy, the error in measurement of y.

Notes:

1. An error is taken to be positive when the measured value is greater than the actual value and negative when it is less.
2. Percentage error in y is given by $\left(\frac{dy}{y}\right)\times 100$.

Rate of change

While defining the derivative, we have seen that derivative is the 'rate of change'. This can be applied to motion of bodies to determine their velocity and acceleration.

Velocity If we have s, the distance covered by a body expressed as a function of t, i.e., $s = f(t)$, then rate of change of s is called velocity (v). $v = \frac{ds}{dt} = f'(t)$.

Acceleration Rate of change of velocity is defined as acceleration. Since $v = f'(t)$ itself is a function of t, we can write $v = f'(t)$.

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, i.e., acceleration is the second derivative of the function $s = f(t)$.

Maxima and minima

A function takes a maximum value or a minimum value when the slope of the tangent of the curve at that point is zero, i.e., when the first derivative of the function is zero. If $y = f(x)$, then y is maximum or minimum at the point $x = x_1$ if $\left(\frac{dy}{dx}\right)_{x=x_1} = 0$.

Thus we can find the value of x_1 by equating $\frac{dy}{dx} = 0$.

As mentioned above that y can have a maximum or a minimum value at $x = x_1$. Whether y is a maximum value or minimum is governed by the sign of the second derivative. The function y has a minimum value if the second derivative is positive. In other words, y is maximum at $x = x_1$ if $\frac{d^2y}{dx^2} < 0$ at $x = x_1 \cdot y$ is minimum at $x = x_1$ if $\frac{d^2y}{dx^2} > 0$ at $x = x_1$. $\left(\frac{dy}{dx}\right)_{x=x_1} = 0$. in both the cases discussed above.

1. If $f'(c) = 0$ and $f''(c)$ is negative, then $f(x)$ is maximum for $x = c$
2. If $f'(c) = 0$ and $f''(c)$ is positive, then $f(x)$ is minimum for $x = c$
3. If $f'(c) = f''(c) = \cdots = f^{r-1}(c) = 0$ and $f^r(c) \neq 0$, then
   (i) If r is even, then $f(x)$ is maximum or minimum for $x = c$ according as $f^r(c)$ is negative or positive.
   (ii) If r is odd, then there is neither maximum nor a minimum for $f(x)$ at $x = c$.

Rolle's theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$;
2. f is differentiable on (a, b) and
3. $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

Lagrange's mean value theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b) then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Another form If f is defined on $[a, a + h]$ such that

1. f is continuous on $[a, a + h]$.
2. f is differentiable on $(a, a + h)$ then there exists atleast one $\theta \in (0, 1)$ such that $f(a + h) = f(a) + hf'(a + \theta h)$.

Cauchy's mean value theorem Let f and g be two functions defined on $[a, b]$ such that

1. f and g are continuous on $[a, b]$
2. f and g are differentiable on (a, b)
3. $g'(x) \neq 0$ for any $x \in (a, b)$ then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Series expansions of some standard functions

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$
4. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$
5. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$
6. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots + \frac{(-1)^{n-1} x^n}{n}$
7. $(1 + x^{-1} = 1 - x + x^2 - x^3 + \cdots$
8. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
9. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
10. $(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{1.3}{2.3}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot x^3 + \cdots$

11. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{(2n-1)}x^{2n-1} + \cdots$

12. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \cdots$

Example 6: For the function $f(x) = x(x^2 - 1)$ test for the applicability of Rolle's theorem in the interval $[-1, 1]$ and hence find c such that $-1 < c < 1$.

Solution: Given $f(x) = x(x^2 - 1)$

1. f is continuous in $[-1, 1]$
2. f is differentiable in $(-1, 1)$
3. $f(-1) = f(1) = 0$

$\therefore$ $f(x)$ satisfies all the properties of Rolle's theorems
$\therefore$ We can find a number c such that $f'(c) = 0$, i.e., $f'(x) = 3x^2 - 1$

$$f(c) = 0 \quad \Rightarrow \quad 3c^2 - 1 = 0 \quad \Rightarrow \quad \pm\sqrt{\frac{1}{3}}$$

$$\Rightarrow \quad c = \sqrt{\frac{1}{3}}$$

Example 7: If $f(x) = 2x^2 + 3x + 4$, then find the value of θ in the mean value theorem.

Solution: $f(a) = 2a^2 + 3a + 4$

$f(a + h) = 2(a^2 + 2ah + h^2) + 3a + 3h + 4$
$f(a + h) - f(a) = 4ah + 2h^2 + 3h = 2(2ah + h^2) + 3h$

$$\frac{f(a+h) - f(a)}{h} = 2(2a + h) + 3$$

$$= 4\left(a + \frac{h}{2}\right) + 3 \qquad (1)$$

Now $f'(x) = 4x + 3, f'(a + \theta h)$
$= 4a + 4h\theta + 3$ (2)

Comparing (1) and (2) we have $4\left(a + \frac{h}{2}\right) + 3$

$$= 4a + 4h\theta + 3 \quad \Rightarrow \quad a + h\theta = a + \frac{h}{2}$$

$$\Rightarrow \quad \theta = \frac{1}{2}$$

Partial Differentiation

Let u be a function of two variables x and y. Let us assume the functional relation as $u = f(x, y)$. Here x alone or y alone or both x and y simultaneously may be varied and in each case a change in the value of u will result. Generally the change in the value of u will be different in each of these three cases. Since x and y are independent, x may be supposed to vary when y remains constant or the reverse.

The derivative of u wrt x when x varies and y remains constant is called the partial derivative of u wrt x and is denoted by $\frac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

Total Differential Coefficient

If u be a continuous function of x and y and if x and y receive small increments Δx and Δy, u will receive, in turn, a small increment Δu. This Δu is called total increment of u.

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

du is called the total differential of u. If $u = f(x, y, z)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$

Implicit function

If the relation between x and y be given in the form $f(x, y) = c$ where c is a constant, then the total differential coefficient wrt x is zero.

Homogeneous functions

Let us consider the function $f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$. In this expression the sum of the indices of the variable x and y in each term is n. Such an expression is called a homogeneous function of degree n.

Euler's theorem

If $f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

This is known as Euler's theorem on homogeneous function.

11. The nth derivatives of some special functions:

(i) $\frac{d^n}{dx^n}x^n = n!$

(ii) $\frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n}$ s (m being a positive integer more than n)

(iii) $\frac{d^n}{dx^n}e^{ax} = a^ne^{ax}$

(iv) $\frac{d^n}{dx^n}\left(\frac{1}{x+a}\right)=\frac{(-1)^n\cdot n!}{(x+a)^{n+1}};\ x\neq -a$

(v) $\frac{d^n}{dx^n}\log(x+a)=\frac{(-1)^{n-1}(n-1)!}{(x+a)^n};\ (x+a)>0$

(vi) $\frac{d^n}{dx^n}\sin(ax+b)=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

(vii) $\frac{d^n}{dx^n}\cos(ax+b)=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

(viii) $\frac{d^n}{dx^n}(e^{ax}\sin bx)=(a^2+b^2)^{n/2}\,e^{ax}\sin(bx+n\tan^{-1}b/a)$

(ix) $\frac{d^n}{dx^n}(e^{ax}\cos bx)$

$=(a^2+b^2)^{\frac{n}{2}}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)$

(x) $\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$

where $\theta=\tan^{-1}(x/a)$

(xi) $\frac{d^n}{dx^n}(\tan^{-1}x)=(-1)^{n-1}(n-1)!\sin^n\theta\cdot\sin n\theta.$

where $\theta=\cot^{-1}x$

Maxima and minima for function of two variables

A function $f(x, y)$ is said to have a local maximum at a point (a, b), if $f(a+h, b+k)\le f(a, b)$ for all small values of h and k, i.e., $f(x, y)$ has a local maximum at (a, b), if $f(a, b)$ has a highest value in a neighbourhood of (a, b).

Similarly, $f(x, y)$ is said to have a local minimum at a point (a, b), if $f(x, y)$ has least value at (a, b) in a neighbourhood of (a, b).

Procedure to obtain maxima and minima

Let $f(x, y)$ be a function of two variables for which we need to find maxima and minima.

1. Find $f_x=\frac{\partial f}{\partial x}$ and $f_y=\frac{\partial f}{\partial y}$
2. Take $f_x=0$ and $f_y=0$ and solve them as simultaneous equations to get pairs of values for x and y, which are called stationary points.
3. Find $r=f_{xx}=\frac{\partial^2 f}{\partial x^2}$ $s=f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ and

$t=f_{yy}=\frac{\partial^2 f}{\partial y^2}$ and find $rt-s^2$.

4. At a stationary point, say (a, b)
   (a) If $rt-s^2>0$, then (a, b) is called an extreme point of $f(x, y)$ at which $f(x, y)$ has either maximum or minimum which can be found as follows.

Case I: If $r<0$, then $f(x, y)$ has a local maximum at $a, b)$

Case II: If $r>0$, then $f(x, y)$ has a local minimum at (a, b).
(b) If $rt-s^2<0$, then (a, b) is called as saddle point of $f(x, y)$ where $f(x, y)$ has neither maximum nor minimum at (a, b).

Example 8: Find the stationary points of the function $f(x, y)=x^2y+3xy-7$ and classify them into extreme and saddle points.

Solution: Given $f(x, y)=x^2y+3xy-7$

$\therefore f_x=\frac{\partial f}{\partial x}=2xy+3y$ and $f_y=\frac{\partial f}{\partial y}=x^2+3x$

Now $f_x=0\Rightarrow 2xy+3y=0$ and $f_y=0$
$\Rightarrow x^2+3x=0$

$\Rightarrow\quad y=0$ and $x=\frac{-3}{2};\ x(x+3)x=0$ and $x=-3$

But for $x=\frac{3}{2}$, $f_y\neq 0$

$\therefore$ The stationary points of $f(x, y)$ are $(0, 0)$ and $(-3, 0)$
Now $r=f_{xx}=2y$; $s=f_{xy}=2x+3$ and $t=f_{yy}=0$
And $rt-s^2=2y\times 0-(2x+3)^2=-(2x+3)^2$
$\therefore rt-s^2<0$ at $(0, 0)$ as well as $(-3, 0)$
Hence the two stationary points $(0, 0)$ and $(-3, 0)$ are saddle points where $f(x, y)$ has neither maximum not minimum.

Example 9: Find the maximum value of the function $f(x, y, z)=z-2x^2-3y^2$ where $3xy-z+7=0$.

Solution: Given $f(x, y, z)=z-2x^2-3y^2$ (3)
Where $3xy-z+7=0$ (4)
$\Rightarrow z=3xy+7$ (5)
Substituting the value of z in (3), we have
$f=3xy+7-2x^2-3y^2$

$$\therefore f_x=\frac{\partial f}{\partial x}=3y-4x \text{ and } f_y=\frac{\partial f}{\partial y}=3x-6y$$

$f_x=0\Rightarrow 3y-4x=0$ and $f_y=0\Rightarrow 3x-6y=0$
$f_x=0$ and $f_y=0$ only when $x=0$ and $y=0$
$\therefore$ The stationary point is $(0, 0)$

Now $r=f_{xx}=\frac{\partial^2 f}{\partial x^2}=-4;\ s=f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=3$ and

$t=f_{yy}=\frac{\partial^2 f}{\partial y^2}=-6$

$\therefore rt-s^2=(-4)(-6)-3^2=24-9=15>0$ and $r=-4<0$
$\therefore f$ has a maximum value at $(0, 0)$
For $x=0$, $y=0$, from (5), $z=3\times 0\times 0+7\Rightarrow z=7$
$\therefore$ The maximum value exists for $f(x, y, z)$ at $(0, 0, 7)$ and that maximum value is $f(x, y, z)_{at\ (0, 0, 7)}=7-2\times 0^2-3\times 0^2=7$

Indefinite Integrals

If $f(x)$ and $g(x)$ are two functions of x such that $g'(x) = f(x)$, then the integral of $f(x)$ is $g(x)$. Further, $g(x)$ is called the antiderivative of $f(x)$.

The process of computing an integral of a function is called Integration and the function to be integrated is called integrand.

An integral of a function is not unique. If $g(x)$ is any one integral of $f(x)$, then $g(x) + c$ is also its integral, where C is any constant termed as constant of integration.

Definite Integrals

The difference in the values of an integral of a function $f(x)$ for two assigned values say a, b of the independent variable x, is called the Definite Integral of $f(x)$ over the interval $[a, b]$ and is denoted by $\int_a^b f(x)dx$

The number 'a' is called the lower limit and the number 'b' is the upper limit of integration.

Fundamental Theorem of Integral Calculus

If $f(x)$ is a function of x continuous in $[a, b]$, then $\int_a^b f(x)dx = g(b) - g(a)$ where $g(x)$ is a function such that $\frac{d}{dx} g(x) = f(x)$.

Properties of definite integrals

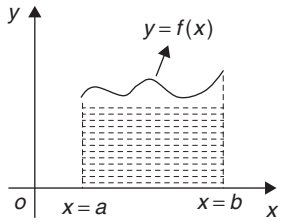
1. If $f(x)$ is a continuous function of x over $[a, b]$, and c belongs to $[a,b]$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
2. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_a^b Kf(x)dx = K\int_a^b f(x)dx$.
3. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
4. If $f(x)$ is continuous in some neighbourhood of a, then $\int_a^a f(x)dx = 0$.
5. If $f(x)$ and $g(x)$ are continuous in $[a, b]$, then $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
6. $\int_a^b f(x)dx = \int_a^b f(z)dz = \int_a^b f(t)dt$
7. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
8. $\int_{-a}^a f(x) = 0$, if $f(x)$ is odd
9. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even
10. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$
    $= 0$ if $f(2a-x) = -f(x)$
11. $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if $f(a+x) = f(x)$

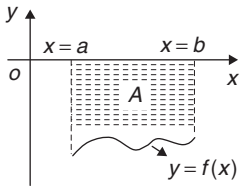
Applications of integration

Area as a definite integral

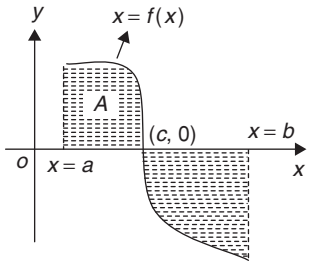
1. The area enclosed by a curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is given by:

$$A = \int_a^b |f(x)|dx = -\begin{cases} \int_a^b f(x)dx, & \text{if } f(x) \geq 0,\ a \leq x \leq b \\ \int_a^b f(x)dx, & \text{if } f(x) \leq 0,\ a \leq x \leq b \end{cases}$$

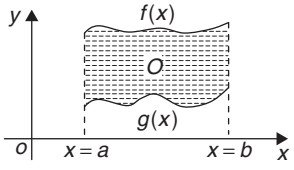


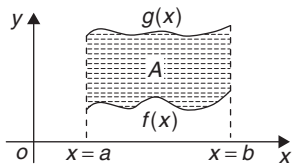


2. Similarly, the area enclosed by the curve $x = g(y)$, the lines $y = c$ and $y = d$ and the y-axis is $A = \int_c^d |g(y)dy|$
3. when $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $c \leq x \leq b$, then the area enclosed by the curve $y = f(x)$, the lines $x = a$ an $x = b$ and the x-axis is $A = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$



4. The area enclosed by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by





$$A = \int_a^b |f(x) - g(x)|\,dx = \begin{cases} \int_a^b (f(x) - g(x))dx, & \text{if } f(x) \geq g(x), \\ & a \leq x \leq b \\ \int (g(x) - f(x))dx, & \text{if } f(x) \leq g(x); \\ & a \leq x \leq b \end{cases}$$

Example 10: Find the area enclosed by the curve $y = x^3$, the line $y = 2$ and the y-axis in first quadrant?

Solution: The area bounded by $y = x^3$, $y = 2$ and the y-axis is the area OAB as shown in the figure.

So, the region OAB is bounded by the curve $x = y^{\frac{1}{3}}$, the lines $y = 0$ and $y = 2$ and the y-axis and $x = y^{\frac{1}{3}} \geq 0$, $y \in [0, 2]$

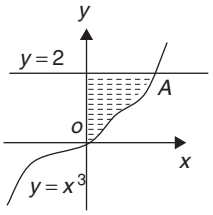
$\therefore$ The required area

$$= \left[\int_{y=0}^{2} y^{\frac{1}{3}} dy = \frac{3}{4} y^{\frac{4}{3}}\right]_0^2$$

$$= \frac{3}{4} \times 2^{\frac{4}{3}}$$

$$= \frac{3}{2^{\frac{2}{3}}}$$

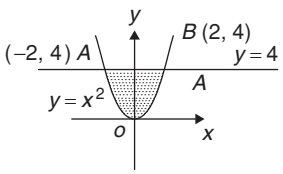
$$= \frac{3}{\sqrt[3]{4}}$$



Example 11: Find the area enclosed by the curve $y = x^2$ and line $y = 4$?

Solution: The area enclosed by the curve $y = x^2$ and the line $y = 4$ is the region OAB.

$\therefore$ The region OAB is bounded by line $y = 4$ and the curve $y = x^2$ from $x = -2$ to $x = 2$ and $4 \geq x^2$ for all $x \in [-2, 2]$



$\therefore$ The required area $= \int_{x=-2}^{2} (4 - x^2)\,dx$

$$= 2\int_0^2 (4 - x^2)dx \quad (\because \quad 4 - x^2 \text{as even})$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3}$$

Rectification

The process of determining the length of arcs of plane curves is called Rectification. The length of the arc can be calculated by any one of the methods given below.

Cartesian equations Let $y = f(x)$ be a function of x. The length of arc between the points with x-coordinates 'a' and 'b' is given by $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, provided $\frac{dy}{dx}$ is continuous on $[a, b]$.

Note: If the equation of the curve is given in the form $x = f(y)$, then the length of the arc between the points with y-coordinates 'c' and 'd' is given by

$$S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \text{ provided } \frac{dx}{dy} \text{ is continuous on } [c, d]$$

Parametric equations Let $x = f(t)$ and $y = g(t)$ be parametric functions of 't'.The length of the arc between the points $\{f(t_1), g(t_1)\}$ and $\{f(t_2), g(t_2)\}$ is given by

$$\int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt} + \frac{dy}{dt}\right]}\,dt \text{ provided } \frac{dx}{dt} \text{ and } \frac{dy}{dt} \text{ are both continuous on } [t_1, t_2].$$

Polar equations Let $r = f(\theta)$ be a function of θ, the length of the arc between the points $\{f(\theta_1), \theta_1\}$ and $\{f(\theta_2), \theta_2\}$ is given by $S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ provided $\frac{dr}{d\theta}$ is continuous along the arc.

If the equation of the curve is given in the form $\theta = f(r)$, then the length of the arc between the points $(r_1, f(r_1))$, $(r_2, f(r_2))$ is given by

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr \text{ provided } \frac{d\theta}{dr} \text{ is continuous along the arc.}$$

Improper integrals

Consider definite integral $\int_a^b f(x)dx$ (6)

If $f(x)$ is a function defined in a finite interval $[a, b]$ and $f(x)$ is continuous for all x which belongs to $[a, b]$

Then (6) is called proper integral.

If $f(x)$ is violated, at least one of these conditions then the integral is known as improper integral. These improper integrals are classified into three kinds.

Improper integral of the first kind In a definite integral if one or both limits of integration are infinite then it is an improper integral of first kind.

1. $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$
   (Singularity at upper limit)
2. $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$
3. $\int_{-\infty}^\infty f(x)dx = \lim_{\substack{b\to\infty\\ a\to-\infty}}\int_a^b f(x)dx.$ Or
4. $\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx.$

   Or $= \lim_{r\to\infty}\int_{-r}^r f(x)dx.$

Convergent: If the limits of the above integral exists or finite then the integral is said to be converge.

Divergent: If the limits do not exist then they are said to be Divergent.

Note:

1. Geometrically for $f(x) \geq 0$, the improper integral $\int_a^\infty f(x)dx$ denotes the area of an unbounded region lying between the curve $y = f(x)$ the ordinate $x = a$ and x-axis.
2. Let $f(x)$ and $g(x)$ be non-negative functions and $0 \leq f(x) \leq g(x)$ for $x \geq a$. If $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ is also converges and $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$.

   Similarly let $0 \leq g(x) \leq f(x)$ if $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ also diverges.

That is, the convergent or divergent of an improper integral by comparing it with a simple integral.

Improper integral of the second kind

Consider $\int_a^b f(x)dx$ (7)

If both the limits of (7) are finite $f(x)$ is undefined or discontinuous at a point in between a and b, then (7) is known as Improper integral of second kind.

This can be evaluated as follows.

Let $f(x)$ be undefined at a point c which belongs to (a, b) then

$$\int_a^b f(x)dx = \lim_{\epsilon\to0}\int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to0}\int_{c+\epsilon}^b f(x)dx.$$

If these limits exist then it is convergent otherwise it is divergent.

Improper integral of third kind If the limits of the integral are infinite or $f(x)$ may be discontinuous or both then the improper integral is known as third kind.

Notes:

1. $\int_1^\infty \frac{1}{x^p}dx$ is convergent when $p > 1$ and it is divergent when $p \leq 1$. This result is used in comparison test for testing the convergence or divergence of the integral of first kind.
2. $\int_a^c \frac{1}{(x-c)^p}dx$ is convergent for $p < 1$ and is divergent for $p \leq 1$. This is used for convergence or divergence of an improper integral of second kind.

Example 12: Examine $\int_1^\infty \frac{dx}{x^p}$ for convergence/divergence.

Solution:

Consider $\int_1^k \frac{dx}{x^p} = \int_1^k x^{-p}dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^k$ if $p \neq 1$

And $\Rightarrow$ $[\log x]_1^k$ if $p = 1$

Case I: If $p = 1$, $\int_1^k \frac{dx}{x} = \log k - \log 1 = \log k \to \infty$ when $k \to \infty$ it does not tend to a finite limit.

$\therefore$ It is divergent.

Case II: If $p \neq 1 \int_1^k \frac{dx}{x^p} = \frac{1}{1-p}[k^{1-p}]$ it converges

If $p > 1$ and diverges if $p \leq 1$.

Multiple integrals

Double Integrals: Integration of $f(x, y)$ over a region R in xy-plane is called a double integral.

$$\iint_R f(x, y)dR = \int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2} f(x, y)dxdy$$

Order of integration in a double integral Order of integration depends on the nature of limits of the variables.

Case 1: If the limits of y are function of x, say $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the limits of x are constants, say $x_1 = a$ and $x_2 = b$, where a and b are constants, then integrate wrt y first treating x as constant and then integrate wrt x.

$$\text{That is, } \iint_R f(x, y)dR = \int_{x_1=a}^{x_2=b}\begin{pmatrix} y_2 = f_2(x) \\ \int f(x, y)dy \\ y_1 = f_1(x)\end{pmatrix}dx$$

Case 2: If the limits of x are function of y, say $x_1 = g_1(y)$ and $x_2 = g_2(y)$ and the limits of y are constants, say $y_1 = c$ and y_2

$= d$, then integrate wrt x first treating y as constant and then integrate wrt y.

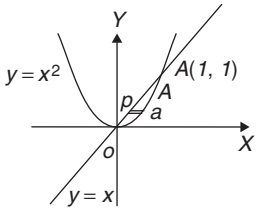
$$\text{That is, } \iint_R f(x, y)dR = \int_{y_1=c}^{y_2=d} \begin{pmatrix} x_2 = g_2(y) \\ \int f(x, y)dx \\ x_1 = g_1(y) \end{pmatrix} dy$$

Case III: If both the variables x and y have constant limits, then one can follow any order of integration.

Change of order of integration Evaluation of some of the double integrals can be made simple by changing the order of integration. In change of order of integration, we take the limits of the variables for the given region of integration in such a way that the order of integration reverses.

Example 13: Evaluate $\int_{x=0}^{1}\int_{y=x^2}^{x} \frac{x}{y} e^{-\frac{x^2}{y}} dydx$

Solution: Let $I = \int_{x=0}^{1}\int_{y=x^2}^{x} \frac{x}{y} e^{-\frac{x^2}{y}} dydx$ (8)



Evaluation of this integral can be made simple by changing the order integration.

From the limits of x and y given, the region of integration is the region bounded by the line $y = x$ and the parabola $y = x^2$ as shown in figure.

Now by changing the order of integration, we first integrate wrt x, along the horizontal strip PQ from $P(x = y)$ to $Q(x = \sqrt{y})$ and then

We integrate wrt y from $0(y = 0)$ to $A(y = 1)$

$$\therefore I = \int_{x=0}^{1}\int_{y=x^2}^{x} \frac{x}{y} e^{-\frac{x^2}{y}} dxdy$$

$$\therefore I = \int_{y=0}^{1}\left(\int_{x=y}^{\sqrt{y}} \frac{x}{y} e^{-\frac{x^2}{y}} dx\right) dy \quad (9)$$

$$\text{Put } \frac{x^2}{y} = t \Rightarrow \frac{2x}{y}dx = dt$$

$$\Rightarrow \frac{x}{y}dx = \frac{1}{2}dt$$

$$x = y \Rightarrow t = \frac{y^2}{y} = y \text{ and } x = \sqrt{y} \Rightarrow t\frac{(\sqrt{y})^2}{y} = 1$$

$\therefore$ (9) Becomes

$$I = \int_{y=0}^{1}\left(\int_{t=y}^{1} e^{-1}\frac{1}{2}dt\right)dy$$

$$= \int_{y=0}^{1}(e^t)\Big]_{t=y}^{1} dy = \int_{y=0}^{1}[-e^{-1} + e^{-y}]dy$$

$$= -ye^{-1} - e^{-y}$$

$$= (-e^{-1} - e^{-1}) - (0 - e^{-0})$$

$$= 1 - 2e^{-1} = \frac{e-2}{e}$$

Triple integrals Integration of a function $f(x, y, z)$ over a 3-dimensional region V is called the triple integral.

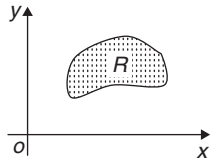
$$\int_V\int\int f(x, yx, z)dv = \int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2}\int_{z=z_1}^{z_2} f(x, y, z)dxdydz$$

Like double integrals, in triple integrals also the order of integration depends on the nature of the limits of the variables.

Applications of double and triple integrals

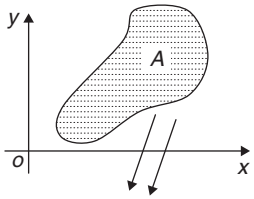
1. Area of the region R in xy-plane is given by

   $$\text{Area of } R = \int_R\int dxdy$$



2. Volume of the solid Revolution:
   (i) The volume of the solid of revolution obtained by revolving the area A about x-axis is

   $$\text{Volume} = V = \int_A\int 2\Pi y dxdy$$

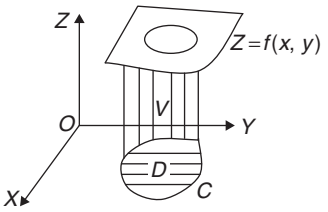


   (ii) The volume of the solid of revolution obtained by revolving the area A about y-axis is

   $$\text{Volume} = V = \int_R\int 2\Pi x dxdy$$

   (iii) Volume under the surface as a double integral: The volume V of the solid under the surface $Z = f(x, y)$ and above the xy-plane with the projection of $Z = f(x, y)$ on xy plane as its base is

   $$\text{Volume} = \iint_D f(x, y)dxdy$$



(iv) Volumes as a triple integral: The volume of the 3-dimensional region V is given by $\int_v\iiint dxdydz$

Example 14: Find the volume under the surface $x + 2y + z = 4$ and above the circle $x^2 + y^2 = 4$ in the xy-plane.

Solution:

Given surface is $x + 2y + z = 4$

$\Rightarrow z = 4 - x - 2y$ (10)

Let D be the region bounded by the circle $x^2 + y^2 = 4$ in xy-plane

$\therefore$ In D, y varies from $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$ and x varies from $x = -2$ to $x + 2$.

$\therefore$ The volume under the surface $x + 2y + z = 4$ and above the circle $x^2 + y^2 = 4$ in xy-plane is

$$V = \int_D\int zdxdy = \int_{x=-2}^{2}\int_{y=-2\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x-2y)dxdy \qquad (11)$$

Evaluation of this double integral can be made simple by changing it into polar coordinates.

In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$\therefore J = r$

Also, in the circle $x^2 + y^2 = 4$, r varies from $r = 0$ to $r = 2$ and θ varies from $\theta = 0$ to $\theta = 2\pi$

$\therefore$ From (11),

$$V = \iint_D(4-x-2y)dxdy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}(4 - r\cos\theta - 2r\sin\theta)\,|J|\,drd\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}(4 - r\cos\theta - 2r\sin\theta)rdrd\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=0}^{2}(4r - r^2\cos\theta - 2r^2\sin\theta)dr\right)d\theta$$

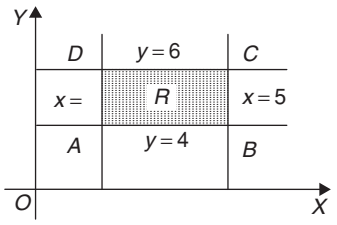
$$= \int_{\theta=0}^{2\pi}\left[2r^2 - \frac{r^3}{3}\cos\theta - \frac{2r^3}{3}\sin\theta\right]_{r=0}^{2}d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[8 - \frac{8}{3}\cos\theta - \frac{16}{3}\sin\theta\right]d\theta$$

$$= 8\theta - \frac{8}{3}\sin\theta + \frac{16}{3}\cos\theta\Big]_{\theta=0}^{2\pi} = 16\Pi$$

Example 15: Find the volume generated by the revolution of the rectangle formed by the lines $x = 2$, $x = 5$, $y = 4$ and $y = 6$ about x-axis.

Solution: The volume of the solid generated by revolving the rectangle $ABCD$ about x-axis $= V = \int_R\int 2\Pi ydxdy$



$$= \int_{x=2}^{5}\int_{y=4}^{6}2\pi\,ydxdy = \left(\int_{x=2}^{5}dx\right)\left(\int_{y=4}^{6}2\pi\,ydy\right)$$

$$= \left(x\right]_{x=2}^{5}\left)\left(\pi y^2\right]_{y=4}^{6}\right) = 3\times 20\pi = 60\pi$$

Change of variables Evaluation of some of the double (or) triple integrals can be made simple by changing the variables.

1. **In a double integral:** Let a double integral $\int_{R_{xy}}\iint f(x, y)dxdy$ in x and y is to be converted into the variables u and ϑ

   where $x = \phi(u, \vartheta)$ and $y = \Psi(u, \vartheta)$. Then

   $$\int_{R_{xy}}\int f(x, y)dxdy = \int_{R_{u\vartheta}}\int_1 f(\phi(u, \vartheta), \psi(u, \vartheta))\,|J|\,dud\vartheta$$

   Where $$J = \frac{\partial(x, y)}{\partial(u, \vartheta)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} \end{vmatrix}$$

   Is the Jacobian of x and y wrt u and ϑ and R_u^1 is the region of integration in u, ϑ-plane corresponding R_{xy} in xy-plane.

2. **In a triple integral:** Let a triple integral $\int_{R_{xyz}}\iiint f(x, y, z)\,dxdydz$ in x, y and z is to be converted into the variables u, ϑ and w, where $x = \phi(u, \vartheta, \omega)$, $y = \Psi(u, \vartheta, \omega)$ and $z = h(u, \vartheta, \omega)$

   Then $$\int_{R_{xyz}}\iint f(x, y, z)dxdydz = \int_{R_{1u\vartheta\omega}}\int\int f(\phi(u, \vartheta, \omega),$$
   $$\psi(u, \vartheta, \omega), h(u, \vartheta, \omega))\,|J|\,dud\vartheta d\omega$$

   where $$J = \frac{\partial(x, y, z)}{\partial(u, \vartheta, \omega)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \boldsymbol{\omega}} \end{vmatrix}$$

Jacobian of x, y and z wrt u, ϑ and ω and $R^1_{u\omega}$ is the region of integration in u, ϑ, ω, coordinate system corresponding to the region R_{xyz} in xyz coordinate system.

Fourier Series

Periodic function

A function $f(x)$ is said to be periodic if $f(x + a) = f(x)$ for all x. The least value of a is called the period of $f(x)$.

Example: $\sin x$, $\cos x$ are periodic functions with period 2π.

Note:

1. $f(x)$ and $g(x)$ are periodic functions with period k then a $f(x) + bg(x)$ is also a periodic function with period k.
2. If $f(x)$ is a periodic function with period k, then the period of $f(a x)$ is $\frac{k}{a}$.
3. If the periods of functions $f(x)$, $g(x)$ and $h(x)$ are a, b, c, respectively, then the period of $f(x) + g(x) + h(x)$ is the LCM of a, b and c.

Euler's formula for the Fourier coefficients Let $f(x)$ is a periodic function whose period is 2π and is integrable over a period. Then $f(x)$ can be represented by trigonometric series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a_o an, b_n are called Fourier coefficients and these are obtained by

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx \text{ for } n = 1,\ 2,\ 3$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx \text{ for } n = 1,\ 2,\ 3$$

Example: Obtain the Fourier series expansion of $f(x) = e^x$ in $(0, 2\pi)$

Solution: $a_o = \frac{1}{\pi}\int_o^{2\pi} f(x)dx$

$$= \frac{1}{\pi}\int_o^{2\pi} e^x dx$$

$$\frac{1}{\pi}e^x\Big]_o^{2\pi} = \frac{1}{\pi}(e^{2\pi} - 1) \qquad (12)$$

$$a_n = \frac{1}{\pi}\int_o^{2\pi} f(x)\cos nxdx$$

$$= \frac{1}{\pi}\int_o^{2\pi} e^x \cos nxdx$$

We know that $\int e^{ax}\cos bxdx$

$$= \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx]$$

$$\therefore \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{e^{2\pi}}{1+n^2}(\cos 2\pi n - 1)$$

$$b_n = \frac{1}{\pi}\int_o^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_o^{2\pi} e^x \sin nxdx$$

$$= \frac{1}{\pi}\frac{e^x(\sin nx - n\cos nx]_o^{2\pi}}{1+n^2}$$

$$\left(\int e^{ax}\sin bxdx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

$$\frac{1}{\pi}\frac{1}{1+n^2}(n - e^{2\pi}n\cos 2\pi n)$$

$$= \frac{n}{\pi(1+n^2)}(1 - e^{2\pi}\cos 2\pi n)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$= \frac{1}{2\pi}(e^{2\pi} - 1) + \sum_{n=1}^{\infty}$$

$$\left[\frac{1}{\pi}\frac{}{1+n^2}(e^{2\pi}\cos 2\pi n - 1) + \frac{n}{\pi(1+n^2)}(1 - e^{2\pi}\cos 2\pi n)\right]$$

Even and odd functions

Even function: A function $f(x)$ is said to be even if $f(–x) = f(x)$ for all x.

Example: x^2, $\cos x$

Odd function: A function $f(x)$ is said to be odd if $f(–x) = –f(x)$ for all x

Example: x^3, $\sin x$

Notes:

1. The sum of two odd functions is odd.
2. The product of an odd function and an even function is odd.
3. Product of two odd functions is even.

Fourier series for odd function and even function

Case I:

1. Let $f(x)$ is an even function in $(–\pi, \pi)$. Then the Fourier series of the even function contains only cosine terms and is known as Fourier cosine series and it is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where}$$

$$= a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

Case II:
If $f(x)$ is an odd function, then the Fourier series of an odd function contains only sine terms, and is known as Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Example: Expand the function $f(x) = \frac{\pi^2}{24} - \frac{x^2}{8}$ in Fourier series in the interval $(-\pi, \pi)$.

Solution: $f(x) = \frac{\pi^2}{24} - \frac{x^2}{8}$

$$f(-x) = \frac{\pi^2}{24} - \frac{(-x)^2}{8} = \frac{\pi^2}{24} - \frac{x^2}{8} = f(x)$$

$\therefore f(x)$ is an even function.

$$f(x) = a_o + \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\pi^2}{24} - \frac{x^2}{8} dx$$

$$= \frac{1}{\pi}\frac{\pi^2 x}{24} - \frac{x^3}{24}\Big]_0^x = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdxs$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\pi^2}{24} - \frac{x^2}{8}\right)\cos nxdx$$

$$= \frac{2}{\pi}\int_{o}^{\pi}\left(\frac{\pi^2}{24} - \frac{x^2}{8}\right)\cos nxdx$$

$$= \frac{2}{\pi}\left\{\left(\frac{\pi^2}{24} - \frac{x^2}{8}\right) - \left(\frac{\sin nx}{n}\right)\Big]_o^{\pi} \right.$$
$$\left. -\frac{1}{8}\int_0^{\pi}\left(\frac{-\sin nx}{x}\right)dx \right\}$$

$$= -\frac{2}{\pi}\int_o^{\pi}\frac{2x}{8}\frac{\sin nx}{n}dx$$

$$= \frac{2}{\pi}\left[\frac{2}{8n}\left\{\frac{-x\cos nx}{n}\right]_o^{\pi} - \int_o^{\pi}\frac{\cos nx}{n}dx\right.$$

$$= \frac{-4}{8\pi n^2}(\pi\cos n\pi)$$

$$= \frac{-1}{2n^2}(\text{con } n\pi),\ n = 1, 2, 3\ = \frac{(-1)^{n+1}}{2n^2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^2}\cos nx$$

$$= \frac{1}{2}\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

Function of any period (P = 2L) If the function $f(x)$ is of period $P = 2L$ has a Fourier series, then $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

Where the Fourier coefficients are as follows

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

Fourier series of even and odd functions Let $f(x)$ be an even function in $(-L, L)$, then the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

Where $a_o = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_o^L f(x)dx$

$$a_n = \frac{2}{L}\int_o^L f(x)\cos\frac{n\pi x}{L}dx.$$

Let $f(x)$ be an odd function in $(-L, L)$ then Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$$

Where $b_n = \frac{2}{L}\int_o^L f(x)\sin\frac{n\pi x}{L}dx$

Vector Calculus

If $\bar{r}$ is the position vector of a point P, having co-ordinates (x, y, z), then $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along OX, OY, OZ respectively, then:

$$|\bar{r}| = |x\bar{i} + y\bar{j} + z\bar{k}| = \sqrt{x^2 + y^2 + z^2}.$$

Given any vector $\bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$ its direction ratios are a, b, c and its direction cosines are given by:

$$\ell = \frac{a}{|\bar{v}|},\ m = \frac{b}{|\bar{v}|},\ n = \frac{c}{|\bar{v}|} \text{ and } \ell^2 + m^2 + n^2 = 1$$

Linear combinations

A vector $\bar{r}$ is said to be a linear combination of the vectors $\bar{a}, \bar{b}, \bar{c} \ldots$ etc. if there exist scalars x, y, z such that $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c} + \cdots$

Test of collinearity

Three points A, B, C with position vectors $\bar{a}, \bar{b}, \bar{c}$ respectively are collinear if f there exist scalars x, y, z not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = 0,$ where $x + y + z = 0$

Test of coplanarity

Four points A, B, C and D with position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar if there exist scalars x, y, z and u (not all zero) such that $x\bar{a} + y\bar{b} + z\bar{c} + u\bar{d} = 0,$ where $x + y + z + u = 0$

Linear dependence and independence

A system of vectors $\bar{a}, \bar{b}, \bar{c} \ldots$ is said to be linearly independent (L.I.) if $x\bar{a} + y\bar{b} + z\bar{c} + \cdots = 0$

$\Rightarrow x = y = z \cdots = 0$

If $\bar{a}, \bar{b}, \bar{c} \ldots$ is a system of vectors which is not LI, then they are linearly dependent (L.D) and for such system of vectors there exist scalars $x, y, z \cdots$ (not all zeros) such that $x\bar{a} + y\bar{b} + z\bar{c} + \cdots = 0$

Note: Every non-zero vector is LI.
Every set of four or more vectors is LD.
Every pair of non-zero non-collinear vectors is LI.
Every pair of collinear vectors is LD.
Three non-coplanar vectors are LI.
Three coplanar vectors are LD.

Vector variable

A variable of the form $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is called a vector variable and x, y, z are scalar variables.

Scalar function If t is a scalar variable on a range $a \le t \le b$ and a function f defined as $f = f(t)$ for $t \in [a, b]$ is called a scalar function of t.

Example: $f(t) = 9t^3 + 4t^2 + 7,$
$f(t) = \sin t + 5\cos t + e^t$, etc.

Vector function If t is a scalar variable defined on a domain $[a, b]$, and a function $\bar{F}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is called a vector function of the scalar variable t.

Note: t is generally taken as 'time'.

Differentiation If $\bar{F}(t)$ is a continuous single valued vector function of the variable t, then the derivative of $\bar{F}(t)$ is defined as $\dfrac{d\bar{F}}{dt} = \underset{\Delta t \to 0}{\text{Lt}} \dfrac{F(t + \Delta t) - F(t)}{\Delta t}$ where Δt is a small increment in t.

One can also look at second and higher order derivatives in a similar way.

Differentiation formula

1. The derivative of a constant vector with respect to any scalar variable is 0.
2. $\dfrac{d}{dt}[\bar{F}(t) \pm \bar{G}(t)] = \dfrac{d\bar{F}}{dt} \pm \dfrac{d\bar{G}}{dt}.$
3. $\dfrac{d}{dt}[s(t)\bar{F}(t)] = s(t) \cdot \dfrac{d\bar{F}}{dt} + \dfrac{ds}{dt} \cdot \bar{F}$
4. Chain rule: $\dfrac{d\bar{F}}{dt} = \dfrac{d\bar{F}}{du} \times \dfrac{du}{dt},$ where $\bar{F} = \bar{F}(u).$
5. Dot and cross products:

$$\frac{d}{dt}(\bar{F} \cdot \bar{G}) = \bar{F} \cdot \frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt} \cdot \bar{G},$$

$$\frac{d}{dt}(\bar{F} \times \bar{G}) = \bar{F} \times \frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt} \times \bar{G}.$$

6. Partial derivatives: If $\bar{F}$ is vector function dependent on $\bar{F} = \bar{F}(x, y, z),$ then partial derivative of $\bar{F}$ with respect to x is defined as $\dfrac{\partial \bar{F}}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{\bar{F}(x + \Delta x, y, z) - \bar{F}(x, y, z)}{\Delta x}.$

Likewise, one can also define $\dfrac{\partial \bar{F}}{\partial y}$ and $\dfrac{\partial \bar{F}}{\partial z}$ can be defined.

It is also possible to define higher order partial derivatives as:

$$\frac{\partial^2 \bar{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{F}}{\partial x}\right), \quad \frac{\partial^2 \bar{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \bar{F}}{\partial y}\right).$$

$$\frac{\partial^2 \bar{F}}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{F}}{\partial z}\right), \text{ etc}$$

Differential vectors

1. If $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$ then

$$d\bar{F} = dF_1\hat{i} + dF_2\hat{j} + dF_3\hat{k}$$

2. $d(\bar{F} \cdot \bar{G}) = \bar{F} \cdot d\bar{G} + d\bar{F} \cdot \bar{G}$
3. $d(\bar{F} \times \bar{G}) = \bar{F} \times d\bar{G} + d\bar{F} \times \bar{G}$
4. If $\bar{G} = \bar{G}(x, y, z)$ then

$$d\bar{G} = \frac{\partial \bar{G}}{\partial x}dx + \frac{\partial \bar{G}}{\partial y}dy + \frac{\partial \bar{G}}{\partial z}dz$$

Vector differential operators ∇ is to be read as del or nabla

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ is called Laplacian.}$$

Gradient of a scalar function

If $\phi(x, y, z)$ is a scalar function, then $\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ. One can also write the gradient of ϕ using the ∇ operator as grad

$$\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}=\nabla\phi$$

Now $\nabla\phi$ denotes a vector field.

Notes:

1. If ϕ is a constant, then $\nabla\phi=0$
2. If a vector $\bar{G}(x, y, z)$ is defined at all points in a region we say $\bar{G}$ is a vector field. A vector field is said to be irrational if G = grad ϕ for some scalar function ϕ.
3. Gradient can be used in finding directional derivative. (An example is discussed in worked examples section)
4. $\nabla\phi$ also gives the normal to the surface $\phi(x, y, z)=C$.
5. If $\nabla^2\phi=0$, the function is called the harmonic function.

Divergence of vector

$\bar{F}(x, y, z)$ be a vector field which is differentiable at each point (x, y, z) in some region of space, i.e., $\bar{F}$ is differentiable vector field. The scalar product of the vector operator ∇ and $\bar{F}$ gives a scalar which is termed as divergence.

$$\nabla\cdot\bar{F}=\hat{i}\cdot\frac{\partial\bar{F}}{\partial x}+\hat{j}\cdot\frac{\partial\bar{F}}{\partial y}+k\cdot\frac{\partial\bar{F}}{\partial z}$$

Note: If div $(\bar{F})$ or $\nabla\cdot\bar{F}=0$, then $\bar{F}$ is called 'solenoidal'

Curl of a vector

Let $\bar{F}(x, y, z)$ is a vector field defined for all (x, y, z) in a certain region of space and is differentiable, i.e., $\bar{F}$ is a differential vector field. The cross product of the vector operator ∇ with the vector $\bar{F}$ is termed as curl $\bar{F}$.

$$\text{curl } F=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix};\ \bar{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$$

Note: If curl $\bar{F}=0$, then $\bar{F}$ is said to be irrotational.

Standard results

1. div $(\phi f)=\phi$ div $f+f\cdot$grad ϕ or $\nabla\cdot\phi f=\phi\nabla\cdot f+f\cdot\nabla\phi$
2. curl $(\phi f)=\nabla\phi\times f+\phi$ curl f
3. div $(f\times g)=f\cdot$curl $g-g\cdot$ curl f
4. $\nabla\cdot\nabla\phi=$ div (grad ϕ) or $\nabla\cdot\nabla\phi=\nabla^2\phi$
5. curl (grad ϕ) = 0 or $\nabla\times(\nabla\phi)=0$, i.e., curl of a gradient equals 0.
6. div (curl f) = 0 or $\nabla\cdot(\nabla\times f)=0$
7. curl (curl f) = grad (div f) $-\nabla^2 f$(or) $\nabla\times(\nabla\times f)=\nabla(\nabla\cdot f)-\nabla^2 f$

Integration

Line integral

Let $\bar{F}(x, y, z)$ be a vector function defined on a region of space and let C be curve in that region, the integral $\int_c\bar{F}\cdot d\bar{r}$ is called the line integral.

For Riemann Integration,

$\int_{x=a}^{x=b} f dx$ the limits of integration are along the line segment joining $(a, 0)$, $(b, 0)$, where $a<b$.

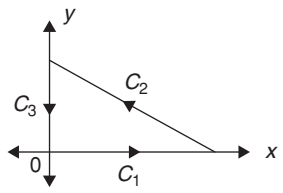
Here instead of line, we integrate along the curve C.

Circulation

The line integral around a closed curve C denoted by $\int_c F\cdot dr$ is called circulation of F around C.

Example: Evaluate $F\cdot dr$, where $F=xy\hat{i}+y^2\hat{j}$ along the triangle $x=0$, $y=0$ and $x+y=1$ in the first quadrant.

Solution:



$$\int_c F\cdot dr=\int_{C_1}xydx+y^2dy+\int_{C_2}xydx+y^2dy+\int_{C_3}xydx+y^2dy$$

C_1	C_2	C_3
$y=0$	$y=1-x$	$x=0$
$0<x<1$	$0<x<1$	$dx=0$
$dy=0$	$dy=-dx$	$1<y<0$

$$=\int_{x=0}^{1}[x(0)dx+0+\int_{x=0}^{1}x(1-x)dx+(1-x)^2(-dx)+\int_1^0 y^2dy$$

$$=\int_0^1(x-x^2-1-x^2+2x)dx+\int_1^0 y^2dy$$

$$=\int_0^1(-2x^2+3x-1)dx-\int_0^1 y^2dy$$

$$=\left(-\frac{2}{3}+\frac{3}{2}-1-\frac{1}{3}\right)=-\frac{1}{2}.$$

Surface Integral Let S be a closed surface, then the normal surface integral $\int_s FNds$ is called the flux of F over S.

Cartesian form Let $F(r)=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$, where, F_1, F_2, F_3, are continuous and differentiable functions of x, y, z. If

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ be the direction cosines of the unit normal N, then

$$N = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma.$$

$$\therefore \int_S F \cdot N ds = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma) ds$$

But then $ds\cos\alpha$, $ds\cos\beta$ and $ds\cos\gamma$ are the projections of ds on yz, zx and xy planes. If dx, dy, dz are the differentials along the areas then

$ds\cos\alpha = dy\,dz$; $ds\cos\beta = dz\,dx$; $ds\cos\gamma = dx\,dy$.

$$\therefore \int_S F \cdot N ds = \int\int_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Note: If R_1 is the projection of S on xy-plane, then

$$\int_S F \cdot N ds = \int_{R_1}\int F \cdot N \frac{dxdy}{\cos\gamma}$$

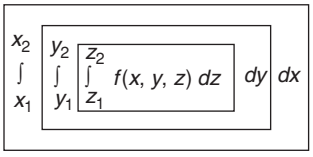
$$= F \cdot N \cdot \frac{dxdy}{|N \cdot \hat{k}|} (|N \cdot \hat{k}| = \cos\gamma)$$

Equivalently,

$$\int_S F \cdot N ds = \int\int_{R_2} F \cdot N \frac{dydz}{|N \cdot \hat{i}|} = \int\int_{R_3} F \cdot N \frac{dzdx}{|N \cdot \hat{j}|}$$

Volume integral

$$\int_{x_1}^{x^2}\int_{y_1}^{y^2}\int_{z_1}^{z^2} f(x, y, z) dzdydx$$



Gauss divergence theorem

If $\bar{F}$ is continuously differentiable vector function in the region bounded by a surface S, then the $\int_S F \cdot N ds = \int_V \text{div } F dv$ where N is the unit normal to the surface.

Greens theorem: If P and Q are scalar point functions, possessing continuous derivatives of the first order, in a region S of the xy plane bounded by a closed curve C then $\int_C P dx + Q dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

Stokes theorem If S is an open surface bounded by a closed curve C and f is a continuously differentiable vector point function, then $\int_C \bar{F} \cdot d\bar{r} = \int_S \text{curl } \bar{F} \cdot N ds$, where N is unit normal drawn at any point on the surface.

Example 14: If $\phi \equiv x^3 + y^3 + z^3 - 3xyz$, then find the value of grad ϕ at (2, 1, 1).

Solution: grad $\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ (by definition)

$$= \hat{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)$$

$$+ \hat{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz).$$

$$= 3[\hat{i}(x^2 - yz) + \hat{j}(y^2 - xz) + \hat{k}(z^2 - xy)]$$

$\therefore$ grad ϕ at (2, 1, 1)

$$= 3[\hat{i}(4-1) + \hat{j}(1-8) + \hat{k}(1-2)]$$

$$= 9\hat{i} - 21\hat{j} - 3\hat{k}$$

Examples 15: If $f = \hat{i}t^2 + 2bt\hat{j} + \hat{k}$, then find the value of b if (i) magnitude of f is constant (ii) Direction of f is constant

Solution:

(i) When magnitude of f is constant, $f \cdot \frac{df}{dt} = 0.$

$$\Rightarrow (\hat{i}t^2 + 2bt\hat{j} + \hat{k}) \cdot \frac{d}{dt}(\hat{i}t^2 + 2bt\hat{j} + \hat{k}) = 0$$

$$\Rightarrow 2t^3 + 4b^2t = 0 \Rightarrow b^2 + t^2 = 0 \Rightarrow b = 0.$$

(ii) When direction of f is constant, $f \times \frac{df}{dt} = 0.$

$$\Rightarrow (\hat{i}t^2 + 2bt\hat{j} + \hat{k}) \times \frac{d}{dt}(\hat{i}t^2 + 2bt\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & 2bt & 1 \\ 2t & 2b & 0 \end{vmatrix} = 0$$

$$\Rightarrow -2\hat{i}b - \hat{j}(-2t) + \hat{k}(2bt^2 - 4bt^2) = 0 \Rightarrow b = 0.$$

Example 16: If $p = x^2y\hat{i} - x^3\hat{j} + xyz^2\hat{k}$, then find div p and curl p.

Solution:

(i) div $p = \nabla \times p$

$$= \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(-x^3) + \frac{\partial}{\partial z}(xyz^2)$$

$$= 2xy - 0 + 2xyz = 2xy\,(1 + z)$$

(ii) curl $p = \nabla \times p = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -x^3 & xyz^2 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xyz)^2 - \frac{\partial}{\partial z}(-x^3)\right) - \left(\hat{j}\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial Z}(x^2y)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(-x^3)-\frac{\partial}{\partial y}(x^2y)\right)$$
$$= xz^2\hat{i}-yz^2\hat{j}+\hat{k}(-3x^2-x^2)$$
$$= xz^2\hat{i}-yz^2\hat{j}-4x^2\hat{k}$$

Example 17: Find the value of r if $p = xy^2\hat{i}+xyz^2\hat{j}+(r-2)xyz^3\hat{k}$ is solenoidal at (1, − 1, 1).

Solution: p is solenoidal $\Rightarrow$ div $p = 0 \Rightarrow \nabla\cdot p = 0$

$$\Rightarrow \frac{\partial p_1}{\partial x}+\frac{\partial p_2}{\partial y}+\frac{\partial p_3}{\partial z}=0$$
$$\Rightarrow \frac{\partial}{\partial x}xy^2+\frac{\partial}{\partial y}(xyz^2)+\frac{\partial}{\partial z}[(r-2)xyz^3]=0$$
$\Rightarrow y^2 + xz^2 + (r - 2)\ 3xyz^2 = 0$ at (1, −1, 1), div $p = 0$
$\Rightarrow (-1)^2 + (1)^2 + (r - 2)\ 3\ (1)\ (-1)\ (1)^2 = 0$
$$\Rightarrow 1+r-3r+6=0 \Rightarrow r=\frac{7}{2}.$$

Example 18: Find the value of a, if $P = (y^2 + 2xz)\hat{i} + (z^2 + 2xy)\hat{j} + (x^2 + ayz)\hat{k}$ is irrotational.

Solution: The vector P is irrotational

$\Rightarrow$ curl $P = 0 \Rightarrow \nabla \times P = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz & z^2+2xy & ayz+x^2 \end{vmatrix} = 0$$
$$= \hat{i}\left(\frac{\partial}{\partial y}(x^2+ayz)-\frac{\partial}{\partial z}(2xy+z^2)\right)$$
$$-\hat{j}\left(\frac{\partial}{\partial x}(x^2+ayz)-\frac{\partial}{\partial z}(y^2+2xz)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}(z^2+2xy)-\frac{\partial}{\partial y}(y^2+2xz)\right)=0$$
$$\Rightarrow \hat{i}(az-2z)+\hat{j}(2x-2x)-\hat{k}(2y-2y)=0$$
$$\Rightarrow \hat{i}\,z(a-2)=0=0\,\hat{i} \Rightarrow z(a-2)=0$$
$\Rightarrow z = 0$ or $a - 2 = 0 \Rightarrow a = 2$

Example 19: Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at (1, − 2, − 1).

Solution: Let $f = xy^2z - 3x - z^2 = 0$ and $g = 3x^2 - y^2 + 2z - 1 = 0$.

$\therefore$ grad $f = \hat{i}(y^2z-3)+\hat{j}(2xyz)+\hat{k}(xy^2-2z)$

grad $g = \hat{i}(6x)+\hat{j}(-2y)+\hat{k}(B)$

But, angle between two surfaces at a point is equal to angle between the normals to the surfaces at that point.

$\therefore$ Let $\bar{n}_1 = \text{grad}\, f$ at (1, − 2, 1) and $\bar{n}_2 = \text{grad}\, g$ at (1, − 2, 1) respectively

$\therefore$ $\bar{n}_1 = (\text{grad}\, f)(1, -2, 1)$
$$= \hat{i}[(-2)^2 1-3]+\hat{j}[2(A)(-2)1]$$
$$+\hat{k}[1(-2)^2-2(A)]=\hat{i}-4\hat{j}+2\hat{k}$$
$\bar{n}_2 = (\text{grad}\, g)(1, -2, 1)$
$$= \hat{i}[6\ (A)]+\hat{j}[-2(-2)]+\hat{k}(B)=6\hat{i}+4\hat{j}+2\hat{k}$$

$\therefore$ Let the angle between the normals $\bar{n}_1$ and $\bar{n}_2$ be θ.

So, $\bar{n}_1\cdot\bar{n}_2 = |n_1||n_2|\cos\theta \Rightarrow 6-16+4$
$$= (\sqrt{1+16+4})(\sqrt{36+16+4})\cos\theta$$
$$\therefore \cos\theta = \left|\frac{-6}{\sqrt{21}\sqrt{56}}\right| = \left|\frac{-3}{7\sqrt{6}}\right| = \frac{3}{7\sqrt{6}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$$

Example 20: If $\bar{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ evaluate $\oint \bar{F}\cdot d\bar{r}$ along the straight line C from (0, 0, 0) to (1, 2, 3).

Solution: The equation of the line joining (0, 0, 0) and (1, 2, 3) is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=(t)$.

Then along the line C, $x = t$, $y = 2t$, $z = 3t$.

$\therefore \bar{r} = x\hat{i}+y\hat{j}+z\hat{k} = t\,\hat{i}+2t\,\hat{j}+3t\hat{k}$ $dr = \hat{i}+2\hat{j}+3\hat{k}$

Given $\bar{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$

And along C $\bar{F} = [t^2 + (2t)^2]\hat{i} - 2t(2t)\hat{j} = 5t^2\hat{i} - 4t^2\,\hat{j}$

$\therefore F\cdot dr = (5t^2 - 8t^2 + 0)\,dt = -3t^2\,dt$

at (0, 0, 0), $t = 0$ and at (1, 2, 3), $t = 1$.

$$\therefore \int_c F\cdot dr = \int_{t=0}^{1} -3t^2 dt = \left(\frac{-3t^3}{3}\right)_0^1 = -1$$

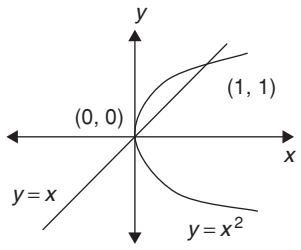
Example 21: Evaluate by Green's theorem (C) $(xy + y^2)\,dx + x^2\,dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Here $P = xy + y^2 \quad \therefore \quad \frac{\partial P}{\partial y} = x + 2y$

$$Q = x^2 \quad \therefore \quad \frac{\partial Q}{\partial x} = 2x$$

Hence by Green's theorem,

$$\int_C (xy+y^2)dx + x^2dy = \iint_S (2x - x - 2y)dx\,dy$$
$$= \iint_S (x-2y)dx\,dy = \int_{x=0}^{1}\left[\int_{y=x}^{x^2}(x-2y)dy\right]dx$$



$$= \int_{x=0}^{1} [xy - y^2]_{y=x}^{x^2} = \int_{x=0}^{1} (x^3 - x^4)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^5}{5} \right)_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

Example 22: Evaluate $\int_c F \cdot dr$ by Stokes theorem,

If $F = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, where c is the rectangle formed $x = \pm a$, $y = 0$ and $y = b$.

Solution: $\bar{F} = (x^2 + y^2)\hat{i} - 2xyj$

By Stokes theorem,

$$\int (\nabla \times \bar{F}) \cdot \bar{N}\, ds = \int_C \bar{F} \cdot d\bar{r}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$$

$$\therefore \int (\nabla \times \bar{F}) \cdot \bar{N}\, ds$$

$$= \int (-4ky) \cdot N\, ds = \int -4y\, (N \cdot k)\, ds \int_R \int -4y\, dx\, dy$$

Since $N \cdot k\, ds = dx\, dy$

And R is the region bounded by the rectangle.

$$= \int_{x=-a}^{a} \int_{y=0}^{b} (-4y)dx\, dx = \int_{-a}^{a} \left[\frac{-4y^2}{2} \right]_0^b dx$$

$$= -2\int_{-a}^{a} (b^2 - 0)dx = -2b^2 [x]_{-a}^{a} = -4ab^2.$$

EXERCISE

Practice Problems

Directions for questions 1 to 67: Select the correct alternative from the given choices.

1. $\lim_{x \to \infty} \{3x - \sqrt{9x^2 - x}\} =$ _____.

(A) $\frac{1}{6}$ (B) 3

(C) 6 (D) None of these

2. $\lim_{x \to 0} \left[\frac{24\cos x - 24 + 12x^2 - x^4}{24x^6} \right] =$

(A) $\frac{1}{720}$ (B) $\frac{-1}{120}$

(C) $\frac{1}{120}$ (D) $\frac{-1}{720}$

3. Evaluate $\lim_{x \to 2.7} (x - [x])$, where $[x]$ is the greatest integer less than or equal to x

(A) −0.3 (B) 0.7

(C) 4.7 (D) 2

4. $\lim_{x \to 0} \left(\frac{2^x + 3^x}{2} \right)^{\frac{1}{x}} =$

(A) 1 (B) $\sqrt{3}$

(C) $\sqrt{6}$ (D) $\sqrt{2}$

5. $\lim_{x \to 2} |x - 2| + [x - 2] =$

(A) 0 (B) Only left limit exists

(C) Only right limit exists (D) Limit does not exist

6. $f(x) = \begin{cases} 5x - 4 & 0 < x \le 1 \\ 4x^2 - 3x & 1 < x < 2 \end{cases}$ at $x = 1$

(A) Left hand continuous at $x = 1$

(B) Right hand continuous at $x = 1$

(C) Continuous at $x = 1$

(D) None of these

7. Check the continuity of the following function

$$f(x) = \begin{cases} \frac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ a^2 & \text{when } x = 0 \end{cases} \text{ at } x = 0$$

(A) Continuous at $x = 0$

(B) Discontinuous at $x = 0$

(C) Discontinuous of first kind

(D) None of these

8. If $f(x) = \begin{cases} 7 & x < 5, \\ ax + b & 5 < x < 7, \\ 11 & x > 7 \end{cases}$ is continuous on R,

Then the values of a and b are

(A) $a = 2, b = 3$ (B) $a = -2, b = 3$

(C) $a = 3, b = -2$ (D) $a = 2, b = -3$

9. Find values of a, b, c so that $f(x)$

$$= \begin{cases} a\cot^{-1}(x-3), & 0 \le x < 3 \\ bx, & x = 3 \\ c\tan^{-1}\left(\dfrac{1}{x-3}\right), & 3 < x < 4 \\ \cos^{-1}(4-x) + a\pi, & 4 \le x \le 5 \end{cases}$$

is continuous in the interval [0, 6].

(A) $a = \dfrac{2}{3},\ b = \dfrac{\pi}{9},\ c = \dfrac{2}{3}$

(B) $a = -\dfrac{2}{3},\ b = -\dfrac{\pi}{9},\ c = -\dfrac{2}{3}$

(C) $a = \dfrac{3}{2},\ b = \dfrac{\pi}{9},\ c = -\dfrac{2}{3}$

(D) $a = -\dfrac{3}{2},\ b = -\dfrac{\pi}{9},\ c = \dfrac{2}{3}$

10. Let f$(x) = \max(1 - x, x^2 - 1)$. Then f is:

(A) Not continuous at $x = 1, -2$

(B) Continuous and differentiable everywhere

(C) Not differentiable at $x = -2, 1$.

(D) Continuous but not differentiable at $x = 1, -1$

11. Consider the function $f(x)$:

$= \dfrac{1}{x-1} + \dfrac{1}{3-x}$ defined in the interval [1, 3]

(P) f is continuous on [1, 3]

(Q) f is differentiable on (1, 3)

(R) There exists $c \in (1, 3)$ such that $f'(c) = 0$ which of the above statements are true?

(A) P, Q only (B) Q, R only

(C) P, R only (D) P, Q, R

12. A function f: $R \to R$ is such that $f(x + y) = f(x).f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. If $f(x)$ is differentiable and $f'(0) = 2$, then

(A) $f'(x) = 2f(x)$ (B) $f(x) = 2f'(x)$

(C) $f(x) = f'(x)$ (D) $f'(x) = -f(x)$

13. The function $f(x) = x(1 - x\cot x) - \dfrac{1}{x}$ in $\left(0, \dfrac{\pi}{2}\right)$ has:

(A) Only one minimum

(B) Only one maximum

(C) No extrema

(D) One maximum and one minimum

14. Which of the following statement(s) is/are true?

(A) $y = x^2$ has a minimum value at $x = 0$

(B) $y = |x - 3|$ has a minimum value at $x = 3$

(C) The maximum value of the function $y = \dfrac{1}{1 + x^2}$ is 1

(D) All the above

15. The maximum and minimum values of $f(x) = 3\sin^2 x + 4\cos^2 x$ is:

(A) $\{-4, -3\}$ (B) $\{7, 3\}$

(C) $\{4, -3\}$ (D) $\{4, 3\}$

16. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$, where $a > 0$, attains its maximum and minimum at $x = p$ and $x = q$ respectively such that $p^2 = q$, then the value of 'a' is

(A) 2 (B) $\dfrac{1}{4}$

(C) $\dfrac{1}{8}$ (D) 4

Common Data for Questions 17 and 18: The sum of the hypotenuse and one side of a right angled triangle is given as units.

17. When the area is maximum the ratio of the side and the hypotenuse is _____.

(A) 2 : 1 (B) 1 : 3

(C) 1 : 2 (D) 2 : 3

18. When the area is maximum, find the angle between the hypotenuse and the other side is _____.

(A) 60° (B) 30°

(C) 45° (D) None of these

19. Consider $f(x) = |x^2 - 3|,\ 0 \le x \le \sqrt{6}$ and $g(x) = \begin{cases} 3^x, & 0 \le x \le 1, \\ 4 - x, & 1 < x \le 3 \end{cases}$. Then Rolle's theorem can be applied in the respective intervals

(A) To both $f(x)$ and $g(x)$

(B) Only to $f(x)$

(C) Only to $g(x)$

(D) Neither to $f(x)$ nor to $g(x)$

20. If the function $f(x) = px^2 + qx^2 + rx + s$ on [0, 1], satisfies the mean value theorem, then the value of c in the interval (0, 1) is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$

(C) $\dfrac{2}{3}$ (D) $\dfrac{2}{3\sqrt{3}}$

21. $f(x) = \dfrac{x^2}{x+1}$ increases in

(A) (–2, 0) (B) [–4, –2]

(C) $(-\infty, -2] \cup [0, \infty)$ (D) $(-\infty, -2) \cup (0, \infty)$

22. If $u = \left(\dfrac{\sqrt[4]{x} + \sqrt[4]{y}}{\sqrt[6]{x} + \sqrt[6]{y}}\right)^6$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} =$

(A) $\dfrac{u}{2}$ (B) $\dfrac{4}{u}$

(C) $4u$ (D) $6u$

23. If $u = \log(x + \sqrt{x^2 + y^2})$, then $e^u \frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} =$

(A) 0 (B) 1
(C) –1 (D) u

Common Data for Questions 24 and 25:

Let $u = \tan f = \dfrac{x^{\frac{5}{2}} + y^{\frac{5}{2}} + z^{\frac{5}{2}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}}$

24. The value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ is equal to

(A) 2 tan$f$ (B) tan$f$
(C) 2 sin2f (D) sin 2f

25. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} =$

(A) tan 2f (B) cot 2f
(C) cos 2f (D) sin 2f

26. The stationary points of the function $f(x, y) = x^3 + y^4 - 27x + 32y + 100$ is/are

(A) (3, 2), (3, −2) (B) (−3, 2), (−3, −2)
(C) (3, 2), (−3, −2) (D) (3, −2), (−3, −2)

27. For the function $f(x, y) = 2x^2 + 4y^2 + 4xy + 2x + 10y + 7$

(A) Local maximum exists, but no local minimum
(B) Local minimum exists, but no local maximum
(C) Neither local minimum nor local maximum exists
(D) Both local minimum and local maximum exists

28. For the function xyz, if $x + y + z = 3$, then the local maximum occurs for xyz at the point _____

(A) $\left(4, \frac{1}{2}, \frac{1}{2}\right)$ (B) (5, –1, –1)
(C) (1, 1, 1) (D) (7, –3, –1)

29. The ratio of the dimensions of a rectangular box of volume 64 cubic units and open at the top that requires least material for its construction is

(A) 2 : 2 : 1 (B) 2 : 4 : 5
(C) 2 : 3 : 4 (D) 1 : 2 : 3

30. Area bounded by the curve $y^2 = x$ and the line $x = 3$ is _____ sq units:

(A) $2\sqrt{3}$ (B) $4\sqrt{3}$
(C) $6\sqrt{3}$ (D) $8\sqrt{3}$

31. Area bounded by the curve$y = -3x^2$, $x = 2$ and the two coordinate axes is _____ sq units.

(A) 2 (B) 3
(C) 6 (D) 8

32. The volume of the solid obtained by revolving the area bounded by the parabola $y^2 = x - 4$, x – axis and the lines $x = 4$ and $x = 7$, about x – axis is _____ cubic unit

(A) $\frac{9}{2}\pi$ (B) $\frac{11}{2}\pi$
(C) $\frac{13}{2}\pi$ (D) $\frac{15}{2}\pi$

33. The length of arc of s the curvey = ln (cos x) from $x = 0$ to $x = \frac{\pi}{4}$ is _____.

(A) ln $(1 + \sqrt{2})$ (B) ln $(\sqrt{2} - 1)$
(C) ln $(2 + \sqrt{3})$ (D) ln $(2 - \sqrt{3})$

34. $\int_1^\infty \frac{1}{x^{1.0001}}dx =$ _____.

(A) 1000 (B) 100000
(C) 10000 (D) 1000000

35. $\int_0^3 \frac{1}{(x-2)^{\frac{4}{5}}}dx =$ _____.

(A) $5 - 2^{\frac{1}{5}}$ (B) $5 + 2^{\frac{1}{5}}$
(C) $5(1-2)^{\frac{1}{5}}$ (D) $5\left[1 + 2^{\frac{1}{5}}\right]$

36. $\int_{-1}^{1} \frac{1}{x^{\frac{2}{5}}}dx =$ _____

(A) 5 (B) $\frac{10}{3}$
(C) $\frac{5}{2}$ (D) $\frac{5}{3}$

37. $\int_3^\infty \frac{4x+7}{x^6+10}dx$ is _____.

(A) Convergent
(B) Divergent
(C) Cannot be determined
(D) None of these

38. $\int_1^p \int_2^q \frac{dxdy}{xy} =$ _____.

(A) $\log p \cdot \log\left(\frac{q}{2}\right)$ (B) $\log \pi \log \theta$
(C) $\log\left(\frac{p}{2q}\right)$ (D) $\log\left(\frac{q}{2p}\right)$

39. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dxdy}{\sqrt{1-x^2-y^2}}$.

(A) $\frac{\pi}{4}$ (B) 0
(C) $\frac{\pi}{2}$ (D) 1

40. $\int_0^{\log t}\int_0^{x} e^{x+y}\,dxdy =$ _____.

(A) $\frac{t^2}{2}-t+\frac{1}{2}$ (B) $\frac{t^2-t}{2}$

(C) $\frac{t^2-1}{2}$ (D) $\frac{t^2+t-1}{2}$

41. The value of $\int_0^2\int_{e^x}^{e^2}\frac{dydx}{\ell ny^2}$ is:

(A) e^2-1 (B) $\frac{1}{2}(e^2-1)$

(C) e^2+1 (D) $\frac{1}{2}(e^2+1)$

42. By changing the order of integration, the integral $\int_2^{\infty}\int_0^{x-2} f(x,y)dxdy$ becomes _____

(A) $\int_0^{\infty}\int_2^{y+2} f(x,y)dxdy$ (B) $\int_0^{\infty}\int_{y+2}^{2} f(x,y)dxdy$

(C) $\int_0^{\infty}\int_0^{\infty} f(x,y)dxdy$ (D) $\int_1^{\infty}\int_{y+2}^{1} f(x,y)dxdy$

43. By changing the variables in the double integral $\int_R\int\frac{dxdy}{xy}$, where $x=e^{u+v}$ and $y=uv$, it changes to $\int_R\int\phi(u,v)dudv$ then $\phi(u,v)$ is

(A) $\frac{R}{(e^{u+v})(uv)}$ (B) $\frac{e^{u+v}}{uv}$

(C) $\frac{1}{v}+\frac{1}{u}$ (D) $\frac{1}{v}-\frac{1}{u}$

44. The area bounded by the circle $x^2+y^2=6$ and the parabola $y=x^2$ is given by

(A) $\int_{x=-2}^{2}\int_{y=\sqrt{x}}^{\sqrt{x^2-6}} dydx$

(B) $\int_{x=-\sqrt{2}}^{\sqrt{2}}\int_{y=\sqrt{x}}^{\sqrt{6-x^2}} dydx$

(C) $\int_{x=-2}^{2}\int_{y=\sqrt{x}}^{\sqrt{x^2-6}} (x^2+y^2)dydx$

(D) $\int_{x=-\sqrt{2}}^{\sqrt{2}}\int_{y=\sqrt{x}}^{\sqrt{6-x^2}} (y-x^2)dxdy$

45. The volume of the region under the plane $12x+4y-32=0$, $z\ge 0$, and above the region bounded by $x=1$, $x=3$, $y=x^2$ and $y=4x^2$ is _____ cubic units:

(A) 240 (B) 484

(C) 724 (D) 860

46. The volume of the solid generated by revolving the triangle formed by the lines $y=1$, $y=x$ and $x+y=4$ about x – axis is _____ cubic units:

(A) $\frac{2\pi}{3}$ (B) $\frac{4\pi}{3}$

(C) 2π (D) $\frac{8}{3}\pi$

47. Let $f(x)=\begin{cases}0 & \text{if } -\pi\le x\le 0\\ x^3 & \text{if } 0<x\le\pi\end{cases}$

Which is a periodic function with period 2π, then $a_0=$ _____.

(A) $\frac{\pi^3}{4}$ (B) $\frac{\pi^3}{8}$

(C) $\frac{\pi^3}{12}$ (D) $\frac{\pi^3}{16}$

48. The value of the fourier coefficient a_n for $n\ge 2$ for $f(x)=x\sin x$ in $(-\pi,\pi)$ is _____.

(A) $\cos\frac{(n-1)\pi}{n-1}\cos nx$

(B) $\cos\frac{2(n-1)\pi}{n-1}\cos nx$

(C) $\frac{1}{2}+\sum_{n=1}^{\infty}\cos nx$

(D) $\left[\cos\frac{(n-1)\pi}{n-1}-\cos\frac{(n+1)\pi}{n+1}\right]$

Common Data for Questions 49 to 50:
Let $f(x)=x^2$ in the interval $(-\pi,\pi)$.

49. The Fourier series of $f(x)$ is _____.

(A) $\frac{\pi^2}{3}$

(B) $\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\cos nx$

(C) $\sum_{n=1}^{\infty}\cos nx$

(D) $\frac{\pi^2}{3}+\sum_{n=1}^{\infty}4\frac{(-1)^n}{n^2}\cos nx$

50. The value of $\frac{\pi^2}{12}$ is

(A) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ (B) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

(C) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ (D) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

51. If $\bar{r}$ is the position vector of a particle which passes along the curve $x=3\sin 4t$, $y=3\cos 4t$, and $z=5t$ $(t>0)$. The magnitude of its velocity and acceleration respectively are

(A) 13, 45 (B) 12, 48
(C) 13, 48 (D) 12, 45

52. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|r| = r$, then $\nabla r^n =$

(A) $n(n-1)r^{n-1}\bar{r}$ (B) $n(n-2)r^{n-2}\bar{r}$

(C) $n \cdot r^{n-2} \cdot \bar{r}$ (D) $n(n-1)\bar{r}$

53. The directional derivative of $f = x^{3y} + y^3 z + z^3 x$ in the direction of $i + 2j + 2k$ at (0, 1, – 1) is

(A) $\frac{5}{3}$ (B) $\frac{4}{3}$

(C) $\frac{-4}{3}$ (D) $\frac{-5}{3}$

54. The magnitude of maximum directional derivative of $\phi = 2xy^2 - xyz + y^2 z$ in the direction from the point (1, – 1, 1) is

(A) 62 (B) $\sqrt{52}$
(C) $\sqrt{62}$ (D) $\sqrt{56}$

55. The values of constants a, b, c respectively such that the directional derivative of the scalar function $\phi = ax^3 + by^3 + cz^3$ at (1, – 1, –1) has maximum magnitude of 3 in the direction parallel to y – axis is

(A) 0, 1, 0 (B) 1, 0, 1
(C) 0, 1, 1 (D) 1, 1, 0

56. The directional derivative of a scalar point function is a function of

(A) Only direction (B) Only position
(C) Either A or B (D) Both A and B

57. The values of div $\bar{r}$ and curl $\bar{r}$ respectively when $\bar{r} = 2x\bar{i} - y\bar{j} + 3z\bar{k}$ is:

(A) 4; $\bar{i}$ (B) 0, $\bar{0}$

(C) 4, 4 k (D) 4, $\bar{0}$

58. If $\bar{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, then the value of $\bar{F} \cdot \text{curl } \bar{F} =$

(A) 0 (B) 3
(C) 2 (D) 1

59. For what value of p the vector $f = (2x + 3y)i + (z + 2y)j + (x - pz)k$ is solenoidal?

(A) 4 (B) –4
(C) 2 (D) 0

60. For what values of p, q and r the vector $f = (x + ry - z)i + (3x - y + qz)j + (px + y - z)k$ is irrotational?

(A) $p = 1, q = -1, r = 3$
(B) $p = -1, q = 1, r = 3$
(C) $p = -1, q = 1, r = -3$
(D) $p = 1, q = 1, r = -3$

61. If $\nabla\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$, then $\phi(x, y, z) =$

(A) $xyz + f(y, z)$; $f \neq$ constant
(B) $xyz + g(x, z)$; $g \neq$ constant
(C) $xyz + h(x, y)$; $h \neq$ constant
(D) $xyz + k$; k is a constant

62. If $\bar{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, compute the line integral $\int_C \bar{F} \cdot d\bar{r}$ where C is $y = x^3$ in the xy-plane joining (1, 1) and (2, 8).

(A) 35 (B) –32
(C) 12 (D) 18

63. Compute $\int_S x^2 y^2 ds$ around the circle $x = \cos t$ and $y = \sin t$.

(A) $\frac{\pi}{4}$ (B) 0

(C) $\frac{\pi^3}{2}$ (D) π

64. If $\bar{F} = y^2\hat{i} - 2xy\hat{j}$, compute the circulation $\int_C \bar{F} \cdot d\bar{r}$ where C is the rectangle bounded by $y = 0$, $y = 1$, $x = 0$ and $x = 2$.

(A) 3 (B) 4
(C) –4 (D) –3

65. Find the area of the region in the first quadrant bounded by the curves $y = 4x$, $y = \frac{1}{x}$ and $y = \frac{x}{4}$ using green's theorem.

(A) log 2 (B) $\frac{1}{2}\log 2$
(C) log 4 (D) log 16

66. Evaluate $\iint_s F \cdot nds$ where $F = 2xz\bar{i} - yz\bar{j} + yx\bar{k}$ where S is the cube bounded by $x = 0$, $x = 3$, $y = 0$, $y = 3$ and $z = 0$, $z = 3$.

(A) $\frac{27}{2}$ (B) $\frac{81}{4}$

(C) $\frac{27}{4}$ (D) $\frac{81}{2}$

67. If S is a closed surface bounding a volume V and n is unit normal to the surface 'S' then $\iint_S \bar{r}\,\bar{n} ds =$

(A) 4 V
(B) 3 V
(C) 2 V
(D) V

Previous Years' Questions

1. If $S = \int_1^{\infty} x^{-3} dx$, then S has the value: **[2005]**

(A) $\frac{-1}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{2}$ (D) 1

2. For the function $f(x) = x^2 e^{-x}$, the maximum occurs when x is equal to: **[2005]**

(A) 2 (B) 1

(C) 0 (D) –1

3. For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1, 3) is: **[2005]**

(A) $\sqrt{\frac{13}{9}}$ (B) $\sqrt{\frac{9}{2}}$

(C) $\sqrt{5}$ (D) $\frac{9}{2}$

4. The Fourier series for the function $f(x) = \sin^2 x$ is: **[2005]**

(A) $\sin x + \sin 2x$ (B) $1 - \cos 2x$

(C) $\sin 2x + \cos 2x$ (D) $0.5 - 0.5 \cos 2x$

5. $x(t)$ is a real valued function of a real variable with period T. Its trigonometric Fourier Series expansion contains no terms of frequency $\omega = 2\pi \frac{(2k)}{T}$; $k = 1, 2,$ …Also, no sine terms are present. Then $x(t)$ satisfies the equation. **[2006]**

(A) $x(t) = -x(t - T)$

(B) $x(t) = x(T - t) = -x(-t)$

(C) $x(t) = x(T - t) = -x\left(t - \frac{T}{2}\right)$

(D) $x(t) = x(t - T) = x\left(t - \frac{T}{2}\right)$

6. The expression $V = \int_0^H \pi R^2 (1 - h/H)^2 dh$ for the volume of a cone is equal to: **[2006]**

(A) $\int_0^R \pi R^2 (1 - h/H)^2 dr$

(B) $\int_0^R \pi R^2 (1 - h/H)^2 dh$

(C) $\int_0^H 4\pi rH\left(1 - \frac{r}{R}\right) dh$

(D) $\int_0^H 4\pi rH\left(1 - \frac{r}{R}\right)^2 dr$

7. A surface $S(x, y) = 2x + 5y - 3$ is integrated once over a path consisting of the points that satisfy $(x+1)^2 + (y-1)^2 = \sqrt{2}$. The integral evaluates to: **[2006]**

(A) $17\sqrt{2}$ (B) $\frac{17}{\sqrt{2}}$

(C) $\frac{\sqrt{2}}{17}$ (D) 0

8. Divergence of the vector field $V(x, y, z) = -(x \cos xy + y)i + (y \cos xy)j + (\sin z^2 + x^2 + y^2)k$ is **[2007]**

(A) $2z \cos z^2$ (B) $\sin xy + 2z \cos z^2$

(C) $x \sin xy - \cos z$ (D) None of these

9. The integral $\frac{1}{2\pi}\int_0^{2\pi} \sin(t - \tau)\cos\tau\, d\tau \;\cos t\, d\tau$ equals: **[2007]**

(A) $\sin t \cos t$ (B) 0

(C) $\left(\frac{1}{2}\right)\cos t$ (D) $\left(\frac{1}{2}\right)\sin t$

10. Consider function $f(x) = (x^2 - 4)^2$ where x is a real number. Then the function has: **[2008]**

(A) Only one minimum (B) Only two minima

(C) Three minima (D) Three maxima

11. $f(x, y)$ is a continuous function defined over $(x, y) \in [0, 1] \times [0, 1]$. Given the two constraints, $x > y^2$ and $y > x^2$, the volume under $f(x, y)$ is **[2009]**

(A) $\int_{y=0}^{y=1}\int_{x=y^2}^{x=\sqrt{y}} f(x, y)dxdy$

(B) $\int_{y=x^2}^{y=1}\int_{x=y^2}^{x=1} f(x, y)dxdy$

(C) $\int_{y=0}^{y=1}\int_{x=0}^{x=1} f(x, y)dxdy$

(D) $\int_{y=0}^{y=\sqrt{x}}\int_{x=0}^{x=\sqrt{y}} f(x, y)dxdy$

12. $F(x, y) = (x^2 + xy)\hat{a}_x + (y^2 + xy)\hat{a}_y$. It's line integral over the straight line from $(x, y) = (0, 2)$ to $(x, y) = (2, 0)$ evaluates to: **[2009]**

(A) –8 (B) 4

(C) 8 (D) 0

13. The value of the quantity P, where $P = \int_0^1 xe^x dx$, is equal to: **[2010]**

(A) 0 (B) 1

(C) e (D) $1/e$

14. Divergence of the three-dimensional radial vector field $\bar{r}$ is: **[2010]**

(A) 3 (B) $\frac{1}{r}$

(C) $\bar{i} + \bar{j} + \bar{k}$ (D) $3(\bar{i} + \bar{j} + \bar{k})$

15. At $t = 0$, the function $f(t) = \frac{\sin t}{t}$ has: **[2010]**

(A) A minimum

(B) A discontinuity

(C) A point of inflection
(D) A maximum

16. The two vectors [1, 1, 1] and [1, a, a^2] where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$ are: **[2011]**
(A) Orthonormal (B) Orthogonal
(C) Parallel (D) Collinear

17. The function $f(x) = 2x - x^2 + 3$ has **[2011]**
(A) a maxima at $x = 1$ and a minimum at $x = 5$
(B) a maxima at $x = 1$ and a minima at $x = -5$
(C) Only a maxima at $x = 1$
(D) Only a minima at $x = 1$

18. The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is: **[2012]**
(A) 21 (B) 25
(C) 41 (D) 46

19. The direction of vector $\bar{A}$ is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla.A = 0$ is: **[2012]**
(A) −2 (B) 2
(C) 1 (D) 0

20. The curl of the gradient of the scalar field defined by $V = 2x^2y + 3y^2z + 4z^2x$ is: **[2013]**
(A) $4xya_x + 6yza_y + 8zxa_z$
(B) $4a_x + 6a_y + 8a_z$
(C) $(4xy + 4z^2)\,ax + (2x^2 + 6yz)\,ay + (3y^2 + 8zx)az$
(D) $\bar{0}$

21. Given a vector field $F = y^2\,xa_x - yz\,a_y$, the line integral $\int F \cdot d$ evaluated along a segment on the x-axis from $x = 1$ to $x = 2$ is: **[2013]**
(A) −2.33 (B) 0
(C) 2.33 (D) 7

22. A function $y = 5x^2 + 10x$ is defined over an open interval $x = (1, 2)$. At least at one point in this interval, $\frac{dy}{dx}$ is exactly: **[2013]**
(A) 20 (B) 25
(C) 30 (D) 35

23. Let $f(x) = x\,e^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is: **[2014]**
(A) e^{-1} (B) e
(C) $1 - e^{-1}$ (D) $1 + e^{-1}$

24. Let g: $[0, \infty) \to [0, \infty)$ be a function defined by $g(x) = x - [x]$, where $[x]$ represents the integer part of x. (That is, it is the largest integer which is less than or equal to x).
The value of the constant term in the Fourier series expansion of $g(x)$ is _____ **[2014]**

25. The line integral of function $F = yzi$, in the counter-clockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is: **[2014]**
(A) $-\pi$ (B) $-\pi$
(C) π (D) 2π

26. Minimum of the real values function $f(x) = (x-1)^{\frac{2}{3}}$ occurs at x equal to: **[2014]**
(A) $-\infty$ (B) 0
(C) 1 (D) ∞

27. To evaluate the double integral $\int_0^8 \left(\int_{\frac{y}{2}}^{\left(\left(\frac{y}{2}\right)+1\right)} \left(\frac{2x-y}{2}\right) dx \right) dy$, we make the substitution $u = \left(\frac{2x-y}{2}\right)$ and $v = \frac{y}{2}$.
The integral will reduce to: **[2014]**
(A) $\int_0^4 \left(\int_0^2 2u\,du \right) dv$ (B) $\int_0^4 \left(\int_0^1 2u\,du \right) dv$
(C) $\int_0^4 \left(\int_0^1 u\,du \right) dv$ (D) $\int_0^4 \left(\int_0^2 u\,du \right) dv$

28. The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval $[-3, 3]$ is: **[2014]**
(A) 20 (B) 28
(C) 16 (D) 32

29. A particle, starting from origin at $t = 0$ s, is travelling along x-axis with velocity:
$v = \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$ m/s. At $t = 3$ s, the difference between the distance covered by the particle and the magnitude of displacement from the origin is _____. **[2014]**

30. Let $\nabla.(f\,v) = x^2y + y^2z + z^2x$, where f and v are scalar and vector fields respectively. If $v = yi + zj + xk$, then $v.\nabla f$ is **[2014]**
(A) $x^2y + y^2z + z^2x$ (B) $2xy + 2yz + 2zx$
(C) $x + y + z$ (D) 0

31. The volume enclosed by the surface $f(x, y) = e^x$ over the triangle bounded by the lines $x = y$; $x = 0$; $y = 1$ in the xy plane is _____. **[2015]**

32. The maximum value attained by the function $f(x) = x(x - 1)\,(x - 2)$ in the interval [1, 2] is _____ **[2016]**

33. Let $S = \sum_{n=0}^{\infty} n\alpha^n$ where $|\alpha| < 1$ the value of α in the range $0 < \alpha < 1$, such that $S = 2\ \alpha$ is ____. **[2016]**

34. The value of the line integral **[2016]**
$$\int_c \left(2xy^2dx + 2x^2ydy + dz\right)$$
along a path joining the origin (0, 0, 0) and the point (1, 1, 1) is

(A) 0 (B) 2
(C) 4 (D) 6

35. The value of the integral $2\int_{-\infty}^{\infty}\left(\frac{Sin2\pi t}{\pi t}\right)dt$ is equal to **[2016]**

(A) 0 (B) 0.5
(C) 1 (D) 2

36. The line integral of the vector field $F = 5xz\hat{i} + \left(3x^2 + 2y\right)\hat{j} + x^2z\hat{k}$ along a path from (0, 0, 0) to (1, 1, 1) parameterized by (t, t^2, t) is _____. **[2016]**

Answer Keys

Exercise

Practice Problems

1. A	**2.** D	**3.** B	**4.** C	**5.** D	**6.** C	**7.** A	**8.** D	**9.** B	**10.** C
11. B	**12.** A	**13.** C	**14.** D	**15.** D	**16.** A	**17.** C	**18.** B	**19.** D	**20.** A
21. D	**22.** A	**23.** A	**24.** A	**25.** D	**26.** D	**27.** B	**28.** C	**29.** A	**30.** B
31. D	**32.** A	**33.** A	**34.** C	**35.** D	**36.** B	**37.** A	**38.** A	**39.** C	**40.** A
41. B	**42.** B	**43.** D	44. B	**45.** C	**46.** D	**47.** A	**48.** D	**49.** D	**50.** D
51. C	**52.** C	**53.** D	**54.** C	**55.** A	**56.** D	**57.** D	**58.** A	**59.** A	**60.** B
61. D	**62.** A	**63.** A	**64.** C	**65.** C	**66.** D	**67.** B			

Previous Years' Questions

1. C	**2.** A	**3.** C	**4.** D	**5.** C	**6.** D	**7.** D	**8.** A	**9.** D	**10.** B
11. A	**12.** D	**13.** B	**14.** A	**15.** B	**16.** B	**17.** C	**18.** C	**19.** D	**20.** D
21. B	**22.** B	**23.** A	**24.** 0.5 to 0.5		**25.** B	**26.** C	**27.** B	**28.** B	
29. 2 to 2	**30.** A	**31.** 0.70 to 0.76		**32.** 0	**33.** 0.2929		**34.** B	**35.** D	
36. 4.4167									

Chapter 3

Differential Equations

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Differential equations
- Order of a differential equation
- Degree of a differential equation
- Linear differential equation
- Solution of a differential equation
- First order equation
- Second order equation
- Homogeneous differential equations
- Methods to find the integrating factors
- Linear equations
- Bernoulli's equation
- Cauchy's homogenous linear equations
- Method of variation of parameters
- Partial Differential Equation (PDE)
- Classification of second order homogenous linear equation

Introduction

Familiarity with various methods used in evaluating indefinite integrals or finding anti-derivatives of functions [or, in other words, evaluating $\int f(x)\,dx$] is a pre-requisite.

Differential Equations

An equation involving derivatives of a dependent variable with respect to one or more independent variables is called a differential equation. The equation may also contain the variables and/or their functions and constants. If there is only one independent variable, the corresponding equation is called an ordinary differential equation. If the number of independent variables is more than one, the corresponding equation is called a partial differential equation.

Examples:

1. $\dfrac{dy}{dx} = x^4 + e^{-x} + y$

2. $x^2 \dfrac{d^2y}{dx^2} + 3\left(\dfrac{dy}{dx}\right)^2 + 3y^4x = \sin x + 6$

3. $\dfrac{dy}{dx} + 5y = x^3 - \tan x$

4. $\dfrac{d^2y}{dx^2} + 4y = 0$

5. $\left(\dfrac{d^3y}{dx^3}\right)^2 + 5\left(\dfrac{dy}{dx}\right)^4 + e^{2xy} = 6$

6. $\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} + \dfrac{3dy}{dx} + 9y = 16x^2$

7. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 8u$

8. $\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} = 10$

9. $\dfrac{\partial^2 u}{\partial y^2} = 25\dfrac{\partial^2 u}{\partial x^2}$

10. $\dfrac{\partial^4 u}{\partial x^4} + 6\dfrac{\partial^2 u}{\partial x^2 \partial y^2} + \dfrac{\partial^4 u}{\partial y^4} = e^{3xy}$

We note that in the examples above, equations (1) to (6) are ordinary differential equations while equations (7) to (10) are partial differential equations. We refer to these examples later on in this chapter.

Certain geometrical results may also be expressed as differential equations

Illustration 1 Consider a family of parallel lines having the same slope. If k represents this slope, we may interpret the family of these parallel lines as curves having the same slope. As $\dfrac{dy}{dx}$ represents the slope of the tangent to a curve at any point (x, y), we may say that the differential equation $\dfrac{dy}{dx} = k$ represents a family of parallel lines.

Illustration 2 The differential equation $y\frac{dy}{dx} = k$ (a constant) may be said to represent the family of curves having the length of subnormal equal k at every point (x, y) on the curve. (We may note that the family of curves is the family of parabolas).

Our study is confined to ordinary differential equations. In what follows, differential equation means ordinary differential equations.

Order of a differential equation

It is defined as the order of the highest derivative present in the equation. Examples (1), (3) are first order; (2), (4) are second order and (5), (6) are third.

Degree of a differential equation

The degree of a differential equation is defined as the degree of the highest order derivative present in the equation.

(It is assumed that the various order differential co-efficients or derivatives present in the equation are made free from fractional powers).

Examples (1), (2), (3), (4), (6) are first degree while example (5) is second degree.

Consider the differential equation,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 4\frac{d^3y}{dx^3}.$$

Taking the square on both sides (to free it from fractional powers) the differential equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5} = 16\left(\frac{d^3y}{dx^3}\right)^2.$$

This is a third order second degree differential equation.

Linear differential equation

If, in a differential equation, the dependent variable and the derivatives appear only in the first degree and there is no term involving products of the above or containing functions of the dependent variable, it is called **linear differential equation.**

1. $\frac{dy}{dx} + Py = Q$ (where P and Q are functions of only x) is an example of a first order linear differential equation.
2. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ where P, Q, R are functions of only x; $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$ where a_1, a_2 are constants and $f(x)$ is a function of x, are examples of second order linear differential equation.

Similarly, we can have nth order linear differential equation.

$$P_0\frac{D^2y}{dx^n} + P_1\frac{d^{n-1}}{dx^{n-1}} + P_2\frac{d^{n-2}}{dx^{n-2}} + \cdots$$
$$+P_{n-1}\frac{dy}{dx} + P_ny = Q$$

where $P_0, P_1, P_2, \ldots, P_n, Q$ are functions of x or constants. If an equation is not linear, it is called a non-linear differential equation. In examples, 1, 3, 4, 6 are linear differential equations, while examples 2 and 5 are non-linear differential equations.

Solution of a Differential Equation

A function $y = f(x)$ or $F(x, y) = 0$ is called a solution of a given differential equation if it is defined and differentiable (as many times as the order of the given differential equation) throughout the interval where the equation is valid, and is such that the equation becomes an identity when $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are replaced by $f(x)$, $f'(x)$, $f''(x), \ldots$ respectively.

[In the case of $F(x, y) = 0$ one has to get $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ by successive differentiation of $F(x, y) = 0$ with respect to x].

Examples:

1. $y = e^{7x}$ is a solution of $\frac{dy}{dx} = 7y$, since on substitution of $y = e^{7x}$, both left and right sides of the differential equation become identical. We find that $y = e^{7x}, 3e^{7x}, \frac{-1}{2}e^{7x}, \ldots$ or, in general, $y = Ce^{7x}$, where C is an arbitrary constant represents solutions of $\frac{dy}{dx} = 7y$.
2. $y^2 - x^2 = 4$ is a solution of the differential equation $\frac{dy}{dx} = \frac{x}{y}$. Also, $y^2 - x^2 = 5$, $y^2 - x^2 = -10$, … or, in general, $y^2 - x^2 = C$ where C is an arbitrary constant represents solutions of $\frac{dy}{dx} = \frac{x}{y}$.

In both the above examples, we could represent the solutions of the differential equations which involve an arbitrary constant denoted by C. We now define the general solution of a first order differential equation.

The *general solution* of a first order differential equation is a relation between x and y involving one arbitrary constant such that the differential equation is satisfied by this relation or, the general solution of a first order differential equation is a one parameter family of curves where the parameter is the arbitrary constant. By assigning particular values to the arbitrary constant, we generate *particular solutions* of the equation.

In example (1) above, $y = Ce^{7x}$ represents the general solution of the differential equation $\frac{dy}{dx} = 7y$ and the

solutions $y = e^{7x}$, $y = 3e^{7x}$, … are its particular solutions. The general solution represents a family of exponential curves.

In example (2) above, $y^2 - x^2 = C$ represents the general solution of the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and the solutions $y^2 - x^2 = 4$, $y^2 - x^2 = 5$, … are its particular solutions. The general solution in this case represents a family of rectangular hyperbolas.

3. $y = 2e^{-3x} + 5e^{6x}$ is a solution of the second order differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 18y = 0$.
   (which can be verified by actual substitution). Also, $y = 4e^{-3x} - 10e^{6x}$, $e^{-3x} + e^{6x}$, … or, in general, $y = Ae^{-3x} + Be^{6x}$ where A and B are arbitrary constants represents solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 18y = 0$.
4. $y = 2\cos 4x + 3\sin 4x$ or, in general, $y = A\cos 4x + B\sin 4x$ where A and B are arbitrary constants represents solutions of $\frac{d^2y}{dx^2} + 16y = 0$.

In example (3), the general solution is $y = Ae^{-3x} + Be^{6x}$ and in example (4), the general solution is $y = A\cos 4x + B\sin 4x$.

By assigning particular values to the arbitrary constants one can generate particular solutions.

From examples (3) and (4), we infer that the general solution of a second order differential equation is a relation between x and y involving two arbitrary constants such that the differential equation is satisfied by this relation or the general solution of a second order differential equation is a two-parameter family of curves where the parameters are the arbitrary constants.

To sum up, the general solution of an nth order differential equation is a relation between x and y involving n arbitrary constants, such that the differential equation is satisfied by this relation or the general solution of an nth order differential equation is an n-parameter family of curves where the parameters are the arbitrary constants. For the first and second order differential equations, we have

First order equation

One parameter family of curves:

Representation: Relation between x and y involving one arbitrary constant, say C.

Eliminate: Eliminate C to obtain a DE representing the given curve.

Second order equation

Two-parameter family of curves:

Representation: Relation between x and y involving two arbitrary constants, say A and B

Elimination: Eliminate A and B to obtain a DE representing the two-parameter family of curves.

We shall work out a few examples to illustrate the formation of differential equations.

Solved Examples

Example 1: Form the differential equation representing the one-parameter family of curves

$$x^3 - Ay = 0.$$

Solution: Given, $x^3 - Ay = 0$ (1)

$$Ay = x^3$$

$$A\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad A = \frac{3x^2}{\frac{dy}{dx}} \tag{2}$$

Substituting A in the equation (1), we have

$$x^3 - \frac{3x^2}{\frac{dy}{dx}} \cdot y = 0 \quad \Rightarrow \quad x\frac{dy}{dx} - 3y = 0.$$

Example 2: Obtain the differential equation of all the circles in the first quadrant, which touch the co-ordinate axes.

Solution: The equation of any circle in the first quadrant, which touches the co-ordinate axes may be represented as $(x-h)^2 + (y-h)^2 = h^2$, differentiating with respect to x,

$$2(x-h) + 2(y-h)\frac{dy}{dx} = 0$$

or
$$h = \frac{x + y\frac{dy}{dx}}{\left(1 + \frac{dy}{dx}\right)}$$

Substituting the above expression for h in the equation of the circle

$$\left(x - \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2 + \left(y - \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2 = \left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2$$

or
$$(x-y)^2\left(\frac{dy}{dx}\right)^2 + (x-y)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

or
$$(x-y)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left[x + y\left(\frac{dy}{dx}\right)\right]^2.$$

Initial value problems A first order differential equation with a condition that $y = y_0$ when $x = x_0$ [written as $y(x_0) = y_0$)] is known as an initial value problem for e.g.,

1. $\frac{dy}{dx} = \frac{x}{y}$; $y(0) = 1$
2. $\frac{dy}{dx} + 2xy = x^3$; $y(1) = 6$
3. $\frac{dy}{dx} + \frac{3y}{x} = e^x$; $y(0) = 4$

To solve such problems, we first obtain the general solution and find that particular value of the arbitrary constant in the general solution which satisfies the condition $y(x_0) = y_0$. This means that the solution of an initial value problem is a particular solution of the given differential equation.

First order first degree equations The general form of the equation will be $\frac{dy}{dx} = f(x, y)$. In the example, we explain how to solve equations of the above type (or how to obtain the general solutions of such equations).

Separable equations (or variables separable type) Here, the given differential equation can be reduced to the form $f(y)dy = g(x)dx$. [Recall that $\frac{dy}{dx}$ may be thought as the ratio of the differential of y to the differential of x]. Direct integration of the relation with respect to the variable on each side gives general solution or, in other words, the general solution of the differential equation above may be written as $\int f(y)\,dy = \int g(x)\,dx + C$, where C is an arbitrary constant.

Example 1: Solve: $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$.

Solution:

$$\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{1}{\sqrt{1+y^2}}dy = \frac{1}{\sqrt{1+x^2}}dx$$

Integrating on both sides,

$$\int \frac{1}{\sqrt{1+y^2}}dy = \int \frac{1}{\sqrt{1+x^2}}dx.$$

$$\sin h^{-1}y = \sin h^{-1}x + c.$$

Example 2: Solve: $(x - xy^2)\frac{dy}{dx} + (y + x^2y) = 0$.

Solution: $(x - xy^2)\frac{dy}{dx} + (y + x^2y) = 0$

$$(x - xy^2)\,dy + (y + x^2y)\,dx = 0$$

$$x(1 - y^2)\,dy + y(1 + x^2)\,dx = 0$$

$$\frac{1-y^2}{y}dy + \frac{1+x^2}{x}dx = 0$$

Integrating on both sides,

$$\int\left(\frac{1}{y} - y\right)dy + \int\left(\frac{1}{x} + x\right)dx = 0$$

$$\log y - \frac{y^2}{2}\log x + \frac{x^2}{2} = \log C$$

$$\log_e \frac{xy}{C} = \frac{y^2 - x^2}{2}\quad \frac{xy}{c} = e^{\left(\frac{y^2-x^2}{2}\right)}$$

$$\Rightarrow \quad xy = c\,e^{\left(\frac{y^2-x^2}{2}\right)}$$

Example 3: Solve the initial value problem

$$y^2\frac{dy}{dx} = x^2e^{y^3} \cdot y(1) = (0)$$

Solution: Given: $y^2\frac{dy}{dx} = x^2e^{y^3}$

$$y^2e^{-y^3}\,dy = x^2dx.$$

$$\int y^2e^{-y^3}\,dy = \int x^2dx$$

Let $e^{-y^3} = t \quad \Rightarrow \quad e^{-y^3} \cdot -3y^2dy = dt$

$$-\frac{1}{3}\int dt = \int x^2dx$$

$$\frac{-1}{3}t = \frac{x^3}{3} + c$$

$$-\frac{1}{3}e^{-y^3} = \frac{x^3}{3} + c.$$

Given: When $x = 1$, $y = 0$;

$$-\frac{1}{3}e^{\circ} = \frac{1}{3} + c$$

$$c = -\frac{2}{3}$$

$\therefore$ The solution is $-\frac{1}{3}e^{-y^3} = \frac{x^3}{3} - \frac{2}{3}$.

$$x^3 + e^{-y3} - 2 = 0.$$

Homogeneous Differential Equations

The given differential equation will be of the form $f(x, y)\,dy = g(x, y)\,dx$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions in x and y of the same degree.

Definition

A function $F(x, y)$ in x and y is a homogeneous function in x and y of degree n(n, a rational number), if $F(x, y)$ can be expressed as $x^n\phi\left(\frac{y}{x}\right)$ or $y^n\psi\left(\frac{x}{y}\right)$.

1. $x^3 + 4x^2y - y^3 = x^3\left(1 + \frac{4y}{x} - \frac{y^3}{x^3}\right)$ is a homogeneous function in x and y of degree 3.
2. $x^3\tan\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree 3.
3. $\frac{x+y}{2x-3y}$ is a homogeneous function in x and y of degree 0.

We change the dependent variable y to v by the substitution $y = vx$. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

On substitution y and $\frac{dy}{dx}$ in the given homogeneous equation, it reduces to the variables separable form.

Example 4: Solve: $x^2\frac{dy}{dx} = x^2 + 7xy + 9y^2$.

Solution: $x^2\frac{dy}{dx} = x^2 + 7xy + 9y^2$

$$\frac{dy}{dx} = 1 + \frac{7y}{x} + 9\left(\frac{y}{x}\right)^2$$

Put $y = xv \quad \Rightarrow \quad \frac{dy}{dx} = v + \frac{xdv}{dx}$

$$v + \frac{xdv}{dx} = 1 + 7v + 9v^2$$

$$x\frac{dv}{dx} = 9v^2 + 6v + 1$$

$$\frac{1}{9v^2 + 6v + 1}dv = \frac{1}{x}dx$$

Integrating on both sides,

$$\int \frac{1}{9v^2 + 6v + 1}dv = \int \frac{1}{x}dx$$

$$= \int \frac{1}{(3v+1)^2}dv = \int \frac{1}{x}dx \quad -\frac{1}{3(3v+1)} = \log x + \log c$$

$$= -\frac{1}{3\left(\frac{3y}{x} + 1\right)} = \log_e cx = \frac{-x}{9y + 3x} = \log_e cx$$

where C is an arbitary constant.

Example 5: Solve $x\frac{dy}{dx} = y + x\sin\left(\frac{y}{x}\right)$

Solution: Given: $x\frac{dy}{dx} = y + x\sin\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right) \qquad (3)$$

Put $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting in (3), we get

$$v + x\frac{dv}{dx} = v + \sin v$$

$$\Rightarrow \quad \frac{xdv}{dx} = \sin v = \frac{1}{\sin v}dv = \frac{1}{x}dx$$

$$= \int \text{cosec}\, v\, dv = \int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \log(\text{cosec}\, v - \cot v) = \log x + \log c$$

$$\Rightarrow \quad \text{cosec}\, v - \cot v = cx$$

$$\text{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = cx.$$

Example 6: Solve $3y^2\, dx + (2xy + 3x^2)\, dy = 0$.

Solution: $3y^2\, dx + (2xy + 3x^2)\, dy = 0$.

$$\frac{dy}{dx} = \frac{-3y^2}{2xy + 3x^2}$$

Put $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{-3v^2}{2v + 3}$$

$$x\frac{dv}{dx} = \frac{-3v^2}{2v + 3} - v$$

$$x\frac{dv}{dx} = \frac{-3v^2 - 2v^2 - 3v}{2v + 3}$$

$$\frac{2v + 3}{-5v^2 - 3v}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \frac{2v + 3}{v(5v + 3)}dv + \frac{1}{x}dx = 0$$

Integrating on both sides,

$$\Rightarrow \quad \int\left[\frac{1}{v} - \frac{3}{5v + 3}\right]dv + \int\frac{1}{x}\, dx = 0$$

$$\Rightarrow \quad \log v - \frac{3}{5}\log(5v + 3) + \log x = \log c.$$

$$\Rightarrow \quad 5\log v - 3\log(5v + 3) + 5\log x = 5\log c.$$

$$\Rightarrow \quad \log\frac{v^5}{(5v + 3)^3}x^5 = \log C^5$$

$$\Rightarrow \quad \frac{y^5}{\left(5\frac{y}{x} + 3\right)^3} = c_1, \text{ where } c_1 = c^5$$

$$\Rightarrow \quad \frac{y^5x^3}{(5y + 3x)^3} = c \quad \Rightarrow \quad x^3y^5 = c_1(5y + 3x)^3$$

Exact differential equations

If M, as well as N, is a function in x and y, then the equation $Mdx + Ndy = 0$ is said to be an exact differential equation, if there exist a function $f(x, y)$ such that

$$d(f(x, y)) = Mdx + Ndy.$$

That is, $$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Example: $3x^2ydx + x^3dy = 0$ is an exact differential equation since there exists a function x^3y such that

$$d(x^3y) = 3x^2ydx + x^3dy$$

The necessary and sufficient condition for an equation of the form $Mdx + Ndy = 0$ to be an exact equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution of the exact differential equation

$$Mdx + Ndy = 0 \text{ is } U + \int \phi(y)\, dy = C$$

where $U = \int Mdx$ and $\phi(y) = N - \frac{\partial u}{\partial y}$

Or $\int^{x} Mdx + \int$ (terms of N not containing x)$dy = C$

Here $\int^{x} M\,dx$ denotes integration of M with reference to x treating y as a constant.

Example 7: Find the solution of

$$(3x - 2y + 5)dx + (3y - 2x + 7)dy = 0.$$

Solution: $M = 3x - 2y + 5$, $N = 3y - 2x + 7$

$$\frac{\partial M}{\partial y} = -2 \frac{\partial N}{\partial x} = -2. \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is an exact equation.
The solution is

$$\int^{x} Mdx + \int \text{ (the terms of } N \text{ not containing } x)\, dy = \int C$$

$$\int^{x} (3x - 2y + 5)\, dx + \int (3y + 7)\, dy = C$$

$$\frac{3x^2}{2} - 2yx + 5x + \frac{3y^2}{2} + 7y = C$$

Example 8: Find the solution of $(e^y + 1)\cot x\, dx + e^y \log(\sin x)\, dy = 0$.

Solution: Given $(e^y + 1)\cot x\, dx + e^y \log(\sin x)dy = 0$
Let $M = (e^y + 1)\cot x$ and $N = e^y \log(\sin x)$

$$\frac{\partial M}{\partial y} = e^y \cot x \text{ and } \frac{\partial N}{\partial x} = e^y \cot x$$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The given equation is an exact equation.
The solution is

$$\int^{x} Mdx + \int$$

(the terms of N not containing x)$dy = C$

$$\therefore \quad \int^{x} (e^y + 1)\cot x\, dx + \int 0\, dy = C$$

$$(e^y + 1) \log(\sin x) = C$$

Integrating factors: Let us say $M(x, y)dx + N(x, y)\, dy = 0$ be a non-exact differential equation. If it can be made exact by multiplying it by a suitable function $\mu(x, y)$, then $\mu(x, y)$ is called an integrating factor.

Methods to Find the Integrating Factors

Method 1

If $Mdx + Ndy = 0$ is a homogeneous differential equation and $Mx + Ny \neq 0$ then $\frac{1}{Mx + Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Method 2

If the differential equation $Mdx + Ndy = 0$ is of the form y, $f(x, y)dx + x\, g(x, y)dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Method 3

In the equation $Mdx + Ndy = 0$,

if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$, then $e^{\int f(x)\, dx}$ is an integrating factor of the given equation.

Similarly if $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ then $e^{\int g(y)\, dy}$ is an integrating factor of the given equation

Example 9: Find the solution of $xy^2dx + (y + y^2)dy = 0$.

Solution: Given $xy^2dx + (y + y^2)dy = 0$ (4)

$$Mdx + Ndy = 0$$

$$M = xy^2; \quad N = y + y^2$$

$$\frac{\partial M}{\partial y} = 2xy \text{ and } \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^2}[-2xy]$$

$$= -2/y = g(y)$$

Integrating factor is $e^{\int g(y)dy}$

$$= e^{\int \frac{-2}{y} dy} = e^{-2\log y\, dy} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

Multiplying (4) by $1/y^2$, we get $\frac{xy^2 dx}{y^2} + \left(\frac{y + y^2}{y^2}\right)dy = 0$

$$xdx + \left(\frac{1}{y} + 1\right)dy = 0$$

Integrating on both sides, we get $\frac{x^2}{2} + \log y + y = C$

Linear Equations

Consider the linear differential equation $\frac{dy}{dx} + Py = Q$ (5)
where P and Q are functions of only x.

Example 10: Solve $\sin x \frac{dy}{dx} + y\cos x = 1$.

Solution: $\sin x \frac{dy}{dx} + y\cos x = 1$

$$\frac{dy}{dx} + \cot xy = \text{cosec}\, x.$$

This is a linear equation in y
Here, $P = \cot x$, $Q = \text{cosec}\, x$.

$$\int p dx = \int \cot x\, dx = \log(\sin x)$$
$$\text{IF} = e^{\int p dx} = e^{\log \sin x} = \sin x.$$

$\therefore$ The general solution is Y. IF $= \int \text{QIF}\, dx + c$

$$y \sin x = \int \text{cosec}\, x \cdot \sin x\, dx + c$$
$$y \sin x = \int dx + c$$
$$y \sin x = x + c.$$

Example 11: Solve: $(1+x^4)\frac{dy}{dx} + 4x^3 y = \sin^3 x$.

Solution: Given: $(1+x^4)\frac{dy}{dx} + 4x^3 y = \sin^3 x$

$$\frac{dy}{dx} + \frac{4x^3}{1+x^4} y = \frac{\sin^3 x}{1+x^4}$$

It is a linear differential equation in y.

Here, $P = \frac{4x^3}{1+x^4}$ and $Q = \frac{\sin^3 x}{1+x^4}$

$$\int P dx = \int \frac{4x^3}{1+x^4} dx = \log(1+x^4)$$

$\text{IF} = e^{\int p dx} = e^{\log(1+x^4)} = 1+x^4$

General solution y. IF $= \int Q$ IF $dx + c$.

$$y(1+x^4) = \int \frac{\sin^3 x}{1+x^4}(1+x^4)\, dx + c$$
$$= \int \sin^3 x\, dx + C = \int \frac{3\sin x - \sin 3x}{4} dx + c$$
$$y(1+x^4) = \frac{\cos 3x}{12} - \frac{3}{4}\cos x + c$$
$$12y(1+x^4) = \cos 3x - 9\cos x + c$$

Bernoulli's equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's equation, where P, Q are continuous functions in x.

Example 12: Solve $\frac{dy}{dx} + xy = -(3xy^2)$.

Solution: Given $\frac{dy}{dx} + xy = -(3xy^2)$

Dividin the equation with y^2 we get

$$y^{-2}\frac{dy}{dx} + xy^{-1} = -3x \quad (6)$$

Let $y^{-1} = u \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$

The equation (6) becomes $\frac{-du}{dx} + xu = -3x$

$$\frac{du}{dx} - xu = 3x$$

The above equation is a linear differential equation in u.

$\therefore$ $\text{IF} = e^{\int p dx} = e^{-\int x dx} = e^{\frac{-x^2}{2}}$

$\therefore$ solution is u. IF $= \int \text{QIF}\, dx$

$$= u.e^{\frac{-x^2}{2}} = \int 3xe^{\frac{-x^2}{2}} dx.$$

$$= -\int 3e^{-t} dt \text{ when } t = \frac{-x^2}{2}$$

$$= \frac{-3e^{-t}}{-1} = 3e^{-t}$$

$$u.e^{\frac{-x^2}{2}} = 3e^{\frac{-x^2}{2}} + C$$

$$\frac{1}{y} = 3 + Ce^{\frac{x^2}{2}}$$

$$y = \frac{1}{3 + Ce^{\frac{x^2}{2}}}$$

Second order linear differential equations with constant co-efficients

The standard form of a second order linear differential equation with constant co-efficients is

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = F(x) \quad (7)$$

where a_0, a_1, a_2 are real constants and $F(x)$ is a function of only x. The second order equation,

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \quad (8)$$

represents the corresponding homogeneous equation.

We infer that $y = u(x) + v(x)$ is the general solution of the given equation (7). Thus, the general solution of (8) is the sum of the general solution of the corresponding homogeneous equation (8) and a particular solution of the given equation (7). $y = u(x)$ is called the *complementary function* of (II) and $y = v(x)$ is called a particular integral of (7). The general solution of (7) is given by $y = u(x) + v(x)$.

= [complementary function] + [particular integral]
= CF + PI (in short).

We may now summarize the nature of the complementary function of (8) as follows:

Roots of the auxiliary equation $a_0m^2 + a_1m + a_2 = 0$	Complementary function of (1), or general solution of (2)
1. Roots, real and distinct, say m_1, m_2	$y = c_1e^{m_1x} + c_2e^{m_2x}$
2. Roots, real and equal, say each equals m_1	$y = (c_1 + c_2x)e^{m_1x}$
3. Roots, complex, say $\alpha \pm i\beta$	$y = e^{\alpha x}\{c_1\cos \beta x + c_2 \sin \beta x\}$
4. Roots, complex and repeated, say $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$	$y = e^{\alpha x}[(c_1 + c_2x)\cos \beta x + (c_3 + c_4x)\sin \beta x$

Example 13: Obtain the complementary function of the equation $\frac{d^2y}{dx^2} - \frac{7dy}{dx} + 6y = x^4$.

Solution: $\frac{d^2y}{dx^2} - \frac{7dy}{dx} + 6y = x^4$

$\Rightarrow \quad (D^2 - 7D + 6)y = x^4$

Auxiliary equation is $m^2 - 7m + 6 = 0$

$m = 1, 6.$

$\therefore$ The complementary function of the given equation is

$$y = c_1e^x + c_2e^{6x}$$

Example 14: Obtain the general solution of the equation $\frac{d^2y}{dx^2} - 10 \cdot \frac{dy}{dx} + 25y = 0.$

Solution: Given: $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

$\Rightarrow \quad (D^2 - 10D + 25)y = 0$

Auxiliary equation is $m^2 - 10m + 25 = 0$

The roots are $(m) = 5, 5$

$\therefore$ The general solution of the equation is $(c_1 + c_2x)e^{5x}$.

Example 15: Obtain the complementary function of the equation $\frac{d^2y}{dx^2} - 6 \cdot \frac{dy}{dx} + 10y = e^{3x}$.

Solution: Given: $\frac{d^2y}{dx^2} - 6 \cdot \frac{dy}{dx} + 10y = e^{3x}$

$\Rightarrow \quad (D^2 - 6D + 10)y = e^{3x}$

Auxiliary equation is $m^2 - 6m + 10 = 0$

$$m = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$\therefore$ The complementary function is given by $xy = e^{3x}(C_1\cos x + C_2\sin x)$.

To find a particular integral of (1) or to find a particular solution of the equation (1)

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$$

We may write the above as $(a_0D^2 + a_1D + a_2)y = F(x)$ or $f(D)\ y = F(x)$ where $f(D)$ stands for $(a_0D^2 + a_1D + a_2)$. Particular integral y is that function of x independent of arbitrary constants such that $f(D)$ on y or $f(D)\ y$ yields $F(x)$. This is symbolically represented as $y = \frac{1}{f(D)}\{F(x)\}$.

Case I: $F(x) = e^{kx}$ where k is a constant.

We have $D(e^{kx}) = ke^{kx}$, $D^2(e^{kx}) = k^2e^{kx}$ … or, in general, $g(D)\ (e^{kx}) = g(k)\ e^{kx}$ where $g(D)$ is a polynomial in D, in particular, $f(D)\ \{e^{kx}\} = f(k)\ e^{kx}$.

Since $\frac{1}{f(D)}e^{kx}$ is that function of x which when operated by $f(D)$ gives e^{kx}, it is clear that $\frac{1}{f(D)}e^{kx} = \frac{1}{f(k)}e^{kx}$ provided $f(k) \neq 0$. $f(k)$ reduces to zero when one or both the roots of the auxiliary equation $a_0m^2 + a_1m + a_2 = 0$, is k.

1. Suppose one of the roots is k.
   This is a linear equation and the particular solution of the above equation is xe^{kx}. Therefore, particular integral $= \frac{1}{a_0(k - m_0)}xe^{kx}$.
2. Suppose both the roots of the auxiliary equation are k. Then, particular integral
   Particular integral in this case is given by $y = \frac{x^2}{2}e^{kx}$.

Example 16: Solve the differential equation.

$$(D^2 + 5D + 6)y = e^{-4x}$$

Solution: $(D^2 + 5D + 6)y = e^{-4x}$

Auxiliary equation is $m^2 + 5m + 6 = 0$.

$$(m + 3)\ (m + 2) = 0.$$

$\therefore$ Roots are $m = -3, -2$.

Complementary function $c_1e^{-3x} + c_2e^{-2x}$.

Particular integral $= \frac{1}{D^2 + 5D + 6} \cdot e^{-4x}$

$$= \frac{1}{(-4)^2 + 5(-4) + 6}e^{-4x} = \frac{e^{-4x}}{2}$$

$\therefore$ General solution is

$$y = c_1e^{-3x} + c_2e^{-2x} + \frac{e^{-4x}}{2}.$$

Example 17: Solve $(3D^2 - D - 10)y = 6e^{2x}$

Solution: Given $(3D^2 - D - 10)y = 6e^{2x}$

Auxiliary equation $3m^2 - m - 10 = 0$

$$m = 2, -5/3.$$

$\therefore$ Complementary function is

$$\text{CF} = c_1 e^{2x} + C_2 e^{\frac{5}{3}x}$$

$$\text{PI} = \frac{1}{3D^2 - D - 10} 6e^{2x}$$

$$= \frac{1}{(D-2)(3D+5)} 6e^{2x}$$

$$= 6\frac{1}{D-2}\left[\frac{1}{3D+5}e^{2x}\right] = 6\frac{1}{D-2}\frac{1}{11}e^{2x}$$

$$= \frac{6}{11}\frac{1}{D-2}e^{2x} = \frac{6}{11}xe^{2x}$$

$\therefore$ General solution is

$$y = c_1 e^{2x} + c_2 e^{-\frac{5}{3}x} + \frac{6}{11}xe^{2x}$$

Example 18: Solve $(D^2 - 12D + 36)y = e^{6x}$

Solution: Given: $(D^2 - 12D + 36)y = e^{6x}$

Auxiliary equation is $m^2 - 12m + 36 = 0$.

$$m^2 - 12m + 36 = 0.$$

$$m = 6, 6$$

Complementary function (CF) $= (c_1 + c_2 x)e^{6x}$

$$\text{PI} = \frac{1}{D^2 - 12D + 36}e^{6x} = \frac{1}{(D-6)^2}e^{6x}$$

$$= \frac{x^2}{2!}e^{6x}$$

$\therefore$ General solution is $y = \text{CF} + \text{PI}$

$$= (C_1 + C_2 x)e^{6x} + \frac{x^2}{2!}e^{6x0}$$

Case II: $F(x) = \sin kx$ or $\cos kx$ where k is a constant.

We have $D\{\sin kx\} = k\cos kx$

$$D^2\{\sin kx\} = -k^2 \sin kx$$

Similarly, $D^2\{\cos kx\} = -k^2 \cos kx$

If $g(D^2)$ is a polynomial in D^2,

$g(D^2)\{\sin kx \text{ or } \cos kx\} = g(-k^2)\sin kx$ or $g(-k^2)\cos kx$.

Hence, $\frac{1}{g(D^2)}\sin kx = \frac{1}{g(-k^2)}\sin kx$ and $\frac{1}{g(D^2)}\cos kx$

$= \frac{1}{g(-k^2)}\cos kx$, provided $g(-k^2) \neq 0$.

We shall illustrate the above technique by considering two examples.

Example 19: Find the particular integral of the equation $(D^2 + 16)y = \cos 3x$.

Solution: $\text{PI} = \frac{1}{D^2 + 16}\cos 3x = \frac{1}{-(3)^2 + 16}\cos 3x = \frac{\cos 3x}{7}$

Example 20: Find the particular integral of the equation $(D^2 - 5D + 6)y = \sin 3x$.

Solution: $\text{PI} = \frac{1}{D^2 - 5D + 6}\sin 3x$

$$= \frac{1}{-3^2 - 5D + 6}\sin 3x$$

$$\frac{1}{-5D - 3}\sin 3x$$

$$= -\frac{5D - 3}{(5D+3)(5D-3)}\sin 3x$$

$$= \frac{(5D-3)}{-(25D^2 - 9)}\sin 3x = \frac{3 - 5D}{25 \times -9 - 9}\sin 3x$$

$$= \frac{1}{-234}[(3 - 5D)\sin 3x]$$

$$= \frac{-1}{234}[3\sin 3x - 5D\sin 3x]$$

$$= \frac{-1}{234}[3\sin 3x - 15\cos 3x]$$

$$\text{PI} = \frac{15\cos 3x}{234} - \frac{3\sin 3x}{234}$$

Example 21: Solve the equation $(D^2 + 16)y = \sin 4x$.

Solution: Given: $(D^2 + 16)y = \sin 4x$

Auxiliary equation $m^2 + 16 = 0$

$$m = \pm 4i$$

$\therefore$ $\text{CF} = C_1 \cos 4x + C_2 \sin 4x$

$$PI = \frac{1}{D^2 + 16}\sin 4x.$$

$$= \frac{x}{2.4}\sin 4x$$

$$\left(\therefore \frac{1}{D^2 + k^2}\sin kx = -\frac{x}{2kx}\cos kx\right)$$

$$= -\frac{x}{8}\cos 4x$$

General solution is $y = \text{CF} + \text{PI}$

$$= c_1 \cos 4x + c_2 \sin 4x - \frac{x}{8}\cos 4x.$$

Cauchy's Homogenous linear equations

An equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots p_n y = Q(x) \qquad (9)$$

where $p_1, p_2, \ldots, p_n$ are constants is called Cauchy's linear equation. To convert the above equation into linear differential equation with constant co-efficients, we substitute $x = e^z$ = or $z = \log x$

Example 22: Solve: $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} - 3y = 0$

Solution: Let $x = e^z$ or $z = \log x$

Then $x \frac{dy}{dx} = \theta y;\ x^2 \frac{d^2 y}{dx^2} = \theta(\theta - 1) y$

The above equation becomes

$$[\theta(\theta - 1) + 3\theta - 3] y = 0$$

$$\lfloor \theta^2 + 2\theta - 3 \rfloor y = 0$$

$$A.E = m^2 + 2m - 3 = 0$$

$$= (m + 3)(m - 1) = 0$$

$$m = -3, 1$$

$\therefore$ $$y = c_1 e^{-3z} + c_2 e^z$$

$$Y = c_1 x^{-3} + c_2 x$$

Method of variation of parameters

An equation of the form $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = R(x)$, where $P(x)$, $Q(x)$ and $R(x)$ are real valued functions of x, is called the linear equation of the second order with variable co-efficients.

The above equation is solved by the method of variation of parameters.

The method is explained below:

1. Find the solution of $\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + Qy = 0$ and let the solution be $Y_c = C_1 U(x) + C_2 V(x)$
2. Write particular solution as follows $y_p = AU(x) + BV(x)$

   Where $A = \int \frac{-VR}{W} dx$

   $$B = \int \frac{-UR}{W} dx$$

   Where $W = \begin{vmatrix} U & V \\ \frac{dU}{dx} & \frac{dV}{dx} \end{vmatrix} = U \frac{dV}{dx} - V \frac{dU}{dx}$ is called the Wronskian of U and V.
3. Then the solution is $y_c + y_p$

   $$y = C_1 U(x) + C_2 V(x) + AU(x) + BV(x)$$

Example 23: Solve the equation $(D^2 + 4)y = \sec 2x$ by variation of parameters.

Solution: Given $(D^2 + 4)y = \sec 2x$

$$\text{AE} = m^2 + 4 = 0\ m^2 = \pm 2i$$

$$\text{CF} = y_c = C_1 \cos 2x + C_2 \sin 2x$$

$$\therefore U(x) = \cos 2x;\ V(x) = \sin 2x$$

$$Yp = AU(x) + BV(x)$$

$$W = U \frac{dV}{dx} - V \frac{dU}{dx}$$

$$= \cos 2x \frac{d}{dx}(\sin 2x) - \sin 2x . \frac{d}{dx}(\cos 2x)$$

$$= 2\cos^2 2x + 2 \sin^2 2x = 2$$

$$A = -\int \frac{VR}{W} dx = -\int \frac{\sin 2x . \sec 2x}{2} dx$$

$$= -\int \frac{\tan 2x}{2} dx = \frac{1}{4} \log(\cos 2x)$$

$$B = \int \frac{UR}{W} dx = \int \frac{\cos 2x . \sec 2x}{2} dx = \frac{1}{2} x$$

$\therefore$ $$y_p = \frac{1}{4} [\log(\cos 2x)] \cdot \cos 2x + \frac{1}{2} x \sin 2x$$

$\therefore$ $$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x +$$

$$\frac{1}{4} [\log(\cos 2x)] \cos 2x + \frac{1}{2} x \sin 2x$$

Example 24: Solve the equation $y'' + 4y' + 4y = x^3 e^{2x}$

Solution: Given equation

$$(D^2 + 4D + 4) y = x^3 e^{2x}$$

$$m^2 + 4m + 4 = 0$$

$$(m + 2)^2 = 0 \quad \Rightarrow \quad m = -2$$

$$y_C = C_2 \times e^{-2x} + C_1 e^{-2x}$$

$$U(x) = e^{-2x} \quad \text{and} \quad V(x) = xe^{-2x}$$

$$Y_p = AU(x) + BV(x)$$

$$A = -\int \frac{VR}{W} dx,\ B = \int \frac{UR}{W} dx$$

$$W = u \frac{dv}{dx} - v \frac{du}{dx} = e^{-2x} \frac{d}{dx}(xe^{-2x}) - xe^{-2x} \frac{d}{dx}(e^{-2x})$$

$$= e^{-2x}\left[e^{-2x} - 2xe^{-2x}\right] + 2xe^{-2x}e^{-2x} = e^{-4x}$$

$$A = -\int \frac{UR}{\frac{udv}{dx} - \frac{vdu}{dx}}dx = -\int \frac{xe^{-2x}x.^3e^{2x}}{e^{-4x}}dx$$

$$= -\int x^4 e^{4x}dx = -\left[x^4\frac{e^{4x}}{4} - 4\int x^3\frac{e^{4x}}{4}dx\right]$$

$$= -\left[x^4\frac{e^{4x}}{4} - \left(x^3\frac{e^{4x}}{4} - \frac{3}{4}\int x^2e^{4x}dx\right)\right]$$

$$= -\left[x^4\frac{e^{4x}}{4} - x^3\frac{e^{4x}}{4} + \frac{3}{4}\left(x^2\frac{e^{4x}}{4} - 2\int x\frac{e^{4x}}{4}dx\right)\right]$$

$$= -x^4\frac{e^{4x}}{4} + \frac{x^3e^{4x}}{4} - 3\frac{x^2e^{4x}}{16} + 6\frac{xe^{4x}}{16\times4} - 6\frac{e^{4x}}{16\times16}$$

$$B = \int \frac{UR}{W}dx = \int \frac{e^{-2x}x^3e^{2x}}{e^{-2x}}dx = \int x^3e^{4x}dx$$

$$= x^3\frac{e^{4x}}{4} - \frac{3}{4}\left[x^2\frac{e^{4x}}{4} - 2\frac{xe^{4x}}{16} + \frac{e^{4x}}{32}\right]$$

$$y_d = y_c + y_p = AU(X) + BV(x) + C_1e^{-2x} + C_2xe^{-2x}$$

$$= C_1e^{-2x} + C_2xe^{-2x} - 4x\ \frac{e^{2x}}{4} + \frac{x^3e^{2x}}{4} - \frac{3x^2e^{2x}}{16}$$

$$= \frac{6xe^{2x}}{64} - \frac{3e^{2x}}{3128} + \frac{x^4e^{2x}}{4} - \frac{3x^3e^{2x}}{16} + \frac{3}{32}x^2e^{2x} - \frac{3}{128}e^{2x}x$$

$$= c_1e^{-2x} + c_2xe^{-2x} - \frac{1}{16}x^3e^{2x} - \frac{3}{32}x^2e^{2x}\ \frac{9}{128} - \frac{xe^{2x}}{128} + \frac{9}{128}e^{2x}$$

Partial Differential Equation (PDE)

An equation involving two or more independent variables x, y and a dependent variable z and its partial derivatives is called a partial differential equation.

$$\therefore\quad f\left(x, y, z, \frac{\partial z}{\partial z}, \frac{\partial z}{\partial y}\ldots\right) = 0.$$

Standard notation

$$\frac{\partial z}{\partial x} = p = z_x, \quad \frac{\partial z}{\partial y} = q = z_y$$

$$\frac{\partial^2 z}{\partial x^2} = r = z_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = t = z_{yy}$$

$$\frac{\partial 2z}{\partial x\partial y} = z_{xy} = s$$

Formation of partial differential equations

Partial differential equation can be formed by two ways.

1. By eliminating arbitrary constants.
2. By eliminating arbitrary functions.

Formation of PD.E by eliminating arbitrary constants Let the functions $f(x, y, z, a, b) = 0$
where a, b, are arbitrary constants.

Differentiating this partially wrt, x and y eliminate a, b from these equations we get as

$f(x, y, z, p, q) = 0$ which is partial differential equation of first order.

Example 25: $z = ax^2 - by^2$, a, b are arbitrary constants.

Solution: Given

$$z = ax^2 - by^2 \tag{10}$$

Differentiating z partially wrt x,

$$\frac{\partial z}{\partial x} = 2ax \quad\Rightarrow\quad p = 2ax \quad\Rightarrow\quad a = \frac{p}{2x}$$

differentiate z partially wrt y,

$$\frac{\partial z}{\partial y} = 2by, \text{ i.e., } q = -2by$$

$$\Rightarrow\quad b = \frac{-q}{2y}$$

Substitute the values of a and b in (10) we get

$$z = \frac{p}{2x}x^2 + \frac{q}{2y}, y^2$$

$2z = px + qy$ which is a partial differential equation of order 1.

Formation of PDE by eliminating arbitrary function Consider $z = f(u)$ (11)

z is an arbitrary function in u and u is function in x, y, z .

Now differentiate (11) wrt x, y partially by chain rule we get

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \tag{12}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} \tag{13}$$

by eliminating the orbitary functions from (11), (12), (13) we get a PDE of first order

Formation of PDE two arbitrary functions are involved When two arbitrary functions are involved, we differentiate the given equation two times and eliminate the two arbitrary functions from the equation obtained.

Example 26: $z = \dfrac{f(x)}{g(y)}$

Solution: Given $z = \dfrac{f(x)}{g(y)}$

$$p = z_x = \frac{f^1(x)}{g(y)} \tag{14}$$

$$q = z_y = \frac{-f(x)}{[g(y)]^2} \cdot g'(y) \qquad (15)$$

$$z = \frac{f(x)}{g(y)}$$

$$\frac{\partial z}{\partial x} = \frac{f^1(x)}{g(y)}$$

$$S = \frac{\partial^2 z}{\partial x \partial y} = \frac{-f^1(x)}{[g(y)]^2} \cdot g'(y) \qquad (16)$$

$$(29) \times (30) = pq = \frac{f'(x)}{g(y)} \cdot \left(\frac{-f(x) \cdot g'(y)}{[g(y)]^2} \right) = -s.z$$

$\because \quad pq + sz = 0$

Forming PDE by the elimination of arbitrary function of specific functions Consider $f(u, v) = 0$

Where u, v are the functions in x, y, z.

Differentiate the above equation wrt x and y by chain rule and eliminate the $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$ and convert them in the form $Pp + Qq = R$, which is a first order linear PDE where P, Q, R functions x, y, z.

Linear equation of first order

Linear equation of first order is $Pp + Qq = R$. This is also called Lagrange's equation, where P, Q, R are the functions in x, y, and z.

Procedure for solving Lagrange equations Take the auxiliary equation as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Solving any two equations and take the solutions as u and v. The complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

Example 27: $(z - y)p + (x - z)\, q = y - x$.

Solution: Auxiliary equation

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}.$$

Using the multipliers of x, y, z, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(z-y) + y(x-z) + z(y-x)}$$

$$= xdx + ydy + zdz = 0$$

$$\therefore \quad x^2 + y^2 + z^2 = 0$$

and also $\dfrac{dx + dy + dz}{z - y + x - z + y - z} = 0$

$$dx + dy + dz = 0,\ x + y + z = 0.$$

$\therefore$ The required solution is $x^2 + y^2 + z^2 = f(x + y + z)$.

Non-linear equation of first order

There are four types of non linear equations of first order.

Type I:

$$f(p, q) = 0.$$

The given equation contain only p and q then the solution is taken as $z = ax + by + c$.

Example 28: Solve $2p + 3q = 5$

Solution: Given $2p + 3q = 5$

$Z = ax + by + c$.

Where $2a + 3b = 5$

$$b = \frac{5 - 2a}{3}$$

$\therefore$ The solution is $z = ax + \left(\frac{5 - 2a}{3} \right) y + c$.

Type II:

$$f(z, p, q) = 0$$

When the equation is not containing x and y then to solve the equation assume $u = x + ay$ and substitute $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$

Solve the resulting equation and replace u by $x + ay$

Type III:

$$f(x, p) = f(y, q).$$

The equation is not containing z.

Assume $f(x, p) = a$ and $f(y, q) = a$.

Solve the equations for p and q and then write the solution

Example 29: Solve $p^2 - q^2 = x^2 - y^2$.

Solution: $p^2 - q^2 = x^2 - y^2$

$p^2 - x^2 = -y^2 + q^2$

Let $p^2 - x^2 = a^2 = -y^2 + q^2$

$p^2 = a^2 + x^2 \qquad q^2 = y^2 + a^2$

$p = \sqrt{a^2 + x^2} \qquad q = \sqrt{a^2 + y^2}$

$\therefore$ Take $dz = pdx + qdy$

Integrating on both sides, $\int dz = \int pdx + \int qdy$

$$z = \int \sqrt{a^2 + x^2}\,dx + \int \sqrt{a^2 + y^2}\,dy$$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \sin^{-1} h \frac{x}{a} + \frac{y}{2}\sqrt{a^2 + y^2}$$

$$+ \frac{a^2}{2} \sin^{-1} h \frac{y}{a} + b.$$

Type IV:

$$z = px + qy + f(p, q)$$

The equation in the above form is Clairaut's equation. The solution is $z = ax + by + f(a, b)$.

Classification of Second Order Homogenous Linear Equation

A second order linear homogeneous PDE of the form

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+F\phi(x,y)=0 \quad (17)$$

Where A, B, C, D, E and F are either functions of x and y only or constants is called

1. a parabolic equation, if $B^2-4AC=0$
2. an elliptic equation, if $B^2-4AC<0$
3. a hyperbolic equation, if $B^2-4AC>0$

Examples:

1. Consider the one dimensional heat equation

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0$$

Comparing it with (1), we have
$A=c^2$, $B=0$ and $C=0$
$\therefore B^2-4AC=0^2-4\times c^2\times 0=0$
$\therefore$ One dimensional heat equation is parabolic.
Similarly, it can be easily observed that

2. One dimensional wave equation
$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$ is hyperbolic ($B^2-4AC>0$) and
3. The Laplace equation
$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is elliptic ($B^2-4AC<0$)

Method of separation of variables

Consider a PDE involving a dependent variable u and two independent variables x and y. In the method of separation of variables, we find a solution of the PDE in the form of a product of a function of x and a function of y, i.e., we write

$$u(x,y)=X(x).\,Y(y) \quad (18)$$

Then $\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(xy)=x'y;\ \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(xy)=xy'$

$\frac{\partial^2 u}{\partial x^2}=x''y,\ \frac{\partial^2 u}{\partial x\partial y}=x'y',\ \frac{\partial^2 u}{\partial y^2}=xy''$ and so on

Here $x'=\frac{dX}{dx};\ y'=\frac{dY}{dy};\ x''=\frac{d^2X}{dx^2};\ y''=\frac{d^2Y}{dy^2}$.

Substituting these in the given PDE, separating x and its derivatives from $y$and its derivatives, finding solutions for x and y and substituting them in (18), we get the solution of the given PDE

This is best explained through the examples given as follows:

Example 30: Solve $xp+yq=0$ by the method of separation of variables.

Solution: For the PDE

$$xp+yq=0 \quad (19)$$

Let $$z=X(x).\,Y(y) \quad (20)$$

By solving,

$$\therefore \quad p=\frac{\partial z}{\partial x}=x'y \text{ and } q=\frac{\partial z}{\partial y}=xy'$$

Substituting these in (19)

$$xX'y+yXY'=0$$

$$\Rightarrow xX'Y=-yXY'$$

$$\Rightarrow \quad x\frac{X'}{X}=-y\frac{Y'}{Y} \quad (21)$$

In (21), as LHS is a function of x alone and RHS is a function of y alone, they are equal only if each of them is equal to some constant

$$\therefore \quad x\frac{X'}{X}=-y\frac{Y'}{Y}=k(\text{say}) \quad (22)$$

Where k is a constant

From (22), $x\frac{X'}{X}=k \quad \Rightarrow \quad xX'=kX$

$$\Rightarrow \quad x\frac{dX}{dx}=kx$$

$$\Rightarrow \quad \frac{dX}{X}=k\cdot\frac{dx}{X}$$

Integrating on both sides we have,

$$\int\frac{dx}{X}=k\int\frac{dx}{x}$$

$$\Rightarrow \quad \log x=k\log x+\log c_1$$

$$\Rightarrow \quad \log x=\log x^k c$$

$$\Rightarrow \quad X=c_1x^k \quad (23)$$

Again from (22), $-y\frac{Y'}{Y}=k$

$$\Rightarrow \quad -yY'=kY$$

$$\Rightarrow \quad y\frac{dY}{dy}=-kY$$

$$\Rightarrow \quad \frac{dY}{Y}=-k\frac{dy}{y}$$

Integrating on both sides,

$$\int\frac{dy}{y}=-k\int\frac{dy}{y}$$

$$\Rightarrow \quad \log y=-k\log y+\log C_2$$

$$\Rightarrow \quad \log y=\log y^{-k}C_2$$

$$\Rightarrow \quad y=C_2y^{-k} \quad (24)$$

Substituting (23) and (24) in (20), we get the solution of (19) as,

$$Z = (C_1 x^k)\,(C_2 y^{-k})$$
$$= C_1 C_2\, x^k y^{-k}$$
$$\therefore \quad Z = C\left(\frac{x}{y}\right)^k \text{ where } C = C_1 C_2$$

Example 31: Solve the PDE $u_x + u_t = 3u$; $u(0, t) = 4e^t$ by the method of separation of variable.

Solution: Let $u = X(x).\ T(t)$ (25)

be the solution of the PDE

$$U_x + u_t = 3u \tag{26}$$
$$U = XT \quad \Rightarrow \quad u_x = \frac{\partial u}{\partial x} = X'T \text{ and } u_t$$
$$= \frac{\partial u}{\partial t} = XT^1$$

Substituting these in (25), we get

$$X'T + XT' = 3XT$$

Dividing throughout by XT, we have

$$\frac{X'}{X} + \frac{T'}{T} = 3$$
$$\Rightarrow \quad \frac{X'}{X} = \frac{-T'}{T} + 3 = k \qquad (26), \text{ (say)}$$

From (26), $\quad \frac{X_1}{X} = k \quad \Rightarrow \quad X' = KX$

$$\Rightarrow \quad X' - KX = 0 \tag{27}$$

Which is a linear equation with its auxiliary equation being

$$m - k = 0 \quad \Rightarrow \quad m = k$$

Hence its solution is $X = C_1\, e^{kx}$ (28)

Again from (26), $\frac{-T^1}{T} + 3 = k$

$$\Rightarrow \quad \frac{T^1}{T} = 3 - k$$
$$\Rightarrow \quad T' = (3 - k)T$$
$$\Rightarrow \quad T' - (3 - k)\,T = 0 \tag{29}$$

Which is a linear equation with its auxiliary equation being $m - (3 - k) = 0$

$$\Rightarrow \quad m = 3 - k$$

$\therefore$ The solution of (29) is $T = C_2\, e^{(3-k)t}$ (30)

Substituting (28) and (30) in (27), we get the general solution of given PDE (25) as

$$U = X.T = (C_1 e^{kx})\,(C_2 e^{(3-k)t})$$
$$= C_1 C_2 e^{kx + (3-k)t}$$
$$\therefore\ u = ce^{kx+(3-k)t};\ \text{where } c = c_1\, c_2$$
$$\therefore \qquad u(x, t) = ce^{kx+(3-k)t} \tag{31}$$

Given $u(0, t) = 4e^t$

$\therefore$ From (31), $u(0, t) = ce^{(3-k)t} = 4e^t$

Comparing on both sides, we get

$$C = 4,\ 3 - k = 1$$
$$\Rightarrow \quad c = 4;\ k = 2$$

Substituting these in (31), we get the required solution of (26) as

$$U(x, t) = 4e^{2x+t}$$

EXERCISE

Practice Problems

Directions for questions 1 to 45: Select the correct alternative from the given choices.

1. The order and degree of the DE $\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3} = \frac{d^2y}{dx^2}$ respectively are

(A) 1, 2 (B) 3, 2
(C) 2, 2 (D) 2, 3

2. The differential equation whose solution is $y = mx + \frac{4}{m}$, where 'm' is the parameter is

(A) $x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + 4 = 0.$

(B) $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} + 4 = 0.$

(C) $x\frac{dy}{dx} - y + 4 = 0.$

(D) $x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 4 = 0.$

3. Obtain the differential equation, whose solution is given by $xy = ae^x + be^{-x}$, where a, b are arbitrary constants.

(A) $xy_1 - 2y_1 + xy = 0$ (B) $xy_2 + 2y_1 = xy$
(C) $y_2 + 2y_1 = xy$ (D) $xy_2 - 2y_1 = xy$

4. The differential equation of all rectangular hyperbolas with asymptotes as coordinate axes is

(A) $xy' + xy = 0$ (B) $1 + y' = 0$
(C) $y + xy' = 0$ (D) $x + yy' = 0$

5. Find the solution of $\tan y \sec^2 x\, dx + \tan x \sec^2 y dy = 0$ when $x = y = \frac{\pi}{4}$.

(A) $\tan x \tan y = 1$ (B) $\cot x \tan y = 1$
(C) $\tan x \cot y = 1$ (D) $\cot x \cot y = 1$

6. The general solution of the DE $(e^x + 1)\, ydy = (y + 1)e^x dx$ is

(A) $\log(e^x + 1) - \log(y + 1) + c = 0$
(B) $\log(e^x + 1) = y - \log(y + 1) + c$
(C) $\log(e^x - 1) + \log(y + 1) + c = 0$
(D) $\log\left(\frac{e^x}{y+1}\right) = c$

7. Solve $(x+y)^2 \frac{dy}{dx} = k^2$.

(A) $y = \tan^{-1}(x+y)$

(B) $y = \sin^{-1}\left(\frac{x+y}{k}\right) + c$

(C) $y = k\tan^{-1}\left(\frac{x+y}{k}\right) + c$

(D) $y = \cot^{-1}\left(\frac{x+y}{k}\right) + c$

8. The general solution of $\frac{dy}{dx} = \frac{x-y}{x+y}$ is

(A) $x^2 + xy + y^2 = k$ (B) $x^2 - y^2 = k$

(C) $x^2 - 2xy - y^2 = k$ (D) $x^2y^2 = k$

9. The general solution of $\frac{dy}{dx} = \frac{x-2y+1}{2x-4y+3}$ is

(A) $x^2 - 4xy - 6y = c$

(B) $x^2 - 4xy + 4y^2 + 2x - 6y = c$

(C) $x^2 + 4xy + 4y^2 + 2x - 6y = c$

(D) $x^2 + 4xy - x + 6y = c$

10. Consider the differential equation $\cos y \frac{dy}{dx} + 3x^2 \sin y = x^2$. To convert the above equation into linear form the substituted variable is

(A) $z = \cos y$ (B) $z = \operatorname{cosec} y$

(C) $z = \sin y$ (D) $z = \sec y$

11. The general solution of $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$ is

(A) $e^x + y^2 = c$ (B) $e^{xy} + y^2 = c$

(C) $e^{y^2} + xy = c$ (D) $e^y + xy = c$

12. The integrating factor of the equation $(x^2 + xy - y^2)dx + (xy - x^2)\,dy = 0$ is

(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^3}$

(C) x^2 (D) x^3

13. Consider the differential equation $\cos y \frac{dy}{dx} + 3x^2 \sin y = x^2$.

The solution of the above equation is

(A) $\sin y\, e^{-x^3} = e^{x^3} + c$

(B) $\sin y\, e^{x^3} = 3e^{x^3} + c$

(C) $\sin y = e^{x^3} + c$

(D) $\sin y\, e^{x^3} = \frac{1}{3}e^{x^3} + c$

14. The solution of $(1+x)\frac{dy}{dx} - xy = 1 - x$ satisfying the initial conditions at $x = 0$ and $y = 1$ is

(A) $1 + x = y + e^x$ (B) $y(1+x) = x + e^x$

(C) $x + y = e^x$ (D) $x(1+y) = ce^x$

15. The general solution of $x\frac{dy}{dx} + y = y^2 \log x$ is

(A) $y = \log x + cx$ (B) $y = x + c\log x$

(C) $\frac{1}{y} = 1 + cx$ (D) $\frac{1}{y} = 1 + cx + \log x$

Common Data for Questions 16 and 17: Consider the differential equation $\frac{dy}{dx} + y\cot x = y^2 \sin x$

16. The integrating factor of the above equation is

(A) $\operatorname{cosec} x$ (B) $\sin x$

(C) $\cos x$ (D) $\sec x$

17. The solution of the above equation when $x = \frac{\pi}{2}$, $y = 1$ is

(A) $y\operatorname{cosec} x - x = \frac{\pi+2}{2}$ (B) $\frac{\operatorname{cosec} x}{y} + x = \frac{\pi+2}{2}$

(C) $y\operatorname{cosec} x + x = \frac{\pi-2}{2}$ (D) $\frac{\operatorname{cosec} x}{y} - x = \frac{\pi+2}{2}$

18. The solution of the differential equation $2xy\,dy + (x^2 + y^2 + 1)dx = 0$ is

(A) $x^3 + xy^2 + 3x = c$ (B) $x^3 + 3xy^2 + x = c$

(C) $\frac{x^3}{3} + xy^2 + x = c$ (D) $3x^2 + y^2 + 2x = c$

19. The solution of the differential equation $(3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$ is

(A) $x^3y + x^2y = c$ (B) $x^3y + x^2y^2 = c$

(C) $x^2y + xy^2 = c$ (D) $2xy(x+y) = c$

20. The general solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$ is

(A) $y = c_1e^{2x} + c_2e^x$ (B) $y = (c_1 + c_2x)\,e^{4x}$

(C) $y = c_1e^{2x} + c_2e^{4x}$ (D) $y = (c_1 + c_2x)e^{2x}$

21. The solution of $(aD^2 + bD + c)\,y = 0$ whose auxiliary equation has its discriminant as zero and has 5 as one of its roots is

(A) $y = c_1e^{5x} + c_2e^{5x}$ (B) $y = c_1e^x + c_2e^x$

(C) $y = (c_1 + c_2x)\,e^{5x}$ (D) $y = c_1 + c_2x$

22. The particular integral of $(D^2 - 4D + 3)\,y = e^{3x}$ is

(A) $\frac{xe^{3x}}{2}$ (B) e^{3x}

(C) $\frac{1}{2}e^{3x}$ (D) xe^{2x}

23. The particular integral of $(D^3 - 4D^2)\,y = 6$ is

(A) x^2 (B) $\frac{3}{4}x^2$

(C) $-\frac{3}{4}x^2$ (D) $\frac{-x^2}{4}$

24. The particular integral of DE $(D^3 - 5D^2 + 7D - 3)y = 2e^x \cosh 2x$ is

(A) $\frac{e^{3x} + e^{-x}}{4}$ (B) $\frac{e^{x} - e^{-3x}}{4}$

(C) $\frac{1}{16}\{xe^{3x} + e^{x}\}$ (D) $\frac{1}{16}\{4xe^{3x} - e^{-x}\}$

25. The particular of integral of $(D^2 + 3D + 2)y = \cos 2x$ is

(A) $3 \sin 2x - \cos 2x$ (B) $\frac{3\sin 2x - \cos 2x}{20}$

(C) $\frac{\cos 2x - 3\sin 2x}{10}$ (D) $\frac{\cos x - \sin 2x}{40}$

26. The particular integral of $(D^2 - D)\, y = x^2 - 2x + 4$ is

(A) $x^3 - 8x + 4$ (B) $-x^3 + 4x - 4$

(C) $\frac{x^3}{3} + 8x - 4$ (D) $\frac{-x^3}{3} - 4x - 4$

27. Solve $(D^2 - D - 2)y = e^{2x} \sin x$.

(A) $y = c_1e^{x} + c_2e^{-x} + e^{2x}(3\cos x + \sin x)$

(B) $y = c_1e^{-x} + c_2e^{2x} + e^{2x}(\cos x + 3\sin x)$

(C) $y = c_1e^{-x} + c_2e^{2x} + e^{2x}(\cos x - 3\sin x)$

(D) None of these

28. If $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are two solutions of a second order linear differential equation, then the Wronskian W of y_1 and y_2 is _____

(A) e^{4x} (B) xe^{4x}

(C) $2e^{4x}$ (D) $2xe^{4x}$

29. The complementary function of the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 5e^{3x}$ is $y_c = c_1\, e^{-2x} + c_2e^{-3x}$ using the method of variation of parameters, its particular is found to be $y_p = A(x)\, e^{-2x} + B(x)\, e^{-3x}$. Then $A(x) =$

(A) $5e^{5x}$ (B) e^{5x}

(C) $\frac{1}{5}e^{-5x}$ (D) e^{-5x}

Common Data for Questions 30 and 31: Consider the equation $(D^2 + 5D + 6)y = e^{5x}$

30. Solving the above equation by using variation of parameters A can be calculated using the formula _____

(A) $A = \int \frac{UR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$

(B) $A = -\int \frac{VR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$

(C) $A = \int \frac{UR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$

(D) $A = -\int \frac{VR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$

31. In solving the above equation by variation of parameters method $B =$

(A) e^{7x} (B) $\frac{e^{3x}}{3}$

(C) $\frac{e^{7x}}{7}$ (D) e^{3x}

32. The solution of the DE $(D^2 + 1)\, y = 0$ given $x = 0, y = 2$ and $x = \frac{\pi}{2}$, $y = -2$ is

(A) $y = \sin x - \cos x$ (B) $y = 2(\cos x - \sin x)$

(C) $y = 2 \cos x \sin x$ (D) $y = 2(e^{x} + e^{-x})$

33. Solve the equation $3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2$

(A) $y = C_1x^{-3} + C_2x^{-1} + \frac{x^3}{7}$

(B) $y = C_1x^{3} + C_2x + \frac{x^2}{7}$

(C) $y = C_1x^{\frac{1}{3}} + C_2x^{-1} + \frac{x}{7}$

(D) $y = C_1x^{\frac{-1}{3}} + C_2x + \frac{x^2}{7}$

34. The *PI* of the equation $\left(\frac{x^2d^2y}{dx^2} + 7x\frac{dy}{dx} + 9\right)y = x^4 \log x$ is

(A) $x^4(\log x - 2)$ (B) $\frac{x^4}{49}$

(C) $\frac{x^4 \log x}{49} - \frac{2}{343}x^4$ (D) $\frac{x^4 \log x}{49}$

35. The order and degree of the $\frac{\partial^2 z}{\partial x^2} + 3xy\left(\frac{\partial z}{\partial x}\right)^2 + 5\frac{\partial z}{\partial y} = 8$ are

(A) 1, 1 (B) 1, 2

(C) 2, 1 (D) 2, 2

36. The differential equation whose solution is $z = (x - a)(y - b)$ is _____.

(A) $pq = 2z$ (B) $pq = z$

(C) $p = 2zq$ (D) $p = zq$

37. Solve $(2p + 1)\, q = pz$

(A) $py - xq = z$ (B) $py + xq = z$

(C) $px + yq = z$ (D) $px - yq = z$

38. Solution of $\frac{\partial^2 z}{\partial x^2} = \cos(xy)$ is _____.

(A) $y^2 + \cos(xy) = \phi(y) + z$

(B) $z = \cos(xy) + x\, \phi(y)$

(C) $z = \cos(xy)$

(D) None of these

39. The solution of $x^2p + y^2q = (x + y)\,z$ is _____.

(A) $f(xy, x - y) = 0$.

(B) $f\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$

(C) $f(zx, z - x) = 0$

(D) None of these

40. The solution of $x\,(x^2 + 3y^2)p - y\,(3x^2 + y^2)\,q = 2z\,(y^2 - x^2)$ is _____.

(A) $f\left(\frac{xy}{z}, (x^2 + y^2)z\right) = 0$

(B) $f(xyz, x^2 + y^2) = 0$

(C) $f(xy, (x^2 + y^2)z) = 0$

(D) $f(xz, (x^2 + y^2)z) = 0$

41. Form a PDE of $z = (x - y)\,\phi\,(x^2 - y^2)$

(A) $a \log (z - a) = x - ay + b$

(B) $2a \log (z + a) = ay + b$

(C) $2a \log (z - a) = x + ay + b$

(D) $a \log (z + a) = 3x + ay + b$

42. The solution of $q^2x(1 + y^2) = py^2$ is _____.

(A) $z = a\,(1 + y^2)$

(B) $z = \frac{ax^2}{2} - a(a + y^2) + b$

(C) $z = \frac{ax^2}{2} + \sqrt{a(1 + y^2)} + b$

(D) $z = \frac{ax}{2} + \sqrt{a(1 + y^2)} + b$

43. Solve $pqz = q^2\,(yp + q^2) + p^2\,(xq + p^2)$.

(A) $z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$

(B) $z = ax - by$

(C) $z = ax + by + \frac{a}{a^3} + \frac{b}{b^3}$

(D) None of these

44. In the process of solving the partial differential equation $\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{dy^2} = 0$ by the method of separation of variables, the linear differential equation involving the independent variable 'X' is _____ (Here k is a constant).

(A) $\frac{d^2X}{dx^2} + kX(x) = 0$

(B) $\frac{d^2X}{dx^2} - kX(x) = 0$

(C) $\frac{d^2X}{dx^2} + k\frac{dX}{dx} + k^2X(x) = 0$

(D) $\frac{d^2X}{dx^2} - k\frac{dX}{dx} + 2kX(x) = 0$

45. The second order partial differential equation

$3x^2\frac{\partial^2 u}{\partial x^2} - 6xy\frac{\partial^2 u}{\partial x \partial y} + 3y^2\frac{\partial^2 u}{\partial y^2} - 5\frac{\partial u}{\partial x} + 7\frac{\partial u}{\partial y} = 6x^2y$ is _____

(A) Elliptic equation

(B) Parabolic equation

(C) Hyperbolic equation

(D) Depends on the value of x and y.

Previous Years' Questions

1. The solution of the first order differential equation $\dot{x}(t) = -3x(t), x(0) = x_0$ is: **[2005]**

(A) $x(t) = x_0e^{-3t}$

(B) $x(t) = x_0e^{-3}$

(C) $x(t) = x_0e^{\frac{-t}{3}}$

(D) $x(t) = x_0e^{-t}$

2. For the equation $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 5$, the solution $x(t)$ approaches which of the following values at $t \to \infty$ **[2005]**

(A) 0

(B) $\frac{5}{2}$

(C) 5

(D) 10

3. For the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$, the solution is **[2010]**

(A) $x(t) = 2e^{-6t} - e^{-2t}$

(B) $x(t) = 2e^{-2t} - e^{-4t}$

(C) $x(t) = -e^{-6t} + 2e^{-4t}$

(D) $x(t) = e^{-2t} + 2e^{-4t}$

4. With k as a constant, the possible solution for the first order differential equation $\frac{dy}{dx} = e^{-3x}$ is **[2011]**

(A) $\frac{-1}{3}e^{-3x} + k$

(B) $-\frac{1}{3}e^{3x} + k$

(C) $-3e^{-3x} + k$

(D) $-3e^{-x} + k$

5. With initial condition $x(1) = 0.5$, the solution of the differential equation $t\frac{dx}{dt} + x = t$ is **[2012]**

(A) $x = t - \frac{1}{2}$

(B) $x = t^2 - \frac{1}{2}$

(C) $x = \frac{t^2}{2}$

(D) $x = \frac{t}{2}$

6. The solution for the differential equations $\frac{d^2x}{dt^2} = -9x$, with initial conditions $x(0) = 1$ and $\frac{dx}{dt}|_{t=0} = 1$, is **[2014]**

(A) $t^2 + t + 1$ (B) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$

(C) $\frac{1}{3}\sin 3t + \cos 3t$ (D) $\cos 3t + t$

7. Consider the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. Which of the following is a solution to this differential equation for $x > 0$? **[2014]**

(A) e^x (B) x^2

(C) $\frac{1}{x}$ (D) In x

8. A moving average function is given by $y(t) = \frac{1}{T}\int_{t-T}^{t} u(\tau)\,d\tau$. If the input u is a sinusoidal signal of frequency $\frac{1}{2T}$ Hz, then in steady state, the output y will lag u (in degree) by _____. **[2015]**

9. A solution of the ordinary differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $y(1) = -\frac{1-3e}{e^3}$. The value of $\frac{dy}{dt}(0)$ is ____. **[2015]**

10. The signum function is given by **[2015]**

$$\text{sgn}(x) = \begin{cases} \frac{x}{|x|}; x \neq 0 \\ 0; x = 0 \end{cases}$$

The Fourier series expansion of sgn(cos(*t*)) has

(A) only sine terms with all harmonics.

(B) only cosine terms with all harmonics.

(C) only sine terms with even numbered harmonics.

(D) only cosine terms with odd numbered harmonics.

11. The Laplace transform of $f(t) = 2\sqrt{\frac{t}{\pi}}$ is $s^{-3/2}$. The Laplace transform of $g(t) = \sqrt{\frac{1}{\pi}}$ is **[2015]**

(A) $3s^{-5/2}/2$ (B) $s^{-1/2}$

(C) $s^{1/2}$ (D) $s^{3/2}$

12. A differential equation $\frac{di}{dt} - 0.2i = 0$ is applicable over $-10 < t \pm 10$. If $i(4) = 10$, then $i(-5)$ is _____. **[2015]**

13. A function $y(t)$ such that $y(0) = 1$ and $y(1) = 3e^{-1}$ is a solution of the differential equation

$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ then $y(2)$ is_____. **[2016]**

(A) $5e^{-1}$ (B) $5e^{-2}$

(C) $7e^{-1}$ (D) $7e^{-2}$

14. A power system has 100 buses including 10 generator buses, for the load flow analysis using Newton – Raphson, method in polar coordinates, the size of the Jacobian is **[2016]**

(A) 189×189 (B) 100×100

(C) 90×90 (D) 180×180

15. Let $y(x)$ be the Solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with initial conditions $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$. Then the value of y(1) is _____. **[2016]**

Answer Keys

Exercise

Practice Problems

1. C	**2.** A	**3.** B	**4.** C	**5.** A	**6.** B	**7.** C	**8.** C	**9.** B	**10.** C
11. B	**12.** B	**13.** D	**14.** B	**15.** D	**16.** A	**17.** B	**18.** C	**19.** B	**20.** C
21. C	**22.** A	**23.** C	**24.** D	**25.** B	**26.** D	**27.** D	**28.** A	**29.** B	**30.** B
31. C	**32.** B	**33.** D	**34.** C	**35.** C	**36.** B	**37.** B	**38.** D	**39.** B	**40.** A
41. C	**42.** C	**43.** A	**44.** B	**45.** B					

Previous Years' Questions

1. A	**2.** B	**3.** B	**4.** A	**5.** D	**6.** C	**7.** C	**8.** 90°	**9.** –3	**10.** D
11. B	**12.** 1.6 to 1.7		**13.** B	**14.** A	**15.** 7.389				

Chapter 4

Complex Variables

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Complex number
- Modulus and argument of a complex Number
- Functions of a complex variable
- Continuity
- Differentiability
- Analytic function
- Entire function
- Elementary functions
- Complex integration
- Cauchy's integral theorem
- Morera's theorem [converse of Cauchy's theorem]
- Taylor's series expansion of $f(z)$
- Laurent's series expansion of $f(z)$
- Singular points
- Cauchy's residue theorem

Complex Number

A number of the form $x + iy$, where $x, y \in R$, the set of real numbers, and $i = \sqrt{-1}$ is called a complex number.

Notes:

1. If $z = x + iy$, then Ix is termed as the real part of z denoted by $R(z)$ and y is termed of the imaginary part of z, denoted by $I(z)$.
2. $z = x + iy$ can also be written as an ordered pair notation (x, y).

Modulus and Argument of a Complex Number

The modulus of a complex number $z = x + iy$ is the real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$.

The argument or amplitude of a complex number $z = x + iy$, when $(x, y) \neq (0, 0)$ is the value $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and is denoted by arg (z).

Note:
The arg (z) satisfying the inequality $-\pi < \theta \leq \pi$ is called the principal argument, denoted by Arg (z).

Functions of a Complex Variable

Complex functions are functions which assign complex numbers for complex numbers. Let C be the set of complex numbers.

A function $f : C \to C$ is a rule which associates with $z \in C$ a unique $w \in C$, written as $w = f(z)$.

Since $z = x + iy$, $w = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$ (u and v are functions of x and y

Note:
A complex valued function does not have to be defined on the whole of C; it may be defined over a non-empty subset D of C. 'D' is called the domain of definition of the function 'f'.

Solved Examples

Example 1: Consider $F(z) = z^2 + iz$. When expressed in terms of real and imaginary parts

Solution:
$$f(z) = (x + iy)^2 + i(x + iy)$$
$$= x^2 - y^2 + 2ixy + ix - y$$
$$= x^2 - y^2 - y + i(x + 2xy)$$

Here $u(x, y) = x^2 - y^2 - y$ and $v(x, y) = x + 2xy$

Limits: Let $w = f(z)$ be any function of z defined in a bounded closed domain D, then the limit of $f(z)$ as z approaches z_o is denoted by 'l' and written as $\lim_{z \to z_0} f(z) = l$, i.e., for every $\in > 0$, there exists $d > 0$ such that $|f(z) - l| < \in$ whenever $|z - z_o| < \delta$, where $\in$ and δ are arbitrary small positive real numbers.

Notes:

1. Here z should approach z_o through all possible curves.
2. 'l' is the simultaneous limit of $f(z)$ as $z \to z_{0.}$

Example 2: $\lim_{z \to 10} \frac{z^3 - 10^3}{z - 10} = \frac{2^3 - 10^3}{2 - 10}$

Solution:
$$\lim_{z \to 10} \frac{z^3 - 10^3}{z - 10} = \text{Lim}_{z \to 10}(z^2 + 10z + 100)$$
$$= 100 + 100 + 100 = 300$$

(using standard limits as in real variable calculus)

Example 3: Show that $\lim_{(x,y)\to(0,0)}\left(\frac{x^2y}{x^4+y^2}\right)$ does not exist.

Solution:

Path I:

$$\lim_{x\to 0,\, y\to 0}\frac{x^2y}{x^4+y^2}=\operatorname*{Lt}_{y\to 0}0=0$$

Path II:

$$\lim_{y\to 0,x\to 0}\frac{x^2y}{x^4+y^2}\lim_{x\to 0}0=0$$

Path III: along any straight line through the origin. Let $y = mx$

$$\lim_{y\to mx,\,x\to 0}\left(\frac{x^2y}{x^4+y^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{mx^3}{x^4+m^2x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{mx}{x^2+m^2}\right)=0$$

Path IV: Let $y =$

$$\lim_{y\to mx^2,\,x\to 0}\left(\frac{x^2y}{x^4+y^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{mx^4}{x^4+m^2x^4}\right)$$

$=\lim_{x\to 0}\left(\frac{m}{1+m^2}\right)$, which depends on the values of m.

Therefore the limit does not exist.

Continuity

A function $w = f(z)$ defined in the bounded closed domain D, is said to be continuous at a point Z_0, if $f(z_0)$ is defined, $\operatorname*{Lt}_{z\to z_0} f(z)$ exists and $\operatorname*{Lt}_{z\to z_0} f(z)=f(z_0)$

Notes:

1. A function $f(i)$ is said to be continuous in a domain if it is continuous at every point of the domain.
2. A function $f(z)$ is not continuous at z_0 implies either $f(z_0)$ does not exist or $\operatorname*{Lt}_{z\to z_0} f(z)$ does not exists or $\operatorname*{Lt}_{z\to z_0} f(z)\ \#\ f(z_0)$
3. If $f(z) = u + iv$ is continuous, then both u and v are continuous.
4. If $f(z)$ and $g(z)$ are two continuous functions in D, then $f+g$, $f-g$, fg and f/g $(g(z) \uparrow 0)$ are all continuous.

Differentiability

A function $f(z)$ defined in a domain D is said to be differentiable at a point z_0, if the limit $f'(z_0)=\lim_{\Delta z\to 0}\left(\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right)$

$=\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$ exists

The limit $f'(z_0)$ is known as the derivative of $f(z)$ at z_0.

Analytic Function

A function $f(z)$ is said to be analytic at a point z_0, if f is differentiable not only at z_0, but also at every point in some neighbourhood of z_0.

Notes:

1. Analytic functions are also called as holomorphic, regular or monogenic functions.
2. A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.
3. Sum difference product and quotient of analytic functions are also analytic.

Entire Function

A function which is analytic every where (i.e., at every point Z on the complex plane) is known as an entire function.

For example, All polynomial functions are entire functions.

Standard results on complex differentiation

1. $\frac{dc}{dz}=0$, where 'c' is a complex constant.
2. $\frac{d}{dz}[f\pm g]=\frac{df}{dz}+\frac{dg}{dz}$
3. $\frac{d}{dz}[c\,f(z)]=c.\frac{df}{dz}$
4. $\frac{d}{dz}[f\cdot g]=f.\frac{dg}{dz}+g.\frac{df}{dz}$
5. $\frac{d}{dz}\left(\frac{f}{g}\right)=\frac{g.\frac{df}{dz}-f\frac{dg}{dz}}{g^2}$
6. $\frac{d}{dz}z^n=n\cdot z^{n-1}$
7. If $f(z) = G(h(z))$

   Then $\frac{d}{dz}(f(z))=G'(h(z))\cdot h'(z)$ (termed as chain rule]

The necessary condition for f(z) to be analytic

If a function $f(z) = (x, y) + iv(x, y)$ is differentiable at any point $z = x + iy$, then the first order partial derivatives of u and v exists at this point and satisfies the equations.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad (1)$$

and

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad (2)$$

Equations (1) and (2) are referred as Cauchy–Riemann Equations (C–R equations).

The sufficient condition for f (z) to be analytic

A continuous function $f(z)$ is analytic in a domain D if the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists, are also continuous and satisfy the Cauchy–Riemann equations at each point on the domain D.

Laplace equation

An equation having the second order partial derivatives of the form $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace equation. ∇^2 is called the Laplacian Operator.

Harmonic function

A function $u(x, y)$ is said to be a harmonic function if it satisfies the Laplace equation i.e., $\nabla^2 u = 0$

Note: The real and imaginary parts of an analytic function are harmonic.

Conjugate harmonic function: If $f(z) = u + iv$ is an analytic function, then 'v' is the conjugate harmonic function of 'u' and vice versa.

Cauchy–Riemann equations in polar coordinates If $f(z) = u(r, \theta) + iv(r, \theta)$ where $Z = re^{i\theta}$, then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{d\theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$ are called Cauchy–Riemann equations in polar form.

Methods of constructing an analytic function

1. If the real part $u(x, y)$ of function $f(z)$ is given, then

   Step I: find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

   Step II: Write $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ [using C–R equations]

   Step III: Replace 'x' by z and 'y' by '0'.

   Step IV: Integrate $f'(z)$ with respect to z.

   The above method is known as Milne–Thompson method.
2. If the real part $u(x, y)$ of an analytic function $f(z)$ is given, to find the harmonic conjugate, find $f(z)$ using Milne–Thompson method and then separate the real and imaginary parts by putting $z = x + iy$. Then the imaginary part $v(x, y)$ is the harmonic conjugate of $u(i, y)$.

Complex potential function If $w = \phi(x, y) + i\Psi(x, y)$ is analytic then it is called complex potential function. Its real part $\phi(x, y)$ is known as velocity potential function and its imaginary part $y(x, y)$ is known as stream function.

Note: Both ϕ and Ψ satisfy Laplace equation.

Elementary Functions

1. Complex exponential function a^s: $e^z = e^x (\cos y + i\sin y)$
2. Trigonometric Functions:

   (i) $\cos t = \frac{1}{2}(e^{iz} + e^{-iz})$

   (ii) $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

   (iii) $\tan z = \frac{\sin z}{\cos z}$, if $\cos z \neq 0$.

   (iv) $\cot z = \frac{\cos z}{\sin z}$, if $\sin z \neq 0$

   (v) $\sec z = \frac{1}{\cos z}$, if $\cos z \neq 0$

   (vi) $\operatorname{cosec} z = \frac{1}{\sin z}$, if $\sin z \neq 0$

Notes:

1. e^z, $\sin z$ and $\cos z$ are entire functions.
2. $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$.
3. $\cot z$ and $\operatorname{cosec} z$ are analytic except at the points where $\sin z = 0$
4. $e^{iz} = \cos z + i\sin z$.

3. Hyperbolic Functions

   (i) $\sinh z = \frac{e^z - e^{-z}}{2}$

   (ii) $\cosh z = \frac{e^z + e^{-z}}{2}$

   We can write the other hyperbolic functions like above.
4. Relationship among trigonometric complex and hyperbolic functions:

   (i) $\cosh(iz) = \cos z$
   (ii) $\sinh(iz) = i\sin z$
   (iii) $\cos(iz) = \cos hz$
   (iv) $\sin(iz) = i\sin hz$
   (v) $\tan(iz) = i\tan hz$

Find the following limits:

Example 4: (i) $\underset{z\to\infty}{\text{Lt}}\left(\frac{z}{3+iz}\right)$

(ii) $\underset{z\to\infty}{\text{Lt}}\left(\sqrt{z+4i} - \sqrt{z+i}\right)$

Solution:

(i) $\underset{z\to\infty}{\text{Lt}}\frac{z}{3+iz}$, Let $z = 1/a \Rightarrow a = 1/z$ as $z \to \infty$; $a \to 0$

$\therefore$ The given Limit becomes

$$\lim_{a\to 0}\frac{1/a}{3+i/a}=\lim_{a\to 0}\frac{1}{3a+i}=\frac{1}{i}=-i$$

(ii) $$\lim_{z\to\infty}\frac{\left(\sqrt{z+4i}-\sqrt{z+i}\right)\left(\sqrt{z+4i}+\sqrt{z+i}\right)}{\sqrt{z+4i}+\sqrt{z+i}}$$

$$=\lim_{z\to\infty}\frac{\left(\sqrt{z+4i}\right)^2-\left(\sqrt{z+i}\right)^2}{\sqrt{z+4i}+\sqrt{z+i}}$$

$$=\lim_{z\to\infty}\frac{3i}{\sqrt{z+4i}+\sqrt{z+i}}$$

Let $z = 1/a \quad\Rightarrow\quad a = 1/z$, as $z\to\infty$, $a\to 0$

$$\therefore\quad \lim_{a\to 0}\frac{3i}{\sqrt{\frac{1}{a}+4i}+\sqrt{\left(\frac{1}{a}+i\right)}}$$

$$=\lim_{a\to 0}\frac{3\sqrt{ai}}{\sqrt{1+4ai}+\sqrt{1'1+ai}}=0$$

Example 5: Show that the following limits does not exist:

(i) $$\lim_{z\to 0}\frac{\left[\mathrm{Re}\,z-\mathrm{Im}\,z\right]^2}{|z|^2}$$

(ii) $$\lim_{z\to 0}\left[\frac{1}{1-e^{1/x}}-iy\right]$$

Solution:

(i) If we consider any straight line path i.e., $y = mx$,

$$\lim_{z\to 0}\frac{\left[\mathrm{Re}\,z-\mathrm{Im}\,z\right]^2}{|z|^2}=\lim_{z\to 0}\frac{(x-y)^2}{x^2+y^2}$$

$$\Rightarrow\quad \lim_{z\to 0}\frac{(x-mx)^2}{x^2+m^2x^2}=\lim_{z\to 0}\frac{x^2(1-m)^2}{x^2(1+m^2)}$$

$$=\frac{(1-m)^2}{1+m^2}$$

Since the limit depends on 'm', it is not unique.

$\therefore$ The limit does not exist.

(ii) $$\lim_{z\to 0}\left[\frac{1}{1-e^{1/x}}-iy\right]$$

If we consider the path $y\to 0$, followed by $x\to 0$

$$\Rightarrow\quad \lim_{x\to 0}\left[\frac{1}{1-e^{1/x}}\right]=\begin{vmatrix}0, & x>0\\ 1. & x<0.\end{vmatrix}$$

$\therefore$ The limit is not unique. The limit does not exist.

Example 6: Show that the function $f(z)$ is not continuous at $z = 0$, where:

(i) $$f(z)=\begin{vmatrix}\dfrac{\mathrm{Im}(z)}{|z|^2} & z\neq 0\\ 0, & z=0\end{vmatrix}$$

(ii) $$f(z)=\begin{vmatrix}\dfrac{\mathrm{Re}(z)^2}{|z|^2}, & z\neq 0\\ 0, & z=0\end{vmatrix}$$

Solution:

(i) $$\lim_{z\to 0}\frac{\mathrm{Im}(z)}{|z|}=\lim_{y\to 0,\,x\to 0}\frac{y}{\sqrt{x^2+y^2}}.$$

If we consider the path $y = mx$

$$\Rightarrow\quad \lim_{x\to 0}\frac{mx}{\sqrt{x^2+x^2m^2}}$$

$$=\lim_{x\to 0}\frac{mx}{x\sqrt{1+m^2}}=\frac{m}{\sqrt{1+m^2}}$$

Since the limit value depends on m, the limit does not exist at $z = 0$. It is not continuous at $z = 0$.

(ii) $$\lim_{z\to 0}\frac{\mathrm{Re}(z^2)}{|z|^2}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^2-y^2}{x^2+y^2}$$

Considering the path $y = mx$,

$$\Rightarrow\quad \lim_{x\to 0}\frac{x^2-m^2x^2}{x^2+m^2y^2}=\frac{1-m^2}{1+m^2}$$

Which also depends on m. Hence the limit does not exist and therefore $f(z)$ is not continuous at $z = 0$.

Example 7: Show that the function:

$$f(z)=\begin{vmatrix}\dfrac{x^3(1+i)-y^3(1-i),}{x^2+y^2} & z\neq 0\\ 0, & z=0.\end{vmatrix}$$

Satisfies Canchy–Riemann equations at $z = 0$ but $f'(0)$ does not exist.

Solution: We have $u(x,y)=\dfrac{x^3-y^3}{x^2+y^2}$ and

$$v(x,y)=\frac{x^3+y^3}{x^2+y^2}$$

When $(x, y) \neq (0, 0)$

Also given that $f(0) = 0 \quad\Rightarrow\quad u(0, 0) = V(0, 0) = 0.$

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{-y}{y}=-1.$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{y}{y}=1.$$

We observe that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=1 \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-1$$

$\therefore$ C–R equations are satisfied at the origin.

But $\underset{z\to 0}{\text{Lt}}\dfrac{f(z)-f(0)}{z}$

$$=\underset{z\to 0}{\text{Lt}}\frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$=\underset{z\to 0}{\text{Lt}}\frac{(1+i)\left[x^3+iy^3\right]}{(x^2+y^2)(x+iy)}\frac{(x-iy)}{(x-iy)}$$

$$\Rightarrow \quad \underset{z\to 0}{\text{Lt}}\frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)^2}$$

We consider the path $y=mx$. We get

$$\underset{x\to 0}{\text{Lt}}\frac{(1+i)(x^3+m^3x^3)(x-imx)}{(x^2+m^2x^2)^2}$$

$$\Rightarrow \quad \underset{x\to 0}{\text{Lt}}\frac{(1+i)(1+im^3)(1-im)x^4}{(1+m^2)^2x^4}$$

$$=\frac{(1+i)(1+im^3)(1-im)}{(1+m^2)^2}$$

This depends on the value of m. Therefore the limit does not exist. Hence $f'(0)$ does not exist.

$\therefore$ For the given function $f(z)$, though Canchy–Riemann equations are satisfied at the origin, it is not analytic at the origin.

Example 8: Show that the function $f(z)=\dfrac{z}{z+1}$ is analytic at $z=\infty$

Solution: Given that $f(z)=\dfrac{z}{z+1}$

$$\Rightarrow \quad f(1/z)=\frac{\frac{1}{z}}{1/z+1}=\frac{1}{z+1}$$

We know that $f(1/z)$ is differentiable at $z=0$ and at all points in its neighbourhood.

$\therefore$ $f(1/z)$ is analytic at $z=0$

$\Rightarrow$ $f(z)$ is analytic at $z=\infty$.

Example 9: If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=e^{-x}[(x-y)\sin y-(x+y)\cos y]$ then find u, v and the analytic function $f(z)$.

Solution: $u-v=e^{-x}[(x-y)\sin y-(x+y)\cos y]$

$$\Rightarrow \quad \frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}$$

$=-e^{-x}[(x-y)\sin y-(x+y)\cos y]+e^{-x}[\sin y-\cos y]$

$=e^{-x}[(1-x+y)\sin y+(x+y-1)\cos y]$ (3)

Also $\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial y}=e^{-x}[x\cos y-(\sin y+y\cos y)+x\sin y-(\cos y-y\sin y)]$

$=e^{-x}[(x+y-1)\sin y+(x-y-1)\cos y]$ (4)

Using C–R equations, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

And substituting in (3) and (4), we get

$$\frac{\partial u}{\partial x}=u_x=e^{-x}[(1-x)\sin y+y\cos y]=\frac{\partial v}{\partial y} \qquad (5)$$

And $$\frac{\partial v}{\partial x}=V_x=e^{-x}[(1-x)\cos y-y\sin y]=-\frac{\partial u}{\partial y} \qquad (6)$$

Integrating (5) and (6) partially wrt x, we get respectively,

$$u(x,y)=e^{-x}[x\sin y-y\cos y]+\psi_1(y)$$
$$V(x,y)=e^{-x}[y\sin y+x\cos y]+\psi_2(y)$$

Where $\psi_1(y)$ and $\psi_2(y)$ are arbitrary functions of z. substituting in equation (5) and (6), we obtain respectively, $e^{-x}[(1-x)\sin y+y\cos y]=e^{-x}[\sin y+y\cos y-x\sin y]+\psi_2'(y)$ and $e^{-x}[(1-x)\cos y-y\sin y]=e^{-x}\left[a\cos y-\cos y+y\sin y\right]+\psi_1'(y)$ on simplifying, we get $\psi_2'(y)=0$ and $\psi_1'(y)=0$.

Which gives $\psi_1'(y)=k_1$ and $y_2(y)=k_2$

Where k_1 and k_2 are arbitrary constants.

Hence we have

$u(x,y)=e^{-x}(x\sin y-y\cos y)+k_1$ and $V(x,y)=e^{-x}(y\sin y+x\cos y)+k_2$.

and $f(z)=u(x,y)+iv(x,y)$

$=e^{-x}[x\sin y-y\cos y]+k_1+i[e^{-x}(y\sin y+x\cos y]+k_2$

$=e^{-x}[(x+iy)\sin y-i(a+iy)\cos y]+k_1+ik_2$

$=i(x+iy)e^{-x}e^{-iy}+k_1+ik_2$

$=i(x+iy)e^{-}(x+iy)+k_1+ik_2$

$=ize^{-z}+k$, where $k=k_1+ik_2$ is a complex constant.

Example 10: Show that the function $u(r,\theta)=r^2\cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

Solution: We have $z=re^{i\theta}$ and given that $u(r,\theta)=r^2\cos 2\theta$.

$$\Rightarrow \quad \frac{\partial u}{\partial r}=2r\cos 2\theta, \frac{\partial^2 u}{\partial r^2}=2\cos 2\theta.$$

Also $\dfrac{\partial u}{\partial \theta}=-2r^2\sin 2\theta$ and

$$\frac{\partial^2 u}{\partial \theta^2}=-4r^2\cos 2\theta.$$

Since $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

$$=2\cos 2\theta+\frac{1}{r}(2r\cos 2\theta)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$=4\cos 2\theta-4\cos 2\theta=0$,

$u(r, \theta)$ satisfies the Laplace equation in polar form and therefore it is harmonic.
Also from C–R equations we know that

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = r\cdot(2r\cos 2\theta) = 2r^2\cos 2\theta.$$

Integrating $\frac{\partial v}{\partial \theta}$. Partially wrt θ, we get $v(r, \theta) = r^2 \sin 2\theta + \psi(r)$, where $\psi(r)$ is an arbitrary function of r.

Also using C–R equations, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

We get, $2r\sin 2\theta + \psi'(r) = -\frac{1}{r}(-2r^2\sin 2\theta)$

$\Rightarrow \quad \psi'(r) = 0 \Rightarrow \psi(r) = c$ a constant.

$\therefore v(r, \theta) = r^2 \sin 2\theta + c.$

$\therefore f(z) = u(r, \theta) + iv(r, \theta)$

$= r^2\cos 2\theta + i\,[r^2 \sin 2\theta + c]$

$= r^2[\cos 2\theta + i \sin 2\theta] + ic = r^2 e^{2i\theta} + ic.$

Example 11: Show that the given function $u(x, y) = \sinh x\cos y$ is harmonic. Find the corresponding harmonic function $v(x, y)$ and construct the analytic function $f(z) = u + iv$.

Solution: $u(x, y) = \sinh x\cos y.$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \cosh x\cos y, \frac{\partial^2 u}{\partial x^2} = \sinh x\cos y.$$

Also $\frac{\partial u}{\partial y} = -\sinh x \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -\sinh x\cos y.$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{`}u \text{ is harmonic.}$$

Since $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = \frac{\partial u}{\partial x} - \frac{i\partial u}{\partial y} \text{ [using C–R equations]}$$

$$= \cosh x\cos y - i(-\sinh x\sin y)$$

Using Milne–Thompson method, replacing x by z and y by 0, we get $f'(z) = \cosh z$
Integrating wrt z, we get $f(z) = \sinh z + ic$

$$f(z) = \sinh z + ic = \sinh(x + iy) + ic$$
$$= \sinh x\cos y + i\cosh x\sin y + ic$$
$$\therefore v(x, y) = \sin y\cosh x + c$$

Example 12: Show that $v(x, y) = \arg(z)$, $z \neq 0$, is harmonic. Find the corresponding conjugate harmonic function $u(x, y)$ and construct the analytic function $f(z) = u + iv$.

Solution: Given $v(x, y) = \arg(z)$

$$v(x, y) = \tan^{-1}(y/x)$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)(0) + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow v(x, y)$ satisfies Laplace equation and hence harmonic.
Since $f(z) = u + iv$

$$f(z) = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = \frac{\partial v}{\partial y} + \frac{i\partial v}{\partial x} \text{ [using C–R equation]}$$

$$= \frac{x}{x^2+y^2} + i\left[\frac{-y}{x^2+y^2}\right]$$

Using Milne–Thompson method, replacing x by z and y by '0', we get

$$f'(z) = \frac{z}{z^2} = \frac{1}{z}$$

$\Rightarrow$ On integrating, $f(z) = \log z + ic$

$$f(z) = \log(re^{i\theta}) + ic = \log r + i\theta + ic$$
$$= \log\sqrt{x^2+y^2} + i[\theta + c]$$
$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + ic$$

$$\therefore u(x, y) = \frac{1}{2}\log(x^2+y^2)$$

Example 13: Find the constants p, q, r such that the function $f(z) = x - 2py + i(qx - ry)$ is analytic. Express $f(z)$ in terms of z.

Solution: From the given function, $u(x, y) = x - 2py$ and $v(x, y) = qx - ry$.
Since u and v are continuous every where, they will be analytic if C–R equations are satisfied.

$$\text{i.e., } \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = -2p, \frac{\partial v}{\partial x} = q \text{ and } \frac{\partial v}{\partial y} = r$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = -r \Rightarrow r = -1$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2p = -q$$

Where p is arbitrary $\Rightarrow q = 2p$
$\therefore$ The required analytic function is
$f(z) = x - 2py + i(qx - ry)$

$$= x - 2py + i\,(2px + y)$$
$$= [x + iy] + 2pi[x + iy]$$
$$f(z) = z + 2piz.$$

Complex integration

Let $f(z)$ be a function of complex variable defined in a domain D. Also let 'c' be the closed curve in the domain D.

Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$

(OR)

$$f(z) = u + iv \text{ and } dz = dx + idy.$$

$$\int_c f(z)dz = \int_c (u+iv)\,dz$$
$$= \int_c (u+iv)\,(dx+idy)$$
$$= \int_c (udx - vdy) + i\int_c (udy + vdx)$$

$\int_c f(z)dz$ is determined as contour integral.

Example 14: Evaluate $\int_c f(z)dz$ where $f(z) = z^2$ and c is the straight line joining $z = 0$ and $z = 4 + i$.

Solution: If we integrate $\int_0^{4+i} z^2 dz$ formally,

We get $\left[\frac{z^3}{3}\right]_0^{4+i} = \frac{(4+i)^3}{3} = \frac{52+47i}{3}$.

But, Let us find the parametric equations of the curve c, i.e., the straight line joining $z = 0$ and $z = 4 + i$, that is O (0, 0) and A (4, 1).

Slope of $OA = \frac{1-0}{4-0} = \frac{1}{4}$.

$\therefore$ Equation of $OA = y - 0 = \frac{1}{4}(x-0)$

$\Rightarrow \quad x = 4y. \Rightarrow \quad dx = 4\,dy.$

Taking 'y' itself as the parameter,

$$\int z^2 dz = \int_{y=0}^{1} (x+iy)^2 (dx+idy)$$
$$= \int_{y=0}^{1} (4y+iy)^2 (4dy+idy)$$
$$= \int_{y=0}^{1} (16y^2 - y^2 + 8iy^2)(4+i)dy.$$
$$= (4+i)\int_{y=0}^{1} (15y^2 + 8iy^2)dy.$$
$$= (4\ +\ i)[15+8i]\left[\frac{y^3}{3}\right]_0^1 = \frac{(4+i)(15+8i)}{3}$$
$$= \frac{60+15i+32i-8}{3} = \frac{52+47i}{3}.$$

Notes:

1. The value of the integral in both the above cases is the same.
2. But we cannot expect such a thing to happen in all the cases.

   The value of the integral $\int_c f(z)dz$ depends on the contour c.

Example 15: Evaluate $\int_c \bar{z}dz$, where c is the straight line joining $z = 0$ and $z = 2 + i$.

Solution: $\int_c \bar{z}dz = \int_0^1 (x-iy)(dx+idy)$

$$= \int_0^1 (2y-iy)\,(2dy+idy)$$
$$= (2+i)\int_0^1 (2-i)(y)dy.$$
$$(4-i^2)\int_0^1 ydy = 5\left[\frac{y^2}{2}\right]_0^1 = \frac{5}{2}.$$

[Remark: Let us suppose 'c' is the path consisting of two line segments joining $z = 0$, $z = 2$ and $z = 2$ and $z = 2 + i$. That is, let us suppose 'c' is the curve consisting of OA and OB.

$$\therefore \int_{OAB} \bar{z}dz = \int_{OA} \bar{z}dz + \int_{AB} \bar{z}dz.$$
$$= \int_{x=0}^{x=2} (x-iy)(dx+idy) + \int_{y=0}^{y=1} (x-iy)(dx+idy).$$
$$= \int_{x=0}^{x=2} (x)(dx) + \int_{y=0}^{y=1} (2-iy)(idy).$$
$$= \left[\frac{x^2}{2}\right]_0^2 + i\left[2y\frac{iy}{2}\right]_0^1 = 2 + i[2 - i/2]$$
$$= 2i + 2 + 1/2 = 2i + 5/2.]$$

Note: Therefore it is very clear that the value of the integral depends on the contour c.

Cauchy's integral theorem

If $f(z)$ is analytic in a simply-connected region R, then $\int_c f(z)dz = 0$ for every closed contour c contained in R.

Example 16: If 'c' is any simple closed curve, evaluate $\int_c f(z)dz$, if f $(z) =$

(A) $2z^2 + 18$ (B) e^{12^z}

(C) $\cos 8z$ (D) $\sin 4z + 8z^5$

Solution: All these functions are analytic every where and hence in particular on and within any closed curve c.

Hence by Cauchy's theorem $\int_c f(z)dz = 0$.

Notes:

1. If $f(z)$ is analytic in the region R, then the line integral of $f(z)$ is independent of the path joining any two points of that region.
2. If $f(z)$ is analytic in a doubly-connected region R bounded by two simple closed curves c_1 and c_2 where c_2 is in the inside of c_1, then $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$.

Example 17: Evaluate $\int_c \frac{dz}{z}$, where c is the circle, $|z-3|=2$.

Solution: $f(z)=\frac{1}{z}$ is analytic at all points except the origin, but the circle $|z-3|=2$ does not pass through the origin and also O does not lie in c. Hence $f(z)$ is analytic on and in c. Hence by Cauchy's theorem.

$$\int_c f(z)dz = 0.$$

Example 18: Evaluate $\int_c \frac{z^2}{z-5}dz$, where 'c' is the circle $|z|=2$.

Solution: $f(z)=\frac{z^2}{z-5}$ is not analytic at $z = 5$. But this point lies outside the circle $|z|=2$. Hence $f(z)$ is analytic at all points interior and on the closed curve c. Hence by Cauchy's theorem $\int_c f(z)dz = 0$.

The cauchy's integral formula

If $f(z)$ be analytic within and on a closed contour c inside a simply-connected domain and if z_o is any point in the middle of C, then $f(z_o)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_o}dz.$, the integral being taken in the positive sense around c.

Generalisation of cauchy's integral formula If $f(z)$ is analytic on and within a simple closed curve C and if z_o is any point within c, then

$$f^n(z_o)=\frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_o)^{n+1}}dz.$$

Morera's Theorem [Converse of Cauchy's Theorem]

If a function $f(z)$ is continuous throughout the simple connected domain D and if $\int_c f(z)dz = 0$ for every closed contour c in D, then $f(z)$ is analytic in D.

Taylor's Series Expansion of f(z)

Let $f(z)$ be analytic at all points within a circle c_o with centre a and radius r. Then at each point z within c_o,

$$f(z)=f(A)+f'(A)(z-a)+\frac{f'(a)(z-a)^2}{2!}+\cdots+$$

$$\frac{f^n(a)}{n!}(z-a)^n.+\cdots\infty$$

Example 19: Find the Taylor expansion for e^z about the point $z = 3$.

Solution: Put $z - 3 = w$, then $e^z = e^{w+3} = e^3 \cdot e^w = e^3$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}=e^3\sum_{n=0}^{\infty}\frac{(z-3)^n}{n!}.$$

Laurent's Series Expansion of f(z)

If $f(z)$ is analytic in the ring shaped region $r_2 < |z-a| < r_1$, and 'c' be any simple closed curve around $z = a$ in the region of analycity, then $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$,

Where $a_n=\frac{1}{2\pi i}\oint_c \frac{f(p)}{(p-a)^{n+1}}dp.$

Example 20: Expand $e^{\frac{z}{z-2}}$ as a Laurent series about $z = 1$.

Solution: Put $z - 2 = w$. Then

$$e^{\frac{z}{z-2}}=e^{\frac{2+w}{w}}=e^{1+\frac{2}{w}}=e^1\cdot e^{\frac{2}{w}}$$

$$=e\left[1+\sum_{n=1}^{\infty}\frac{2^n w^{-n}}{n!}\right]$$

$$=e\left[1+\sum_{n=1}^{\infty}\frac{(z-2)^{-n}}{n!}\right].$$

Example 21: Expand $f(z)=\frac{z^2-1}{(z+1)(z+4)}; 2\le|z|\le 3$, as Laurent's series.

Solution: $f(zi)=\frac{z^2-1}{(z+1)(z+4)}$ ceases to be analytic only for $z = -1$ and $z = -4$.

$\therefore$ In the region $2 \le |z| \le 3$, $f(z)$ is analytic. Resolving $f(z)$ into partial fractions,

$$f(z)=1-\frac{5}{z+4}.$$

Since $|z|\le 3, \left|\frac{z}{4}\right|<1$, hence

$$\frac{1}{z+4}=\frac{1}{4}\left[1+\frac{z}{4}\right]^{-1}=\frac{1}{4}\left[1-\frac{z}{4}+\frac{z^2}{16}-\cdots\infty\right]$$

$$\therefore\ \frac{z^2-1}{(z+1)(z+4)}=1-\frac{5}{z+4}=1-\frac{5}{4}\left[1-\frac{z}{4}+\frac{z^2}{16}-\cdots\infty\right]$$

Which is the required Laurent's expansion.

Zero of an analytic function

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.

Zero of mth order

If an analytic function $f(z)$ can be expressed in the form $f(z) = (z-p)^m \Psi(z)$, where $\Psi(p) \neq 0$ and $\Psi(p)$ is analytic then $z = p$ is called zero of mth order of $f(z)$.

Note: Zero of order one is known as simple zero.

Example 22:

1. $f(z) = (z-2)^5$, then $z = 2$, is a zero of order 5 of $f(z)$.
2. $f(z) = \sin z$, then $z = 0, \pm\pi, \pm 2\pi, \ldots$ are simple zeros of $f(z)$.
3. $f(z) = \dfrac{2}{3-z}$, then $z = °$ is a simple zero of $f(z)$.

Singular Points

A singular point (or singularity) of a function $f(z)$ is the point at which $f(z)$ ceases to be analytic.

Types of singularity

There are two types of singularities

1. Isolated singularity and
2. Non-isolated singularity.

Isolated singularity If $f(z)$ is analytic at each point in the deleted neighbourhood of a singularity $z = z_0$, which contains no other singularity then $z = z_0$ is said to be an isolated singularity of $f(z)$.

Example:

1. $f(z) = \dfrac{1}{z}$ is analytic everywhere except at $z = 0$.

   Hence $z = 0$ is an isolated singularity.

2. $f(z) = \dfrac{3}{\sin z}$ has an infinite number of isolated singularities at $z = \pm\pi, \pm 2\pi,$ .... etc.

3. $f(z) = \dfrac{z-1}{z^3(z^2+9)}$ has three isolated singularities at $z = 0, z = \pm 3i$.

Isolated singularities are further classified into the three categories.

1. **Removable singularity:** If a single valued function $f(z)$ is not defined at $z = a$, but $\lim\limits_{z\to a} f(z)$ exists, then the singularity $z = a$ is known as removable singularity.

   For example: The function $f(z) = \dfrac{\sin z}{z}$ which is not defined at $z = 0$.

   For $z \neq 0$, we have

   $$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

   $$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$ which contains no negative powers of z.

   If we set $f(z) = 1$ at $z = 0$, then $f(z)$ becomes analytic at $z = 0$. Thus the function $f(z)$ has a removable singularity at $z = 0$.

2. **Poles:** If $z = a$ is an isolated singular point of an analytic function $f(z)$, then $f(z)$ can be expanded in Lauren't series about $z = a$.

   i.e., $f(z) = \sum\limits_{n=0}^{\infty} a_n (z-a)^n + \sum\limits_{n=1}^{\infty} \dfrac{b_n}{(z-a)^n}$.

   Here the negative powers of $z - a$, i.e., $\sum\limits_{n=1}^{\infty} \dfrac{b_n}{(z-a)^n}$ is known as the principal part of the Laurent's series of $f(z)$.

   If the principal part contains a finite number of terms say m, then $z = a$ is called the pole of order m.

Note: Pole of order 'one' is called a simple pole.

Example:

1. $f(z) = \dfrac{z^3}{(z-2)(z+3)^4}$ has a simple pole at $z = 2$ and a pole of order 4 at $z = -3$.

2. $f(z) = \dfrac{1-e^{2z}}{z^4}$ has a pole of order 3 at $z = 0$, since expanding,

   $$f(z) = \frac{1-e^{2z}}{z^4}$$

   $$= \frac{1-\left[1 + \dfrac{2z}{1!} + \dfrac{(2z)^2}{2} + \cdots\right]}{z^4}$$

   $$= -\left[\frac{2}{z^3} + \frac{2}{z^2} + \frac{4}{3}\cdot\frac{1}{z} + \frac{2}{3} + \frac{4z}{15} + \cdots\right]$$

   The highest negative power of z is -3.

   $\therefore$ $f(z)$ has a pole of order 3 at $z = 0$.

3. **Essential singularity** In the Laurent's series expansion of $f(z)$ at $z = a$, if the principal part of $f(z)$ contains an infinite number of terms, i.e., $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is infinite, then the point $z = a$ is called an essential singularity of $f(z)$.

   For example, sin $1/z$ has an essential singularity at the origin.

   For, the Laurent's series expansion of sin $1/z$ about the origin is $\frac{1}{z} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots\infty$ which contains infinite powers of $1/z$.

   Also $f(z) = e^{1/z}$ has essential singularity at $z = 0$, since the principal part of $e^{1/z}$ contains infinite term having negative powers of $(z - 0)$.

Non-isolated singularity

Definition: If $z = a$ is a singularity, such that every neighbourhood of 'a' contains another singularity, then $z = a$ is a non-isolated singularity. We understand the definition of non-isolated singularity through an example.

Let $f(z) = \log z$, which has $z = 0$ as a singularity. But $\log z$ is not analytic on the negative real axis. Since every neighbourhood of $z = 0$ contains points on the negative real axis, $z = 0$ is not an isolated singularity of $f(z)$.

We say that for $\log z$, $z = 0$ is a non-isolated singularity.

Note: We consider only isolated singularities in our problems.

Alternative explanation: If $z = a$ is an isolated singular point of an analytic function $f(z)$, then $f(z)$ can be expanded in Laurent's series about the point = a.

That is, $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$.

The series of negative powers of $(z - a)$ is known as principal part of the Laurent's series.

1. If the principal part contains a finite number of terms, say m, i.e., $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m + 1)$ then the singular point at $z = a$ is called the pole of order m.
2. If the principal part of $f(z)$ contains infinite number of terms, i.e., $a_n \neq 0$ for infinitely many negative integers n, then the point $z = a$ is called the essential singularity of $f(z)$.
3. If the principal part of $f(z)$ contains no term i.e., $a_n = 0$ for $n \leq -1$, then the singularity at $z = a$ is called the removable singularity of $f(z)$.

Residue at an isolated singularity

The coefficient of $\frac{1}{z-a}$ in the Laurent's expansion of $f(z)$ about the isolated singularity at $z = a$ is called the residue of $f(z)$ at that point.

$\therefore$ Residue of $f(z)$ at $z = a$ is $\frac{1}{2\pi i}\int_c f(z)dz$ which implies $\int_c f(z)dz = 2\pi i$ [Residue of $f(z)$ at $z = a$] where 'c' is the closed curve containing 'a' and no other singularity.

Calculation of residues

1. Residue at a simple pole.
   Let $z = a$ is a simple pole of $f(z)$
   Then $[\text{Res } f(z)] = \underset{z\to a}{\text{Lt}} (z-a) f(z)$.

2. Residue at a pole of order 'm':
   Let $z = a$ be the pole of order m of $f(z)$
   Then $[\text{Res } f(z)]_{z=a}$

$$= \underset{z\to a}{\text{Lt}} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

3. Residue at infinity: When $f(z)$ has an essential singularity at $z = a$, then the residue of $f(z)$ at $z = a$ is found as the coefficient of $\frac{1}{z-a}$ in its Laurent's expansion.

Example 23: Find the residues of $f(z) = \frac{1}{(z+2)(z+5)}$ at the poles.

Solution:
The given function $f(z)$ has two simple poles at $z = -2$ and $z = -5$

$$\therefore \quad [\text{Res } f(z)]_{z=-2} = \underset{z\to -2}{\text{Lt}} (z+2) f(z)$$

$$= \underset{z\to -2}{\text{Lt}} \frac{1}{z+5} = \frac{1}{-2+5} = \frac{1}{3}.$$

$$\text{Also } [\text{Res } f(z)]_{z=-5} = \underset{z\to -5}{\text{Lt}} (z+5) f(z)$$

$$= \underset{z\to 5}{\text{Lt}} \frac{1}{-z+2} = \frac{1}{-5+2} = \frac{-1}{3}.$$

Example 24: Find the residue of $f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2-4)}$ at each pole.

Solution:

$$f(z) = \frac{z(z-2)}{(z+1)^2(z+2)(z-2)} = \frac{z}{(z+1)^2(z+2)}$$

$\therefore$ Poles of $f(z)$ are -2 and -1. -1 is a pole of order 2 and -2 is a simple pole.

$$\therefore \quad [\text{Res } f(z)]_{z=-2} = \underset{z\to -2}{\text{Lt}} (z+2) f(z)$$

$$= \underset{z\to -2}{\text{Lt}} \frac{z}{(z+1)^2} = \frac{-2}{(-2+1)^2} = -2.$$

$$[\text{Res } f(z)]_{z=-1} = \frac{1}{(2-1)!}\underset{z\to -1}{\text{Lt}}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \underset{z\to -1}{\text{Lt}}\frac{d}{dz}\left[\frac{z}{z+2}\right] = \underset{z\to -1}{\text{Lt}}\frac{(z+2)(1)-z(1)}{(z+2)^2}$$

$$= \underset{z\to -1}{\text{Lt}}\frac{2}{(z+2)^2} = \frac{2}{(-1+2)^2} = 2.$$

Example 25: Find the residue of the function $f(z) = \dfrac{1-e^z}{z^3}$ at the poles.

Solution:

$z = 0$ is the singular point of $f(z)$.

Expanding

$$f(z) = \frac{1-e^z}{z^3} = \frac{1-\left[1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]}{z^3}$$

$$= -\left[\frac{1}{z^2}+\frac{1}{2z}+\frac{1}{6}+\frac{z}{24}+\cdots\right]$$ $z = 0$ is a pole of order 2, and the $[\text{Res } f(z)]_{z=0}$ = co-efficient of $\dfrac{1}{z-0} = \dfrac{1}{2}$.

Cauchy's residue theorem

If $f(z)$ is analytic within and on a closed curve c, except at a finite number of poles within c, then $\int_c f(z)dz = 2\pi i$ (Sum of the residues at the poles within c).

Example 26: Evaluate $\int_c \dfrac{e^{2z}}{(z-1)^3}dz$. Where c is the circle $|z-1| = 2$

Solution:

The given function $\dfrac{e^{2z}}{(z-1)^3}$ has only one singular point at $z = 1$ which is a pole of order 3. Also $z = 1$ is inside the given circle.

$\therefore$ $[\text{Res } f(z)]_{z=1}$

$$= \underset{z\to 1}{\text{Lt}}\frac{1}{(3-1)!}\frac{d^2}{dz^2}[(z-1)^3\cdot f(z)]$$

$$= \frac{1}{2}\underset{z\to 1}{\text{Lt}}\frac{d^2}{dz^2}[e^{2z}] = \frac{1}{2}\underset{z\to 1}{\text{Lt}}[(4e^{2z})]$$

$$= \frac{1}{2}\times 4e^2 = 2e^2.$$

$\therefore$ By residue theorem,

$$\int_c \frac{e^{2z}}{(z-1)^3}dz = 2\pi i[2e^2] = 4e^2 i\pi.$$

Example 28: Obtain the Taylor series expansion of $f(z) = \dfrac{1}{z^2+(1+2i)z+2i}$, about $z = 0$. Also find the radius of convergence.

Solution: We have,

$$\frac{1}{z^2+(1+2i)z+2i} = \frac{1}{(z+2i)(z+1)}.$$

$f(z) = \dfrac{1}{1-2i}\left[\dfrac{1}{z+2i}-\dfrac{1}{z+1}\right]$. The function is analytic at $z = 0$.

$$\therefore \frac{1}{z+2i} = \frac{1}{2i}\left[1+\frac{z}{2i}\right]^{-1}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n, \left|\frac{z}{2i}\right| < 1\right]$$

or $|z| < 2 \Rightarrow R_1 = 2$ and $\dfrac{1}{z+1} = (1+z)^{-1}$

$$= \sum(-1)^n z^n, \ |z| < 1, \ R_2 = 1$$

$$\therefore \quad f(z) = \frac{1}{1-2i}\left[\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n - \sum_{n=0}^{\infty}(-1)^n z^n\right]$$

$$= \frac{1}{1-2i}\left[\sum_{n=0}^{\infty}(-1)^n z^n\left[\left(\frac{1}{2i}\right)^{n+1}-1\right]\right]$$

The radius of convergence $R = \min(R_1, R_2) = 1$. This is the distance between the point $z = 0$ and $z = -1$ at which $f(z)$ is not analytic.

Example 29: Find the nature and location of singularities of the function:

1. $\dfrac{2e^{2z}}{z+\sin z}$
2. $\dfrac{z^5+2z^2+1}{z^3+5z^2+2}$
3. $\dfrac{3e^{5z}}{2z-\sin 2z}$
4. $\dfrac{\sin 2z}{z^p}$, $p \geq 2$ is a positive integer.
5. $\dfrac{e^{\frac{p}{z-r}}}{e^{\frac{z}{r}}-1}$

Solution:

1. $f(z) = \dfrac{2e^{2z}}{z+\sin z}$

The denominator of $f(z) = z + \sin z$

$$= z + [z - z^3/3! + z^5/5! - \cdots]$$

$$= 2z - \frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$$

$$= z\left[2-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right]$$

$= z\phi(z)$, where

$$\phi(z) = 2 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \text{ and}$$

$$\phi(0) \neq 0.$$

$$\therefore \quad f(z) = \frac{2e^{2z}}{z + \sin z} = \frac{2e^{2z}}{z \cdot \phi(z)}, \phi(z) \neq 0.$$

$\therefore$ $z = 0$ is a simple pole of $f(z)$.

2. Let $f(z) = \dfrac{z^5 + 2z^2 + 1}{z^3 + 5z^2 + 2}$ writing $z = 1/a$

$\Rightarrow \quad a = 1/z \quad \Rightarrow \quad$ as $z \to \infty$, $a \to 0$.

$f(1/a)$

$$= \frac{1/a^5 + 2/a^2 + 1}{1/a^3 + 5/a^2 + 2} = 1/a^2 \left[\frac{1 + 2a^3 + a^5}{1 + 5a + 2a^3}\right]$$

$= 1/a^2\, \phi(a)$, $\phi(0) \neq 0$.

$\therefore$ $f(1/a)$ has a pole of order 2 at $a = 0$.

Hence the function $f(z)$ has a pole of order 2 at $z = \infty$.

3. Let $f(z) = \dfrac{3e^{5z}}{2z - \sin 2z}$

Writing the denominator of $f(z) = 2z - \sin z$

$$= 2z - \left[2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right]$$

$$= \frac{(2z)^3}{3!} - \frac{(2z)^5}{5!} + \cdots -$$

$$= z^3 \left[\frac{8}{3!} - \frac{32z^2}{5!} + \cdots\right] = z^3 \phi(z),$$

Where $\phi(z) = \dfrac{8}{3!} - \dfrac{32z^2}{5!} + \cdots$, and $\phi(0) \neq 0$.

$$\therefore \quad f(z) = \frac{3e^{5z}}{z^3 \phi(z)}.$$

Hence $z = 0$ is a pole of order 3.

4. $f(z) = \dfrac{\sin 2z}{z^p} = \dfrac{1}{z^p}\left[2z - \dfrac{(2z)^3}{3!} + \dfrac{(2z)^5}{5!} - \cdots\right]$

$$= \frac{1}{z^{p-1}}\left[2 - \frac{8z^2}{3!} + \frac{32z^4}{5!} - \cdots\right] = \frac{1}{z^{p-1}}[\phi(z)]$$

Where $\phi(z) = 2 - \dfrac{8z^2}{3!} + \dfrac{32z^4}{5!} - \cdots$ and $\phi(0) \neq 0$

Hence $f(z)$ has a pole of order $p - 1$ at $z = 0$.

5. Let $f(z) = \dfrac{e^{\frac{p}{z-r}}}{e^{\frac{z}{r}} - 1} = \dfrac{e^{\frac{p}{z-r}}}{e^{1 + \frac{z-r}{r}} - 1}$

$$\therefore \quad f(z) = \frac{1 + \dfrac{p}{z-r} + \dfrac{p^2}{2!(z-r)^2} + \dfrac{p^3}{3!(z-r)^3} + \cdots}{e \cdot \left[1 + \dfrac{z-r}{r} + \dfrac{(z-r)^2}{2!r^2} + \cdots\right] - 1}$$

$$= \left[1 + \frac{p}{z-r} + \frac{p^2}{2!(z-r)^2} + \cdots\right]$$

$$\left[-\left[1 - e\left(1 + \frac{z-r}{4} + \frac{(z-r)^2}{2!r^2} + \cdots\right)\right]^{-1}\right]$$

$$= -\left[1 + \frac{p}{z-r} + \frac{p^2}{2!(z-r)^2} + \cdots\right]$$

$$\left[1 + e\left(1 + \frac{z-r}{r} + \frac{(z-r)^2}{2!r^2} + \cdots\right)\right.$$

$$\left. + e^2\left(1 + \frac{z-r}{r} + \frac{(z-r)^2}{2!r^2} + \cdots\right)^2 + \cdots + \infty\right]$$

It is very clear from the above expansion of $f(z)$ that it contains an infinite number of terms containing negative powers of $1/z - r$. Hence $z = 4$ is an isolated essential singularity of $f(z)$.

Exercise

Practice Problems

Directions for questions 1 to 34: Select the correct alternative from the given choices.

1. The imaginary part of cosec$z$ is _____.

(A) $\dfrac{2\cos x \sinh y}{\cosh 2y - \cos 2x}$

(B) $\dfrac{-2\cos x \sinh y}{\cosh 2y - \cos 2x}$

(C) $\dfrac{-2\cos x \sinh y}{\cosh 2y + \cos 2x}$

(D) None of these

2 The real part of tanh$z$ is _____.

(A) $\dfrac{-\sinh 2x}{\cosh 2x + \cosh 2y}$

(B) $\dfrac{\sinh 2x}{\cosh 2x - \cosh 2y}$

(C) $\dfrac{\sinh 2x}{\cosh 2x + \cosh 2y}$

(D) None of these

3 The imaginary part of log sin$z$ is:
(A) $\tan^{-1}(\cot x \operatorname{sech} y)$
(B) $\tan^{-1}(\cot x \tanh y)$
(C) $\tan^{-1}\left(\dfrac{\cot x}{\tanh y}\right)$
(D) None of these

4. If $z = x + iy$, then $\overline{\cos z}$ = _____.
(A) cos$z$ (B) tanh$z$
(C) $\cos \bar{z}$ (D) $\sec \bar{z}$

5. The principal value of log $(-i)$ is _____
(A) $\log 1 - \dfrac{\pi}{2} i$ (B) $\log 1 + \dfrac{\pi}{2} i$
(C) $-\log 1 - \dfrac{\pi}{2} i$ (D) None of these

6. The function $f(z) = 5z^2$ is:
(A) Continuous but not differentiable
(B) Differentiable only at some points
(C) Analytic only at $z = 0$
(D) Entire function

7. The function $f(z)\dfrac{z+3}{z^2(z^2+1)}$ ceases to be analytic at:
(A) $z = 0$ (B) $z = i$
(C) $z = -i$ (D) All of these

8. The function $f(z) = \dfrac{xy^5}{x^2 + y^8} - \dfrac{ix^2y^4}{x^2 + y^8}$, for $z \neq 0$, $= 0$ for $z = 0$, is:
(A) Does not satisfy Cauchy–Riemann equation
(B) Is analytic everywhere
(C) Satisfies C–R equations but not analytic at the origin
(D) None of these

9. The rational function $f(z) = \dfrac{(z-2i)^4}{(z^2 - 4z + 5)^2}$
(A) Analytic every point
(B) Not analytic at $z = 3 \pm 2i$
(C) Analytic every where except at $z = 2 \pm i$
(D) None of these

10. The harmonic conjugate of $u = y^2 - x^2 - x$ is:
(A) $-2xy - y$ (B) $2xy + y$
(C) $2xy - y$ (D) $-2xy + y$

11. Find the analytic function whose real part is $\dfrac{2y}{x^2 + y^2}$.
(A) $\dfrac{2i}{z} + C$ (B) $iz + C$
(C) $\dfrac{-1}{z^2} + C$ (D) $\dfrac{1}{z} + C$

12. The analytic function $f = u + iv$ where $u = 12xy - 9x + 6$ is:
(A) $-3z + 2z^2i + C$ (B) $9z + 6z^2i + C$
(C) $-9z - 6z^2i + C$ (D) None of these

13. Given that $u = e^{3(x^2 - y^2)} \cos 6xy$ is a harmonic function, the corresponding analytic function is:
(A) $e^{z^2} + C$ (B) $e^{2z} + C$
(C) $e^{-z} + C$ (D) $e^{3z^2} + C$

14. If $f(z) = -e^x (\cos py - i \sin py)$ is analytic, then the value of p is:
(A) 1 (B) 0
(C) -1 (D) None of these

15. Which among the following is an example for entire function?
(A) $z + 2\bar{z}$ (B) $\log 2z$
(C) e^{2z} (D) None of these

16. u^2 is a harmonic function, only when 'u' is _____
(A) Any harmonic function
(B) A constant harmonic function
(C) Not a constant function
(D) None of these

17. The loci of the points z satisfying $\arg \dfrac{2z-1}{2z+1} = \dfrac{\pi}{3}$ is _____.
(A) A circle
(B) A straight line
(C) Interior of a circle
(D) Exterior of a circle

18. If z_1, z_2 and z_3 are complex numbers on the complex-plane and if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$, then the triangle whose vertices are z_1, z_2 and z_3 is _____.
(A) An isosceles right triangle
(B) An equilateral triangle
(C) Scalene triangle
(D) None of these

19. Evaluate $\int_c \dfrac{\cos \pi z}{z^2 - 2z - 3} dz$ where c is a $|z| = \dfrac{5}{2}$.
(A) πi (B) $2\pi i$
(C) $\pi i/2$ (D) None of these

20. Evaluate $\int_c \dfrac{z - 4\, dz}{z^2 + 4z + 8}$, where c is a circle $|z| = 2$.
(A) πi (B) $2\pi i$
(C) 0 (D) None of these

21. Evaluate $\int_c \dfrac{2z}{(z-1)(z-3)^2} dz$, where c is the circle $|z - 3| = \dfrac{1}{3}$,
(A) πi (B) $-\pi i$
(C) $2\pi i$ (D) $-2\pi i$

22. Evaluate $\int_c \left[\frac{e^{-z}}{z^2}+\frac{z^3}{(z-i)^2}\right]dz$, Where c: $|z| = 3/2$

(A) $-8\pi i$ (B) $8\pi i$
(C) $-6\pi i$ (D) $-2\pi i$

23. Evaluate $\int_c \frac{\log z\, dz}{(z-2)^2}$, where c: $|z-3| = 2$

(A) $-\pi i$ (B) πi
(C) $2\pi i$ (D) None of these

24. Find the Laurent series expansion of the function $f(z)=\frac{z^2-8z+2}{(z-2)(z-4)(z+3)}$ in the region $5<|z+3|<7$.

(A) $\frac{1}{z+3}\sum_{n=0}^{\infty}\left(\frac{5}{z+3}\right)^n-\frac{1}{7}\sum_{n=0}^{\infty}\left(\frac{z+3}{7}\right)^n+\frac{1}{z+3}$

(B) $\frac{1}{z+3}\sum_{n=0}^{\infty}\left(\frac{5}{z+3}\right)^n+\frac{1}{7}\sum_{n=0}^{\infty}\left(\frac{z+3}{7}\right)^n+\frac{1}{z+3}$

(C) $\frac{1}{z+3}\sum_{n=0}^{\infty}\left(\frac{5}{z+3}\right)^n+\frac{1}{7}\sum_{n=0}^{\infty}\left(\frac{z+3}{7}\right)^n-\frac{1}{z+3}$

(D) None of these

25. $f(z)=\frac{\cos z}{2(z-\pi)^2}$ has the pole of order:

(A) 1 (B) 2
(C) 3 (D) 0

26. Location and nature of the singularities of the function $f(z)=\frac{z^3+8}{(z+2)^4}$ is:

(A) $z = -2$, pole of order 4
(B) $z = -2$, pole of order 3
(C) $z = -2$, isolated singular point
(D) None of these

27. Nature and location of the singularities of $f(z)=(z+5)\sin\left(\frac{1}{z+1}\right)$:

(A) Removable singularity $z = -1$
(B) An essential singularity $z = -1$
(C) An isolated essential singularity $z = -1$
(D) Simple pole $z = -1$

28. The nature of the singularities of $f(z)=\frac{z+3}{z^2}\sin\left(\frac{1}{z-2}\right)$ is _____.

(A) $z = -3$, isolated essential singularity,
$z = 0$, pole of order two.
(B) $z = 0$, pole of order two
$z = 2$, isolated essential singularity
(C) $z = 2$, pole of order two
$z = 0$, non-isolated essential singularity.
(D) None of these

29. What kind of singularity the function $f(z) = z^2 \operatorname{cosec} z$ at $z = \infty$ has?

(A) Isolated singularity.
(B) Isolated essential singularity.
(C) Non-isolated essential singularity.
(D) None of these

30. The nature and location of singularities of the function e^{-1/z^2} is _____.

(A) $z = 0$ is the pole of order 2
(B) No singularities
(C) $z = 0$ is the isolated essential singularity
(D) $z^2 = 0$ is the non-isolated essential singularity.

Let $f(z)=\frac{e^{-2z}}{(2z-3)^4}$

31. The nature and singularity of the function $f(z)$ is _____.

(A) $z = 3$ is a pole of order 4.
(B) $z = 3/2$ is a pole of order 4.
(C) $z = 2$ is a pole of order 3.
(D) None of these.

32. For the above function, calculate the residue at the pole.

(A) $[\text{Res } f(z)]_{z=3/2}$ is $-1/12e^3$
(B) $[\text{Res } f(z)]_{z=2/3}$ is $-1/12e$
(C) $[\text{Res } f(z)]_{z=2/3}$ is $-1/12e^2$
(D) $[\text{Res } f(z)]_{z=3/2}$ is $-1/e^3$

33. Find the residues corresponding to the poles of $\frac{1}{(9z^2-4)^3}$.

(A) 1/2, –1/2 (B) 1, –1
(C) 1/4, –1/4 (D) None of these

34. Evaluate $\int_0^{2\pi}\frac{d\phi}{5+3\cos\phi}$.

(A) π (B) πi
(C) $\pi/2$ (D) None of these

Previous Years' Questions

1. The value $\oint_c \frac{dz}{(1+z^2)}$ where c is the contour $|z - i/2| = 1$ is: **[2007]**

(A) $2\pi i$ (B) π

(C) $\tan^{-1}z$ (D) $\pi i \tan^{-1}z$

2. Given $F(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If c is a counterclockwise path in z-plane such tha $|z+1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is: **[2012]**

(A) –2 (B) –1

(C) 1 (D) 2

3. $x = \sqrt{-1}$, then the value of x^x is: **[2012]**

(A) $e^{-\pi/2}$ (B) $e^{\pi/2}$

(C) x (D) 1

4. Square roots of $-i$, where $i = \sqrt{-1}$, are **[2013]**

(A) $i, -i$

(B) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

(C) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

(D) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

5. $\int \frac{z^2-4}{z^2+4}dz$ evaluated anticlockwise around the circle $|z - i| = 2$, where $i = \sqrt{-1}$, is: **[2013]**

(A) -4π (B) 0

(C) $2 + \pi$ (D) $2 + 2i$

6. Let S be the set of points in the complex plane corresponding to the unit circle. (That is $S = \{z: |z| = 1\}$). Consider the function $f(z) = z\,z^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane: **[2014]**

(A) Unit circle

(B) Horizontal axis line segment from origin to (1, 0)

(C) The point (1, 0)

(D) The entire horizontal axis

7. What is the average of all multiples of 10 from 2 to 198? **[2014]**

(A) 90 (B) 100

(C) 110 (D) 120

8. Integration of the complex function $f(z) = \frac{z^2}{z^2-1}$, in the counter clockwise, around $|z - 1| = 1$, is: **[2014]**

(A) $-\pi i$ (B) 0

(C) πi (D) $2\pi i$

9. Given $f(z) = g(z) + h(z)$, where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE? **[2015]**

(A) If $f(z)$ is differentiable at z_0, then $g(z)$ and $h(z)$ are also differentiable at z_0.

(B) If $g(z)$ and $h(z)$ are differentiable at z_0, then $f(z)$ is also differentiable at z_0.

(C) If $f(z)$ is continuous at z_0, then it is differentiable at z_0.

(D) If $f(z)$ is differentiable at z_0, then so are its real and imaginary parts.

10. The value of the integral

$$\oint \frac{2z+5}{\left(z-\frac{1}{2}\right)\left(z^2-4z+5\right)}dz$$

Over the contour $|z| = 1$, taken in the anti-clockwise direction, would be **[2016]**

(A) $\frac{24\pi i}{13}$ (B) $\frac{48\pi i}{13}$

(C) $\frac{24}{13}$ (D) $\frac{12}{13}$

11. Consider the function $f(z) = z + z^*$ where z is a complex variable and z^* denotes its complex conjugate. Which one of the following is TRUE? **[2016]**

(A) $f(z)$ is both continuous and analytic

(B) $f(z)$ is continuous but not analytic

(C) $f(z)$ is not continuous but is analytic

(D) $f(z)$ is neither continuous nor analytic

Answer Keys

Exercise

Practice Problems

1. B	**2.** D	**3.** B	**4.** C	**5.** A	**6.** D	**7.** D	**8.** C	**9.** C	**10.** A
11. A	**12.** C	**13.** D	**14.** C	**15.** C	**16.** B	**17.** A	**18.** B	**19.** C	**20.** C
21. B	**22.** A	**23.** B	**24.** B	**25.** B	**26.** B	**27.** C	**28.** B	**29.** C	**30.** B
31. B	**32.** A	**33.** A	**34.** C						

Previous Years' Questions

1. B	**2.** C	**3.** A	**4.** B	**5.** A	**6.** C	**7.** B	**8.** C	**9.** B	**10.** B
11. B									

Chapter 5

Probability and Statistics

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Probability
- Definition of probability
- Addition theorem of probability
- Conditional probability
- Advanced probability
- Mathematical expectation
- Some special discrete distributions
- Some special continuous distributions
- Measures of central tendencies
- Measures of dispersion
- Correlation
- Regression
- Covariance

PROBABILITY

The word PROBABILITY is used, in a general sense, to indicate a vague possibility that something might happen. It is also used synonymously with chance.

Random experiment

If the result of an experiment conducted any number of times under essentially identical conditions, is not certain but is any one of the several possible outcomes, the experiment is called a trial or a random experiment. Each of the out comes is known as an event.

Example:

1. Drawing 3 cards from a well shuffled pack is a random experiment while getting an Ace or a King are events.
2. Throwing a fair die is a random experiment while getting the score as '2' or an odd number' are events.

Mutually exclusive events If the happening of any one of the events in a trial excludes or prevents the happening of all others, then such events are said to be mutually exclusive.

Example: The events of getting a head and that of getting a tail when a fair coin is tossed are mutually exclusive.

Equally likely events Two events are said to be equally likely when chance of occurrence of one event is equal to that of the other.

Example: When a die is thrown, any number from 1 to 6 may be got. In this trial, getting any one of these events are equally likely.

Independent events Two events E_1 and E_2 are said to be independent, if the occurrence of the event E_2 is not affected by the occurrence or non-occurrence of the event E_1.

Example: Two drawings of one ball each time are made from a bag containing balls.

Here, we have two events drawing a ball first time (E_1) and drawing a ball second time (E_2). If the ball of the first draw is replaced in the bag before the second draw is made, then the outcome of E_2 does not depend on the outcome of E_1. In this case E_1 and E_2 are Independent events.

If the ball of the first draw is not replaced in the bag before the second draw is made, then the outcome of E_2 depends on the outcome of E_1. In this case, events E_1 and E_2 are Dependent events.

Compound events When two or more events are in relation with each other, they are known as compound events.

Example: When a die is thrown two times, the event of getting 3 in the first throw and 5 in the second throw is a compound event.

Definition of Probability

If an event E can happen in $\boldsymbol{m}$ ways and fail in $\boldsymbol{k}$ ways out of a total of n ways and each of them is equally likely, then the probability of happening E is $m/(m + k) = m/n$ where $n = (m + k)$.

In other words, if a random experiment is conducted n times and m of them are favourable to event E, then the probability of happening of E is $P(E) = m/n$. Since the event does not occur ($n - m$) times, the probability of non-occurrence of E is $P(\bar{E})$.

$$P(\bar{E}) = \frac{n-m}{n} = -m/n = 1 - P(E)$$

Therefore, $P(E) + P(\bar{E}) = 1$.

Notes:

1. Probability $[P(E)]$ of the happening of an event E is known as the probability of success and the probability $[P(\bar{E})]$ of the non-happening of the event is the probability of failure.
2. If $P(E) = 1$, the event is called a certain event and if $P(E) = 0$ the event is called an impossible event.
3. Instead of saying that the chance of happening of an event is m/n, we can also say that the odds in favour of the event are m to $(n - m)$ or the odds against the event are $(n - m)$ to m.

Addition Theorem of Probability

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This result follows from the corresponding result in set theory. If $n(X)$ represents the number of elements in set X, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$.

Example: If a die is rolled, what is the probability that the number that comes up is either even or prime?

A = The event of getting an even number = {2, 4, 6}

B = The event of getting a prime = {2, 3, 5}

$A \cup B = \{2, 3, 4, 5, 6\}$

$A \cap B = \{2\}$

$P(A) = \frac{3}{6}$, $P(B) = \frac{3}{6}$, $P(A \cup B) = \frac{5}{6}$ and $P(A \cap B) = \frac{1}{6}$. We can verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Solved Examples

Example 1: When a cubical dice is rolled, find the probability of getting an even integer.

Solution: When a dice is rolled, the number of possible out comes is 6. The number of favourable outcomes of getting an even integer is 3.

The required probability $= \frac{3}{6} = \frac{1}{2}$

Example 2: If a card is drawn from a pack of cards, find the probability of getting a queen.

Solution: When a card is drawn, the number of possible outcomes is 52. The number of favourable outcomes of getting a queen card is 4.

The required probability $= \frac{4}{52} = \frac{1}{13}$

Example 3: A bag contains 5 green balls and 4 red balls. If 3 balls are picked from it at random, then find the odds against the three balls being red.

Solution: The total number of balls in the bag = 9. Three balls can be selected from 9 balls in ${}^9C_3$ ways.

Three red balls can be selected from 4 red balls in ${}^4C_3$ ways.

Probability of picking three red balls

$$= \frac{{}^4C_3}{{}^9C_3} = \frac{4}{84} = \frac{1}{21};\ P(\bar{E}) = \frac{20}{21}$$

Odd against the three balls being red are

$$= P(\bar{E}) : P(E) = \frac{20}{21} : \frac{1}{21} = 20 : 1$$

Example 4: When two dice are rolled together, find the probability of getting at least one 4.

Solution: Let E be the event that at least one dice shows $4 \cdot \bar{E}$ be the event that no dice shows 4. The number of favourable outcomes of $\bar{E}$ is $5 \times 5 = 25 \cdot P(\bar{E}) = \frac{25}{36}$

$$\therefore\ P(E) = 1 - P(\bar{E}) = 1 - \frac{25}{36} = \frac{11}{36}.$$

Example 5: When two dice are rolled together find the probability that total score on the two dice will be 8 or 9?

Solution: When two dice are rolled, the total number of outcomes = $6 \times 6 = 36$.

Favourable outcomes for getting the sum 8 or 9 are {(2, 6), (6, 2), (3, 5), (5, 3), (4, 4), (3, 6), (6, 3), (4, 5), (5, 4)}, i.e., the total number of favourable outcomes = 9.

The required probability $= \frac{9}{36} = \frac{1}{4}$.

Example 6: If two cards are drawn simultaneously from a pack of cards, what is the probability that both will be jacks or both are queens?

Solution: Here two events are mutually exclusive, $P(J \cup Q) = P(J) + P(Q)$. Probability of drawing two jacks is $P(J) = \frac{{}^4C_2}{{}^{52}C_2}$

Probability of drawing two queens is $P(Q) = \frac{{}^4C_2}{{}^{52}C_2}$

$$P(J \cup Q) = P(J) + P(Q)$$

$$= \frac{{}^4C_2}{{}^{52}C_2} + \frac{{}^4C_2}{{}^{52}C_2} = 2 \cdot \frac{{}^4C_2}{{}^{52}C_2} = \frac{2}{221}.$$

Example 7: When two cards are drawn from a pack of cards, find the probability that the two cards will be kings or blacks.

Solution: The probability of drawing two kings $= \frac{{}^4C_2}{{}^{52}C_2}$

The probability of drawing two black cards is $= \frac{{}^{26}C_2}{{}^{52}C_2}$

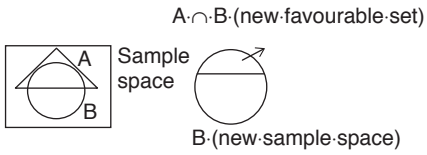
The probability of drawing two black kings is $\frac{{}^{2}C_{2}}{{}^{52}C_{2}}$

$\therefore$ The required probability

$$= \frac{{}^{4}C_{2}}{{}^{52}C_{2}} + \frac{{}^{26}C_{2}}{{}^{52}C_{2}} - \frac{{}^{2}C_{2}}{{}^{52}C_{2}} = \frac{55}{221}$$

Conditional Probability

Let S be a finite sample space of a random experiment and A, B are events, such that $P(A) > 0$, $P(B) > 0$. If it is known that the event B has occurred, in light of this we wish to complete the probability of A, we mean conditional probability of A given B. The occurrence of event B would reduce the sample space to B, and the favourable cases would now be $A \cap B$.



Notation The conditional probability of A given B is denoted by $P(A/B)$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

Notes:

1. This definition is also valid for infinite sample spaces.
2. The conditional probability of B given A is denoted by $P\left(\frac{B}{A}\right)$ and $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$

Multiplication theorem

Let A and B be two events of certain random experiment such that A occurs only when B has already occurred. Then, for the conditional event $\frac{A}{B}$, the total possible outcomes are the outcomes favourable to the event B and its favourable outcomes are the outcomes favourable to both A and B.

So, $P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)}$

$$= \frac{n(A \cap B)}{n(S)} \times \frac{n(S)}{n(B)} = P(A \cap B) \times \frac{1}{P(B)}$$

That is, $P\left(\frac{A}{B}\right) \cdot P(B) = P(A \cap B)$

This is called the multiplication theorem on probability.

Example 8: A letter is selected at random from the set of English alphabet and it is found to be a vowel. What is the probability that it is e?

Solution: Let A be the event that the letter selected is e and B be the event that the letter is a vowel. Then, $A \cap B = \{e\}$ and $B = \{a, e, i\, o, u\}$

So, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{1}{26}\right)}{\left(\frac{5}{26}\right)} = \frac{1}{5}$

Independent events In a random experiment, if A, B are event such that $P(A) > 0$, $P(B) > 0$ and if $P(A/B) = P(A)$ or $P(B/A) = P(B)$ (conditional probability equals to unconditional probability) then we say A, B are independent events.

If A, B are independent $P(A \cap B) = P(A)\, P(B)$.

Example: Two coins are tossed one after the other and let A be the event of getting tail on second coin and B be the event of getting head on first coin then find $P(A/B)$

Solution: Sample space = $\{HH, HT, TH, TT\}$, $A = \{HT, TT\}$ and $B = \{HH, HT\}$, $(A \cap B) = \{HT\}$

$$\therefore \quad P(A) = \frac{2}{4} = \frac{1}{2} \quad \text{and} \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Thus $P(A/B) = P(A)$

$\therefore$ Logically too we understand that occurrence or non-occurrence of tail in 2nd coin.

Baye's rule

Suppose $A_1, A_2, \ldots, A_n$ are n mutually exclusive and exhaustive events such that $P(A_i) \neq 0$. Then for $i = 1, 2, 3, \ldots, n$,

$$P\left(\frac{A_i}{A}\right) = \frac{P(A_i) \cdot P\left(\frac{A}{A_i}\right)}{\sum_{k=1}^{n} P(A_K) P\left(\frac{A}{A_K}\right)}$$

Where A is an arbitrary event of S.

Example 9: Akshay speaks the truth in 45% of the cases, In a rainy season, on each day there is a 75% chance of raining. On a certain day in the rainy season, Akshay tells his mother that it is raining outside. What is the probability that it is actually raining?

Solution: Let E denote the event that it is raining and A denote the event that Akshay tells his mother that it is raining outside.

Then, $P(E) = \frac{3}{4}$, $P(\bar{E}) = \frac{1}{4}$

$$P\left(\frac{A}{E}\right)=\frac{45}{100}=\frac{9}{20} \text{ and } P\left(\frac{A}{\bar{E}}\right)=\frac{11}{20}$$

By Baye's Rule, we have

$$P\left(\frac{E}{A}\right)=\frac{P(E)P\left(\frac{A}{E}\right)}{P(E)P\left(\frac{A}{E}\right)+P(\bar{E})P\left(\frac{A}{\bar{E}}\right)}$$

$$=\frac{\frac{3}{4}\times\frac{9}{20}}{\frac{3}{4}\times\frac{9}{20}+\frac{1}{4}\times\frac{11}{20}}=\frac{27}{38}$$

Advanced Probability

Random variable

A random variable is a real valued function defined over the sample space (discrete or continuous).

A **discrete random variable** takes the values that are finite or countable. For example when we consider the experiment of tossing of 3 coins, the number of heads can be appreciated as a discrete random variable (X). X would take 0, 1, 2 and 3 as possible values.

A continuous random variable takes values in the form of intervals. Also, in the case of a **continuous random variable** $P(X = c) = 0$, where c is a specified point.

Heights and weights of people, area of land held by individuals, etc., are examples of continuous random variables.

Probability mass function (PMF)

If X is a discrete random variable, which can take the values $x_1, x_2 \ldots$ and $f(x)$ denote the probability that X takes the value x_i, then $p(x)$ is called the PMF of X.

$p(x_i) = P(x = x_i)$. The values that X can take and the corresponding probabilities determine the probability distribution of X. We also have

1. $p(x) \geq 0$;
2. $\Sigma p(x) = 1$.

Probability density function (PDF)

If X is a continuous random variable then a function $f(x)$, $x \in I$ (interval) is called a PDF. The probability statements are made as $P(x \in I) = \int_I f(x)dx$

We also have,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

The probability $P(X \leq x)$ is called the cumulative distribution function (CDF) of X and is denoted by $F(X)$. It is a point function. It is defined for discrete and continuous random variables.

The following are the properties of probability distribution function $F(x)$,

1. $F(x) \geq 0$.
2. $F(x)$ is non-decreasing i.e., for $x > y$, $F(x) \geq F(y)$.
3. $F(x)$ is right continuous.
4. $F(-\infty) = 0$ and $F(+\infty) = 1$.

Also,

5. $P(a < x \leq b) = F(b) - F(a)$. For a continuous random variable
6. $Pr\{x < X \leq x + dx\} = F(x + dx) - F(x) = f(x)\,dx$; where dx is very small
7. $f(x) = \frac{d}{dx}[F(x)]$ where;

   (i) $f(x) \geq 0 \ \forall \ x \in R$.

   (ii) $\int_R f(x)dx = 1$.

Mathematical Expectation [$E(X)$]

Mathematical Expectation is the weighted mean of values of a variable.

If X is a random variable which can assume any one of the values $x_1, x_2, \ldots, x_n$ with the respective probabilities $p_1, p_2, \ldots, p_n$, then the mathematical expectation of X is given by $E(X) = p_1x_1 + p_2x_2 + \cdots + p_n x_n$

For a continuous random variable,

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx \text{ where } f(x) \text{ is the } p.d.f. \text{ of } X.$$

Some Special Discrete Distributions

Discrete Uniform Distribution

A discrete random variable defined for values of x from 1 to n is said to have a uniform distribution if its probability mass function is given by

$$F(x)=\begin{cases}\frac{1}{n};\ \text{for } x=1,\ 2,\ 3...,\ n\\ 0,\ \text{otherwise}\end{cases}$$

- The cumulative distribution function $F(x)$ of the discrete uniform random variable x is given by

$$F(x)=\begin{cases}0,\ \text{for } x<1\\ \frac{x}{n};\ \text{for } 1\leq x\leq n\\ 1;\ \text{for } x>1\end{cases}$$

- Mean of $x = \mu = \frac{n+1}{2}$
- Variance of $x = \sigma^2 = \frac{n^2-1}{12}$

Binomial Distribution

An experiment which is made of n independent trials, each of which resulting in either 'success' with probability 'p' or 'failure' with probability 'q' ($q = 1 - p$), then the probability distribution for the random variable X when represents the number of success is called a binomial distribution. The probability mass function,

$$p(x) = b(x; n, p) = {}^{n}C_{x}\, p^{x}\, q^{n-x};\ x = 0, 1, 2, \ldots n$$

Example: Hitting a target in 5 trials

Here the random variable (X) represents the number of trials made for hitting the target, i.e., $x = 0$ or 1 or 2 or 3 or 4 or 5.

We have a set of 5 trials $n = 5$

Each trial may hit the target termed to be success (p) or not termed to be failure (q), which are independent.

$\therefore$ This is an example for Binomial distribution.

Properties of binomial distribution

1. $E(X) = np$ (mean)
2. $V(X) = E(X^2) - (E(X))^2 = npq$; (variance) (mean > variance.)
3. SD $(X) = \sqrt{npq}$
4. Mode of a binomial distribution lies between $(n + 1)\ p - 1 \le x \le (n + 1)p$
5. If $X_1 \sim b(n_1, p)$ and $X_2 \sim b\ (n_2, p)$ and if X_1 and X_2 are independent, then $X_1 + X_2 \sim b\ (n_1 + n_2, p)$ where (n, p) is the pmf of binomial distribution.

Poisson distribution A random variable X is said to follow a Poisson distribution with parameter λ, $\lambda > 0$, if it assumes only non-negative values and its probability mass function is given by

$$p(x) = p(x;\ \lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda X}{x!} & : x = 0,\ 1,\ 2, \ldots \\ 0 & \lambda > 0 \\ & \text{otherwise} \end{cases}$$

In a binomial distribution if n is large compared to p, then np approaches a fixed constant say λ. Such a distribution is called poisson distribution (limiting case of binomial distribution)

Properties of Poisson distribution

1. $E(X) = \sum_{x} x \cdot \dfrac{e^{-\lambda}\lambda^{x}}{x!} = \lambda$
2. $V(X) = E(X^2) - (E(X))^2 = \lambda$

   SD $(X) = \sqrt{\lambda}$

   $\therefore$ Mean = λ = Variance
3. Mode of a Poisson distribution lies between $\lambda - 1$ and λ
4. If $X_1 \sim P\ (\lambda_1)$ and $X_2 \sim P\ (\lambda_2)$, and X_1, X_2 independent then $X_1 + X_2 \sim P\ (\lambda_1 + \lambda_2)$.

Some Special Continuous Distributions

Continuous Uniform Distribution or Rectangular Distribution

A continuous random variable x defined on $[a, b]$ is said to have a uniform distribution, if its probability density function is given by

$$F(x) = \begin{cases} \dfrac{1}{b-a};\ \text{for } x \in [a,\ b] \\ 0;\ \text{otherwise} \end{cases}$$

- The cumulative distribution function of the continuous uniform random variable x is given by

$$F(x) = \begin{cases} 0;\ \text{if } x \le a \\ \dfrac{x-a}{b-a};\ \text{if } a < x < b \\ 1;\ \text{if } x \ge b \end{cases}$$

- Mean of $x = \mu = \dfrac{a+b}{2}$
- Variance of $x = \sigma^2 = \dfrac{(b-a)^2}{12}$

Normal Distribution

A continuous random variable X is said to have a normal distribution with parameters μ and σ^2 if its density function is given by the probability density function,

$$f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} & -\infty < x < \infty \\ & -\infty < \mu < \infty \\ & \sigma > 0 \\ 0 & \text{otherwise} \end{cases}$$

It is denoted as $X \sim N\ (\mu, \sigma^2)$.

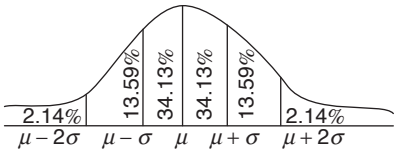


Figure 1 The graphical representation of normal distribution.

Properties of normal distribution

1. The function is symmetrical about the value μ.
2. It has a maximum at $x = \mu$.
3. The area under the curve within the interval ($\mu \pm \sigma$) is 68%. That is, $P(\mu - \sigma \le x \le \mu + \sigma) = 0.68$.
4. A fairly large number of samples taken from a 'Normal' population will have average, median and mode nearly the same, and within the limits of average $\pm 2 \times$ S.D., there will be 95% of the values.

5. $E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \mu.$
6. $V(X) = \sigma^2$; S.D $(X) = \sigma$
7. For a normal distribution,
   Mean = Median = Mode.
8. All odd order moments about mean vanish for a normal distribution.
   That is, $\mu_{2n+1} = 0 \forall = n = 0, 1, 2, \ldots$
9. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, X_1, X_2 independent, then,
   $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
   Also, $X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$
10. If $\mu = 0$ and $\sigma^2 = 1$, we call it as standard normal distribution. The standardization can be obtained by the transformation,

$$z = \frac{x-\mu}{\sigma}. \quad \text{Also, } \frac{X-\mu}{\sigma} \sim N(0, 1).$$

Exponential Distribution

A continuous random variable x is said to have an exponential distribution if its probability density function $f(x)$ is given by,

$$F(x) = \begin{cases} \lambda e^{-\lambda x}; \text{ for } x > 0 \\ 0; \text{ otherwise} \end{cases}$$

Here λ is the parameter of the exponential distribution and $\lambda > 0$.

The cumulative distribution function $F(x)$ of an exponential distribution with λ as parameter is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}; \text{ if } x > 0 \\ 0, \text{ otherwise} \end{cases}$$

$$\text{Mean} = \mu = \frac{1}{\lambda}$$

$$\text{Variance} = \sigma^2 = \frac{1}{\lambda^2}$$

Example 10: An unbiased die is thrown at random. What is the expectation of the number on it?

Solution. Let X denotes the number on the die, which can take the values 1, 2, 3, 4, 5 or 6;

Probability of each will be equal to $\frac{1}{6}$

X	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_x xP(X = x)$$
$$= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$
$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{6 \times 7}{6 \times 2} = \frac{7}{2}$$
$$= 3.5.$$

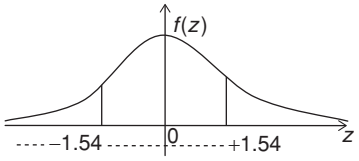
Example 11: In a city 5 accidents take place in a span of 25 days. Assuming that the number of accidents follows the Poisson distribution, what is the probability that there will be 3 or more accidents in a day? (Given $e^{-0.2} = 0.8187$)

Solution. Average number of accidents per day $= \frac{5}{25} = 0.2$;
$\therefore \lambda = 0.2$

Probability (3 or more accidents per day)
$= 1 - P$ (2 or less accidents)
$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$
$= 1 - [e^{-0.2} + 0.2e^{-0.2} + 0.02e^{-0.2}]$
$= 1 - e^{-0.2}[1.22] = 1 - 0.99814 = 0.001186.$

Example 12: What is the area under the normal curve to the left of $Z = -1.54$ (given area between 0 and $-1.54 = 0.4382$)?

Solution: Required area = 0.5 – 0.4382 = 0.0618



Example 13: A family consists of five children. If the random variable (X) represents the number of boys in that family then,

1. Find the expected value $E(X)$ of X.
2. Find the variance of X.

Solution: This situation can be modelled as binomial distribution.

$$X \sim b\left(5, \frac{1}{2}\right); \; E(X) = np = 5 \times \frac{1}{2} = 2.5$$
$$V(X) = npq = 5 \times \frac{1}{2} \times \frac{1}{2} = 1.25$$

Example 14: Ram and Shyam play a game in which their chances of winning are in the ratio 2 : 3. Find Shyam's chance of winning at least 3 games out of five games played.

Solution: $P(\text{Shyam wins}) = \frac{3}{5}$;

$$P(\text{Shyam loses}) = \frac{2}{5}$$

Let x denote the number of games won by Shyam.

$P(\text{Shyam wins at least 3 games}) = P(X \geq 3)$

$$= \sum_{x=3}^{5} 5C_x \left(\frac{3}{5}\right)^x \left(\frac{2}{5}\right)^{5-x} = \sum_{x=3}^{5} 5C_x \frac{3^x 2^{5-x}}{5^5}$$

$$= \frac{3^3}{5^5}[5C_3 2^2 + 5C_4 \times 3 \times 2 + 1 \times 3^2 \times 1]$$

$$= \frac{27 \times 79}{3125} = 0.68$$

Example 15: The PDF of a random variable X is

$$f(x) = \begin{cases} \left(\frac{1}{10}\right) e^{\left(-\frac{x}{10}\right)}; & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \le 10)$? (given $e^{-1} = 0.3679$)

Solution: $P(X \le 10) = \int_0^{10} f(x)\,dx = \int_0^{10} \frac{1}{10} e^{-\frac{x}{10}} dx$

$$= \frac{1}{10}\left(\frac{e^{-\frac{x}{10}}}{-\frac{1}{10}}\right)_0^{10} = 1 - e^{-1} = 0.6321$$

Joint distribution of random variables joint probability mass function

Let x and y be two discrete random variables on the same sample space S with the range space of x as $R_x = \{x_1, x_2, \ldots, x_m\}$ and the range space of y as,

$R_y = \{y_1, y_2, \ldots, y_n\}$ and $P_x(x)$ and $P_y(y)$ as the probability mass functions of x and y. Then the joint probability mass function $P_{xy}(x, y)$ of the two dimensional random variable (x, y) on the range space $R_x \times R_y$ is defined as,

$$P_{xy}(x_i, y_j) = \begin{cases} P(x = x_i, y = y_j), & \text{for } (x_i, y_j) \in R_x \times R_y \\ 0, & \text{otherwise} \end{cases}$$

This joint probability mass function can be represented in the form of a table as follows

x \ y	y_1	y_2	$y_3 \ldots$	y_n	$\sum_{j=1}^{n} P_{xy}(x_i, y_j)$
X_1	$P_{xy}(x_1, y_1)$	$P_{xy}(x_1, y_2)$	$P_{xy}(x_1, y_3) \ldots$	$P_{xy}(x_1, y_n)$	$P_x(x_1)$
X_2	$P_{xy}(x_2, y_1)$	$P_{xy}(x_2, y_2)$	$P_{xy}(x_3, y_3) \ldots$	$P_{xy}(x_3, y_n)$	$P_x(x_2)$
X_3	$P_{xy}(x_3, y_1)$	$P_{xy}(x_3, y_2)$	$P_{xy}(x_3, y_3) \ldots$	$P_{xy}(x_3, y_n)$	$P_x(x_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_m	$P_{xy}(x_m, y_1)$	$P_{xy}(x_m, y_2)$	$P_{xy}(x_m, y_3) \ldots$	$P_{xy}(x_m, y_n)$	$P_x(x_m)$
$\sum_{i=1}^{m} P_{xy}(x_i, y_j)$	$P_y(y_1)$	$P_y(y_2)$	$P_y(y_3) \ldots$	$P_y(y_n)$	

From the above table, it can be easily observed that the marginal probability mass functions of x and y namely $P_x(x)$ and $P_y(y)$ respectively can be obtained from the joint probability mass function $P_{xy}(x, y)$ as

$$P_x(x_i) = \sum_{j=1}^{n} P_{xy}(x_i, y_j), \text{ for } i = 1, 2, \cdots m$$

And

$$P_y(y_j) = \sum_{j=1}^{m} P_{xy}(x_i, y_j) P_{xy}(x_i, y_j) \text{ for } j = 1, 2, 3, \cdots n$$

- $P_{xy}(x_i, y_j) \ge 0 \ \forall\ i, j$
- $\sum_{i=1}^{m} \sum_{j=1}^{n} P_{xy}(x_i y_j) = 1$
- The cumulative joint distribution function of the two dimensional random variable (x, y) is given by $F_{xy}(x, y) = P(X \le x, Y \le y)$.

Joint probability density function

Let X and Y are two continuous random variables on the same sample space S with $f_x(x)$ and $f_y(y)$ as the probability density functions respectively. Then a function $f_{xy}(x, y)$ is called the joint probability density function of the two dimensional random variable (X, Y) if the probability that the point (x, y) will lie in the infinitesimal rectangular region of area $dxdy$ is $f_{xy}(x, y)\,dxdy$,

That is,

$$P\left(x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx,\ y - \frac{1}{2}dy \le Y \le y + \frac{1}{2}dy\right)$$

$$= f_{XY}(x, y)\,dxdy$$

- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dxdy = 1$
- The marginal probability density functions $f_X(x)$ and $f_Y(y)$ of the two continuous random variables x and y are given by,

$$f_x(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy \text{ and } f_y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

- The cumulative joint distribution function $F_{XY}(x, y)$ of the two dimensional random variable (x, y) (where x and y are any two continuous random variables defined on the same sample space) is given by,

$$F_{XY}(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy.$$

Conditional probability functions of random variables

Let x and y be two discrete (continuous) random variables defined on the same sample space with joint probability mass (density) function $f_{XY}(x, y)$, then

1. the conditional probability mass (density) function $f_{\frac{X}{Y}}\left(\frac{x}{y}\right)$ of x, given $Y = y$ is defined as

$$F_{\frac{X}{Y}}\left(\frac{x}{y}\right) = \frac{f_{XY}(x, y)}{f_Y(y)}, \text{ where } f_Y(y) \neq 0 \text{ and}$$

2. the conditional probability mass (density) function $f_{\frac{Y}{X}}\left(\frac{y}{x}\right)$ of y, given $X = x$ is defined as $f_{\frac{Y}{X}}\left(\frac{y}{x}\right)$

$$= \frac{f_{XY}(x, y)}{f_X(x)} \text{ where } f_X(x) \neq 0$$

Independent random variables

Two discrete (continuous) random variables X and Y defined on the same sample space with joint probability mass (density) function $P_{XY}(x, y)$ are said to be independent, if and only if,

$$P_{XY}(x, y) = P_X(x)\, P_Y(y)$$

Where $P_X(x)$ and $P_Y(y)$ are the marginal probability mass (density) functions of the random variables X and Y respectively.

Note: If the random variables X and Y are independent then

$$P_{xy}(a \le X \le b, c \le Y \le d) = P_x(a \le X \le b)\, P_y(c \le Y \le d)$$

STATISTICS

Statistics is basically the study of numeric data. It includes methods of collection, classification, presentation, analysis and inference of data. Data as such is qualitative or quantitative in nature. If one speaks of honesty, beauty, colour, etc., the data is qualitative while height, weight, distance, marks, etc. are quantitative.

The present course aims to systematically study statistics of quantitative data. The quantitative data can be divided into three categories

1. Individual series
2. Discrete series and
3. Continuous series.

Individual series

Examples:

1. Heights of 8 students
   5.0, 4.9, 4.5, 5.1, 5.3, 4.8, 5.1, 5.3 (in feet)
2. The weight of 10 students
   46, 48, 52, 53.4, 47, 56.8, 52, 59, 55, 52 (in kgs)

Discrete series

Example:

x : Number of children in a family

f : Number of families

Total number of families = 50

x	0	1	2	3	4
f	8	10	19	8	5

Continuous series

Example: Total number of students = 50

Class Interval (CI)	Frequency (f)
0–10	8
10–20	12
20–30	13
30–40	10
40–50	7

In order to analyze and get insight into the data some mathematical constants are devised. These constants concisely describe any given series of data. Basically we deal with two of these constants,

Measures of central tendencies These tell us about how the data is clustered or concentrated. They give the central idea about the data. The measures are

1. Arithmetic mean or mean
2. Geometric mean
3. Harmonic mean
4. Median
5. Mode

The first three are mathematical averages and the last two are averages of position.

Measures of dispersion It is possible that two sets of data may have the same central value, yet they may differ in spread. So there is a need to study about the spread of the data.

The measures we deal with are,

1. Range
2. Quartile deviation or semi inter-quartile range
3. Mean deviation
4. Standard deviation (including variance)

The formulae for each of the above mentioned measures is listed for each of the series in what follows.

Measures of Central Tendencies

Arithmetic mean (A.M or $\bar{x}$)

1. Individual series:
$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\Sigma x_i}{n}$$
2. Discrete series:
$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$
where $x_1, x_2, \ldots, x_n$ are n distinct values with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively.
3. Continuous series:
$$\bar{x} = \frac{f_1m_1 + f_2m_2 + \cdots + f_nm_n}{f_1 + f_2 + \cdots + f_n} = \frac{\Sigma f_i m_i}{\Sigma f_i}$$
where $f_1, f_2, f_3, \ldots, f_n$ are the frequencies of the classes whose mid-values are $m_1, m_2, \ldots, m_n$ respectively.

Some important results based on AM

1. The algebraic sum of deviations taken about mean is zero.
2. Its value is based on all items.
3. Mean of first n natural numbers is $\frac{(n+1)}{2}$.
4. Arithmetic mean of two numbers a and b is $\frac{(a+b)}{2}$.
5. If b is A.M. of a and c then a, b, c are in arithmetic progression.

Combined mean If x_1 and x_2 are the arithmetic means of two series with n_1 and n_2 observations respectively, the combined mean, $\bar{x}_c = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

Median

If for a value the total frequency above (or below) it is half of the overall total frequency the value is termed as median. Median is the middle-most item.

Individual series If $x_1, x_2, \ldots, x_n$ are arranged in ascending order of magnitude then the median is the size of $\left(\frac{n+1}{2}\right)$th item.

Some results based on median

1. Median does not take into consideration all the items.
2. The sum of absolute deviations taken about median is least.
3. Median is the abscissa of the point of intersection of the cumulative frequency curves.
4. Median is the best suited measure for open end classes.

Mode The most frequently found item is called mode. Being so, it is easy and straight forward to find for individual and discrete series.

Empirical formula

- For moderately symmetrical distribution, Mode = 3 median – 2 mean
- For a symmetric distribution, Mode = Mean = Median. This formula is to be applied in the absence of sufficient data. Given any two, of the mean, median or mode the third can be found.

Measures of Dispersion

Range

The range of a distribution is the difference between the greatest and the least values observed.

Some important results based on range

1. Range is a crude measure of dispersion as it is based only on the value of extreme observations.
2. It is also very easy to calculate.
3. It does not depend on the frequency of items.

Quartile deviation (QD)

$$\text{QD} = \frac{Q_3 - Q_1}{2}$$

Individual series The numbers are first arranged in ascending or descending order, then we find the quartiles Q_1 and Q_3 as

$Q_1 \rightarrow$ size of $(n + 1)/4$th item

$Q_3 \rightarrow$ size of $3(n + 1)/4$th item

The first quartile (or the lower quartile) Q_1 is that value of the variable, which is such that one-quarter of the observations lies below it. The third quartile Q_3 is that value of the variable, which is such that three-quarters of the observations lie below it.

Mean deviation (MD)

It is defined as the arithmetic mean of the deviation from origin, which may be either mean or median or mode.

Individual series

$$\text{MD} = \frac{|x_1 - A| + |x_2 - A| + \cdots + |x_n - A|}{n}$$

where $x_1, x_2, \ldots, x_n$ are the n observations and A is the mean or median or mode.

Some results based on MD

1. Mean deviation depends on all items.
2. By default, mean deviation is to be computed about mean.
3. Mean deviation about the median is the least.
4. Mean deviation of two numbers a and b is $\frac{|a-b|}{2}$.

Standard deviation (SD)

Standard deviation is referred to as root mean squared deviation about the mean.

Individual series

$$SD(\sigma) = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

where $x_1, x_2, \ldots, x_n$ are n observations with mean as x

Alternatively $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$ is a useful formula for computational purpose.

Some results based on SD

1. The square of standard deviation is termed as variance.
2. SD is the least mean square deviation.
3. If each item is increased by a fixed constant the SD does not alter or SD is independent of change of origin.
4. Standard deviation depends on each and every data item.
5. For a discrete series in the form $a, a+d, a+2d, \ldots$ (AP), the standard deviation is given by $\text{SD} = d\sqrt{\frac{n^2-1}{12}}$, where n is number of terms in the series.

Co-efficient of variation (CV)

Co-efficient of variation (CV) is defined as, $\text{CV} = \frac{\text{SD}}{\text{AM}} \times 100$.

This is a relative measure, which helps in measuring the consistency. Smaller the co-efficient of variation, greater is the consistency.

Example 16: For the individual series, compute the mean, median and mode 8, 11, 14, 17, 20, 23, 26, 29.

Solution: Mean $= \bar{x} = \frac{\Sigma x_i}{n} = \frac{8+11+\cdots+29}{8} = 18.5$

Median: As the numbers are in ascending order and the number 17 and 20 being middle terms.

$$\text{Median} = \frac{17+20}{2} = \frac{37}{2} = 18.5$$

Mode: As no term can be regarded as 'most often found', mode is not-defined. However using empirical formula,

Mode = 3 median – 2 mean

= 3(18.5) – 2(18.5) = 18.5.

Example 17: The arithmetic mean of 8, 14, x, 20 and 24 is 16; then find x?

Solution: $\bar{x} = \frac{8+14+x+20+24}{5} = 16$

$\Rightarrow \quad \bar{x} = 80 - 66 = 14.$

Example 18: Calculate standard deviation of first five prime numbers.

Soluyion: Given set of observations {2, 3, 5, 7, 11}

$$\frac{\Sigma x^2}{n} = \frac{208}{5}$$

$$\frac{\Sigma x}{n} = \frac{28}{5}$$

$$\therefore \text{SD} = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\frac{208}{5} - \left(\frac{28}{5}\right)^2} = 3.2$$

Example 19: In a series of observations, co-efficient of variation is 25 and mean is 50. Find the variance.

Solution: Co-efficient of variation: $\text{CV} = \frac{\text{SD}}{\bar{x}} \times 100$

$$\Rightarrow \quad \text{SD} = \frac{\text{CV}}{100} \cdot \bar{x}$$

$$= 50 \times \frac{25}{100} = 12.5$$

Variance = $(12.5)^2 = 156.25$.

Correlation

The degree of relationship (linear) that exists between two variables or whether a change in one variable is accompanied by a direct or inverse change in the other is called correlation.

Example: Heights and weights

Scatter diagram

The collection of x and y values plotted on a graph paper gives a certain trend. The trend offers a crude way of understanding the relation ship.

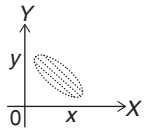


Figure 2 Decreasing trend (–ve correlation)

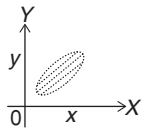


Figure 3 Increasing trend (+ve correlation)

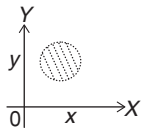


Figure 4 No trend exhibited

Spearman's rank correlation co-efficient

$$r = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

Properties

1. $-1 \le r \le +1$
2. If $r = \pm 1$, then there exist a perfect correlation between x and y.
3. If two variables are independent, then $r = 0$.
4. r is not affected by change of scale and origin.

Karl Pearson's correlation co-efficient

$$r(x, y) = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{N\sigma_x\sigma_y}$$

or
$$= \frac{\text{covariance } (x, y)}{\sigma_x\sigma_y}$$

or
$$r = \frac{\Sigma x_i y_i - N\bar{x}\bar{y}}{N\sigma_x\sigma_y}$$

where σ_x, σ_y are standard deviations of x and y and N is the total number of observations.

The shortcut method is $r(x, y)$

$$= \frac{\Sigma dxdy}{\sqrt{\Sigma dx^2}\sqrt{\Sigma dy^2}}$$

Principle of least squares

The curve of best fit for the given pairs of observations is the curve that makes the sum of squares of the differences between the expected and the observed as the least.

Curve fitting

To fit a line $y = a + bx$ for given n-points (x_i, y_i), $1 \le i \le n$ according to principle of least square fitting, we solve the normal equations for a and b given as

$$\Sigma y_i = na + b\,\Sigma x_i \quad (1)$$
$$\Sigma\, x_i\, y_i = a\,\Sigma x_i + b\,\Sigma x_i^2 \quad (2)$$

Note: The line $y = a + bx$ passes through $(\bar{x}, \bar{y})$.

To fit a parabola $y = a + bx + cx^2$ for given n-points (x_i, y_i), $1 \le i \le n$ according to principle of least square fitting, we solve the normal equations for a, b and c that is given as

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$
$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3$$
$$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$$

To fit a curve of the form $y = ax^b$, for given n points (x_i, y_i), we solve the normal equations for a and b that is given as $\Sigma Y_i = B\Sigma x_i + nA$ and $\Sigma x_i y_i = B\Sigma x_i^2 + A\Sigma x_i$

where $Y_i = \log y_i$; $x_i = \log x_i$,

$A = \log a$ and $B = b$

Regression

The average relationship between two or more variables one dependent and the other independent is called regression.

1. The regression line of y on x is
$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$
2. The regression line of x on y is
$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$
where σ_x, σ_y are standard deviation of variables x and y and r is the correlation co-efficient of x and y.
3. The term $r \cdot \frac{\sigma_x}{\sigma_y}$ and $r \cdot \frac{\sigma_y}{\sigma_x}$ are called regression co-efficients of 'x on y' and 'y on x' respectively.
4. The correlation co-efficient 'r' is the geometric mean of the regression co-efficients.
5. The regression co-efficients are of same signs, either both are positive or both or negative.
6. The point of intersection of regression lines in $(\bar{x}, \bar{y})$.

Covariance

The covariance between two variables x and y of n pairs of observations (x_1, y_1) $(x_2, y_2) \ldots x_n, y_n)$ is defined as follows

$$\text{Cov } (x, y) = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n}$$

where,

$\bar{x}$ is mean of x_i

$\bar{y}$ is mean of y_i

or

$$\text{Cov } (x, y) = \frac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}$$

Example 20: The regression lines of a bi-variate are given by $L_1 \equiv 4x + 3y + 7 = 0$ and $L_2 \equiv 3x + 4y + 8 = 0$.

1. Identify the line of y on x.
2. Identify the line of x on y.
3. Find the mean of x and mean of y.
4. Find the correlation co-efficient.

Solution: Let the line L_1 be y on x and the line L_2 be x on y.

We write

$$y = -\frac{4}{3}x - \frac{7}{3}$$

$$b_{yx} = -\frac{4}{3}$$

$$x = -\frac{4}{3}y - \frac{8}{3}$$

$$b_{xy} = -\frac{4}{3}$$

Now $r = \sqrt{b_{xy}\ xb_{yx}} = \sqrt{\frac{16}{9}}$, where r is the correlation co-efficient. As the limits of r is given by $|r| \le 1$, a contradiction has arised. So L_1 should be x on y and L_2 should be y on x.

1. The regression line of y on x is

$$y = -\frac{3}{4}x - 2$$

$$b_{yx} = -\frac{3}{4}$$

2. The regression line of x on y is

$$x = -\frac{3}{4}y - \frac{7}{4}$$

$$b_{xy} = -\frac{3}{4}$$

3. The mean of x and y is the point of intersection of the 2 lines

$$\bar{x} = -417, \text{ and } \bar{y} = -\frac{11}{7}$$

4. $r^2 = b_{xy}\, b_{yx}$

$$= \frac{-3}{4} \times \frac{-3}{4} = \frac{9}{16}$$

$$r = \pm\frac{3}{4} = \pm 0.75.$$

We retain the –ve sign as b_{xy} and b_{yx} are negative. Hence $r = -0.75$.

Example 21: Find cov(x, y) between x and y from the following data:

$\Sigma x = 20, \Sigma y = 40, \Sigma xy = 210$ and $n = 10$

Solution: $\text{cov}\ (x, y) = \dfrac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}$

$$= \frac{210}{10} - \frac{20}{10} \cdot \frac{40}{10}$$

$$= 21 - 8 = 13$$

Example 22: Find the covariance between x and y for the following observation (x, y).

x	3	11	21	11	18	13	16	17
y	5	16	13	17	18	11	12	9

Solution:

x_i	y_i	$x_i y_i$
3	5	15
11	16	176
21	13	273
11	17	187
18	18	324
13	11	143
16	12	192
17	9	153
Σx_i 110	Σy_i 101	$\Sigma x_i y_i$ 1463

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{110}{8}$$

$$\bar{y} = \frac{\Sigma y_i}{n} = \frac{101}{8}$$

$$\Sigma x_i y_i = 1463$$

$$\therefore\ \text{cov}\ (x, y) = \frac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}$$

$$= \frac{1463}{8} - \frac{110}{8} \cdot \frac{101}{8}$$

$$= \frac{11704 - 11110}{64} = \frac{594}{64} = 9.28125.$$

Example 23: Calculate the rank correlation co-efficient for the following data

x	116	135	121	131	125	129
Y	131	133	129	132	128	126

Solution:

x	$R_x = R_1$	y	$R_y = R_2$	$R_1 - R_2 = d$	d^2
116	6	131	3	3	9
135	1	133	1	0	0
121	5	129	4	1	1
131	2	132	2	0	0
125	4	128	5	–1	1
129	3	126	6	–3	9

$$\Sigma d^2 = 20$$

$$\rho = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)} = 1 - \frac{6(20)}{6\ (36 - 1)}$$

$$= 1 - \frac{120}{210} = \frac{3}{7} = 0.429$$

Example 24: Find the Karl Pearson's co-efficient of correlation for the following data.

x	3	8	10	12	17
y	2	9	10	11	18

Solution:

x	y	$dx = x - \bar{x}$	$dy = y - \bar{y}$	dx^2	dy^2	dxdy
3	2	– 7	– 8	49	64	56
8	9	– 2	– 1	4	1	2
10	10	0	0	0	0	0
12	11	2	1	4	1	2
17	18	7	8	49	64	56

$$\bar{x} = \frac{\Sigma x}{5} = \frac{50}{5} = 10$$

$$\bar{y} = \frac{\Sigma y}{5} = \frac{50}{5} = 10$$

$$r = \frac{\Sigma dxdy}{\sqrt{\Sigma dx^2}\sqrt{\Sigma dy^2}}$$

$$= \frac{116}{\sqrt{106}\sqrt{130}}$$

$$= \frac{116}{(10.29)\times(11.40)}$$

$$= \frac{116}{117.30} = 0.988$$

EXERCISE

Practice Problems

Directions for questions 1 to 63: Select the correct alternative from the given choices.

1. If eight unbiased coins are tossed together, then the probability that the number of heads exceeds the number of tails is

(A) $\frac{31}{128}$ (B) $\frac{1}{2}$

(C) $\frac{93}{256}$ (D) $\frac{57}{256}$

2. If four fair dice are rolled together, then the probability that the total score on the four dice is less than 22 is

(A) $\frac{26}{27}$ (B) $\frac{3}{432}$

(C) $\frac{427}{432}$ (D) $\frac{83}{108}$

3. A bag contains five red balls, three black balls and one white ball. If three balls are drawn from the bag, the probability that the three balls are of different colours is

(A) $\frac{23}{28}$ (B) $\frac{5}{28}$

(C) $\frac{3}{28}$ (D) None of these

4. From a box containing 18 bulbs, of which exactly 1/3rd are defective, five bulbs are chosen at random to fit into the five bulb holders in a room. The probability that the room gets lighted is

(A) $1-\frac{{}^{6}C_5}{{}^{18}C_5}$ (B) $\frac{{}^{6}C_5}{{}^{18}C_5}$

(C) $\frac{{}^{12}C_5}{{}^{18}C_5}$ (D) $1-\frac{{}^{12}C_5}{{}^{18}C_5}$

5. A five digit number is formed using the digits 0, 1, 2, 3, 4 and 5 at random but without repetition. The probability that the number so formed is divisible by 5 is

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$

(C) $\frac{4}{25}$ (D) $\frac{9}{25}$

6. If ten students are to be seated in a row, then the probability that two particular students never sit together is

(A) $\frac{2}{5}$ (B) $\frac{4}{5}$

(C) $\frac{1}{5}$ (D) $\frac{3}{5}$

7. A puzzle in logic was given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. The probability that the problem being solved is

(A) $\frac{29}{32}$ (B) $\frac{31}{32}$

(C) $\frac{1}{8}$ (D) $\frac{7}{8}$

8. If A and B are two events of an experiment such that $P(A \cup B) = \frac{3}{4}$, $P(A) = \frac{7}{20}$, then find $P(B)$ given that

(i) A and B are mutually exclusive

(A) $\frac{1}{4}$ (B) $\frac{1}{5}$

(C) $\frac{3}{5}$ (D) $\frac{2}{5}$

(ii) A and B are equally likely

(A) $\frac{7}{20}$ (B) $\frac{3}{4}$

(C) $\frac{2}{5}$ (D) $\frac{13}{20}$

(iii) A and B are independent events

(A) $\frac{7}{13}$ (B) $\frac{8}{13}$

(C) $\frac{6}{13}$ (D) $\frac{2}{5}$

9. The probability that a square selected at random from a 8×8 chessboard is of size 3×3 is

(A) $\frac{8}{51}$ (B) $\frac{14}{17}$

(C) $\frac{3}{17}$ (D) $\frac{25}{204}$

10. X and Y are independent events. The probability that both X and Y occur is 1/8 and the probability that neither of these occur is 3/8. The probability of occurrence of X can be

(A) $\frac{2}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{3}$ (D) $\frac{3}{4}$

11. A letter is taken randomly from the word SISTER, and another letter is taken randomly from the word RESIST the probability that the two letters are same is

(A) $\frac{2}{3}$ (B) $\frac{3}{5}$

(C) $\frac{7}{36}$ (D) $\frac{2}{9}$

12. I had to type a 6 character password. The probability that I make a mistake in typing a character is 0.3. The password that I typed turned out to be wrong. Find the probability that only the last character that I entered is wrong.

(A) $\frac{(0.7)^5 \times (0.3)}{1-(0.7)^6}$ (B) $\frac{(0.7)^6}{1-(0.7)^6}$

(C) $\frac{(0.7)^5}{1-(0.7)^6}$ (D) $\frac{(0.7)(0.3)^5}{1-(0.7)^6}$

13. There are two groups X and Y. Each group contains three different types of people. In group X, there are 10 Indians, 8 Americans and 7 Japanese. In group Y, there are 8 Indians, 6 Japanese and 6 Americans. If two people are selected at random from the two groups, both selected are Indians then what is the probability that they are from group X?

(A) $\frac{56}{113}$ (B) $\frac{57}{103}$

(C) $\frac{57}{113}$ (D) $\frac{57}{123}$

14. An unbiased coin is tossed a person gets ₹30 if the coin shows head, and he loses ₹15 if the coin shows tail. If three coins are tossed, the probability that the person gets ₹45 is

(A) $\frac{3}{8}$ (B) $\frac{1}{2}$

(C) $\frac{1}{10}$ (D) $\frac{1}{25}$

15. A and B pick a card at random from a well shuffled pack of cards, one after the other replacing it every time till one of them gets a spade. The person who picks a spade is declared the winner. If A begins the game, then the probability that B wins the game is

(A) $\frac{5}{9}$ (B) $\frac{4}{9}$

(C) $\frac{3}{7}$ (D) $\frac{4}{7}$

16. If three unbiased dice are rolled simultaneously in a random experiment, the sum of the numbers showing up on them is 14. What is the probability of an event of showing up 6 on any one of the dice?

(A) $\frac{7}{15}$ (B) $\frac{11}{15}$

(C) $\frac{4}{5}$ (D) $\frac{3}{5}$

17. $P(A \cup B) = \frac{2}{3}$, $[P(\bar{A} \cup \bar{B}) \cap A] = \frac{1}{3}$ and $P(\bar{A} \cap B) + P(A \cap \bar{B}) = \frac{1}{2}$, then what is the value of $P\left(\frac{A}{B}\right)$?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) $\frac{1}{6}$ (D) $\frac{1}{5}$

18. If $P(A) = \frac{3}{5}$, $P(B^C) = \frac{6}{7}$ and $P(A \cap B) = \frac{1}{4}$, then find $P\left(\frac{A^c}{B^c}\right)$.

(A) $\frac{17}{60}$ (B) $\frac{71}{120}$

(C) $\frac{19}{60}$ (D) $\frac{29}{60}$

19. If two events A and B are such that $P(\bar{A}) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.2$, then $P\left(\frac{B}{A \cup \bar{B}}\right)$ is

(A) $\frac{3}{5}$ (B) $\frac{2}{5}$

(C) $\frac{1}{4}$ (D) $\frac{4}{5}$

20. A cinema historian noted that for a brief period, all movies released were either directed by Nolan or starred Bale. Also no movie directed by Nolan starred Bale. The probability that a movie was directed by Nolan is 0.5, and the probability that a movie starred Bale is 0.5. The probability that a movie is a hit if directed by Nolan is 0.6, while the probability that a movie is a hit given that Bale acted in it is 0.4. Given that a movie is a hit, find the probability that it is directed by Nolan.

(A) 0.4 (B) 0.5

(C) 0.6 (D) 0.7

21. A random variable X has mean 2 and $E(X^2) = 6$. Then the standard deviation of X is

(A) $\sqrt{6}$ (B) $\sqrt{2}$

(C) 2 (D) 4

22. The standard deviation of $3x + 2$ is 4; then the variance of x is

(A) 12 (B) $\frac{4}{3}$

(C) $\frac{16}{9}$ (D) Cannot be determined

Common Data for Questions 23 and 24: Probability mass function of a variate x is as follows.

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$

23. $k =$

(A) $\frac{2}{15}$ (B) $\frac{1}{15}$

(C) $\frac{3}{15}$ (D) $\frac{4}{15}$

24. $P(x \geq 3) =$

(A) $\frac{1}{3}$ (B) $\frac{4}{15}$

(C) $\frac{3}{5}$ (D) $\frac{5}{7}$

Common Data for Questions 25 and 26: A variate x has the probability distribution as

x:	4	8	12
$P(X = x)$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{1}{15}$

25. Values of $E(x)$ and $E(x^2)$ respectively are

(A) $\frac{104}{15}, \frac{160}{3}$ (B) $\frac{102}{15.}, \frac{150}{3}$

(C) $\frac{21}{3}, \frac{160}{5}$ (D) $\frac{104}{15}, \frac{151}{3}$

26. The value of $E[(3x + 2)^2]$ is _____.

(A) 675.2 (B) 560.2

(C) 134.56 (D) 567.2

27. For a random variable x, the PDF

$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Find $E(x)$.

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$

(C) $\frac{3}{8}$ (D) $\frac{3}{4}$

28. In the random experiment of drawing a card from 15 cards numbered 1 to 15, if x is the random variable defined by the number appeared on the card, then the expectation of x is

(A) 8 (B) 7

(C) 6 (D) 5

29. If $X(n, p)$ follows a binominal distribution with $n = 6$ such that $9P[X = 4] = P[X = 2]$, then $p =$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) 1 (D) $\frac{1}{4}$

Common Data for Questions 30 and 31: Let ABC be a bulb manufacturing company. The probability that a bulb manufactured by ABC defective is 0.25.

30. What is the probability that in a sample of 8 bulbs at most one will be defective?

(A) $\frac{11}{3}\left(\frac{3}{4}\right)^8$ (B) $11\left(\frac{3}{4}\right)^8$

(C) $\frac{11}{3}\left(\frac{3}{4}\right)^7$ (D) $\frac{1}{4^8}$

31. At least 3 will be defective?

(A) $1-\left(\frac{3}{4}\right)^5\left(\frac{25}{48}\right)$ (B) $1-\left(\frac{3}{4}\right)^8\left(\frac{61}{9}\right)$

(C) $1-\left(\frac{3}{4}\right)^8\left(\frac{24}{48}\right)$ (D) $1-\left(\frac{3}{4}\right)^6\left(\frac{1}{48}\right)$

32. A random variable X follows a Poisson distribution such that $P[X = 1] = P[X = 2]$. Its mean and variance are, respectively,

(A) 1, 1 (B) 2, 2

(C) $\sqrt{3}$, 2 (D) $\sqrt{2}$, $\sqrt{2}$

33. The probability that a person hits a target is 0.003. What is the probability of hitting the target with 2 or more bullets if the number of shots is 2000?
(A) $1 - e^{-6}$ (B) $1 - e^{6}$
(C) $1 - 7e^{6}$ (D) $1 - 7e^{-6}$

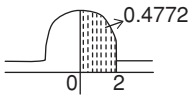
34. If X is a continuous random variable with PDF $f(x) = \frac{1}{4}$ if $-2 \le x \le 2$ and $f(x) = 0$ elsewhere, the mean of x is
(A) 1 (B) 1.5
(C) 2 (D) 0

35. If x is a uniformly distributed random variable in [1, 4] then $P\left(x > \frac{3}{2}\right)$ is
(A) $\frac{1}{6}$ (B) $\frac{1}{2}$
(C) $\frac{5}{6}$ (D) $\frac{1}{4}$

36. If x is a uniformly distributed random variable in [2, 5] then $E(x^2)$ is
(A) 2 (B) 8
(C) 13 (D) 15

37. If the life time of bulbs (in months) is exponential with mean 5 months, then the probability that the bulb lasts for at least 7 months is
(A) 0.2466 (B) 0.7534
(C) 0.4932 (D) 0.5068

38. x is a normal variate with mean 35 and variance 25 probability of $31 \le x < 45$ is $(-0.8 \le z < 0 = 0.2881)$



(A) 0.6735 (B) 0.7563
(C) 0.7653 (D) 0.5736

39. Let X_1 and Y_1 be two discrete random variables with joint probability mass function as given below

X \ Y	2	3	$P(X_1 = x_i)$
1	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$
4	$\frac{4}{5}$	$\frac{8}{15}$	$\frac{4}{5}$
$P(Y_1 = y_i)$	$\frac{1}{3}$	$\frac{2}{3}$	

Let X_2 and Y_2 be two discrete random variables with joint probability mass function as given below

X_2 \ Y_2	0	4	7	$P(X_2 = x_i)$
–1	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{7}$
3	$\frac{4}{21}$	$\frac{2}{7}$	$\frac{2}{21}$	$\frac{4}{7}$
$P(Y_2 = y_i)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	

Which of the following statements is TRUE about the random variables x_1, x_2, y_1 and y_2?
(A) Only x_1 and y_1 are independent
(B) Only x_2 and y_2 are independent
(C) x_1 and y_1 are independent as well as x_2 and y_2 are independent
(D) Neither x_1 and y_1 are independent nor x_2 and y_2 are independent

40. If X and Y are two independent random variables with expectations 3 and 4 respectively. Then the expectation of xy is
(A) 1 (B) 7
(C) 12 (D) 16

41. If X and Y are two independent random variables that are uniformly distributed over the same interval [2, 5] then $P\left(X \le \frac{11}{4},\ Y \ge \frac{11}{3}\right)$ is
(A) $\frac{1}{9}$ (B) $\frac{2}{9}$
(C) $\frac{1}{3}$ (D) $\frac{4}{7}$

42. The mean of cubes of first 10 natural numbers is
(A) 305 (B) 300
(C) 302.5 (D) 310

43. The mean of 25 observations was found to be 38. It was later discovered that 23 and 38 were misread as 25 and 36, then the mean is
(A) 32 (B) 36
(C) 38 (D) None of these

44. If the mean of a set of 12 observations is 10 and another set of 8 observations is 12, then the mean of combined set is
(A) 12.6 (B) 10.8
(C) 12.8 (D) 10.6

45. The mode of a distribution of 13 and its mean is 4 then its median is
(A) 7 (B) 9
(C) 8 (D) 11

46. Consider the non-decreasing series of the numbers, 1, 8, 8, 13, 14, 14, x, y, 18, 20, 31, 34, 38 and 40. If the median of the series is 15, then the mode of the series is

(A) 14 (B) 16
(C) 18 (D) Cannot be determined

47. The standard deviation of 5, 5, 5, 5, 5, 5, 5, 13 is
(A) $2\sqrt{2}$ (B) $\sqrt{6}$
(C) 5 (D) $\sqrt{7}$

48. If the standard deviation of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 is M, then the standard deviation of 101, 102, 103, 104, ... and 111 is
(A) M (B) $100 + M$
(C) $100 - M$ (D) $M - 100$

49. The arithmetic mean of five observations is 6.4 and the variance is 8.24. If three of the observations are 3, 4, 8, then find the other two observations.
(A) 6, 11 (B) 10, 7
(C) 8, 9 (D) 5, 12

50. Compute the covariance for the following data

x	4	8	12	5	9	16
y	1	7	4	15	10	5

(A) 4.17 (B) – 4.17
(C) – 3.15 (D) 3.15

51. When $n = 10$, $\Sigma x = 50$, $\Sigma y = 50$, $\Sigma x^2 = 500$, $\Sigma y^2 = 410$ and $\Sigma xy = 300$ then the cov(x, y) is _____.
(A) 2 (B) 3
(C) 4 (D) 5

52. The correlation co-efficient of the above data is _____.
(A) 0.25 (B) 0.5
(C) 0.35 (D) 0.4

53. Find the Karl Pearson's co-efficient of correlation for the following data.

X	50	60	70	80	90	100	110
Y	20	30	40	50	60	70	80

(A) $r = 0.8$ (B) $r = 0.1$
(C) $r = 0$ (D) $r = 1$

54. The co-efficient of correlation of a data of price and demand is –0.82, then which of the following Is true?
(A) When price is decreased, demand is decreased.
(B) When price is increased, demand is increased.
(C) When price is increased then the demand is decreased.
(D) None of these

55. Which of the following statement/s is/are true? (r is co-efficient of correlation)
(i) If $r = 0$ the regression lines are perpendicular.
(ii) If $r = +1$ or $r = -1$ the regression lines coincide.
(A) i alone (B) ii alone
(C) Both i and ii (D) None of these

56. Given the marks obtained in Graphics and Network Theory by 6 students. Find the rank correlation co-efficient

Graphics	38	64	32	63	73	52
Network Theory	74	64	38	73	48	75

(A) –0.135 (B) –0.085
(C) 0.085 (D) 0.015

57. Find the value of a if $y = a + bx$ is a line fit for the following data according to the principle of least squares.

x	–3	–2	0	1	2
y	–4	–2	0	2	4

(A) 10 (B) $\frac{13}{9}$
(C) 0 (D) 3

58. For the curve 'C' fitted according to principle of least squares, which of the following statements is/are true?
(i) C passes through origin.
(ii) C passes through all the points.
(iii) C passes through at least one of the points.
(A) i alone (B) i and ii alone
(C) ii and iii alone (D) None of these

59. For the following data, find the regression line of y on x:

x	3	2	5	12	8
y	1	8	9	2	15

(A) $0.03x - y + 6.82 = 0$ (B) $0.07x + y + 6.82 = 0$
(C) $0.03x + y - 6.82 = 0$ (D) $0.03x + y + 6.82 = 0$

60. Out of the two regression lines $3x + 5y = 25$ and $4x + 3y = 12$, the line of regression of x on y is _____.
(A) $x = 5 - \frac{5}{3}y$ (B) $y = 5 - \frac{3}{5}x$
(C) $x = 3 - \frac{3}{4}y$ (D) $y = 4 - \frac{3}{4}x$

61. If the regression co-efficients are given by $b_{yx} = 1.5$ and $b_{xy} = 0.5$ then the angle between two regression lines is _____.
(A) $\tan^{-1}\left(\frac{5}{16}\right)$ (B) $\tan^{-1}\left(\frac{1}{4}\right)$
(C) $\tan^{-1}\left(\frac{1}{8}\right)$ (D) $\tan^{-1}\left(\frac{3}{4}\right)$

62. The lines of regression are $5x + 7y = 35$ and $4x + 9y = 36$.
Find the co-efficient b_{yx}.
(A) $-\frac{5}{7}$ (B) $-\frac{9}{7}$
(C) $-\frac{7}{5}$ (D) $-\frac{4}{9}$

63.

	Mean	S.D
x	20	5
y	80	10

The correlation co-efficient between x and y = 0.6.

The estimate value of y when x = 60 is

(A) 140 (B) 120

(C) 128 (D) 160

Previous Years' Questions

1. If P and Q are two random events, then which of the following is TRUE: **[2005]**

(A) Independence of P and Q implies that probability $(P \cap Q) = 0$

(B) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)

(C) If P and Q are mutually exclusive, then they must be independent

(D) Probability $(P \cap Q) \leq$ Probability (P)

2. A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is: **[2005]**

(A) $\frac{1}{8}$ (B) $\frac{1}{2}$

(C) $\frac{3}{8}$ (D) $\frac{3}{4}$

3. Two fair dice are rolled and the sum r of the numbers turned up is considered **[2006]**

(A) $Pr\ (r > 6) = \frac{1}{6}$

(B) $Pr\left(\frac{r}{3} \text{ is an integer}\right) = \frac{5}{6}$

(C) $Pr\left(r = 8 \middle| \frac{r}{4} \text{ is an integer}\right) = \frac{5}{9}$

(D) $Pr\left(r = 6 \middle| \frac{r}{5} \text{ is an integer}\right) = \frac{1}{18}$

4. A loaded dice has following probability distribution of occurrences

Dice value	1	2	3	4	5	6
Probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is **[2007]**

(A) same as that of occurrence of 3, 4, 5

(B) same as that of occurrence of 1, 2, 5

(C) $\frac{1}{128}$ (D) $\frac{5}{8}$

5. X is a uniformly distributed random variable that takes values between 0 and 1. The value of $E\{X^3\}$ will be **[2008]**

(A) 0 (B) $\frac{1}{8}$

(C) $\frac{1}{4}$ (D) $\frac{1}{2}$

6. Assume for simplicity that N people, all born in April (a month of 30 days), are collected in a room. Consider the event of at least two people in the room being born on the same date of the month, even if in different years, e.g., 1980 and 1985. What is the smallest N so that the probability of this event exceeds 0.5? **[2009]**

(A) 20 (B) 7

(C) 15 (D) 16

7. A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is **[2010]**

(A) $\frac{1}{3}$ (B) $\frac{3}{7}$

(C) $\frac{1}{2}$ (D) $\frac{4}{7}$

8. Two independent random variables X and Y are uniformly distributed in the interval [−1, 1]. The probability that max[X, Y] is less than ½ is **[2012]**

(A) $\frac{3}{4}$ (B) $\frac{9}{16}$

(C) $\frac{1}{4}$ (D) $\frac{2}{3}$

9. A fair coin is tossed till a head appears for the first time. The probability that the number of required tosses is odd, is **[2012]**

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) $\frac{2}{3}$ (D) $\frac{3}{4}$

10. A continuous random variable X has a probability density function $f(x) = e^{-x}$,

$0 < x < \infty$. Then $P\{X > 1\}$ is **[2013]**

(A) 0.368 (B) 0.5

(C) 0.632 (D) 1.0

11. A fair coin is tossed n times, the probability that the difference between the number of heads and tails is $(n-3)$ is **[2014]**

(A) 2^{-n} (B) 0

(C) ${}^{n}C_{n-3}2^{-n}$ (D) 2^{-n+3}

12. Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is _____. **[2014]**

13. Let X be a random variable with probability density function

$$f(x) = \begin{cases} 0.2, & \text{for } |x| \le 1 \\ 0.1, & \text{for } 1 < |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The probability $P(0.5 < X < 5)$ is_____ **[2014]**

14. Lifetime of an electric bulb is a random variable with density $f(x) = kx^2$, where x is measured in years. If the minimum and maximum lifetimes of bulb are 1 and 2 years respectively, then the value of k is _____. **[2014]**

15. The mean thickness and variance of silicon steel laminations are 0.2 mm and 0.02 respectively. The varnish insulation is applied on both the sides of the laminations. The mean thickness of one side insulation and its variance are 0.1 mm and 0.01 respectively. If the transformer core is made using 100 such varnish coated laminations, the mean thickness and variance of the core respectively are **[2014]**

(A) 30 mm and 0.22 (B) 30 mm and 2.44

(C) 40 mm and 2.44 (D) 40 mm and 0.24

16. A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx; & \text{for } 0 < x < 1 \\ 0; & \text{otherwise} \end{cases}$$

If the expected value $E[X] = 2/3$, then $Pr[X < 0.5]$ is _____. **[2015]**

17. Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is **[2015]**

(A) 5/11 (B) ½

(C) 7/13 (D) 6/11

18. Two coins R and S are tossed. The 4 joint events H_RH_S, T_RT_S, H_RT_S, T_RH_S have probabilities 0.28, 0.18, 0.30, 0.24, respectively, where H represents head and T represents tail. Which one of the following is TRUE? **[2015]**

(A) The coin tosses are independent.

(B) R is fair, S is not.

(C) S is fair, R is not.

(D) The coin tosses are dependent.

19. Candidates were asked to come to an interview with 3 pens each. Black, blue, green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is _____. **[2016]**

20. Let the probability density function of a random variable, X, be given as: **[2016]**

$$f_x(x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}(-x)$$

Where $u(x)$ is the unit step function

Then the value of 'a' and Probe $\{X \le 0\}$, respectively, are

(A) $2, \frac{1}{2}$ (B) $4, \frac{1}{2}$

(C) $2, \frac{1}{4}$ (D) $4, \frac{1}{4}$

Answer Keys

Exercise

Practice Problems

1. C	**2.** C	**3.** B	**4.** A	**5.** D	**6.** B	**7.** A	**8.** (i) D (ii) A (iii) B		
9. C	**10.** B	**11.** D	**12.** A	**13.** C	**14.** A	**15.** C	**16.** C	**17.** B	**18.** B
19. B	**20.** C	**21.** B	**22.** C	**23.** B	**24.** C	**25.** A	**26.** D	**27.** D	**28.** A
29. D	**30.** A	**31.** B	**32.** B	**33.** D	**34.** D	**35.** C	**36.** C	**37.** A	**38.** C
39. C	**40.** C	**41.** A	**42.** C	**43.** C	**44.** B	**45.** A	**46.** D	**47.** D	**48.** A
49. A	**50.** B	**51.** D	**52.** A	**53.** D	**54.** C	**55.** C	**56.** B	**57.** C	**58.** D
59. A	**60.** C	**61.** C	**62.** D	**63.** C					

Previous Years' Questions

1. D	**2.** B	**3.** C	**4.** C	**5.** C	**6.** B	**7.** C	**8.** B	**9.** C	**10.** A
11. B	**12.** 0.13 to 0.15		**13.** 0.35 to 0.45		**14.** 0.4 to 0.5		**15.** –	**16.** 0.25	**17.** D
18. D	**19.** 0.2	**20.** A							

Chapter 6

Numerical Methods

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Numerical methods
- Methods for finding the real roots
- Bisection method
- Regula falsi method
- The secant method
- Newton–Raphson method
- Solutions of linear system of equations
- Gauss elimination method
- Gauss–Jordan elimination method
- Numerical integration
- Simpson's three eight rule

Numerical Methods

We encounter problems in Engineering Mathematics for which analytical methods are not available to find solutions. Further, it may be sufficient in engineering applications to find approximate solutions. The numerical methods offer us approximate solutions

1. Methods for finding zeros of an algebraic or transdental equations
2. Numerical solutions of ordinary differential equation.

Methods for Finding the Real Roots (Zeros) of $f(x) = 0$

The equation of the form $f(x) = 0$ is called an Algebraic or Transcendental according as $f(x)$ is purely a polynomial in x or contains some other functions such as exponential, logarithmic and trigonometric functions etc.

Examples:

1. $x^9 + 8x^5 - 4x^3 - 11x + 3 = 0 \rightarrow$ Algebraic equation
2. $10x^4 - \log(x^2 - 3) + e^{-x}\sin x + \tan^2 x = 0 \rightarrow$ Transcendental equation

In this chapter, we obtain the solution of an equation $f(x) = 0$, i.e., we mean to find the zeros of $f(x)$.

We shall now discuss few methods to find the real roots of both algebraic (with numerical coefficients) and transcendental equations.

We first find an approximate value of the root of the given equation and then successively improve it to some desired degree of accuracy.

We start with an initial approximate value, say x_0, and then find the better approximations successively $x_1, x_2, x_3 \ldots, x_n$ by repeating the same method.

If the successive approximations at each step of a method approach the root more and more closely, we say that the method converges.

The Intermediate Value Theorem

If a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root say a between a and b of the equation

$f(x) = 0$, such that $f(a) = 0$

Note 1: Root 'α' of $f(x) = 0$, will be unique in (a, b) if $f'(x)$ has the same sign in (a, b) (i.e., $f'(x) > 0$ or $f'(x) < 0$ in $a < x < b$)

Relations between Roots and Coefficients

An nth order equation has n roots. Corresponding to every root, there is a factor. If α is a root of $f(x) = 0$, then $x - \alpha$ is a factor of $f(x)$. Sometimes $(x - \alpha)^2$ may also be a factor. In such a case, α is said to be a double root. Similarly equations can have triple roots, quadruple roots and roots of multiplicity m. If m is the greatest value of k, for which $(x - \alpha)^k$ is a factor of $f(a)$, then α is said to be a root of multiplicity m. If all the roots are counted by taking their multiplicity into account, the number of roots is equal to n, the degree of the equation.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) are the roots of $f(x) = 0$, then

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$
$$= a_n [x^n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots + (-1)^n S_n]$$

Where S_1 = the sum of the roots
S_2 = the sum of the products of the roots taken 2 at a time
S_3 = the sum of the product of the roots taken 3 at a time and so on.
S_n = the 'sum' of the product of the roots taken n (or all) at a time. Thus, S_n is a single term.
$S_n = \alpha_1 \alpha_2 \ldots \alpha_n$

Roots of Equations and Descartes' Rule

If the coefficients are all real and the complex number z_1, is a root of $f(x) = 0$, then the conjugate of z_1, *viz*, $\bar{z}_1$ is also a root of $f(x) = 0$. Thus, for equations with real, coefficients, complex roots occur in pairs.

A consequence of this is that any equation of an odd degree must have at least one real root.

The number of positive roots is at the most equal to the number of sign changes. It could also be less than that by 2, 4…, i.e., if there are k sign changes in $f(x)$, the number of positive roots could be k, $k-2$, $k-4$, …

This is called **Descartes' Rule of Signs**. This rule can be extended to negative roots as follows. The number of negative roots of $f(x) = 0$ is equal to the number of positive roots of $g(x) = f(-x) = 0$

For example, consider $f(x) = x^5 - 3x^3 + 6x^2 - 28x + 24$. There are 4 sign changes in $f(x)$

$\therefore$ The number of positive roots could be 4, 2 or 0.

Consider $g(x) = f(-x)(-x)^5 - 3(-x)^3 + 6(-x)^2 - 28(-x) + 24 = -x^5 + 3x^3 + 6x^2 + 28x + 24$

There is only one sign change in $f(-x)$.

$\therefore$ The number of negative roots of $f(x) = 0$ is 1. (It can't be $-1, -3, \ldots$)

The following table shows the various possibilities for the roots.

Negative	Positive	Complex
1	4	0
1	2	2
1	0	4

We have considered one specific equation and this specific equation has 5 specific roots. We can use more advanced techniques to find the actual roots. But even without that, using only Descartes Rule, we expect exactly one of the 3 situations shown in the table above.

Solved Example

Example 1: Find the nature of roots of the equation, $f(x) = x^3 + x - 2 = 0$.

Solution: There is only 1 change of sign in $f(x)$.
We know that when $f(x)$ has r changes of sign then $f(x)$ has $r, r-2, r-4, \ldots$ positive roots.
$\therefore$ $f(x) = 0$ has one positive root.
Now $f(-x) \equiv -x^3 - x - 2 = 0$. $q = 0$
Since there is no change of sign in $f(-x)$, $f(x)$ has no negative roots. The number of complex roots is even.
$\therefore$ The equation has one positive root, and two complex roots. Hence $f(x) = 0$ has 1 real root and two complex roots.

Example 2: How many non real-roots does the equation $x^4 - 2x^2 + 3x - 2 = 0$ have?

Solution: Let $f(x) = x^4 - 2x^2 + 3x - 2$
$f(x)$ has 3 sign changes
$\therefore$ $f(x)$ has 3 or 1 positive roots.

$$f(-x) = x^4 - 2x^2 - 3x - 2$$

$\therefore$ $f(-x)$ has one sign change
$\therefore$ $f(x)$ has exactly one negative root.
As the sum of the co-efficient of $f(x)$ is zero,
$x = 1$ is a root $f f(x) = 0$
$\therefore$ $f(x) = (x-1)(x^3 + x^2 - x + 2) = (x-1) f_1(x)$. By trial, $f_1(-2) = 0$
$\therefore$ $f_1(x) = (x+2)(x^2 - x + 1)$
We can see that $x^2 - x + 1 = 0$ has two non-real roots.
$\therefore$ $f(x)$ has one positive, one negative and two non-real roots.

Bisection Method (Bolzano Method) or (Halving Method)

Consider the equation $f(x) = 0 \rightarrow (1)$

If $f(x)$ is continuous between a and b and $f(a) f(b) > 0$, then there exists one root between a and b. Let $f(a)$ be negative and $f(b)$ be positive. The bisection method isolates the root in $[a, b]$ by halving process, approximately dividing the given interval $[a, b]$ into two, four, eight, etc. equal parts.

Thus, the first approximation to the root is given by

$$x_0 = \frac{a+b}{2}$$

$$a \text{—} \text{—} \underset{\frac{a+b}{2}}{\text{—•—}} \text{—} \text{—} b$$

If $f(x_0) = 0$, then i_0 is a root, otherwise the root lies either between a and x_0 or x_0 and b depending on whether $f(x_0)$ is positive or negative. We again bisect the interval and repeat the process until the root is obtained to desired accuracy.

Example 3: Find a real root of the equation $f(x) = x^3 - 2x^2 + 3x - 1$ on the interval (0, 1) using bisection method with four iterations.

Solution: We have $f(0) = -1 < 0$ and

$$f(1) = 1 - 2 + 3 - 1 = 1 > 0$$

$\therefore$ A root lies between 0 and 1

$\therefore$ The first approximation to the root is $\frac{0+1}{2} = 0.5$. Now $f(0.5) = (0.5)^3 - 2(0.5)^2 + 3(0.5) - 1 = 0.125 > 0$ and $f(0) < 0$

$\therefore$ The root lies between 0 and 0.5. The second approximation to the root is $\frac{0+0.5}{2} = 0.25$

Now $f(0.25) = (0.25)^3 - 2(0.25)^2 + 3(0.25) - 1$
$= -0.359 < 0$ and $f(0.5) > 0$

$\therefore$ The root lies between 0.25 and 0.5.

$\therefore$ The third approximation to the root is $\frac{0.25+0.5}{2} = \frac{0.75}{2} = 0.375.$

Now $f(0.375) = (0.375)^3 - 2(0.375)^2 + 3(0.375) - 1$
$= -0.103 < 0$ and $f(0.5) > 0$

$\therefore$ The root lies between 0.375 and 0.5.

$\therefore$ The fourth approximation to the root is $\frac{0.375+0.5}{2}$

$$= \frac{0.875}{2} = 0.4375$$

Convergence of Bisection Method

If $x_1, x_2, x_3 \ldots, x_n$ is the sequence of midpoints obtained by bisection method, then $|c - x_n| \le \frac{b-a}{2^n}$, $n = 1, 2, 3 \ldots$ where 'c' is between a and b

Note: In bisection method, the convergence is very slow but definite. The order of convergence is linear or of first order.

Regula Falsi Method or (The Method of False Position)

In this method, to find the real root of the equation $f(x) = 0$, we consider a sufficiently small interval (a, b), $a < b$ such that $f(a)$ and $f(b)$ will have opposite signs. This implies a root lies between a and b according to intermediate value theorem. Also the curve $y = f(x)$ will meet the x-axis at some point between $A\,[a, f(a)]$ and $B\,[b, f(b)]$.

The equation of the chord joining $A\,[a, f(a)]$ and $B\,[b, f(b)]$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \qquad (1)$$

The point of intersection of the chord (1) with x-axis is given by $y = 0$ in (1)

$$-f(a) = \frac{f(b) - f(a)}{b - a}(x - a),$$

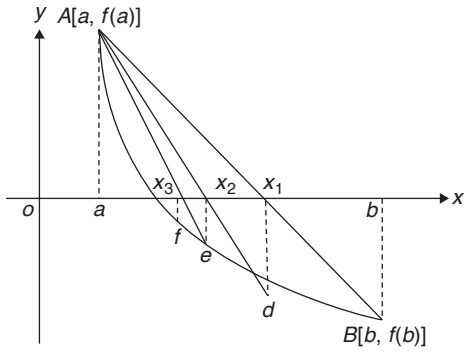
$$\Rightarrow \quad x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$\therefore$ The first approximation $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$ (2)

If $f(x_1) = 0$, then x_1 is the root. If $f(x_1) \neq 0$ and if $f(x_1)$ and $f(a)$ have opposite signs, the second approximation

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)} \qquad (3)$$

Proceeding in the same way, we get x_3, x_4 and so on. Geometrically, the required root is shown in the figure below.



$y = f(x)$

Note: This method is faster than the first order fixed point iteration.

Convergence of Regula Falsi Method

The order of convergence of the method of false position is greater than 1.

The Secant Method

This method is quite similar to that of Regula–Falsi method except for the condition $f(a)\,f(b) < 0$. The interval at each iteration may not contain the root. Let the initial limits of the interval be 'a' and 'b'.

The formula for successive approximation general form is

$$x_{n+1} = x_n + \frac{(x_n - x_{n-1}) f(x_n)}{f(x_{n-1}) - f(x_n)}$$

In case at any stage $f(x_n) = f(x_{n-1})$ the method fails.

Notes:

1. This method does not converge always, but Regula–Falsi method always converges.
2. If it converges, it converges with order 1.62 approximately, which is more rapidly than the Regula–Falsi method.

Example 4: Find a root for $2e^x \sin x = 3$ using Regula–Falsi method and correct to three decimal places with three iterations.

Solution: Let $f(x) = 2e^x \sin x - 3$

$$f(0) = -3 < 0, f(1) = 2e' \sin 1 - 3$$
$$= 1.5747 > 0$$

$\therefore$ A root lies between 0 and 1.

Here $a = 0$ and $b = 1$

$\therefore$ The first approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0(1.5747) - 1(-3)}{1.5747 - (-3)}$$

$$= \frac{3}{4.5747} = 0.6557$$

Now $f(0.6557) = 2e^{0.6557}\sin(0.6557) - 3$
$= -0.6507 < 0$ and $f(1) > 0$

$\therefore$ The root lies between 0.6557 and 1.

The second approximation x_2

$$= \frac{(0.6557)(1.5747) - 1(-0.6507)}{1.5747 - (-0.6507)}$$

$$= \frac{1.0325 + 0.6507}{2.2254} = \frac{1.6832}{2.2254} = 0.7563$$

Now $f(0.7563) = -0.0761 < 0$ and $f(1) > 0$

$\therefore$ The root lies between 0.7563 and 1

$\therefore$ The third approximation to the root x_3

$$= \frac{(0.7653)(1.5747) - 1(-0.0761)}{1.5747 - (-0.0761)}$$

$$= \frac{1.1909 + 0.0761}{1.6508} = 0.7675$$

$\therefore$ The best approximation to the root upto three decimal places is 0.768

Newton–Raphson Method

Let x_0 be the approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root. Then $f(x_1) = 0$

$$\Rightarrow \quad f(x_0 + h) = 0 \rightarrow (1)$$

Expanding (1) using Taylor's theorem,
We get $f(x_0) + hf'(x_0) + \cdots = 0$

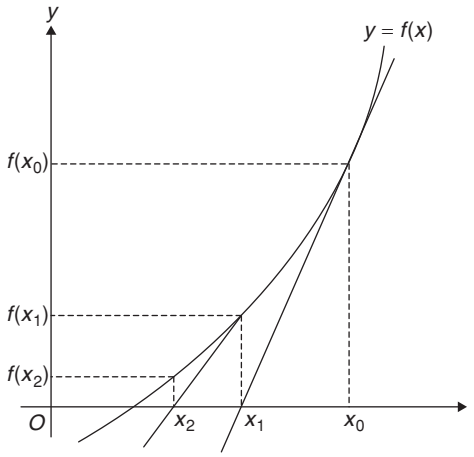
$$\Rightarrow \quad h = \frac{-f(x_0)}{f'(x_0)}$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now x_1 is the better approximation than x_0. Proceeding this way, the successive approximations $x_2, x_3 \ldots x_{n+1}$ are given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

This is called Newton–Raphson formula.

Geometrical interpretation of Newton–Raphson formula



The geometrical meaning of Newton–Raphson method is a tangent is drawn at the point $[x_0, f(x_0)]$ to the curve $y = f(x)$. It cuts the x-axis at x_1 which will be a better approximation of the root. Now drawing another tangent at $[x_1, f(x_1)]$ which cuts the x-axis at x_2 which is a still better approximation and the process can be continued till the desired accuracy has been achieved.

Convergence of Newton–Raphson method

The order of convergence of Newton–Raphson method is 2 or the convergence is quadratic.

It converges if $|f(x).\, f''(x)| < |f'(x)|^2$. Also this method fails if $f'(x) = 0$

Newton's iterative formula to find bth root of a positive real number a

The iterative formula is given by

$$x_{n+1} = \frac{1}{b}\left\{(b-1)x_n + \frac{a}{x_n^{b-1}}\right\}$$

Newton's iterative formula to find a reciprocal of a number N

The iterative formula is given by

$$x_{i+1} = x_i\,(2 - x_i N)$$

Example 5: Find a real root of the equation $-4x + \cos x + 2 = 0$, by Newton–Raphson method upto four decimal places assuming $x_0 = 0.5$

Solution: Let $f(x) = -4x + \cos x + 2$ and

$$f'(x) = -4 - \sin x$$

Also $f(0) = 1 + 2 = 3 > 0$ and

$$f(1) = -4 + \cos 1 + 2 = -1.4596 < 0$$

So, a root lies between 0 and 1.

Given $x_0 = 0.5$

$\therefore$ The first approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{[-4(0.5) + \cos(0.5) + 2}{-4 - \sin(0.5)}$$

$$= (0.5) - \frac{[-2 + 2 + \cos(0.5)}{-4 - \sin 0.5}$$

$$= 0.5 - \frac{0.8775}{-4.4794}$$

$$= 0.5 + 0.1958 = 0.6958$$

Example 6: Obtain the cube root of 14 using Newton–Raphson method.

Solution: We know that, the iterative formula to find $\sqrt[b]{a}$ is

$$x_{n+1} = \frac{1}{b}\left\{(b-1)x_n + \frac{a}{x_n^{\,b-1}}\right\}$$

Here $b = 3$ and $a = 14$ and let $x_0 = 2.5$

$$\therefore \quad x_1 = \frac{1}{3}\left\{2x_0 + \frac{14}{x_0^{\,2}}\right\}$$

$$x_1 = \frac{1}{3}\left\{2(2.5) + \frac{14}{(2.5)^2}\right\}$$

$$= \frac{1}{3}\left\{5 + \frac{14}{6.25}\right\} = \frac{1}{3}\{5 + 2.24\} = 2.413$$

$$x_2 = \frac{1}{3}\left\{2(2.413) + \frac{14}{(2.413)^2}\right\}$$

$$= \frac{1}{3}\left\{4.826 + \frac{14}{5.822569}\right\}.$$

$$= \frac{1}{3}4.826 + 2.4044\} = 2.410$$

$\therefore$ The approximate cube root of 14 is 2.41.

Example 7: Find the reciprocal of 24 using Newton–Raphson method.

Solution: The iterative formula to find $\frac{1}{N}$ is $x_{n+1} = x_n(2 - x_n N)$

Let $x_0 = 0.041$. Then $x_1 = x_0(2 - x_0\,(24))$

$$\Rightarrow \quad x_1 = (0.041)\,(2 - (24)\,(0.041))$$
$$= 0.04165$$

$x_2 = (0.0416)\,(2 - (24)\,(0.04165)] = 0.04161$, similarly proceeding we get $x_3 = 0.041666$

$\therefore$ The reciprocal of 24 is 0.04166.

Numerical solutions of ordinary differential equations

The following methods are discussed on the numerical solution of ordinary differential equations.

Single-step methods:

1. Taylor's series Method
2. Picard's Method of successive approximation

Multi-step methods:

1. Euler's Method
2. Modified Euler's Method
3. Runge–Kutta Method
4. Predictor-corrector Methods [Milne's and Adam's]

Taylor's series method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with initial condition } y(x_0) = y_0 \qquad (4)$$

Let $y = f(x)$ be the solution of (4)

Writing the Taylor series expansion of $f(x)$ at x_0

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots$$

$$\Rightarrow \quad y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots$$

Put $x = x_1$, we get

$$y_1 = y_0 + (x_1 - x_0)y_0' + \frac{(x_1 - x_0)^2}{2!}y_0'' + \cdots$$

If we take $h = x_1 - x_0$

$$\Rightarrow \quad y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots$$

$\therefore$ In general,

$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \cdots$ will be the iterative formula.

Example 8: Given $\frac{dy}{dx} = x - y^2$ with the initial condition $y(0) = 1$

Find $y(0.1)$ using Taylor series method

Solution: $f(x, y) = x - y^2$

$x = 0.1, x_0 = 0, y_0 = 1, h = 0.1$

$y' = x - y^2 \quad \Rightarrow \quad y'(0) = x_0 - y_0^{\,2} = -1;$

$y'' = 1 - 2yy' \Rightarrow \quad y''(0) = 1 - 2y_0y_0'$

$= 1 - 2\,(1)(-1) = 3$

$y''' = -2yy'' - 2(y')^2$

$\Rightarrow \quad y'''(0) = -2\,(1)\,(3) - 2\,(-1)^2$

$= -6 - 2 = -8$

By Taylor's formula,

$$y(0.1) = y_1 = y_0 + hy''(0) + \frac{h^2}{2!}y''(0) + \frac{h^3}{3!}y'''(0) \quad \Rightarrow \quad y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \cdots$$

$$= 1 - 0.1 + 0.015 - 0.0013 + \cdots$$

$$y_1 = 0.9137$$

Picard's method of successive approximation

Given the differential equation $\frac{dy}{dx} = f(x, y)$ (5)

Integrate (5) from x_0 to x, we get

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow \quad y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow \quad y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\,dx \qquad (6)$$

Put $y = y_0$, we get the first approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similarly, put $y = y_1$ in $f(x, y)$ of (6), the second approximation will be

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

Proceeding this way, in general

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

Example 9: Given $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$

Evaluate y (0.1) by Picard's Method upto three approximations.

Solution: $f(x, y) = 1 + xy$

$$x_0 = 0, y_0 = 1$$

The first approximation $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$$1 + \int_{x_0}^{x} 1 + xy_0\,dx = 1 + \int_0^x 1 + x\,dx$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

at $x = 0.1$, $y_1 = 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$

The second approximation y_2

$$= y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$\Rightarrow \quad y_2 = 1 + \int_0^x 1 + xy_1\,dx$$

$$\Rightarrow \quad y_2 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right]dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

at $x = 0.1$

$$y_2 = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8}$$

$$y\,(0.1) = 1.10534$$

The third approximation $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$

$$\Rightarrow \quad y_3 = 1 + \int_0^x 1 + xy_2\,dx$$

$$1 + \int_0^x \left(1 + x\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]\right)dx$$

$$1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

At $x = 0.1$,

$$y_3 = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

$$= 1 + 0.1 + 0.005 + 0.0003 + 0.0000125 + 0.0000006 + 0.00000002$$

$$y_3 = 1.105313$$

Multi-step Methods

Euler's method For the differential equation $\frac{dy}{dx} = f(x, y)$

With initial condition $y(x_0) = y_0$, the Euler's iteration formula is

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}),\ n = 1, 2, 3 \ldots$$

Note: The process is very slow and to obtain accuracy, h must be very small, i.e., we have to divide $[x_0, x_n]$ into a more number of subintervals of length 'h'.

Example 10: Solve $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$, find $y(0.5)$ by Euler's method choosing $h = 0.25$

Solution: $f(x, y) = \dfrac{y - x}{y + x}$

$$x_0 = 0, y_0 = 1, h = 0.25$$

Euler's iteration formula,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

Put $n = 1$,

$$x_1 = 0.25 \quad \Rightarrow \quad y_1 = y(0.25) = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.25)\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + (0.25)\frac{1 - 0}{1 + 0} = 1.25$$

Put $n = 2$

$$x_2 = 0.5 \quad \Rightarrow \quad y_2 = y(0.5) = y_1 + hf(x_1, y_1)$$

$$= (1.25) + (0.25)\left[\frac{y_1 - x_1}{y_1 + x_1}\right]$$

$$= 1.25 + (0.25)\left[\frac{1.25 - 0.25}{1.25 + 0.25}\right]$$

$$= 1.25 + 0.166666 = 1.4166$$

$$\therefore \quad y(0.5) = 1.4166$$

Modified Euler's method For the differential equation $\dfrac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$, the Modified Euler's iteration formula is

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{n-1})]$$

Note: To find y_n, we proceed to find the approximations $y_n^{(0)}, y_n^{(1)}, y_n^{(2)} \ldots$ until the two successive approximations are approximately equal.

Example 11: Find y for $x = 0.1$ using modified Euler's method for the differential equation $\dfrac{dy}{dx} = \log(x + y)$ with $y(0) = 1$

Solution: Given $f(x, y) = \log(x + y)$

$$x_0 = 0, y_0 = 1, h = 0.1$$

To find y_1, $x_1 = 0.1$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)\log(0 + 1) = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= y_0 + \frac{h}{2}\left[\log(x_0 + y_0) + \log\left(x_1 + y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1)]$$

$$= 1 + \frac{0.1}{2}[\log 1 + \log 1.1] = 1.0047$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= y_0 + \frac{h}{2}\left[\log(0 + 1) + \log\left(x_1 + y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1.0047)] = 1.0049$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1.0049)] = 1.0049$$

$$= 1.0049$$

$$\therefore \quad y_1 = 1.0049$$

Runge–Kutta methods

First order Runge–Kutta Method $y_1 = y_0 + hy_0'$ [same as Euler's method]

Second order Runge–Kutta method The formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$

Third order Runge–Kutta Method The formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$k_3 = hf(x_0 + h, y_0 + k')$ where $k' = hf(x_o + h, y_o + k_1)$

Fourth-order Runge–Kutta Method The formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Example 12: Given $\dfrac{dy}{dx} = x^2 + y^2$, $y(1) = 1.2$

Find $y(1.05)$ applying fourth order Runge–Kutta method

Solution: $f(x, y) = x^2 + y^2, x_0 = 1, y_0 = 1.2$

$h = 0.05$

$$\therefore \quad k_1 = hf(x_0, y_0) = (0.05)\,[x_0^2 + y_0^2]$$
$$= (0.05)\,[1^2 + (1.2)^2] = 0.122$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.05)\,[f(x_0 + 0.025, y_0 + 0.061]$$
$$= (0.05)\,[f(1.025, 1.261)]$$
$$= (0.05)\,[(1.025)^2 + (1.261)^2] = 0.1320$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.05)f(1 + 0.025, 1.2 + 0.066)]$$
$$= (0.05)f(1.025, 1.266)$$
$$= (0.05)\,[(1.025)^2 + (1.266)^2] = 0.1326$$
and $k_4 = hf(x_0 + h, y_0 + k_3)$
$$= (0.05)f(1 + 0.05, 1.2 + 0.1326)$$
$$= (0.05)f(1.05, 1.3326)$$
$$= (0.05)\,[(1.05)^2 + (1.3326)^2] = 0.1439$$
$$\therefore \quad y_1 = y(1.05) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1.2 + \frac{1}{6}[0.122 + 2(0.1320) + 2(0.1326) + 0.1439]$$
$$= 1.2 + \frac{1}{6}[0.7951] = 1.3325$$

Predictor–corrector methods

Milne's predictor formula

$$y_{n+1}^p = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$$

Milne's corrector formula

$$y_{n+1}^c = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}^p\right]$$

Adams–Bashforth predictor formula

$$y_{n+1}^p = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

Adams–Moulton corrector formula

$$y_{n+1}^c = y_n + \frac{h}{24}\left[9f_{n+1}^p + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

EXERCISE

Practice Problems

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Three of the roots of the equation $x^4 + \ell x^3 + mx^2 + nx + 24 = 0$ are 3, 1 and –2. Which of the following could be the value of $\ell + m - n$?
(A) 0 (B) 1
(C) 2 (D) 3

2. If one of the roots of the equation $x^3 + 5x^2 - 12x - 36 = 0$ is thrice another root, then the third root is:
(A) – 6 (B) 3
(C) –2 (D) – 89/13

3. If the equation $x^6 + 5x^5 + 11x^4 + 34x^2 + 20x + 24 = 0$ has exactly four non-real roots, then the number of negative roots is:
(A) 1 (B) 0
(C) 3 (D) 2

4. A student finds, by trial, two negative and one positive root(s) of the equation $x^5 + 5x^4 + 2802x + 3024 = 103x^3 + 329x^2$. How many non–real roots does the equation have?
(A) 0 (B) 1
(C) 2 (D) 4

5. If the equation $3x^4 - 13x^3 + 7x^2 + 17x + a - 10 = 0$ has exactly three positive roots, then a can be:
(A) 11 (B) 4
(C) 13 (D) 12

6. If two of the roots of the equation $x^3 + 3x^2 - 10x - 24 = 0$ are such that one is twice the other, then the third root is
(A) –4 (B) –3
(C) –2 (D) 3

7. Which of the following is an algebraic equation?
(A) $4x^3 - 3x^2 + 10x - 1 = 0$
(B) $\sin x + xe^x = 0$
(C) $x^4 + \log x - 11 = 0$
(D) $x^2e^{-x} = 3$

8. The Transcendental equation of the following is:
(A) $x^2 - 2x^5 + 9 - x = 0$
(B) $4x - \sin^2 x + \log x = 0$
(C) $2x^3 - 3x^2 + 12x = 0$
(D) None of these

9. Let $f(x) = x^3 - x - 5 = 0$. By bisection method first two approximations x_0 and x_1 are 1.5 and 2.25 respectively, then x_2 is:
(A) 1.625 (B) 1.875
(C) 1.999 (D) None of these

10. The second approximation of the positive root of the equation $2x - \cos x = 0$ between 0 and 0.5 using Bisection method is:
(A) 0.405 (B) 0.258
(C) 0.375 (D) None of these

11. Find the fourth approximation of the root of the equation $x^3 + x - 11 = 0$, between 2 and 3, using Bisection method.
(A) 1.925 (B) 2.832
(C) 2.5215 (D) 2.0625

12. If the first two approximations x_0 and x_1 to a root of $x^3 - x - 4 = 0$ are 1.666 and 1.780 respectively, then find x_2 by Regula–Falsi method.
(A) 1.974 (B) 1.794
(C) 1.896 (D) None of these

13. Find the second approximation to the root of the equation $2x - 5 = 3 \sin x$ between (2, 3) using the method of false position.
(A) 2.2523 (B) 2.012
(C) 2.8804 (D) None of these

14. The third approximation to the root of $x^2 - 2\log_e x - 10 = 0$ by the method of false position is:
(A) 3.5392 (B) 3.7856
(C) 3.9310 (D) None of these

15. For $N = 28$ and $x_0 = 5.5$, the first approximation to $\sqrt{N}$ by Newton's iteration formula is:
(A) 5.295 (B) 5.582
(C) 5.396 (D) None of these

16. Find the second approximation to the cube root of 24 correct to three decimal places using Newton's iterative formula.
(A) 2.695 (B) 2.885
(C) 3.001 (D) None of these

17. Find the reciprocal of 22 using Newotn-Raphson method.
(A) 0.0454545 (B) 0.4504504
(C) 0.54054 (D) None of these

18. If the first approximation of the root of $x^3 - 3x - 5 = 0$ is $(x_0 =)2$, then find x_1 by Newton–Raphson method.
(A) 2.2806 (B) 2.2790
(C) 2.3333 (D) None of these

19. Find the first approximation of the real root by Newton–Raphson method for $x^4 + x^3 - 7x^2 - x + 5 = 0$ by taking $x_0 = 2$.
(A) 2.066 (B) 2.981
(C) 2.819 (D) None of these

20. If $\frac{dy}{dx} = 1 - 3xy^2$, $y(0) = 0$, then by Taylor's method $y(0.1) =$
(A) 0.02 (B) 0.001
(C) 0.05 (D) 0.1

21. If $\frac{dy}{dx} = 2x + y$, $y(0) = 1$, the Picard's approximate of y upto second degree terms is:
(A) $1 + x + x^2$ (B) $1 + x + \frac{x^2}{2}$
(C) $1 - x - \frac{x^2}{2}$ (D) None of these

22. If $y_0 = 1, f(x_0, y_0) = 1.2, f(x_1, y_1^{(0)}) = 1.9312, h = 0.3$, by modified Euler's formula $y_1^{(1)} =$
(A) 1.4696 (B) 1.2015
(C) 1.325 (D) 1.525

23. Using Euler's modified method, find a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with $y(0) = 1$ at $x = (0.2)$.
(A) 1.3902 (B) 1.2309
(C) 1.3092 (D) None of these

24. Find k_1, by Runge–Kutta method of fourth order if $\frac{dy}{dx} = 2x + 3y^2$ and $y(0.1) = 1.1165$, $h = 0.1$.
(A) 0.3993 (B) 0.9393
(C) 0.3939 (D) None of these

25. If $y(0) = 1$, $h = 0.5$ and $k_1 = 0.8$, find k_2 by Runge–Kutta method of order 4, given $y' = \frac{y^2 - 3x^2}{y^2 + 3x^2}$.
(A) 0.2416 (B) 0.0052
(C) 0.6184 (D) 0.4126

26. Find y (0.8) by Milne's predictor formula, given $\frac{dy}{dx} = x - y^2$, $y_2 = 0.0795$, $y(0.6) = 0.1762, y_0 = 0.0000$, $y_1' = 0.1996, y_2' = 0.3937, y_3' = 0.5689, h = 0.2$
(A) 0.9304 (B) 0.4930
(C) 0.3049 (D) None of these

27. For the above problem find $y(0.8)$ using Milne's corrector formula.
(A) 0.3046 (B) 0.4036
(C) 0.436 (D) None of these

28. Find using the Adams-Bash forth corrector formula y (0.4), for the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$, given $y(0.1) = 1.01$, $y(0.2) = 1.022$, y (0.3) $= 1.023, y_0' = 0, y_1' = 0.0505, y_2' = 0.1022, y_3' = 0.1535$
(A) 1.5418 (B) 1.0410
(C) 1.4100 (D) None of these

29. For the above differential equation find $y(0.5)$ using Adams–Bashforth predictor formula
(A) 1.00463 (B) 1.06463
(C) 1.00599 (D) None of these

30. The Runge–Kutta methods has the error of order _____
(A) 1 (B) 3
(C) 5 (D) 2

Previous Years' Questions

1. The algebraic equation $F(s) = s^5 - 3s^4 + 5s^3 - 7s^2 + 4s + 20$ is given. $F(s) = 0$ has: **[2006]**

(A) A single complex root with the remaining roots being real

(B) One positive real root and four complex roots, all with positive real parts

(C) One negative real root, two imaginary roots, and two roots with positive real parts

(D) One positive real root, two imaginary roots and two roots with negative real parts

2. The differential equation $\frac{dx}{dt} = \frac{1-x}{\tau}$ is discretized using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discrete time equation? **[2007]**

(A) 1 (B) $\frac{\tau}{2}$

(C) τ (D) 2τ

3. Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then, after one step of Newton's method, estimate x_1, of the solution will be given by **[2008]**

(A) 0.71828

(B) 0.36784

(C) 0.20587

(D) 0.00000

4. A differential equation $\frac{dx}{dt} = e^{-2t}u(t)$ has to be solved using trapezoidal rule of integration with a step size $h = 0.01$ *s*. Function $u(t)$ indicates a unit step function. If $x(0-) = 0$, then value of x at $t = 0.01$ s will be given by **[2008]**

(A) 0.00099 (B) 0.00495

(C) 0.0099 (D) 0.0198

5. A cubic polynomial with real coefficients: **[2009]**

(A) Can possibly have no extreme and no zero crossings

(B) May have up to three extreme and up to 2 zero crossings

(C) Cannot have more than two extreme and more than three zero crossings

(D) Will always have an equal number of extreme and zero crossings

6. Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton–Raphson's method is given by **[2009]**

(A) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$ (B) $x_{k+1} = x_k - \frac{117}{x_k}$

(C) $x_{k+1} = x_k - \frac{x_k}{117}$

(D) $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

7. Roots of the algebraic equation $x^3 + x^2 + x + 1 = 0$ are **[2011]**

(A) $(+1, +j, -j)$ (B) $(+1, -1, +1)$

(C) $(0, 0, 0)$ (D) $(-1, +j, -j)$

8. Solution of the variables x_1 and x_2 for the following equations is to be obtained by employing the Newton–Raphson iterative method.

Equation (i) $10x_2 \sin x_1 - 0.8 = 0$

Equation (ii) $10X_2^2 - 10x_2 \cos x_1 - 0.6 = 0$

Assuming the initial values $x_1 = 0.0$ and $x_2 = 1.0$, the Jacobian matrix is **[2011]**

(A) $\begin{bmatrix} 10 & -0.8 \\ 0 & 0.6 \end{bmatrix}$ (B) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & -0.8 \\ 10 & 0.6 \end{bmatrix}$ (D) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$

9. When the Newton–Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is **[2013]**

(A) -0.82 (B) 0.49

(C) 0.705 (D) 1.69

10. The function $f(x) = e^x - 1$ is to be solved using Newton–Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at 2nd iteration is_____ **[2014]**

Answer Keys

Exercise

Practice Problems

1. D	**2.** B	**3.** D	**4.** A	**5.** B	**6.** D	**7.** A	**8.** B	**9.** B	**10.** C
11. D	**12.** B	**13.** C	**14.** A	**15.** A	**16.** B	**17.** A	**18.** C	**19.** A	**20.** D
21. A	**22.** A	**23.** B	**24.** C	**25.** D					
26. C	**27.** A	**28.** B	**29.** B	**30.** C					

Previous Years' Questions

1. C	**2.** D	**3.** A	**4.** C	**5.** C	**6.** A	**7.** D	**8.** B	**9.** C

10. 0.05 to 0.07

Chapter 7

Transform Theory

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Laplace transform
- Some important (theorems) properties of laplace transform
- Inverse Laplace transforms
- Fourier transforms
- Convolution of two functions
- Z-transform
- Inverse Z-transform
- Difference equations
- Order of a difference equation
- Linear difference equation

Laplace Transform

Let $f(t)$ be a given function defined for all $t \geq 0$. The Laplace transform of $F(t)$ is denoted by $L\{f(t)\}$ or $L\{f\}$ and is defined as $L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$

Here L is Laplace transform operator. $F(t)$ is the determining function depends on it. $F(s)$ is the function to be determined called generating function. e^{-st} is called kernel of the transform.

Some standard results of Laplace transforms are given below.

1. $L\{e^{at}\} = \dfrac{1}{s-a}, s > a$
2. $L\{e^{-at}\} = \dfrac{1}{s+a},$
3. (i) Let k be a constant $L\{k\} = \dfrac{k}{s}$

   (ii) $L\{1\} = \dfrac{1}{s}, s > 0$
4. $L\{t^n\} = \dfrac{n!}{s^{n+1}}, S > 0$
5. $L\{\cos at\} = \dfrac{s}{s^2 + a^2}, s > 0$
6. $L\{\sin at\} = \dfrac{a}{s^2 + a^2}, s > 0$
7. $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}, s > |a|$
8. $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}, s > |a|$
9. $L\{t^n \cdot e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$
10. $L\left\{\dfrac{1}{t^n} f(t)\right\} = \int_s^\infty \cdots_{(n \text{ times})} \int_s^\infty F(s)ds$

Solved Examples

Example 1: Find the Laplace transform of the function

$$F(x) = 5e^{2x} + 7e^{-3x}$$

Solution: $L\{F(x)\} = L(5e^{2x} + 7e^{-3x})$

$$= 5L(e^{2x}) + 7L(e^{-3x})$$

$$L\{F(t)\} = 5 \cdot \frac{1}{s-2} + 7 \cdot \frac{1}{s+3}$$

$$= \frac{5}{s-2} + \frac{7}{s+3}$$

Example 2: Find $L\{F(t)\}$ where $F(t) = 0, 0 < t < 1$

$= 1, 1 < t < 2$

$= t, t > 2$

Solution: As the given function is not defined at $t = 0$, 1 and 2

$$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)dt$$

$$= \int_0^1 e^{-st} \cdot 0dt + \int_1^2 e^{-st} \cdot 1dt + \int_2^\infty e^{-st} \cdot tdt$$

$$= \int_1^2 e^{-st}\,dt + \int_2^\infty e^{-st}\cdot t\,dt$$

$$= \frac{e^{-st}}{-s}\Big]\int_1^2 + t\cdot\frac{e^{-st}}{-s}\Big]\int_2^\infty - \int_2^\infty \frac{e^{-st}}{-s}\cdot dt$$

$$= -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s} + \frac{1}{s}\frac{e^{-st}}{-s}\Big]_2^\infty$$

$$= \frac{-e^{-2s}}{s} + \frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s} + \frac{1}{s^2}e^{-2s}$$

$$= \frac{e^{-2s}}{s}\left(1+\frac{1}{s}\right) + \frac{e^{-s}}{s}$$

Example 3: Find the Laplace Transform of the function

$$F(t) = \sin 2t,\ 0 < t < \pi$$
$$= 0,\ t > \pi$$

Solution: $L\{F(t)\} = \int_0^\infty e^{-st}\cdot F(t)dt$

$$= \int_0^\pi e^{-st}\cdot \sin 2t\ dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$$

$$= \int_0^\pi e^{-st}\sin 2t\,dt$$

$$= \frac{e^{-st}}{s^2+4}[-s\sin 2t - 2\cos 2t]\Big]_0^\pi$$

$$= \frac{2(1-e^{-\pi s})}{s^2+4}$$

Example 4: Find the Laplace transform of the function $F(t) = (\sin t + \cos t)^2$

Solution: $L\{\sin t + \cos t)^2\} = L\{1+\sin 2t\} = L\{1\} + L\{\sin 2t\}$

$$= \frac{1}{s} + \frac{2}{s^2+4}$$

Some Important (Theorems) Properties of Laplace Transform

Linear property

Let f and g be any two functions of t and a_1, a_2 are constants, then $L\{a_1 f(t) + a_2 g(t)\} = a_1 L\{f(t)\} + a_2 L\{g(t)\}$

Shifting property

If $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t) = F(s-a)$

Example: $L\{e^{at}\cos ct\} = \dfrac{s-a}{(s-a)^2+c^2}$

Change of scale property

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \left|\frac{1}{a}\right| F\left(\frac{s}{a}\right)$

Example: We know $L\{e^{at}\} = \dfrac{1}{s-a} = F(s)$

Then $L\{be^{at}\} = \dfrac{1}{|b|}F\left(\dfrac{s}{b}\right)$

$$= \frac{1}{|b|}\frac{1}{\frac{s}{b}-a} = \frac{1}{|b|}\cdot\frac{b}{s-ab}$$

Differentiation theorem

If derivatives of $f(t)$ are continuous and $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = Sf(S) - f(o)$ and

$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-3}f''(0)\ldots f^{n-1}(0)$

$= s^nF(s)\sum_{r=0}^{n-1} s^{n-1-r}\cdot f^r(0)$ (f^r represents rth derivative of f)

Multiplication theorem

If $L\{f(t)\} = F(s)$ then $L\{t\cdot f(t)\} = -F'(s)$

And $L\{t^n\cdot f(t)\} = (-1)^n\dfrac{d^n}{ds^n}[F(s)]$

Division theorem

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{1}{t}f(t)\right\} = \int_s^\infty F(s)ds$

Transforms of integrals (theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(u)du\right\} = \dfrac{1}{s}F(s)$

Example 5: Find the Laplace transform of $te^{-2t}\sin^2 t$.

Solution: $L\{\sin^2 t\} = \dfrac{1}{2}L\{1-\cos 2t\} = \dfrac{1}{2}\left\{\dfrac{1}{s} - \dfrac{s}{s^2+4}\right\}$

$$\therefore\ L\{t\cdot\sin^2 t\} = (-1)\frac{d}{ds}\left(\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}\right)$$

(Using multiplication theorem)

$$L\{t\sin^2 t\} = \frac{-1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+4)-s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{1}{2s^2} + \frac{4-s^2}{(s^2+4)^2}$$

$$L\{e^{-2t}\cdot t\sin^2 t\} = \frac{1}{2(s+2)^2} + \frac{4-(s+2)^2}{[(s+2)^2+4]^2}$$

(Using shifting property)

$$= \frac{1}{2(s+2)^2} - \frac{4s+s^2}{(s^2+4s+8)^2}$$

Example 6: Find the Laplace transform of $\dfrac{\sin 2t - \cos 2t}{t}$.

Solution: $L\{\sin 2t - \cos 2t\} = \dfrac{2}{s^2+4} - \dfrac{s}{s^2+4}$

$$L\left\{\frac{\sin 2t-\cos 2t}{t}\right\}=\int_s^\infty\left(\frac{2}{s^2+4}-\frac{s}{s^2+4}\right)ds$$

(Using division property)

$$=\frac{2}{2}\left(\tan^{-1}\frac{s}{2}\right)_s^\infty-\frac{1}{2}[\log(s^2+4)]_s^\infty$$
$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}+\frac{1}{2}\log(s^2+4)$$
$$=\cot^{-1}\frac{s}{2}+\frac{1}{2}\log(s^2+4)$$

Example 7: Find the Laplace transform of $\int_0^t\frac{\sin 2u}{u}du$

Solution: $L\{\sin 2u\}=\frac{2}{s^2+4}$ and $L\left\{\frac{\sin 2u}{u}\right\}=\int_s^\infty\frac{2}{s^2+4}ds$

(Using division theorem)

$$\frac{2}{2}\tan-1\frac{s}{2}\Big]_s^\infty=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}=\cot^{-1}\frac{s}{2}$$
$$\therefore L\left\{\int_0^t\frac{\sin 2u}{t}dt\right\}=\frac{1}{s}\cot^{-1}\frac{s}{2}$$

(Using transform of integration theorem)

Inverse Laplace Transforms

If $F(s)$ is the Laplace transform of the function $f(t)$, i.e., $L\{f(t)\}=F(s)$ then $f(t)$ is called the inverse Laplace transform of the function $F(s)$ and is written as $(t)=L^{-1}\{F(s)\}$. Here L^{-1} is called inverse Laplace transformation operator.

Some important standard results for inverse Laplace transform.

1. $L^{-1}\left(\frac{1}{s}\right)=1$
2. $L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!}$ where n is a positive integer

   or $L^{-1}\left(\frac{1}{s^n}\right)=\frac{t^{n-1}}{(n-1)!}$
3. $L^{-1}\left(\frac{1}{s-a}\right)=e^{at}$
4. $L^{-1}\left(\frac{1}{(s-a)^n}\right)=\frac{e^{at}t^{n-1}}{(n-1)!}$
5. $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$
6. $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$
7. $L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$
8. $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$
9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$
11. $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}(\sin at-at\cos at)$
12. $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}=\frac{1}{2a}t\sin at$

To find the inverse Laplace transform we use the following methods.

Using the following properties

1. If $L^{-1}\{F(s)\}=f(t)$, then $L^{-1}\{F(s-a)\}=e^{at}f(t)$
2. If $L^{-1}\{F(s)\}=f(t)$ and $f(0)=0$;
   Then
   (i) $L^{-1}\{sF(s)\}=\frac{d}{dt}(f(t))$
   (ii) $L^{-1}\{s^nF(s)\}=\frac{d^n}{dt^n}(f(t))$; if $f(0)=f'(0)=\ldots f^{(n-1)}(0)=0$
3. If $L^{-1}\{F(s)\}=f(t)$, then
   (i) $L^{-1}\left\{\frac{F(s)}{s}\right\}=\int_\circ^t f(t)dt$
   (ii) $L^{-1}\left\{\frac{F(s)}{s^2}\right\}=\int_\circ^t\left\{\int_\circ^t f(t)dt\right\}$

Convolution theorem

Let $f(t)$ and $g(t)$ be two functions and $L^{-1}\{f(s)\}=f(t)$ and $L^{-1}\{G(s)\}=g(t)$, then $L^{-1}\{f(s)\cdot G(s)\}=\int_0^t f(x)g(t-x)dx$ It is denoted by $f(t)*g(t)$ here $*$ represents convolution.

Unit step function

This function is defined as $u(t-a)H(t-a)=\begin{cases}0 & t<a\\ 1 & t\geq a\end{cases}$

the Laplace transform of $H(t-a)=L\{H(t-a)\}$
$=\int_0^\infty e^{-st}u(t-a)dt=\frac{e^{-as}}{s}$

Note: This is also called as Heavisides unit function

Periodic function

If $f(t)$ is a periodic function with period a i.e., $f(t+a)=f(t)$, then

$$L\{f(t)\}=\frac{\int_0^a e^{-st}f(t)dt}{1-e^{-sa}}$$

Using partial fractions

If $F(s)$ is of the from $\frac{G(s)}{H(s)}$ where G and H are polynomials in S then break $F(s)$ into partial fractions and manipulate term by term.

Heavisides expansion formula

Let $F(s)$ and $G(s)$ be two polynomials in 's' where $F(s)$ has degree less than that of $G(s)$. If $G(s)$ has n distinct zeros ar, $r = 1, 2, 3, \ldots n$ i.e., $G(s) = (s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)$, then

$$L^{-1}\left[\frac{F(s)}{G(s)}\right] = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G^1(\alpha_r)} e^{\alpha_r t}$$

Transform of Special Functions

Bessel function

$$J_0(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Then $L\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}}$

Error function

Error function is denoted as erf (t)

$$erf(t)\ er\ f(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2}\, dt,$$

Then $L\{erf(\sqrt{x})\} = \frac{1}{s\sqrt{s+1}}$

Complex inversion (theorem) formula

If $f(t)$ has a continuous derivative and is of exponential order and $L\{f(t)\} = F(s)$ then $L^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} F(s) ds,\ t > 0 \text{ and } f(t) = 0 \text{ for } t < 0$$

Notes:

1. The above result is also known as Brounwich's integral formula
2. The integration is to be performed along a line $s = r$ in the complex plane where $s = x + iy$. The real number r is chosen so that $p = r$ lies to the right of all the singularities.

The gamma function

If $n > 0$, then the gamma function is defined by

$$\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u}\, du$$

Exponential Integral

The exponential integral is denoted by

$$E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du$$

Example 8: Evaluate $L^{-1}\left\{\frac{e^{2-3s}}{(s+2)^{5/2}}\right\}$

Solution: We have $L^{-1}\left\{\frac{1}{(s+2)^{5/2}}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$

$$= e^{-2t} \frac{t^{\frac{5}{2}-1}}{\Gamma\left(\frac{5}{2}\right)} = \frac{2t^{\frac{3}{2}} e^{-2t}}{3\sqrt{\pi}}$$

$$\therefore L^{-1}\left\{\frac{e^{2-3s}}{(s+2)^{5/2}}\right\} = e^2 L^{-1}\left\{\frac{e^{-3s}}{(s+2)^{5/2}}\right\}$$

$$= \left\{\begin{matrix} e^2 \frac{4}{3\sqrt{\pi}} (t-3)^{3/2} \cdot e^{-2(t-3)}, t > 3 \\ 0, t < 3 \end{matrix}\right\}$$

$$= \frac{4}{3\sqrt{\pi}} (t-3)^{3/2} e^{-2(t-4)},\ t > 3 = 0, t < 3$$

$$= \frac{4}{3\sqrt{\pi}} (t-3)^{3/2} e^{-2(t-4)},\ H(t-3)$$

(When expressed in terms of Heaviside unit step function)

Example 9: Evaluate $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Solution: $L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$

$$= L^{-1}\left\{\frac{3(s-1)}{(s-1)^2-4} + \frac{10}{(s-1)^2-4}\right\}$$

$$= 3L^{-1}\left(\frac{s-1}{(s-1)^2-4}\right) + 10L^{-1}\left\{\frac{1}{(s-1)^2-4}\right\}$$

$$= 3e^t L^{-1}\left\{\frac{s}{s^2-2^2}\right\} + 10L^{-1}\left\{\frac{1}{s^2-2^2}\right\} e^t.$$

$$= 3e^t \cosh 2t + 5e^t \sinh 2t = 4e^{3t} - e^{-t}.$$

Example 10: Evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Solution: $L^{-1}\left\{\frac{1}{s^2} \cdot \frac{s}{(s^2+4)^2}\right\}$

Let $F_1(s) = \frac{1}{s^2}$ and $F_2(s) = \frac{s}{(s^2+4)^2}$ so that

$$L^{-1}\{F_1(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = f_1(t)$$

And $L^{-1}F_2(s) = L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

$$= \frac{t \cdot \sin 2t}{4} = f_2(t) \text{ (say)}$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\} = L^{-1}\{F_1(s)\cdot F_2(s)\}$$

$$= \int_0^t f_2(x) f_1(t-x)dx$$

$$= \int_0^t \frac{x}{4}\sin 2x(t-x)dx$$

$$= \frac{t}{4}\int_0^t x\sin 2x dx - \frac{1}{4}\int_0^t x^2\sin 2x dx$$

$$= \frac{t}{4}\left(-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right)_0^t$$

$$-\frac{1}{4}\left(-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)_0^t$$

$$= \frac{1}{16}(1 - t\sin 2t - \cos 2t)$$

Application of Laplace Transforms to Solutions of Differential Equations

Solution of ordinary differential equations with constant co-efficients:

Consider a linear differential equation with constant co-efficients

$$(D^n + C_1D^{n-1} + C_2D^{n-2} + \cdots + (C_{nt})\, y = F(t) \qquad (1)$$

where $F(t)$ is a function of the independent variable t

Let $(0) = A_1, y'(0) = A_2, \ldots, y^{n-1}(0) = A_{n-1}$ (2)

be the given initial or boundary conditions where A_1, A_2 ... A_{n-1} are constants.

By taking the Lapalce transform on both sides of (1) and using the conditions (2), we obtain an algebraic equation known as subsidiary equation from which $y(s) = L\{y(t)\}$ is determined. The required solution is obtained by finding the inverse Laplace transform of $y(s)$

Example 11: Solve $(D + 3)^2 y = 9e^{-3t}$, $y(0) = -1$ and $y'(0) = 9$.

Solution: The given equation can be written as

$(D^2 + 6D + 9)\, y = 9e^{-3t}$

applying Laplace transform we get

$L\{y''\} + 6L\{y'\} + 9L\{y\} = 9L\{e^{-3t}\}$

Or $s^2L\{y\} - sy(0) - y'(0) + 6[sL\{y\} - y(0)]$

$$+ 9L\{y\} = \frac{9}{s+3}$$

Or $s^2L\{y\} + s - 9 + 6sL\{y\} + 6 + 9L\{y\} = \frac{9}{s+3}$

$$= (s^2 + 6s + 9)\, L\{y\} = \frac{9}{s+3} - s + 3$$

$$(s+3)^2 L\{y\} = \frac{18 - s^2}{s+3}$$

$$L\{y\} = \frac{18 - s^2}{(s+3)^3}$$

$$\therefore y = L^{-1}\left\{\frac{9 - (s+3)^2 + 6(s+3)}{(s+3)^3}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{9 - s^2 + 6s}{s^3}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{9}{s^3}\right\} - L^{-1}\left\{\frac{1}{s}\right\} + 6L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$y = e^{-3t}\left(9\cdot\frac{t^2}{2!} - 1 + 6t\right)$$

$\therefore$ The required solution is

$$y = \frac{e^{-3t}}{2}(9t^2 + 12t - 2)$$

Fourier Transforms

Integral Transform

An integral transform $F(s)$ of the function $f(t)$ is defined by an integral of the form $\int_a^b k(s,t) f(t)dt = F(s)$ where $k(s, t)$, is a function of two variables s and t, and is called the KERNEL of the integral transform.

Fourier Transform

The Fourier transform of the Function $f(x)$ is defined as $F(p)$ and is given by

$F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$, and the inverse Fourier transform of $F(p)$ is $f(x)$ and is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)e^{-ipx}dp$$

Sufficient conditions for the existence of Fourier transform

1. $f(x)$ is piecewise continuous on every finite interval
2. $f(x)$ is absolutely integrable on the x-axis

The finite and infinite Fourier sine transforms

1. The infinite Fourier sine transform of $f(x)$ where $0 < x < \infty$ is $Fs(p)$ which is defined to be

   $F_s(p) = \int_0^{\infty} f(x)\sin px\, dx$ and

   $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(p)\sin px\, dp$ is called the inverse infinite Fourier transform of $Fs(p)$.

2. The finite Fourier sine transform of $f(x)$ when $0 < x < l$, is defined as

$F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$, where $n \in Z$, and

$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\cdot\sin\left(\frac{n\pi x}{l}\right)$ is called the inverse finite Fourier sine transform of $Fs(x)$.

The finite and infinite Fourier cosine transform

1. The infinite Fourier cosine transform of $f(x)$ where $0 < x < \infty$ is defined as

$F_c(p) = \int_0^{\infty} f(x)\cos pxdx$ and

$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(p)\cos pxdp$ is called the inverse infinite Fourier cosine transform of $F_c(p)$.

2. The finite Fourier cosine transform of $f(x)$ when $0 < x < l$ is defined as

$F_c(x) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$, and

$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{l}\right)$ where $n \in z$, is the inverse finite Fourier cosine transform of $F_c(n)$

Properties of Fourier Transform

Linearity property

If $F(p)$ and $G(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then, $F(af(x) + bg(x)] = aF(p) + bG(p)$, where a and b are constants.

Change of scale property

If $F(p)$ is the Fourier transform of $f(x)$, then $F[f(ax)] = \frac{1}{a}F\left(\frac{p}{a}\right), a > 0$

Shifting property

If $F(p)$ is the Fourier transform of $f(x)$, the $F(f(x - a)] = e^{ipa}\cdot F(p)$

Modulation property

If $F(p)$ is the Fourier transform of (x), then

$$F[f(x)\cos ax] = \frac{1}{2}[F(p + a) + F(p - a)]$$

Fourier Integrals of f(x)

1. The Fourier integral of $f(x)$ is defined as

$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\cos p(t - x) f(t)\, dtdp$

The Fourier sine integral of $f(x)$ is defined as

$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin px\int_0^{\infty} f(t)\sin ptdtdp$

2. The Fourier cosine integral of $f(x)$ is defined as

$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos px\int_0^{\infty} f(t)\cos ptdtdp$

Some other important results

1. If $F_s(p)$ and $F_c(p)$ are the Fourier sine and cosine transforms of $f(x)$ respectively, then

(i) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(p + a) + F_s(p - a)]$

(ii) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(p + a) - F_s(p - a)]$

(iii) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(p - a) - F_c(p + a)]$

(iv) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a + p) + F_c(a - p)]$

2. $F\{x^n f(x)\} = (-i)^n\frac{d^n}{dp^n}[F(p)]$

3. $F\left\{\frac{d^n}{dx^n}f(x)\right\} = (-ip)^n\cdot F(p)$

4. (i) $F_s\{f'(x)\} = -pF_c(p)$
(ii) $F_c\{f'(x)\} = -f(0) + p\cdot F_s(p)$

Some miscellaneous points

1. The Fourier transform and its inverse are called a transform pair
2. The graph of $(p, |F(p)|)$ is called the amplitude spectrum of $f(x)$, where $F(p)$ is the Fourier transform of $f(x)$ and 'p' is called the frequency of the transform
3. Gamma function is defined as $\Gamma(n) = \int_0^x e^{-x}x^{n-1}dx$

Notes:

1. $\Gamma(1/2) = \sqrt{\pi}$
2. $\Gamma(n + 1) = n\,\Gamma(n)$ for, $n > 0$
3. $\Gamma(n + 1) = n!$ When, n is a positive integer
4. $\Gamma(p)\Gamma(1 - p) = \frac{\pi}{\sin p\pi}[0 < p < 1]$

4. Beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx; [m,n > 0]$$

Notes:

1. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
2. $\Gamma(m)\Gamma(m + 1/2) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$
3. $\int_0^{\frac{\pi}{2}}\sin^p x\cos^q xdx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Example 15: Find the Fourier transform of

$$f(x) = \begin{cases}2, & |x| < 2\\ 0, & |x| > 2\end{cases}$$

Solution: We have, $F(f(x)) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-\infty}^{-2} e^{ipx} f(x)dx + \int_{-2}^{2} e^{ipx} f(x)dx + \int_{2}^{\infty} e^{ipx} f(x)dx$$

$$= \int_{-2}^{2} (2)e^{ipx}dx = 2\left[\frac{e^{ipx}}{ip}\right]_{-2}^{2}$$

$$= \frac{2}{ip}[e^{2ip} - e^{-2ip}] = \frac{4}{ip}\sin h(2ip)$$

$$= \frac{2}{p}\sin 2p [\because \sin h(ix) = i\sin x$$

$$\therefore F[f(x)] = \frac{8\sin p\cos p}{p}$$

Example 16: Find the Fourier sine and cosine transforms of e^{-mx}, $m > 0$, and obtain the values of

1. $\int_0^{\infty} \frac{\cos px}{m^2 + p^2} dp$
2. $\int_0^{\infty} \frac{p\sin px}{m^2 + p^2} dp$

Solution: $F_s(e^{-mx}) = \int_0^{\infty} e^{-mx} \sin px\, dx$

$$= \left[\frac{e^{-mx}}{m^2 + p^2}\{-m\sin px - p\cos px\}\right]_0^{\infty}$$

$$F_s\{e^{-mx}\} = \frac{p}{m^2 + p^2}$$

Also $F_c\{e^{-mx}\} = \int_0^{\infty} e^{-mx} \cos px\, dx$

$$= \left[\frac{e^{-mx}}{m^2 + p^2}(-m\cos px + p\sin px)\right]_0^{\infty}$$

$$\therefore \quad F_c(e^{-mx}) = \frac{m}{m^2 + p^2}$$

Deductions:

1. Using inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(f(x))\sin pxdp$$

$$e^{-mx} = \frac{2}{\pi}\int_0^{\infty} F_s(e^{-mx})\sin px\, dp$$

$$\Rightarrow \quad e^{-mx} = \frac{2}{\pi}\int_0^{\infty} \frac{p}{m^2 + p^2}\sin px\, dp$$

$$\therefore \quad \int_0^{\infty} \frac{p\sin px}{m^2 + p^2} dp = \frac{\pi}{2}e^{-mx}$$

2. Similarly, using Fourier cosine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(f(x))\cos pxdp$$

$$e^{-mx} = \frac{2}{\pi}\int_0^{\infty} F_c(e^{-mx})\cos px\, dp$$

$$\Rightarrow \quad e^{-mx} = \frac{2}{\pi}\int_0^{\infty} \frac{m}{m^2 + p^2}\cos px\, dp$$

$$\therefore \quad \int_0^{\infty} \frac{\cos px}{m^2 + p^2} dp = \frac{\pi}{2m}e^{-mx}$$

Example 17: Find the Fourier transform of $e^{-5|x|}$ and hence deduce that $F\left\{x \cdot e^{-5|x|}\right\} = \frac{20ip}{(25 + p^2)^2}$

Solution: By definition, we have

$$F\{e^{-5|x|}\} = \int_{-\infty}^{\infty} e^{ipx} \cdot e^{-5|x|}dx$$

$$= \int_{-\infty}^{\infty} e^{-5|x|}[\cos px + i\sin px]dx$$

$$= \int_{-\infty}^{\infty} e^{-5|x|}\cos pxdx + i\int_{-\infty}^{\infty} e^{-5|x|}\sin pxdx$$

$$= 2\int_0^{\infty} e^{-5|x|}\cos pxdx + 0$$

$$= 2\left[\frac{5}{5^2 + p^2}\right] = \frac{10}{25 + p^2}$$

[$\therefore$ The first integrand is an even function and the second is an odd function]

Deduction:
We know that

$$F\{x^n f(x)\} = (-i)^n \frac{d^n F(p)}{dp^n}$$

$$\therefore \quad F\{xe^{-5|x|}\} = (-i)\frac{d}{dp}F[e^{-5|x|}]$$

$$= (-i)\frac{d}{dp}\left[\frac{10}{25 + p^2}\right]$$

$$= -i(10)\left[\frac{-1}{(25 + p^2)^2}\right](2p)$$

$$= \frac{20pi}{(25 + p^2)^2}$$

Hence proved.

Example 18: Find the finite Fourier cosine transform of $f(p) = \frac{\pi}{3} - p + \frac{p^2}{2\pi}$, where $x \in (0, \pi)$

Solution: By def, $F_c(x) = \int_0^l f(p)\cos\frac{n\pi p}{l} dp$

$$= \int_0^{\pi}\left(\frac{\pi}{3} - p + \frac{p^2}{2\pi}\right)\cos npdp$$

Integrating by parts,

$$= \left[\left(\frac{\pi}{3} - p + \frac{p^2}{2\pi}\right)\frac{\sin np}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin np}{n}\left(-1 + \frac{p}{\pi}\right)dp$$

$$= \frac{1}{n^2} - \frac{1}{n^2\pi}\int_0^{\pi} \cos np\,dp = \frac{1}{n^2} \text{ if } n > 0$$

If $n = 0$,

$$F_c(n) = \int_0^{\pi}\left(\frac{\pi}{3} - p + \frac{p^2}{2\pi}\right)dp = \frac{\pi^2}{3} - \frac{\pi^2}{2} + \frac{\pi^2}{6} = 0$$

Convolution of Two Functions

The convolution or falting of any two integrable functions $f(x)$ and $g(x)$ over the interval $(-\infty,\infty)$ is denoted by $f * g$ is defined as

$$h(x) = f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

Convolution theorem of fourier transforms

If $F(p)$ and $G(p)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms i.e., $[f(x) * g(x)] = F(p)\,G(p)$

Parseval's identity of fourier transforms

If $F(p)$ and $G(p)$ are Fourier Transforms of $f(x)$ and $g(x)$ respectively, then

1. $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\overline{G(p)}dp = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$
2. $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2 dp = \int_{-\infty}^{\infty} |f(x)|^2 dx$

Where bar represents the complex conjugate

Notes:

1. $\frac{2}{\pi}\int_0^{\infty} F_c(p)G_c(p)dp = \int_0^{\infty} f(x)g(x)dx$
2. $\frac{2}{\pi}\int_0^{\infty} F_s(p)G_s(p)dp = \int_0^{\infty} f(x)g(x)dx$
3. $\frac{2}{\pi}\int_0^{\infty} |F_c(p)|^2 dp = \int_0^{\infty} |f(x)|^2 dx$
4. $\frac{2}{\pi}\int_0^{\infty} |F_s(p)|^2 dp = \int_0^{\infty} |f(x)|^2 dx$

Where F_s and F_C denote the Fourier sine transform and the Fourier cosine transform

Z-transform

The technique of Z-transform is useful in solving difference equations and can be applied in the area of communication systems, digital signal processing, control systems, linear time in variant systems and digital fitters.

The Z-transforms are defined not on functions of continuous arguments but on special functions, i.e., sequences [sequence is a function whose domain is the set of natural numbers]. Of the discrete integer-valued arguments, i.e., $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

The Z-transform must be applied on sequences defined only for $n \geq 0$ and the sequence for $n < 0$, if any do not affect the Z-transform.

Definition: If the function $f(n)$ defined for discrete values of $n = 0, 1, 2, 3, \ldots$ and $f(n) = 0$ for $n < 0$, then the Z-transform $Z[f(n)] = \sum_{n=0}^{\infty} f(n)\cdot z^{-n} = f(z)$ for all Z for which $F(z)$ converges. This Z-transform is known as one sided or unilateral Z-transform.

Notes:

1. If the Z-transform is defined for both positive and negative values of n, it is said to be Bilateral Z-transform
2. The region in which $F(z)$ converges is said to be the region of convergence (ROC) of the Z-transform.

Some standard Z-transforms

1. $Z(1) = \sum_{n=0}^{\infty} \cdot 1 \cdot z^{-n} = \sum_{n=0}^{\infty} \frac{1}{z^n} = 1 + \frac{1}{z} + \cdots$

$$= \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

2. $Z(a^n) = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots = \left(1 - \frac{a}{z}\right)^{-1} = \frac{z}{z-a}.$$

3. $Z(\ 1)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n} = 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots$

$$= \left(1 + \frac{1}{z}\right)^{-1} = \frac{z}{z+1}.$$

4. $Z(n) = \sum_{n=0}^{\infty} n \cdot z^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right] = \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} = \frac{z}{(z-1)^2}.$$

5. $Z(na^n) = \sum_{n=0}^{\infty} n \cdot a^n \cdot z^{-n} = \sum_{n=0}^{\infty} n\left(\frac{a}{z}\right)^n$

$$= \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= \frac{a}{z}\left[1 - \frac{a}{z}\right]^{-1} = \frac{a}{z}\left[\frac{z^2}{(z-a)^2}\right] = \frac{az}{(z-a)^2}.$$

6. **Recurrence formula**

$$Z(n^p) = -z \cdot \frac{d}{dz} z(n^{p-1})$$

Note:

1. put $p = 1$, $Z(n) = \frac{z}{(z-1)^2}$.
2. put $p = 2$, $Z(n^2) = \frac{z^2 + z}{(z-1)^3}$
3. put $p = 3$, $Z(n^3) = \frac{z(z^2 + 4z + 1)}{(z-1)^4}$.

7. $Z\left(\frac{1}{n!}\right) = e^{1/z}$

8. $Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$.

9. $Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot z^{-n} = z \cdot \log\left(\frac{z}{z-1}\right)$.

10. $Z\left(\frac{1}{(n+1)!}\right) = Z\left[e^{1/z} - 1\right]$

11. Z-transform of discrete unit step function: The discrete unit step function $f(n)$ is given by,

$$f(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0. \end{cases}$$

$$\therefore Z[f(n)] = \frac{z}{z-1}, |z| < 1.$$

12. **Z-transform of discrete unit impulse function:** The discrete unit impulse function

$$\phi(n) \begin{cases} = 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\therefore Z[\phi(n)] = 1.$$

13. $Z(\cos n\theta) = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$

14. $Z(\sin n\theta) = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

15. $Z(a^n \cos n\theta) = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$

16. $Z(a^n \sin n\theta) = \frac{az \sin\theta}{z^2 - 2az\cos\theta + a^2}$

Properties of Z-transforms

1 **Linearity property:** If a, b are any constants and $f(n)$ and $g(n)$ be any discrete functions, then $Z[af(n) + bg(n)] = aZ[(n)] + bZ[g(n)]$

2 **Change of scale property (or) Dampling rule:** If $Z[f(n)] = F(z)$ then

(i) $Z[a^{-n} f(n)] = F(az)$ and

(ii) $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$.

3. **Shifting properties:**

If $z[f(n)] = F(z)$, then

(i) First shifting property: $Z[f(n-k)] = z^{-k} F(z)$.

(ii) Second shifting property:

$$Z[f(n+k)] = z^k [F(z) - f(o) - f(1)z^{-1} - f(2) \cdot z^{-2} \ldots - f(k-1) \cdot z^{-(k-1)}]$$

Initial value theorem

If $Zf(n) = F(z)$, then $\underset{z \to \infty}{\text{Lt}} F(z) = f(0)$.

Note:

$$f(1) = \underset{z \to \infty}{Lt}\, z[F(z) - f(0)]$$

$$f(2) = \underset{z \to \infty}{\text{Lt}}\, z^2[F(z) - f(0) - f(1) \cdot z^{-1}]$$

$$f(3) = \underset{z \to \infty}{\text{Lt}}\, z^3[F(z) - f(0) - f(1) \cdot z^{-1} - f(2) \cdot z^{-2}]$$ and so on.

Final value theorem

If $Z[f(n)] = F(z)$ then $\underset{n \to \infty}{\text{Lt}} f(n) = \underset{z \to 1}{\text{Lt}} (z-1)F(z)$.

Multiplication and division by n

If $Z[f(n)] = F(z)$ then

1. $Z[nf(n)] = -z \cdot \frac{d}{dz}[F(z)]$
2. $Z\left[\frac{f(n)}{n}\right] = -\int \frac{F(z)}{z}\, dz$.

Inverse Z-transform

It is the process of determining the sequence which generates the given Z-transform.

The inverse Z-transform of $F(z)$ is denoted by $Z^{-1}[F(z)] = f(n)$.

Methods of Evaluation of inverse Z-transform

Method I: To find the inverse of $F(z)$ we can make use of the known standard formulae.

For Example, we know that $Z(n) = \frac{z}{(z-1)^2}$.

$$\therefore Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n.$$

Method II: Partial fraction method: This method for inverting the rational transforms consists of resolving $\frac{F(z)}{z}$ into partial fractions, multiplying the resulting expansion by z and then finding the inverse of the same.

Method III: Power-series method: This is the simplest of all the methods. The given $F(z)$ can be expanded into a power series of the form $F(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n}$ which converges in the given R.O.C. i.e., $Z^{-1}[F(z)] = \alpha_n$.

Method IV: Contour Integration Method: By this method, the poles of the function $F(z).\ z^{n-1}$ within the contour drawn in the region of convergence and residues at these poles are to be determined.

Thus $f(n) = \frac{1}{2\pi i}\oint F(z)\cdot z^{n-1}dz.$

Method V: Convolution Theorem: If $Z^{-1}[F(z)] = f(n)$ and $Z^{-1}[G(z)] = g(n)$ then $Z^{-1}[F(z).\ G(z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)\cdot g(n-m)$ where * denotes the convolution operation.

Difference Equations

The difference equations are used for the treatment of discrete processes just as the differential equations are used for dealing with continuous processes.

Definition: An equation which involves difference functions at one or more values of the argument is said to be a difference equation.

Example: $\Delta y_{(n+1)} + y_{(n)} = 5$ … (1) is a difference equation. Here Δ is 'increment' and (+1), (n) are arguments of the equation. Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(+1)}$ which is the forward difference operator, (1) takes the form $y_{n+2} - y_{n+1} + y_n = 5$.

Order of a difference equation

The order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

For Example: for the equation $y_{n+2} - y_{n+1} - y_n = 4$. The order is

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2.$$

Linear difference equation

A linear difference equation is that in which y_{n+1}, y_{n+2} etc occur in the first degree only and are not multiplied together. A linear difference equation with constant co-efficients is of the form $y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + \ldots + a_k y_n = f(n)$ where a_1, a_2, … a_k are constants.

Applications to difference equations The Z-transforms are quite useful for solving linear difference equations.

Working Procedure To solve a given linear difference equation with constant coefficients by Z-transforms.

Step 1: Take Z-transform on both sides of the given difference equation using the following formulae and then apply the given conditions.

Formulae: $Z[y_{n+1}] = z[y(z) - y_0]$

$$Z[y_{n+2}] = z^2[y(z) - y_0 - y_1 z^{-1}]$$

$z[y_{n+3}] = z^3[y(z) - {}_0 - y_1 z^{-1} - y_2 z^{-2}]$ and so on.

Step 2: Simplify the above equation and find $F(z)$ and then resolve into partial fractions, if necessary.

Step 3: Take the inverse-Z-transform on both sides of $F(z)$ and find the solution y_n which is the desired solution.

Example 19: Find $z[(\cos k\theta + i\sin k\theta)^n]$.

Hence evaluate (a) $Z(\cos kn\theta)$ (b) $Z(\sin kn\theta)$.

Solution: $Z[(\cos k\theta + i\sin k\theta)^n] = Z[(e^{ki\theta})^n]$

$$= \frac{z}{z - e^{ki\theta}}\left[\because Z(a^n) = \frac{z}{z-a}\right]$$

Using Demoivre's theorem,

$$Z\left[\cos kn\theta + i\sin kn\theta\right] = \frac{z}{z - e^{ki\theta}} \times \frac{z - e^{-ki\theta}}{z - e^{-ki\theta}}$$

$$= \frac{z(z - e^{-ki\theta})}{z^2 - z(e^{ki\theta} + e^{-ki\theta}) + 1}$$

$$= \frac{z\left(z - \cos k\theta + i\sin k\theta\right)}{z^2 - 2z\cos k\theta + 1}$$

$$= \frac{z\left(z - \cos k\theta\right) + \left(zi\sin k\theta\right)}{z^2 - 2z\cos k\theta + 1}$$

Comparing the real and imaginary parts,

$$Z(\cos kn\theta) = \frac{z^2 - z\cos k\theta}{z^2 - 2zk\cos\theta + 1} \text{ and}$$

$$Z(\sin kn\theta) = \frac{z\sin k\theta}{z^2 - 2zk\cos\theta + 1}.$$

Example 20: Find the Z-transform of

(i) $(-7)^n$ and (ii) $\frac{5^n}{n!}e^{-5}$.

Solution:

(i) we know that

$$Z(a^n) = \frac{z}{z-a}.$$

Put a = –7,

$$\Rightarrow \quad Z[(-7)^n] = \frac{z}{z-(-7)} = \frac{z}{z+7}.$$

(ii) $Z\left[\frac{5^n}{n!}e^{-5}\right]$

$$\sum_{n=0}^{\infty}\frac{5^n}{n!}e^{-5}\cdot z^{-n} = e^{-5}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{z}\right)^n}{n!}$$

$$= e^{-5}\left[1 + \frac{1}{1!}\left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + \cdots\right]$$

$$= e^{-5}\,e^{5/z} = e^{5\,[1/z\,-\,1]}.$$

Example 21: Using the linearity property, find the Z-transform of $7\sin\frac{n\pi}{3} + 9t^2 - 3n^2$.

Solution: $Z\left[7\sin\frac{n\pi}{3}+9t^2-3n^2\right]$

$$= 7Z\left[\sin\frac{n\pi}{3}\right]+9t^2\cdot Z(1)-3Z(n^2)$$

$$= 7\frac{z\cdot\sin\frac{\pi}{3}}{z^2-2z\cos\frac{\pi}{3}+1}+9t^2\left(\frac{z}{z-1}\right)-3\frac{z^2+z}{(z-1)^3}$$

$$= \frac{7z\cdot\sqrt{\frac{3}{2}}}{z^2-2z\left(\frac{1}{2}\right)+1}+\frac{9t^2z}{z-1}-\frac{3(z^2+z)}{(z-1)^3}$$

$$= \frac{7\sqrt{3}z}{2(z^2-z+1)}+\frac{9zt^2}{z-1}-\frac{3(z^2+z)}{(z-1)^3}.$$

Example 22: Using the change of scale property (or damping rule) show that $z\,[n^2\cdot e^{-mn}]=\dfrac{z\cdot e^{-m}(z+e^{-m})}{(z-e^{-m})^3}$

Solution: Let $f(n)=n^2$ then $Z[f(n)]=\dfrac{z^2+z}{(z-1)^3}=F(z)$.

By damping rule,

$$Z[n^2\cdot e^{-mn}]=Z[(e^m)^{-n}\cdot n^2]=F[e^m\cdot z]$$

$$=\frac{(e^m\cdot z)^2+e^mz}{(e^mz-1)^3}=\frac{e^{2m}z[z+e^{-m}]}{e^{3m}(z-e^{-m})^3}$$

$$=\frac{ze^{-m}(z+e^{-m})}{(z-e^{-m})^3}$$

Example 23: Find the Z-transform of 3^{3p+4}.

Solution: Let $f(p)=3^{3p+4}=3^{3p}\cdot 81$

$$\therefore Z[f(p)]=Z[81\cdot 3^{3p}]=81\cdot Z[3^{3p}]=81\sum_{p=0}^{\infty}3^{3p}z^{-p}$$

$$=81[1+3^3\cdot z^{-1}+3^6z^{-2}+3^9\cdot z^{-3}+\cdots]$$

$$=81\,[1+3^3\cdot z^{-1}+(3^3\cdot z^{-1})^2+(3^3\cdot z^{-1})^3+\cdots]$$

$$=\frac{81}{1-3^3\cdot z^{-1}}=\frac{81}{1-\frac{27}{z}}=\frac{81z}{z-27}.$$

Example 24: Find $Z[5\cdot 8^n-11\cdot n]$ and deduce $Z[5\cdot 8^{n+4}-11(n+4)]$ using shifting theorem.

Solution: Let $f(n)=5\cdot 8^n-11\cdot n$. By linearity property,

$$Z\,[5\cdot 8^n-11n]=5\cdot Z\,[8^n]-11\cdot Z[n]$$

$$=5\cdot\frac{z}{z-8}-11\frac{z}{(z-1)^2}=\frac{5z}{z-8}-\frac{11z}{(z-1)^2}=F(z).\ \text{(Say)}$$

Also, $f(0)=5, f(1)=29, f(2)=298, f(3)=2527$. By shifting theorem, $Z[f(n+4)]$

$$=z^4\,[F(z)-f(0)-f(1)\,z^{-1}-f(2)\,z^{-2}-f(3)\cdot z^{-3}]$$

$$\therefore z\,[f(n+4)]$$

$$=Z[5\cdot 8^{n+4}-11(n+4)]$$

$$=z^4\left[\frac{5z}{z-8}-\frac{11z}{(z-1)^2}-5-\frac{29}{z}-\frac{298}{z^2}-\frac{2527}{z^3}\right].$$

Example 25: If $f(z)=\dfrac{5z^2-8z+6}{(z-1)^3}$, $|z|>1$, then find $f(2)$ using initial value theorem.

Solution: Given $f(z)=\dfrac{z^2\left[5-\frac{8}{z}+\frac{6}{z^2}\right]}{z^3\left[1-\frac{1}{z}\right]^3}$.

$$\therefore\ f(z)=\frac{1}{z}\left[\frac{5-8z^{-1}+6z^{-2}}{(1-z^{-1})^3}\right]$$

$\therefore$ By initial value theorem, $f(0)=\underset{z\to\infty}{\text{Lt}}\ f(z)=0$.

$$f(1)=\underset{Z\to\infty}{\text{Lt}}\ z[f(z)-f(0)]$$

$$=\underset{z\to\infty}{\text{Lt}}\ z\left[\frac{5-\frac{8}{z}+\frac{6}{z^2}}{z\left(1-\frac{1}{z}\right)}-0\right]=5.$$

$$f(2)=\underset{z\to\infty}{\text{Lt}}\ z^2[f(z)-f(0)-f(1)\cdot z^{-1}]$$

$$=\underset{z\to\infty}{\text{Lt}}\ z^2\left[\frac{5z^2-8z+6}{(z-1)^3}-0-\frac{5}{z}\right]$$

$$=\underset{z\to\infty}{\text{Lt}}\ z^2\left[\frac{5z^3-8z^2+6z-5(z-1)^3}{z(z-1)^3}\right]$$

$$=\underset{z\to\infty}{\text{Lt}}\ z^2\left[\frac{5z^3-8z^2+6z-5z^3+15z^2-15z+5}{z(z-1)^3}\right]$$

$$=\underset{z\to\infty}{\text{Lt}}\ z^2\,\frac{7z^2-9z+5}{z(z-1)^3}$$

$$=\underset{z\to\infty}{\text{Lt}}\ z^4\,\frac{7-\frac{9}{z}+\frac{5}{z^2}}{z^4\left(1-\frac{1}{z}\right)^3}=7.\quad\therefore\quad f(2)=7.$$

EXERCISE

Practice Problems

Directions for questions 1 to 50: Select the correct alternative from the given choices.

1. Laplace transform of $2\sin^2 2t =$ _____.

(A) $\frac{1}{s}+\frac{1}{s^2+16}$ (B) $\frac{s}{s^2+16}$

(C) $\frac{1}{s}-\frac{s}{s^2+16}$ (D) $\frac{1}{s}-\frac{1}{s^2+16}$

2. The Laplace transform of $(t+1)^3$ is _____.

(A) $\frac{6-6s+3s^2-s^3}{s^3}$ (B) $\frac{6+6s+3s^2+s^3}{s}$

(C) $\frac{6(1+s+s^2+s^3)}{s^4}$ (D) $\frac{6+6s+3s^2+s^3}{s^4}$

3. Laplace transform of $e^{-2t}\cos^2 2t$ is _____.

(A) $\frac{s^2+4s+12}{(s+2)(s^2+4s+20)}$ (B) $\frac{4s^2+12}{(s+2)(s^2+4s+20)}$

(C) $\frac{s^2-4s+12}{(s-2)(s^2+4s+20)}$ (D) $\frac{s^2-4s-12}{(s+2)(s^2-4s+12)}$

4. The value of $L\{\sinh 3t\cos 3t\}$ _____.

(A) $\frac{s^2+18}{s^4+81}$ (B) $\frac{s^2+18}{s^4+324}$

(C) $\frac{3(s^2-18)}{s^4+324}$ (D) $\frac{3(s^2+18)}{s^4-324}$

5. The value of $L\{t^2\cos 3t\}$ is _____.

(A) $\frac{s^2-27}{(s^2+9)^4}$ (B) $\frac{2s(s^2-27)}{(s^2+9)^3}$

(C) $\frac{s^3-27}{(s^2+9)^4}$ (D) $\frac{s(s^3-27)}{(s^2+9)^3}$

6. Laplace transform of $\frac{\cos 4t}{t}$ _____.

(A) $\frac{64}{s^2+16}$ (B) $\frac{16}{(s^2+16)^2}$

(C) $\frac{8}{(s^2+16)^2}$ (D) Does not exist

7. If $f(t)=t;\ 0<t<3$ and $f(t+3)=f(t)$, then $L\{f(t)\}$ is

(A) $\frac{1}{s^2(1-e^{-3s})}[1+e^{3s}+e^{-3s}]$

(B) $\frac{1}{s(1-e^{-3s})}[1-e^{-3s}+se^{-3s}]$

(C) $\frac{1}{s^2(1-e^{-3s})}[1-e^{-3s}-3se^{-3s}]$

(D) $\frac{1}{s(1-e^{-3s})}[1-e^{-3s}-se^{-3s}]$

8. The value of $\int_0^\infty \frac{e^{-4t}-e^{-8t}}{t}dt$ is _____.

(A) log 2 (B) log 4

(C) log 8 (D) log 6

9. $\int_0^\infty t.e^{-2t}\sin 3t dt =$ _____.

(A) $\frac{5}{169}$ (B) $\frac{10}{169}$

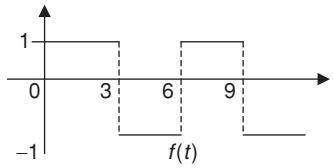
(C) $\frac{6}{169}$ (D) $\frac{12}{169}$

10. The Laplace transform of the function defined by f $f(t)=\begin{cases}2, & 0<t<1\\ 1, & t>1\end{cases}$ is _____.

(A) $\frac{2-e^{-s}}{s}$ (B) $\frac{2-e^{-s}}{2}$

(C) $\frac{2+e^{-s}}{s}$ (D) $\frac{2+e^{-s}}{2}$

11. Represent the following in terms of a periodic function between $t=0$ to $t=6$



(A) $f(t)=3,\ t<3$
$=-3,\ t>3$

(B) $f(t)=1,\ t<3$
$=-1,\ t>3$

(C) $f(t)=1,\ 0\le t<3,$
$=-1,\ 3\le t<6$

(D) None of these

12. The Laplace transform of the function in the above question is _____.

(A) $\frac{1-e^{-3s}}{1+e^{-3s}}$ (B) $\frac{1}{s}\left(\frac{1-e^{-3s}}{1+e^{-3s}}\right)$

(C) $\frac{1+e^{-3s}}{1-e^{-3s}}$ (D) $\frac{1}{s}\left(\frac{1+e^{-3s}}{1-e^{-3s}}\right)$

13. $L\{tJ_0(at)\} =$ _____.

(A) $\frac{s}{(s^2+a^2)^{\frac{3}{2}}}$ (B) $\frac{s}{(s^2-a^2)^{\frac{3}{2}}}$

(C) $\frac{1}{(s^2+a^2)^{\frac{3}{2}}}$ (D) None of these

14. $\int_0^\infty J_0(t)dt =$ _____.

(A) 0 (B) 1
(C) s (D) None of these

15. $L\{erf(3\sqrt{t})\} =$ _____.

(A) $\frac{1}{s\sqrt{s+3}}$ (B) $\frac{9}{s\sqrt{s+3}}$
(C) $\frac{3}{s\sqrt{s-9}}$ (D) $\frac{3}{s\sqrt{s+9}}$

16. $L\{e^{2t}erf\sqrt{t}3\} =$ _____.

(A) $\frac{3}{s\sqrt{s+9}}$ (B) $\frac{9}{(s-2)\sqrt{s+9}}$
(C) $\frac{3}{(s-2)\sqrt{s+7}}$ (D) $\frac{9}{(s+2)\sqrt{s+7}}$

17. The inverse Laplace transform of $\left(\frac{9}{s^{\frac{9}{2}}}\right)$ is _____.

(A) $\frac{16}{105}\sqrt{\frac{t^7}{\pi}}$ (B) $\frac{8}{15}\sqrt{\frac{t^5}{\pi}}$
(C) $\frac{16}{35}\sqrt{\frac{t}{\pi}}$ (D) $\frac{8}{105}\sqrt{\frac{t^7}{\pi}}$

18. The value of $L^{-1}\left\{\frac{8}{3s-2}-\frac{4+2s}{16s^2-25}\right\}$ is _____.

(A) $\frac{8}{3}\sinh\frac{5t}{4}-\cosh\frac{5t}{4}$

(B) $\frac{8}{3}e^{\frac{2}{3t}}-\sinh\frac{5t}{4}-\cosh\frac{5t}{4}$

(C) $\frac{8}{3}e^{\frac{2}{3t}}-\frac{1}{5}\sinh\frac{5t}{4}-\frac{1}{8}\cosh\frac{5t}{4}$

(D) None of these

19. The value of $L^{-1}\left\{\left(\frac{\sqrt{s}+2}{s}\right)^2+\frac{5}{2s+1}\right\}$ is

(A) $1+4t+\frac{5}{2}e^{\frac{-1}{2t}}+4\sqrt{\frac{t}{\pi}}$

(B) $t+\frac{5}{2}e^{\sqrt{t}}+\frac{4}{\sqrt{\pi}}$

(C) $1-4t-\frac{5}{2}e^{-t}-4\sqrt{\frac{t}{\pi}}$

(D) $1-t+\frac{5}{2}e^{\frac{-t}{2}}+4\sqrt{\frac{t}{\pi}}$

20. The inverse Laplace transform of $\frac{1}{s^2-8s+20}$ is _____.

(A) $\frac{e^{2t}}{2}\sin 2t$ (B) $\frac{e^{4t}}{2}\sin 2t$
(C) $e^{4t}\sin 2t$ (D) $e^{4t}\sin 4t$

21. The inverse Laplace transform of $\frac{e^{-3s}}{(s-4)^5}$ when expressed in terms of Heaviside unit step function is _____.

(A) $\frac{1}{6}t^4e^{4(t-3)}H(t-3)$

(B) $\frac{1}{24}(t-3)^4e^{4t}H(t-3)$

(C) $\frac{1}{24}(t-3)^4e^{4(t-3)}H(t-3)$

(D) $\frac{1}{24}t^4e^{4t}H(t-3)$

22. The inverse Laplace transforms of $\frac{1}{s^3\left(s^2+4\right)}$ is

(A) $\frac{1}{16}\left(2t^2+\cos 2t-1\right)$ (B) $2t^2-\cos 2t-1$
(C) $\frac{1}{16}(1-\cos 2t-4t^2)$ (D) $\frac{1}{8}(2+\cos 2t-4t^2)$

23. The inverse Laplace transforms of $\frac{2s^3-13s^2+34s-15}{(s^2-1)(s-3)^2}$ is _____.

(A) $e^t-2e^{-t}+3e^{3t}$ (B) $e^{-t}-2e^t-3e^{-3t}$
(C) $e^t+2e^{-t}-e^{3t}(1-3t)$ (D) None of these

24. The value of $L^{-1}\left\{\log\frac{s-4}{s+3}\right\}$ is

(A) $e^{4t}-e^{-3t}$ (B) $\frac{1}{t}(e^{4t}-e^{-3t})$
(C) $\frac{1}{t}(e^{-3t}-e^{4t})$ (D) $t(e^{-3t}-e^{4t})$

25. The value of $L^{-1}\left\{\frac{1}{s}\log\frac{s-4}{s+3}\right\}$ is

(A) $\int_0^t\frac{e^{4x}-e^{-3x}}{x}dx$ (B) $\int_0^t(e^{4x}-e^{-3x})$
(C) $\int_0^t x(e^{-3x}-e^{4x})$ (D) $\int_0^t\frac{e^{-3x}-e^{4x}}{x}dx$

26. Using convolution theorem, the value of $\int_0^t\sin x\cos(t-x)dx$ is _____.

(A) $\frac{t}{2}\cos t$ (B) $\frac{t}{2}\sin t$
(C) $t\sin\frac{t}{2}$ (D) $t\cos\frac{t}{2}$

27. The value of $L^{-1}\left\{\frac{1}{\sqrt{s}(s-2)}\right\}$ _____.

(A) $e^{t}erf(\sqrt{t})$ (B) $e^{\sqrt{2t}}erf\sqrt{2t}$

(C) $\frac{e^{2t}}{\sqrt{2}}erf(\sqrt{2t})$ (D) $e^{-2t}erf(\sqrt{2t})$

28. The value of 2 * 2 * 2 *… upto n times is

(A) $\frac{2^{n-1}t^{n-1}}{n-1}$ (B) $\frac{2^{n}t^{n-1}}{n-1}$

(C) $\frac{2^{n}t^{n-1}}{(n-1)!}$ (D) $\frac{2^{n}t^{n}}{n!}$

29. Solve $(D^4 - 16)\,y = 1, y = y' = y'' = y''' = 0$.

(A) $y = \frac{-1}{16} - [\cosh 2t + \sinh 2t]$

(B) $y = \frac{1}{32}(1 - \cosh 2t + \cos 2t)$

(C) $y = \frac{-1}{16} + \frac{1}{32}(\cosh 2t - \sin t)$

(D) $y = \frac{-1}{16} + \frac{1}{32}(\cosh 2t + \cos 2t)$

30. Solve $(D^2 - 5D + 6)\,y = 1 + e^{-2t}, y = 1, y' = 0$ when $t = 0$.

(A) $y = \frac{1}{20}e^{-2t} + \frac{11}{4}e^{2t} - \frac{59}{30}e^{3t}$

(B) $y = \frac{1}{6} - \frac{1}{20}e^{-2t} + \frac{11}{4}e^{2t} - \frac{28}{15}e^{3t}$

(C) $y = \frac{1}{6} - \frac{1}{20}e^{2t} + \frac{11}{4}e^{-2t} + \frac{59}{30}e^{3t}$

(D) $y = \frac{1}{6} - \frac{1}{20}e^{2t} - \frac{11}{4}e^{-2t} + \frac{59}{30}e^{3t}$

31. Find the Fourier transform of

$$f(x) = \begin{cases} r, -m < x < 0, r > 0 \\ s,\ 0 < x < m,\ s > 0 \\ 0, \text{otherwise} \end{cases}$$

(A) $\frac{1}{ip}[(r-s) + s \cdot e^{ipm} - re^{-ipm}]$

(B) $\frac{1}{ip}[(r+s) + s \cdot e^{im} + r \cdot e^{-ipm}]$

(C) $\frac{1}{ip}[(r-s) + s \cdot e^{ipm} + r \cdot e^{ipm}]$

(D) None of these

32. Find the Fourier sine transform of $\frac{3}{2x}$.

(A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{6}$ (D) None of these

33. The finite Fourier cosine transform of $f(y) = \left(\frac{\pi - y}{\pi}\right)^2$ where $0 < y < \pi$ for $p > 0$ and $p = 0$ are respectively,

(A) $\frac{2}{\pi^2 p}, \frac{\pi}{6}$ (B) $\frac{2}{\pi p}, \frac{\pi}{2}$

(C) $\frac{2}{\pi p^2}, \frac{\pi}{3}$ (D) None of these

34. Find the finite Fourier cosine transform of $f(x) = \frac{\pi}{6} - 2x + \frac{x^2}{\pi}$ when $p > 0$.

(A) $\frac{3}{p^2}$ (B) $\frac{2}{p^2}$

(C) $\frac{1}{p^2}$ (D) None of these

35. In the above question, the finite Fourier cosine transform when $p = 0$ is

(A) $\frac{\pi^2}{2}$ (B) $\frac{\pi^3}{3}$

(C) $\frac{-\pi^2}{2}$ (D) None of these

36. $\frac{d}{ds}F_s[\{f(x)\}] =$

(A) $F_c[xf(x)]$ (B) $F_s[xf(x)]$

(C) $F_c[f(x)]$ (D) None of these

37. If $F(s)$ is the Fourier transform of $f(x)$, then $F[f(x)\cos 3x] =$

(A) $\frac{1}{2}[F(s+3) + F(s+6)]$

(B) $\frac{1}{2}[F(s+3) + F(s-3)]$

(C) $\frac{1}{2}[F(s+3) - F(s-3)]$

(D) None of these

38. If $f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 3 - x, & 1 < x < 2, \\ 0, & x > 2 \end{cases}$, then $F_s[f(x)] =$

(A) $\frac{3\sin p - p\cos 2p - \sin 2p}{p^2}$

(B) $\frac{3\sin p + p\cos 2p + p\sin 2p}{p^2}$

(C) $\frac{3\cos p - p\sin 2p - p\sin p}{p^2}$

(D) None of these

39. Find the Fourier sine transform of $3e^{-2x} + 4e^{-5x}$.

(A) $\frac{4p}{p^2+4} + \frac{3p}{p^2+25}$ (B) $\frac{3p}{p^2+4} - \frac{4p}{p^2+25}$

(C) $\frac{3p}{p^2+4} + \frac{4p}{p^2+25}$ (D) None of these

40. If $Fs(p) = p^m e^{-cp}$, find $f(x)$.

(A) $\frac{2}{\pi} m! \frac{\sin m\theta}{(x^2+c^2)^{\frac{m}{2}}}$ (B) $\frac{2}{\pi} m! \frac{\sin(m+1)\theta}{(x^2+c^2)^{\frac{m+1}{2}}}$

(C) $\frac{2}{\pi} m! \frac{\sin(m+1)\theta}{(x^2+c^2)^{m+1}}$ (D) None of these

41. $\frac{2}{\pi}\int_0^\infty F_s(t)G_s(t)dt =$

(A) $\int_0^\infty f(x)g'(x)dx$ (B) $\int_0^\infty f(x)f(x)dx$

(C) $\int_0^\infty f(x)g(x)dx$ (D) $\int_0^\infty |g(x)|^2 dx$

42. $Z\left(\frac{1}{(n+2)!}\right) =$ _____.

(A) $z^2\left[e^{\frac{1}{z}} - 1 - \frac{1}{z}\right]$ (B) $z^2\left[e^{\frac{1}{z}} + 1 + \frac{1}{z}\right]$

(C) $z^2\left[\frac{1}{z} + 1 - e^{\frac{-1}{z}}\right]$ (D) None of these

43. If $Z[\sin m\alpha] = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$ (for $m \geq 0$), then for $m \geq 0$, $Z[e^m \cdot \sin m\alpha] =$ _____.

(A) $\frac{z\sin\alpha}{z^2 - 2ze\sin\alpha + e^2}$

(B) $\frac{ze\sin\alpha}{z^2 + 2ze\cos\alpha - e^2}$

(C) $\frac{ze\cos\alpha}{z^2 - 2ze\sin\alpha + e^2}$

(D) $\frac{ze\sin\alpha}{z^2 - 2ze\cos\alpha + e^2}$

44. $Z[\sin(4n+7)] =$ _____.

(A) $\frac{z^2\sin 7 + z\sin 3}{z^2 - 2z\cos 4 + 1}$ (B) $\frac{z^2\sin 7 - z\sin 3}{z^2 + 2z\cos 4 + 1}$

(C) $\frac{z^2\sin 7 + z\sin 3}{z^2 + 2z\cos 4 + 1}$ (D) $\frac{z^2\sin 7 - z\sin 3}{z^2 - 2z\cos 4 + 1}$

45. $Z[(\cos m\theta + i\sin m\theta)^n] =$

(A) $\frac{2(z - \cos m\theta) - iz\sin m\theta}{z^2 - 2z\cos m\theta + 1}$

(B) $\frac{z(z - \cos m\theta) + iz\sin m\theta}{z^2 - 2z\cos m\theta + 1}$

(C) $\frac{z(z + \cos m\theta) - iz\sin m\theta}{z^2 - 2z\cos m\theta + 1}$

(D) None of these

46. Using the above result, the values of $Z(\cos 5n\theta)$ and $Z(\sin 3n\theta)$ respectively are _____.

(A) $\frac{z(z - \cos 5\theta)}{z^2 - 2z\cos 5\theta + 1}, \frac{z\sin 3\theta}{z^2 - 2z\cos 3\theta + 1}$

(B) $\frac{z(z - \cos 3\theta)}{z^2 - 2z\cos 3\theta + 1}, \frac{z\sin 5\theta}{z^2 - 2z\cos 5\theta + 1}$

(C) $\frac{z(z + \cos 5\theta)}{z^2 + 2z\cos 5\theta + 1}, \frac{z\sin 3\theta}{z^2 + 2z\cos 3\theta + 1}$

(D) None of these

47. If $F(z) = \frac{7z^2 - 5z + 18}{(z-2)^4}$, then find $f(3)$.

(A) 61 (B) 15

(C) 51 (D) None of these

48. Evaluate the inverse Z-transform of $\frac{3z}{5z-3}$

(A) $(3/5)^n$ (B) $(3/5)^{n+1}$

(C) $(3/5)^{n-1}$ (D) None of these

49. $Z^{-1}\left[\frac{z^2}{z^2 - 11z + 18}\right] =$ _____.

(A) $\frac{1}{7}[9^{n+1} + 2^{n+1}]$ (B) $\frac{1}{7}[9^{n-1} + 2^{n-1}]$

(C) $\frac{1}{7}[9^{n+1} - 2^{n+1}]$ (D) None of these

50. If '$*$' represents the convolution operator, the value of $\frac{1}{j!} * \frac{1}{j!} * \frac{1}{j!} * \frac{1}{j!} =$ _____.

(A) $\frac{4^j}{j!}$ (B) $\frac{4^{j+1}}{(j+1)!}$

(C) $\frac{4^{j-1}}{(j-1)!}$ (D) None of these

Previous Years' Questions

1. The Laplace transform of a function $f(t)$ is $F(s)=\dfrac{5s^2+23s+6}{s(s^2+2s+2)}$. As $t \to \infty$, $f(t)$ approaches **[2005]**

(A) 3 (B) 5

(C) $\dfrac{17}{2}$ (D) ∞

2. If $u(t)$ is the unit step and $\delta(t)$ is the unit impulse function, the inverse z-transform of $F(z)=\dfrac{1}{z+1}$ for $k \geq 0$ is: **[2005]**

(A) $(-1)^k \delta(k)$ (B) $\delta(k)-(-1)^k$

(C) $(-1)^k U(k)$ (D) $U(k)-(-1)^k$

3. A function $y(t)$ satisfies the following differential equation $\dfrac{dy(t)}{dt}+y(t)=\delta(t)$:

Where $\delta(t)$ is the delta function. Assuming zero initial condition, and denoting the unit step function by $u(t)$, $y(t)$ can be of the form **[2008]**

(A) e^t (B) e^{-t}

(C) $e^t u(t)$ (D) $e^{-t}u(t)$

4. Let $x(t)=\text{rect}\left(t-\dfrac{1}{2}\right)$ (where rect $(x)=1$ (For $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$ and zero otherwise). Then if $\text{sinc}(x)=\dfrac{\sin(\pi x)}{\pi x}$, the Fourier Transform of $x(t)+x(-t)$ will be given by: **[2008]**

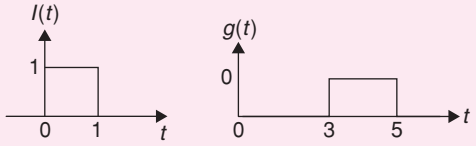
(A) $\text{sinc}\left(\dfrac{\omega}{2\pi}\right)$

(B) $2\,\text{sinc}\left(\dfrac{\omega}{2\pi}\right)$

(C) $2\,\text{sinc}\left(\dfrac{\omega}{2\pi}\right)\cos\left(\dfrac{\omega}{2}\right)$

(D) $\text{sinc}\left(\dfrac{\omega}{2\pi}\right)\sin\left(\dfrac{\omega}{2}\right)$

Common Data for Questions 5 and 6: Given $f(t)$ and $g(t)$ as shown below:



5. $g(t)$ can be expressed as: **[2010]**

(A) $g(t)=f(2t-3)$ (B) $g(t)=f\left(\dfrac{t}{2}-3\right)$

(C) $g(t)=\left(2t-\dfrac{3}{2}\right)$ (D) $g(t)=\left(\dfrac{t}{2}-\dfrac{3}{2}\right)$

6. The Laplace transform of $g(t)$ is: **[2010]**

(A) $\dfrac{1}{s}(e^{3s}-e^{5s})$ (B) $\dfrac{1}{s}(e^{-5s}-e^{-3s})$

(C) $\dfrac{e^{-3s}}{s}(1-e^{-2s})$ (D) $\dfrac{1}{s}(e^{5s}-e^{3s})$

7. Let $X(s)=\dfrac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal $x(t)$. Then, $x(0^+)$ is **[2014]**

(A) 0 (B) 3

(C) 5 (D) 21

8. Consider a signal defined by

$$x(t)=\begin{cases} e^{j10t} & \text{for } |t| \leq 1 \\ 0 & \text{for } |t|>1 \end{cases}$$

Its Fourier Transform is **[2015]**

(A) $\dfrac{2\sin(\omega-10)}{\omega-10}$ (B) $2e^{j10}\dfrac{\sin(\omega-10)}{\omega-10}$

(C) $\dfrac{2\sin\omega}{\omega-10}$ (D) $e^{j10\omega}\dfrac{2\sin\omega}{\omega}$

9. The Laplace Transform of $f(t) = e^{2t}\ sin(5t)\ u(t)$ is _____. **[2016]**

(A) $\dfrac{5}{s^2-4s+29}$ (B) $\dfrac{5}{s^2+2}$

(C) $\dfrac{s-2}{s^2-4s+29}$ (D) $\dfrac{5}{s+5}$

10. Let $f(x)$ be a real, periodic function satisfying $f(-x) = -f(x)$. The general from of its Fourier series representation would be **[2016]**

(A) $f(x)=a_0+\sum_{k=1}^{\infty}a_k\,Cos(kx)$

(B) $f(x)=\sum_{k=1}^{\infty}b_k\,Sin(kx)$

(C) $f(x)=a_0+\sum_{k=1}^{\infty}a_{2k}\,Cos(kx)$

(D) $f(x)=\sum_{k=0+1}^{\infty}a_{2k+1}Sin\,(2k+1)x$

Answer Keys

Exercise

Practice Problems

1. C	**2.** D	**3.** A	**4.** C	**5.** B	**6.** D	**7.** C	**8.** A	**9.** D	**10.** A
11. C	**12.** B	**13.** A	**14.** B	**15.** D	**16.** C	**17.** A	**18.** C	**19.** A	**20.** B
21. C	**22.** A	**23.** C	**24.** C	**25.** D	**26.** B	**27.** C	**28.** C	**29.** D	**30.** B
31. A	**32.** B	**33.** C	**34.** B	**35.** C	**36.** A	**37.** B	**38.** A	**39.** C	**40.** B
41. C	**42.** A	**43.** D	**44.** D	**45.** B	**46.** A	**47.** C	**48.** B	**49.** C	**50.** A

Previous Years' Questions

1. A	**2.** C	**3.** D	**4.** C	**5.** D	**6.** C	**7.** B	**8.** A	**9.** A	**10.** B

TEST
ENGINEERING MATHEMATICS

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. If 2, 3 and 5 are the eigen values of a 3×3 matrix A, then the eigen values of adjoint of A are _____
(A) 2, 3, 5 (B) 2, 6, 15
(C) 3, 10, 15 (D) 6, 10, 15

2. If a system of n linear equations in n unknowns $AX = B$, has a unique solution, then which of the following is TRUE?

P: Rank of the augmented matrix $[A/B]$ = Rank of $A = n$

Q: Determinant of A is non-zero real number.

R: Rank of the augmented matrix $[A/B]$ = Rank of $A < n$
(A) P only (B) P and R only
(C) Q and R only (D) P and Q only

3. If a fair coin is tossed 5 times, then what is the probability of getting at least two heads or at most two tails?
(A) $\frac{13}{16}$ (B) $\frac{3}{16}$
(C) $\frac{11}{32}$ (D) $\frac{21}{32}$

4. If the system of equations,

$x + 3y - 2z = 3$
$5x + 5y - z = 4$
$3x + ay + 3z = b$

is inconsistent, then
(A) $a = -1; b \neq -2$ (B) $a \neq -1; b = -2$
(C) $a = -1; b = -2$ (D) $a \neq -1; b \neq -2$

5. Let X and Y be two normal random variables with the following data

	X	Y
Mean	3	?
Standard Deviation	4	3

If $P(X \geq 5) = P(Y \leq 4)$, then the mean of Y is ____
(A) 5 (B) 5.5
(C) 6 (D) 6.5

6. The iterative formula to find a root of the equation $f(x) = 0$ by secant method is _____

(A) $x_{n+1} = \frac{x_n + x_{n-1}}{2}$

(B) $x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

(C) $x_{n+1} = x_n - \frac{f(x_n)}{f^{\#}(x_n)}$

(D) $x_{n+1} = \frac{x_n f(x_n) - x_{n-1} f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

7. For positive real values of x, the function $\ln(2x^3 - 15x^2 + 36x + 2)$ has a local maximum at
(A) 2 (B) 3
(C) 2.5 (D) 3.5

8. The call duration (in seconds) of 8 consecutive calls on a mobile number is 54, 53, 52, 56, 60, 57, 50, 57 respectively. What are the median and the mode for this data?
(A) 54 and 57 respectively
(B) 56 and 54 respectively
(C) 55 and 57 respectively
(D) 55 and 56 respectively

9. Evaluate $\int \frac{e^{3x}}{\sqrt{4 + e^{3x}}}\, dx$.
(A) $\ln(4 + e^{3x}) + c$ (B) $\frac{2}{3}\sqrt{(4 + e^{3x})} + c$
(C) $\frac{2}{3\sqrt{(4 + e^{3x})}} + c$ (D) None of these

10. Out of 300 students that appeared for GATE from an engineering college, $\frac{1}{3}rd$ are of EC stream, 40% are of ME stream and the rest are of CS stream. If one of them got 4th rank and another of them got 6th rank in GATE, then what is the probability that none of them are from EC or ME streams?
(A) 0.070 (B) 0.546
(C) 0.825 (D) 0.347

11. While numerically solving the differential equation $\frac{dy}{dx} = 1 + x + xy + y$, $y(0) = 2$ using Runge–Kutta method of second order with a step size 0.1, the value of y at $x = 0.2$ correct to four decimal places is _____.
(A) 2.6435 (B) 2.7366
(C) 2.5769 (D) 2.8433

12. Which of the following complex functions is not analytic at all values of $Z = x + iy$ at which $|Z|$ is finite
(A) $\sin Z$ (B) $\cos Z$
(C) $\tan Z$ (D) e^Z

13. The value of $\int_0^{\frac{\pi}{4}} \sin^3 4x \cos^4 2x\, dx$ is _____
(A) $\frac{1}{10}$ (B) $\frac{1}{15}$
(C) 4 (D) $\frac{1}{40}$

14. Given the Laplace transform,

$L[\cos at] = \frac{s}{s^2 + a^2}$, then $L[t \cos 3t]$ is _____

(A) $\frac{s^2-9}{s^2+9}$ (B) $\frac{1}{s^2+9}$

(C) $\frac{s^2-9}{(s^2+9)^2}$ (D) $\frac{(s^2-9)^2}{(s^2+9)^2}$

15. The possible values for eigen values of an idempotent matrix of any order are _____

(A) 0 or 1 (B) –1 or 1

(C) 0 or –1 (D) None of these

16. The area of the region bounded by the curves $y^2 = x - 2$ and $y = x - 8$ is

(A) $\frac{125}{3}$ (B) $\frac{125}{2}$

(C) $\frac{125}{6}$ (D) $\frac{75}{2}$

17. The slope of the tangent at (x, y) to a curve $y = f(x)$ passing through $\left(1, \frac{\pi}{4}\right)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then the equation of the curve is

(A) $\tan\frac{y}{x} - 2 + x = 0$ (B) $\tan y = x$

(C) $\tan\left(\frac{y}{x}\right) = \log\left(\frac{e}{x}\right)$ (D) $\tan\left(\frac{x}{y}\right) = \log\frac{x}{e}$

18. If $\bar{i} + \bar{j} + \bar{k}, \bar{i} + 3\bar{j} + 5\bar{k}, 7\bar{i} + 9\bar{j} + 11\bar{k}$ are three consecutive vertices of a parallelogram, then the angle between the diagonals is

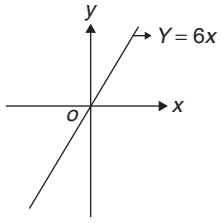
(A) $\cos^{-1}\left(\frac{17}{\sqrt{700}}\right)$ (B) $\sin^{-1}\left(\frac{17}{\sqrt{700}}\right)$

(C) $\cos^{-1}\left(\frac{17}{\sqrt{325}}\right)$ (D) $\cos^{-1}\left(\frac{11}{\sqrt{175}}\right)$

19. Evaluate $L = \int_{-1}^{2} (y^2 + 1)dx$ along the straight line given in the plot below.

(A) 98 (B) 111

(C) 121 (D) 132



20. The value of x, that satisfies the differential equation $\frac{dx}{dt} = 3t^2, x(0) = 3$

(A) is always finite.

(B) is finite when t is negative only.

(C) is finite when t is positive only.

(D) is finite, when t is finite only.

21. If z_1 and z_2 are any two complex numbers, such that $|Z_1| = 6$ and $\frac{1}{|Z_2|} = 5$, then $\left|\frac{Z_1}{Z_2}\right|$ is, (where $|Z|$ denotes the modulus of a complex number z) _____.

(A) 11 (B) 1

(C) $\frac{6}{5}$ (D) 30

22. The inverse Laplace transform of $\frac{s+3}{s^2+6s+13}$ _____

(A) $e^{3t}\cos 2t$ (B) $e^{-3t}\cos 2t$

(C) $e^{2t}\cos 3t$ (D) $e^{-3t}\cos 3t$

23. Identify the set of functions, which are linearly independent?

(i) $\sin x$, $\sin^2 x$ and $\cos^2 x$

(ii) $\cos 2x$, $\sin^2 x$ and $\cos^2 x$

(iii) $\cos 3x$, $\cos^3 x$ and $\cos x$

(iv) $\cos x$, $\sin x$ and $\tan x$

(A) (i) only (B) (ii) and (iii) only

(C) (i) and (iv) only (D) (iii) and (iv) only

24. Evaluate $\oint \bar{F} \cdot d\bar{r}$, where $\bar{F} = -y\bar{i} + x\bar{j}$ along the boundary of the rectangle formed by the lines $x = \pm 2$ and $y = \pm 3$ in xy-plane in anticlockwise direction.

(A) 12 (B) 24

(C) 36 (D) 48

25. If $z = a$ is an essential singularity of $f(z)$, then the residue of $f(z)$ at $z = a$ is given by _____

(A) The co-efficient of $(z - a)$ in the Laurent's series expansion of $f(z)$ about $z = a$

(B) The co-efficient of $(z - a)^{-1}$ in the Laurent's series expansion of $f(z)$ about $z = a$.

(C) The constant term in the Laurent's series expansion of $f(z)$ about $z = a$.

(D) The co-efficient of highest negative power of $(z - a)$ in the Laurent's series expansion of $f(z)$ about $z = a$.

Answer Keys

1. D	**2.** D	**3.** A	**4.** A	**5.** B	**6.** B	**7.** A	**8.** C	**9.** B	**10.** A
11. B	**12.** C	**13.** A	**14.** C	**15.** A	**16.** C	**17.** C	**18.** D	**19.** B	**20.** D
21. D	**22.** B	**23.** C	**24.** D	**25.** B					

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Electrical Engineering

PART III

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UNIT II:	Analog and Digital Electronics	3.91
UNIT III:	Electric Circuits and Fields	3.359
UNIT IV:	Electrical Machines	3.515
UNIT V:	Power Electronics and Drives	3.643
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Signals and Systems

UNIT I

Chapter 1: Introduction to Signals and Systems 3.5

Chapter 2: Laplace Transform 3.36

Chapter 3: z-Transform 3.50

Chapter 4: Continuous Time Fourier Analysis 3.67

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Chapter 1

Introduction to Signals and Systems

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Classification of signals
- Basic operations on signals
- Amplitude scaling
- Time scaling
- Elementary signals
- Exponentially damped sinusoidal signals
- Systems and classification of systems
- Linear time invariant systems
- Interconnected systems
- Properties of continuous time LTI systems
- Invertibililty
- Discrete time LTI systems

Introduction

A signal is defined as a function of one or more variables that conveys information on the nature of a physical phenomenon.

When the function depends on a single variable, the signal is said to be one dimensional (Example: Speech signal), otherwise it is said to be multidimensional (Example: Image).

A system is defined as an entity that manipulates one or more signals to accomplish a function, there by yielding new signals, (Example: Communication system).

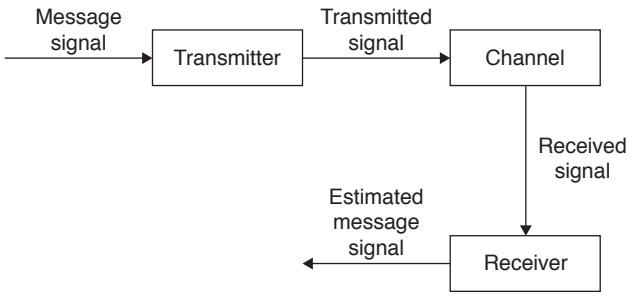


Figure 1 Elements of communication system

The transmitter changes the message signal into a form suitable for transmission over the channel.

The receiver process the channel output to produce an estimate of the message signal.

Classification of Signals

Continuous-time and discrete-time signals

A signal $x(t)$ is said to be a continuous-time signal if it is defined for all time t.

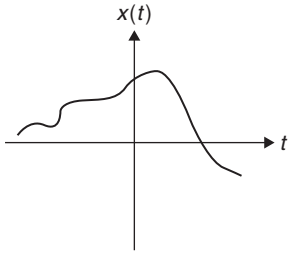


Figure 2 Continuous-signal

The above waveform represents an example of a continuous-time signal whose amplitude or value varies continuously with time.

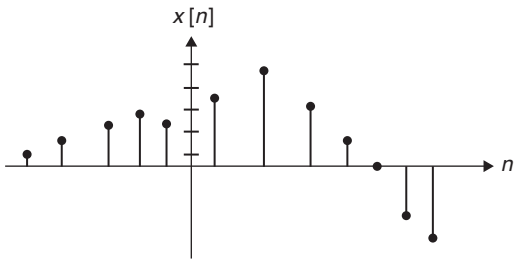


Figure 3 Representation of $x(t)$ as discrete time signal $x[n]$

A discrete time signal is defined only at discrete instants of time. Thus the independent variable has discrete values only, which are usually uniformly spaced. A discrete time signal is often derived from a continuous-time signal by sampling it at a uniform rate.

Let T_s denote the sampling period and n denotes an integer. Then sampling a continuous-time signal $x(t)$ at time $t = nT_s$ yields a sample with values $x(nT_s)$, we can write $x[n] = x(nT_s)$, $n = 0, \pm1, \pm2, \ldots$

Even and odd signals

A continuous-time signal $x(t)$ is said to be an even signal if $x(-t) = x(t)$ for all t. The signal $x(t)$ is said to be an odd signal if $x(-t) = -x(t)$ for all t. A discrete-time signal $x[n]$ is even if $x[-n] = x[n]$ and $x[n]$ is odd if $x[-n] = -x[n]$.

Examples of even signal $x(t) = \cos t$, odd signal $x(t) = \sin t$. An odd signal must necessarily be 0 at $t = 0$ or $n = 0$ since $x(-t) = -x(t)$, or $x[-n] = -x[n]$, so $x(0) = 0$ or $x[0] = 0$.

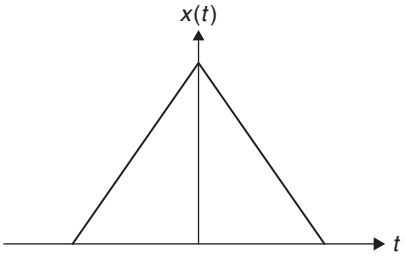


Figure 4 Even continuous-time signal

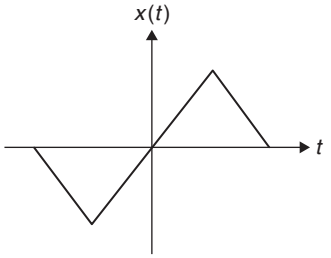


Figure 5 Odd continuous-time signal

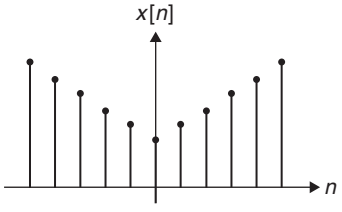


Figure 6 Even discrete-time signal

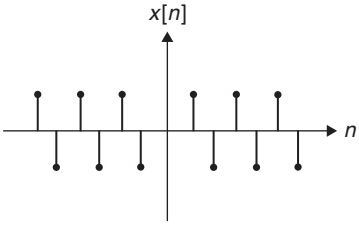


Figure 7 Odd discrete-time signal

Even signals are symmetric about vertical axis or time origin. Where as odd signals are anti-symmetrical about the time origin, (same in case of discrete time signals). Any signal $x(t)$ can be expressed as sum of even and odd components.

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(t)]$$

In case of complex valued signal, a complex valued signal $x(t)$ is said to be conjugate symmetric if $x(-t) = x*(t)$. Where $x*(t)$ is complex conjugate

If $x(t) = a(t) + jb(t)$

$$x(-t) = a(-t) + jb(-t)$$

then,

$$x^*(t) = a(t) - jb(t)$$

By comparing above equations, A complex valued signal $x(t)$ is conjugate symmetric if its real part is even $[a(t) = a(-t)]$, and its imaginary part is odd $[b(-t) = -b(t)]$. (A similar remark applies for discrete signals too).

Example 1: The even and odd components of the signal $x(t) = e^{-2t} \sin t$ are?

Solution: By replacing t with $-t$, $x(-t) = e^{2t}$.

We have,

$$\sin(-t) = -e^{2t} \sin t.$$

Then,

$$x_e(t) = \frac{1}{2}[x(t) + (-t)] = \frac{1}{2}[e^{-2t} \sin t - e^{+2t} \sin t]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{-2t} \sin t + e^{2t} \sin t]$$

Now,

$$x_e(t) = \sin t \frac{1}{2}[e^{-2t} - e^{2t}] = -\sin h(2t) \sin t$$

$$x_o(t) = \sin t \frac{1}{2}[e^{-2t} + e^{2t}] = \cos h(2t) \sin t$$

Periodic and non-periodic signals

If $x(t) = x(t + T)$ for all t, where T is a positive constant, then $x(t)$ is said to be periodic.

If this condition satisfies for $T = T_0$, then it also satisfied for $T = 2T_0, 3T_0, 4T_0 \ldots$.

The smallest value of T that satisfies $x(t) = x(t + T)$ is called fundamental period of T and its reciprocal $f = \frac{1}{T_0}$ is called fundamental frequency.

Frequency measured in Hertz (Hz) and the Angular frequency measured in radians per second is defined by $\omega = 2\pi f = \frac{2\pi}{T_0}$.

A discrete time signal $x[n]$ is periodic with period N, (where N is a positive integer) if it is unchanged by a time shift of N.

$$x[n] = x[n + N] \text{ for integer } n, N.$$

And $x[n]$ is also periodic with period $2N$, $3N\ldots$. The fundamental period N_0 is the smallest positive value of N for which of the above equation holds.

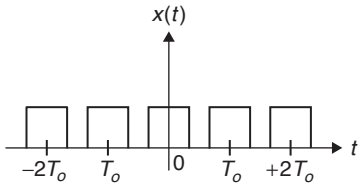


Figure 8 Continuous-time periodic signal with T_0.

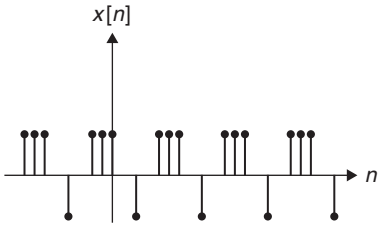


Figure 9 Discrete-time periodic signal with $N_0 = 4$

A signal $x(t)/x[n]$ that is not periodic will be referred to as an non-periodic signal.

Deterministic signals and random signals

The signal that can be completely specified by a mathematical equation is called deterministic signal, otherwise random signal.

Examples of deterministic signals.

$$x(t) = 2 - 2t\, u(t),\ x(t) = \sin \omega t,$$
$$x[n] = \cos \omega n,\ x[n] = a^n\, u[n]$$

A random signal is a signal about which there is uncertainty before it occurs. Such random signals are generally characterized by their mean, mean square values.

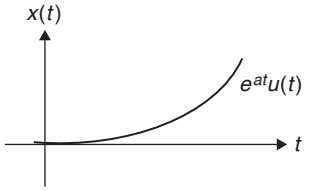


Figure 10 Deterministic signal



Figure 11 Random signal

Examples of random signals: the electrical noise generated in the amplifier, and interference component in television receiver.

Energy signals and power signals

Total energy of the continuous-time signal $x(t)$ is

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

and its time average power

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

The time averaged power of a periodic signal $x(t)$ of fundamental period T is given by $P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$.

In case of discrete-time signals the integrals are replaced by the corresponding sums.

$$E = \sum_{n=-\infty}^{\infty} x^2[n],$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

The average power of periodic signal $x(n)$ with fundamental period N is

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

A signal is referred to as an energy signal if the total energy of the signal satisfies the condition $0 < E < \infty$.

A signal is said to be as a power signal if the average power of the signal satisfies the condition $0 < P < \infty$.

The energy and power clarification of signals are mutually exclusive, an energy signal has zero time averaged power where as a power signal has infinite energy.

It is of interest to note that periodic signals and random signals are usually viewed as power signals, where as signals that are both deterministic and non-periodic are usually viewed as energy signals.

Causal, non-causal, and anti-causal signals

A signal is said to be causal if it is defined only for $t > 0$.

Example: $u(t)$, $Ae^{bt}\, u(t)$, $\sin \omega t \cdot u(t)$.

A signal is said to be non-causal if it is defined for either $t \le 0$ or for both $t \le 0$ and $t > 0$.

Example: Ae^{bt}, for all t, $\sin \omega t$, for all t.

When a non-causal signal is defined only for $t \le 0$ it is called anti-causal signal.

Example: $Ae^{bt}\, u(-t)$, $\sin\omega t \cdot u(-t)$.

Example 2: What is the fundamental period of the discrete time signal $x[n] = e^{j(2\pi/6)n} + e^{j(3\pi/4)n}$?

Solution: The first exponential fundamental period $N_1 = m\left(\frac{2\pi}{2\pi/6}\right) = 6$ m will be integer for $m = 1$, so fundamental period $N_1 = 6$.

The second exponential, the fundamental period $N_2 = m\left(\frac{2\pi}{3\pi/4}\right) = m \cdot \frac{8}{3}$ will be integer for $m = 3$, so fundamental period, $N_2 = 8$.

The least common multiple for these fundamental periods is 24, i.e., the fundamental period of $x[n]$.

Example 3: Find the fundamental period (in sec) and fundamental frequency (in rad/sec) for the following signals.

- $v(t) = 20\sin 100t + 10\cos 300t + 5\sin\left(500t + \frac{\pi}{3}\right)$
- $v(t) = 3\cos 200t + 4\cos 300t + 6\sin 500t$

Solution:

- $v(t) = 20\sin 100t + 10\cos 300t + 5\sin\left(500t + \frac{\pi}{3}\right)$

  For the above signal, we can consider the individual time periods are $T_1 = \frac{2\pi}{100}$, $T_2 = \frac{2\pi}{300}$, and $T_3 = \frac{2\pi}{500}$.

  The LCM of T_1, T_2, T_3—least common multiple is $\frac{2\pi}{100} = T_0$ is the fundamental time period in seconds. $T_0 = \frac{2\pi}{100}$, the fundamental frequency is $\omega_0 = \frac{2\pi}{T_0} = 100$ rad/sec.
- $v(t) = 3\cos 200t + 4\cos 300t + 6\sin 500t$

  For the above signal, we can consider the individual time periods are $T_1 = \frac{2\pi}{200}$, $T_2 = \frac{2\pi}{300}$, and $T_3 = \frac{2\pi}{500}$.

  The LCM of $(T_1, T_2, T_3), = \frac{2\pi}{100} = T_0$ is the fundamental period. [$\therefore$ Check $v(t + T_0) = v(t)$]. So fundamental frequency $(\omega_0) = \frac{2\pi}{T_0} = 100$ rad/sec.

Basic Operations on Signals

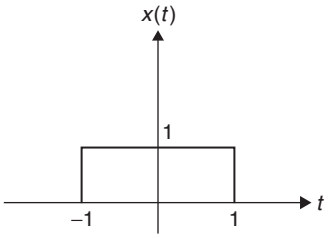
Operations Performed on Dependent Vvariables

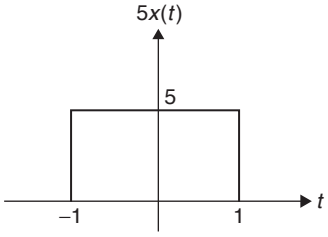
Amplitude scaling

$$y(t) = Cx(t),$$
$$y[n] = Cx[n].$$

The value of $y(t)$ is obtained by multiplying the corresponding value of $x(t)$ by the scalar C for each instant of time t.

A physical example of a device that performs amplitude scaling is an electronic amplifier.

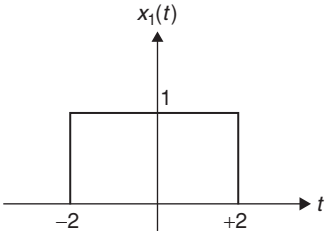


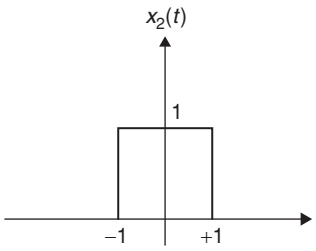


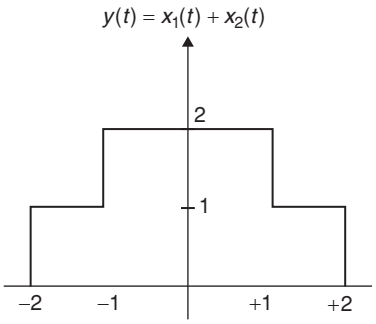
Addition

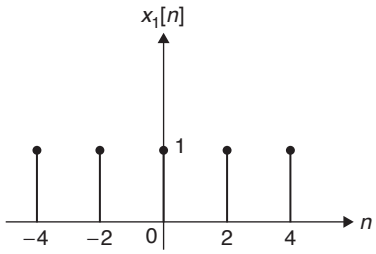
$$y(t) = x_1(t) + x_2(t)$$
$$y[n] = x_1[n] + x_2[n]$$

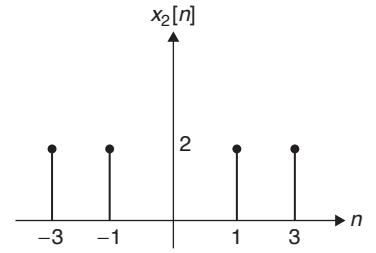
$x_1(t)$ and $x_2(t)$ are continuous-signals then $x_1(t) + x_2(t)$ denote addition of the two signals.

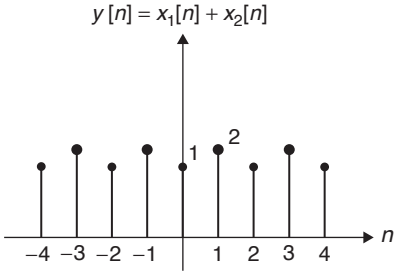












Multiplication

$$y[n] = x_1[n] \cdot x_2[n],$$

$$y(t) = x_1(t) \cdot x_2(t)$$

If $x_1(t)$ and $x_2(t)$ are two continuous-time signals, then their product $y(t) = x_1(t)\, x_2(t)$. The value of $y(t)$ at time 't' is given by the product of the corresponding values of $x_1(t)$ and $x_2(t)$.

For example in AM modulated signal, the message signal will be multiplied by the carrier signal; the resultant changes the amplitude of carrier as per message signal.

Differentiation

$$y(t) = \frac{d}{dt}x(t),$$

$$y[n] = x[n] - x[n-1]$$

The derivative of $x(t)$ with respect to time is defined as $y(t) =$.

Example: The voltage developed across inductor is given by $v(t) = L\dfrac{di(t)}{dt}$.

Integration

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

The Integral of $x(t)$ with respect to time 't' is defined as $y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$

Example: The voltage across capacitor is given by

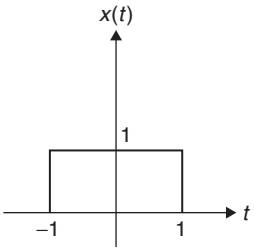
$$\vartheta(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau.$$

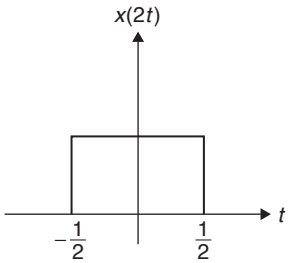
Operations Performed on the Independent variable

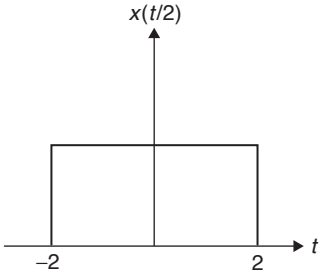
Time scaling

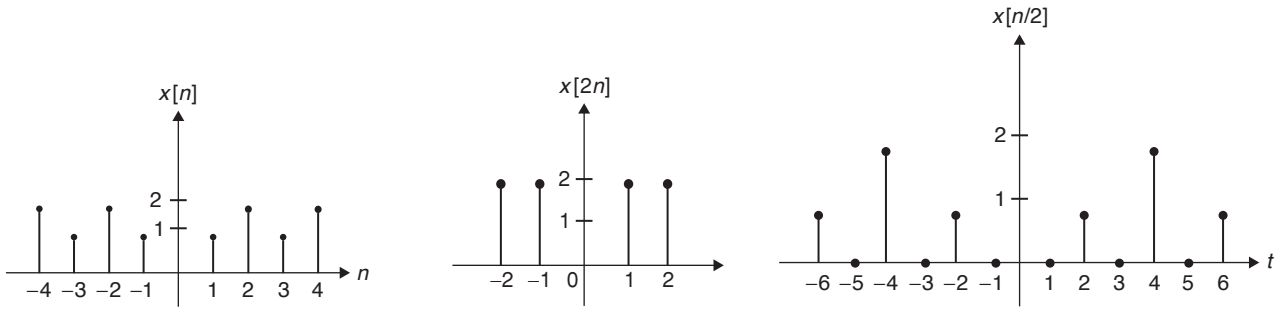
$$y(t) = x(\text{at})$$

If $a > 1$, the signal $y(t)$ is a compressed version of $x(t)$, if $0 < a < 1$ the signal $y(t)$ is an expanded (stretched) version of $x(t)$.





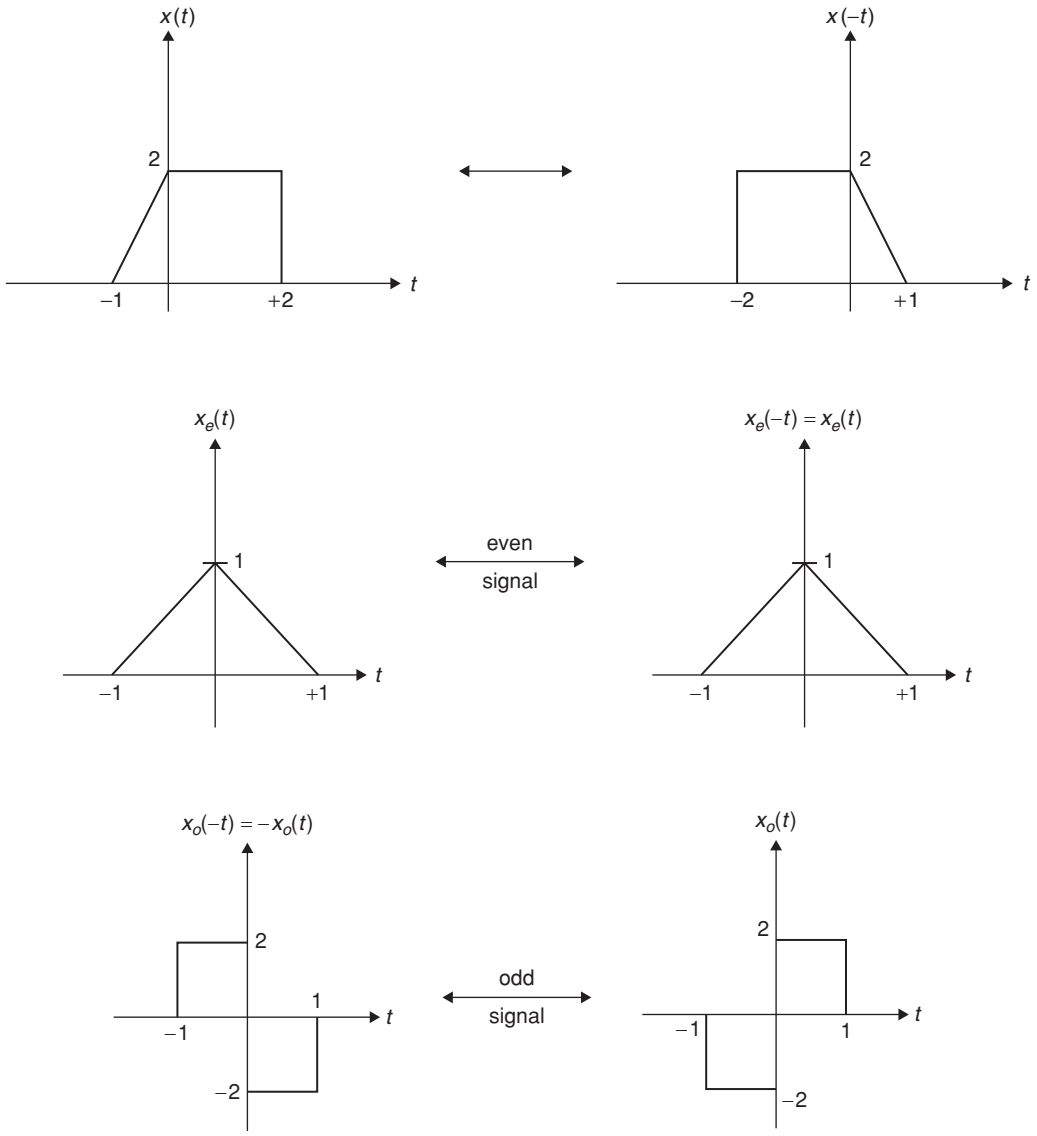




Reflection

$$y(t) = x(-t)$$

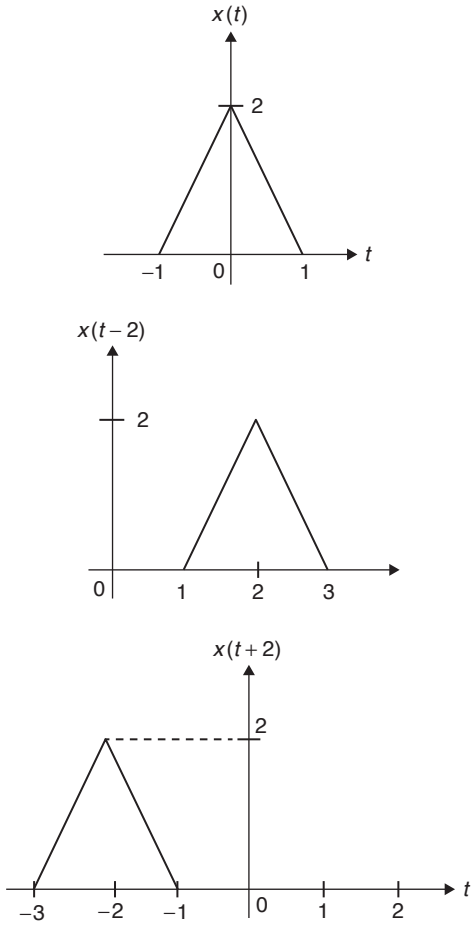
Even signal $[x(-t) = x(t)]$ is same as reflected signal odd signal $[x(-t) = -x(t)]$ is negative of its reflection. Reflected signal can be obtained by replacing t by $-t$.



Time shifting

$$y(t) = x(t - t_0)$$

If $t_0 > 0$, the waveform of $y(t)$ is obtained by shifting $x(t)$ toward the right, relative to the time axis, if $t_0 < 0$, $x(t)$ is shifted to the left.



Precedence rule for time shifting and time scaling

If $y(t) = x(at - b)$, to obtain $y(t)$ from $x(t)$, the proper order is based on the fact that the scaling operation always replaces 't' by 'at', while time shifting operation always replaces 't' by '$t - b$'.

Hence time shifting operation is performed first on $x(t)$, resulting in an intermediate signal $\vartheta(t) = x(t - b)$.

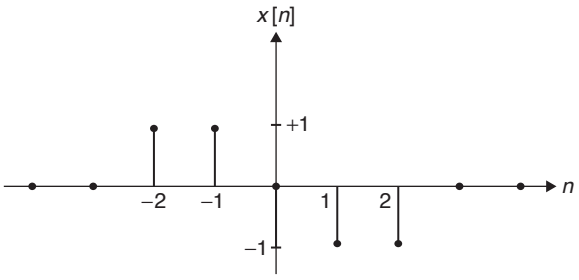
Next time scaling operation is performed on $\vartheta(t)$ resulting is desired output $y(t) = \vartheta(at) = x(at - b)$.

Incorrect way is first time scaling $[x(at)]$ and then applying time shifting results is $x[a(t - b)] \neq x[at - b]$. Similar approach we follow for discrete time signals.

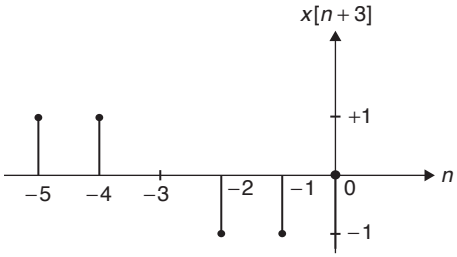
Example 4: The discrete time signal

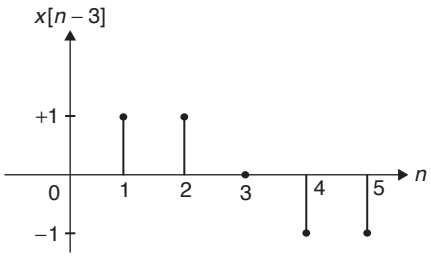
$$x[n] = \begin{cases} 1, & n = -1, \quad -2 \\ -1, & n = 1, \quad 2 \\ 0, & n = 0, \quad [n] > 2 \end{cases}$$

Then $x[n + 3]$, and $x[n - 3]$ are?

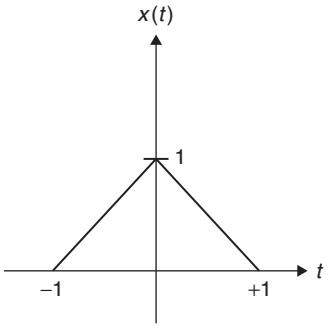


Solution:

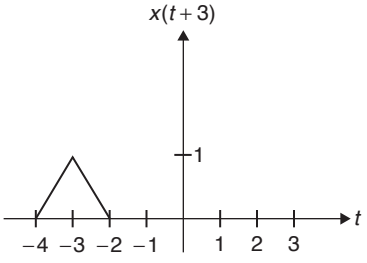


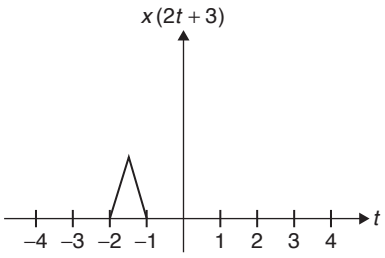


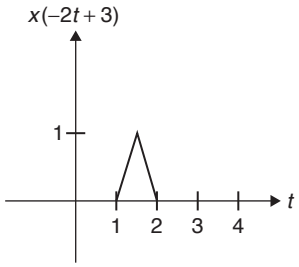
Example 5: A triangular pulse $x(t)$ is depicted in figure sketch $y(t) = x(-2t + 3) + x(2t + 3)$.

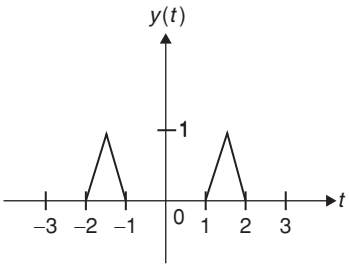


Solution:

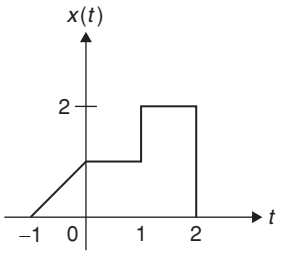




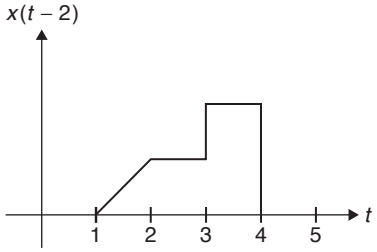




Example 6: For the signal $x(t)$ depicted in figure, draw the signals $x(t-2)$, $x(2t+1)$ $x(3-t)$, $x(-t-1)$, $[x(-2t) + x(2t)]$?

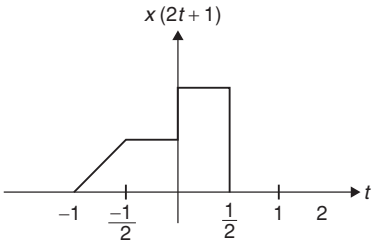


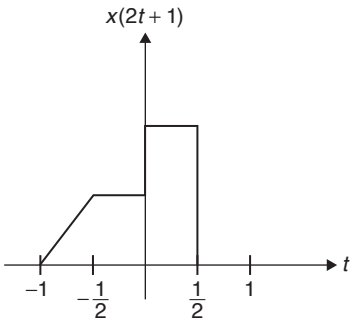
Solution:



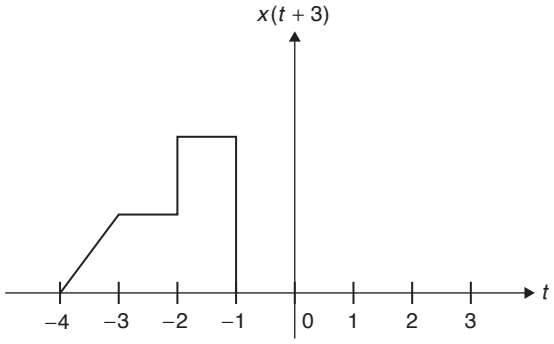
$x(t-2)$ is right shift by 2. To find out $x(2t+1)$, scaling by 2 replaces t by $2t$, similarly shifting by + 1 replaces t by $t+1$.

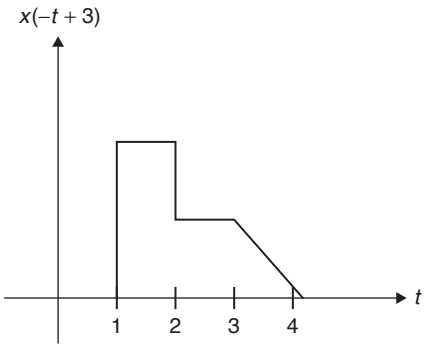
To get $x(2t+1)$, first we need to perform shifting $x(t+1)$ then scaling $x(2t+1)$ but first scaling results in $x(2t)$, and then shifting results in $x(2(t+1) = x(2t+2)$ which is not desired.



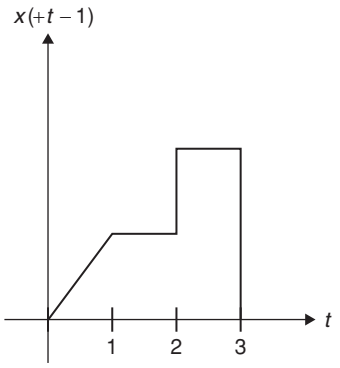


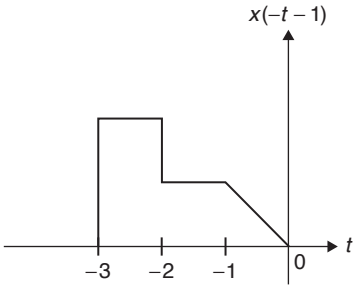
$x(3-t)$, first shifting $x(t+3)$, then scaling by –1, i.e., $x(-t+3)$.

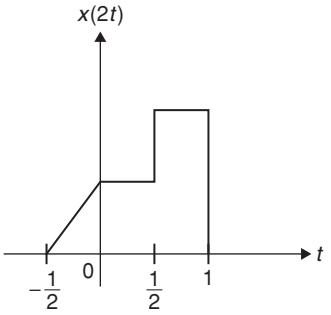


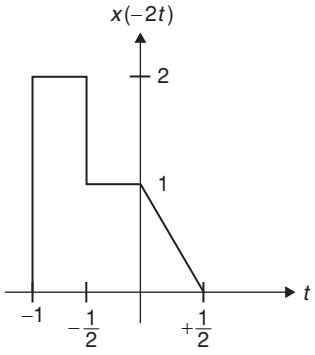


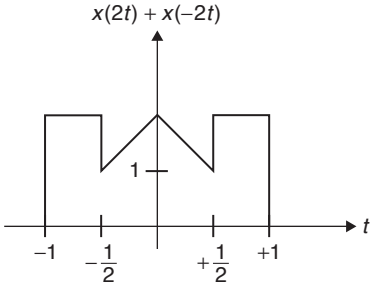
For $x(-t-1)$ first shifting $x(t-1)$ then scaling $x(-t-1)$.



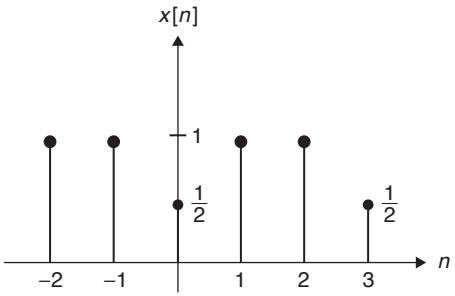




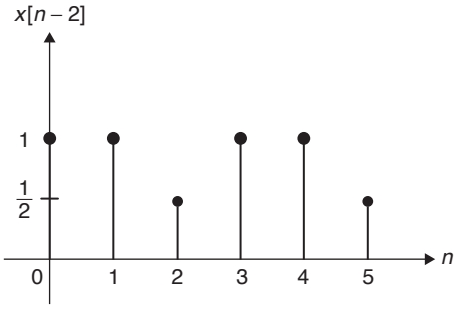


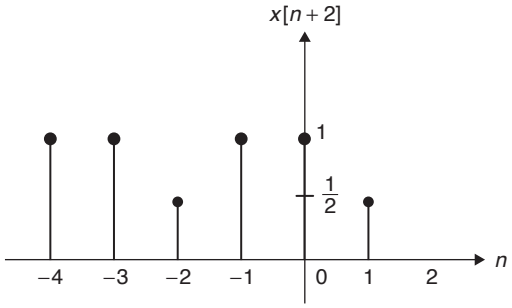


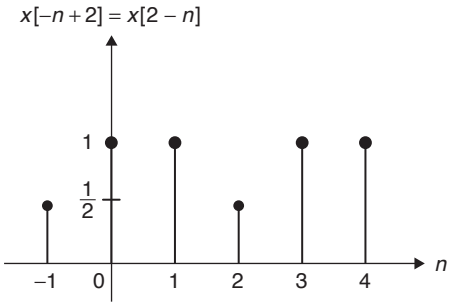
Example 7: For the discrete-time signal shown in figure, draw the following signals $x[n-2]$, $x[2-n]$ $x[3n+2]$, $x[n/2]$, $x[n]+x[-n]$.

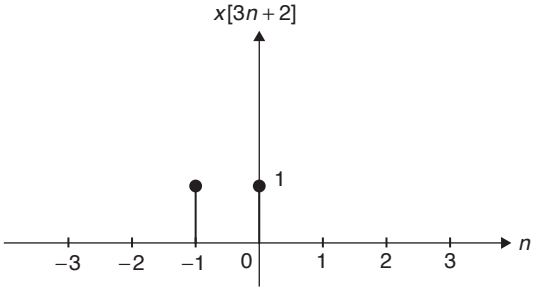


Solution:



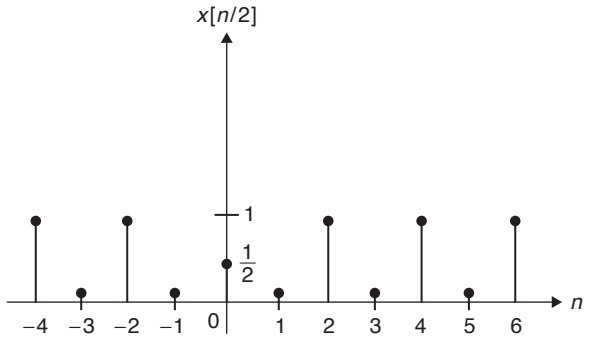






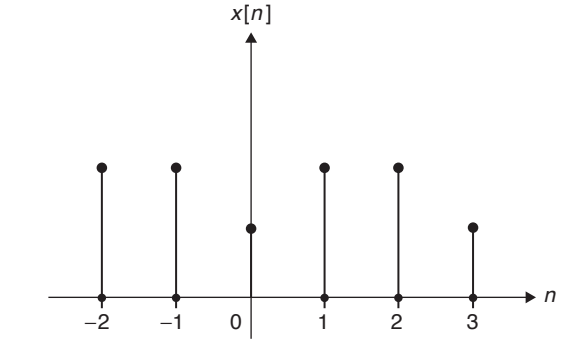
$x[3n+2]$ will be defined in the range of $-2 \le 3n+2 \le 3, -4 \le 3n \le 1, -4/3 \le n < 0/3$ and $-1 \le n \le 0$

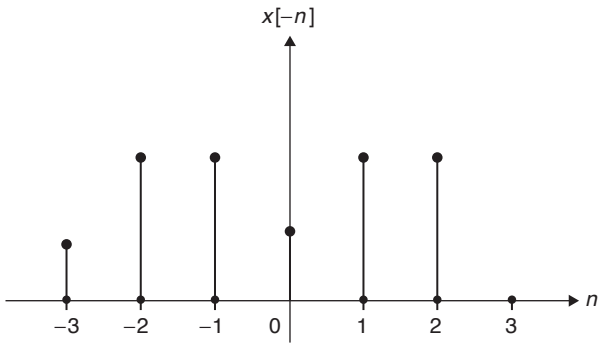
$$x(3n+2),\ x(3\times 0+2) = x(2) = 1$$
$$x(3x-1+2) = x(-1) = 1$$

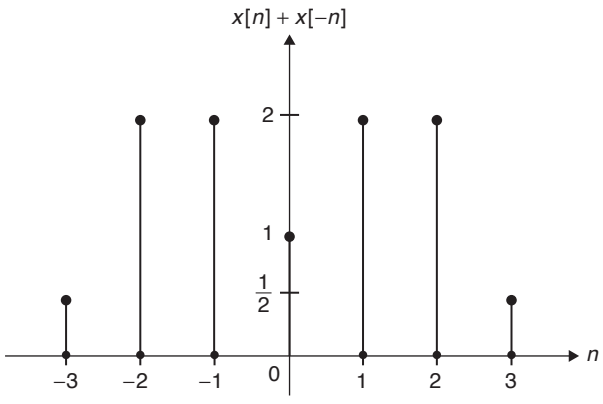


For $x[n/2]$, the range is $-2 \le n/2 \le 3 \Rightarrow -4 \le n \le 6$. Now, $x[n/2]$ for $n = 4$ is $x[-4/2] = x[-2] = 1$.

For $n = -3$, $x[-3/2] = 0$. Which is not defined and so on….







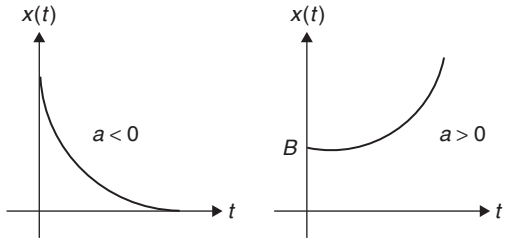
Elementary Signals

Exponential Signals

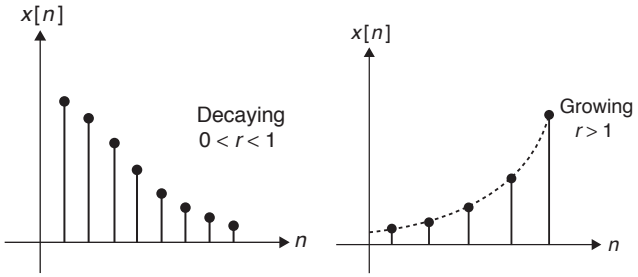
$$x(t) = Be^{at}$$

a, B are real parameters, B is the amplitude of the exponential signal measured at $t = 0$, when $a < 0$, it is decaying exponential.

When $a > 0$, it is growing exponential



In discrete time, real exponential signal $x[n] = Br^n, r = e^{\alpha}$,



For $r < 0$, the discrete time exponential $x[n]$ assumes alternating signs for the n, r^n is positive for n even and negative for n odd.

Sinusoidal Signals

$$x(t) = A\cos(\omega t + \phi)$$

A sinusoidal signal is an example of periodic signal $x(t+T) = x(t)$, where $T = \dfrac{2\pi}{\omega}$

Consider discrete-time signal,

$$x[n] = A\cos(\Omega n + \phi).$$

It is periodic signal

$$x[n+N] = x[n],$$

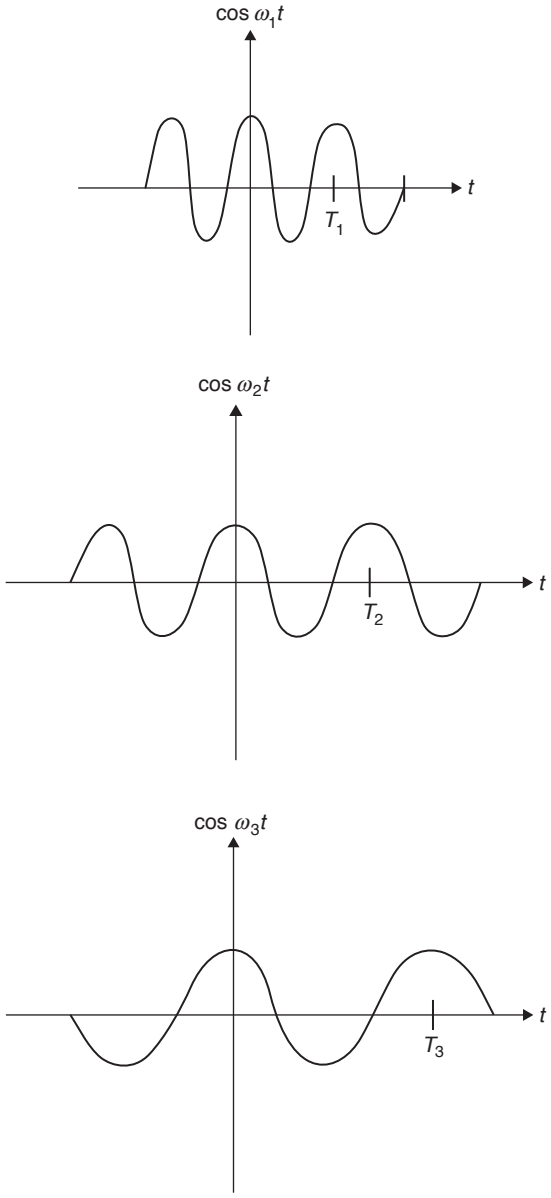
where $N = \dfrac{2\pi m}{\Omega}$, m, N integers.

The angular frequency Ω must be a rational multiple of 2π. Relation between sinusoidal and complex exponential signals is $e^{j\theta} = \cos\theta + j\sin\theta$. $x(t) = e^{j\omega_0 t}$ is a complex exponential signal. An important property of this signal is that it is periodic, $e^{j\omega_0 t} = e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T}$ for periodicity $e^{j\omega_0 T} = 1$.

This is true for the fundamental period $T_0 = \dfrac{2\pi}{|\omega_0|}$. Thus the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$ (from Euler's relations) we can

observe from the relation $T_0 = \frac{2\pi}{|\omega_0|}$, that fundamental period T_0 is inversely proportional to the fundamental frequency ω_0.



For the above 3 sinusoidal, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

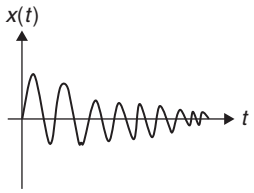
In case of discrete-time complex exponentials $e^{j\omega_0 n}$ is periodic only if, for some integers M, N the fundamental period will be $N = M\left(\frac{2\pi}{\omega_0}\right)$, for integers M, N, assuming M, N do not have any factors in common.

Exponentially Damped Sinusoidal Signals

The multiplication of a sinusoidal signal by a real valued decaying exponential signal results in a new signal referred to as an exponentially damped sinusoidal signal.

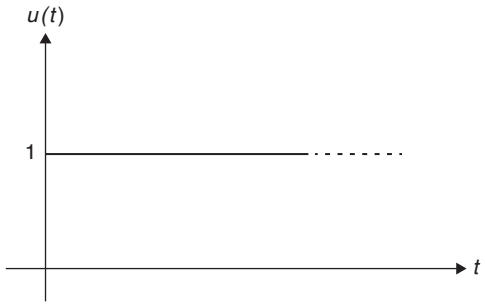
The discrete-time version of the exponentially damped sinusoidal signal is $x[n] = Ar^n \sin[\Omega n + \phi]$. For the signal to decay exponentially with time, the parameter r should lie in the range $0 < |r| < 1$.

For continuous-time, $x(t) = Ae^{-\alpha t} \sin(\omega t + \phi)$ $\alpha > 0$.

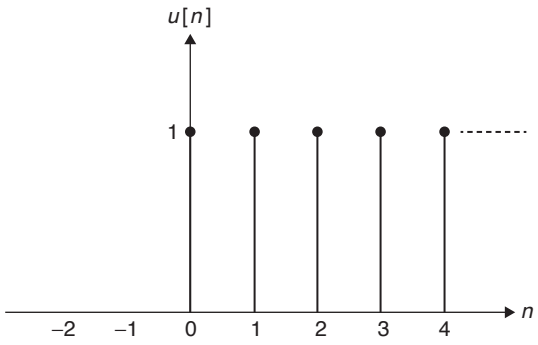


Step Function

$$u(t) = 1, \; t > 0$$
$$= 0, \; t < 0$$
$$= \frac{1}{2}, \; t = 0$$



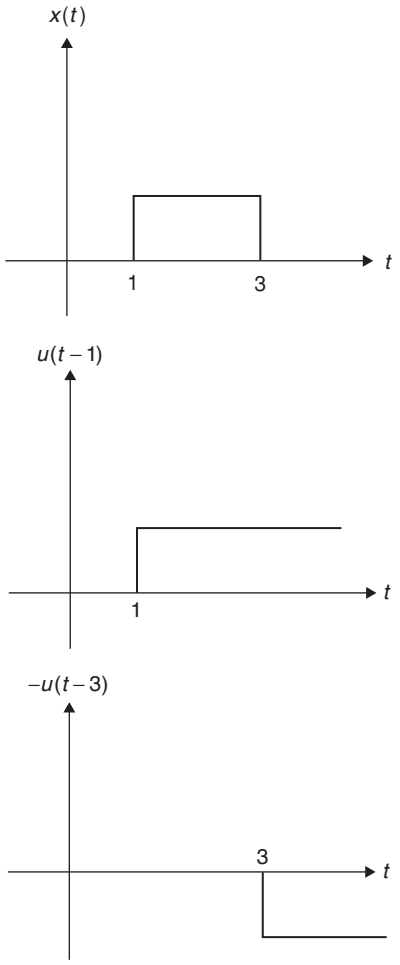
For discrete-time version $u[n] = 1, \; n \geq 0 = 0, \; n < 0$



The unit step function $u(t)$ is a particularly simple signal to apply as a test signal, the unit step function is useful because the output of a system due to a step input reveals a great deal about how quickly the system responds to an abrupt change in the input signal. A similar remark applies to $u[n]$ in the context of discrete-time system.

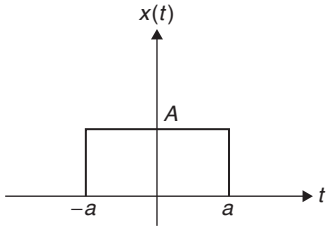
If we want a signal to start at $t = 0$ (so that it has value of zero for $t < 0$) we need to multiply the signal by $u(t)$.

Consider a rectangular pulse from $t = 1$ to $t = 3$, it can be derived from delayed unit impulse functions as shown here:



$$x(t) = u(t-1) - u(t-3)$$

Rectangular Pulse



$$x(t) = A,\ 0 \le |t| \le a$$
$$= 0,\ |t| > a$$

It can be written as $x(t) = A\,u(t+a) - A\,u(t-a)$.

Impulse Function

In discrete time

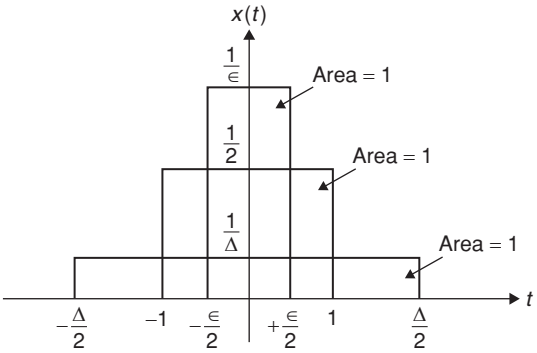
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The continuous-time version of the unit impulse is defined by $\delta(t) = 0$ for $t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

The impulse $\delta(t)$ is also referred to as the Dirac delta function.

We can visualize an impulse function as a tall, narrow, rectangular pulse of unit area, the width of this rectangular public is very small $\in \to 0$, and consequently its height is very large $\frac{1}{\in} \to \infty$



Evolution of a rectangular pulse of unit area into an Impulse of unit strength

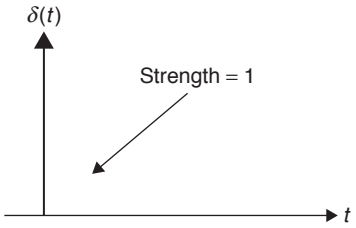


Figure 11 Unit impulse

The impulse $\delta(t)$ and unit step function $u(t)$ are related to each other as $\delta(t) = \frac{d}{dt}[u(t)]$ (or) $\delta[n] = u[n] - u[n-1]$.

$$\text{Conversely } u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{o}^{\infty} \delta(t-\tau)d\tau \text{ (or)}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{k=-\infty}^{n} \delta[k]$$

The properties of impulse function:

1. $\delta(-t) = \delta(t)$
2. $x(t)\delta(t) = x(0)\ \delta(t),\ x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
3. $\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0), \int_{-\infty}^{\infty} x(t)\ \delta(t-t_0)dt = x(t_0)$
4. $\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$

5. $\delta(at) = \frac{1}{a}\delta(t),\ a > 0$

6. $\int_{-\infty}^{\infty} \delta'(t)dt = 0$

7. $\int_{-\infty}^{\infty} f(t)\delta'(t-t_0)dt = \frac{d}{dt} f(t)|_{t=t_0}$

8. $\int_{-\infty}^{\infty} f(t)\delta^n(t-t_0)dt = \frac{d^n}{dt^n} f(t)|_{t=t_0}$

9. $x[n]\ \delta[n] = x[0]\ \delta[n],\ x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

10. $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Example 8: $\frac{\omega^2+1}{\omega^2+9}\delta(\omega-1) = \frac{1^2+1}{1^2+9}(\omega-1) = \frac{1}{5}\delta(\omega-1).$

Solution:

$$\sin\left(t-\frac{\pi}{2}\right)\delta(t) = \sin\left(0-\frac{\pi}{2}\right)\delta(t) = -\delta(t)$$

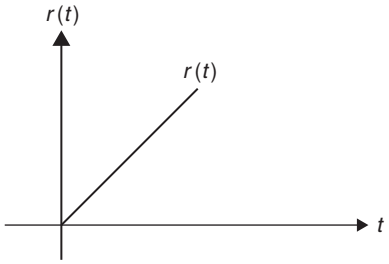
$$\int_{-\infty}^{\infty} \delta(t-4)\cos\left(\frac{\pi t}{4}\right)dt = \cos\left(\frac{\pi}{4}.4\right) = \cos\pi = -1$$

$$\int_{-\infty}^{\infty} e^{-2(x-t)}\delta(2-t)dt = e^{-2(x-2)}$$

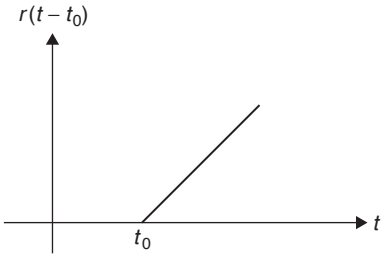
Ramp Function

The impulse function $\delta(t)$ is the derivative of the step function $u(t)$ with respect to time. By the same token, the integral of the step function $u(t)$ is a ramp function of unit slope.

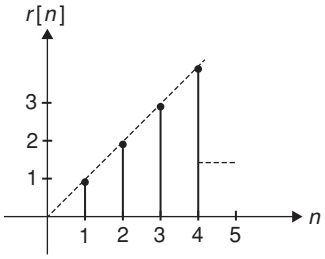
$r(t) = t,\ t \geq 0,\ 0,\ t < 0$ or $r(t) = t\,u(t)$



$r(t-t_0) = t - t_0,\ t \geq t_0$
$= 0,\ t < t_0$



The discrete-time version of the ramp function is defined by

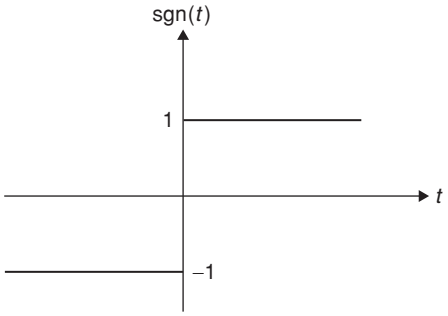


$r[n] = n,\ n \geq 0$
$= 0\ n \leq 0$
(or) $r[n] = n\ u\ [n]$

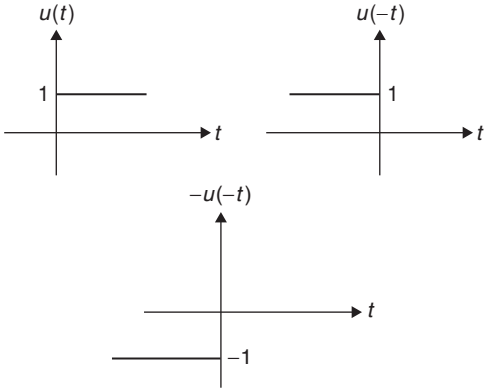
Signum Function

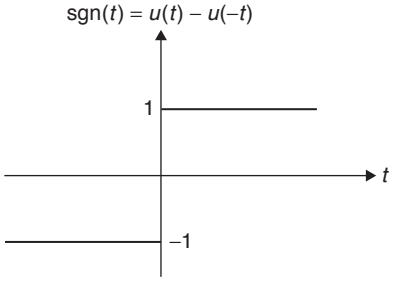
The signum function is defined as

$$\text{sgn}(t) = \begin{cases} -1; & t < 0 \\ 0; & t = 0 \\ +1; & t > 0 \end{cases}$$



The representation is shown in below figure, it can be derived from unit step function $u(t)$ and folded unit step function $u(-t)$.





$\text{sgn}(t) = u(t) - u(-t) = u(t) - [1 - u(t)] = 2u(t) - 1$. In discrete-time

$$\text{sgn}[n] = \begin{cases} -1, & n < 0 \\ 0, & n = 0 \\ +1, & n > 0 \end{cases}$$

$$\text{sgn}[n] = u[n] - u[-n] = 2u[n] - 1$$

Relation between elementary signals:

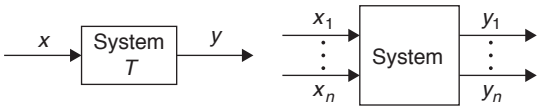
1. $\frac{d}{dt}u(t) = \delta(t), \text{or } u(t) = \int_0^{\infty} \delta(t-\sigma)d\sigma$
2. $x(t)u(t) = 0$ for $t < 0 = x(t)$ for $t \geq 0$
3. $\frac{d}{dt}r(t) = u(t) \text{ (or)} \int_{-\infty}^{t} u(t)dt = r(t)$
4. $\frac{d}{dt}r(t-t_0) = u(t-t_0)$ or $\int_{-\infty}^{t} u(t-t_0)dt = r(t-t_0)$
5. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
6. $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
7. $u(kt - t_0) = u(t - t_0/k), k \neq 0.$

Systems and Classification of Systems

A system is a mathematical model of a physical process that relates the input signal to the output signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transform of x into y.

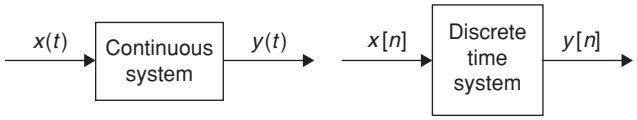
$$y = T\{x\}$$



System with single or multiple input and output signals.

Continuous-time and Discrete-time Systems

If the input and output signals x and y are continuous-time (discrete-time) signals, then the system is called a continuous-time (discrete-time) system.



Where $x(t)/x[n]$ is the input and $y(t)/y[n]$ is the output. (or) we can represent by the notation $x(t) \rightarrow y(t)$ or $x[n] \rightarrow y[n]$.

System with Memory and without Memory

A system is said to be memory less if the output at any time depends on only the input at that same time, otherwise the system is said to be system with memory. Memory less example: $y(t) = kx(t)$, $u[n] = kx[n]$.

The input output relationship of a resistor will be $v(t) = Ri(t)$. Where current $i(t)$ is input and $v(t)$ voltage is output. The output at time $t = t_0$ depends on only input at that instant.

A system is said to possess memory if its output signal depends on past or future values of the input signals. The example for continuous-time system with memory is capacitor, if input is taken to be the current and the output is the voltage.

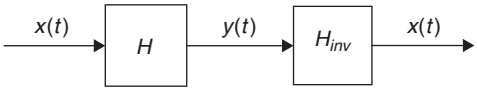
$$V(t) = \frac{1}{C}\int_{-\infty}^{t} i(T)dT \text{ where } C \text{ is capacitance}$$

$y(t) = x^2(t)$ is memory less system, but $y(t) = x(t^2)$ is system with memory,

With Memory example: $y[n] = \sum_{k=-\infty}^{n} x[k]$, $y[n] = x[n-1]$

Invertible and Non-invertible Systems

A system is said to be invertible if distinct inputs leads to distinct outputs or a system is said to be invertible if the input of the system can be recovered from the output. When several different inputs result in the same output (as in rectifier), it is impossible to obtain the input from output, and the system is non invertible.



The system that achieves the inverse operation of obtaining $x(t)$ from $y(t)$ is the inverse system (H_{inv}) for H. For example if H is an ideal integrator, then its inverse system (H_{inv}) is an ideal differentiator.

$$x(t) = H_{inv}\{y(t)\}$$
$$= H_{inv}H\{x(t)\}$$

$H_{inv}H = I$, I is identity operator,

Cascading a system with its inverse system results an identity system. The output of a system described by the identity operator is exactly equal to the input. The property of invertibility is of particular importance in the design of communication systems.

When a transmitted signal propagates through a communication channel, it gets distorted owing to non-ideal frequency response, and finite band width of channel. To compensate this distortion equalizer network will be used which is the inverse of channel.

Invertible system: $y(t) = x(t-z)$, its inverse system is $x(t) = y(t+z)$.

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)d\tau; \text{ inverse system } x(t) = L\frac{d}{dt}y(t).$$

Noninvertible system: $y(t) = x^2(t)$, the condition for invariability is different inputs should produce different output but for this case $-x(t)$ and $+x(t)$ both will lead to same output so non invertible system.

Causal and Non-causal Systems

If the output of system at present time depends on only the present input $x(t)$ or past values of the input, but not on its future values, then the system is causal.

In other words a casual (also known as physical or non anticipative) system is one for which the output at any instant t_0 depends only on the value of input $x(t)$ for $t \le t_0$. In causal system the output cannot start before the input is applied. A system that violates the conditions of causality is called a non-causal (or anticipative) system. The RC circuit is a causal system, since the capacitor voltage responds only to present and past values of the source voltage. $y(t) = x(-t)$ is non-causal because when $t > 0$, consider for $t = +2$, $y(2) = x(-2)$ it seems like a causal system but for $t < 0$, consider for $t = -2$, $y(-2) = x(2)$ have output depends on future value, the causal input output relation should valid for all times in a causal system. So the above system is a non-causal system. $y[n] = x[n] \cos[n+1]$ is a causal system, here $\cos[n+1]$ is a constant that varies with n, but output $y[n]$ depends on $x[n]$ (present input) only, so system is causal.

Example of causal system:

$$y[n] = \sum_{k=-\infty}^{n} x[k], y(t) = t.x(t) y[n] = x[-n]u[n]$$

Example of non-causal system:

$$y[n] = x[n+1],$$

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k], y(t) = x(t+2) - x(t-2)$$

All memory less systems are causal but not vice versa.

Linear Systems and Non-linear Systems

A system is said to be a linear system if it satisfies the following two conditions.

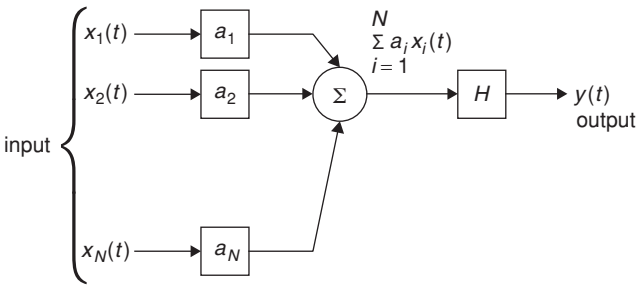
a. **Additivity:** given that $y_1 = T\{x_1\}$, and $y_2 = T\{x_2\}$, then $T\{x_1 + x_2\} = y_1 + y_2$ for any signals x_1 and x_2.
b. **Homogeneity (or scaling):** given that $T\{x\} = y$ then $T\{ax\} = ay$, for any signal x and y, any scalar a.

The above conditions can be combined as $T\{a_1x_1 + a_2x_2\} = a_1y_1 + a_2y_2$. Any system that doesn't satisfy the above condition is called non-linear system. Example for non-linear system:

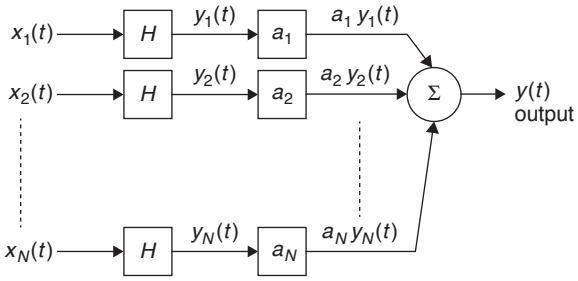
$$y = x^2, y = \sin`x'.$$

From the condition 'b', we can say that a zero input yields zero output for linear systems.

When a system violates either the principle of super position or the property of homogeneity, the system is said to be 'non-linear'.



The combined operation of amplitude scaling and summation, then precedes the system operator H for multiple inputs to produce final output $y(t)$.



The system operator H precedes the amplitude scaling for each input; the resulting outputs are summed to produce the overall output $y(t)$.

If these two configurations produce the same output $y(t)$, then the system operator H is linear.

Consider the system output $y(t)$ and input $x(t) = \sum_{i=1}^{N} a_i x_i(t)$.

$$\text{Then } y(t) = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\} = \sum_{i=1}^{N} a_i H\{x_i(t)\} = \sum_{i=1}^{N} a_i y_i(t).$$

Now, the system will be linear system.

Consider a system by input–output relation as $y(t) = tx(t)$ if $x(t)$ is expressed as weighted sum $x(t) = \sum_{i=1}^{N} a_i x_i(t)$.

$$\text{Then } y(t) = t \cdot \sum_{i=1}^{N} a_i x_i(t) = \sum_{i=1}^{N} a_i t x_i(t)$$

Now, $y_i(t) = t \cdot x_i(t)$ is output due to each input acting individually then $y(t) = \sum_{i=1}^{N} a_i y_i(t)$ so the system is linear.

Consider discrete time system $y[n] = x[n]\, x[n-1]$. Then $y_1[n] = x_1[n]\, x_1[n-1]$, $y_2[n] = x_2[n]\, x_2[n-1]$. If input is $a_1 x_1[n] + a_2 x_2[n]$.

Then $y[n] = (a_1 x_1[n] + a_2 x_2[n])\,(a_1 x_1[n-1] + a_2 x_2[n-1]) \neq a_1 y_1[n] + a_2 y_2[n]$, so system is non-linear.

Consider system $y(t) = x^2(t)$, $y_1(t) = x_1^2(t)$, $y_2(t) = x_2^2(t)$. If $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for linear system $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, but $y_3(t) = x_3^2(t) = [a_1x_1(t) + a_2x_2(t)]^2 \neq a_1x_1^2(t) + a_2 x_2^2(t)$, so system is non-linear.

Consider the system If $y[n] = \text{Re}\{x[n]\}$, assume $x[n] = p[n] + jq[n]$, then $y[n] = p[n]$.

For linearity, the system must satisfy both additively and homogeneity properties and that the signals, as well as any scaling constants, are allowed to be complex.

To check homogeneity property $ay[n] = T\{ax[n]\}$. Here consider scaling constant $a = j$,

$$x_1[n] = jx[n] = jp[n] - q[n]$$

$$y_1[n] = \text{Re}\{x_1[n]\} = -q[n] \neq jy[n]$$

The system violates homogeneity property hence the system is not linear.

Consider the system $y[n] = 5x[n] + 7$, we can see that the system violates zero – in/zero – out property of linear systems. For zero input $x[n] = 0$, $y[n] = 0$ for linear systems, here $y[n] = 7$, so the system is non-linear.

$$\frac{dy(t)}{dt} + 4y(t) = x(t) \text{ is also a linear system,}$$

$$y(t)\frac{dy(t)}{dt} + 4y(t) = x(t) \text{ is non-linear system}$$

Time-Invariant and Time-varying Systems

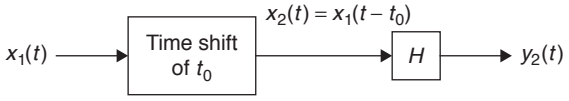
A system is called time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal.

$$T\{x(t - T)\} = y(t - T)(\text{for CT})$$

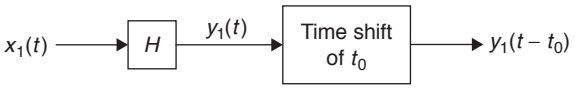
$$T\{x[n - k]) = y(n - k) \text{ (for DT)}$$

A system which does not satisfy above condition is called time-varying system.

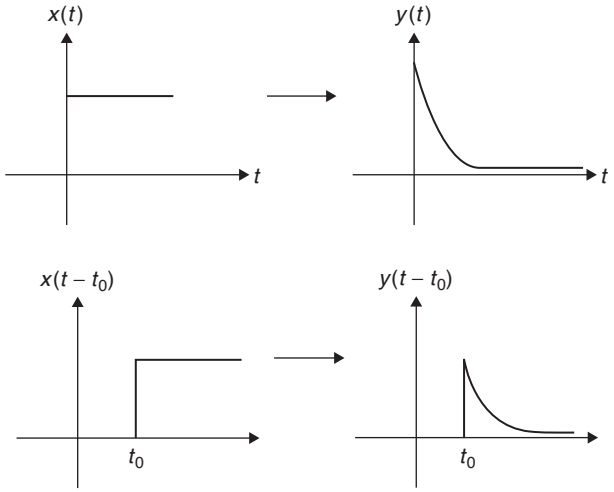
This implies that a time-invariant system responds identically no matter when input signal is applied. In other way, the characteristics of a time-invariant system do not change with time.



Time shift preceding operator H.



Time shift following operator H. If $y_2(t) = y_1(t - t_0)$ then the system is time-invariant.



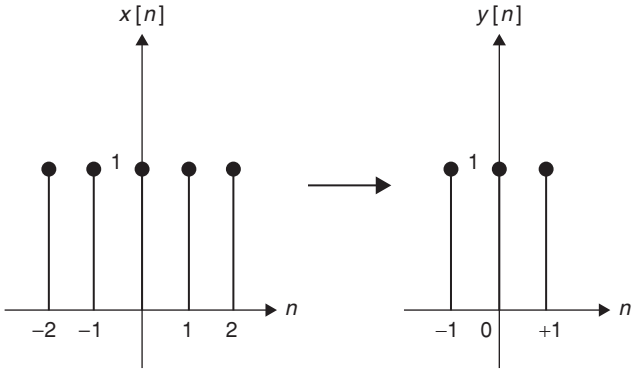
If the system is time-invariant, then the delayed output $y(t - t_0)$ can also be obtained by first delaying the input $x(t)$ before applying it to the system.

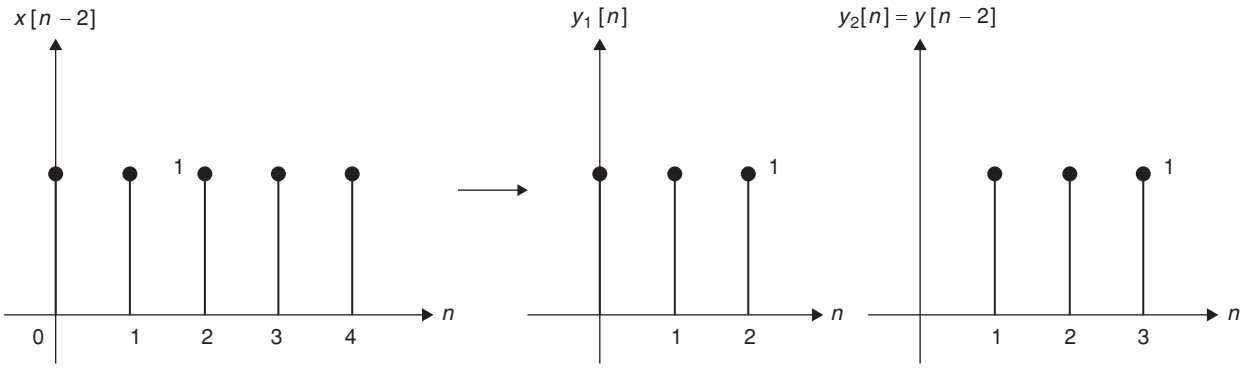
Consider a system specified by $y(t) = e^{-2t}x(t - 1)$ for the delayed input $x(t - t_0)$, output $y_1(t) = e^{-2t}x(t - t_0 - 1)$ but delayed output $y_2(t) = y(t - t_0) = e^{-2(t-t_0)}x(t - t_0 - 1)$ $y_1(t) \neq y_2(t)$, so the system is time varying.

Consider another system $y(t) = (\sin t)\, x(t + 1)$ for delayed input $x(t - t_0)$, output $y_1(t) = \sin t \cdot x(t - t_0 + 2)$ the delayed output. Now, $y_2(t) = y(t - t_0) = \sin(t - t_0)\, x(t - t_0 + 2)$ $y_1(t) \neq y_2(t)$. So the system is time varying.

Consider a discrete-time system $y[n] = \sin(x[n])$, for delayed, input $x[n - n_0]$, output $y_1[n] = \sin(x[n - n_0])$. When output is delayed, $y_2[n] = y[n]–n_0] = \sin(x[n - n_0])$ here $y_1[n] = y_2[n]$, so the system is time-invariant.

Consider a discrete-time system $y[n] = x[2n]$, for delayed input $x[n - n_0]$, output $y_1[n] = x[2n - n_0]$. When output is delayed, $y_2[n] = y[n - n_0] = x[2(n - n_0)]$ $y_1[n] \neq y_2[n]$ so the system is time invariant.





So, the above system is time variant, $y_2[n] \neq y_1[n]$. If the system is linear and also time-invariant, then it is called a linear time-invariant (LTI) system.

Stable Systems

A system is bounded input bounded output (BIBO) stable if for any bounded input x defined by $|x| \leq K_1$. The corresponding output y is also bounded defined by $|y| \leq K_2$, where K_1 and K_2 are finite real constants.

Consider the system $y[n] = \frac{1}{3}(x[n] + x[n-2] + x[n-3])$.

If input is finite value, then $|x[n]| \leq k < \infty$ for all n,

$$\text{Then } |y[n]| = \left|\frac{1}{3}(x[n] + [n-2] + x[n-3])\right|$$

$$\leq \frac{1}{3}(|x[n]| + |x[n-2]| + |x[n-3]|)$$

$$\leq \frac{1}{3}(k_1 + k_1 + k_1) = k_1 < \infty$$

The output is also finite for finite input, so system is BIBO stable.

Consider a continuous-time system $y(t) = a^t x(t)$. If input is finite value $|x(t)| \leq k_1 < \infty$ for all n. Then $|y(t)| = |a^t x(t)| = |a^t|\,|x(t)|$.

With $a > 1$, the multiplying factor a^t diverges for increasing t, so system is unstable for $a > 1$, for $a < 1$, the system will be stable.

Consider the continuous-time system $y(t) = t \cdot x(t)$. Even for bounded input $|x(t)| \leq k < \infty$ the output is unbounded $|y(t)| = |t|\,|x(t)| = |t|k$. So the system is unstable.

Consider the continuous-time system $y(t) = e^{2x(t)}$, when $x(t)$ is finite. $|x(t)| < k < \infty - k < x(t) < k$, then $y(t) = e^{2x(t)}$ i.e., $e^{-2k} < |y(t)| < e^{2k} y(t)$ is also finite. So the system is stable.

Linear Time Invariant Systems

Impulse Response

The Impulse response $h(t)$ of a continuous-time LTI system (represented by T) is defined to be the response of the system, when the input is $\delta(t)$, that is $h(t) = T\{\delta(t)\}$.

In continuous-time case, a true impulse signal having zero width and infinite amplitude cannot physically be generated and is usually approximated by a pulse of large amplitude and brief duration. Thus impulse response may be interpreted as the system behavior in response to a high energy input of extremely brief duration.

Response to an Arbitrary Input $x(t)$

Given the impulse response $h(t)$ we determine the output $y(t)$ due to an arbitrary input signal $x(t)$, by expressing the input as a weighted super position of time-shifted impulses. By linearity and time invariance, the output signal must be a weighted super position of time-shifted impulse responses, this weighted super position is termed as the convolution integral for continuous-time system, and the convolution sum for discrete-time system.

$$y[t] = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Input $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Output

$$y(t) = T\{x(t)\}$$

$$= T\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\; T\{\delta(t-\tau)\}\, d\tau$$

Since the system is time invariant.

$$T\{\delta(t-\tau)\} = h(t-\tau)\ (\because \text{ impulse response}).$$

So

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.$$

Convolution Integral

The convolution of two signals $x(t)$, $h(t)$ is denoted by

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

The output of any LTI system is the convolution of the input and the Impulse response of the system.

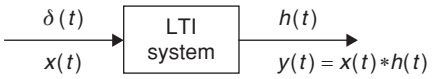


Figure 12 Continuous-time LTI system.

Properties of convolution integral:

1. Commutative $x(t) * h(t) = h(t) * x(t)$
2. Associative $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$
3. Distributive $x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$
4. $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$
5. $x_1(t) * x_2(t) = c(t), x_1(t - T_1) * x_2(t_2 - T_2) = c(t - T_1 - T_2)$
6. Convolution with an impulse $x(t) * \delta(t) = x(t), x(t - t_1) * \delta(t - t_2) = x(t - t_1 - t_2)$
7. The width property: if $x_1(t)$ is defined for T_1 duration and $x_2(t)$ is defined for T_2 duration, then duration (width) of $x_1(t) * x_2(t)$ is $T_1 + T_2$.
8. Causality: If $h(t)$ and $x(t)$ are causal systems then

$$y(t) = x(t) * h(t) = \int_{0-}^{t} x(\tau)h(t-\tau)d\tau, t \geq 0$$

$$= 0, t < 0$$

or

$$y(t) = \int_{0-}^{t} h(\tau)x(t-\tau)d\tau, t \geq 0$$

$$= 0, t < 0$$

Step Response

The step response $s(t)$ of a continuous-time LTI system is defined to be the response of the system when input is $u(t)$: unit step response $s(t) = T\{u(t)\}$ $s(t)$ can be easily determinate by

$$s(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = \int_{0}^{\infty} h(t-\tau)d\tau$$

or

$$h(t) = \frac{d}{dt}s(t).$$

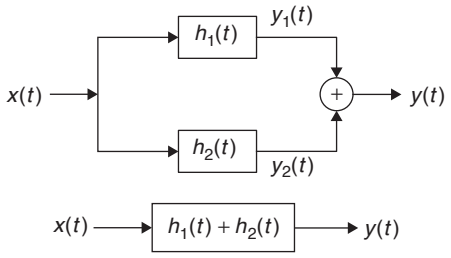
Table 1 *Convolution table*

Sl. No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
1	$x(t)$	$\delta(t - T)$	$x(t - T)$
2	$u(t)$	$u(t)$	$t \cdot u(t)$
3	$e^{at}u(t)$	$e^{bt}u(t)$	$\frac{e^{at} - e^{bt}}{a - b}u(t), a \neq b$
4	$e^{at}u(t)$	$e^{at}u(t)$	$te^{at}u(t)$
5	$e^{at}u(-t)$	$e^{bt}u(-t)$	$\frac{e^{at} - e^{bt}}{b - a}u(-t)$

Interconnected Systems

A larger, more complex system can often be viewed as the interconnection of several smaller subsystems, each of which is easier to characterize. Knowing the characterizations of these subsystems, it becomes simpler to analyze such large systems

Consider two continuous-time LTI systems are in parallel,



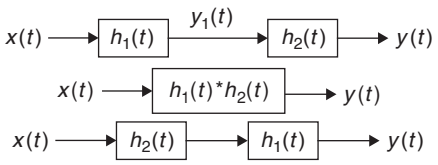
$$y_1(t) = h_1(t) * x(t), y_2(t) = h_2(t) * x(t)$$

$$y(t) = y_1(t) + y_2(t) = h_1(t) * x(t) + h_2(t) * x(t)$$

$$= [h_1(t) + h_2(t)] * x(t)$$

This is another interpretation of distributive property of convolution.

Consider two continuous-time LTI systems in cascade



$$y_1(t) = x(t) * h_1(t)$$

$$y(t) = y_1(t) * h_2(t)$$

$$= x(t) * h_1(t) * h_2(t)$$

$$= x(t) * \{h_1(t) * h_2(t)\}$$

This is another interpretation of associate property and commutative property of convolution.

Properties of Continuous-time LTI Systems

Systems with or without Memory

A system is memory less if its output at any time depends only on the value of the input at that same time.

$$\text{Output: } y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

If $y(t)$ need to be dependent only present input $x(t)$ then $h(\tau)$ $x(t - \tau) = kx(t)$, this is possible when $h(\tau)$ is defined for only $\tau = 0$ i.e., $h(t) = 0$ for $t \neq 0$.

The corresponding Impulse response $h(t)$ of memory less system is simply $h(t) = k\delta(t)$. Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, the continuous-time LTI system has memory.

Causality The output of a causal system depends only on the present and past values of input to the system.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$y(t)$ should not depend on future values, i.e., $h(t) = 0$ for $t < 0$.

For a causal continuous-time LTI system, we have $h(t) = 0, t < 0$.

The output of causal LTI system is

$$y(t) = \int_{0}^{t} h(\tau)x(t-\tau)d\tau = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

Stability A system is bounded input-bounded output (BIBO) stable if the output is guaranteed to be bounded for every bounded input.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$|y(t)| < \infty$, for $|x(t)| < \infty$ then

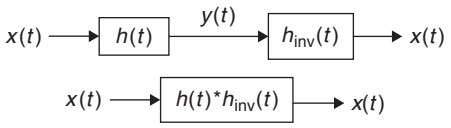
$$|y(t)| = \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty \text{ as } |x(t)| < \infty$$

A continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable.

$$\text{i.e.,} \int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$$

Invertibililty If an LTI system is invertible, then it has an LTI inverse system, when the inverse system is connected in series with original system, it produces an output equal to the input to the first system.



That is $x(t) = x(t) * \{h(t) * h_{inv}(t)\}$. This will be valued only when $h(t) * h_{inv}(t) = \delta(t)$

Discrete-time LTI Systems

1. Impulse response $h[n] = T\{\delta[n]\}$
2. Response to an arbitrary input $x[n]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

3. Convolution $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

4. Properties of convolution sum $x[n] * h[n] = h[n] * x[n]$ is commutative. $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$ is associative. $x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$ is distributive.
5. Shifting property $x_1[n] * x_2[n] = c[n]$, then $x_1[n-n_1] * x_2[n-n_2] = c[n-n_1-n_2]$.
6. Convolution with impulse $x[n]*\delta[n] = x[n]$, $x[n-n_1] * \delta[n-n_2] = x[n-n_1-n_2]$.
7. The width property: if $x_1[n]$ and $x_2[n]$ have finite widths of w_1 and w_2, respectively then the width of $x_1[n]*x_2[n]$ is $w_1 + w_2$. The width of a signal is one less than the number of elements.
8. Causal system output: if $x[n]$ and $h[n]$ are causal, then

$$y[n] = \sum_{k=0}^{\infty} x[k]h[n-k].$$

9. Convolution sum table:

Sl. No.	$x_1[n]$	$x_2[n]$	$x_1[n]*x_2[n]$
1	$\delta[n-k]$	$x[n]$	$x[n-k]$
2	$u[n]$	$u[n]$	$(n+1)\cdot u[n]$
3	$a^n u[n]$	$b^n u[n]$	$\left[\frac{a^{n+1}-b^{n+1}}{a-b}\right]u[n],\ a \neq b$
4	$u[n]$	$nu[n]$	$\frac{n(n+1)}{2}u[n]$
5	$a^n u[n]$	$a^n u[n]$	$(n+1)a^n u[n]$

10. When two LTI Discrete time systems with impulse response $h_1[n]$ and $h_2[n]$ respectively are connected in parallel, the composite parallel system impulse response is $h_1[n] + h_2[n]$, if these system are connected in cascade (in any order), the impulse response of the composite system is

$$h_1[n] * h_2[n] = h_2[n] * h_1[n].$$

11. Step response

$$s[n] = h[n] * u[n] = \sum_{k=0}^{\infty} h[n-k] = \sum_{k=-\infty}^{n} h[k],$$

$$h[n] = s[n] - s[n-1]$$

12. If $h[n] \neq 0$, for $n \neq 0$, the discrete-time LTI system has memory (or) $h[n] = 0$, for $n \neq 0$ system is memory less.
13. Causality condition is $h[n] = 0$ for $n < 0$, causal system output

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} x[k]h[n-k].$$

14. A DT, LTI system is BIBO stable if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.
15. Invertibility: $h[n] * h_{inv}[n] = \delta[n]$.

Table *The steps involved in calculation of convolution.*

Continuous-time Convolution	Discrete-time Convolution
$y(t) = x(t) * h(t) = h(t) *$ $x(t) = \int_{-\infty}^{+\infty} h(\tau) \times (t-\tau) d\tau$ $= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$	$y[n] = x[n] * h[n] = h[n] * x[n]$ $= \sum_{k=-\infty}^{+\infty} \times[k] h[n-k]$ $= \sum_{k=-\infty}^{+\infty} h[k] \times [n-k]$
$x(t) \rightarrow x(\tau)$ or $h(t) \rightarrow h(\tau)$	$h[n] \rightarrow h[k]$ or $x[n] \rightarrow x[k]$
Folding or flipping $x(-\tau)$ or $h(-\tau)$	Folding or flipping $x[-k]$ or $h[-k]$
Shifting $x(t-\tau)$ or $h(t-\tau)$	Shifting $x[n-k]$ (or) $h[n-k]$
Multiplication $x[t-\tau]\ h[\tau]$ or $h[t-\tau] \times [\tau]$	Multiplication $h[k]\ x[n-k]$ or $x[k]\ h[n-k]$
Integration	Summation

Solved Examples

Example 1: Consider the following signals $x(t) = \sin\pi t + \cos 2\pi t$.

$$y(t) = \sin 2t + \sin 5t,\ z(t) = \sin t + \cos\pi t$$

Periodic signals are?

(A) $x(t)$ and $z(t)$ (B) $x(t)$ and $y(t)$

(C) $y(t)$ and $z(t)$ (D) Only $y(t)$

Solution: (B)

$x(t)$ is periodic with period $T = 2$ sec and $y(t)$ is periodic with period $T = 2\pi$ sec. For $z(t)$ fundamental time period can't be determined.

Example 2: Consider the following signals with $z > 0$.

(a) $e^{-zt}\,u(t)$ (b) $e^{zt}u(t)$ (c) $tu(t)$

The energy signals are?

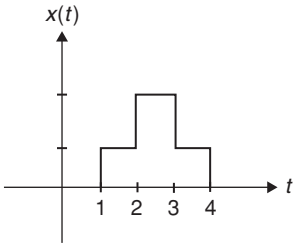
(A) a and b (B) b only

(C) a only (D) all

Solution: (C)

All integrable signals are energy signals. 'a' is only integrable signal and 'b', 'c' are not integrable signals.

Example 3: Consider the voltage waveform $x(t)$



The equation for $x(t)$ is?

(A) $u(t-1) - u(t-2) - u(t-3) - u(t-4)$

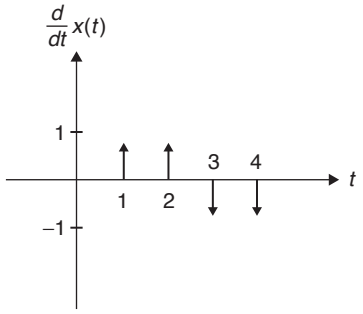
(B) $u(t-1) + u(t-2) - u(t-3) - u(t-4)$

(C) $u(t-1) + u(t-2) + u(t-3) - u(t-4)$

(D) $u(t-1) - u(t-2) + u(t-3) + u(t-4)$

Solution: (B)

Differentiate $x(t)$

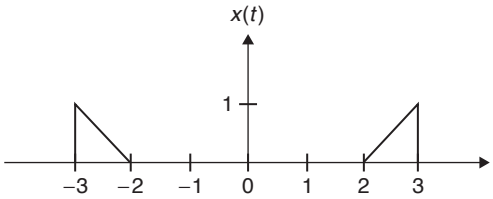


$$\frac{d}{dt}x(t) = \delta(t-1) + \delta(t-2) - \delta(t-3) - \delta(t-4)$$

Now integrate, then

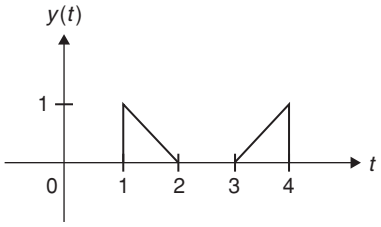
$$x(t) = u(t-1) + u(t-2) - u(t-3) - u(t-4)$$

Example 4: The signal $x(t)$ is depicted as

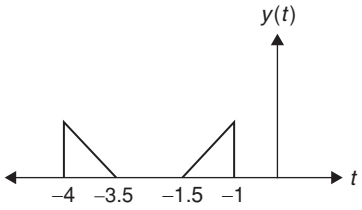


The signal $y(t)$ is related to $x(t)$ as $y(t) = x(2t+5)$, the sketch of $y(t)$ is?

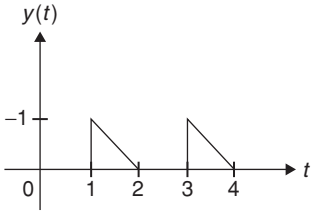
(A)



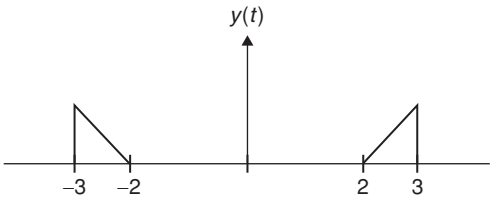
(B)



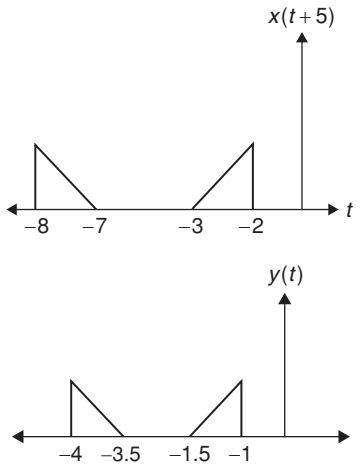
(C)



(D)



Solution: (B)



Example 5: The even part of a function $x(n) = u(n) + u(-n)$ is?

(A) $u(n) - u(-n)$ (B) $u(n) + u(-n)$

(C) $u(-n) - u(n)$ (D) $2u(n)$

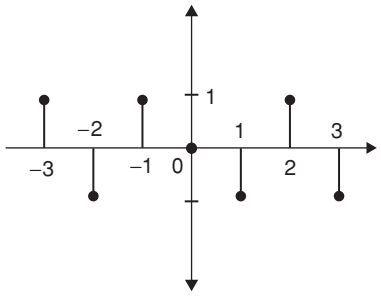
Solution: (B)

$$\text{Even part of } x(n) = \frac{x(n) + x(-n)}{2}$$

$$= \frac{u(n) + u(-n) + u(-n) + u(n)}{2}$$

$$x_1(n) = u(n) + u(-n)$$

Example 6: $x(n)$ is given as



Choose the correct option for $y(n) = x(1-n) - x(n-1)$?

(A) $\{-2, 2, -2, 0, 2, -2, 2\}$

(B) $\{-1, 1, -1, 0, 2, 2, 2\}$

(C) $\{2, -2, 2, 0, 2, -2, 2\}$

(D) $\{-2, 2, -2, 0, -2, 2, -2\}$

Solution: (A)

$$x(n-1) = \{1, -1, \underset{\uparrow}{1}, 0, -1, 1, -1\}$$

$$x(1-n) = \{1, 1, \underset{\uparrow}{-1}, 0, 1, -1, 1\}$$

$$x(1-n) - x(n-1) = \{-2, 2, \underset{\uparrow}{-2}, 0, 2, -2, 2\}$$

Example 7: A discrete time signal

$$x(n) = \cos\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi n}{8} + \frac{1}{2}\right), \text{ the period of } x(n)?$$

(A) Periodic with period 112

(B) Periodic with period 223

(C) Periodic with period 334

(D) Not periodic

Solution: (A)

$N_1 = 14, N_2 = 16$ $N = \text{LCM}(14, 16) = 112$

So $x(n)$ is periodic with period 112.

Example 8: $x(n) = \{1, 2, \underset{\uparrow}{-3}, 2, 1\}, h(n) = \{4, \underset{\uparrow}{-2}, 1\}$

Find $x[n] * h(n)$?

(A) $\{4, 6, -15, \underset{\uparrow}{16}, -3, 0, 1\}$

(B) $\{\underset{\uparrow}{4}, 6, -15, 16, -3, 0, 1\}$

(C) $\{4, 6, 15, \underset{\uparrow}{-16}, 3, 0, 1\}$

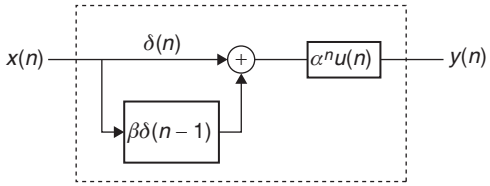
(D) $\{\underset{\uparrow}{4}, 6, 15, -16, 3, 0, 1\}$

Solution: (A)

$x(n) * h(n) = h(n+2) + 2h(n+1) - 3h(n) + 2h(n-1) + h(n-2)$

4	−2	1				
	8	−4	2			
		−12	6	−3		
			8	−4	2	
				4	−2	1
4	6	−15	16	−3	0	1

Example 9:



What is the transfer function $h(n)$

(A) $\alpha^{n+1}u(n) + \beta\alpha^{n-1}u(n-1) = h(n)$

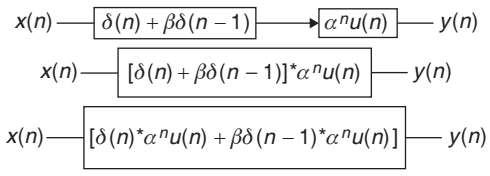
(B) $\alpha^n u(n-1) + \beta\alpha^{n-1}u(n) = h(n)$

(C) $\alpha^n u(n) + \beta\alpha^n u(n-1) = h(n)$

(D) $\alpha^n u(n) + \beta\alpha^{n-1}u(n-1) = h(n)$

Solution: (D)

Blocks are parallel, addition blocks are cascaded convolution so

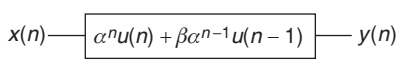


From basic formulae

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x(t) \cdot \delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

$$\int_{-\infty}^{\infty} x(t) * \delta(t - t_0)\,dt = x(t_0)$$



Example 10: The result of $u(t) * h(t)$, when $h(t) = \begin{cases} e^{at}, t < 0 \\ e^{-bt}, t > 0 \end{cases}$

(A) $\frac{1}{a}e^{at} + \frac{1}{a}(1-e^{at})u(t) + \frac{1}{b}(1-e^{-bt})u(t)$

(B) $\frac{1}{a}e^{at} - \frac{1}{a}(1-e^{at})u(t) + \frac{1}{b}(1-e^{bt})u(t)$

(C) $\frac{1}{a}e^{at} - \frac{1}{a}(1-e^{at})u(t) + \frac{1}{b}(1-e^{bt})u(t)$

(D) $\frac{1}{a}e^{-at} + \frac{1}{a}(1-e^{-at})u(t) + \frac{1}{b}(1-e^{bt})u(t)$

Solution: (A)

For $t < 0$, $\int_{-\infty}^{0} e^{at}dt = \frac{e^{at}}{a}$

For $t \geq 0$ $\int_{-\infty}^{0} e^{at}dt + \int_{-\infty}^{t} e^{-bt}dt$

$= \frac{1}{a} + \frac{1}{b} - \frac{1}{b}e^{-bt} = \frac{1}{a} + \frac{1}{b}(1-e^{-bt})$

$= u(t) * h(t) = \frac{e^{at}}{a}(1-u(t)) + \left(\frac{1}{a} + \frac{1}{b}(1-e^{bt})\right)u(t)$

$= \frac{1}{a}e^{at} + \frac{1}{a}(1-e^{at})u(t) + \frac{1}{b}(1-e^{-bt})u(t)$

Example 11: The discrete time input $x(n)$ and output $y(n)$ relation ship as $y(n) = \sqrt{nx(n)}$, then properties of the system are

(A) Linear, time in variant, causal, stable
(B) Non-linear, time in variant, causal, stable
(C) Non-linear, time variant, causal, unstable
(D) Linear, time in variant, causal, unstable

Solution: (C)

$$y_1(n) \to \sqrt{nx_1(n)}$$

$$y_2(n) \to \sqrt{nx_2(n)}$$

$$y_1(n) + y_2(n) \to \sqrt{n(x_1(n) + x_2(n)}$$

$$\neq \sqrt{nx_1(n)} + \sqrt{nx_2(n)}$$

It is not additive so it is non-linear

$$y(n) \to \sqrt{nx(n)}$$

$$y(n-n_0) \to \sqrt{(n-n_0)\times(n-n_0)} \neq \sqrt{nx(n-n_0)}$$

So it is time variant.

At any discrete time $n = n_0$, the response depends only on the excitation at that time. So 'causal' if the excitation is a constant the response is unbounded as 'n' approaches infinity so unstable.

Example 12: A continuous-time linear system with input $x(t)$ and output $y(t)$ yields the following input out pairs:

$$x(t) = e^{-jt} \leftrightarrow y(t) = e^{j3t}$$
$$x(t) = e^{jt} \leftrightarrow y(t) = e^{-j3t}$$

If $x(t) = \sin(t-1)$.

The corresponding $y(t)$ is

(A) $3\sin(t-1)$ (B) $\sin(3t-1)$
(C) $\sin(3t-3)$ (D) $\sin(t-3)$

Solution: (B)

$$\sin(t-1) = \frac{e^{j(t-1)} - e^{-j(t-1)}}{2j} = \frac{e^{-j}e^{jt} - e^{j}e^{-jt}}{2j}$$

Since system is linear

$$y(t) = \frac{e^{-j}e^{j3t} - e^{j}e^{-j3t}}{2j}$$

$$= \frac{e^{j(3t-1)} - e^{-j(3t-1)}}{2j} = \sin(3t-1)$$

Example 13: The rise of cosine pulse $x(t)$ is defined as

$$x(t) = \begin{cases} 1+\cos\omega t, & \frac{-\pi}{\omega} \leq t \leq \frac{\pi}{\omega} \\ 0, & \text{otherwise} \end{cases}.$$

The total energy of $x(t)$ is?

(A) $\frac{\pi}{\omega}$ (B) $\frac{2\pi}{\omega}$
(C) $\frac{3\pi}{\omega}$ (D) $\frac{4\pi}{\omega}$

Solution: (C)

$$\text{Energy} = \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} (1+\cos\omega t)^2 dt$$

$$= 2\int_{0}^{\frac{\pi}{\omega}} \left(\frac{1}{2}\cos 2\omega t + 2\cos\omega t + \frac{3}{2}\right)dt$$

$$= 2\left(\frac{3}{2}\right)\left(\frac{\pi}{\omega}\right) = \frac{3\pi}{\omega}$$

Example 14: A signal $v(n)$ is defined by

$$v(n) = \begin{cases} 1; & n = 1 \\ -1; & n = -1 \\ 0; & n = 0 \text{ and } |n| > 1 \end{cases}$$

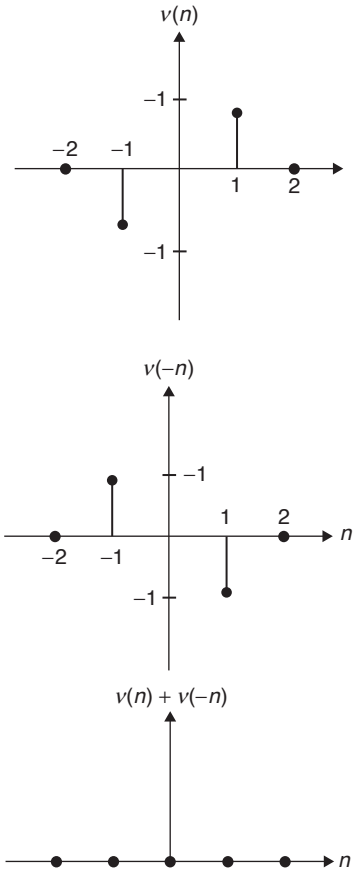
Which is the value of the composite signal defined as $v[n] + v[-n]$?

(A) 0 for all integer values of 'n'
(B) '2' for all integer value of 'n;'
(C) 1 for all integer values of 'n'
(D) –1 for all integer values of 'n'.

Solution: (A)

$$v(n) + v(-n) = 0$$

For all integer values of n



Example 15: The impulse response of a LTI system is given as $h(n)=\left(\frac{-1}{4}\right)^n u(n)$ the step response is

(A) $\frac{1}{4}\left[5+\left(\frac{-1}{5}\right)^n\right]$ (B) $\frac{1}{5}\left[4+\left(\frac{-1}{4}\right)^n\right]$

(C) $\frac{1}{4}\left[5-\left(\frac{-1}{4}\right)^n\right]$ (D) $\frac{1}{5}\left[4-\left(\frac{-1}{5}\right)^n\right]$

Solution: (B)

For $n < 0$, $s(n) = 0$

For $n \geq 0$, $s(n)=\sum_{k=0}^{n}\left(\frac{-1}{4}\right)^k$

$$=\frac{1-\left(\frac{-1}{4}\right)^{n+1}}{1+\frac{1}{4}}$$

$$=\frac{1}{5}\left(4+\left(\frac{-1}{4}\right)^n\right)$$

EXERCISES

Practice Problems 1

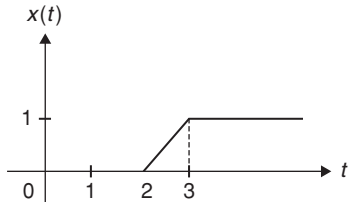
Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. The period of signal $x(t) = \sin t + \cos \sqrt{2}\, t$ is

(A) $\pi/\sqrt{2}$ (B) 2π

(C) 3π (D) not periodic

2. Consider the following signal $x(t)$



The function that describe the $x(t)$ is?

(A) $(t-2)\,u(t-2)+(t-3)\,u(t-3)$

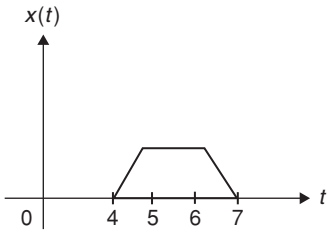
(B) $(t-2)\,u(t-2)-(t-3)\,u(t-3)$

(C) $(t-2)\,u(t-2)+(t+3)\,u(t+3)$

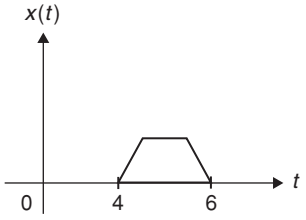
(D) $(t-2)\,u(t-2)-(t+3)\,u(t+3)$

3. A signal is described by $x(t) = r(t-4) - r(t-5) - r(t-6) + r(t-7)$, where $r(t)$ is a unit ramp function starting at $t = 0$. The signal $x(t)$ is represented as?

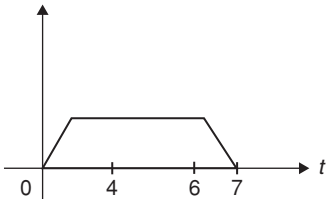
(A)



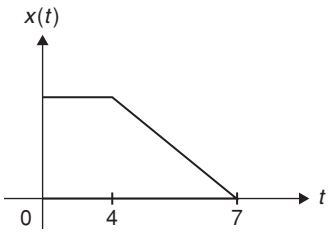
(B)



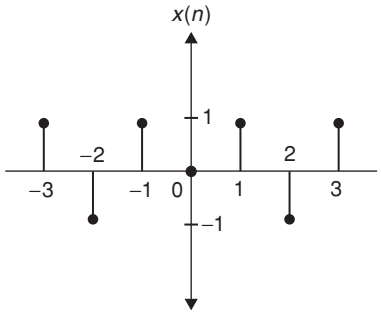
(C)



(D)

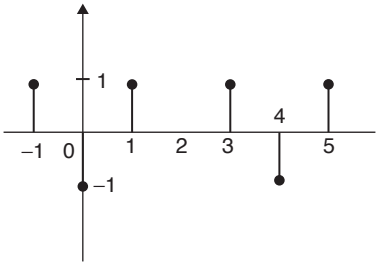


4. $x(n)$ is given as

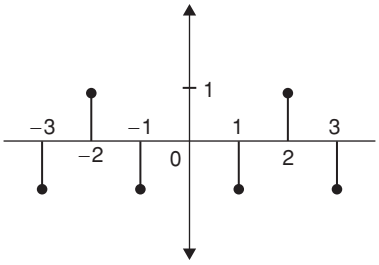


Choose the sketch for the signal $x(2-n)$?

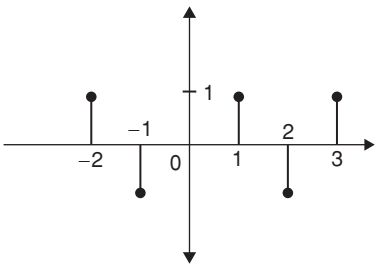
(A)



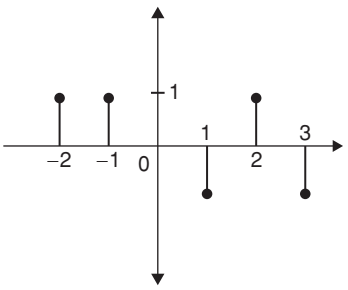
(B)



(C)



(D)

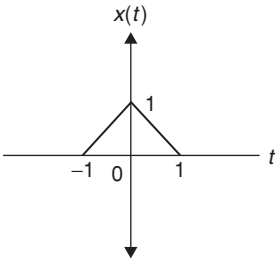


5. A discrete time signal $x(n)$ defined as $x(n) = e^{-j\left(\frac{n}{4}+\pi\right)}$ the period of signal $x[n]$ is?

(A) Periodic with $N = 8$ (B) Periodic with $N = 16$

(C) Periodic with $N = 4$ (D) None of the above

6. The signal $x(t)$ is

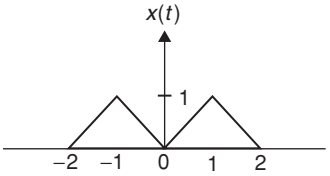


$x(t)$ is applied to a differentiator defined by $y(t) = \frac{dx(t)}{dt}$, The total energy of $y(t)$ is?

(A) 2 (B) 1

(C) 3 (D) 4

7. The signal $x(t)$ is



The total energy of $x(t)$ is?

(A) 4/3 (B) 3/4

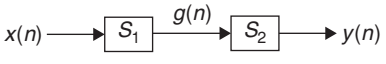
(C) 1 (D) 5/4

8. The sinusoidal signal $x(t) = 3\cos(100t + \pi/4)$ is passed through a square law device defined by the input, output relation $y(t) = x^2(t)$, the DC component in the signal is

(A) 9 (B) 9/2

(C) 9/4 (D) 9/8

9. Consider the following two system S_1 and S_2 as shown



S_1: causal LTI $g(n) = g(n-1) + x(n)$

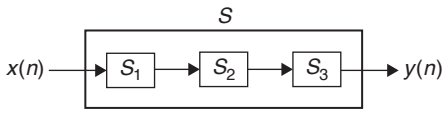
S_2: causal LTI $y(n) = ay(n-1) + bg(n)$

The difference equation for cascaded system is $y(n) = -\frac{1}{4}y(n-2) + \frac{5}{4}y(n-1) + \frac{1}{2}x(n)$ the values of a and b are?

(A) $\frac{1}{4}, \frac{1}{2}$ (B) $\frac{1}{2}, \frac{1}{2}$

(C) $\frac{1}{2}, \frac{1}{4}$ (D) $\frac{1}{4}, \frac{1}{4}$

10. Three discrete time systems S_1, S_2 and S_3 are connected in cascade to form a new system 'S' as



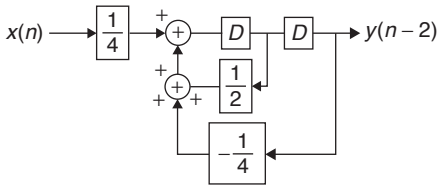
Consider the following statements.

(1) If S_1, S_2 and S_3 are linear, then 'S' is linear.
(2) If S_1, S_2 and S_3 are nonlinear, then 'S' is nonlinear.
(3) If S_1, S_2 and S_3 are causal, then 'S' is causal.
(4) If S_1, S_2 and S_3 are time invariant, then 'S' is the time invariant.

True statements are?

(A) 1, 2, 3 (B) 2, 3, 4
(C) 1, 3, 4 (D) All

11. The system shown as



(A) Stable and causal
(B) Stable but not causal
(C) Causal but unstable
(D) Unstable but not causal

12. The following signal

$y(n) = \sum_{k=n-n_0}^{n+n_0} x[k]$ is

(A) Causal (B) Non-causal
(C) Both (D) Anti-causal

13. $x(n) = \{1, 2, 0, 2, 1\}$, $h(n) = x(n)$ find $x(n) * h(n)$?

(A) $\{1, 4, 4, 4, \underset{\uparrow}{10}, 4, 4, 4, 1\}$ (B) $\{\underset{\uparrow}{1}, 4, 4, 4, 10, 4, 4, 4, 1\}$

(C) $\{1, 2, 3, 4, \underset{\uparrow}{10}, 4, 3, 2, 1\}$ (D) $\{\underset{\uparrow}{1}, 2, 3, 4, 10, 4, 3, 2, 1\}$

14. A Discrete time LTI system with input $u[n]$ produces output $\delta(n)$ then output due to the input $nu(n)$

(A) $u(n)$ (B) $u(n-1)$
(C) $u(n) - u(n-1)$ (D) $u(n) + u(n-1)$

15. A discrete time system has impulse response $h(n) = a^n u(n+2)|a| < 1$ which one of the following statement is correct.

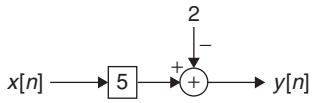
The system is

(A) Stable, causal and memory less
(B) Stable, non-causal and has memory
(C) Stable, non-causal and memory less
(D) Unstable, non-causal and memory less

16. Which one of the following systems having an impulse response, given is stable and causal LTI system?

(A) $h_1(t) = 1$ (B) $h_2(t) = u(t)$
(C) $h_3(t) = \frac{u(t+1)}{t+1}$ (D) $h_4(t) = e^{-2t}u(t)$

17. The discrete time input $x(n)$ and output $y(n)$ relationship as

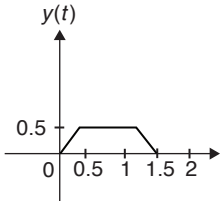


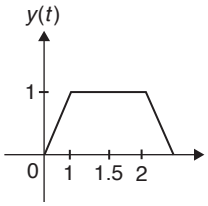
Determine system properties?

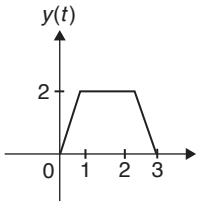
(A) Non-linear, time invariant, causal, stable
(B) Non-linear, time variant, causal, unstable
(C) Linear, time in variant, causal, stable
(D) Non-linear, time in variant, non-causal, stable

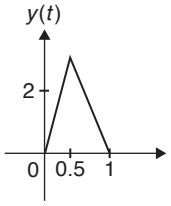
18. Suppose that $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$h(t) = x(2t)$ then $y(t) = x(t) * h(t)$ is

(A)


(B)


(C)


(D)


19. Consider the following systems $y_1(t) = x(t+4)$, $y_2(t) = \int_{-\infty}^{t} x(dT)\, dT$, $y_3(t) = 2\frac{d}{dt}x(t)$

The non invertible system is

(A) $y_1(t)$ and $y_2(t)$ (B) $y_1(t)$ and $y_3(t)$
(C) Only $y_3(t)$ (D) All the above

20. If the signals $f_1(t), f_2(t)$ are orthogonal over the interval (t_1, t_2), then

(A) $\int_{t_1}^{t_2} f_1(t) f_2(t) dt = \frac{\pi}{2}$ (B) $\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0$

(C) $\int_{t_2}^{t_1} f_1(t) * f_2(t) dt = 0$ (D) $\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 1$

21. What are the even and odd components of the signal $f(t) = e^{jt}$

(A) $\cos t, j\sin t$ (B) $\cos t, -j\sin t$
(C) $\sin t, j\cos t$ (D) $-\cos t, j\sin t$

22. A continuous-time LTI system has 'step response' $s(t)$ and impulse response $h(t)$ are related by

(A) $h(t) = \int s(t)dt$ (B) $h(t) = \frac{ds(t)}{dt}$

(C) $h(t) = s(t) + s(t-T)$ (D) Not related

23. $x(t) * u(t-t_0) =$

(A) $\int_{-\infty}^{\infty} x(t)dt$ (B) $\int_{-\infty}^{t_0} x(t)dt$

(C) $\int_{-\infty}^{t_0} x(\tau - t_0)d\tau$ (D) $\int_{-\infty}^{t-t_0} x(\tau)d\tau$

24. The range of value of a and b for stability of an LTI system $h[n] = (-3a)^n$; $n \geq 0$

$= b^{-n}$, $n < 0$

(A) $0 < |a| < \frac{1}{3}$; $0 < |b| < 2$

(B) $0 < |a| < \frac{1}{3}$; $0 < |b| < 1$

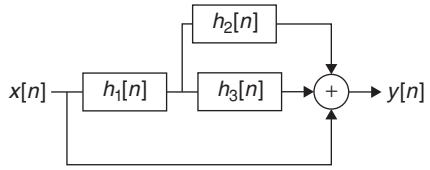
(C) $0 < |a| < \frac{1}{3}$; $0 < |b| < 3$

(D) $0 < |a| < \frac{1}{2}$; $0 < |b| < 2$

25. The system $y(t) = 2\sin t + 3\cos t$ is

(A) Non-linear, time variant, with memory, casual
(B) Non-linear, time invariant, memory less, casual
(C) Linear, time invariant, with memory, causal
(D) Linear, time variant, with memory, non-causal

26. The equivalent impulse response for this system



(A) $(h_1[n] + h_3[n]) * (h_1[n] + h_2[n]) + 1$
(B) $h_1[n] * (h_3[n] + h_2[n] + 1)$
(C) $h_1[n] * (h_3[n] + h_2[n]) + 1$
(D) $(h_3[n] + h_1[n]) * h_2[n] + 1$

27. The system $h[n] = \delta[n] + \delta[n+2] + \delta[n-2]$

(A) Causal and stable
(B) Causal but not stable
(C) Not causal but stable
(D) Neither causal nor stable

28. Given $x[n]$ defined in range $-6 \leq n \leq 3$, $y[n]$ in range $-2 \leq n \leq 7$ then the range of their convolution?

(A) $-6 \leq n \leq 7$ (B) $-2 \leq n \leq 3$
(C) $-8 \leq n \leq 7$ (D) $-8 \leq n \leq 10$

29. Response of a system to a complex input $x(t) = e^{j5t}$ is given as $y(t) = 5te^{j5t}$ then the system is

(A) Not LTI (B) LTI
(C) May be LTI (D) Not predictable

30. Impulse response of two LTI systems is given $h_1(t) = e^{-(5-4j)t}u(t)$, $h_2(t) = e^{-3t}\cos 4t \cdot u(t)$ the stable system is

(A) Only $h_1(t)$ (B) Only $h_2(t)$
(C) Both $h_1(t)$ and $h_2(t)$ (D) None of these.

Practice Problems 2

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Determine the fundamental period of these signals.

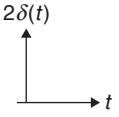
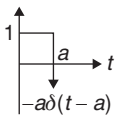
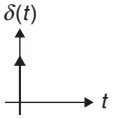
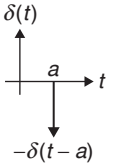
(1) $\cos\frac{\pi}{4}t + \sin\frac{\pi}{3}t$

(2) $\sin\sqrt{2}t + \cos 2t$

(A) 12, $\sqrt{2}$ (B) 12, $\frac{1}{\sqrt{2}}$

(C) 24, non-periodic (D) 24π, non-periodic

2. Match the first order derivatives of group (1) to group (2)

Group (1)		Group (2)	
P.	$u(t)$	1.	
Q.	$u(t) - u(t-a)$ $a > 0$	2.	
R.	$t(u(t) - u(t-a))$ $a > 0$	3.	
S.	Sgn(t)	4.	

(A) P – 1, Q – 4, R – 3, S –2
(B) P – 3, Q – 4, R – 2, S – 1
(C) P – 3, Q – 2, R – 4, S – 1
(D) P – 3, Q – 1, R – 2, S – 4

3. Periodic signals are

(A) $f[n + mN] = f[n]$ (B) $f(t - T) = f(t)$
(C) $f(t + T) = f(t)$ (D) A, B and C

4. Which is correct regarding power, and energy signals respectively?

(A) $0 < P < \infty$, $E = \infty$; $0 < E < \infty$, $P = 0$
(B) $0 < P < \infty$, $E = 0$; $0 < E < \infty$, $P = \infty$
(C) $0 < P < \infty$, $E = 0$; $0 < E < \infty$, $P = 0$
(D) $0 < P < \infty$, $E = \infty$; $0 < E < \infty$, $P = \infty$

5. Find the wrong statement

(A) $\int_{-\infty}^{\infty} \delta(t) = 1$

(B) $\delta(-t) = \delta(t)$

(C) $\sin\left(t - \frac{\pi}{2}\right)\delta(t) = -1$

(D) $\cos t \cdot \delta(t - \pi) = -\delta(t - \pi)$

6. Discrete time unit impulse can be obtained as

(A) $\delta[n] = \frac{du[n]}{dn}$

(B) $\delta[n] = u[n] - u[n-1]$

(C) A or B

(D) $\delta[n] = \int_{-\infty}^{\infty} u[n]dn$

7. For unit impulse signal of zero duration, the magnitude is

(A) 1

(B) °

(C) 0

(D) t

8. Match the following

Group (1)	Group (2)
P. $\int_{-\infty}^{\infty} \delta(t)dt$	1. 0
Q. $\int_{-\infty}^{\infty} x(\lambda)\cdot\delta(t-\lambda)d\lambda$	2. 1
R. $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt$	3. $\frac{1}{\lvert a\rvert}\delta(t)$
S. $\delta(at)$	4. $\lvert a\rvert\delta(t)$
	5. $x(t)$
	6. $x(\lambda)$

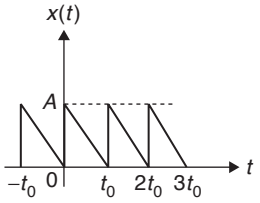
(A) P – 2, Q – 6, R – 5, S – 4

(B) P – 2, Q – 5, R – 6, S – 3

(C) P – 1, Q – 5, R – 6, S – 4

(D) P – 1, Q – 6, R – 5, S – 3

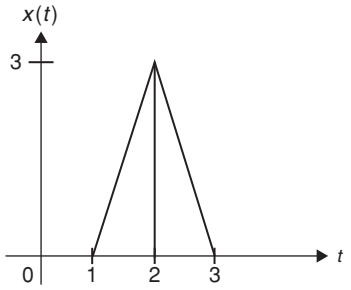
9. Find the value of $x(0)$ for the following sawtooth wave shown



(A) A (B) A/2

(C) Undefined (D) 0

10. The energy of signal is



(A) 3 joules (B) 9 joules

(C) 6 joules (D) 12 joules

11. Find the mutually orthogonal functions below.

(1) $\sin m\omega_0 t$, $\cos m\omega_0 t$ (2) $\sin m\omega_0 t$, $\sin n\omega_0 t$

(3) $\sin m\omega_0 t$, $\tan n\omega_0 t$ (4) $\cos n\omega_0 t$, $\cos m\omega_0 t$

(5) $\tan n\omega_0 t$, $\sec m\omega_0 t$

(A) 1, 2, 4 (B) 1, 2, 4, 5

(C) 2, 4 (D) 1, 3, 5

12. Two complex functions $f_1(t)$ and $f_2(t)$ are orthogonal over the interval (t_1, t_2) if

(A) $\int_{t_1}^{t_2} f_1(t) f_2^*(t)dt = 0$ (B) $\int_{t_1}^{t_2} f_1^*(t) f_2(t)dt = 0$

(C) Both A and B (D) $\int_{t_1}^{t_2} f_1(t) * f_2(t)dt = 0$

13. Which of the following is not correct about the signals $e^{j\omega t}$, $e^{-j\omega t}$?

(1) Both oscillate at same frequency ω

(2) One oscillate at 'ω' frequency, other at '$-\omega$'.

(3) Two phases rotating in opposite directions.

(4) When added, they yield a real function.

(A) 2, 4 (B) 1, 3, 4

(C) 4 (D) 2

14. If $y(t)$ is the out put of a CT–LTI system with input $x(t)$, the output of the system if the input is $\frac{dx(t)}{dt}$.

(A) $-ty(t)$ (B) $ty(t)$

(C) $\frac{d}{dt}y(t)$ (D) $\equiv y(t)dt$

15. If $h(t)$ is the impulse response of casual, linear, time invariant continuous, system. Then the output of the system for input $x(t)$ is

(A) $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ (B) $\int_{0}^{t} x(\tau)h(t-\tau)d\tau$

(C) $\int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau$ (D) $\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$

16. Which of the following system is time-invariant?

(A) $y[n] = x[n] - x[n-1]$ (B) $y[n] = nx[n]$

(C) $y[n] = x[-n]$ (D) $y[n] = n\,x^2[n]$

17. Which of the following system is time-variant?
(A) $y[n] = x[n] - x[n-1]$
(B) $y[n] = x[n] - 6x[n-1]$
(C) $y[n] = a^{x[n]}$
(D) $y[n] = nx^2[n]$

18. Which of the following system is linear?
(A) $y[n] = x^2[n]$ (B) $y[n] = Bx[n] + C$
(C) $y[n] = nx^2[n]$ (D) $y[n] = nx[n]$

19. Which of the following system is casual?
(A) $y[n] = x[n] + 3x[n+4]$
(B) $y[n] = x[n] - x[n-1]$
(C) $y[n] = x[n^2]$
(D) $y[n] = x[2n]$

20. Pick up the non-casual system?
(A) $y[n] = \sum_{m=-\infty}^{n} x[n]$ (B) $y[n] = ax[n]$
(C) $y[n] = x[-n]$ (D) $y[n] = nx[n]$

21. The response of the LTI system whose input $x[n] = \{\underset{\uparrow}{-1}, 2, 3, 1\}$ impulse response
$h[n] = \{1, 2, 1, -1\}$
(A) $\{\underset{\uparrow}{-1}, 0, 6, 10, 3, -2, -1\}$
(B) $\{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$
(C) $\{-1, \underset{\uparrow}{0}, 6, 10, 3, -2, -1\}$
(D) $\{1, 4, \underset{\uparrow}{8}, 8, 3, -2, -1\}$

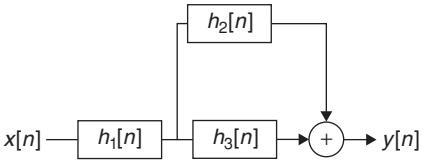
22. The discrete time system
$y[n] = x[n-3] - 4x[n-7]$ is a
(A) Dynamic system
(B) Time varying system
(C) Memory less system
(D) Non-linear system

23. Which of the following system is causal?
(A) $h[n] = n\left(\frac{1}{2}\right)^{n+1} u[n+1]$
(B) $y[n] = x^2[n] - x^2(n+1)$
(C) $y[n] = x[-n] + x[2n+1]$
(D) $h[n] = n\left(\frac{1}{2}\right)^{n+1} u[n]$

24. The system $y[n] = \sqrt{x[n]} + 3\sqrt{x[n+3]}$
(A) Non-linear, time variant, memory, causal, stable
(B) Linear, time invariant, memory less, causal unstable
(C) Non-linear, time invariant, memory less, non-causal, unstable
(D) Non-linear, time invariant, memory, non-causal, stable

25. $x(t) = u(t)$ is
(A) Power signal with power = –1
(B) Energy signal with energy = 1
(C) Power signal with power = 1
(D) Energy signal with energy = –1

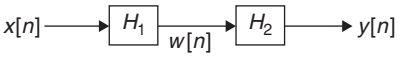
26. The equivalent impulse response of the system



(A) $(h_1[n] + h_3[n]) * (h_1[n] + h_2[n])$
(B) $(h_1[n] * h_3[n]) + h_1[n] * h_2[n]$
(C) $(h_1[n] + h_3[n]) * h_2[n]$
(D) $(h_1[n] + h_2[n]) * h_3[n]$

27. The period of the $x[n] = \cos\frac{\pi n}{4}\cos\frac{\pi n}{3}$
(A) 12
(B) 24
(C) 6
(D) Non-periodic

28. Consider the cascade of following two systems H_1 and H_2



H_1 is given by $w[n] = \frac{1}{4}w(n-1) + x[n]$
Now, H_2 is given by $y[n] = a\,y[n-1] + b\,w[n]$
The difference equation for cascaded system is $y[n] + \frac{1}{4}y\,[n-2] - \frac{5}{4}y(n-1) = x[n]$
Then the value of 'a' is
(A) 1
(B) 1/3
(C) 1/4
(D) 1/2

29. The value of b is
(A) 1
(B) 1/2
(C) 1/3
(D) 1/4

30. Impulse response of
$y[n] = x[n] + 2x[n-1] - 4 \times [n-2]$
(A) $\delta[n] - 2\delta[n-1]$
(B) $\delta[n] + 2\delta[n-1]$
(C) $\delta[n] + 2\delta[n-1] + 4\delta[n-2]$
(D) $\delta[n] + 2\delta[n-1] - 4\delta[n-2]$

Previous Years' Questions

1. A continuous-time system is described by $y(t) = e^{-|x(t)|}$, where $y(t)$ is the output and $x(t)$ is the input, $y(t)$ is bounded **[2006]**
(A) Only when $x(t)$ is bounded
(B) Only when $x(t)$ is non-negative
(C) Only for $t \geq 0$ if $x(t)$ is bounded for $t \geq 0$
(D) Even when $x(t)$ is not bounded

2. $y[n]$ denotes the output and $x[n]$ denotes the input of a discrete-time system given by the difference equation $y[n] - 0.8y[n-1] = x[n] + 1.25x[n+1]$. Its right-sided impulse response is **[2006]**
(A) Causal (B) Unbounded
(C) Periodic (D) Non-negative

3. Let a signal $a_1\sin(\omega_1 t + \phi_1)$ be applied to a stable linear time invariant system. Let the corresponding steady state output be represented as $a_2F(\omega_2 t + \phi_2)$. Then which of the following statements is true? **[2007]**
(A) F is not necessarily a 'sine' or 'cosine' function but must be periodic with $\omega_1 = \omega_2$.
(B) F must be a 'sine' or 'cosine' function with $a_1 = a_2$.
(C) F must be a 'sine' function with $\omega_1 = \omega_2$ and $\phi_1 = \phi_2$.
(D) F must be a 'sine' or 'cosine' function with $\omega_1 = \omega_2$.

4. If $u(t)$, $r(t)$ denote the unit step and unit ramp functions respectively and $u(t) * r(t)$ their convolution, then the function $u(t+1) * r(t-2)$ is given by **[2007]**
(A) $(1/2)(t-1)(t-2)$ (B) $(1/2)(t-1)(t-2)$
(C) $(1/2)(t-1)^2 u(t-1)$ (D) None of these

5. The integral $\frac{1}{2\pi}\int_0^{2\pi} \sin(t-\tau)\cos\tau\, d\tau$ equals **[2007]**
(A) $\sin t \cos t$ (B) 0
(C) $(1/2)\cos t$ (D) $(1/2)\sin t$

6. A signal $e^{-\alpha t}\sin(\omega t)$ is the input to a linear time invariant system. Given K and ϕ are constants, the output of the system will be of the form $Ke^{-\beta t}\sin(\upsilon t + \phi)$ where **[2008]**
(A) β need not be equal to a but α equal to ω
(B) υ need not be equal to ω but β equal to $\propto$
(C) β equal to $\propto$ and υ equal to ω
(D) β need not be equal to $\propto$ and υ need not be equal to ω

7. The impulse response of a causal linear time invariant system is given as $h(t)$. Now consider the following two statements
Statement (I): Principle of superposition holds.
Statement (II): $h(t) = 0$ for $t < 0$.
Which one of the following statements is correct? **[2008]**
(A) Statement (I) is correct and statement (II) is wrong.
(B) Statement (II) is correct and statement (I) is wrong.
(C) Both Statement (I) and statement (II) are wrong.
(D) Both Statement (I) and statement (II) are correct.

8. A system with input $x(t)$ and output $y(t)$ is defined by the input–output relation

$$y(t) = \int_{-\infty}^{-2t} x(t)\,dt$$

The system will be **[2008]**
(A) Causal, time-invariant and unstable.
(B) Causal, time-invariant and stable.
(C) Non-causal, time-invariant and unstable.
(D) Non-causal, time-variant and unstable.

9. Let $x(t)$ be a periodic signal with time period T. Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The fourier series coefficients of $y(t)$ are denoted by b. If $b_k = 0$ for all odd k, then t_0 can be equal to **[2008]**
(A) $\frac{T}{8}$ (B) $\frac{T}{4}$
(C) $\frac{T}{2}$ (D) $2T$

10. A linear time invariant system with an impulse response $h(t)$ produces output $y(t)$ when input $x(t)$ is applied. When the input $x(t-\tau)$ is applied to a system with impulse response $h(t-\tau)$, the output will be **[2009]**
(A) $y(t)$ (B) $y(2(t-\tau))$
(C) $y(t-\tau)$ (D) $y(t-2\tau)$

11. A cascade of 3 linear time invariant systems is causal and unstable. From this, we conclude that **[2009]**
(A) Each system in the cascade is individually causal and unstable.
(B) At least one system is unstable and at least one system is causal
(C) At least one system is causal and all systems are unstable.
(D) The majority are unstable and the majority are causal.

12. The period of the signal $x(t) = 8\sin\left\{0 \cdot 8\pi t + \frac{\pi}{4}\right\}$ is **[2010]**
(A) $0.4\pi s$ (B) $0.8\pi s$
(C) $1.25 s$ (D) $2.5 s$

13. The system represented by the input–output relationship $y(t) = \int_{-\infty}^{5t} x(\tau)\,d\tau,\ t > 0$ is **[2010]**
(A) Linear and causal
(B) Linear but non-causal
(C) Causal but non-linear
(D) Neither linear non-causal

14. At $t = 0$, the function $f(t) = \frac{\sin t}{t}$ has **[2010]**

(A) A minimum (B) A discontinuity
(C) A point of inflection (D) A maximum

15. Given two continuous-time signals $x(t) = e^{-t}$ and $y(t) = e^{-2t}$ which exist for $t > 0$, the convolution $z(t) = x(t) * y(t)$ is **[2011]**

(A) $e^{-t} - e^{2t}$ (B) e^{-3t}
(C) e^{+t} (D) $e^{-1} + e^{-2t}$

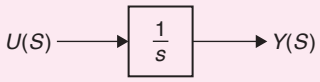
16. The response $h(t)$ of a linear time invariant system to an impulse $\delta(t)$, under initially relax condition is $h(t) = e^{-t} + e^{-2t}$. The response of this system for a unit step input $u(t)$ is

(A) $u(t) + e^{-t} + e^{-2t}$ (B) $(e^{-t} + e^{-2t})\, u(t)$
(C) $(1.5 - e^{-t} - 0.5e^{-2t})$ (D) $e^{-t}\, \delta(t) + e^{-2t}\, u(t)$

17. The input $x(t)$ and output $y(t)$ of a system are related as $y(t) = \int_{-\infty}^{t} x(\tau)\cos(3\tau)d\tau$. The system is **[2012]**

(A) Time-invariant and stable
(B) Stable and not time-invariant
(C) Time-invariant and not stable
(D) Not time-invariant and not stable

18. Assuming zero initial condition, the response $y(t)$ of the system given below to a unit step input $u(t)$ is **[2013]**



(A) $u(t)$ (B) $tu(t)$
(C) $\frac{t^2}{2}u(t)$ (D) $e^{-t}u(t)$

19. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t - 1)$, the output is **[2013]**

(A) $\frac{t^2}{2}u(t)$ (B) $\frac{t(t-1)}{2}u(t-1)$
(C) $\frac{(t-1)^2}{2}u(t-1)$ (D) $\frac{t^2-1}{2}u(t-1)$

20. Which one of the following statements is NOT TRUE for a continuous-time causal and stable LTI system? **[2013]**

(A) All the poles of the system must lie on the left side of the $j\omega$ axis.
(B) Zeros of the system can lie anywhere in the s-plane.
(C) All the poles must lie within $|s| = 1$.
(D) All the roots of the characteristic equation must be located on the left side of the $j\omega$ axis.

21. Two systems with impulse responses $h_1(t)$ and $h_2(t)$ are connected in cascade. Then the overall impulse response of the cascaded system is given by **[2013]**

(A) Product of $h_1(t)$ and $h_2(t)$
(B) Sum of $h_1(t)$ and $h_2(t)$
(C) Convolution of $h_1(t)$ and $h_2(t)$
(D) Subtraction of $h_2(t)$ from $h_1(t)$

22. The impulse response of a continuous-time system is given by $h(t) = \delta(t - 1) + \delta(t - 3)$. The value of the step response at $t = 2$ is **[2013]**

(A) 0 (B) 1
(C) 2 (D) 3

23. The response $y(t)$ to a unit step input is **[2013]**

(A) $\frac{1}{2} - \frac{1}{2}e^{-2t}$ (B) $1 - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$
(C) $e^{-2t} - e^{-t}$ (D) $1 - e^{-t}$

24. $x(t)$ is non-zero only for $T_x < t < T'_x$ and similarly, $y(t)$ is non-zero only $T_y < t < T'_y$. Let $z(t)$ be convolution of $x(t)$ and $y(t)$. Which one of the following statements is TRUE? **[2014]**

(A) $z(t)$ can be non-zero over an unbounded interval.
(B) $z(t)$ is non-zero for $t < T_x + T_y$.
(C) $z(t)$ is zero-outside of $T_x + T_y < t < T'_x + T'_y$.
(D) $z(t)$ is non-zero for $t > T'_x + T'_y$.

25. A discrete system is represented by the difference equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a & a-1 \\ a+1 & a \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

It has initial conditions $X_1(0) = 1$; $X_2(0) = 0$. The pole locations of the system for $a = 1$, are **[2014]**

(A) $1 \pm j0$ (B) $-1 \pm j0$
(C) $\pm 1 + j0$ (D) $0 \pm j1$

26. Consider an LTI system with impulse response $h(t) = e^{-5t}\, u(t)$. If the output of the system is $y(t) = e^{-3t}\, u(t) - e^{-5t}\, u(t)$ then the input, $x(t)$, is given by **[2014]**

(A) $e^{-3t}\, u(t)$ (B) $2e^{-3t}\, u(t)$
(C) $e^{-5t}\, u(t)$ (D) $2e^{-5t}\, u(t)$

27. Consider a continuous-time system with input $x(t)$ and output $y(t)$ given by

$y(t) = x(t) \cos(t)$

This system is **[2016]**

(A) linear and time-invariant
(B) non-linear and time-invariant
(C) linear and time-varying
(D) non-linear and time-varying

28. The value of $\int_{-\infty}^{+\infty} e^{-t}\delta(2t-2)dt, where \delta(t)$, is the Dirac delta function, is **[2016]**

(A) $\frac{1}{2e}$ (B) $\frac{2}{e}$
(C) $\frac{1}{e^2}$ (D) $\frac{1}{2e^2}$

29. Consider a causal LTI system characterized by differential equation $\frac{dy(t)}{dt}+\frac{1}{6}y(t) = 3x(t)$. The response of the system to the input $x(t) = 3e^{-\frac{t}{3}}u(t)$, where u(t) denotes the unit step function, is _____. **[2016]**

(A) $9e^{-\frac{t}{3}}u(t)$

(B) $9e^{-\frac{t}{6}}u(t)$

(C) $9e^{-\frac{t}{3}}u(t)-6e^{-\frac{t}{6}}u(t)$

(D) $54e^{-\frac{t}{6}}u(t)-54e^{-\frac{t}{3}}u(t)$

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** B	**3.** A	**4.** A	**5.** D	**6.** A	**7.** A	**8.** B	**9.** A	**10.** C
11. A	**12.** B	**13.** B	**14.** B	**15.** B	**16.** D	**17.** A	**18.** A	**19.** C	**20.** B
21. A	**22.** B	**23.** D	**24.** B	**25.** B	**26.** C	**27.** C	**28.** D	**29.** D	**30.** C

Practice Problems 2

1. C	**2.** B	**3.** D	**4.** A	**5.** C	**6.** B	**7.** B	**8.** B	**9.** B	**10.** C
11. A	**12.** C	**13.** D	**14.** C	**15.** B	**16.** A	**17.** D	**18.** D	**19.** B	**20.** C
21. A	**22.** A	**23.** D	**24.** D	**25.** C	**26.** B	**27.** B	**28.** A	**29.** A	**30.** D

Previous Years' Questions

1. D	**2.** A	**3.** D	**4.** C	**5.** D	**6.** A	**7.** D	**8.** D	**9.** B	**10.** D
11. B	**12.** D	**13.** B	**14.** B	**15.** A	**16.** C	**17.** D	**18.** B	**19.** C	**20.** C
21. C	**22.** B	**23.** B	**24.** C	**25.** A	**26.** B	**27.** C	**28.** A	**29.** D	

Chapter 2

Laplace Transform

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- The region of convergence
- Poles and zeroes
- Properties of ROC
- Laplace transforms of some elementary signals
- Inverse laplace transform
- Unilateral laplace transform
- The system function
- Stability
- Causal and stable system
- Invertibility
- Systems interconnections

Laplace Transform

A continuous time LTI system with impulse response $h(t)$, the output $y(t)$ of the system to the complex exponential input of the form e^{st} is

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau} \cdot d\tau$$

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

where, $H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$

The function $H(s)$ in above equation is referred to as the Laplace transform of $h(t)$. For a general continuous-time signal $x(t)$, the Laplace transform $X(s)$ is defined as,

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \text{ (bilateral Laplace transform)}$$

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt \text{ (unilateral Laplace transform)}$$

Both unilateral and bilateral transform are same if $x(t) = 0$, for $t < 0$.

Symbolically, the transform will be represented by

$$X(s) = L\{x(t)\}.$$

The Region of Convergence

The range of values of the complex variable 's' for which the Laplace transform converges is called the region of convergence (ROC).

Consider the signal $x(t) = e^{-at}u(t)$ then Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \left| \frac{-1}{s+a} e^{-(s+a)t} \right|_{0}^{\infty} = \frac{1}{s+a}, \text{ Re}(s) > -a,$$

because $\lim_{t \to \infty} e^{-(s+a)t} = 0$, when $\text{Re}(s + a) > 0$ or $\text{Re}(s) > -a$ is the ROC.

The convenient way to display the ROC is shown in the Figure 1, the variable 's' is a complex number, and in the figure, we display the complex plane, generally referred to as the s-plane, the horizontal and vertical axes are sometimes referred to as $\text{Re}\{s\}$ or σ-axis and the $\text{Im}\{s\}$- or $j\omega$-axis, respectively.

The shaded region in the figure represents the set of points in the s-plane corresponding to the ROC.

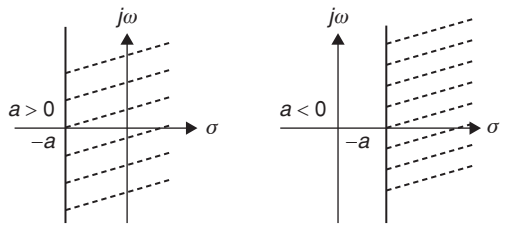


Figure 1 ROC for $X(s)$

The ROC of $X(s)$ is $\text{Re}(s) > -a$. This fact means that the integral defining $X(s)$, exists, only for the values of s in the shaded region, for other values of s, the integral does not converge. For this reason, the shaded region is called ROC or region of existence of $X(s)$.

Poles and Zeroes of $X(s)$

Usually, $X(s)$ will be a rational function in s, that is $X(s)=\frac{a_0}{b_0}\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$, where m, n are positive integers, $X(s)$ is proper rational function if $n > m$, the roots of numerator polynomial z_k are called the zeroes of $X(s)$, as $X(s) = 0$ for those values. Similarly, the roots of denominator p_k are called the poles of $X(s)$ as for those values $X(s)$ is infinite.

Therefore, the poles of $X(s)$ lie outside of ROC, since $X(s)$ does not converge at the poles, but zeroes may lie inside or outside of ROC.

Properties of ROC

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane and ROC does not contain any poles.
2. If $x(t)$ is finite-duration signal and absolutely integrable, i.e.,

   $$x(t) \neq 0,\ t_1 \le t \le t_2 = 0$$

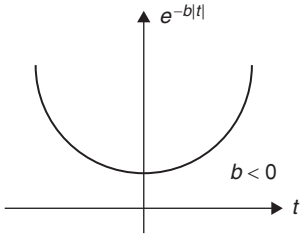
   otherwise, ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.
3. If $x(t)$ is a right-sided signal, that is $x(t) = 0$ for $t < t_1 < \infty$, then ROC is of the form $\mathrm{Re}(s) > \sigma_{max}$, where σ_{max} equals the maximum real part of any poles of $X(s)$, thus ROC is to the right of all the poles of $X(s)$.
4. If $x(t)$ is a left-sided signal, that is $x(t) = 0$ for $t > t_2 > -\infty$, then ROC is of the form $\mathrm{Re}(s) < \sigma_{min}$, where σ_{min} is the minimum real part of any of the poles of $X(s)$, thus ROC is to the left of all the poles of $X(s)$.
5. If $x(t)$ is a two-sided signal, that is $x(t)$ is an infinite-duration signal, then ROC is of the form $\sigma_1 < \mathrm{Re}(s) < \sigma_2$ where σ_1, σ_2 are real parts of the two poles of $X(s)$, thus ROC is a vertical strip in the s-plane between σ_1 and σ_2.

Example 1: Find the Laplace transform of $x(t) = e^{-b|t|}$.

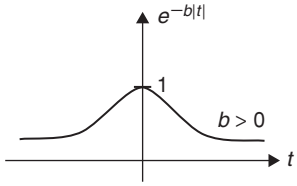
Solution: $x(t)$ be expressed as

$$X(t) = e^{-bt}u(t) + e^{+bt}u(-t)$$

So, the Laplace transform of $e^{-bt}u(t) \leftrightarrow \frac{1}{s+b}$, $\mathrm{Re}\{s\} > -b$ and for $e^{+bt}u(-t) \leftrightarrow \frac{-1}{s-b}$, $\mathrm{Re}\{s\} < +b$. If $b \le 0$, then there is no common region of ROC. Thus, for those values of b, $x(t)$ has no Laplace transform.



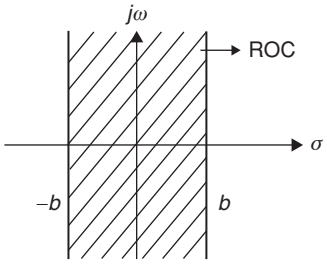
For $b < 0$, the signal is also not finite integrable, so no Laplace transform exists.



For $b > 0$, the signal is integrable, and there is common ROC is $-b < \mathrm{Re}\{s\} < +b$

$$e^{-b|t|} \xleftrightarrow{L} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2},\ -b < \mathrm{Re}\{s\} < +b$$

The corresponding pole zero plot is



Laplace Transforms of Some Elementary Signals

1. Unit impulse function $\delta(t)$

   $$L(\delta(t)] = \int_{-\infty}^{\infty} \delta(t)dt = 1, \text{ all } s$$

   $$L\{\delta(t-T)\} = e^{-sT}, \text{ all } s$$

2. Unit step function $u(t)$

   $$L[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt$$

   $$= \frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}\ \mathrm{Re}(s) > 0$$

3. Some other Laplace transform pairs

   $$L\{-u\ (-t)\} = \frac{1}{s},\ \mathrm{Re}(s) < 0$$

   $$L\{tu\ (t)\} = \frac{1}{s^2},\ \mathrm{Re}(s) > 0$$

   $$L\{t^k u\ (t)\} = \frac{k!}{s^{k+1}},\ \mathrm{Re}(s) > 0$$

x(t)	x(s)	ROC
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re(s) > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re(s) < -a$

(Continued)

x(t)	x(s)	ROC
$te^{-at}\,u(t)$	$\frac{1}{(s+a)^2}$	$Re(s) > -a$
$-te^{-at}\,u(-t)$	$\frac{1}{(s+a)^2}$	$Re(s) < -a$
$\cos \omega_0 t \cdot u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re(s) > 0$
$\sin \omega_0 t \cdot u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
$e^{-at}\cos \omega_0 t \cdot u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
$e^{-at}\sin \omega_0 t \cdot u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
$t^k e^{-at}\,u(t)$	$\frac{k!}{(s+a)^{k+1}}$	$Re(s) > -a$
$\frac{d^n}{dt^n}\delta(n)$	s^n	All s
$u(t) * u(t) * \ldots$ n times	$1/s^n$	$Re\{s\} > 0$

Properties of Laplace Transform

1. **Linearity:** Laplace transform is a linear operator, It holds the principle of super position.

   If $L\{x_1(t)\} = X_1(s)$, ROC $= R_1$

   $L\{x_2(t)\} = X_2(s)$, ROC $= R_2$

   $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s)$, ROC $= R_1 \cap R_2$

2. **Time shifting:** The time shifting property states that delaying a signal by t_0 seconds, amounts to multiplying its transform with e^{-st_0}.

   $L\{x(t)\} = X(s)$, ROC $= R$

   $L\{x(t-t_0)\} = e^{-st_0}\,X(s)$, ROC $= R$

3. **Shifting in *s*-domain:** Shifting in s-domain (or) frequency shifting property states that frequency shift by s_0 is same as multiplying its inverse transform with e^{sot}.

   $$e^{s_o t}x(t) \leftrightarrow x(s-s_0), \text{ROC} = R + \text{Re}(s_0)$$

   If the Laplace transform of $x(t)$ has a pole or zero at $s = a$, then the Laplace transform of $e^{j\omega_0 t}x(t)$ has a pole or zero at $s = a + j\omega_0$.

4. **Time scaling**

   $$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right), \text{ROC} = R/\text{a}$$

   x(at) is the signal $x(t)$, time compressed by the factor a, and $X\left(\frac{s}{a}\right)$ is $X(s)$ expanded along the s-scale by the same factor a.

   The scaling property states that time compression of a signal by a factor 'a' causes expansion of its Laplace transform in the s-scale by the same factor.

   Similarly, time expansion $x(t)$ causes compression of $X(s)$ in the s-scale by the same factor.

5. **Time reversal**

   $$x(-t) \leftrightarrow X(-s), \text{ROC} = -R$$

6. **Differentiation in time-domain**

   $$\frac{d}{dt}[x(t)] \leftrightarrow sX(s) - x(0^-), \text{ROC} \supset R$$

   Repeated application of this property yields

   $$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

   $$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX(s) - \sum_{k=1}^{n} s^{n-k}\,\frac{d^{k-1}x(0^-)}{dt^{k-1}}$$

7. **Differentiation in the *s*-domain:** The dual of the time-differentiation property is the frequency-differentiation property.

   $$-tx(t) \leftrightarrow \frac{d}{ds}X(s), R^1 = R$$

8. **Integration in the time domain**

   $$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)dt,$$

   ROC $= R \cap \{\text{Re}(s) > 0\}$

9. **Time convolution**

   $$x_1(t) * x_2(t) \leftrightarrow X_1(s) \cdot X_2(s), \text{ROC} \supset R_1 \cap R_2.$$

   We can apply the time-convolution property to the LTI system input–output relationship $y(t) = x(t) * h(t)$ to obtain $Y(s) = X(s)\,H(s)$.

   $$H(s) = \frac{Y(s)}{X(s)} = \text{transfer function } H(s).$$

10. **Time differentiation**

    $$\frac{d}{dt}x(t) \leftrightarrow sX(s) - x(0^-)$$

    $$\frac{d^2}{dt^2}x(t) \leftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$$

    $$\frac{d^n}{dt^n}x(t) \leftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x' (0^-) - \cdots - SX^{(n-2)}(0) - X^{n-1}(0^-)$$

11. **Frequency integration:** The dual of the time-integration property is the frequency-integration property.

    $$\frac{x(t)}{t} \leftrightarrow \int_{s}^{\infty} X(p)dp$$

12. Frequency convolution

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

13. Initial value theorem

$$x(0^+) = \underset{s\to\infty}{\text{Lt}}\, sX(s)$$

14. Final value theorem

$$\underset{s\to 0}{\text{Lt}}\, sX(s) = x(\infty)$$

It is desirable to know the values of $x(t)$ as $t \to 0$ and $t \to \infty$ (initial and final values of $x(t)$) from the knowledge of its Laplace transform $X(s)$, then initial and final value theorems provide such information.

15. Conjugation property: $x(t) \leftrightarrow X(s)$ with ROC = R, then conjugate of $x(t)$, $x^*(t) \leftrightarrow X^*(s^*)$ with ROC = R. When $x(t)$ is real, $x(t) = x^*(t)$, so

$$X(s) = X^*(s^*).$$

Consequently if $x(t)$ is real, and if $X(s)$ has a pole/zero at $s = s_0$, then $X(s)$ also has pole or zero at the complex point $s = s^*_0$.

Inverse Laplace Transform

$$x(t) \leftrightarrow X(s)$$
$$x(t) = L^{-1}\{X(s)\}$$
$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

In this integral, the real 'σ' is to be selected such that if the ROC of $X(s)$ is $\sigma_1 < \text{Re}(s) < \sigma_2$, then $\sigma_1 < \sigma < \sigma_2$.

Unilateral Laplace Transform

The Laplace transforms for the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$ are identical except for their regions of convergence. Therefore, for a given $X(s)$, there may be more than one inverse transform, depending on the ROC. If we restrict all our signals to the causal type, such an ambiguity does not arise. There is only one inverse transform of $X(s) = \frac{1}{(s+a)}$, namely $e^{-at}u(t)$. To find $x(t)$ from $X(s)$, we need not even specify ROC. If all signals are restricted to the causal type, then for a given $X(s)$, there is only one inverse Laplace transform $x(t)$.

The unilateral Laplace transform is a special case of the bilateral Laplace transform in which all signals are restricted to being causal.

The unilateral Laplace transform $X(s)$ of $x(t)$ is

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt.$$

Example 1: What is the Laplace transform of the signal $x(t) = e^{-3t}\,u(t) + e^{-t}(\cos 2t)\,u(t)$?

Solution: $x(t) = \{e^{-3t} + \frac{1}{2}(e^{2jt} + e^{-2jt})\cdot e^{-t}\}u(t)$

$$= e^{-3t}u(t) + \frac{1}{2}e^{(-1+2j)t}u(t) + \frac{1}{2}\cdot e^{(-1-2j)t}u(t)$$

By taking Laplace transform

$$X(s) = \frac{1}{s+3} + \frac{1}{2}\cdot\frac{1}{s+(1+2j)} + \frac{1}{2}\cdot\frac{1}{s+(1-2j)}$$ and the ROC is $\text{Re}\{s\} > -3$ for first term, for second and third terms $\text{Re}\{s\} > -1$.

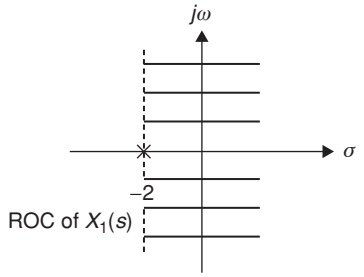
For all three Laplace transforms to converge simultaneously, we must have $\text{Re}\{s\} > -1$

Consequently, $X(s) = \frac{1}{s+3} + \frac{1}{2}\cdot\frac{2s+2}{s^2+2s+5}$

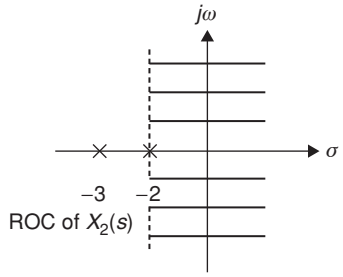
$$= \frac{1}{s+3} + \frac{s+1}{s^2+2s+5},\ \text{Re}\{s\} > -1$$

Example 2: Consider $X_1(s) = \frac{1}{s+2}$ $\text{Re}\{s\} > -2$ and $X_2(s) = \frac{1}{(s+2)(s+3)}$, $\text{Re}\{s\} > -2$. Then what is the ROC of $X(s) = X_1(s) - X_2(s)$?

Solution: $X_1(s) = \frac{1}{s+2}, \text{Re}\{s\} > -2$



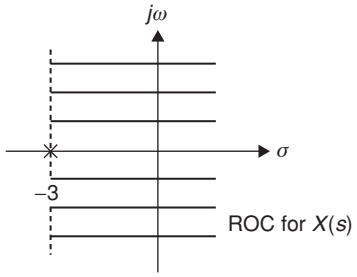
$$X_2(s) = \frac{1}{(s+2)(s+3)},\ \text{Re}\{s\} > -2$$



$$X(s) = X_1(s) - X_2(s)$$
$$= \frac{1}{s+2} - \frac{1}{(s+2)(s+3)} = \frac{s+3-1}{(s+2)(s+3)}$$
$$= \frac{(s+2)}{(s+2)(s+3)} = \frac{1}{s+3},\ \text{Re}\{s\} > -3$$

Thus, in the linear combination of $X_1(s)$ and $X_2(s)$, the pole at $s = -2$ is cancelled by a zero at $s = -2$. The intersection of the ROC for $X_1(s)$ and $X_2(s)$ is $\text{Re}\{s\} > -2$.

However, since the ROC is always bounded by a pole or infinity, for this example, the ROC of $X(s)$ can be extended to the left to be bounded by the pole at $s = -3$, as a result of the pole-zero cancellation at $s = -2$.



Example 3: What is the inverse Laplace transform of $X(s) = \dfrac{2s^2 + 5}{s^2 + 3s + 2}$?

Solution: $X(s) = \dfrac{2s^2 + 5}{s^2 + 3s + 2} = 2 + \dfrac{A}{(s+1)} + \dfrac{B}{(s+2)}$

where, $A = (s+1)X(s)\big|_{s=-1} = \dfrac{2+5}{-1+2} = 7$

$$B = (s+2)X(s)\big|_{s=-2} = \frac{8+5}{-2+1} = -13$$

So, $X(s) = 2 + \dfrac{7}{s+1} - \dfrac{13}{s+2}$

By inverse Laplace transform

$$x(t) = 2\delta(t) + 7e^{-t}u(t) - 13e^{-2t}u(t)$$

Example 4: Find the inverse Laplace transform of $X(s) = \dfrac{s + 2 + 4e^{-3s}}{(s+3)(s+1)}$.

Solution: Observe the exponential term e^{-3s} in numerator of $X(s)$, indicating time delay.

$$X(s) = \frac{s+2}{(s+3)(s+1)} + \frac{4e^{-3s}}{(s+3)(s+1)}$$
$$= X_1(s) + X_2(s)\, e^{-3s}$$

So, $X_1(s) = \dfrac{s+2}{(s+1)(s+3)} = \dfrac{1}{2}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{1}{s+3}$

By inverse Laplace transform

$$x_1(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Similarly, $X_2(s) = \dfrac{4}{(s+1)(s+3)} = \dfrac{2}{s+1} - \dfrac{2}{s+3}$

By inverse Laplace transform

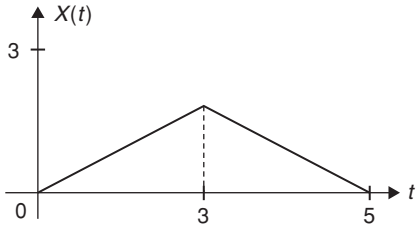
$$x_2(t) = 2e^{-t}u(t) - 2e^{-3t}u(t)$$

$$X(s) = X_1(s) + X_2(s)\, e^{-3s}$$

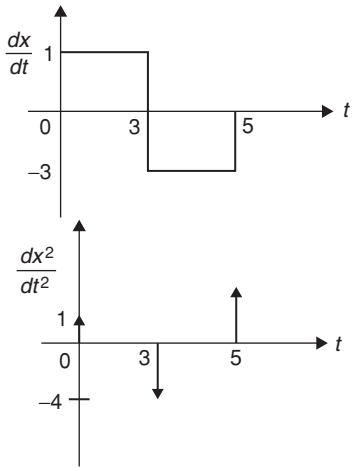
So, $x(t) = x_1(t) + x_2(t-3)$

$$= \frac{1}{2}[e^{-t} + e^{-3t}]u(t) + 2[e^{-(t-3)} - e^{-3(t-3)}]u(t-3)$$

Example 5: Find the Laplace transform of the signal $x(t)$ depicted in figure.



Solution: By considering the derivative of $x(t)$



$$\Rightarrow \frac{d^2x}{dt^2} = \delta(t) - 4\delta(t-3) + 3\delta(t-5)$$

By applying Laplace transform

$$L\left\{\frac{d^2x}{dt^2}\right\} = s^2\, X(s) = 1 - 4e^{-3s} + 3e^{-5s}$$

$$X(s) = \frac{1}{s^2}(1 - 4e^{-3s} + 3e^{-5s})$$

Analysis of Continuous Time LTI System

The System Function

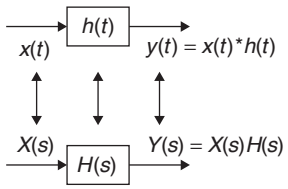
The output of continuous time LTI system equals to the convolution of the input $x(t)$ with the impulse response $h(t)$ that is

$$y(t) = x(t) * h(t)$$

By applying Laplace transform

$$Y(s) = X(s) \cdot H(s)$$

and $$H(s) = \frac{Y(s)}{X(s)}$$



The Laplace transform $H(s)$ of $h(t)$ is referred to as the system function (or the transfer function) of the system.

Characteristics of Continuous Time LTI System

Many properties of continuous time LTI systems can be closely associated with the characteristics of $H(s)$ in s-plane and ROC.

Causality

For causal continuous time LTI system, we have $h(t) = 0$, $t < 0$ Since h(t) is right handed signal, the corresponding $H(s)$, will have ROC Re(s) > σ_{max}.

In other words, for a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane, to the right of the right most pole.

Similarly, for Anti causal $h(t) = 0$, $t > 0$ and $H(s)$ ROC will be Re(s) < σ_{min} (left-half plane).

Stability

A continuous time LTI system is BIBO stable if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$.

The corresponding $H(s)$, ROC contains the $j\omega$-axis or an LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis [i.e., Re$\{s\} = 0$].

Causal and Stable System

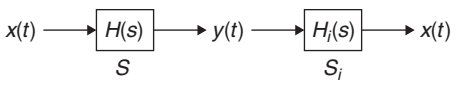
If the system is both causal and stable, then the poles of $H(s)$ must lie in the left half of s-plane.

Since ROC is of form Re(s) > $\sigma_{max,}$ and $j\omega$-axis is included in ROC, we must have $\sigma_{max} < 0$. In other terms, a causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane, i.e., all of the poles have negative real parts.

Invertibility

If $H(s)$ is the transfer function of a system S, then S_i, its inverse system has a transfer function. $H_i(s)$ given by

$$H_i(s) = \frac{1}{H(s)}$$



The cascade of systems S and S_i is an identity system with impulse response $\delta(t)$.

Implying $H(s) \cdot H_i(s) = 1$

Consider an ideal differentiator system $y(t) = \frac{d}{dt}x(t)$ its Laplace transform $Y(s) = s\,X(s)$ (for causal system $x(0^-) = 0$).

So, $H(s) = \frac{Y(s)}{X(s)} = s$ is transfer function.

For an ideal integrator system with zero initial state

$$y(t) = \int_0^t x(\tau)d\tau \rightarrow Y(s) = \frac{1}{s}X(s)$$

Transfer function $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$

From the above discussion, an ideal integrator and its inverse, an ideal differentiator have transfer functions $1/s$ and s respectively, leading to $H(s) \cdot H_i(s) = 1$.

Examples 6: Check the causality of the system function

$$H(s) = \frac{e^s}{s+1},\ \text{Re}\{s\} > -1.$$

Solution: The ROC is to the right of the right most pole. Therefore, impulse response must be one sided.

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}\ \text{Re}\{s\} > -1$$

By using time-shift property

$$x(t - t_0) \leftrightarrow e^{-t_0 s}X(s)$$

$$e^{-(t+1)}u(t+1) \leftrightarrow \frac{e^s}{s+1}\ \text{Re}\{s\} > -1$$

So, impulse response $h(t) = e^{-(t+1)}\,u(t+1)$ which is non zero for $-1 < t < 0$, hence the system is not causal.

Note: The causality implies that the ROC is to the right of the right most pole, but the converse is not in general true, unless the system function is rational.

Example 7: Consider an LTI system with system function $H(s) = \frac{s-1}{(s+2)(s-3)}$. Check the stability and causality for different ROCs.

Solution: $H(s) = \frac{s-1}{(s+2)(s-3)}$

$$= \frac{3}{5}\cdot\frac{1}{s+2} + \frac{2}{5}\frac{1}{s-3}$$

Possible ROCs are Re$\{s\} > 3$, Re$\{s\} < -2$ and $-2 <$ Re $\{s\} < 3$.

When ROC is Re$\{s\} > 3$,

$$h(t) = \frac{3}{5}e^{-2t}u(t) + \frac{2}{5}e^{+3t}u(t)$$

$$= \frac{1}{5}\{3e^{-2t} + 2e^{3t}\}u(t)$$

The system is anti-causal, and unstable.

When ROC is $-2 < \text{Re}\{s\} < 3$,

$$\text{Re}\{s\} < 3 \rightarrow -e^{3t}u(-t) \leftrightarrow \frac{+1}{s-3}$$

$$\text{Re}\{s\} > -2 \rightarrow e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}$$

So, $h(t) = \frac{3}{5} \cdot e^{-2t}u(t) - \frac{2}{5}e^{3t}u(-t)$

The system is non-causal and stable.

When ROC is $\text{Re}\{s\} < -2$,

$$h(t) = -\frac{3}{5} \cdot e^{-2t}u(-t) - \frac{2}{5}e^{3t}u(-t)$$

The system is anti-causal and unstable.

System Function for LTI Systems Described by Linear Constant-coefficient Differential Equation

An LTI system with input $x(t)$, and output of $y(t)$ in differential equation form

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Applying Laplace transform and using differentiation property

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)\Big|_{\text{Zero initial condition}}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Examples 8: Consider an LTI system for which the input $x(t)$ and output $y(t)$ satisfy the linear constant co-efficient differential equation $\frac{dy(t)}{dt} + 2y(t) = x(t)$ with zero initial conditions. Find the impulse response of the system (1) when it is causal system and (2) when it is anti causal system.

Solution: $\frac{dy(t)}{dx} + 2y(t) = x(t)$

By applying Laplace transform $sY(s) + 2Y(s) = X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$$

The above expression is system function, but we don't know the ROC of the system. When the system is causal, the ROC can be inferred to be to the right of the right most pole, which is this case corresponds to $\text{Re}\{s\} > -2$.

Then $h(t) = e^{-2t}\,u(t)$

When the system is anti causal, the ROC associate with $H(s)$ would be $\text{Re}\{s\} < -2$. Then $h(t) = -e^{-2t}\,u(-t)$.

Example 9: If input to an LTI system is $x(t) = e^{-2t}u(t)$ then the output is $y(t) = [e^{-t} - e^{-3t}]u(t)$. What is the transfer function of the system, and the differential equation of the system with zero initial conditions?

Solution: By Laplace transform for $x(t)$, $y(t)$ we get $X(s) = \frac{1}{s+2}$, $\text{Re}\{s\} > -2$ and $Y(s) = \frac{1}{s+1} - \frac{1}{s+3}$, $\text{Re}\{s\} > -1$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2/(s+1)(s+3)}{1/s+2} = \frac{2(s+2)}{(s+1)(s+3)}$$

According to convolution property, the ROC of $Y(s)$ must include at least the inter sections of the ROCs of $X(s)$ and $H(s)$.

So, $H(s)$ ROC should be $\text{Re}\{s\} > -1$ and $H(s)$ is causal.

Since both poles of $H(s)$ have negative real parts it follows that the system is stable.

We can write $\frac{Y(s)}{X(s)} = \frac{2(s+2)}{(s+1)(s+3)} = \frac{2s+4}{s^2+4s+3}$

$$s^2Y(s) + 4sY(s) + 3 \cdot Y(s) = 2s\,X(s) + 4X(s)$$

$\frac{d^2}{dt^2}y(t) + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$ is the differential equation of the system specified.

Example 10: From the following information about an LTI system, find the system function of that LTI system:

1. The system is causal.
2. The system function is rational and has only one pole at $s = -1$ and $s = 3$
3. if $x(t) = 1$, then output $y(t) = 0$
4. The value of the impulse response at $t = 0^+$ is 2.

Solution: Consider the system function $H(s) = \frac{P(s)}{Q(s)}$, and $Q(s) = (s + 1)\,(s - 3)$.

$$H(s) = \frac{P(s)}{(s+1)(s-3)}$$

$y(t)$ is the output of the system for input $x(t) = e^{st}$, then $y(t) = H(s)\,e^{st}$, here for $x(t) = 1 = e^{0t}$.

$$y(t) = H(0)e^{0t} = 0,\ H(0) = 0$$

Which means that $H(s)$ must have a zero at $s = 0$. So, $P(S) = s \cdot P^1(s)$.

The value of impulse response at

$$t = 0 + h(t)\Big|_{t=0^+} = \underset{s\to\infty}{\text{Lt}}\, sH(s)$$

$$= \underset{s\to\infty}{\text{Lt}}\, \frac{s^2P^1(s)}{s^2 - 2s - 3} = 2$$

So, $P^1(s) = 2$, i.e., it is a constant

So, $H(s) = \dfrac{2s}{(s+1)(s-3)}$

And the system is unstable as it has a pole at $s = 3$ with positive real part.

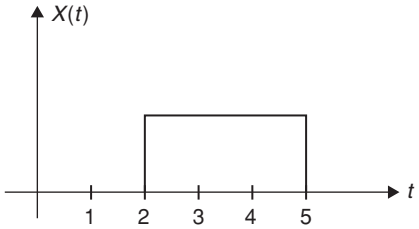
Example 11: A system is described by the differential equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y(t) = x(t)$.

Let $x(t)$ be a rectangular pulse given by

$$x(t) = \begin{cases} 1 & 2 < t < 5 \\ 0 & \text{otherwise} \end{cases}$$

Assuming zero initial conditions, the Laplace transform of $y(t)$ is?

Solution: $x(t) = \begin{cases} 1 & 2 < t < 5 \\ 0 & \text{otherwise} \end{cases}$



$$x(t) = u(t-2) - u(t-5)$$

By taking Laplace transform

$$X(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

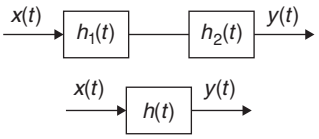
For the given differential equation, by considering Laplace transform $s^2Y(s) + 4sY(s) + 3 = X(s)$

$$Y(s) = \frac{X(s)}{s^2 + 4s + 3} = \frac{e^{-2s} - e^{-5s}}{s(s+1)(s+3)}.$$

Systems Interconnections

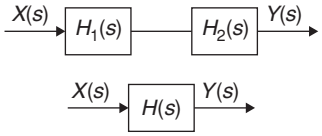
Two systems in cascade,

(a)



$h(t) = h_1(t) * h_2(t)$

(b)

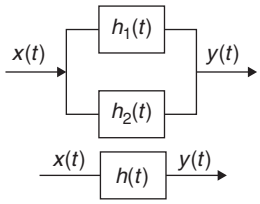


$H(s) = H_1(s) \cdot H_2(s)$

(i) Time-domain, (ii) s-domain

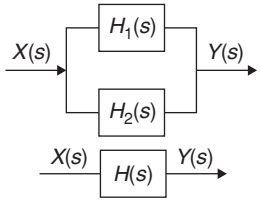
Two systems in parallel

(a)



$h(t) = h_1(t) + h_2(t)$

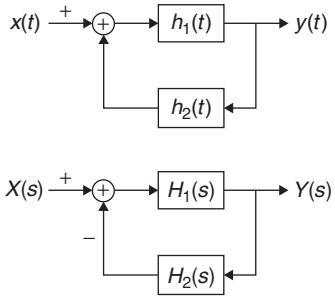
(b)



$H(s) = H_1(s) + H_2(s)$

(i) Time-domain, (ii) s-domain

Feedback interconnection of two LTI systems.



$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Solved Examples

Example 1: $X(s) = e^{-s}\left[\dfrac{-2}{s(s+2)}\right]$. What are the initial and final values of $x(t)$?

(A) -1 and 0 (B) 0 and 1

(C) 0 and -1 (D) 1 and 1

Solution: (C)

Initial value theorem

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} se^{-s}\left[\frac{-2}{s(s+2)}\right] = 0$$

Final value theorem $x(\infty) = \lim_{s\to 0} sX(s)$

$$= \lim_{s\to 0} se^{-s}\frac{-2}{s(s+2)} = -1$$

Example 2: Using Laplace transform find out output signal. when $h(t) = e^{-2t}u(t)$, $t > 0$; $x(t) = e^{-6t}u(t)$, $t > 0$; $y(t) = ?$

(A) $\left(\dfrac{1}{4}e^{-2t} - \dfrac{1}{4}e^{-6t}\right)u(t) = y(t)$, $\text{Re}(s) > -2$

(B) $\left(\frac{1}{4}e^{-2t}+\frac{1}{4}e^{-6t}\right)u(t)= y(t)$, $\text{Re}(s) > 2$

(C) $\left(-\frac{1}{4}e^{-2t}-\frac{1}{4}e^{6t}\right)u(t)= y(t)$, $\text{Re}(s) > -2$

(D) $\left(\frac{1}{4}e^{-2t}-\frac{1}{4}e^{-6t}\right)u(t)= y(t)$, $\text{Re}(s) > -2$

Solution: (A)

$$h(t) = e^{-2t}\,u(t)$$

$$x(t) = e^{-6t}\,u(t)$$

$$X(s),\ x(t) \rightarrow \boxed{h(t)/H(s)} \rightarrow Y(s),\ y(t)$$

Time response

$$\text{output } y(t) = h(t) * x(t)$$

Frequency response

$$Y(s) = X(s)\,H(s)$$

$$Y(s) = \frac{1}{s+6}\cdot\frac{1}{s+2},\ \text{Re}(s) > -2 \text{ (or) } \sigma > -2$$

$$Y(s) = \frac{A}{s+6}+\frac{B}{s+2}$$

By using factorial method

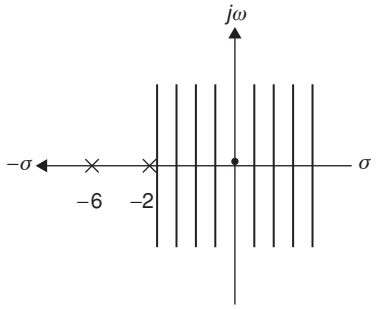
$$A = \underset{s\to -6}{\text{Lt}}\,(s+6)Y(s)$$

$$= \underset{s\to -6}{\text{Lt}}\,(s+6)\frac{1}{(s+2)(s+6)} = \frac{-1}{4}$$

$$B = \underset{s\to -2}{\text{Lt}}\,(s+2)\frac{1}{(s+2)(s+6)} = \frac{1}{4}$$

$$Y(s) = \frac{\frac{-1}{4}}{s+6}+\frac{\frac{1}{4}}{s+2},\ \sigma > -2$$

$$y(t) = \left(\frac{1}{4}e^{-2t}-\frac{1}{4}e^{-6t}\right)u(t),$$



Right most signal stable and causal.

Example 3: If Laplace transform of a signal $Y(s) = \frac{1}{s(s-1)}$, then its final value is

(A) 1 (B) −1

(C) 0 (D) ∞

Solution: (B)

Final value theorem

$$y(\infty) = \underset{s\to 0}{\text{Lt}}\,s\cdot Y(s) = \underset{s\to 0}{\text{Lt}}\,\frac{s}{s(s-1)} = -1$$

Example 4: For the system shown below, $x(t) = (\sin t)\,u(t)$. In steady state, the response $y(t)$ will be

$$x(t) \rightarrow \boxed{\frac{1}{s+1}} \rightarrow y(t)$$

(A) $\frac{1}{\sqrt{2}}\sin t$ (B) $\frac{1}{\sqrt{2}}\sin\left(t-\frac{\pi}{4}\right)$

(C) $\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$ (D) $\frac{1}{\sqrt{2}}\cos t$

Solution: (B)

$$H(s) = \frac{1}{s+1}$$

$$x(t) = (\sin t)u(t) \leftrightarrow X(s) = \frac{1}{s^2+1}$$

$$Y(s) = H(s)\cdot X(s) = \frac{1}{(s+1)(s^2+1)}$$

$$= \frac{1}{2}\left[\frac{1}{s+1}-\frac{s-1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]$$

$$y(t) = \frac{1}{2}[e^{-t}-\cos t+\sin t]u(t)$$

at steady state, $y(t) = \frac{1}{2}[\sin t-\cos t]u(t)$

$$= \frac{1}{\sqrt{2}}\sin\left(t-\frac{\pi}{4}\right)$$

Example 5: Unit step response of a system starting from rest is given by $c(t) = 1 - e^{-2t}$, for $t \geq 0$. The transfer function of the system is

(A) $\frac{s}{s+2}$ (B) $\frac{1}{s+2}$

(C) $\frac{s}{s-2}$ (D) $\frac{2}{s+2}$

Solution: (D)

Unit step response $c(t) = 1 - e^{-2t}$, for $t \geq 0$

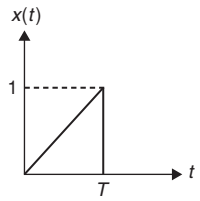
Impulse response $h(t) = \frac{d}{dt}c(t) = 2e^{-2t}$, for $t \geq 0$. So, transfer function $H(s) = \frac{2}{s+2}$.

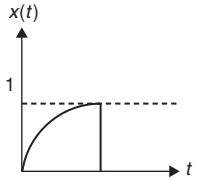
EXERCISES

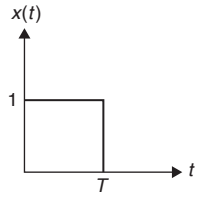
Practice Problems I

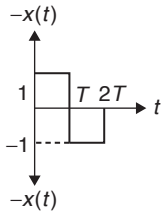
Directions for questions 1 to 22: Select the correct alternative from the given choices.

1. Inverse Laplace transform of $X(s)$ is $x(t)$. When $X(s)=\dfrac{1-e^{-sT}}{s}$, shape of $x(t)$ is?

(A) 

(B) 

(C) 

(D) 

2. Match the following:

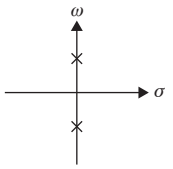
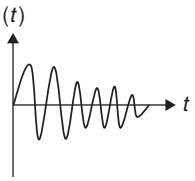
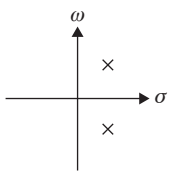
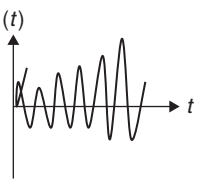
List I	List II
(a) $\dfrac{10}{s(s+10)}$	1. $10\,\delta(t)$
(b) $\dfrac{10}{s^2+100}$	2. $(e^{-10t}\cos 10t)u(t)$
(c) $\dfrac{(s+10)}{(s+10)^2+100}$	3. $(\sin 10t)u(t)$
(d) 10	4. $(1-e^{-10t})u(t)$

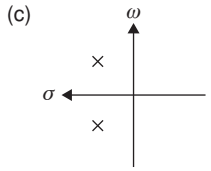
Codes:

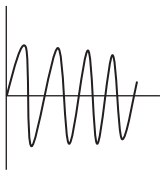
	a	b	c	d
(A)	3	4	1	2
(C)	3	4	1	2

	a	b	c	d
(B)	4	3	2	1
(D)	4	3	1	2

3. Match the following:

List I	List II
Pole-zero plot	Response
(a) 	1. 
(b) 	2. 

(c) 

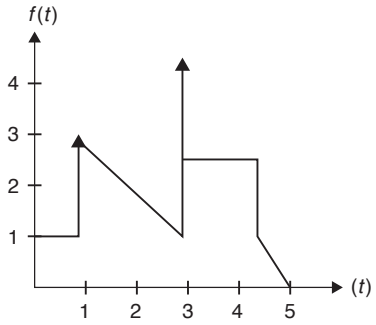
3. 

	a	b	c
(A)	1	2	3
(C)	2	3	1

	a	b	c
(B)	3	2	1
(D)	3	1	2

Common Data for Questions 4 and 5:

4. Resolve $f(t)$ in terms of step, impulse and ramp functions and also find out Laplace transform.



(A) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$

$+\dfrac{1}{s^2}[e^{-s}-e^{-3s}+4e^{-4s}-e^{-5s}]+2e^{-3s}$

(B) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$

$+\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]+2e^{-3s}$

(C) $\dfrac{1}{s}[1+2e^{-s}+3e^{-3s}-e^{-4s}]$

$-\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]+2e^{-3s}$

(D) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$

$-\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]-2e^{-3s}$

5. From above figure find out the initial value.

(A) 0 (B) 4
(C) 2 (D) 1

Common Data for Questions 6 and 7: The impulse response $h(t)$ of a LTI continuous time system is given by $h(t) = \exp(-2t)\,u(t)$ where $u(t)$ denotes the unit step function.

6. The frequency response $H(\omega)$ of this system in terms of angular frequency 'ω' is given by $H(\omega)$

(A) $\frac{1}{1+\omega}$ (B) $\frac{\sin\omega}{\omega}$

(C) $\frac{1}{2+j\omega}$ (D) $\frac{3\omega}{2+\omega}$

7. The output of this system to the sinusoidal input $x(t) = 2\cos(2t)$ for all time 't' is

(A) 0 (B) $2^{-0.25}\cos(2t-0.125\pi)$

(C) $2^{-0.5}\cos(2t-0.125\pi)$ (D) $2^{-0.5}\cos(2t-0.25\pi)$

8. The impulse response of LTI system is $h(t) = e^{-a|t|}$, the step respons e is

(A) $\frac{1}{a}[e^{at}-e^{-at}u(t)-e^{at}u(t)+2u(t)]$

(B) $\frac{1}{a}[e^{-at}+e^{at}u(t)-e^{-at}u(t)-2u(t)]$

(C) $\frac{1}{a}[e^{at}+e^{-at}u(t)-e^{at}u(t)+2u(t)]$

(D) $\frac{1}{a}[e^{at}+e^{-at}u(t)-e^{at}u(t)-2u(t)]$

9. Laplace transform of $(t^2 - 2t)u(t-1)$ is

(A) $\frac{2e^{-s}(1-s)}{s^3}$ (B) $\frac{2e^{s}(1-s)}{s^3}$

(C) $\frac{2e^{-s}(1+s)}{s^3}$ (D) $\frac{2e^{s}(1+s)}{s^3}$

10. The convolution of $x_1(t) = u(t+3)$ and $x_2(t) = \delta(t-5)$ is

(A) $u(t)$ (B) $u(t-2)$

(C) $u(t+2)$ (D) $u(2)$

11. Laplace transform of $x(t) = \sin^2 t \cdot u(t)$ is

(A) $\frac{4}{s(s^2+2)}$ (B) $\frac{2}{s(s^2+2)}$

(C) $\frac{2}{(s^2+4)}$ (D) $\frac{2}{s(s^2+4)}$

12. The inverse Laplace transform of $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]$ is

(A) $\cos^2 t$ (B) $\sin^2 t$

(C) $\cos^2 2t$ (D) $\sin^2 2t$

Linked answer questions 13 and 14:

13. The inverse Laplace transform of $\ln\left(\frac{s-a}{s-b}\right)$ is

(A) $(e^{bt}-e^{at})u(t)$ (B) $\frac{1}{t}(e^{-bt}-e^{-at})u(t)$

(C) $\frac{1}{t}(e^{bt}-e^{at})u(t)$ (D) $\frac{1}{t}(e^{-bt}+e^{-at})u(t)$

14. The inverse Laplace transform of $s\ln\left(\frac{s-a}{s-b}\right)$ is

(A) $\frac{1}{t^2}[(bt-1)e^{bt}+(at-1)e^{at}]\cdot u(t)$

(B) $\frac{1}{t^2}[(bt-1)e^{bt}-(at-1)e^{at}]\cdot u(t)$

(C) $\frac{1}{t^2}[(bt+1)e^{bt}-(at+1)e^{at}]\cdot u(t)$

(D) $\frac{1}{t^2}[(bt+1)e^{-bt}-(at-1)e^{-at}]\cdot u(t)$

15. Laplace transform of $e^{-3t}u(t) * tu(t)$

(A) $\frac{-1}{s^2(s+3)}$ (B) $\frac{1}{s^2(s-3)}$

(C) $\frac{1}{s^2(s+3)}$ (D) $\frac{-1}{s^2(s-3)}$

16. Poles and zeros of $e^{2t}u(t) + e^{-3t}u(-t)$

(A) Zero at $\frac{-5}{2}$, poles at 2, 3

(B) Zero at $\frac{5}{2}$, poles at -2, -3

(C) No common ROC

(D) Zero at $\frac{-5}{2}$, poles at -2, -3

17. The initial and final values of $X(s) = \frac{2s}{s^2+2s+2}$ are

(A) 2 and 1 (B) 2 and 0

(C) 1 and 2 (D) 2 and 2

18. A system is formed by connecting two systems in cascade. The impulse response of these systems are $e^{-2t}u(t)$, $2e^{-t}u(t)$. The impulse response of over all system is

(A) $2(e^{-t}-e^{-2t})u(t)$ (B) $2(e^{-t}+e^{-2t})u(t)$

(C) $(e^{-t}-e^{-2t})u(t)$ (D) $2(e^{-t}\cdot e^{-2t})u(t)$

19. $H(s)$ is the transfer function.

$H(s) = \frac{s}{s^2-s-2}(\sigma < -1)$. Then function is

(A) Causal and stable

(B) Causal and unstable

(C) Non-causal and stable

(D) Non-causal and unstable

20. A continuous time LTI system is described by

$$\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=2\frac{d}{dt}x(t)+4x(t).$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is

(A) $(e^{-t}-e^{-3t})u(t)$ (B) $(e^{t}+e^{3t})u(t)$

(C) $(e^{-t}+e^{-3t})u(t)$ (D) $(e^{t}-e^{3t})u(t)$

21. A linear time invariant, causal, continuous time system has a natural transfer function with sample poles at $s = -2$ and $s = -4$, and one sample zero at $s = -1$. A unit step $u(t)$ is applied at the input of the system. At steady state the output has constant value of 1. The impulse response of this system is

(A) $-4e^{-2t}u(t)$ (B) $12e^{-4t}u(t)$

(C) $(-4e^{-2t} + 12e^{-4t})u(t)$ (D) None

22. Unit impulse response of a system is $f(t) = e^{-t}$, for $t \geq 0$, for this system, the steady state value of the output for unit step input is equal to

(A) 1 (B) –1

(C) 0 (D) ∞

Practice Problems 2

Directions for questions 1 to 17: Select the correct alternative from the given choices.

1. Match the function with ROC.

P.	$e^{-at}u(t)$	**1.**	$\sigma > 0$
Q.	$-e^{-at}u(-t)$	**2.**	$\sigma > a$
R.	$te^{at}u(t)$	**3.**	$\sigma < -a$
S.	$t^n u(t)$	**4.**	$\sigma > -a$

(A) P – 3, Q – 4, R – 2, S – 1

(B) P – 4, Q – 3, R – 2, S – 1

(C) P – 3, Q – 4, R – 1, S – 2

(D) P – 4, Q – 3, R – 1, S – 2

2. The initial $x(0)$, and final value $x(\infty)$ of $X(s) = \dfrac{1}{s(s-1)}$ is

(A) 1, 0 (B) 0, –1

(C) 2, 1 (D) 1, 2

3. If $x(t)$ and $X(s)$ are Laplace transfom pairs then Laplace transform of $e^{at}x(t)$ is

(A) $e^{as}X(s)$ (B) $X(s)$

(C) $X(s-a)$ (D) $X(s+a)$

4. If $x(t)$ and $X(s)$ are Laplace transform pairs then Laplace transform of $t \times(t)$ is

(A) $\int_0^\infty X(s)ds$ (B) $\dfrac{-d}{ds}X(s)$

(C) $\dfrac{\partial}{\partial s}X(s)$ (D) $X(s)\,e^{-s}$

5. Laplace transform of $\left[\dfrac{\sin \pi t}{t}\right]u(t)$ is

(A) $\tan^{-1}\left(\dfrac{s}{\pi}\right)$ (B) $\cos^{-1}\left(\dfrac{s}{\pi}\right)$

(C) $\cot^{-1}\left(\dfrac{s}{\pi}\right)$ (D) $\sin^{-1}\left(\dfrac{s}{\pi}\right)$

6. Laplace transform of a rectified half sine wave with period 2π, amplitude 1 is

(A) $\dfrac{(1+e^{-\pi s})}{s^2+1}$ (B) $\dfrac{(1-e^{-\pi s})}{s^2+1}$

(C) $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$ (D) $\dfrac{1}{(s^2+1)(1+e^{-\pi s})}$

7. Laplace transforms of $x(t) = e^{at}u(t)$, $y(t) = -e^{at}u(-t)$ have

(A) same transforms but different ROCs

(B) different transforms but same ROCs

(C) same transforms and same ROCs

(D) different transforms and different ROCs

8. Laplace transform and ROC of $e^{-5t}u(-t+3)$ is

(A) $-\dfrac{e^{-3(s+5)}}{s+5}$, $\text{Re}(s) < 5$ (B) $\dfrac{e^{-3(s+5)}}{s+5}$, $\text{Re}(s) > 5$

(C) $-\dfrac{e^{-3(s+5)}}{s+5}$, $\text{Re}(s) > 5$ (D) $\dfrac{e^{-3(s+5)}}{s+5}$, $\text{Re}(s) < 5$

9. Match the Laplace transform pairs

P.	$x(at)$	**1.**	$asX(s)$
Q.	$x(t-a)$	**2.**	$X(s-a)$
R.	$e^{at} \cdot x(t)$	**3.**	$e^{-sa}X(s)$
S.	$\dfrac{d}{dt}[ax(t)]$	**4.**	$\dfrac{1}{a}X\left(\dfrac{s}{a}\right)a > 0$

(A) P – 3, Q – 2, R – 1, S – 4

(B) P – 4, Q – 2, R – 1, S – 3

(C) P – 2, Q – 4, R – 3, S – 1

(D) P – 4, Q – 3, R – 2, S – 1

10. Poles and zeros of $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$ are

(A) Zero at $\dfrac{-5}{2}$, poles at 2, 3

(B) Zero at $\dfrac{5}{2}$, poles at –2, –3

(C) No common ROC

(D) Zero at $\dfrac{-5}{2}$, poles at –2, –3

11. Inverse Laplace transform of $\dfrac{2s}{s+2}$ is

(A) $2\delta(t) - 4e^{-2t}u(t)$ (B) $2\delta(t) + 4e^{-2t}u(t)$

(C) $2\delta(t) - 4e^{2t}u(t)$ (D) $2\delta(t) + 4e^{2t}u(t)$

12. Given $H(s) = e^{-5s}$, what is the impulse response of the system?

(A) $u(t-5)$ (B) $\delta(t-5)$

(C) $e^{-5t}u(t)$ (D) $t \cdot e^{-5t}u(t)$

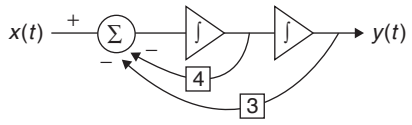
13. Convolution of $a^n u[n]$ and $b^n u[n]$ is

(A) $\dfrac{b^{n+1} + a^{n+1}}{a+b}u[n]$ (B) $\dfrac{b^{n+1} - a^{n+1}}{b-a}u[n]$

(C) $\frac{b^{n-1}-a^{n-1}}{b-a}u[n]$ (D) $\frac{b^{n-1}+a^{n-1}}{b-a}u[n]$

Linked answer questions 14 and 15:

14. For a linear time invariant system whose block diagram is shown here with input $x(t)$ and output $y(t)$



The transfer function is

(A) $\frac{1}{s^2+4s+3}$ (B) $\frac{1}{s+4}$

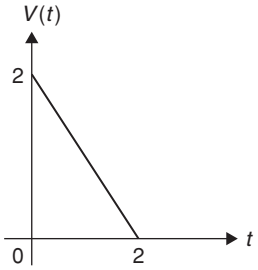
(C) $\frac{1}{s-3}$ (D) $\frac{1}{s^2-4s-3}$

15. The step response of the system is

(A) $\frac{-1}{2}e^{-t}+\frac{1}{6}e^{-3t}$ (B) $\frac{1}{3}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$

(C) $\frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}$ (D) $\frac{1}{3}-\frac{1}{2}e^{t}+\frac{1}{6}e^{-3t}$

16. Express Laplace transform of



(A) $\frac{-2}{s}+\frac{1}{s^2[1-e^{-2s}]}$ (B) $\frac{2}{s}-\frac{1}{s^2}[1+e^{-2s}]$

(C) $\frac{-2}{s}-\frac{1}{s^2[1-e^{-2s}]}$ (D) $\frac{2}{s}-\frac{1}{s^2}[1-e^{-2s}]$

17. Given $L\{f(t))\}=\frac{s+2}{s^2+1}$, $L\{g(t)\}=\frac{s^2+1}{(s+3)(s+2)}$,

$h(t)=\int_0^t f(\tau)g(t-\tau)d\tau$, then $L\{h(t)\}$ is

(A) $\frac{1}{s+2}$ (B) $\frac{1}{s+1}$

(C) $\frac{1}{s+3}$ (D) $\frac{s}{s+3}$

Previous Years' Questions

1. The running integrator, given by $y(t)=\int_0^H \pi R^2 \times(t)dt$ or $Y(S)=\int_{-\alpha}^{t}\times(\alpha)d\alpha$ **[2006]**

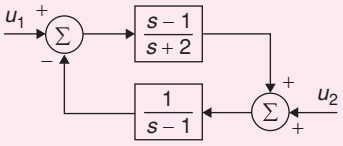
(A) has no finite singularities in its double sided Laplace transform $Y(s)$.

(B) produces a bounded output for every causal bounded input.

(C) produces a bounded output for every anticausal bounded input.

(D) has no finite zeroes in its double sided Laplace transform $Y(s)$.

2. The system shown in the figure is **[2007]**



(A) stable

(B) unstable

(C) conditionally stable

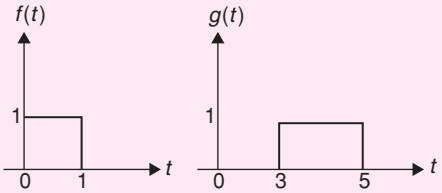
(D) stable for input u_1, but unstable for input u_2

3. For the system $\frac{2}{(s+1)}$ the approximate time taken for a step response to reach 98% of its final value is **[2010]**

(A) 1 s (B) 2 s

(C) 4 s (D) 8 s

Common Data for Questions 4 and 5: Given $f(t)$ and $g(t)$ as shown below:



4. $g(t)$ can be expressed as **[2010]**

(A) $g(t)=f(2t-3)$ (B) $g(t)=f\left(\frac{t}{2}-3\right)$

(C) $g(t)=f\left(2t-\frac{3}{2}\right)$ (D) $g(t)=f\left(\frac{t}{2}-\frac{3}{2}\right)$

5. The Laplace transform of $g(t)$ is **[2010]**

(A) $\frac{1}{S}(e^{3s}-e^{5s})$ (B) $\frac{1}{S}(e^{-5s}-e^{-3s})$

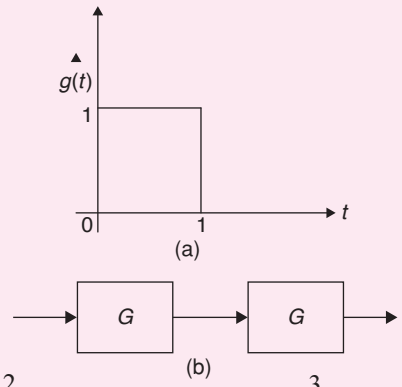
(C) $\frac{e^{-3s}}{S}(1-e^{-2s})$ (D) $\frac{1}{S}(e^{5s}-e^{3s})$

6. Let the Laplace transform of a function $f(t)$ which exists for $t > 0$ be $F_1'(s)$ and the Laplace transform of its delayed version $f(t-\tau)$ be $F_2(s)$. Let $f_1''(s)$ be the complex conjugate of $F_1(s)$ with the Laplace variable set as $s = \sigma + j\omega$.

If $G(s) = \dfrac{F_2(s) \cdot F_1'(s)}{|F_1(s)|^2}$, then the inverse Laplace transform of $G(s)$ is **[2011]**

(A) an ideal impulse $\delta(t)$
(B) an ideal delayed impulse $\delta(t-\tau)$
(C) an ideal step function $u(t)$
(D) an ideal delayed step function $u(t-\tau)$

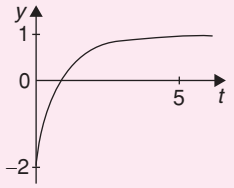
7. The impulse response $g(t)$ of a system G, is as shown in Figure (a). What is the maximum value attained by the impulse response of two cascaded blocks of G as shown in Figure (b)? **[2015]**

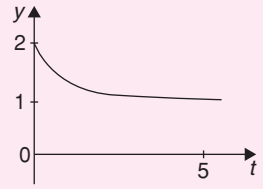


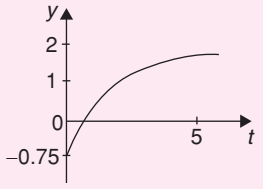
(A) $\dfrac{2}{3}$ (B) $\dfrac{3}{4}$

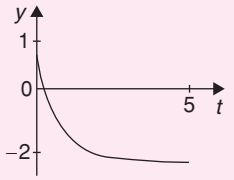
(C) $\dfrac{4}{5}$ (D) 1

8. The unit step response of a system with the transfer function $G(s) = \dfrac{1-2s}{1+s}$ is given by which one of the following waveforms? **[2015]**

(a)


(b)


(c)


(d)


Answer Keys

Exercises

Practice Problems 1

1. C	**2.** B	**3.** B	**4.** C	**5.** D	**6.** C	**7.** D	**8.** A	**9.** A	**10.** B
11. D	**12.** A	**13.** C	**14.** B	**15.** C	**16.** C	**17.** B	**18.** A	**19.** D	**20.** A
21. C	**22.** A								

Practice Problems 2

1. B	**2.** B	**3.** C	**4.** B	**5.** C	**6.** C	**7.** A	**8.** A	**9.** D	**10.** D
11. A	**12.** B	**13.** B	**14.** A	**15.** C	**16.** D	**17.** C			

Previous Years' Questions

1. C	**2.** D	**3.** C	**4.** D	**5.** C	**6.** B	**7.** D	**8.** A

Chapter 3

z-Transform

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Sampling
- Sampling theorem
- The *z*-transform
- Region of convergence
- *z*-transform properties
- Inverse *z*-transform
- Analysis of discrete-time LTI systems
- Causal and stable systems
- The unilateral *z*-transform
- Relationship between *z* and Laplace transform
- Direct form-I structure
- Direct form-II structure of IIR system

Sampling

Let $x(t)$ be any continuous signal periodically sampled at equal sample time period T_s to get discrete time signal represented by $x[n]$



$x[n] = x(nT_s)$
$n = 0, \pm 1, \pm 2, \pm 3 \ldots \infty$
T_s = Sampling time period

Sampling Theorem

Let $x(t)$ is strictly band limited signal with Fourier spectrum.

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_m$$

Then $x(t)$ can be recovered without loss of information from its sample signal $x[n] = x(nT_s)$, if the sampling frequency ω_s is greater than twice of the maximum frequency ω_m (of $x(t)$)

$$\omega_s > 2\omega_m.$$

Minimum sampling frequency $\omega_s = 2\omega_m$ is called Nyquist sampling rate.

In case of sampling frequency $\omega_s < 2\omega_m$.

$X_p(j\omega)$ which is spectrum of $x_p(t) = x(nT_s)$ no longer contains all information of $X(j\omega)$; hence $x(t)$ is not fully recoverable from sampler $x_p(t)$ and there is definite loss of original signal $x(t)$.

The loss of information in such sampling case ($\omega_s < 2\omega_m$) is due to super imposition of high frequency components onto the less-frequency component. This phenomena is called frequency folding or aliasing.

Due to aliasing, high frequency noise components get mixed with signal frequencies between 0 to $\omega_s/2$.

To prevent high frequency noise components having frequency higher than $\omega_s/2$ an analog filter is used before sampler to attenuate signals with frequency higher than $\omega_s/2$. Such filters are called anti-aliasing filters.

The z-Transform

A discrete time LTI system with impulse response $h[n]$, the output $y[n]$ of the system to the complex exponential input of the form z^n is $y[n] = T\{z^n\}$

$$= h[n] * z^n$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$= H(z)\, z^n$$

where, $H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n}$

The variable z is generally complex valued.

When $|z|$ is not restricted to unity, the summation is referred to as the z-transform of $h[n]$.

Definition

For general discrete time signal $x[n]$, the z transfer function $X(z)$ is defined as $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$.

The variable is generally complex valued and is expressed in polar form as $z = re^{j\Omega}$.

where 'r' is the magnitude of z and 'Ω' is the angle of 'z'

$x[n]$ and $X(z)$ are said to form a z transform pair as $x[n] \leftrightarrow X(z)$

Region of Convergence

The range of values of the complex variable z for which the z-transform converges is called the region of convergence.

Consider the sequence $x[n] = a^n u[n]$

$$\text{z-transform of } x[n],\ X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For the convergence this summation should be finite.

This ROC is the range of values of z for which $|az^{-1}| < 1$, or $|z| > |a|$ then,

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}},\ |z| > |a| = \frac{z}{z-a}\ |z| > |a|.$$

There is one zero, at $z = 0$, and one pole at $z = a$, ROC and the pole zero plot for the examples are shown in figure.

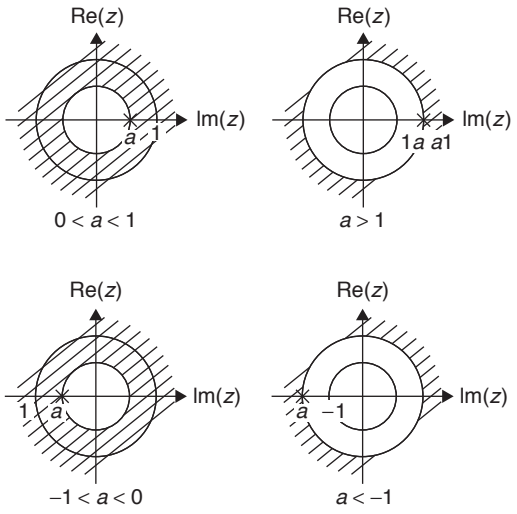


Figure 1 ROC of the form $|z| > |a|$

Now let $x[n] = -a^n u[-n-1]$, then z-transform

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n.$$

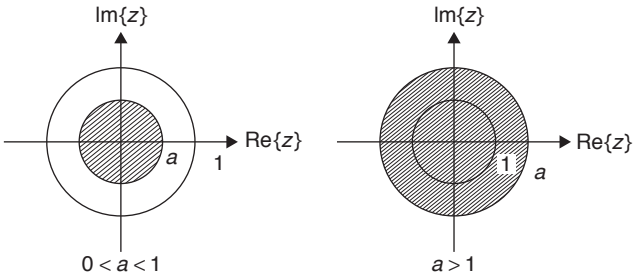
By taking $n = -n$,

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

If $|a^{-1}z| < 1$, then the above summation converges, i.e., for $|z| < |a|$

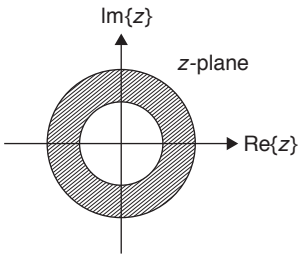
$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{z}{z-a} = |z| < |a|$$

The pole zero plots are



Properties of the ROC

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin. In some cases, the inner boundary can extend inward to the origin, in which case the ROC becomes a disc. In other case, the outer boundary can extend outward to infinity.



2. ROC does not contain any poles; at poles, $X(z)$ is infinite and therefore, by definition it does not converge.
3. If $x[n]$ is a finite sequence, then $X(z)$ converges for every value of z, then ROC is the entire z-plane except possibly $z = 0$ and/ or $z = \infty$.
4. If $x[n]$ is a right-handed sequence, ($x[n] = 0$, for $n < N_1 < \infty$), then ROC is of the form $|z| > r_{max}$ or $\infty > |z| > r_{max}$ where r_{max} is the largest magnitude of any of the poles of $X(z)$. Thus ROC is the exterior of the circle $|z| = r_{max}$.
5. If $x[n]$ is a left-handed sequence ($x[n] = 0$, for $n > N_2 > -\infty$), then ROC is of the form $|z| < r_{min}$ or $0 < |z| < r_{min}$ where r_{min} is the smallest magnitude of any of the poles of $X(z)$, thus ROC is interior of the circle $|z| = r_{min}$.
6. If $x[n]$ is two-sided sequence, ($x[n]$ is infinite duration sequence). Then the ROC is of the form $r_1 < |z| < r_2$ where r_1, r_2 are the magnitudes of the two poles of $X(z)$. It forms like a ring.

7. Relationship between the ROC and the time extent of a signal

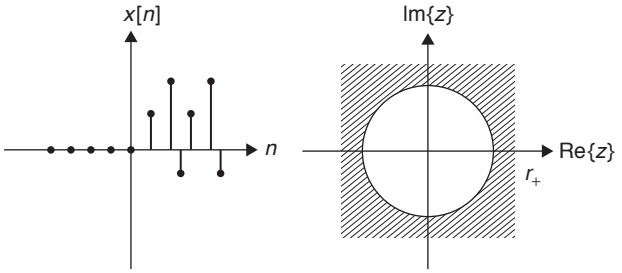


Figure 2 A right sided signal has an ROC of form $|z| > r_+$

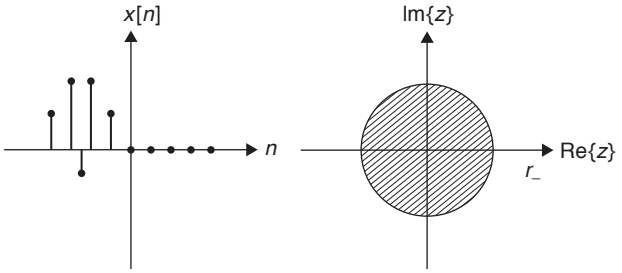


Figure 3 A left sided signal has an ROC of form $|z| < r_-$

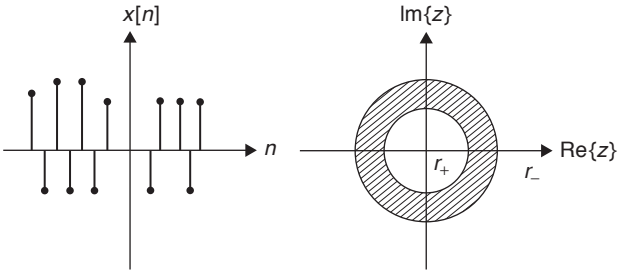


Figure 4 A two sided signal has an ROC of form $r_+ < |z| < r_-$

z-TRANSFORMS OF THE SOME COMMON SEQUENCES

For unit impulse signal $\delta[n]$, its z-transform is given by

$$\delta[n] \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

ROC is entire z-plane including $z = 0$, and $z = \infty$.
For delayed unit impulse

$$\delta[n-n_0] \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$$

ROC is entire z-plane, expect $z = 0$.
For advanced unit impulse

$$\delta[n+n_0] \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n+n_0] z^{-n} = z^{n_0}$$

ROC is entire z-plane, except $z = \infty$

$x(n)$	$X(z)$	ROC
$\delta(n)$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 ($m > 0$) or ∞($m < 0$)
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$
$(r^n \text{Cos}\, \Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$
$a^n \quad 0 < n < N-1$ $0 \quad$ Otherwise	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Example 1: What are the ROC, and z-transforms of the signal $x[n] = 5\left(\frac{1}{4}\right)^n u[n] - 4\left(\frac{1}{3}\right)^n u[n]$?

Solution:
$$X(z) = \sum_{n=-\infty}^{\infty} \left\{5\left(\frac{1}{4}\right)^n u[n] - 4\left(\frac{1}{3}\right)^n u[n]\right\} z^{-n}$$

$$= 5\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n z^{-n} - 4\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n}$$

$$= 5\sum_{n=0}^{\infty}\left(\frac{1}{4}z^{-1}\right)^n - 4\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n \quad (1)$$

$$= \frac{5}{1-\frac{1}{4}z^{-1}} - \frac{4}{1-\frac{1}{3}z^{-1}} = \frac{1-\frac{5}{3}z^{-1}+z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{1-\frac{2}{3}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{z\left(z-\frac{2}{3}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$$

For convergence of $X(z)$, both sums in the equation (1) should converge, which requires $\left|\frac{1}{4}z^{-1}\right| < 1$, and $\left|\frac{1}{3}z^{-1}\right| < 1$ or equivalently $|z| > \frac{1}{4}$ and $|z| > \frac{1}{3}$.

By combining these two regions, the ROC will be $|z| > \frac{1}{3}$.

Example 2: What is the z-transform and ROC of the signal

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1, a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution: $X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-(az^{-1})} = \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z-a}$$

$$= \frac{1}{z^{N-1}} \frac{(z-a)(z^{N-1} + K_1 z^{N-2} + \cdots + a^{N-1})}{z-a}$$

$$= \frac{1}{z^{N-1}} (z^{N-1} + K_1 z^{N-2} + \cdots + a^{N-1})$$

There is a pole of order $N-1$ at $z = 0$, and ROC is all values of z except $z = 0$

Example 3: What is the z-transform of $x[n] = b^{|n|}\ b > 0$

Solution: $x[n] = b^n u[n] + b^{-n} u[-n-1]$

$$b^n u[n] \xleftrightarrow{z} \frac{1}{1-bz^{-1}},\ |z| > b,$$

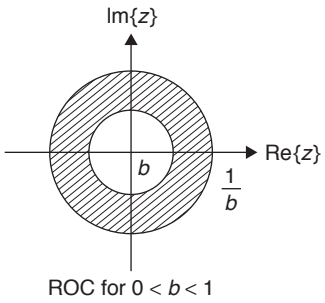
$$\left(\frac{1}{b}\right)^n u[-n-1] \xleftrightarrow{z} \frac{-1}{1-b^{-1}z^{-1}},\ |z| < \frac{1}{b}$$

When $b > 1$, there is no common ROC for the above functions, but for $0 < b < 1$, the ROCs of above two equations will overlap and thus the z-transform for composite sequence is

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}},\ b < |z| < \frac{1}{b}$$

$$= \frac{z}{z-b} - \frac{z}{z-b^{-1}},\ b < |z| < \frac{1}{b}$$

$$= \frac{b^2-1}{b} \frac{z}{(z-b)(z-b^{-1})},\ b < |z| < \frac{1}{b}$$



z-Transform Properties

1. Linearity
   $x_1[n] \leftrightarrow X_1[z]$, ROC = R_1
   $x_2[n] \leftrightarrow X_2[z]$, ROC = R_2
   $a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(z) + a_2X_2(z)$
   $R' \supset R_1 \quad R_2$
2. Time shifting
   $x_1[n] \leftrightarrow X(z)$, ROC = R
   $x[n - n_0] \leftrightarrow z^{-n_0}X(z)$, $R^1 = R \cap \{0 < |z| < \infty\}$
3. Multiplication by z_0^n
   $z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right),\ R' = |z_0|R$
   $a^n x[n] \leftrightarrow X(a^{-1}z),\ R' = |a|R$
   $e^{+j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0 n}),\ R' = R$
4. Time reversal
   $x[-n] \leftrightarrow X\left(\frac{1}{z}\right), R' = \frac{1}{R}$
5. Multiplication by 'n'
   $nx[n] \leftrightarrow z - \frac{dx(z)}{dz},\ R' = R$
6. Accumulation
   $\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z),\ R' \supset R \cap \{|z| > 1\}$
7. Convolution
   $x_1[n] * x_2[n] \leftrightarrow X_1(z)\, X_2(z),\ R' \supset R_1 \cap R_2$
8. Conjugation
   $x^*[n] \leftrightarrow X^*[z^*]$ with ROC = R
9. First difference
   $x[n] - x[n-1] \leftrightarrow (1 - z^{-1})\, X(z)$,
   $R' \supset R \cap \{|z| > 0\}$
10. Initial value theorem
    if $x[n] = 0$, for $n < 0$, then $x[0] = \underset{z\to\infty}{\text{Lt}}\ X(z)$
11. Final value theorem
    $\underset{n\to\infty}{\text{Lt}}\ x[n] = \underset{z\to 1}{\text{Lt}}(z-1)X(z)$, Poles of $(z-1)\ X(z)$ must lie inside the unit circle.
12. Time Expansion
    $x_k[n] = x[n/k] \leftrightarrow X(z^k),\ R' = R^{1/k}$, for $n, k, \frac{n}{k}$ are integers.

Inverse z-Transform

Inversion of the z-transform to find the sequence $x[n]$ from its z-transform $X(z)$ is called inverse z-transform.

$$x[n] = z^{-1}\{X(z)\} \text{ or}$$

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Where C is a counter clockwise contour of integration enclosing the origin.

Example 4: Consider the z-transform $X(z)$

$$= \frac{3-\frac{11}{15}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.$$

Find the inverse z-transform for different ROCs

(i) $|z| > \frac{1}{3}$

(ii) $\frac{1}{5} < |z| < \frac{1}{3}$

(iii) $|z| < \frac{1}{5}$

Solution: $X(z) = \dfrac{3-\frac{11}{15}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}$

$$X(z) = \frac{z^{-1}\left(3z-\frac{11}{15}\right)}{z^{-2}\cdot\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)} = \frac{1}{z^{-1}}\left[\frac{2}{z-\frac{1}{3}}+\frac{1}{z-\frac{1}{5}}\right]$$

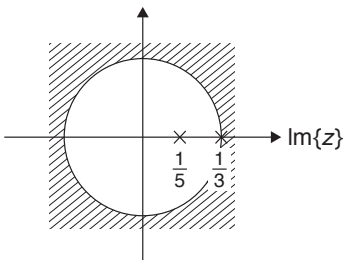
$$X(z) = \frac{2}{1-\frac{1}{3}z^{-1}}+\frac{1}{1-\frac{1}{5}z^{-1}}$$

(i) When ROC is $|z| > \frac{1}{3}$

There are two poles $z = \frac{1}{3}$ and $\frac{1}{5}$, and ROC lies outside outermost pole, so the inverse z-transform is a right sided sequence

$$\left(\frac{1}{5}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}}, |z| > \frac{1}{5}$$

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$$



For the above two functions ROC can be combined as $|z| > \frac{1}{3}$

So, $x[n] = 2\left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{5}\right)^n u[n]$.

(ii) When ROC is $\frac{1}{5} < |z| < \frac{1}{3}$

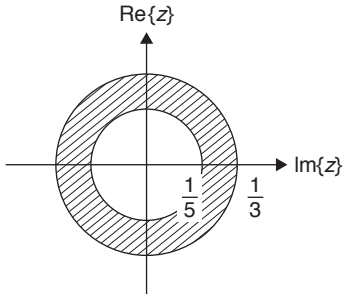
ROC is $|z| < \frac{1}{3}$ and $|z| > \frac{1}{5}$

So, the functions

$$\left(\frac{1}{5}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}} \; |z| > \frac{1}{5}$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \; |z| < \frac{1}{3}.$$

By combining above two functions



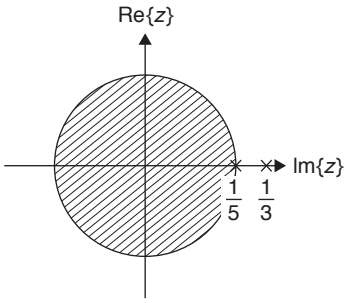
we get $x[n] = \left(\frac{1}{5}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$

(iii) When ROC is $|z| < \frac{1}{5}$; in this case the ROC is inside the both poles

$$-\left(\frac{1}{5}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}} \; |z| < \frac{1}{5}$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \; |z| < \frac{1}{3}$$

By combining the above two equations



we get $x[n] = -2\left(\frac{1}{3}\right)^n u[-n-1] - \left(\frac{1}{5}\right)^n u[-n-1]$

Example 5: What is the inverse z-transform of

$$X(z) = \log(z + a) - \log z, |z| > |a|$$

Solution: $X(z) = \log (z + a) - \log z, |z| > |a|$

$$= \log\left(\frac{z+a}{z}\right), |z| > |a|$$

$$X(z) = \log (1 + az^{-1}), |z| > |a|$$

By using multiplication by n property,

$$n \cdot x[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz} \quad (2)$$

$$\frac{dX(z)}{dz} = \frac{a}{1+az^{-1}} \cdot (-z^{-2})$$

$$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}} |z| > |a| \quad (3)$$

$$(-a)^n u[n] \xleftrightarrow{z} \frac{1}{1+az^{-1}}, |z| > |a|$$

$$a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}}, |z| > |a|$$

By using time shifting property

$$a(-a)^{n-1} u[n-1] \xleftrightarrow{z} z^{-1}\frac{a}{1+az^{-1}}, \quad |z| > |a| \quad (4)$$

So from equations (2), (3) and (4)

$$nx[n] = a(-a)^{n-1} u[n-1]$$

$$\text{or } x[n] = \frac{-(-a)^n u[n-1]}{n}$$

Example 6: What is the inverse z-transform of $X(z) = 5z^4 + 3z^2 + 2 + z^{-1} + 4z^{-3}$ $0 < |z| < \infty$

Solution: $\delta[n] \xleftrightarrow{z} 1$

for any $x[n]$
$x[n - n_0] \leftrightarrow z^{-n_0} X(z)$
So $\delta[n + 4] \leftrightarrow z^4$
$\delta[n + 2] \leftrightarrow z^2$
$\delta [n - 1] \leftrightarrow z^{-1}$
$\delta[n - 3] \leftrightarrow z^{-3}$
$X(z) = 5z^4 + 3z^2 + 2 + z^{-1} + 4z^{-3}$
Then $x[n] = 5\delta[n + 4] + 3\delta[n + 2] + 2$
$\delta[n] + \delta[n - 1] + 4\ \delta[n - 3]$

Example 7: Inverse z-transform for $X(z) = \frac{z^{-1}}{(1-az^{-1})^2}, |z| > |a|$

Solution: $a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} |z| > |a|$

Consider $x[n] = a^n\, u[n]$

$$n\,x[n] \xleftrightarrow{Z} -z\frac{d}{dz}\{X(z)\}$$

$$-z\frac{d}{dz}\left\{\frac{1}{1-az^{-1}}\right\} = -z\frac{-az^{-2}}{(1-az^{-1})^2} = \frac{+az^{-1}}{(1-az^{-1})^2}$$

$$\frac{n}{a}x[n] \xleftrightarrow{z} \frac{z^{-1}}{(1-az^{-1})^2}$$

So the inverse z-transform of $\frac{z^{-1}}{(1-az^{-1})^2}$

is $\frac{n}{a}x[n] = \frac{n}{a} \cdot a^n u[n] = n \cdot a^{n-1} u[n]$

Analysis of Discrete-time LTI Systems

The system function of a discrete-time LTI systems: The output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$ that is $y[n] = x[n] * h[n]$

By applying z-transform $Y(z) = X(z)\, H(z)$

$$H(z) = \frac{Y(z)}{X(z)}$$

The transform $H(z)$ of $h[n]$ is referred to as the system function (or transfer function) of the system

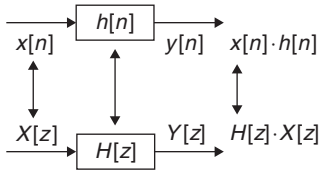


Figure 5 Impulse response and system function

Characterization of Discrete-time LTI Systems

Causality

For causal discrete-time LTI system we have $h[n] = 0, n < 0$.

Since $h[n]$ is right-sided signal, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must be of the form $|z| > r_{max}$, exterior of circle, including infinity.

Similarly if system is anti-causal, ($h[n] = 0, n \geq 0$), the ROC of $H(z)$ must be of the form $|z| < r_{min}$

Stability

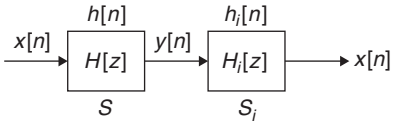
A discrete-time LTI system is BIBO stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ corresponding requirement on $H(z)$ is that ROC of $H(z)$ contains the unit circle (i.e., $|z| = 1$)

Causal and stable systems: If the system is both causal and stable, then all of the poles of $H(z)$ must lie inside the unit circle of the z-plane because the ROC is of the form $|z| > r_{max}$ and since unit circle is included in ROC we must have $r_{max} < 1$.

Invertibility

If $H(z)$ is the transfer function of a system S, the S_i, its inverse system has a transfer function $H_i(z)\frac{1}{H(z)}$.

The inverse system S_i undoes the operation of S, hence if $H(z)$ is placed in cascade with $H_i(z)$, the transfer function of the composite system is unity.



$H(z) \cdot H_1(z) = 1$, i.e., $h[n] * h_i[n] = \delta[n]$. The zeros of $H(z)$ are the poles of $H_i(z)$ and the poles of $H(z)$ are the zeros of $H_i(z)$ as $H_i(z)\frac{1}{H(z)}$. If $H(z)$ is both stable and causal, then all poles of $H(z)$ will lie inside the unit circle. If $H_i(z)$ is to be both stable and causal, then all poles of $H_i(z)$ have to lie inside the unit circle, the poles of $H_i(z)$ = zeros of $H(z)$. So for the systems $H(z)$ and $H_i(z)$ to be stable and causal, all the poles and zero of $H(z)$ must lie inside the unit circle. A system with all poles and zeros inside the unit circle is termed as minimum phase system.

System Function of LTI System Described by Linear Constant Coefficient Difference Equations

A distinct time LTI system for which input $x[n]$ and output $y[n]$ satisfy the general linear constant coefficient difference equation is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

By applying z-transform using time shift property

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

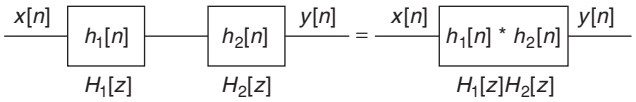
So, $H(z)$ is always rational.

The difference equation by itself does not provide information about which ROC to associate with the algebraic expression $H(z)$. An additional constraint, such as the causality or stability of the system, however serves to specify the region of convergence. For example if the system is causal, the ROC will be outside the outermost pole. If the system is stable, the ROC must include the unit circle.

Systems Interconnections

For two LTI system ($h_1[n]$, $h_2[n]$) in cascade, the overall impulse response $h[n]$ is given by $h[n] = h_1[n] * h_2[n]$

Corresponding system function $H(z) = H_1(z) \cdot H_2(z)$

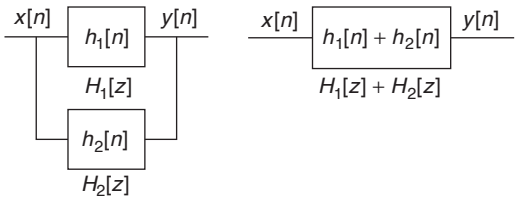


The impulse response of a parallel combination of two LTI systems is given by

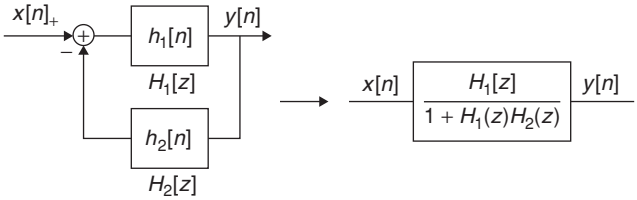
$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

When two systems are connected in feedback inter-connection, as shown in figure, then the overall system function for the feedback system



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$



Examples 8: Consider the LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equations

$$y[n] - \frac{1}{3}y[n-1] = x[n] + \frac{1}{5}x[n-1]$$

Then what is the impulse response of the system when (i) system is stable, (ii) system is anti causal?

Solution: Given difference equations is

$$y[n] - \frac{1}{3}y[n-1] = x[n] + \frac{1}{5}x[n-1]$$

By considering z-transform

$$Y(z) - \frac{1}{3}z^{-1}Y(z) = X(z) + \frac{1}{5}z^{-1}X(z)$$

$$Y(z)\left(1-\frac{1}{3}z^{-1}\right)=X(z)\left(1+\frac{1}{5}z^{-1}\right)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\left(1+\frac{1}{5}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$$

For ROC we need to consider the system conditions

1. When the system is stable, it includes unit circle $|z|>\frac{1}{3}$ (and the system is causal too)

$$H(z)=\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{1}{5}\frac{z^{-1}}{1-\frac{1}{3}z^{-1}}\quad |z|>\frac{1}{3}$$

By considering inverse z-transform

$h[n]=\left(\frac{1}{3}\right)^n u[n]+\frac{1}{5}\left(\frac{1}{3}\right)^{n-1}u[n-1]$ is the impulse response, when the system is causal (or) stable.

2. When the system is anti causal, $|z|<\frac{1}{3}$

$$H(z)=\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{1}{5}\frac{z^{-1}}{1-\frac{1}{3}z^{-1}},\quad |z|<\frac{1}{3}$$

By considering inverse z-transform

$$h[n]=-\left(\frac{1}{3}\right)^n u[-n-1]-\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^{n-1}u[-(n-1)-1]$$

$$h[n]=-\left(\frac{1}{3}\right)^n u[-n-1]-\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^{n-1}u[-n]$$

is the impulse response when the system is anti causal

The Unilateral z-Transform

The unilateral (or one-sided) z-transform $X_I(z)$ of a sequence $x[n]$ is defined as $X_I(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$ and differs from the bilateral transform in that the summation is carried over only $n\geq 0$.

ROC of $X_I(z)$ is always outside a circle in the z-plane. Time shifting property of unilateral z-transform if $x[n]\leftrightarrow X_I[z]$ then for $m\geq 0$

$$x[n-m]\leftrightarrow z^{-m}X_I(z)+z^{-m+1}x[-1]+z^{-m+2}x[-2]+\cdots+x[-m]$$

$$x[n+m]\leftrightarrow z^m X_I(z)-z^m x[0]-z^{m-1}x[1]\supset -zx[m-1]$$

The unilateral z-transform, that is particularly useful in analyzing causal systems specified by linear constant coefficient difference equations with non zero initial conditions

$$\text{Time delay } x[n-1]\xleftrightarrow{\text{(unilateralz)}}z^{-1}X(z)+x[-1]$$

$$\text{Time delay } x[n+1]\xleftrightarrow{\text{(unilateralz)}}zX(z)-zx[0]$$

Relationship between z and Laplace Transform

For a general signal $x(t)$ is sampled at sampling rate $\frac{1}{T}$ to get discrete value $x(kT)$ which has z-transform

$$X(z)=\sum_{k=-\infty}^{\infty}x(kT)z^{-k}$$

The same general signal $x(t)$ can be considered as the impulse sampled at the same rate $\frac{1}{T}$ and may be represented as

$$x(t)=\sum_{k=-\infty}^{\infty}x(kT)\delta(t-kT)$$

Laplace transform of above equation is

$$X(s)=\sum_{k=-\infty}^{\infty}x(kT)e^{-ksT}$$

If $e^{sT}=z$, we can write as

$$X(s)=\sum_{k=-\infty}^{\infty}x(k)z^{-k}=X(z)$$

Thus, $z=e^{sT}$

$\ln z=sT$

$$s=\frac{1}{T}\ln z$$

$$X(s)=X(z)\big|_{z=e^{Ts}}$$

Structures for Realization of IIR Systems

In time domain the representation Nth order IIR system is

$$y(n)=-\sum_{k=1}^{N}a_k y(n-k)+\sum_{k=0}^{M}b_k x(n-k)\qquad (5)$$

And the z-domain representation of an Nth order IIR system is,

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

In the above representation the value of M gives the number of zeros and N gives the number of poles of the system.

Types of structures for realizing the IIR systems are

1. **Direct form-I structure:** Consider the difference equation governing an IIR system.

Consider the equation (1)

$$y(n)+a_1y(n-1)+a_2y(n-2)+\cdots a_Ny(n-N)=b_0x(n)+b_1x(n-1)+b_2x(n-2)+\cdots b_Mx(n-M)$$

On taking z-transform both sides, we get,

$$Y(z)=-a_1z^{-1}Y(z)-a_2z^{-2}Y(z)-\cdots-a_Nz^{-N}Y(z)+b_0X(z)+b_1z^{-1}X(z)+\cdots+b_Mz^{-M}X(z)$$

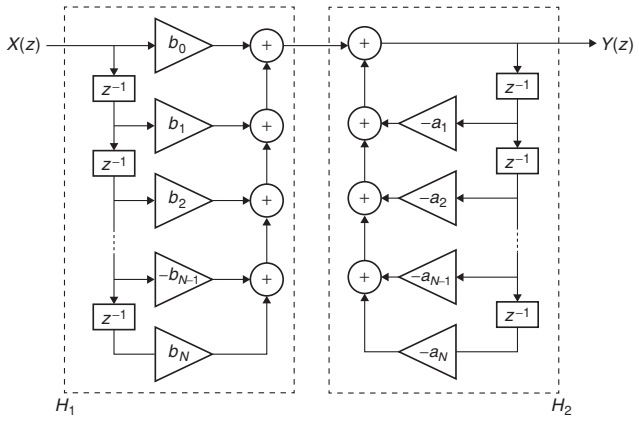


Figure 6 Direct form-I structure of IIR system

This provides a direct relation between time domain and z-domain equations.

⇒ This required more memory elements (z^{-1}).

2. **Direct form-II structure of IIR system:** This structure can be realized which uses less number of delay elements or memory elements than the direct form-I. Consider the general difference equation governing an IIR system.

$$Y(n)-\sum_{k=1}^{N} a_k y(n-k)+\sum_{k=0}^{M} b_k x(n-k)$$

On taking z-transform of the above equation.

We can obtained,

$Y(z)\,[1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_Nz^{-N}] = X(z)\,[b_0 + b_1z^{-1} + b_2z^{-2} \cdots + b_Mz^{-M}]$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots + b_Mz^{-M}}{1 + a_1z^{-1} + \cdots + a_Nz^{-N}}$$

$$\text{Let } \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\frac{Y(z)}{W(z)} \tag{6}$$

$$\text{Where, } \frac{W(z)}{X(z)} = \frac{1}{1 + a_1z^{-1} + a_2z^{-2} + \cdots a_Nz^{-N}} \tag{7}$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1z^{-1} + b_2z^{-2} + \cdots b_Mz^{-M} \tag{8}$$

On cross multiplying equation (6) and (7) we get,

$W(z) = X(z) - a_1z^{-1}W(z) - a_2z^{-2}W(z) \cdots - a_Nz^{-N}W(z)$ and (9)

$Y(z) = b_0W(z) + b_1z^{-1}W(z) + \supset + b_Mz^{-M}W(z)$ (10)

The equation (9) and (10) represents the IIR system in z-domain and it can be realized by a direct structure called direct form-II.

In direct form-II the number of delays is equal to order of the system.

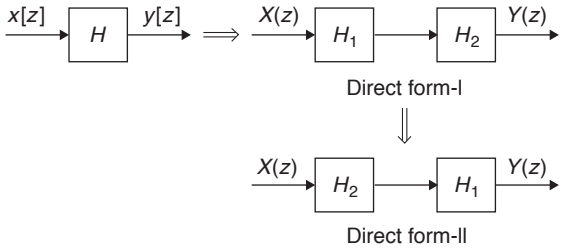


Figure 7 Conversion of Direct form-I to Direct form-II.

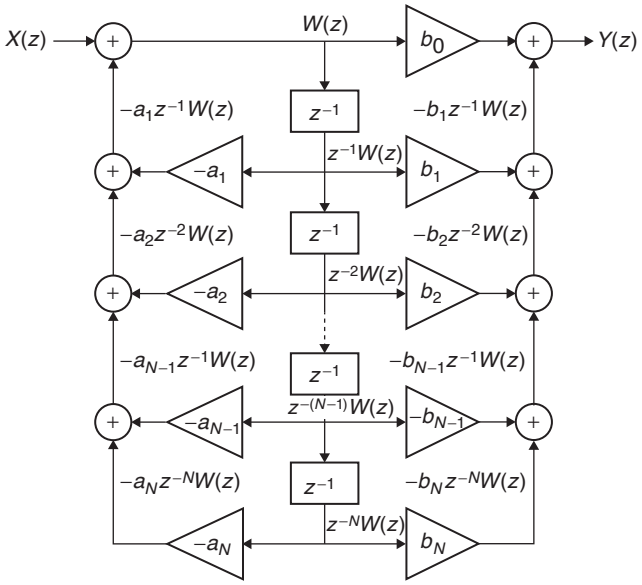
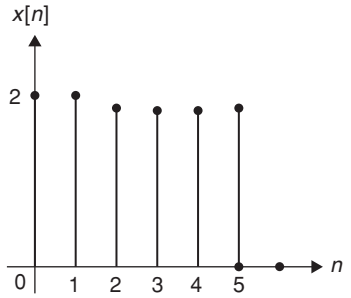


Figure 8 Direct form-II structure of IIR system

Solved Examples

Example 1: What is the z-transform of the signal shown in figure?



Solution: Here $x[0] = x[1] = \cdots = x[5] = 2$, and $x[6] = x[7] = \cdots = 0$

Now, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$= 2 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \frac{2}{z^4} + \frac{2}{z^5}$$

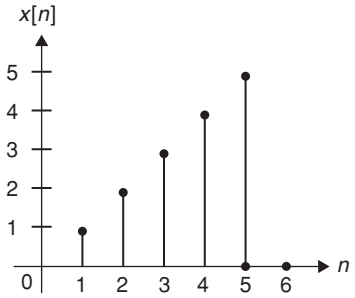
$$= \frac{2}{z^5}(z^5 + z^4 + z^3 + z^2 + z^4 + 1) \text{ for all } z \neq 0$$

Or $X(z) = 2\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5}\right)$

This expression in geometric progression, the summation of n terms with initial value a is $S_n = \frac{a(r^n - 1)}{r - 1}$

So $X(z) = 2\left(\frac{1\left(\left(\frac{1}{z}\right)^6 - 1\right)}{\left(\frac{1}{z} - 1\right)}\right) = \frac{2z}{z-1}(1 - z^{-6})$

Example 2: Find the z-transform of the signal $x[n]$ depicted in the figure



Soluton: The signal $x[n]$ can be expressed as

$x[n] = n$ for $0 \le n \le 5$

Now, $x[n] = n\{u[n] - u[n-6]\}$

$= nu[n] - nu[n-6]$

$= nu[n] - (n - 6 + 6)u[n-6]$

$= nu[n] - (n-6)u[n-6] - 6u[n-6]$

we have $u[n] \leftrightarrow \frac{1}{1 - Z^{-1}}$

$$nu[n] \leftrightarrow \frac{z^{-1}}{(1 - z^{-1})^2}$$

$x[n-k] \leftrightarrow z^{-k} X(z)$

So applying z-transform

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{z^6}\frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{z^6}\frac{6}{(1-z^{-1})}, |z| > 1$$

$$X(z) = \frac{z^{-1} - z^{-7} - 6z^{-6}(1 - z^{-1})}{(1-z^{-1})^2} = \frac{5z^{-7} - 6z^{-6} + z^{-1}}{(1-z^{-1})^2}$$

$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2}$$

Example 3: A discrete LTI system has the difference equation $y[n+1] - 2\,y[n] = x[n+1]$, if the initial conditions are $y[-1] = -1$, and the input $x[n] = \left(\frac{1}{3}\right)^n u[n]$ then find output $y[n]$.

Solution: Given $y[n+1] - 2y[n] = x[n+1]$

$\Rightarrow y[n] - 2y[n-1] = x[n]$

$\Rightarrow y[n]\, u[n] \leftrightarrow Y(z)$

$$y[n-1]u[n] \leftrightarrow \frac{1}{z}Y(z) + y[-1] = \frac{1}{z}y(z) + (-1)$$

$x[-1] = x[-2] \supset = x[-n] = 0$, $x[n]$ is causal input

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$\Rightarrow X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

By taking the z-transform for the difference equation

$$Y(z) - 2\left[\frac{1}{2}Y(z) - 1\right] = X(z)$$

$$Y(z)\left[1 - \frac{2}{z}\right] + 2 = X(z)$$

Now, $Y(z) = \frac{X(z) - 2}{1 - 2z^{-1}} = \frac{\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right) - 2}{1 - 2z^{-1}}$

$$= \frac{1 - 2 + \frac{2}{3}z^{-1}}{(1 - 2z^{-1})\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{-1 + \frac{2}{3}z^{-1}}{(1 - 2z^{-1})\left(1 - \frac{1}{3}z^{-1}\right)}$$

Now we have, $Y(z) = \frac{z(-3z+2)}{(z-2)(3z-1)} = z\left[\frac{A}{z-2} + \frac{B}{3z-1}\right]$

$$= z\left[\frac{-4}{5}\frac{1}{z-2} - \frac{3}{5}\frac{1}{(3z-1)}\right] = \frac{-1}{5}\left[\frac{4}{1 - 2z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}}\right]$$

$$\Rightarrow y[n] = \frac{-1}{5}\left[4.2^n u[n] + \left(\frac{1}{3}\right)^n u[n]\right]$$

$$= -0.2\left[2^{n+2} + \left(\frac{1}{3}\right)^n\right]u[n]$$

Example 4: For a causal system specified by the transfer function $H(z) = \frac{z}{z - 0.4}$, find the zero-state response to input

$$x[n] = (0.5)^n\, u[n] + 2\, 2^n\, u[-n-1]$$

Soluiton: Zero state response means, $y[n]$ with zero initial conditions

$$X(z)\frac{1}{1 - 0.5z^{-1}} - \frac{2}{1 - 2z^{-1}} = \frac{z(z - 2 - 2z + 1)}{(z - 0.5)(z-2)}$$

$$= \frac{-z(z+1)}{(z-0.5)(z-2)}$$

The ROC of $X(z)$ is for the first term $|z| > 0.5$ and for the second term $|z| < 2$.

The ROC for $X(z)$ is common region given by $0.5 < |z| < 2$ and $X|z| = \frac{-z(z+1)}{(z-0.5)(z-2)}$

Then $Y(z) = X(z)H(z) = \dfrac{-z^2(z+1)}{(z-0.4)(z-0.5)(z-2)}$

As $H(z)$ is causal system ROC is $|z| > 0.4$, yhe common region of convergence of $X(z)$ and $H(z)$ is $0.5 < |z| < 2$, so

$$Y(z) = \frac{-z^2(z+1)}{(z-0.4)(z-0.5)(z-2)}$$

$$\text{Now, } \frac{Y(z)}{z} = \frac{-z(z+1)}{(z-0.4)(z-0.5)(z-2)}$$

$$= \frac{-7}{2}\frac{1}{(z-0.4)} + \frac{5}{(z-0.5)} - \frac{5}{2}\frac{1}{z-2}$$

$$\text{So, } Y(z) = \frac{-7}{2}\frac{z}{z-0.4} + 5.\frac{z}{z-0.5} - \frac{5}{2}\frac{z}{z-2},$$

$0.5 < |z| < 2$

$$y[n] = \left[\frac{-7}{2}(0.4)^n + 5(0.5)^n\right]u[n] + \frac{5}{2}2^n u[-n-1]$$

Example 5: Find the z-transform of the signal $x[n]$

$$= \left\{n\left(\frac{-1}{3}\right)^n u[n]\right\} * (5)^n u[-n]$$

Solution: We can write $x[n] = w[n] * y[n]$

$$w[n] = n\left(\frac{-1}{3}\right)^n u[n]$$

$$\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1+\frac{1}{3}z^{-1}} \text{ with ROC}$$

$$|z| > \frac{1}{3}$$

$$w[n] = n\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z} W(z)$$

$$= -z\frac{d}{dz}\left\{\frac{1}{1+\frac{1}{3}z^{-1}}\right\}$$

$$w(z) = -z\frac{-1}{\left(1+\frac{1}{3}z^{-1}\right)^2}\left(\frac{-1}{3}z^{-2}\right)$$

$$= \frac{-1}{3}\frac{z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2} \text{ with ROC } |z| > \frac{1}{3}$$

$$y[n] = (5)^n u[-n] = \left(\frac{1}{5}\right)^{-n} u[-n]$$

$$\left(\frac{1}{5}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{5}}, \text{ ROC } |z| > \frac{1}{5}$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{2}\right), \text{with ROC } \frac{1}{R_x}$$

$$\text{So, } Y(z) = \frac{\frac{1}{z}}{\frac{1}{z}-\frac{1}{5}} \text{ ROC } \frac{1}{|z|} > \frac{1}{5}$$

$$= \frac{-z^{-1}}{\frac{1}{5}-z^{-1}} |z| < 5$$

By applying convolution $y[n] * w[n]$, and z-transform we have $X(z) = Y(z)\ W(z)$

$$= \frac{-z^{-1}}{\frac{1}{5}-z^{-1}} \cdot \frac{-1}{3}\frac{z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2}$$

The common ROC is $|z| > \frac{1}{3}$, and $|z| < 5$

$$\text{Also, } X(z) = \frac{5}{z-5} \cdot \frac{1}{3}\frac{z}{\left(z+\frac{1}{3}\right)^2}$$

$$= \frac{5}{3}\frac{z}{(z-5)\left(z+\frac{1}{3}\right)^2} \text{ with ROC } \frac{1}{3} < |z| < 5$$

Example 6: Find the transfer function and impulse response of a causal LTI system, if the input to the system is $x[n] = u[n]$ and output is $y[n] = \delta[n]$

Soluton: The z-transform of inputs and outputs

$$X(z) = \frac{1}{1-z^{-1}}, \text{ ROC } |z| > 1,$$

$Y(z) = 1$, ROC for all z

Then transfer function $H(z)$

$$= \frac{Y(z)}{X(z)} = \frac{1}{1/(1-z^{-1})}$$

$= 1 - z^{-1}$, with ROC all z except at $z = 0$

By taking inverse z-transform,

$h[n] = \delta[n] - \delta[n-1]$ is the impulse response.

Example 7: An LTI system is described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{18}y[n-2] = x[n] + \frac{1}{20}x[n-1] - \frac{1}{20}x[n-2].$$

Does a stable and causal LTI inverse system exists?

Solution: $y[n]-\frac{1}{2}y[n-1]+\frac{1}{18}y[n-2]=x[n]$
$+\frac{1}{20}x[n-1]-\frac{1}{20}x[n-2]$

By considering z-transform

$$Y(z)-\frac{1}{2}z^{-1}Y(z)+\frac{1}{18}z^{-2}Y(z)=X(z)+\frac{z^{-1}}{20}X(z)-\frac{1}{20}z^{-2}X(z)$$

Now, $H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{20}z^{-1}-\frac{1}{20}z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{18}z^{-2}}=\frac{z^2+\frac{1}{20}z-\frac{1}{20}}{z^2-\frac{1}{2}z+\frac{1}{18}}$

$$=\frac{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}=\frac{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}$$

Then, $H_i(z)=\frac{1}{H(z)}=\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{4}\right)}$

Is the transfer function of inverse system. All the poles and zeros of $H(z)$ lie inside the unit circle, so stable and causal inverse system exists and this is a minimum phase system.

Example 8: A discrete time LTI model for two path communication channel is $y[n] = x[n] + ax[n-1]$, find the difference equation description of the inverse system.

Solution: $y[n] = x[n] + ax[n-1]$, by taking z-transform $Y(z) = X(z) + az^{-1}X(z)$

Transfer function $H(z)=\frac{Y(z)}{X(z)}=1+a\,z^{-1}$

The inverse system has transfer function

$$H_i(z)=\frac{1}{H(z)}=\frac{1}{1+az^{-1}}=\frac{Y_i(z)}{X_i(z)}$$

Now, $Y_i(z) + az^{-1}Y_i(z) = X_i(z)$,
By taking inverse z-transform

$$y[n] + ay[n-1] = x[n]$$

The inverse system is both stable and causal if $|a| < 1$.

EXERCISES

Practice Problems I

Directions for questions 1 to 22: Select the correct alternative from the given choices.

Common Data for Questions 1 to 3:

1. $x[n] = a^n u[n]$ where a is real. z-transform of $x[n]$ is

(A) $\frac{z}{z-a}, |z| < |a|$ (B) $\frac{z}{z-a}, |z| > |a|$
(C) $\frac{z}{z+a}, |z| > |a|$ (D) $\frac{z}{z+a}, |z| < |a|$

2. z-transform of $a^{n+1}u[n+1]$ is

(A) $\frac{z^2}{z-a}, |a| < |z| < \infty$ (B) $\frac{z^2}{z+a}, |a| < |z| < \infty$
(C) $\frac{z}{z-a}, |a| < |z| < \infty$ (D) $\frac{z^2}{z-a}, |a| > |z| < \infty$

3. z-transform of $n\,a^n\,u[n]$ is

(A) $\frac{-az}{(z-a)^2}, |z| > |a|$ (B) $\frac{az}{(z-a)}, |z| > |a|$
(C) $\frac{az}{(z-a)^2}, |z| > |a|$ (D) $\frac{az}{(z-a)^2}, |z| < |a|$

4. Inverse z-transform of $\log(1 - az^{-1})$, $|z| > |a|$ is

(A) $\frac{1}{n}a^n u[n-1]$ (B) $\frac{-1}{n}a^n u[n-1]$
(C) $\frac{-1}{n}a^n u[n+1]$ (D) $\frac{1}{n}a^n u[n+1]$

5. Find the initial and final values of z-domain signal $\frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$

(A) 10, 1 (B) 1, 10
(C) 0, 10 (D) 0, ∞

6. The transfer function of a system is $H(z) = 1 - 3z^{-1}$, the system response for input $x[n]$ is

(A) $x[n] - 0.3\,x[n-1]$ (B) $x[n] + 0.3x[n-1]$
(C) $x[n] - 3x[n+1]$ (D) $x[n] - 3x[n-1]$

7. Unilateral z-transform of $\left(\frac{1}{3}\right)^n u(n)+\left(\frac{1}{5}\right)^n u(n)$ and ROC are respectively?

(A) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/5)}; |z|>\frac{1}{5}$
(B) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/5)}; |z|>\frac{1}{3}$
(C) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/3)}; \frac{1}{5}>|z|>\frac{1}{3}$
(D) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/3)}; \frac{1}{5}<|z|<\frac{1}{3}$

8. Find $f(0)$, $f(1)$, $f(2)$ and $f(3)$, if

$$F(z)=\frac{z^2+2}{z^3-3z^2+3z-1}$$

(A) 0, 9, 31, 16 (B) 0, 1, 3, 8
(C) 1, 3, 3, 1 (D) 0, 1, 2, 4

9. The minimum Nyquist sampling rate $\frac{\sin t}{\pi t}$ is

(A) $2/\pi$ (B) $1/\pi$
(C) π (D) 2π

10. Match the following:

List I	List II
$x(n)$	ROC
(1) $\delta(n)$	(a) All z
(2) $u(n)$	(b) $\|z\| > 1$
(3) $-u(-n-1)$	(c) All z except 0
(4) $\delta(n-m)$	(d) $\|z\| < 1$
1 2 3 4	1 2 3 4
(A) a b c d	(B) b a c d
(C) a b d c	(D) b a d c

11. $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=X(n)$ where $x(n)$ and $y(n)$ are the input and output of the systems then determine system function.

(A) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z|>\frac{1}{2}$

(B) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z|<\frac{1}{2}$

(C) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z|>\frac{1}{2}$

(D) $\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z|>\frac{1}{2}$

12. For a z-transform $X(z)=\frac{z(2z-5/6)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$. Find out the sequences when the region of convergence is $|z| > 1/2$

(A) $\left[\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n u(n)\right]$

(B) $\left(\frac{1}{2}\right)^n u(n)-\left(\frac{1}{3}\right)^n u(-n-1)$

(C) $\left[\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^n\right]u(n)$

(D) $\left(\frac{1}{2}\right)^n u(n)+\left(\frac{1}{3}\right)^n u(-n-1)$

13. Find the sequence when region of convergence $\left|\frac{1}{3}\right|<|z|<\frac{1}{2}$ for above $X(z)$,

(A) $-\left(\frac{1}{2}\right)^n u(-n-1)+\left(\frac{1}{3}\right)^n u(n)$

(B) $\left(\frac{1}{2}\right)^n u(-n-1)+\left(\frac{1}{3}\right)^n u(n)$

(C) $\left(\frac{1}{2}\right)^n u(n)-\left(\frac{1}{3}\right)^n u(-n-1)$

(D) $\left(\frac{1}{2}\right)^n u(n)+\left(\frac{1}{3}\right)^n u(-n-1)$

14. Impulse response for the cascade of LTI systems having impulse response $h_1[n]=\left(\frac{1}{2}\right)^n u[n]$ and $h_2[n]=\left(\frac{1}{4}\right)^n u[n]$ is?

(A) $\left(\frac{1}{4}\right)^n (2n-1)u[n]$ (B) $\left(\frac{1}{4}\right)^n (2^{n+1}-1)u[n]$

(C) $\left(\frac{1}{4}\right)^n \left(2^{n+1}+1\right)u[n]$ (D) $\left(\frac{1}{4}\right)^n (2^n+1)u[n]$

15. What is the Nyquist rate for the signal $x(t) = \cos 2000\pi t + 6\sin 9000\pi t$

(A) 9 kHz (B) 4 kHz
(C) 2 kHz (D) 11 kHz

16. Which of the following is the inverse z-transform of $X(z)=\frac{z}{(z-2)(z-3)}|z|<2$

(A) $[2^n - 3^n]\,u(-n-1)$ (B) $[2^n - 3^n]\,u(n+1)$
(C) $[3^n - 2^n]\,u(-n-1)$ (D) $[2^n - 3^n]\,u(n)$

17. The impulse-response of a relaxed linear time invariant system is $h(n) = \alpha^n u(n)$ with $|\alpha| < 1$. Determine the value of the step response as $n \to \infty$

(A) $\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u(n)$ (B) $\left[\frac{1-\alpha^{n-1}}{1-\alpha}\right]u(n)$

(C) $\left[\frac{1+\alpha^{n+1}}{1-\alpha}\right]u(n)$ (D) $\left[\frac{1+\alpha^{n+1}}{1-\alpha}\right]u(n)$

18. Consider z-transform $X(z) = 5z^2 + 4z^{-1} + 3;\ 0 < |z| < \infty$, the inverse z-transform $x[n]$ is?

(A) $5\delta[n+1]$
(B) $3\delta[n]$
(C) $4\delta[n+2]$
(D) $5\delta[n+2] + 4\delta[n-1] + 3\delta[n]$

19. The transfer function of DT, LTI system is given by

$$H(z) = \frac{2-\frac{3}{4}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

Consider the following statements

S1: System is stable and causal for

$\text{ROC}\,|z| > \frac{1}{2}$

S2: System is stable but not causal for

$\text{ROC}: |z| < \frac{1}{4}$

S3: System is neither stable nor causal for

$\text{ROC}: \frac{1}{4} < |z| < \frac{1}{2}$

Which one of the following statements is valid?

(A) Both *S*1 and *S*2 are true
(B) Both *S*2 and *S*3 are true
(C) Both *S*1 and *S*3 are true
(D) All are true

20. ROC of *z*-transform of the discrete time sequence $x[n]=\left(\frac{1}{3}\right)^n u(n)-\left(\frac{1}{2}\right)^n u[-n-1]$ is

(A) $|z| > \frac{1}{3}$ (B) $|z| < \frac{1}{2}$

(C) $|z|$ for all (D) $\frac{1}{3} < |z| < \frac{1}{2}$

21. A causal LTI system is described by the difference equation $2y[n] = \alpha y[n-1] - 2x[n] + \beta x[n-1]$. The system is stable for which values of α, β?

(A) $|\alpha| > 2, |\beta| < 1$
(B) $|\alpha| < 2$, any β
(C) $|\alpha| < 3, |\beta| > 1$
(D) $|\alpha| > 2$, any β

22. A sequence $x[n]$ with *z*-transform $X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$ is applied as an input to a linear, time-invariant system with impulse response $h[n] = 2\delta[n-3]$, the output at $n = 4$ is

(A) 2 (B) 0
(C) −2 (D) −6

Practice Problems 2

Directions for questions 1 to 21: Select the correct alternative from the given choices.

1. Two DT systems with impulse response $h_1[n] = \delta[n-1]$, $h_2[n] = \delta[n-2]$ are connected in cascade, the overall impulse response of the cascaded system is?

(A) $\delta[n-1]$ (B) $\delta[n-2]$
(C) $\delta[n-3]$ (D) $\delta[n-4]$

2. A system with transfer function $H(z)$ has impulse response $h[n]$, defined as $h[2] = 1$, $h[3] = -1$, and $h[n] = 0$ otherwise, consider following statements

S1: $H(z)$ is a low pass filter

S2: $H(z)$ is an FIR filter

Which one of the following is correct?

(A) Only S2 is true
(B) Both S1 and S2 are false
(C) Both S1 and S2 are true, and S1 is reason for S2
(D) Both S1 and S2 are true, and S2 is a reason for S1

3. *z*-transform $X(Z)$ of a sequence $x[n]$ is given by $X(Z) = \frac{0.5}{1-2z^{-1}}$, it is given that the ROC of $X(Z)$ includes unit circle, the value of $x[0]$ is

(A) 0 (B) 0.5
(C) 1 (D) ∞

4. If ROC of $x_1[n] + x_2[n]$ is $\frac{1}{3} < |z| < \frac{2}{3}$, then the ROC of $x_1[n] - x_2[n]$ includes

(A) $\frac{1}{3} < |z| < 3$ (B) $\frac{2}{3} < |z| < 3$

(C) $\frac{3}{2} < |z| < 3$ (D) $\frac{1}{3} < |z| < \frac{2}{3}$

5. $y(n)=\sum_{k=0}^{n} x[k]$

Which one of the following correctly relate the *z*-transform of the input and output denoted by $X(z)$ and $Y(z)$ respectively?

(A) $Y(z)=\frac{d}{dz}X(z)$ (B) $Y(z)=\frac{X(z)}{1-z^{-1}}$

(C) $Y(z) = X(z)z^{-1}$ (D) $Y(z) = (1-z^{-1})X(z)$

6. $x(n)=\frac{1}{7}\left(\frac{1}{2}\right)^n u(n)-\frac{6}{7}(3)^n u(-n-1)$. Find out ROC of *z*-transform?

(A) $1/2 > |z| < 3$ (B) $|z| > 1/2$
(C) $|z| < 3$ (D) $1/2 < |z| < 3$

7. The range of values a and b for which the *z*-transform with impulse response $h(n) = a^n$, $n \geq 0 = b^n$, $n < 0$ will be stable

(A) $|a| < 1, |b| < 1$ (B) $|a| < 1, |b| > 1$
(C) $|a| > 1, |b| > 1$ (D) $|a| > 1, |b| < 1$

8. The *z*-transform of $a^{-n}\,u[-n-1]$ is

(A) $\frac{az}{1+az}, |z| < \frac{1}{|a|}$ (B) $\frac{az}{1-az}, |z| < \frac{1}{|a|}$

(C) $\frac{-az}{1-az}, |z| < \frac{1}{|a|}$ (D) $\frac{az}{1-az}, |z| > \frac{1}{|a|}$

9. z-transform of $x[n] = \{3, -2, \underset{\uparrow}{4}, 1, 0, -3\}$ is
(A) $3z^2 + 2z + 4 + z^{-1} + 3z^{-2}$, $2 < |z| < 4$
(B) Does not exist
(C) $3z^2 - 2z + 4 + z^{-1} - 3z^{-3}$, $0 < |z| < \infty$
(D) $3z^2 - 2z + 4 + z^{-1} - 3z^{-3}$, $|z| < 0$, $|z| < \infty$

10. z-transform of $u[n + 1]$ is
(A) $\frac{z}{z-1}$ (B) $\frac{z^2}{z-1}$
(C) $\frac{z^{-1}}{1-z^{-1}}$ (D) $\frac{1}{z^{-1}+z^{-2}}$

11. Inverse z-transform of $\frac{2}{z-2}$, $(|z| > 2)$ is
(A) $2^{n-1}\,u[n+1]$ (B) $2^{n+1}\,u[n-1]$
(C) $2^{n}\,u[n+1]$ (D) $2^{n}\,u[n-1]$

12. z-transform and Laplace transfrom are related by
(A) $s = \ln z$ (B) $s = \frac{\ln z}{T}$
(C) $s = T \ln z$ (D) Not related

13. The final value of $x[n]$ is
(A) $\underset{z\to\infty}{\text{Lt}}\, zX(z)$ (B) $\underset{z\to 0}{\text{Lt}}\, zX(z)$
(C) $\underset{z\to\infty}{\text{Lt}}\,(1-z^{-1})\,X(z)$ (D) $\underset{z\to 1}{\text{Lt}}\,(1-z^{-1})\,X(z)$

14. The initial value of $x[n]$, if $X(z) = \frac{3z^2}{(z+3)(z-3)}$
(A) 3 (B) 6
(C) ∞ (D) 0

15. An LTI system defined as $y[n] + 0.5y[n-1] = x[n] + 0.3x[n-1]$. The transfer function in z-plane is
(A) $\frac{z+0.3}{z+0.5}$ (B) $\frac{z-0.3}{z-0.5}$
(C) $\frac{z+0.3}{z-0.5}$ (D) $\frac{z-0.3}{z+0.5}$

16. A signal $x[n] = \{\underset{\uparrow}{1}, -1, 2, 1, 3, 2\}$. The ROC z-transform $X(z)$ is
(A) $|z| > 3$
(B) $-1 < |z| < 3$
(C) $|z| < 3$
(D) the entire z-plane except at $z = 0$

17. The minimum sampling frequency of $x(t) = 5\cos 50\pi t + 2\cos 200\pi t - 10\cos 100\pi t$ is
(A) 100 Hz (B) 200 Hz
(C) 50 Hz (D) Cannot be sampled

18. The minimum number of samples required to exactly describe $x(t) = 10\cos 4\pi t + 4\sin 8\pi t$ is
(A) 4 samples per second.
(B) 2 samples per second.
(C) 8 samples per second.
(D) 16 samples per second.

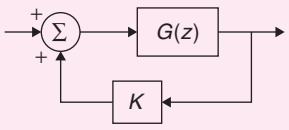
19. Aliasing occurs when sampling frequency ω_s is (ω_m – band limited signal frequency)
(A) $\omega_s = 2\omega_m$ (B) $\omega_s < 2\omega_m$
(C) $\omega_s > 2\omega_m$ (D) 0

20. Interpolation is the process of
(A) Inserting $N-1$ unity sequence values to $x[n]$
(B) Inserting $N-1$ zero sequence values to $x[n]$
(C) Deleting $N-1$ unity sequence values from $x[n]$
(D) Deleting $N-1$ zero sequence values from $x[n]$

21. Given signal $m(t) = \sin 5\pi t + 5\sin 10\pi t$ is sampled instantaneously, the maximum interval between samples is
(A) 0.1 seconds.
(B) 0.2 seconds.
(C) 10 seconds.
(D) None

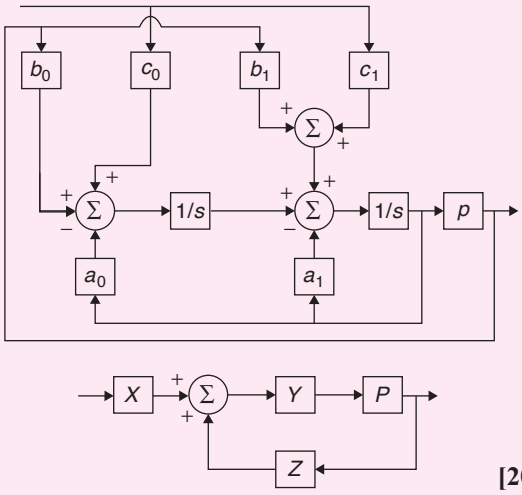
Previous Years' Questions

1. Consider the discrete-time system shown in the figure where the impulse response of $G(z)$ is $g(0) = 0$, $g(1) = g(2) = 1$, $g(3) = g(4) = 0$.



This system is stable for range of values of K **[2007]**
(A) $[-1, 1/2]$ (B) $[-1, 1]$
(C) $[-1/2, 1]$ (D) $[-1/2, 2]$

2. The system shown in figure below can be reduced to the form with



[2007]

(A) $X = c_0 s + c_1, Y = \dfrac{1}{(s^2 + a_0 s + a_1)}, Z = b_0 s + b_1$

(B) $X = 1, Y = \dfrac{(c_0 s + c_1)}{(s^2 + a_0 s + a_1)}, Z = b_0 s + b_1$

(C) $X = c_1 s + c_0, Y = \dfrac{(b_1 s + b_0)}{(s^2 + a_1 s + a_0)}, Z = 1$

(D) $X = c_1 s + c_0, Y = \dfrac{1}{(s^2 + a_1 s + a_0)}, Z = b_1 s + b_0$

3. $X(z) = 1 - 3z^{-1}$, $Y(z) = 1 + 2x^{-2}$ are z-transforms of two signals $x[n]$, $y[n]$ respectively. A linear time invariant system has the impulse response $h[n]$ defined by these two signal as

$H[n] = x[n-1] * y[n]$

Where * denotes discrete time convolution. Then the output of the system for the input $\delta[n-1]$ **[2007]**

(A) has z-transform $z^{-1} X(z)\, Y(z)$

(B) equals $\delta[n-2] - 3\delta[n-3] + 2\delta[n-4] - 6\delta[n-5]$

(C) has z-transform $1 - 3z^{-1} + 2z^{-2} - 6z^{-3}$

(D) does not satisfy any of the above three.

Common Data for Questions 4 and 5:

4. A signal is processed by a causal filter with transfer function $G(s)$. For a distortion free output signal waveform, $G(s)$ must **[2007]**

(A) Provide zero phase shift for all frequency

(B) Provide constant phase shift for all frequency

(C) Provide linear phase shift that is proportional to frequency

(D) Provide a phase shift that is inversely proportional to frequency

5. $G(s) = \alpha z^{-1} + \beta z^{-1}$ is a low-pass digital filter with a phase characteristics same as that of the above question if **[2007]**

(A) $\alpha = \beta$ (B) $\alpha = -\beta$

(C) $\alpha = \beta^{(t/3)}$ (D) $\alpha = \beta^{-(t/3)}$

6. Given a sequence $x[n]$, to generate the sequence $y[n] = x[3 - 4n]$, which one of the following procedures would be correct? **[2008]**

(A) First delay $x[n]$ by 3 sample to generate $z_1[n]$, then pick every 4th sample of $z_1[n]$ to generate $z_2[n]$, and then finally time reverse $z_2[n]$ to obtain $y[n]$

(B) First advance $x[n]$ by 3 samples to generate $z_1[n]$, then pick every 4th sample of $z_1[n]$ to generate $z_2[n]$, and then finally time reverse $z_2[n]$ to obtain $y[n]$

(C) First pick every fourth sample of $x[n]$ to generate $v_1[n]$, time-reverse $v_1[n]$ to obtain $v_2[n]$, and finally advance $v_2[n]$ by 3 sample to obtain $y[n]$

(D) First pick every fourth sample of $x[n]$ to generate $v_1[n]$, time-reverse $v_1[n]$ to obtain $v_2[n]$, and finally delay $v_2[n]$ by 3 samples to obtain $y[n]$

7. $H(z)$ is a transfer function of a real system. When a signal $x[n] = (1 + j)^n$ is the input to such a system, the output is zero. Further, the region of convergence (ROC) of $\left[1 - \frac{1}{2} z^{-1}\right] H(z)$ is the entire z-plane (except $z = 0$). It can then be inferred than $H(z)$ can have a minimum of **[2008]**

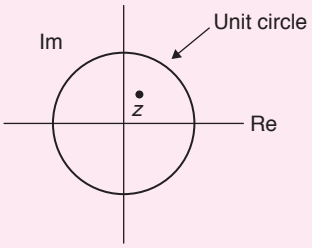
(A) One pole and one zero

(B) One pole and two zeros

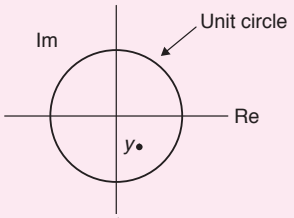
(C) Two poles and one zero

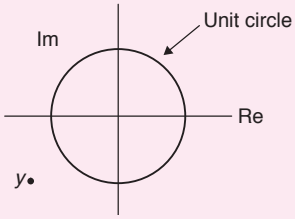
(D) Two poles and two zeros

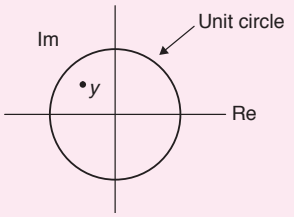
8. A point z has been plotted in the complex plane, as shown in figure below.

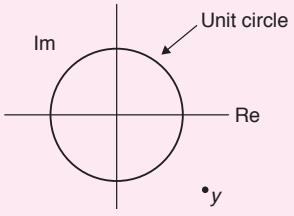


The plot of the complex number $y = \dfrac{1}{z}$ is **[2011]**

(A)


(B)


(C)


(D)


9. Let S be the set of points in the complex plane corresponding to the unit circle. (That is $S = \{z : |z| = 1\}$). Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane **[2014]**
(A) Unit circle
(B) Horizontal axis line segment from origin to (1, 0)
(C) The point (1, 0)
(D) The entire horizontal axis

10. Let $X(z) = \dfrac{1}{1-z^{-3}}$ be the z-transform of a causal signal $x[n]$. Then, the values of $x[2]$ and $x[3]$ are **[2014]**
(A) 0 and 0
(B) 0 and 1
(C) 1 and 0
(D) 1 and 1

11. An input signal $x(t) = 2 + 5\sin(100\pi t)$ is sampled frequency of 400 Hz and applied to the system whose transfer function is represented by

$$\frac{Y(z)}{X(z)} = \frac{1}{N}\left(\frac{1 - z^{-N}}{1 - z^{-1}}\right)$$

Where, N represents the number of samples per cycle. The output $y(n)$ of the system under steady state is **[2014]**
(A) 0 (B) 1
(C) 2 (D) 5

12. Consider a discrete time signal given by

$x[n] = (-0.25)^n\, u[n] + (0.5)^n\, u[-n-1]$

The region of convergence of its Z-transform would be **[2015]**
(A) The region inside the circle of radius 0.5 and centered at origin.
(B) The region outside the circle of radius 0.25 and centered at origin.
(C) The annular region between the two circles, both centered origin and having radii 0.25 and 0.5.
(D) The entire Z plane.

13. The z-Transform of a sequence $x[n]$ is given as $X(z) = 2z + 4 - 4/z + 3/z^2$. If $y[n]$ is the first difference of $x[n]$, then $Y(z)$ is given by **[2015]**
(A) $2z + 2 - 8/z + 7/z^2 - 3/z^3$
(B) $-2z + 2 - 6/z + 1/z^2 + 3/z^3$
(C) $-2z - 2 + 8/z - 7/z^2 + 3/z^3$
(D) $4z - 2 - 8/z - 1/z^2 + 3/z^3$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** C	**4.** B	**5.** C	**6.** D	**7.** A	**8.** B	**9.** B	**10.** C
11. D	**12.** C	**13.** A	**14.** B	**15.** A	**16.** A	**17.** A	**18.** D	**19.** C	**20.** D
21. B	**22.** B								

Practice Problems 2

1. C	**2.** A	**3.** B	**4.** D	**5.** B	**6.** D	**7.** B	**8.** B	**9.** C	**10.** B
11. D	**12.** B	**13.** D	**14.** A	**15.** A	**16.** D	**17.** B	**18.** C	**19.** B	**20.** B
21. A									

Previous Years' Questions

1. A	**2.** D	**3.** B	**4.** C	**5.** A	**6.** D	**7.** A	**8.** D	**9.** C	**10.** B
11. C	**12.** C	**13.** A							

Chapter 4

Continuous Time Fourier Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Introduction
- Approximation of a signal by other signals
- Fourier series representation of periodic signals
- Complex fourier series
- Even and odd Signals
- Harmonic form of fourier series
- Amplitude and phase spectra of periodic signal
- Linearity
- Time shifting
- Periodic convolution
- Parseval's theorem
- Continuous time (C.T.) fourier transform
- Fourier spectra
- Convergence of fourier transform
- Relation between fourier transform and laplace transform
- Common fourier transform pairs

Introduction

There are four distinct fourier representations, each applicable to different class of signals. The Fourier series (FS) applies to continuous-time periodic signals, and the discrete-time Fourier series (DTFS) applies to discrete time periodic signals. Non-periodic signals have Fourier transform representations. The Fourier transform (FT) applies to a signal that is continuous in time and non-periodic. The discrete-time Fourier transform (DTFT) applies to a signal that is discrete in time and non-periodic.

Relationship between time properties of a signal and the appropriate Fourier representation:

Time/property	Periodic	Non periodic
Continuous (t)	Fourier series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-time Fourier series (DTFS)	discrete-time Fourier transform (DTFT)

Approximation of a Signal by Other Signals

Let us consider two signals $f_1(t)$ and $f_2(t)$ we can approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1 < t < t_2)$ as $f_1(t) \simeq C_{12} f_2(t)$ for $(t_1 < t < t_2)$.

We must choose c_{12} such that the error between actual function and the approximated function is minimum and is given by:

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) \cdot f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^{\,2}(t)\,dt}$$

$f_1(t)$ has a component of wave form $f_2(t)$, and this component has a magnitude C_{12}. If C_{12} vanishes, then the signal $f_1(t)$ contains no component of signal $f_2(t)$ and we say the functions are orthogonal to each other over interval (t_1, t_2) if $\int_{t_1}^{t_2} f_1(t) \cdot f_2(t)\,dt = 0$,

similarly in case of complex functions $f_1(t), f_2(t)$ $f_1(t) \simeq C_{12} f_2(t)$ is approximation of $f_1(t)$ in terms of $f_2(t)$, and C_{12} to minimize mean square error magnitude is given by:

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2^{*}(t)\,dt}{\int_{t_1}^{t_2} f_2(t) f_2^{*}(t)\,dt},$$

where $f_2^{*}(t)$ is complex conjugate of $f_2(t)$.

For a set of complex functions $\{g_r(t)\}$ $r = 1, 2 \ldots)$ mutually orthogonal over the interval (t_1, t_2)

$$\int_{t_1}^{t_2} g_m\, g_n^{*}(t)\,dt = \begin{cases} 0, \text{ if } m \neq n \\ K_m, \text{ if } m = n \end{cases}$$

If this set of functions is complete, then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) \cdots + C_r g_r(t) + \cdots$$

where

$$C_r = \frac{1}{K_r}\int_{t_1}^{t_2} f(t) g_r^*(t) dt$$

$$K_r = \int_{t_1}^{t_2} g_r(t) g_r^*(t) dt.$$

Let us consider a set of n functions $g_1(t)$, $g_2(t)$ … $g_n(t)$ which are orthogonal to one another over an interval t_1 to t_2.

$$\int_{t_1}^{t_2} g_j(t) g_k(t) dt = 0, \text{ for } j \neq k \quad \text{and} \quad \text{consider} \int_{t_1}^{t_2} g_j^2(t) dt = K_j$$

An arbitrary function $f(t)$ can be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal functions

$$f(t) \simeq c_1 g_1(t) + c_2 g_2(t) + \cdots + c_k g_k(t)$$
$$+ \cdots + c_n g_n(t) = \sum_{r=1}^{n} c_r g_r(t).$$

For the best approximation, mean square error over the interval has to be minimized then

$$C_r = \frac{\int_{t_1}^{t_2} f(t) g_r(t) dt}{\int_{t_1}^{t_2} g_r^{\,2}(t) dt} = \frac{1}{K_r}\int_{t_1}^{t_2} f(t) g_r(t) dt.$$

From the above discussion we can conclude that, $f(t)$ contains a component of signal $g_r(t)$ and this component has a magnitude C_r. Representation of $f(t)$ by a set of infinite mutually orthogonal functions is called generalized Fourier series representation of $f(t)$.

The representation of square wave by sinusoids.

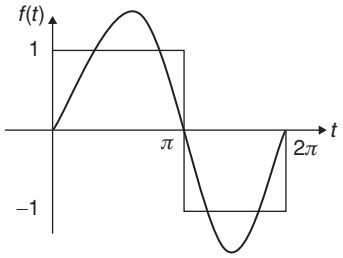


Figure 1 One term approximation

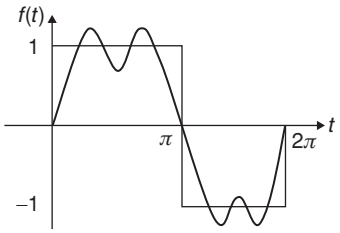


Figure 2 Two term approximation

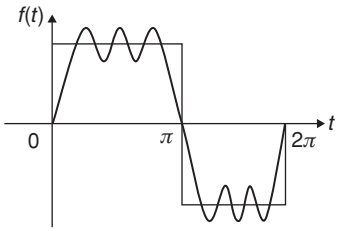


Figure 3 Three term approximation

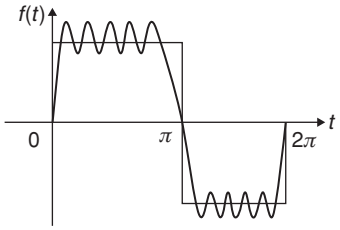


Figure 4 Five term approximation

From the approximation by sinusoids, we can observe that by increasing the number of terms we get nearly square wave.

Some examples of orthogonal functions: $\sin\omega_o t$, $\sin 2\omega_o t$, $\sin 3\omega_o t$, … etc., form orthogonal set over any interval $(t_0, t_0 + 2\pi/\omega_o)$.

Similarly $\cos\omega_o t$, $\cos 2\omega_o t$, $\cos 3\omega_o t$ … etc also form orthogonal set, in the same interval similarly $\cos n\omega_o t$ and $\sin m\omega_0 t$ also orthogonal over the same interval.

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin n\omega_o t \cdot \sin m\omega_o t \, dt = 0$$

$$\int_{t_0}^{t_0+2\pi/\omega_0} \cos n\omega_o t \cdot \cos m\omega_o t \; dt = 0 \quad \text{for all } m \neq n$$

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin m\omega_o t \cdot \cos n\,\omega_o t \; dt = 0$$

So composite set of functions consisting of set $\cos n\omega_o t$ and $\sin n\omega_o t$ for $(n = 0, 1, 2 \ldots)$ forms a complete orthogonal set.

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin^2 n\omega_o t \cdot dt = \int_{t_0}^{t_0+2\pi/\omega_0} \cos^2 n\omega_o t \; dt = \frac{\pi}{\omega_o} = \frac{T_0}{2}.$$

We can represent an arbitrary function $f(t)$ by linear combination of sine and cosine functions which is called as trigonometric Fourier series.

A set of exponential functions $\{e^{jn\omega_o t}\}(n = 0, \pm 1, \pm 2, \pm 3\ldots)$ is orthogonal over an interval $(t_0, t_0 + 2\pi/\omega_o)$ for any value of

$$t_0 \int_{t_0}^{t_0+2\pi/\omega_0} (e^{jn\omega_o t})(e^{jn\omega_o t})^* dt = \frac{2\pi}{\omega_o} = T_0 \text{ for } m = n$$

So we can represent an arbitrary function $f(t)$ by a linear combination of exponential functions over an interval $(t_0, t_0 + 2\pi/\omega_o)$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jn\omega_o t} (t_0 < t < t_0 + 2\pi/w_o).$$

This is called as complex exponential Fourier series representation

$$C_k = \frac{1}{T_0}\int_{t_0}^{t_0+2\pi/\omega_o} f(t) e^{-jk\omega_o t} dt.$$

Fourier Series Representation of Periodic Signals

If $x(t) = x(t + T_0)$, then $x(t)$ said to be periodic, the smallest positive value of T_0 is called fundamental period of $x(t)$.

Examples: $x(t) = \cos(\omega_o t + \phi)$,

$$x(t + T_0) = \cos[\omega_o (t + T_0) + \phi]$$
$$= \cos(\omega_o t + \phi + \omega_o T_0).$$

$x(t + T_0) = x(t)$ for minimum value $\omega_o T_0 = 2\pi$ or for all multiples of 2π, but we need to consider smallest value as fundamental period T_0. So $T_0 = \frac{2\pi}{\omega_o}$ is fundamental period, $x(t) = e^{j\omega_o t}$ are periodic where $\omega_o = \frac{2\pi}{T_0} = 2\pi f_0$ is the fundamental angular frequency.

Consider representing a periodic signal as a weighted super position of complex sinusoidal since the weighted super position must have the same period as the signal, each sinusoid in the superposition must have the same period as the signal, this implies that the frequency of each sinusoid must be an integer multiple of signal's fundamental frequency.

A sinusoid whose frequency is an integer multiple of a fundamental frequency is said to be a harmonic of the sinusoid at the fundamental frequency.

For example: $e^{jk\omega_o t}$ is the Kth harmonic of the $e^{j\omega_o t}$.

Complex Fourier Series

The complex Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 and fundamental frequency ω_0 is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T_0}.$$

C_K: complex Fourier coefficient.

$$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$\int_{T_0}$ denotes integral over one period of T_0.

$c_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$, which indicates C_0 equals the average value of $x(t)$ over one period.

The Fourier series coefficients c_k are known as a frequency-domain representation of $x(t)$ because each Fourier series coefficient is associated with a complex sinusoid of a different frequency.

If $x(t)$ is real then $C_{-K} = C^*_K$

Trigonometric Fourier Series

The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0, (or) fundamental angular frequency ω_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\, dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin k\omega_0 dt.$$

The trigonometric Fourier coefficient a_k, b_k, and complex Fourier coefficient C_K are related as:

$$\frac{a_0}{2} = c_0, a_k = c_k + c_{-k}, b_k = j(c_k - c_{-k})$$

$$c_k = \frac{1}{2}(a_k - jb_k), c_{-k} = \frac{1}{2}(a_k + jb_k)$$

Even and Odd Signals

If a periodic signal $x(t)$ is even, then $b_k = 0$, and its Fourier series contains only cosine terms.

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t, \omega_0 = \frac{2\pi}{T_0}$$

If $x(t)$ is odd, then $a_k = 0$, and its Fourier series contains only sine terms

$$x(t) = \sum_{k=-\infty}^{\infty} b_k \sin k\omega_0 t, \omega_0 = \frac{2\pi}{T_0}$$

Harmonic Form of Fourier Series

Other form of the Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is given by [known as harmonic form Fourier series of $x(t)$]:

$$x(t) = D_o + \sum_{k=1}^{\infty} D_K \cos(k\omega_0 t - \theta_k), \omega_0 = \frac{2\pi}{T_0}$$

The term D_0 is known as dc component, and the term $D_k \cos(k\omega_0 t - \theta_k)$ is referred to as the kth harmonic component of $x(t)$.

The coefficients D_k and θ_k are related to trigonometric Fourier series coefficients a_k and b_k as:

$$D_o = \frac{a_0}{2}, D_K = \sqrt{a_k^2 + b_k^2}, \theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

Dirichlet Conditions

For convergence of Fourier series:

1. $x(t)$ is absolutely integrable over any period
$$\int_{T_0} |x(t)| dt < \infty.$$
2. $x(t)$ has a finite number of maxima and minima with in any field interval of 'T'.
3. $x(t)$ has a finite number of discontinuities with in any finite interval of 'T' and each of these discontinuities is finite.

Table 1 *Effects of symmetry on Fourier coefficients*

Symmetry	Condition	The Trigonometric Fourier Co-efficient to be Zero
Even	$x(t) = x(-t)$	$b_n = 0$, no sine terms will be present
Odd	$x(-t) = -x(t)$	$a_n = 0$, no cos terms will be present
Half wave	$x(t) = -x\left(t \pm \frac{T}{2}\right)$	dc term $a_0 = 0$; a_n, $b_n = 0$ for n even. All even coefficients will be zero

Amplitude and Phase Spectra of Periodic Signal

Complex Fourier coefficients C_k can be expressed as $C_K = |C_K| e^{j\phi_k}$. A plot $|C_k|$ versus angular frequency ω is called amplitude spectrum of $x(t)$. A plot of ϕ_K versus ω is called the phase spectrum of $x(t)$. Both are discrete in nature.

For a real, periodic signal $x(t)$ we have,

$$|C_{-k}| = |C_k|,\ \phi_{-k} = -\phi_k.$$

So amplitude spectrum is an even function of ω, where as phase spectrum is an odd function of ω, for real periodic signal.

Properties of Continuous Time Fourier Series

$x(t)$ is a periodic signal with period T_0 and fundamental frequency $\omega_0 = 2\pi/T_0$, then the Fourier series coefficients of $x(t)$ are denoted by C_k, then we can use the notation $x(t) \underline{\text{FS}} C_k$.

Linearity

If $x(t)$, and $y(t)$ denote two periodic signals with period T_0 and which have Fourier coefficients denoted by a_k and b_k, respectively,

$$x(t) \underline{\text{FS}} a_k, y(t) \underline{\text{FS}} b_k$$

as $x(t)$, and $y(t)$ have the same period T_0, it easily follows that any linear combination of the two signals will also be periodic with period T_0. $z(t) = Ax(t) + By(t)$, then Fourier series co-efficients of $z(t)$ are $C_k = A\ a_k + B\ b_k$

$$z(t) = Ax(t) + By(t) \underline{\text{FS}} C_k = A\ a_k + Bb_k$$

Time Shifting

If $x(t) \underline{\text{FS}} a_k$ then

$$x(t - t_0) \underline{\text{FS}}\ e^{-jk\omega_0 t_0} a_k = e^{-jk}(2\pi/T_0)t_0 a_k$$

When a periodic signal is shifted in time, the magnitude of its Fourier series coefficients remains unaltered, only phase will be changed.

Time Reversal

If $x(t) \underline{\text{FS}} C_k$ than $x(-t) \underline{\text{FS}} C_{-k}$, in other words time reversal applied to a continuous time signal results in a time reversal of the corresponding sequence of Fourier series co-efficients if $x(t)$ is even, then $x(-t) = x(t)$ then its Fourier series coefficients are also even $C_{-k} = C_k$ similarly if $x(t)$ is odd, then $x(-t) = -x(t)$, then so are its Fourier series co-efficients $C_{-k} = -C_k$.

Frequency Shifting

If $x(t) \underline{\text{FS}} C_k$ then

$$e^{jm\omega_0 t} x(t) \underline{\text{FS}} C_{k-m}$$

Time Scaling

If $x(t)$ is periodic with period T_0, and fundamental frequency $\omega_0 = 2\pi/T_0$ then $x(at)$, $a > 0$, is periodic with period $\frac{T_0}{a}$ and fundamental frequency $a\omega_0$ then Fourier series representation of

$$x(at) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The Fourier coefficients C_k have not changed, but the Fourier series representation has changed because of the change in fundamental frequency.

Multiplication

If $x(t) \underline{\text{FS}} a_k$, $y(t) \underline{\text{FS}} b_k$, and $x(t), y(t)$ are periodic with period T_0, then $x(t)$. $y(t)$ is also periodic with period T_0 then

$$x(t) \cdot y(t) \underline{\text{FS}} C_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

C_K can be interpreted as the discrete time convolution of the sequence representing Fourier series co-efficients of $x(t)$ and $y(t)$.

Conjugation and Conjugate Symmetry

If $x(t) \underline{\text{FS}} C_k$, then $x^*(t) \underline{\text{FS}} C_{-k^*}$

If $x(t)$ is real signal then $x(t) = x^*(t)$

so $C_k = C_{-k^*}$ (or) $C_k^* = C_{-k}$

so we can write $|C_k| = |C_{-k}|, \angle C_k = -\angle C_k$

$\text{Re}\{C_k\} = \text{Re}\{C_{-k}\}$, $\text{Im}\{C_k\} = -\text{Im}\{C_{-k}\}$

When $x(t)$ is real $C_k^* = C_{-k,}$

When $x(t)$ is even, then $C_{-k} = C_k$

So when $x(t)$ is real and even $C_k = C_k^*$

And $C_k = C_{-k}$, i.e., the Fourier series coefficients are also real and even

When $x(t)$ is real $C_k^* = C_{-k}$,

And when $x(t)$ is odd $C_{-k} = -C_k$

By combining both, $C_{k^*} = -C_k$

So when $x(t)$ is real and odd, the Fourier series coefficients are also odd and purely imaginary ($\because\ C_k^* = -C_k$)

Periodic Convolution

If $x(t)$ and $y(t)$ are periodic with period T_0, then $x(t)$ $\underset{\sim}{\text{FS}}\, a_k, y(t) \underset{\sim}{\text{FS}}\, b_k$

$$z(t) = \int_{T_0} x(\tau) y(t-\tau) d\tau \underset{\sim}{\text{FS}}\, C_k = T_0 a_k b_k$$

Differentiation

If $x(t) \underset{\sim}{\text{FS}}\, C_k$ is periodic with angular frequency ω_0, then $\frac{dx(t)}{dt} \underset{\sim}{\text{FS}}\, jk\omega_0 c_k$

Integration

If $x(t)$ is finite valued, periodic (ω_0), and if $C_0 = 0$ then $\int_{-\infty}^{t} x(t)\, dt \underset{\sim}{\text{FS}} \frac{1}{jk\omega_0} C_k$

Even Odd Decomposition of Real Signals

If $x(t)$ can be written as $x(t) = x_e(t) + x_0(t)$ as sum of even and odd components of $x(t)$.

$$x(-t) = x_e(t) - x_0(t)$$

$$x_e(t) = \frac{x(t) + x(-t)}{2}, x_0(t) = \frac{x(t) - x(-t)}{2}$$

when $x(t)$ is real, $x(t) \underset{\sim}{\text{FS}}\, C_k$, $x(-t) \underset{\sim}{\text{FS}}\, C_{-k}$

Similarly $C_k = C_{-k}^{*}$ (or) $C_{-k} = C_k^{*}$

So for $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} \underset{\sim}{\text{FS}} \frac{1}{2}\{C_k + C_k^{*}\}$

$$= \text{Re}\{C_k\}$$

For $x_0(t) = \frac{1}{2}\{x(t) - x(-t)\} \underset{\sim}{\text{FS}} \frac{1}{2}\{C_k - C_k^{*}\}$

$$= j\text{Im}\{C_k\}$$

$$x_e(t) \underset{\sim}{\text{FS}}\, \text{Re}\{C_k\}$$

$$x_0(t) \underset{\sim}{\text{FS}}\, j\,\text{Im}\{C_k\}$$

Parseval's Theorem

The average power of periodic signal $x(t)$ over any period is

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |C_K|^2$$

Parseval's theorem states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example 1: Which of the following signal will not have Fourier series representation.

(A) $\sin 4t + \cos 9t$ (B) $1 + \cos 2\pi t$

(C) e^{-j5t} (D) $\cos 4\pi t + \sin 3t$

Solution: Only periodic signals will have Fourier series representation.

For $\sin 4t + \cos 9t \rightarrow$ period is LCM of $\left(\frac{2\pi}{4}, \frac{2\pi}{9}\right) = 2\pi$.

$\cos 2\pi t$ is periodic with period $T_0 = \frac{2\pi}{2\pi} = 1$, and DC component 1 is present in the signal $1 + \cos 2\pi t$ is also periodic.

For e^{-j5t} periodic with period $\frac{2\pi}{5}$. But $\cos 4\pi t + \sin 3t$ is not periodic, $T_1 = \frac{2\pi}{4\pi} = \frac{1}{2}$, $T_2 = \frac{2\pi}{3}, \frac{T_1}{T_2}$ is not rational, so option (D) will not have Fourier series expansion.

Continuous Time (CT) Fourier Transform

The function $X(\omega)$, and $x(t)$ are said to be Fourier transform pairs defined as below

$$x(t) \leftrightarrow X(\omega)$$

The Fourier transform of $x(t)$ is

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

The inverse Fourier transform of $X(\omega)$ is

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Spectra

Fourier transform of $x(t)$ can be expressed as $X(\omega) = |X(\omega)| e^{j\phi(\omega)}$. $X(\omega)$, Fourier transform of non periodic signal $x(t)$ is the frequency domain specification of $x(t)$, and is referred as Fourier spectrum of $x(t)$,

$|X(\omega)|$ is called magnitude spectrum of $x(t)$

$\phi(\omega)$ is called the phase spectrum of $x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

For real signal $x(t)$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

So $X(-\omega) = X^{*}(\omega)$, $|X(\omega)| = |X(-\omega)|$, $\varphi(-\omega) = -\varphi(\omega)$.

Here also, the amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\varphi(\omega)$ is an odd function of ω.

Convergence of Fourier Transform

Dirichlet conditions

1. $x(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$.
2. $x(t)$ has finite number of maxima and minima with in any finite interval.
3. $x(t)$ has a finite number of discontinuities with in any finite interval, and each of these discontinuities is finite.

Relation Between Fourier Transform and Laplace Transform

Fourier transform of $x(t)$ is $X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. The bilateral Laplace transform of $x(t)$ is $X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$. We can observe from these equations, Fourier transform is special case of Laplace transform in which $s=j\omega$. That is, $|X(s)|_{s=j\omega} = F\{x(t)\}$, if we substitute $s=\sigma+j\omega$ in Laplace transform equation:

$$X(\sigma+j\omega)=\int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$=\int_{-\infty}^{\infty}(x(t)e^{-\sigma t})e^{-j\omega t}dt = F\{x(t)e^{-\sigma t}\}.$$

Bilateral Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)e^{-\sigma t}$. If $x(t)$ is absolutely integrable, the Fourier transform of $x(t)$ can be obtained from the laplace transform of $x(t)$ with $S=j\omega$.

Examples 2: Find the Fourier transform of the function $f(t) = e^{-at}u(t)$.

Solution: The Fourier transform does not converge for $a \le 0$, since $\int_{-\infty}^{\infty} e^{-at}u(t)=\infty\, a\le 0$. $f(t)$ is not absolutely integrable for $a \le 0$.

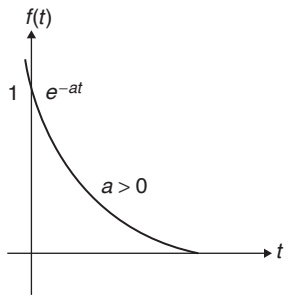
For $a > 0$,

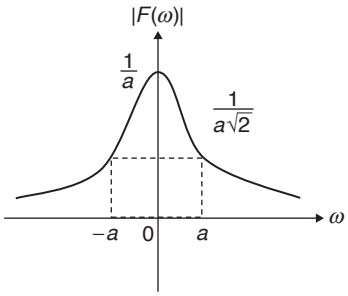
$$F(\omega)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

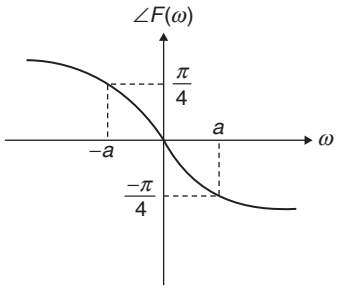
$$=\int_{-\infty}^{\infty} e^{-(a+j\omega)t}dt$$

$$=\frac{-1}{a+j\omega}e^{-a+j\omega)t}\Big|_0^{\infty}$$

$$=\frac{1}{a+j\omega}=\frac{-1}{\sqrt{a^2+\omega^2}}\angle-\tan^{-1}(\omega/a)$$







Examples 3: Find the Fourier transform of the rectangular pulse?

$$x(t)=\begin{cases}1 & |t|<T_0\\ 0 & |t|>T_0\end{cases}$$

Solution: $x(t)$ is absolutely integrable for $T_0<\infty$

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$=\int_{-T_0}^{T_0} 1\cdot e^{-j\omega t}dt$$

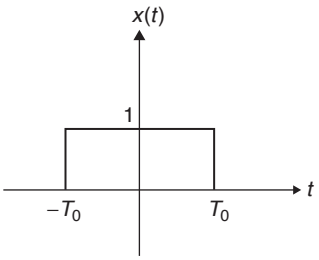
$$=\frac{-1}{j\omega}e^{-j\omega t}\Big|_{-T_o}^{T_o}$$

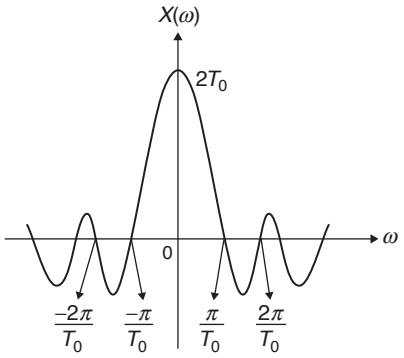
$$=\frac{-1}{j\omega}(e^{-j\omega T_0}-e^{+j\omega T_0})=2\frac{\sin\omega T_0}{\omega}$$

$\sin c$ function can be defined as $\sin c(x)=\frac{\sin\pi x}{\pi x}$

$$X(\omega)=2T_0\frac{\sin\omega T_0}{\omega T_0}=2T_0\sin c(\omega T_0/\pi)$$

when $\omega\to 0, X(\omega)=\lim_{\omega\to 0}\frac{2T_0\sin\omega T_0}{\omega T_0}=2T_0$ (by using L Hospital's rule)





The magnitude spectrum is $|X(\omega)| = 2\left|\frac{\sin(\omega T_0)}{\omega}\right|$.

Phase spectrum is $\angle X(\omega) = \begin{cases} 0, \sin(\omega T_0)/\omega > 0 \\ \pi, \sin \omega T_0/\omega < 0. \end{cases}$

From the above example, we can see that If T_0 increases, then nonzero extent of $x(t)$ increases, and $X(\omega)$ becomes more concentrated about the frequency origin. Similarly when T_0 decreases $x(t)$ width reduces, $X(\omega)$ becomes less concentrated about the frequency origin. The duration of $x(t)$ is inversely related to the width or bandwidth of $X(\omega)$.

Examples 4: Find the Fourier transform of the $f(t) = \delta(t)$.

Solution: $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{j\omega.0} = 1$$

$$\delta(t) \underline{\text{FT}} 1.$$

Time domain signal defined at origin, but frequency components are present from $\omega = -\infty$ to ∞.

Examples 5: What is the inverse Fourier transform of $F(\omega) = 2\pi\delta(\omega)$

Solution: $f(t) = s\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} 2\pi \cdot e^{j0.t} = 1$$

So inverse Fourier transform of $2\pi\delta(\omega)$ is 1.

$$1 \leftrightarrow 2\pi\delta(\omega)$$

This implies that the frequency content of a DC signal is concentrated entirely at $\omega = 0$.

Properties of the Continuous-time Fourier Transform

Linearity:

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

Time shifting: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

By delaying a signal by t_0 seconds $x(t - t_0)$, the magnitude spectrum $|x(\omega)|$ will not be changed, but the phase spectrum $\angle X(\omega)$ will be changed by $-\omega t_0$. So time delay in a signal causes a linear phase shift in its spectrum.

Frequency shifting: $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$

The multiplication of a signal by a factor of $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. We can observe the duality between the time shifting and frequency shifting properties.

The multiplication of a signal $x(t)$ by a sinusoid of frequency ω_0 shifts the spectrum $X(\omega)$ by $\pm\omega_0$

$$x(t)\cos\omega_0 t = \frac{1}{2}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}] \leftrightarrow$$

$$\frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)].$$

Time scaling: $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$.

The scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.

The scaling property implies that if $x(t)$ is wider, its spectrum is narrower, and vice versa, doubling the signal duration halves its bandwidth, and vice versa. This suggests that the band width of a signal is inversely proportional to the signal duration or width (in seconds).

Time reversall: $x(-t) \leftrightarrow X(-\omega)$.

For time scaling property if we consider $a = -1$, we can obtain the inversion property of time and frequency.

Duality $X(t) \leftrightarrow 2\pi x(-\omega)$

Differentiation in the time domain:

$$\frac{d}{dt} x(t) \leftrightarrow j\omega X(\omega),$$

$$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega).$$

Differentiation in the frequency domain:

$$(-jt)x(t) \leftrightarrow \frac{d}{d\omega} X(\omega)$$

Differentiation in time domain corresponds to multiplication by $j\omega$ in the frequency domain.

Integration in time domain:

$$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega).$$

Integration in the time domain corresponds to division by $j\omega$ in the frequency domain and the impulse term reflects the dc or average value that can result from integration.

Convolution:

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega) X_2(\omega).$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

Multiplication:

$$x_1(t)\cdot x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega).$$

Multiplication in the time domain corresponds to convolution in frequency domain. We can observe the duality property between time and frequency domains from above two properties.

Parseval's relations:

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)d\omega,$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega.$$

Parseval's identity says that energy content E of $x(t)$, can be computed by integrating $|X(\omega)|^2$ over all frequencies, for this reason $|X(\omega)|^2$ is often called as energy density spectrum of $x(t)$.

Real signal: If $x(t)$ is real, and let $x(t) = x_e(t) + x_0(t)$, where $x_e(t)$, $x_0(t)$ are even and odd components of $x(t)$, Fourier transform of $x(t)$ is $X(\omega) = A(\omega) + jB(\omega)$ then, $X(-\omega) = X^*(\omega)$. So we can conclude that

$$x_e(t) \xleftrightarrow{\text{FT}} \text{Re}\{X(\omega)\} = A(\omega),$$

$$x_o(t) \xleftrightarrow{\text{FT}} j\text{Im}\{X(\omega)\} = jB(\omega).$$

This shows that the Fourier transform of an even signal is a real function of ω, and Fourier transform of an odd signal is a pure imaginary function of ω. The $X(\omega)$ is complex conjugate symmetric for real signal $x(t)$. When $X^*(\omega) = X(-\omega)$.

By taking the real and imaginary parts of this expression gives Re $\{X(\omega)\}$ = Re$\{X(-\omega)\}$ and Im $\{X(\omega)\}$ = $-$Im$\{X(-\omega)\}$. In other words, if $x(t)$ is real valued, then the real part of the transform is an even function of frequency, while the imaginary part is an odd function of frequency.

This also implies that the magnitude spectrum is an even function of frequency, while the phase spectrum is an odd function.

Table 2 *Common Fourier Transform pairs*

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert}, a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2\frac{\sin\omega a}{\omega}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$sgn(t)$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-KT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\frac{2\pi}{T}$

Fourier Transform Representation of Periodic Signals

If $x(t)$ is a periodic signal with fundamental frequency ω_0, then the Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}dt,\ \omega_0 = \frac{2\pi}{T_0}.$$

Where C_k—complex Fourier coefficient

$$C_k = \frac{1}{T_o}\int_{T_o} x(t)e^{-jk\omega_o t}dt$$

we have $1 \underset{\text{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$. By using frequency shifting property $e^{jk\omega_0 t} \underset{\text{FT}}{\longleftrightarrow} 2\pi\,\delta(\omega - k\omega_0)$.

We can apply linearity property of the Fourier transform to obtain:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \underset{\text{FT}}{\longleftrightarrow} X(\omega) = 2\pi, \sum_{k=-\infty}^{\infty} c_k\delta(\omega - k\omega_0).$$

Thus the Fourier transform of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0. The kth impulse has strength of $2\pi C_k$, C_k is kth Fourier series co-efficient.

Fourier Transform in Terms of Frequency (f)

We can change the variable ω to f (in Hertz), by writing $\omega = 2\pi f$ and $d\omega = 2\pi df$. So $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$, $x(t) = \int_{-\infty}^{\infty} X(2f)e^{j2\pi ft}df$, we can get $X(f)$ by changing ω to $2\pi f$.

Similarly $\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)$,

$$x(t) = e^{-at}u(t) \to X(\omega) = \frac{1}{a + j\omega} \to X(f) = \frac{1}{a + j2\pi f},$$

$$x(t) = e^{-|t|} \to X(\omega) = \frac{2a}{a^2 + \omega^2} \to X(f) = \frac{2a}{a^2 + 4\pi^2 f^2},$$

$$x(t) = u(t) \to X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \to X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f},$$

$$x(t) = e^{-\pi t^2} \to X(\omega) = e^{-\omega^2/4\pi} \to x(f) = e^{-\pi f^2},$$

$$\cos(2\pi at) \to \pi[\delta(\omega + 2\pi a) + \delta(\omega - 2\pi a)]$$

$$\to \frac{1}{2}[d(f + a) + \delta(f - a)]$$

$$\sin(2\pi at) \to j\pi[\delta(\omega + 2\pi a) - \delta(\omega - 2\pi a)]$$

$$\to \frac{j}{2}[\delta(f + a) + \delta(f - a)]$$

Examples 6: Find the Fourier transform of $x(t) = e^{3t}\,u(-t)$

Solution: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

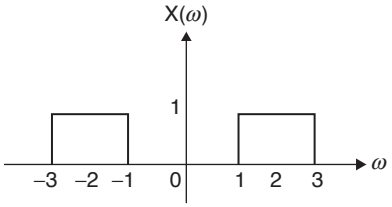
$$= \int_{-\infty}^{0} e^{3t}\,e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{(3-j\omega)}\,dt \text{ replace } t \text{ with } -t$$

$$= -\int_{0}^{\infty} e^{-(3-j\omega)t}dt$$

$$= \frac{-e^{-(3-j\omega)t}}{(3-j\omega)}\Big|_0^{\infty} = \frac{1}{3-j\omega}$$

Examples 7: Find the inverse Fourier transform for the spectra depicted in figure.



Solution: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

So $x(t) = \frac{1}{2\pi}\left[\int_{-3}^{-1} 1e^{j\omega t}d\omega + \int_{1}^{3} 1e^{j\omega t}d\omega\right]$

$$= \frac{1}{2\pi}\left\{\frac{e^{j\omega t}}{jt}\Big|_{-3}^{-1} + \frac{e^{j\omega t}}{jt}\Big|_{1}^{3}\right\}$$

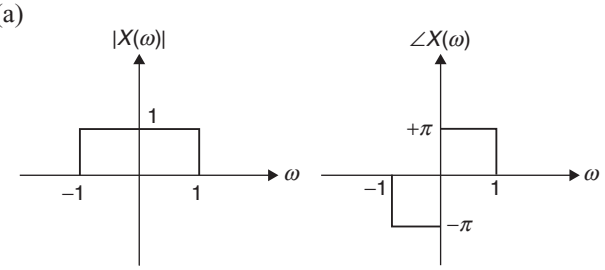
$$= \frac{1}{2\pi jt}(e^{-jt} - e^{-j3t} + e^{j3t} - e^{\cdot jt})$$

$$= \frac{1}{\pi t}\left\{\left(\frac{e^{j3t} - e^{-j3t}}{2j}\right) - \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\right\}$$

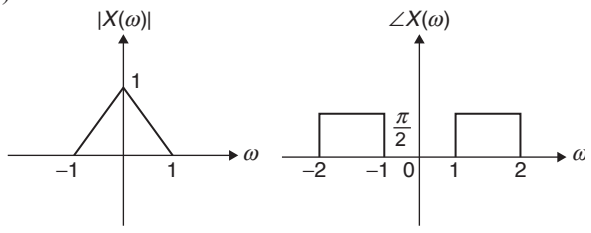
$$= \frac{1}{\pi t}(\sin 3t - \sin t)$$

Examples 8: For the frequency domain signal depicted in figure, find the nature of time domain signal?

(a)



(b)



Solution: For (a) $|x(\omega)|$ is an even function and $\angle X(\omega)$ is an odd function.

So we can say that $x(t)$ is real and odd.

For (b) $|x(\omega)|$ is an even function as well as $\angle X(\omega)$ is also an even function so $x(t)$ is real and even.

Examples 9: Find $x(t)$ for the spectra;

$$X(\omega)=\begin{cases} j\omega & |\omega|<1 \\ 0 & |\omega|>1. \end{cases}$$

Solution: $X(\omega)=\begin{cases} j\omega & |\omega|<1 \\ 0 & |\omega|>1. \end{cases}$

We can see $X(\omega)=j\omega\, z(\omega)$

Where $Z(\omega)=\begin{cases} 1 & |\omega|<1 \\ 0 & |\omega|>1. \end{cases}$

Then we can have $z(t)=\dfrac{1}{\pi t}\sin t$

$z(t)\underline{\text{FT}}\, Z(\omega)$, then $\dfrac{dz(t)}{dt}\leftrightarrow j\omega Z(\omega)=X(\omega).$

The inverse Fourier transform of $X(\omega)$ is

$$x(t)=\frac{d}{dt}z(t)=\frac{d}{dt}\left(\frac{1}{\pi t}\sin t\right),$$

$$x(t)=\frac{1}{\pi t}\cos t-\frac{1}{\pi t^2}\sin t.$$

Examples 10: Find the time domain signal corresponding to the Fourier transform:

$$X(\omega)=\frac{5j\omega+12}{-\omega^2+5j\omega+6}$$

Solution: $X(\omega)=\dfrac{5(j\omega)+12}{(j\omega)^2+5(j\omega)+6}$

$$=\frac{5j\omega+12}{(j\omega+2)(j\omega+3)}=\frac{A}{j\omega+2}+\frac{B}{j\omega+3}$$

$$=\frac{5\times-2+12}{-2+3}\cdot\frac{1}{j\omega+2}+\frac{5\times-3+12}{-3+2}\frac{1}{j\omega+3}$$

$$=\frac{2}{j\omega+2}+\frac{3}{j\omega+3}.$$

By taking inverse Fourier transform.

$$x(t)=2e^{-2t}u(t)+3e^{-3t}u(t)$$

Examples 11: What is inverse Fourier transform of the step function in frequency? ($X(\omega)=u(\omega)$)

Solution: $X(\omega)=u(\omega)$ from the duality property $x(t)\ \underline{\text{FT}}\ X(\omega)$, $X(t)\underline{\text{FT}}\, 2\pi x(-\omega)$.

We have time reversal property $g(t)\leftrightarrow G(\omega)$ then $g(-t)\leftrightarrow G(-\omega)$. So we can write $X(-t)\leftrightarrow 2\pi x(\omega)$

$$\frac{1}{2\pi}X(-t)\leftrightarrow x(\omega)$$

If we consider $x(t)=u(t)$, then $X(\omega)=\pi\delta(\omega)+\dfrac{1}{j\omega}$, then inverse Fourier transform of $u(\omega)$ is

$$\frac{1}{2\pi}X(-t)=\frac{1}{2\pi}\left\{\pi\delta(-t)-\frac{1}{jt}\right\}=\frac{\delta(t)}{2}-\frac{1}{2\pi jt}=\frac{\delta(t)}{2}+\frac{j}{2\pi t}$$

As $\delta(-t)=\delta(t)$

Examples 12: Find $x(t)$ if $X(\omega)=j\dfrac{d}{d\omega}\left\{\dfrac{e^{j3\omega}}{1+j\omega/2}\right\}$.

Solution: Consider $p(t)=e^{-t}u(t)\leftrightarrow P(\omega)=\dfrac{1}{1+j\omega}$,

then $X(\omega)=j\dfrac{d}{d\omega}\{e^{j3\omega}P(\omega/2)\}$.

If $Q(\omega)=P(\omega/2)$ then $q(t)=2p(2t)=2e^{-2t}\,u(2t)=2e^{-2t}u(t)$.

{time scaling $|a|x(at)\leftrightarrow X(\omega/a)$}

So Now $X(\omega)=j\dfrac{d}{d\omega}\{e^{j3\omega}Q(\omega)\}$.

If $R(\omega)=e^{j3}\,Q(\omega)$, then $r(t)=q(t+3)=2e^{-2(t+3)}$

$u(t+3)$ {time shifting $x(t-t_0)\leftrightarrow e^{-j\omega t_0}X(\omega)$}

Now $X(\omega)=j\dfrac{d}{d\omega}\{R(\omega)\}$ then $x(t)=tr(t)$

$=2t\,e^{-2(t+3)}u(t+3)$. [differentiation in frequency domain $tx(t)\leftrightarrow j\dfrac{d}{d\omega}\{x(\omega)\}$].

Examples 13: Find the Fourier transform of the impulse train $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$

Solution: The fundamental period of $x(t)$ is T_0, so $\omega_0=\dfrac{2\pi}{T_0}$ is fundamental frequency.

The Fourier coefficients are:

$$C_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T_0}$$

The Fourier transform of a periodic signal is $X(\omega)$

$$=2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$$

$$=2\pi\sum_{k=-\infty}^{\infty}\frac{1}{T_0}\delta(\omega-k\omega_0)$$

$$=\frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$$

Hence Fourier transform of an impulse train is also an impulse train.

Frequency Response of Continuous-time LTI Systems

Consider a continuous time LTI system with impulse response $h(t)$, and the input is $x(t) = e^{j\omega t}$, then the convolution integral gives the output as

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{j\omega t} \cdot e^{-j\omega\tau} d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$= H(\omega)\, e^{j\omega t}$$

Where we can define $H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega t} d\tau$.

The output of the system is thus a complex sinusoid of the same frequency as the input, multiplied by the complex number $H(\omega)$.

$H(\omega)$ is a function of only the frequency ω, and not the time t, and is termed the frequency response of the continuous-time system.

Output $y(t)$ of a continuous-time LTI system. Equals the convolution of the input $x(t)$, with the Impulse response $h(t)$ $y(t) = x(t) * h(t)$, taking Fourier transform $Y(\omega) = X(\omega)$ $H(\omega)$, $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$.

$H(\omega)$ is called frequency Response of the system. $H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)}$, $|H(\omega)|$ is called magnitude response of the system $\theta_H(\omega)$ is phase response of the system.

The behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$

Input $X(\omega) = |X(\omega)|\, e^{j\theta_H(\omega)}$

$$H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)}$$

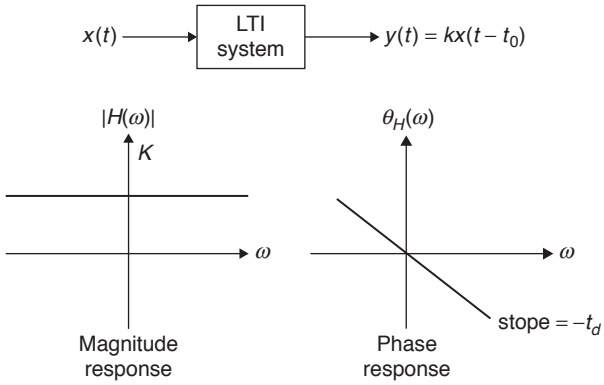
Output $Y(\omega) = |Y(\omega)| e^{j\theta_y(\omega)}$

Then $Y(\omega) = X(\omega) \cdot H(\omega)$. So $|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$

$$\theta_y(\omega) = \theta_x(\omega) + \theta_H(\omega).$$

For an LTI system, with real valued impulse response $h(t)$ and frequency response $H(\omega)$, [i.e., Fourier transform of $h(t)$] the output $y(t)$ for an input $x(t) = A\cos(\omega t - \phi)$ is $y(t)$ $= |H(\omega)| A.\ \cos(\omega t - \phi + \angle\{H(\omega)\})$ the system modifies the amplitude of input sinusoid by $H(\omega)$ and the phase by $\angle\{H(\omega)\}$.

Distortion Less Transmission



For distortion less transmission through LTI system we require, the exact input signal to be reproduced at the output, even its Amplitude is different, or it may be time delayed.

If $x(t)$ is input signal then output, $y(t) = k\, x(t - t_d)$.

By taking Fourier transform $Y(\omega) = k\, e^{-j\omega t_d} X(\omega)$. So we can write for distortion less transmission system must have the frequency response.

$$H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)} = ke^{-j\omega t_d},$$

$$|H(\omega)| = k,\ \theta_H(\omega) = -j\omega t_d.$$

That is amplitude of $H(\omega)$ must be constant over the entire frequency range and the phase of $H(\omega)$ must be linear with frequency.

Amplitude Distortion and Phase Distortion

When the amplitude spectrum $|H(\omega)|$ of the system is not constant within the frequency band of interest, the frequency components of the input signal are transmitted with different amount of gain or attenuation. This effect is called amplitude distortion. When the phase spectrum $\theta_H(\omega)$ of system is not linear with frequency, the output signal has a different wave form than the input signal because of different delays in passing through the system for different frequency components of the input signals. This form of distortion is called as phase distortion.

Equalization

To compensate for linear distortion we may use a network known as equalizer, connected is cascade with the system in question. The equalizer is designed in such a way that, inside the frequency band of interest the over all magnitude and phase responses of this cascade connection approximate the conditions for distortions less transmission to with in prescribed limits.

Consider for example a communication channel with frequency response ($H_C(\omega)$, let an equalizer of frequency response $H_{eq}(\omega)$ be connected in cascade with the channel.

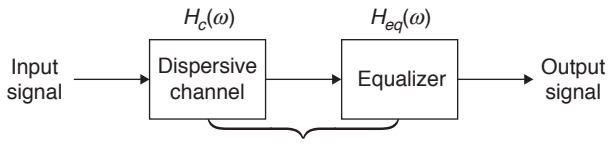


Figure 5 Distortion less transmission system

For overall transmission through the cascade connection to be distortion less, we require $H_c(\omega)\varphi \cdot H_{eq}(\omega) = e^{-j\omega t_d}d$. Where t_d is constant time delay, k is unity here. Therefore the frequency response of the equalizer is inversely related to that of the channel, $H_{eq}(\omega) = \dfrac{e^{-j\omega t_d}}{H_c(\omega)}$.

Example 14: Let $x(t) = \dfrac{1}{\pi t}\sin(2\pi t)$ be the input to a system with impulse response $h(t) = \dfrac{1}{\pi t}\sin(3\pi t)$, then the output of the system $y(t)$ is

Solution: $y(t) = x(t) * h(t)$, We can convert to frequency domain by taking Fourier transform $Y(\omega) = X(\omega)\,H(\omega)$. We have the Fourier transform pair

$$\frac{\sin at}{\pi t} \underline{\text{FT}}\; P_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases},$$

$$x(t) \underline{\text{FT}}\; X(\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases},$$

$$h(t) \underline{\text{FT}}\; H(\omega) = \begin{cases} 1 & |\omega| < 3\pi \\ 0 & |\omega| > 3\pi \end{cases},$$

then

$$Y(\omega) = X(\omega)\cdot H(\omega) \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases},$$

So, we can say $y(t) = \dfrac{1}{\pi t}\sin(2\pi t)$.

Example 15: The output of an LTI system in response to an input $x(t) = e^{-t}u(t)$ is $y(t) = e^{-2t}u(t)$. Find the frequency response and the impulse response of the system?

Solution: $x(t) = e^{-t}u(t) \underline{\text{FT}}\; X(\omega) = \dfrac{1}{j\omega+1}$,

$$y(t) = e^{-2t}u(t) \underline{\text{FT}}\; Y(\omega) = \frac{1}{j\omega+2},$$

$$y(t) = x(t) * h(t) \underline{\text{FT}}\; Y(\omega) = X(\omega)\cdot H(\omega),$$

so $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$.

The frequency response $H(\omega) = \dfrac{j\omega+1}{j\omega+2}$ to find impulse response we need to take inverse Fourier transform for $H(\omega)$

$$H(\omega) = \frac{j\omega+2-1}{j\omega+2} = 1 - \frac{1}{j\omega+2},$$

so $h(t) = \delta(t) - e^{-2t}\,u(t)$,

Example 16: Find the zero state response of a stable LTI system with frequency response $H(\omega) = \dfrac{1}{j\omega+3}$ and the input is $x(t) = e^{-2t}u(t)$.

Solution: Here $X(\omega) = \dfrac{1}{j\omega+2}, H(\omega) = \dfrac{1}{j\omega+3}$,

then $Y(\omega) = H(\omega)\,X(\omega)$

$$= \frac{1}{(j\omega+2)(j\omega+3)} = \frac{1}{j\omega+2} - \frac{1}{j\omega+3}$$

By taking inverse Fourier transform

$$y(t) = (e^{-2t} - e^{-3t})\,u(t).$$

LTI Systems Characterized by Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t),$$

By taking Fourier transforms on both sides:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

They $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$

Which is similar as in laplace transform if $s = j\omega$.

Phase and Group Delays

Whenever a signal is transmitted through a dispersive (i,e., frequency selective) system, some delay is introduced into the output signal in relation to the input signal. The delay is determined by the phase response of the system $\phi = \angle\{H(j\omega)\} = \arg\{H(j\omega)\}$.

Where $H(j\omega)$ is the frequency response of the channel. Suppose that a sinusoidal signal is transmitted through the channel at a frequency ω_c. The signal received at the channel output lags the transmitted signals by $\phi(\omega_c)$ radians. The time delay corresponding to this phase lag is simply equal to $\tau_p = \dfrac{-\phi(\omega_c)}{\omega_c}$.

Where – sign accounts for the lag. However this phase delay is not necessarily the true signals delay, this follows form the fact that a sinusoidal signal has infinite duration, with each cycle exactly like the preceding one. Such a signal does not convey information. When a modulated signals is transmitted through a communication channel, there are two different delays to be considered.

(1) The carrier or phase delay $\tau_p = \dfrac{-\phi(\omega_c)}{\omega_c}$.

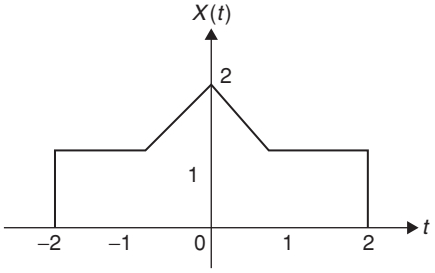
(2) The envelope or group delay

$$\tau_g = \left.\frac{-d\phi(\omega)}{d\omega}\right|_{\omega=\omega_c}.$$

The group delay is the true signal delay.

Solved Examples

Example 1: For the signal $x(t)$ Shown in figure, find $X(0)$ and $\int_{-\infty}^{\infty} X(\omega)d\omega$



Solution: We know the Fourier transform of $x(t)$ is $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, when $\omega = 0, X(0) = \int_{-\infty}^{\infty} x(t)e^{0}dt$ $= \int_{-\infty}^{\infty} x(t)\,dt$ $\int_{-\infty}^{\infty} x(t)\,dt$ is the area under the curve $x(t)$

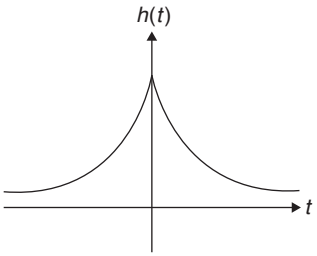
$$X(0) = \int_{-\infty}^{\infty} x(t)dt = 4\times1 + \frac{1}{2}\times1\times1\times2 = 5.$$

Inverse Fourier transform is $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, When $t = 0, \int_{-\infty}^{\infty} X(\omega)d\omega = 2\pi x(0) = 2\pi\times5 = 10\pi$.

Example 2: Fourier transform of a signal $x(t)$ is given by $X(\omega) = \dfrac{\omega^2+28}{\omega^2+16}$, then $x(t)$, is:

Solution: $x(\omega) = \dfrac{\omega^2+28}{\omega^2+16} = \dfrac{\omega^2+16+12}{\omega^2+16} = 1 + \dfrac{12}{\omega^2+16}$

consider $h(t) = \{e^{-a|t|} = \begin{cases} e^{-at}; t>0 \\ e^{at}; t<0 \end{cases}$



$h(t) = e^{-at}\,u(t) + e^{at}\,u(-t)'$ by taking Fourier transform on both sides

$$H(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2},\quad e^{-a|t|} \leftrightarrow \frac{2a}{\omega^2+a^2},$$

here $X(\omega) = 1 + \dfrac{12}{\omega^2+16} = 1 + \dfrac{3}{2}\dfrac{8}{\omega^2+16}$,

By taking inverse Fourier transform

$$x(t) = \delta(t) + \frac{3}{2}e^{-4|t|}.$$

Example 3: Determine the inverse Fourier Transform of $X(\omega) = 4\pi\,\delta(\omega) - 2\pi\,\delta(\omega-2\pi) + 2\pi\,\delta(\omega+2\pi)$

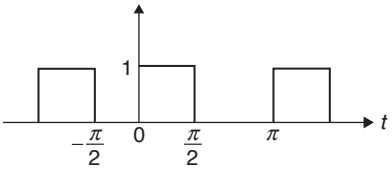
(A) $2[1 - \sin 2\pi t]$ (B) $2j[1 - \sin 2\pi t]$

(C) $2[1 - j\sin 2\pi t]$ (D) $2j[1 - j\sin 2\pi t]$

Solution: (C)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(4\pi\delta(\omega) - 2\pi\delta(\omega-2\pi) + 2\pi\delta(\omega+2\pi))e^{j\omega t}d\omega$$
$$= 2 - e^{j2\pi t} + e^{-j2\pi t}$$
$$= 2 - 2j\left(\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right)$$
$$= 2 - 2j\sin 2\pi t = 2(1 - j\sin 2\pi t).$$

Example 4: Determine the Fourier series co-efficients for periodic signal $x(t)$ as:



(A) $\dfrac{j}{k\pi}[1 - e^{-jk\pi}]$ (B) $\dfrac{1}{jk\pi}[1 + e^{-jk\pi}]$.

(C) $\dfrac{1}{jk\pi}[1 - e^{-jk\pi}]$ (D) $\dfrac{j}{jk\pi}[1 + e^{-jk\pi}]$.

Solution: (C)

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt, T = \pi, w_0 = 2,$$

$$c_k = \frac{2}{T}\frac{(e^{-jk\omega_0 t})_0^{\frac{\pi}{2}}}{-jk\omega_0} = \frac{1}{jk\pi}\left(1 - e^{-jk\omega_0\frac{\pi}{2}}\right),$$

$$= \frac{1}{jk\pi}(1 - e^{-jk\pi}).$$

Example 5: Find the Fourier transform of the signal $x(t) = te^{-2|t-3|}$.

(A) $e^{-3j\omega}\left(\frac{12}{4+\omega^2} - \frac{j8\omega}{(4+\omega^2)^2}\right)$.

(B) $e^{3j\omega}\left(\frac{12}{4+\omega^2} + \frac{j8\omega}{(4+\omega^2)^2}\right)$.

(C) $e^{-3j\omega}\left(\frac{12}{4+\omega^2} + \frac{j8\omega}{(4+\omega^2)^2}\right)$.

(D) $e^{3j\omega}\left(\frac{12}{4+\omega^2} - \frac{j8\omega}{(4+\omega^2)^2}\right)$.

Solution: (A)

$$e^{-2|t|} \to \frac{4}{4+\omega^2},\ x_1(t-3) \to e^{-3j\omega}x_1(\omega),\ tx_2(t) \to \frac{jd}{d\omega}x_2(\omega),$$

$$X(\omega) = j\frac{d}{d\omega}\left|e^{-3j\omega}\frac{4}{4+\omega^2}\right|$$

$$= j\frac{d}{d\omega}\left|e^{-3j\omega}\frac{d}{d\omega}\left(\frac{4}{4+\omega^2}\right) + \left(\frac{4}{4+\omega^2}\right)\frac{d}{d\omega}(e^{-3j\omega})\right)$$

$$= j\left|\frac{-e^{-3j\omega}8\omega}{(4+\omega^2)^2} - 3j\left(\frac{4}{4+\omega^2}\right)e^{-3j\omega}\right)$$

$$= e^{-3j\omega}\left|\frac{12}{4+\omega^2} - \frac{j8\omega}{(4+\omega^2)^2}\right).$$

Example 6: Determine the Inverse Fourier transform of $X(\omega) = \frac{2j\omega+5}{(j\omega+2)^2}$:

(A) $(2e^{-2t} - te^{-2t})\,u(t)$.
(B) $(2e^{-2t} + te^{-2t})\,u(t)$.
(C) $(2e^{-2t} - te^{2t})\,u(t)$.
(D) $(2e^{-2t} - te^{-2t})\,u(t)$.

Solution: (B)

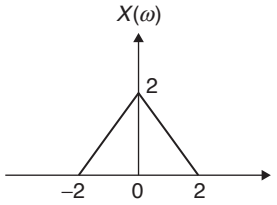
$$X(\omega) = \frac{2j\omega+5}{(j\omega+2)^2} = \frac{2j\omega+4}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^2}$$

$$= \frac{2(j\omega+2)}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^2}$$

$$= \frac{2}{j\omega+2} + \frac{1}{(j\omega+2)^2}$$

Hence, $x(t) = 2e^{-2t}\,u(t) + te^{-2t}\,u(t)$
$x(t) = (2e^{-2t} + te^{-2t})\,u(t) = (2+t)\,e^{-2t}\,u(t)$.

Example 7: Consider the signal $X(\omega)$:



Find $\int_{-\infty}^{\infty}[x(t)]^2\,dt$?

(A) 0
(B) −1
(C) $2/\pi$
(D) $8/3\pi$

Solution: (D)
According to parseavall

$$\int_{-\infty}^{\infty} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X^2(\omega)d\omega = \frac{1}{2\pi}\cdot\frac{16}{3} = \frac{8}{3\pi}$$

Example 8: A real valued continuous-time signal $x(t)$ has a fundamental period $T = 16$, the non zero Fourier series coefficients for $x(t)$ are $x(2) = x(-2) = 1$, $x(4) = x(-4) = j$, the signal $x(t)$ would be?

(A) $\cos\frac{\pi}{4}t + \cos\frac{\pi}{2}t$

(B) $\cos\frac{\pi}{4}t + \cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)t$

(C) $2\left[\cos\frac{\pi}{4}t + \cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]$

(D) $2\left[\cos\frac{\pi}{2}t + \cos\left(\frac{\pi}{4}t + \frac{\pi}{2}\right)\right]$

Solution: (C)

$$x(t) = \sum_{k=-\infty}^{\infty} x(k)e^{jk\omega_0 t}$$

$$= x(-2)e^{-j2\omega_o t} + x(2)e^{j2\omega_o t} + x(-4)e^{-j4\omega_o t} + x(4)e^{j4\omega_o t}$$

$$= e^{-j2\left(\frac{\pi}{8}\right)t} + e^{j2\left(\frac{\pi}{8}\right)t} - e^{+j4\left(\frac{\pi}{8}t\right)} + je^{-j4\left(\frac{\pi}{8}t\right)}$$

$$= 2\left[\frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2}\right] - 2\left[\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}\right]$$

$$= 2\left[\cos\frac{\pi}{4}t - \sin\frac{\pi}{2}t\right]$$

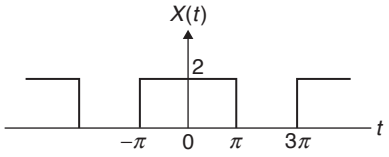
$$= 2\left[\cos\frac{\pi}{4}t + \cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]$$

EXERCISES

Practice Problems 1

Directions for questions 1 to 25: Select the correct alternative from the given choices.

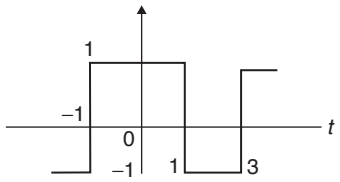
1. Determine the Fourier series coefficient for periodic signal $x(t)$ as:



(A) $\frac{2}{k\pi}\sin\frac{k\pi}{4}$. (B) $\frac{2}{k\pi}\sin\frac{k\pi}{2}$.

(C) $\frac{4}{k\pi}\sin\frac{k\pi}{4}$. (D) $\frac{4}{k\pi}\sin\frac{k\pi}{2}$.

2. Determine the Fourier series coefficient for periodic signal $x(t)$ as:



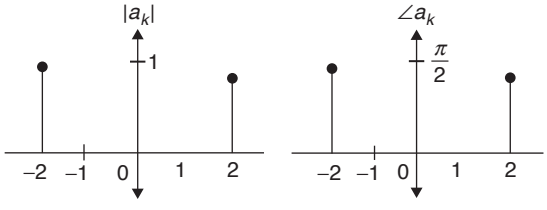
(A) $\frac{1}{2\pi k}[2e^{-jk\pi/2} - e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(B) $\frac{j}{2\pi k}[2e^{jk\pi/2} + e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(C) $\frac{j}{2\pi k}[2e^{-jk\pi/2} - e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(D) $\frac{1}{2\pi k}[2e^{-jk\pi/2} + e^{-3jk\pi/2} - e^{jk\pi/2}]$.

3. Fourier series coefficient of the time domain signal as $\omega_0 = 2\pi$



Determine the corresponding time domain signal.

(A) $e^{j\pi/2}\cos 2\pi t$. (B) $2e^{j\pi/2}\cos 4\pi t$.

(C) $2e^{-j\pi/2}\cos 2\pi t$. (D) $2e^{-j\pi/2}\cos 4\pi t$.

4. Suppose the periodic signal $x(t)$ has fundamental period 'T' and Fourier coefficients a_k, let b_k be the Fourier coefficients of $y(t)$, where $y(t) = \frac{d^2x(t)}{dt^2}$, the Fourier coefficient a_k in terms of b_k will be:

(A) $\frac{-T^2 b_k}{4\pi^2 k^2}$ (B) $\frac{T^2 b_k}{2\pi^2 k^2}$

(C) $\frac{T^2 b_k}{\pi^2 k^2}$ (D) $\frac{2T^2 b_k}{\pi^2 k^2}$

5. Find the Fourier transform of signal $x(t) = e^{-3t}u(t-2)$.

(A) $\frac{e^{-2(3+j\omega)}}{3+j\omega}$. (B) $\frac{e^{-3(2+j\omega)}}{2+j\omega}$.

(C) $\frac{e^{-2(3-j\omega)}}{3-j\omega}$. (D) $\frac{e^{-3(2-j\omega)}}{2-j\omega}$.

6. Determine the Inverse Fourier transform of $X(\omega) = \frac{-j\omega + 6}{\omega^2 - 13j\omega - 36}$.

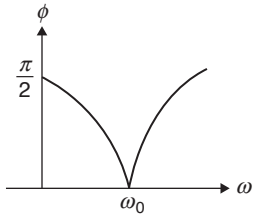
(A) $(3e^{-6t} + 2e^{-4t})u(t)$. (B) $(3e^{-6t} - 2e^{-4t})u(t)$.

(C) $(3e^{-9t} - 2e^{-4t})u(t)$. (D) $(3e^{-9t} + 2e^{-4t})u(t)$.

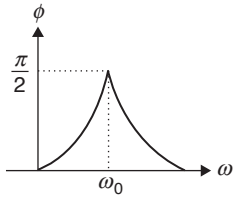
7. Which one of the following represents the phase response of the function?

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \left(\frac{\omega_0}{\theta}\right)s + \omega_0^2}$$

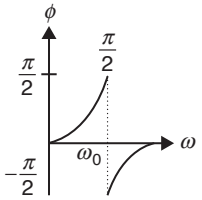
(A)



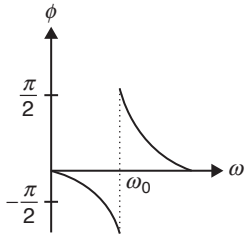
(B)



(C)



(D)

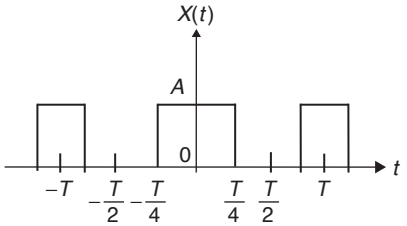


8. Which of the following statements are true?
(1) A Fourier series for an even periodic function will consist entirely of cosine terms.
(2) A Fourier series for an odd periodic function will consist entirely of sine terms.
(3) A Fourier series for an odd periodic function will consist entirely of cosine terms.
(4) A Fourier series for an even periodic function will consist entirely of sine terms.
(A) 3, 4 (B) 1, 2 (C) 1, 3 (D) 2, 4

9. The effect of time displacement τ in a periodic function ($n\omega_0$ frequency):
(A) Frequency spectrum remains same.
(B) Magnetic spectrum remains constant.
(C) Magnetic spectrum shifts by $n\omega_0 t$.
(D) Phase spectrum remains constant.

10. Which of the following is true?
(1) $x(t)$ is real and even function, then the Fourier series coefficients are real and even
(2) $x(t)$ is real and odd function then the Fourier series coefficients are imaginary and odd.
(A) Both (B) Neither (1) nor (2)
(C) Only (1) (D) Only (2)

11. Fourier series of the wave form shown:



(A) $A+\frac{A}{\pi}\left[\cos\omega_0 t-\frac{\cos 3\omega_0 t}{3}+\frac{\cos 5\omega_0 t}{5}\cdots\right]$.

(B) $\frac{A}{2}+\frac{2A}{\pi}\left[\cos\omega_0 t-\frac{\cos 3\omega_0 t}{3}+\frac{\cos 5\omega_0 t}{5}\cdots\right]$.

(C) $\frac{A}{2}+\frac{2A}{\pi}\left[\cos\omega_0 t-\frac{\cos 4\omega_0 t}{4}+\frac{\cos 6\omega_0 t}{6}\cdots\right]$.

(D) $\frac{A}{2}-\frac{2A}{\pi}\left[\cos\omega_0 t-\frac{\cos 3\omega_0 t}{3}+\frac{\cos 5\omega_0 t}{5}\cdots\right]$.

12. Match the following Fourier transform pairs

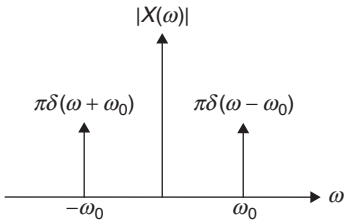
Group (1)		Group (2)	
P.	$e^{-j\omega t_0}$	**1.**	$\delta(t)$
Q.	$e^{-a\|w\|}$	**2.**	$\delta(t-t_0)$
R.	1	**3.**	$\frac{1}{a^2+t^2}$
S.	$2\pi\delta(\omega)$	**4.**	1

(A) P – 4, Q – 3, R – 2, S – 1.
(B) P – 3, Q – 2, R – 1, S – 4.
(C) P – 2, Q – 3, R – 1, S – 4.
(D) P – 2, Q – 3, R – 4, S – 1.

13. A rectangular function defined as:
$f(t) = 1/2,\ 0 < t < \pi,\ -1/2,\ (\pi < t < 2\pi)$, then find approximated function by a wave form sin t over the interval $(0, 2\pi)$ such that the mean square error is minimum.
(A) $\frac{4}{\pi}\sin t$. (B) $\frac{8}{\pi}\sin t$.
(C) $\frac{2}{\pi}\sin t$. (D) $\frac{1}{\pi}\sin t$.

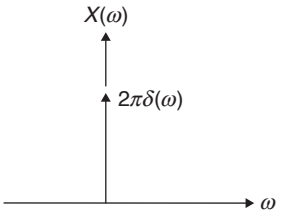
14. Match the following with their magnitude spectrum.
P. $x(t) = 1$

1.



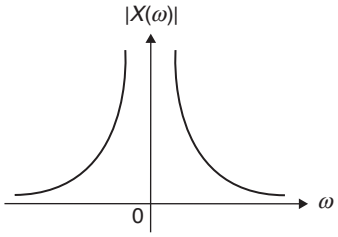
Q. Signum function.

2.



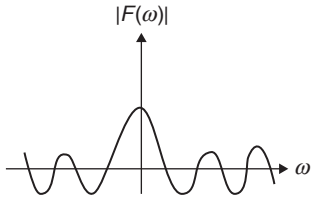
R. Rectangular Pulse.

3.



S. Cosinusoidal signal.

4.



(A) P – 2, Q – 1, R – 4, S – 3.
(B) P – 1, Q – 2, R – 3, S – 4.
(C) P – 2, Q – 3, R – 1, S – 4.
(D) P – 2, Q – 3, R – 4, S – 1.

15. Find the inverse Fourier transform of $\frac{7+j\omega}{(3+j\omega)^2}$

(A) $(4t-1)\,e^{-3t}\,u(t)$ (B) $(4t+1)\,e^{3t}\,u(t)$
(C) $(4+4)\,e^{-3t}\,u(t)$ (D) $(4t+1)\,e^{-3t}\,u(t)$

16. The impulse response of an LTI system is $h(t) = e^{-2t}\,u(t)$, find the response of the system for $x(t) = e^{-6t}\,u(t)$

(A) $\frac{1}{4}(e^{-2t} + e^{-6t})u(t)$

(B) $\frac{1}{4}(e^{-2t} - e^{-6t})u(t)$

(C) $\frac{-1}{4}(e^{-2t} - e^{-6t})u(t)$

(D) $\frac{-1}{4}(e^{-2t} + e^{-6t})u(t)$

Common Data for Questions 17 and 18:

17. Fourier transform of the signal $x(t)$ is $X(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{-j\omega} - 1)$. Find the Fourier transfom of $x(-t)$

(A) $\frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{j\omega} - 1)$.

(B) $\frac{1}{\omega^2}(e^{-j\omega} - j\omega e^{j\omega} - 1)$.

(C) $\frac{1}{\omega^2}(e^{j\omega} + j\omega e^{-j\omega} - 1)$.

(D) $\frac{1}{\omega^2}(e^{-j\omega} - j\omega e^{-j\omega} - 1)$.

18. Find the Fourier transform of $x(-t-1)$:

(A) $\frac{e^{j\omega}}{\omega^2}(e^{-j\omega} + j\omega e^{j\omega} - 1)$;

(B) $\frac{e^{-j\omega}}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1)$;

(C) $\frac{e^{-j\omega}}{\omega^2}(e^{j\omega} + j\omega e^{j\omega} - 1)$;

(D) $\frac{e^{-j\omega}}{\omega^2}(e^{-j\omega} - j\omega e^{j\omega} - 1)$;

19. The Fourier transform of $e^{-t}u(t)$ is $\frac{1}{1+j\omega}$, then Fourier transform of $\frac{1}{1+t}$ is:

(A) $2\pi e^{-j\omega}\,u(j\omega)$ (B) $2\pi e^{j\omega}\,u(j\omega)$
(C) $2\pi e^{-j\omega}\,u(-j\omega)$ (D) $2\pi e^{j\omega}\,u(-j\omega)$

20. Fourier coefficients of $x[n]$ are:
$C_k = \{3, 2+j, 1, 2-j\}$. The value of $x[6]$ is:
(A) 0 (B) 3
(C) $2-j$ (D) $2+j$

21. Frequency response $H(j\omega)$ of causal LTI system of difference equation:

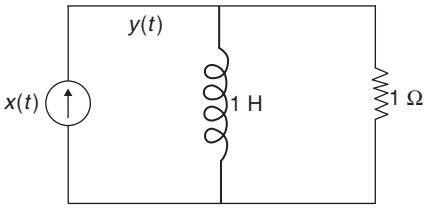
$$y[n] + \frac{1}{3}y[n-1] = x[n]$$

(A) $\frac{1}{1+\frac{1}{3}e^{-j\omega}}$ (B) $\frac{1}{1-\frac{1}{3}e^{-j\omega}}$

(C) $\frac{1}{1-\frac{1}{6}e^{-j\omega}}$ (D) $\frac{1}{1+\frac{1}{4}e^{-j\omega}}$

Common Data for Questions 22 and 23:

22. Consider a causal LTI system implemented as the RL circuit shown—a current source produces an input current $x(t)$, and system output is considered to find the differential equation relating to $x(t)$ and $y(t)$:



(A) $\frac{dy(t)}{dt} - y(t) = x(t)$ (B) $\frac{dy(t)}{dt} = x(t)$

(C) $y(t) = 0$ (D) $\frac{dy(t)}{dt} + y(t) = x(t)$

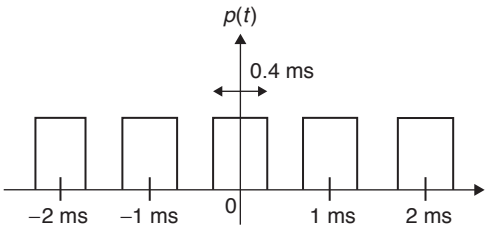
23. The frequency response of this system for input $x(t) = e^{j\omega t}$

(A) $H(j\omega) = 1$ (B) $H(j\omega) = \frac{1}{1+j\omega}$

(C) $H(j\omega) = \frac{1}{j\omega}$ (D) $H(j\omega) = j\omega$

Common Data for Questions 24 and 25:

24. A signal $x(t)$ is multiplied with a rectangular pulse train $p(t)$ as shown. $x(t)$ can be recovered from product $x(t)$. $p(t)$ by using an ideal low pass filter, if $X(j\omega) = 0$ for:



(A) $\omega < 2000\pi$ (B) $\omega < 1000\pi$
(C) $\omega > 1000\pi$ (D) $\omega > 2000\pi$

25. From the above question, the pass band gain of the ideal low pass filter needed to recover $x(t)$ from $x(t)$. $p(t)$ is:
(A) 4 (C) 10
(B) 2 (D) 2.5

Practice Problems 2

Directions for questions 1 to 20: Select the correct alternative from the given choices.

1. Which of the following is not a Dirichlet condition?

(A) $\frac{1}{T}\int_0^T |x(t)|^2\, dt < \infty$

(B) $\frac{1}{T}\int_0^T |x(t)|^2\, dt > 0$

(C) $x(t)$ has finite number of maxima and minima.

(D) $x(t)$ has a finite number of discontinuities.

2. If $f(t)$ is an even function of time period $T/2$, then,

(A) $\int_{-T/2}^{T/2} f(t)dt = 2\int_0^{T/2} f(t)dt$

(B) $\int_0^T f(t)dt = \frac{1}{2}\int_0^{T/2} f(t)dt$

(C) $\int_{-T/2}^{T/2} f(t)dt = 0$

(D) A, B

3. $\int_{-T}^{T} f(t)dt = 0$, $f(t)$ is function of time period $2T$, then $f(t)$ is,

(A) Even function

(B) Odd function

(C) Symmetric function

(D) Can't say

4. The phase spectrum of a function is:

(A) Even function

(B) Symmetric function

(C) Anti symmetric function

(D) Continuous function

5. Power spectrum is symmetrical about.

(A) Horizontal axis

(B) Horizontal axis passing through origin

(C) Vertical axis passing through origin

(D) Vertical axis.

6. The Fourier transform of $u(t)$, unit step function

(A) $\pi\delta(\omega) + \frac{j}{\omega}$ (B) $\pi\delta(\omega) - \frac{j}{\omega}$

(C) $\pi\delta(\omega) - \frac{1}{j\omega}$ (D) $-\pi\delta(\omega) + \frac{1}{j\omega}$

7. Match the Fourier transforms of group (1), to group (2) where $F\{x(t)\} = X(w)$:

Group (1)		Group (2)	
P.	$x(t - t_0)$	1.	$\frac{dX(\omega)}{d\omega}$
Q.	$e^{j\omega_0 t}x(t)$	2.	$X(\omega - \omega_0)$
R.	$\frac{dx(t)}{dt}$	3.	$e^{-j\omega_0 t}X(\omega)$
S.	$(-jt)\, x(t)$	4.	$j\omega\, X(\omega)$

(A) P – 2, Q – 3, R – 1, S – 4

(B) P – 2, Q – 3, R – 1, S – 4

(C) P – 3, Q – 2, R – 1, S – 4

(D) P – 3, Q – 2, R – 4, S – 1

8. Determine the complex exponential Fourier series coefficients of $\sin^2 t$.

(A) $\frac{-1}{4}, \frac{1}{2}, \frac{-1}{4}$ (B) $\frac{1}{2}, \frac{1}{4}, \frac{-1}{2}$

(C) $1, \frac{1}{2}, -1$ (D) None

9. Fourier series of full wave rectified sine wave, $\sin \omega_0 t$ for $t = 0$ to $T/2$ is:

(A) $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2\omega_0 t}{3} + \frac{\cos 4\omega_0 t}{15} + \frac{\cos 6\omega_0 t}{35} + \cdots\right]$

(B) $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 3\omega_0 t}{3} - \frac{\cos 5\omega_0 t}{5} + \frac{\cos 7\omega_0 t}{7} + \cdots\right]$

(C) $\frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2\omega_0 t}{3} - \frac{\cos 4\omega_0 t}{15} + \frac{\cos 6\omega_0 t}{35} + \cdots\right]$

(D) $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\sin 2\omega_0 t}{3} + \frac{\sin 4\omega_0 t}{15} + \frac{\sin 6\omega_0 t}{35} + \cdots\right]$

10. Fourier transform of sinusoidal signal $\sin \omega_0 t$ is:

(A) $\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

(B) $\frac{\omega}{j}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

(C) $j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

(D) $j\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

11. Which of the following signal will not have sine coefficients in Fourier series expansion?

(A) $x(t) = 1 - t$ (B) $x(t) = \cos t$

(C) $x(t) = \frac{1}{1+t}$ (D) $x(t) = (1 - t)^2$

12. The Fourier transform of the signal $e^{-3|t|}$ is:

(A) $\frac{6}{9 - \omega^2}$ (B) $\frac{6}{9 + \omega^2}$

(C) $\frac{6}{\omega^2 - 9}$ (D) $\frac{-6}{\omega^2 + 9}$

13. Find the Fourier transform of $e^{-3|t-t_0|} + e^{3|t+t_0|}$

(A) $\frac{12\cos\omega t_0}{9+\omega^2}$ (B) $\frac{12\sin\omega t_0}{9+\omega^2}$

(C) $\frac{j12\sin\omega t_0}{9-\omega^2}$ (D) $\frac{12\cos\omega t_0}{9-\omega^2}$

14. Fourier transform of a Gaussian pulse is

(A) sin*c* pulse (B) Sin function

(C) Gaussian pulse (D) Impulse train

15. The average power of $x(n)$ in terms of Fourier series coefficients C_k is

(A) $\sum_{k=0}^{\infty}|C_k|^2$ (B) $\frac{1}{N}\sum_{k=0}^{\infty}C_k^{\,2}$

(C) $\frac{1}{N}\sum_{k=0}^{N-1}|C_k|^2$ (D) $\sum_{k=0}^{N-1}|C_k|^2$

16. If we replace 'z' in z transform of $x[n]$ with $e^{j\omega}$ we will get:

(A) Continuous-time Fourier transform

(B) Discrete-time Fourier transform

(C) Hilbert transform

(D) Complex z-transform

17. The function $x(t)$ has the Fourier transform $X(\omega)$ then, $\int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = ?$

(A) $\frac{1}{2\pi}x(\omega)$ (B) $2\pi X(-\omega)$

(C) $2\pi x(-\omega)$ (D) $\frac{1}{2\pi}x(-\omega)$

18. If $F\{x(t)\} = \frac{\omega}{s^2+\omega^2}$, then the value of $\lim_{x\to\infty} x(t)$ is:

(A) Cannot be determined

(B) Zero

(C) Unity

(D) Infinite

19. Match the time domain signals and their Fourier transforms:

List I		List II	
P.	$1/2\pi$	1.	$\delta(\omega-\omega_0)$
Q.	$e^{j\omega_0 t}/2\pi$	2.	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)$
R.	Cos $\omega_0 t$	3.	$\delta(\omega)$
S.	Sin $\omega_0 t$	4.	$j\pi(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

(A) P – 1, Q – 3, R – 4, S – 2;

(B) P – 3, Q – 1, R – 4, S – 2;

(C) P – 3, Q – 1, R – 2, S – 4;

(D) P – 1, Q – 3, R – 2, S – 4;

20. Delaying a signal by t_0 seconds in time domain does:

(A) Not change its amplitude and phase spectrum

(B) Not change its amplitude spectrum but the phase spectrum is changed by $-\omega t_0$

(C) Not change its phase spectrum but the amplitude spectrum is scaled by t_0

(D) Change phase spectrum by ωt_0 and amplitude spectrum by t_0.

Previous Years' Questions

1. The z-transform of a signal $x[n]$ is given by $4z^{-3} + 3z^{-1} + 2 - 6z^2 + 2z^3$. It is applied to a system, with a transfer function $H(z) = 3z^{-1} - 2$. Let the output be $y(n)$. Which of the following is true? **[2009]**

(A) $y(n)$ is non-causal with finite support

(B) $y(n)$ is causal with infinite support

(C) $y(n) = 0; |n| > 3$

(D) $\text{Re}[Y(z)]_{z=e^{j\theta}} = -\text{Re}[Y(z)]_{z=e^{-j\theta}}; \text{lm}[Y(z)]z = e^{j\theta}.$ $\text{lm}[Y(z)]_{z=e^{-j\theta}}; -\pi \le \theta < \pi$

2. The Fourier series coefficient, of a periodic signal $x(t)$, expressed as $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$ are given by $a_{-2} = 2 - j1$; $a_{-1} = 0.5 + j0.2$; $a_0 = j2$; $a_1 = 0.5 - j0.2$; $a_2 = 2 + j1$; and $a_k = 0$; For $|k| > 2$. Which of the following is true? **[2009]**

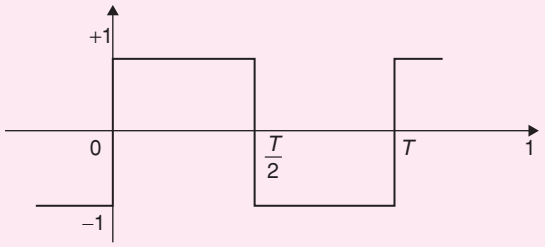
(A) $x(t)$ has finite energy because only finitely many coefficients are non-zero.

(B) $x(t)$ has zero average value because it is periodic.

(C) The imaginary part of $x(t)$ is constant

(D) The real part of $x(t)$ is even.

3. The second harmonic component of the periodic waveform given in the figure has an amplitude of: **[2010]**



(A) 0 (B) 1

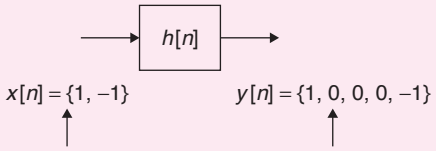
(C) $2/\pi$ (D) $\sqrt{5}$

4. $x(t)$ is a positive rectangular pulse from $t = -1$ to $t = +1$ with unit height as shown in the figure. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$ {where $X(\omega)$ is the Fourier transform of $x(t)$} is: **[2010]**

(A) 2 (B) 2π

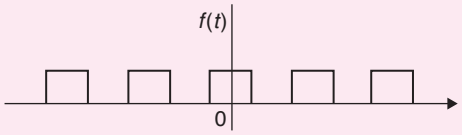
(C) 4 (D) 4π

5. Given the finite length input $x[n]$ and the corresponding finite length output $y[n]$ of an LTI system as shown below, the impulse response $h[n]$ of the system is: **[2010]**



(A) $h[n] = \{1,0,0,1\}$ (B) $h[n] = \{1,0,1\}$
(C) $h[n] = \{1,1,1,1\}$ (D) $h[n] = \{1,1,1\}$

6. The Fourier series expansion $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$ of the periodic signal shown below will contain the following nonzero terms. **[2011]**



(A) a_0 and b_n, $n = 1, 3, 5, \ldots\infty$
(B) a_0 and a_n, $n = 1, 2, 3, \ldots\infty$
(C) a_0, a_n and b_n, $n = 1, 2, 3, \ldots\infty$
(D) a_0 and a_n, $n = 1, 3, 5, \ldots\infty$

7. If $x[n] = (1/3)^{|n|} - (1/2)^n u[n]$, then the region of convergence (ROC) of its z-transforms in the z-plane will be: **[2012]**

(A) $\frac{1}{3} < |z| < 3$ (B) $\frac{1}{3} < |z| < \frac{1}{2}$
(C) $\frac{1}{2} < |z| < 3$ (D) $\frac{1}{3} < |z|$

8. Given $F(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in z-plane such that $|z + 1| = 1$, the value of $\frac{1}{2\pi j}\oint_c f(z)dz$ is: **[2012]**

(A) −2 (B) −1
(C) 1 (D) 2

9. Let $y[n]$ denotes the convolution of $h[n]$ and $g[n]$, where $h[n] = (1/2)^n u[n]$ and $g[n]$ is a Causal sequence. If $y[0] = 1$ and $y[1] = \frac{1}{2}$, then $g[1]$ equals: **[2012]**

(A) 0 (B) $\frac{1}{2}$
(C) 1 (D) 3/2

10. The Fourier transforms of a signal $h(t)$ is $H(j\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. The value of $h(0)$ is: **[2012]**

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$
(C) 1 (D) 2

11. For a periodic signal $v(t) = 30\sin 100t + 10\cos 300t + 6\sin(500t + \pi/4)$, the fundamental frequency in rad/s is: **[2013]**

(A) 100 (B) 300
(C) 500 (D) 1500

12. For a periodic square wave, which one of the following statements is TRUE? **[2014]**

(A) The Fourier series coefficients do not exist.
(B) The Fourier series coefficients exist but the reconstruction converges at no point.
(C) The Fourier series coefficients exist and the reconstruction converges at most points.
(D) The Fourier series coefficients exist and the reconstruction converges at every point.

13. Let $f(t)$ be a continuous-time signal and let $F(\omega)$ be its Fourier transform and $g(t)$ defined by: **[2014]**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$g(t) = \int_{-\infty}^{\infty} F(u)e^{-jut}du$$

What is the relationship between $f(t)$ and $g(t)$?

(A) $g(t)$ would always be proportional to $f(t)$.
(B) $g(t)$ would be proportional to $f(t)$ if $f(t)$ is an even function.
(C) $g(t)$ would be proportional to $f(t)$ only if $f(t)$ is a sinusoidal function.
(D) $g(t)$ would never be proportional to $f(t)$.

14. The following discrete-time equations result from the numerical integration of the differential equations of an un-damped simple harmonic oscillator with state variable x and y. The integration time step is h.

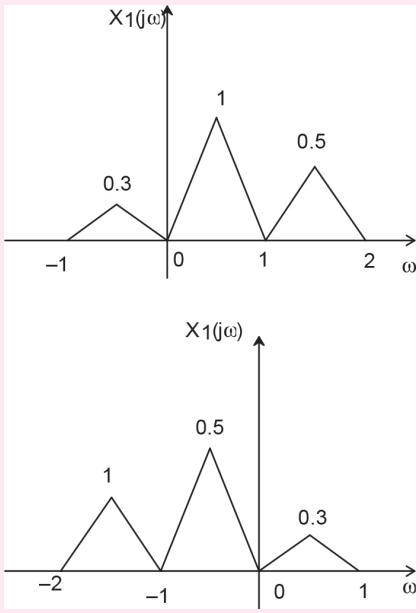
$$\frac{x_{k+1} - x_k}{h} = y_k$$

$$\frac{y_{k+1} - y_k}{h} = -x_k$$

For this discrete-time system, which one of the following statements is TRUE?

(A) The system is not stable for $h > 0$
(B) The system is stable for $h > \frac{1}{\pi}$
(C) The system is stable for $0 < h < \frac{1}{2\pi}$
(D) The system is stable for $\frac{1}{2\pi} < h < \frac{1}{\pi}$

15. Suppose $x_1(t)$ and $x_2(t)$ have the Fourier transforms as shown below.



Which one of the following statements is TRUE? **[2016]**

(A) $x_1(t)$ and $x_2(t)$ are complex and $x_1(t)\ x_2(t)$ is also complex with nonzero imaginary part

(B) $x_1(t)$ and $x_2(t)$ are real and $x_1(t)\ x_2(t)$ is also real

(C) $x_1(t)$ and $x_2(t)$ are complex but $x_1(t)\ x_2(t)$ is real

(D) $x_1(t)$ and $x_2(t)$ are imaginary but $x_1(t)\ x_2(t)$ is real

16. The output of a continuous-time, linear time-invariant system is denoted by $\boldsymbol{T}\{x(t)\}$ where $x(t)$ is the input signal. A signal $z(t)$ is called eigen-signal of the system $\boldsymbol{T}$, when $\boldsymbol{T}\{x(t)\} = xz(t)$, where $x(t)$ is a complex number, in general, and is called an eigenvalue of $\boldsymbol{T}$. Suppose the impulse response of the system $\boldsymbol{T}$ is real and even. Which of the following statements is TRUE? **[2016]**

(A) cos (t) is an eigen-signal but sin (t) is not

(B) cos(t) and sin(t) are both eigen-signals but with different eigenvalues

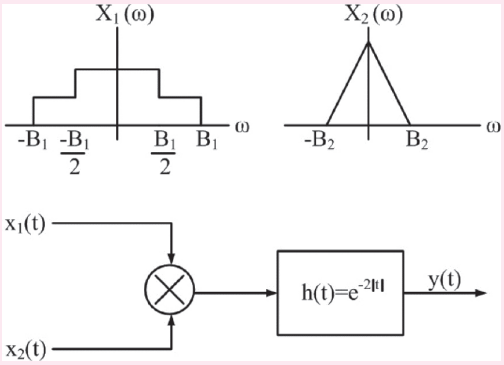
(C) sin(t) is an eigen-signal but cos (t) is not

(D) cos(t) and sin(t) are both eigen-signals but with identical eigenvalues

17. Suppose the maximum frequency in a band-limited signal $x(t)$ is 5kHz. Then, the maximum frequency in $x(t)$ COS$(2000\pi t)$, in kHz, is _____. **[2016]**

18. Let $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$ be two signals whose Fourier Transforms are as shown in the figure below. In the figure $h(t) = e^{-2|t|}$ denotes the impulse response.

For the system shown below, the minimum sampling rate required to sample $y(t)$, so that $y(t)$ can be uniquely reconstructed form its samples, is **[2016]**



(A) $2B_1$ (B) $2(B_1 + B_2)$

(C) $4(B_1 + B_2)$ (D) ∞

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** C	**3.** B	**4.** A	**5.** A	**6.** C	**7.** D	**8.** B	**9.** B	**10.** A
11. B	**12.** C	**13.** C	**14.** D	**15.** D	**16.** B	**17.** A	**18.** C	**19.** D	**20.** A
21. A	**22.** D	**23.** B	**24.** B	**25.** D					

Practice Problems 2

1. B	**2.** A	**3.** B	**4.** C	**5.** C	**6.** B	**7.** D	**8.** A	**9.** A	**10.** C
11. B	**12.** B	**13.** A	**14.** C	**15.** D	**16.** B	**17.** C	**18.** A	**19.** C	**20.** B

Previous Years' Questions

1. A	**2.** A	**3.** A	**4.** D	**5.** C	**6.** D	**7.** C	**8.** C	**9.** A	**10.** D
11. A	**12.** C	**13.** B	**14.** A	**15.** C	**16.** D	**17.** 6	**18.** B		

TEST
SIGNALS AND SYSTEMS

Time: 60 min.

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. The system represented by the input-output relationship $y(t)=\int_{-\infty}^{4t} x(\tau)d\tau, t>0$ is

(A) Linear and causal
(B) Linear but not causal
(C) Causal but not linear
(D) Neither linear nor causal

2. Which of the following cannot be the Fourier series expansion of a periodic signal?

(A) $x(t)=2\cos t+3\cos 3t$
(B) $x(t)=2\cos\pi t+7\cos t$
(C) $x(t)=\cos t+0.5$
(D) $x(t)=2\cos t 1.5\pi t+\sin 3.5\pi t$

3. The transfer function of a system is given by $H(s)=\frac{1}{s^2(s-2)}$. The impulse response of the system is: ($*$ denotes convolution, and $u(t)$ is the unit step function).

(A) $(t^2 * e^{-2t})\,u(t)$ (B) $(t * e^{2t})\,u(t)$
(C) $(t^2e^{-2t})\,u(t)$ (D) $te^{-2t}\,u(t)$

4. $y(n)$ denotes the output and $x(n)$ denotes the input of a discrete time system given by the difference equation $y[n]-0.8\,y[n-1]=x[n]+1.25\,x[n+1]$. Its right sided impulse response is

(A) Anticausal (B) Unstable
(C) Periodic (D) Non negative

5. A linear time invariant system with an impulse response $h(t)$ produces output $y(t)$, when input $x(t)$ is applied. When the input $x(t-\tau)$ is applied to a system with impulse response $h(t-\tau)$, the output will be _____.

(A) $y(t-\tau)$ (B) $y(t)$
(C) $y(t-2\tau)$ (D) $y(2(t-\tau))$

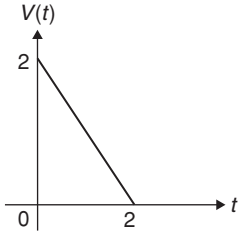
6. Laplace transform of $V(t)$ shown in wave form is

(A) $\frac{-2}{s}+\frac{1}{s^2[1-e^{-2s}]}$

(B) $\frac{2}{s}-\frac{1}{s^2}[1-e^{-2s}]$

(C) $\frac{-2}{s}-\frac{1}{s^2[1-e^{-2s}]}$

(D) $\frac{2}{s}-\frac{1}{s^2[1-e^{-2s}]}$



7. Let $x(t)$ be a periodic signal with time period T. Let $y(t)=x(t-t_0)+x(t+t_0)$. The Fourier series coefficients of $y(t)$ are denoted by b. If $b_k=0$ for all odd K, then t_0 can be equal to

(A) $\frac{T}{8}$ (B) $\frac{T}{4}$
(C) $\frac{T}{2}$ (D) $2T$

8. A signal $x(n)=\sin(\omega_0 n+\phi)$ is the input to a linear time-invariant system having a frequency response $H(e^{j\omega})$. If the output of the system is $Ax(n-n_0)$, then the most general form of $\angle H(e^{j\omega})$ will be

(A) $-n_0\omega_0+\beta$ for any arbitrary real β
(B) $-n_0\omega_0+2\pi k$ for any arbitrary integer k
(C) $n_0\omega_0+2\pi k$ for any arbitrary integer k
(D) $-n_0\omega_0+\phi$

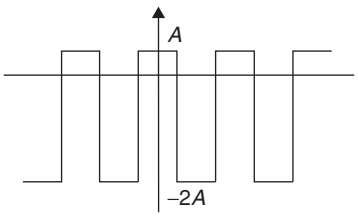
9. A system defined by $y[n]=\sum_{k=-\infty}^{n} x[k]$, where $x[n]$ is causal is an example of

(A) Invertible system
(B) Memory less system
(C) Non invertible system
(D) Averaging system

10. Two discrete time systems with impulse responses $h_1[n]=\delta[n-1]$ and $h_2[n]=\delta[n-2]$ are connected in cascade. The overall impulse response of the cascaded system is

(A) $\delta[n-1]+\delta[n-2]$ (B) $\delta[n-4]$
(C) $\delta[n-3]$ (D) $\delta[n-1]\,\delta[n-2]$

11. A continuous time signal shown below contains



(A) Only cosine terms and zero value for the dc component
(B) Only cosine terms and a positive value for the dc component.
(C) Only cosine terms and a negative value for the dc component.
(D) Only sine terms and a negative value for the dc component.

12. $F\{\ \}$ denotes Fourier transform, $F\{e^{-t}u(t)\}=\frac{1}{1+j2\pi f}$. If so $F\left\{\frac{1}{1+j2\pi t}\right\}$ is

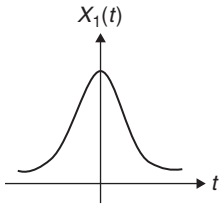
(A) $e^{f}u(f)$ (B) $e^{-f}u(f)$
(C) $e^{f}u(-f)$ (D) $e^{-f}u(-f)$

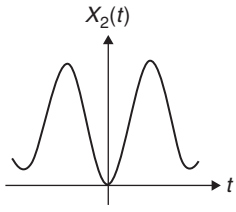
13. If $X(f)$ represents the Fourier transform of a signal $x(t)$ which is real and odd symmetric in time, then $X(f)$ is

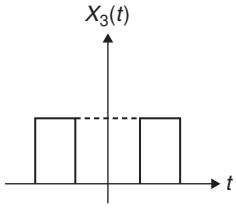
(A) Complex (B) Imaginary
(C) Real (D) Real and non negative

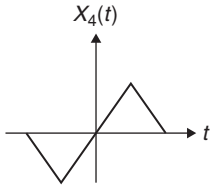
14. Which of the following real signals will not satisfy the condition?

$$\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 0$$

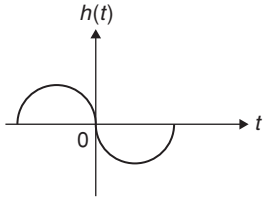
(A) 

(B) 

(C) 

(D) 

15. Consider a system whose input x and output y are related by the equation $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(2\tau)\,d\tau$, where $h(t)$ is as shown in figure, then the system is



(A) Causal, LTI (B) BIBO, causal, LTI
(C) Low pair, LTI (D) BIBO, LTI

16. If $u(t)$ is the unit step and $\delta(t)$ is the unit impulse function, the inverse z-transform of $F(z) = \frac{1}{z+1}$ for $k > 0$ is _____.

(A) $(-1)^k \delta(k)$ (B) $\delta(k) - (-1)^k$
(C) $(-1)^k u(k)$ (D) $u(k) - (-1)^k$

17. An LTI system having transfer function $\frac{s^2+1}{s^2+2s+1}$ and $x(t) = \sin(t+1)$ is in steady state. The output is sampled at a rate ω_s rad/s to obtain the final output $\{Y(k)\}$, which of the following is true?

(A) $Y(\cdot)$ is zero for all sampling frequencies ω_s
(B) $Y(\cdot)$ is non zero for all sampling frequencies for ω_s
(C) $Y(\cdot)$ is non zero for $\omega_s > 2$, but zero for $\omega_s < 2$
(D) $Y(\cdot)$ is zero for $\omega_s > 2$, but non zero for $\omega_s < 2$

18. Given the relationship between the input $u(t)$ and the output $y(t)$ to be $y(t) = \int_0^2 (2+t-\tau)e^{-3(t-\tau)}u(\tau)d\tau$ The transfer function $\frac{Y(s)}{U(s)}$

(A) $\frac{2e^{-2s}}{s+3}$ (B) $\frac{s+2}{(s+3)^2}$
(C) $\frac{2s+5}{s+3}$ (D) $\frac{2s+7}{(s+3)^2}$

Common Data for Questions 19 and 20: The impulse response $h(t)$ of a linear time invariant continuous time system is given by $h(t) = \exp(-2t)u(t)$, where $u(t)$ denotes the unit step function.

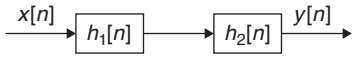
19. The frequency response $H(\omega)$ of this system in terms of angular frequency ω, is given by $H(\omega) =$

(A) $\frac{1}{1+j2\omega}$ (B) $\frac{1}{2+j\omega}$
(C) $\frac{j\omega}{1+j2\omega}$ (D) $\frac{j\omega}{2+j\omega}$

20. The output of this system, to the sinusoidal input $x(t) = 2\cos(2t)$ for all time t, is _____.

(A) $2e^{-0.5}\cos(2t - 0.25\pi)$ (B) $2e^{0.5}\cos(2t + 0.25\pi)$
(C) $\frac{1}{\sqrt{2}}\cos(2t - 0.125\pi)$ (D) $\frac{1}{\sqrt{2}}\cos(2t - 0.25\pi)$

21. Consider the figure given below. The impulse response of the overall system is



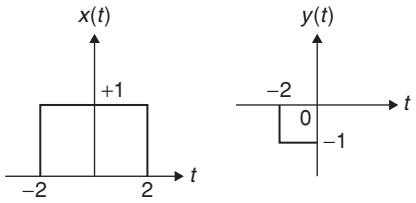
$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ h_2[n] = \left(\frac{1}{4}\right)^n u[n]$$

(A) $\left(\frac{1}{2}\right)^n\left[2-\left(\frac{1}{2}\right)^n\right]u[n]$ (B) $\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^n\right]u[n]$
(C) $\left(\frac{1}{2}\right)^n\left[1-\left(\frac{1}{2}\right)^n\right]u[n]$ (D) $\frac{1}{2}\left[2-\left(\frac{1}{2}\right)^n\right]u[n]$

22. The ROC of the z-transform of the discrete time sequence $x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$ is

(A) $\frac{1}{3} < |z| < \frac{1}{2}$ (B) $|z| > \frac{1}{2}$
(C) $|z| < \frac{1}{3}$ (D) $2 < |z| < 3$

23. Let $x(t)$ and $y(t)$ (with Fourier transforms $X(f)$ and $Y(f)$ respectively) be related as shown in the given figure. Then $Y(f)$ is?



(A) $-\frac{1}{2}X\left[\frac{f}{2}\right]e^{j2\pi f}$ (B) $+\frac{1}{2}X\left[\frac{f}{2}\right]e^{j4\pi f}$

(C) $\frac{1}{4}X\left[\frac{f}{4}\right]e^{j2\pi f}$ (D) $-\frac{1}{2}X\left[\frac{f}{4}\right]e^{-4j\pi f}$

24. To which one of the following difference equations the impulse response, $h(n) = \delta(n + 2) - \delta(n - 2)$ corresponds to?

(A) $y(n+2) = x(n) - x(n-2)$

(B) $y(n-2) = x(n) - x(n-4)$

(C) $y(n) = x(n+2) + x(n-2)$

(D) $y(n) = -x(n+2) + x(n-2)$

25. A stable LTI system has impulse response $h(n) = \begin{cases} a^n & n \geq 0 \\ b^n & n < 0, \end{cases}$ then the values of a and b are?

(A) $|a| < 1$ and $|b| < 1$

(B) $|a| > 1$ and $|b| > 1$

(C) $|a| < 1$ and $|b| > 1$

(D) $|a| > 1$ and $|b| < 1$

Answers Keys

1. B	**2.** B	**3.** B	**4** D	**5.** C	**6.** B	**7.** B	**8.** B	**9.** A	**10.** C
11. C	**12.** C	**13.** B	**14.** A	**15.** D	**16.** B	**17.** A	**18.** D	**19.** B	**20.** D
21. A	**22.** A	**23.** A	**24.** B	**25.** C					

Analog and Digital Electronics

UNIT II

Chapter 1:	Diode Circuits	3.93
Chapter 2:	Bipolar Junction Transistors	3.119
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Chapter 1

Diode Circuits

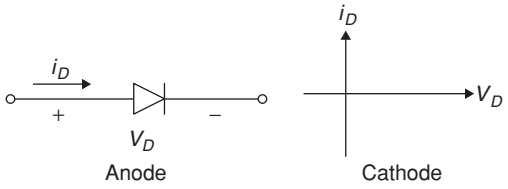
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Small signal equivalent circuit of diode
- Practical PN junction diode
- Large signal diode-models
- Open circuit test
- Short circuit test
- Rectifiers
- Half wave rectifier analysis
- Full wave rectifier
- Peak inverse voltage
- Bridge rectifier
- Clipping circuits
- Clamper circuits
- Voltage multipliers
- Power supplies
- Voltage regulators

Small Signal Equivalent Circuit of Diode

An ideal diode is a two-element device which has the circuit symbol and Volt-ampere characteristic as shown below:



A diode is a two-terminal unipolar device which provides minimum resistance in the forward direction and maximum resistance in the reverse direction.

Ideal diode is an unilateral circuit element, as the current in the device is in one direction only. This behavior is important in switching as it provides an ON-OFF characteristic.

Practical PN Junction Diode

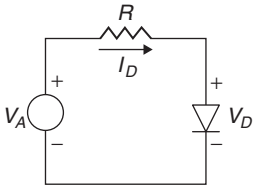


Figure 1 Diode circuit

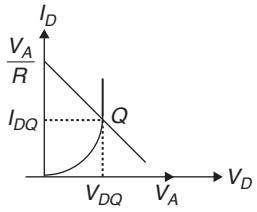


Figure 2 Diode characteristic and load line

From the circuit, $-V_A + I_D R + V_D = 0$ (KVL) $I_D = \frac{-1}{R} V_D + \frac{V_A}{R} A$

This equation defines a straight line, called load line. The load line and diode characteristic must be satisfied simultaneously, at their point of intersection Q, called Quiescent or operating point. The values of current in and voltage across the diode are denoted by I_{DQ}, and V_{DQ} respectively.

Large Signal Diode-Models

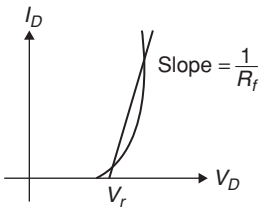


Figure 3 Piecewise linear diode forward characteristic

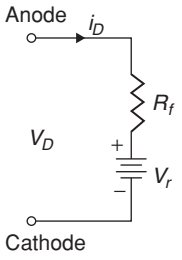


Figure 4 Diode model for forward bias

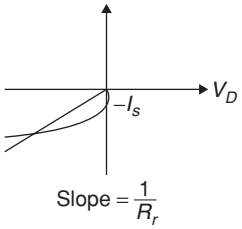


Figure 5 Piecewise linear reverse based characteristic diode

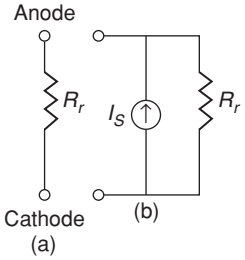
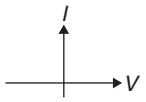
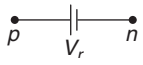
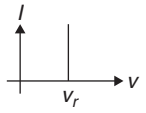
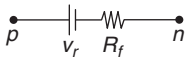
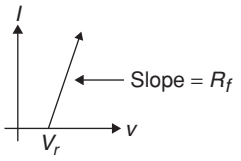


Figure 6 Diode model on piecewise of linear representation (b) Model to include surface leakage

Model	Equivalent Circuit	V-I characteristics
Ideal		
Simplified model		
Piecewise linearmodel		

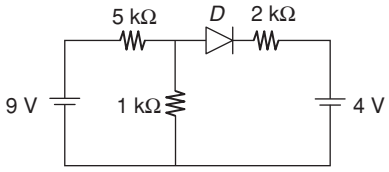
Open circuit test

(i) Replace the diode with open Circuit
(ii) Find the voltage across the diode from '*p*' to '*n*' side, if it is positive, diode is in Forward Biased. If it is negative, diode is in Reverse Biased.

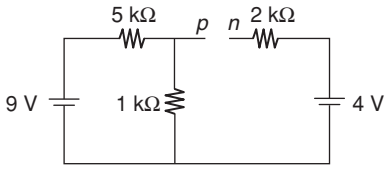
Short circuit test

(i) Replace all the diodes with short Circuit
(ii) Find the current through the diode from '*p*' to '*n*' side say '*I*'.
If $I > 0$, Diode is in forward biased
$I < 0$, Diode is in reverse biased

Example 1: Check whether the diode is in forward biased or not?



Solution: '*D*' is open



$$V_p = \frac{9}{6} = 1.5 \text{ V}$$

$$V_n = 4 \text{ V}$$

$$V_p < V_n$$

Hence '*D*' is in reverse biased condition.

Elementary Diode Applications

Rectifiers

A device, such as the semiconductor diode, which is capable of converting a sinusoidal input waveform into a unidirectional waveform with a non-zero average component is called rectifier.

Half wave rectifier

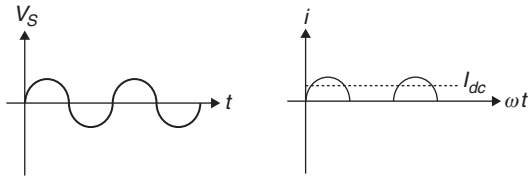


Figure 7 Volt (V_s) and current (i) waveforms

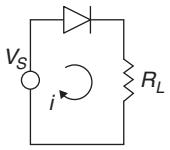


Figure 8 Half wave rectifier

During the first half of cycle (positive cycle). The diode is ON, and current $\frac{V_s}{R_L}$ exists. The diode is off during negative half cycle of V_s so that the current is zero as the current exists for only one half cycle, it is called halfwave rectifier.

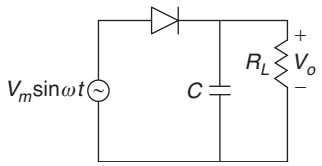


Figure 9 Rectifier with capacitor filter

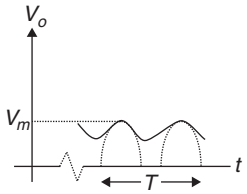


Figure 10 Output voltage waveform

The circuit uses a capacitor '*C*' as a simple filter to convert the waveform to the nearly constant (DC) level. If the time constant $R_L C$ is much greater than the time period of T of input waveforms, the discharge will be slow.

The peak to peak ripple out of any capacitor input filter is $V_R = \frac{I}{fC}$

V_R – peak to peak ripple voltage
I – DC load current
f – ripple frequency
C – capacitance

$$V_{rms} = \frac{V_{PP}}{2\sqrt{2}}$$

If we use real diode, the equivalent circuit is as shown in Figure 11.

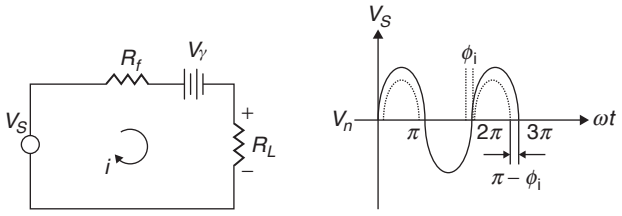


Figure 11 Rectifier equivalent circuit

The current 'i' is obtained by KVL

$$i = \frac{V_s - V_\gamma}{R_L} = \frac{V_m \sin\omega t - V_\gamma}{R_L}$$

as it is greater than zero only when $V_s > V_\gamma$. Thus current waveform does not start when $\omega t = 0$, but has a cutin or ignition angle ϕ, given by $\phi_i = \sin^{-1}\left(\frac{V_\gamma}{V_m}\right)$.

Similarly extinction angle exists at the end of positive cycle its value is $\pi - \phi_i$.

Half Wave Rectifier Analysis

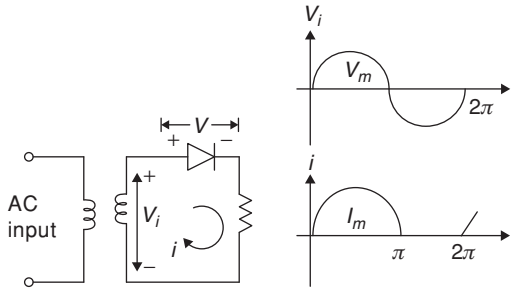


Figure 12 Basic circuit of half wave rectifier

$V_i = V_m \sin\omega t$, has peak value of V_m, which is mostly very high when compared to V_γ of diode, assume $V_\gamma = 0$.

The diode idealized to be a resistance R_f in the ON state, and an open circuit in the OFF state, the currenti in the diode or load R_L is

$$i = I_m \sin\omega t \text{ if } 0 \le \omega t \le \pi$$
$$= 0, \text{ if } \pi \le \omega t \le 2\pi$$

And $I_m = \dfrac{V_m}{R_f + R_L}$

DC current of value (Average current)

$$I_{DC} = \frac{1}{2\pi}\int_0^{2\pi} id(\omega t)$$

For half wave circuit, $I_{DC} = \dfrac{1}{2\pi}\int_0^{\pi} I_m \sin(\omega t)d(\omega t) = \dfrac{I_m}{\pi}$

The DC (average) output voltage is given as

$$V_{DC} = I_{DC} \cdot R_L = \frac{I_m R_L}{\pi} = \frac{V_m}{\pi}$$

The DC voltage across diode is not given by $I_{DC}R_f$, as diode acts as resistance which has two values

R_f in the ON state, and ∞ in the OFF state

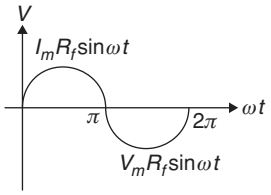


Figure 13 Voltage across the diode

So $V_{DC} = \dfrac{1}{2\pi}\left[\int_0^{\pi} I_m R_f \sin\omega t d(\omega t) + \int_0^{2\pi} V_m \sin(\omega t) d(\omega t)\right]$

$$= \frac{1}{\pi}(I_m R_f - V_m)$$

$$= \frac{1}{\pi}[I_m R_f - I_m(R_f + R_L)]$$

$V_{DC} = \dfrac{-I_m R_L}{\pi}$, so the DC diode voltage is seen to be equal to the negative of average voltage across the load resistor.

The AC current $I_{rms} = \left(\dfrac{1}{2\pi}\int_0^{2\pi} i^2 d(\omega t)\right)^{1/2} = \dfrac{I_m}{2}$

$$V_{rms} = I_{rms} R_L$$

$$V_{rms} = \frac{I_m}{2} R_L = \frac{V_m}{2}$$

Regulation

The variation of DC output voltage as a function of DC load current is called regulation.

$$\% \text{ of regulation } = \frac{V_{\text{noload}} - V_{\text{load}}}{V_{\text{load}}} \times 100\%$$

No load refers to zero current, load indicates the normal load current

Note: ideal power supply has percentage regulation zero

$$I_{DC} = \frac{I_m}{\pi} = \frac{V_m/\pi}{R_f + R_L}, \quad V_{DC} = I_{DC} R_L$$

The DC output voltage varies as

$$V_{DC} = \frac{V_m}{\pi} - I_{DC} R_f$$

Full wave rectifier

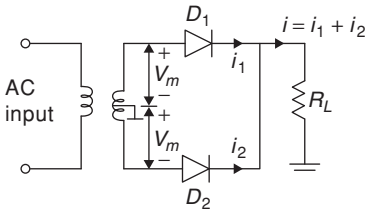


Figure 14 A full wave rectifier circuit

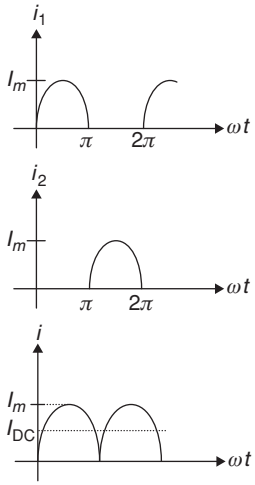


Figure 15 The diode currents i_1 and i_2 and load current i, output voltage $V_o = iR_L$

The circuit, is seen to comprise of two half wave rectifiers, connected so that conduction takes place through one of the diodes at one half cycle.

$$I_{DC} = \frac{2I_m}{\pi}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

$$V_{DC} = \frac{2I_m R_L}{\pi}$$

$$V_{DC} = \frac{2V_m}{\pi} - I_{DC}R_f$$

If we include V_γ, of the diode

$$\theta_1 = \sin^{-1}\frac{V_\gamma}{V_m}, \theta_2 = \pi - \theta_1$$

The average current I_{DC}

$$I_{DC} = \frac{V_m}{\pi(R_f + R_L)}\cos\theta_1 - \frac{\pi - 2\theta_1}{2\pi}\frac{V_\gamma}{R_f + R_L}$$ (half wave rectifier)

$$I_{DC} = \frac{2V_m}{\pi(R_f + R_L)}\cos\theta_1 - \frac{\pi - 2\theta_1}{2\pi}\frac{V_\gamma}{R_f + R_L}$$ (full wave rectifier)

Example 2: A diode with an internal resistance of 10 Ω is used as a rectifier to supply power to a 500 Ω load from a 220 V (rms) source of supply. Calculate (a) the peak load current (b) the DC load current (c) the rms load current d) the DC diode voltage (e) the total input power to the circuit (f) percentage regulation from no-load to the given load.

Solution: A diode is used as rectifier, so half wave rectifier, if two diodes are used, it will be full wave rectifier.

(a) $I_m = \frac{V_m}{R_f + R_L} = \frac{\sqrt{2}V_{rms}}{R_f + R_L} = \frac{\sqrt{2}.220}{(10+500)}$

$= 0.61$ A

(b) $I_{DC} = \frac{I_m}{\pi} = 0.19$ A

(c) $I_{rms} = \frac{I_m}{2} = 0.305$ A

(d) $V'_{DC} = \frac{I_m R_L}{\pi} = -0.61 \times \frac{500}{\pi}$

$= -97.1$ V

(e) Power delivered $= V_{rms} I_{rms}$

$= (220\text{ V})(0.305\text{ A}) = 67.1$ W

(f) % regulation $= \frac{I_{DC}R_L}{\frac{V_m}{\pi} - I_{DC}R_f} \times 100$

$$= \frac{\frac{V_m}{\pi(R_f + R_L)} \cdot R_f}{\left(\frac{V_m}{\pi} - \frac{V_m}{\pi(R_f + R_L)}R_f\right)} \times 100$$

$$= \frac{R_F}{R_L} \times 100$$

As R_L goes from ∞ (no-load) to 500 Ω, percentage regulation will go from 0 to 2%.

Peak Inverse Voltage

For each rectifier circuit there is a maximum voltage to which the diode can be subjected. This potential is called peak inverse voltage (PIV). For half wave rectifier, the peak inverse voltage is V_m and for full wave rectifier PIV is $2V_m$. In full wave circuit, independent of the filter used, the PIV across each diode is twice the maximum transformer voltage (center tap to other end).

Peak inverse voltage is the maximum voltage across the non-conducting diode of rectifier, the voltage must be less than the break down voltage of the diode otherwise the diode will be destroyed.

PIV of half wave rectifier with capacitor input filter PIV = $2V_m$.

Bridge rectifier with capacitor input filter, PIV = V_m

Bridge rectifier

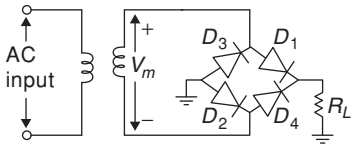


Figure 16 A Bridge rectifier

The bridge rectifier is similar to a full wave rectifier because it produces a full wave output voltage, diodes D_1 and D_2 conduct on the positive half cycle and D_3 and D_4 conduct on the negative half cycle. As result rectified output flows during both half cycles.

Features

(i) The current drawn in both the primary and the secondary of the supply transformer are sinusoidal, and therefore a smaller transformer may be used than for the full wave circuit for the same input.

(ii) A transformer without center tap is used.

The advantage of this type of full wave rectification over the center tapped version in the conventional full wave rectifier, is entire secondary voltage of transformer can be used DC output $(V) = \frac{2V_m}{\pi}$

$$I_{DC} = \frac{2I_m}{\pi}$$

Current through each diode $= I_{diode} = \frac{I_{DC}}{2}$

Example 3: In a bridge rectifier, the input AC source has an amplitude of 5 V, and the load resistance is 100 Ω. The diodes have an on resistance of 10 Ω, V_r = 0.6 V. The DC output voltage is ___

(A) 1.69 V (B) 1.81 V

(C) 2.35 V (D) 2.38 V

Solution: (A)

$$V_{DC} = 2\left[\frac{V_m R_L}{\pi(R_F + R_L)} - V_r\right]$$

$$\cong 2\left[\frac{5}{\pi}\frac{100}{(100+10)} - 0.6\right] = 1.69$$

Table 1 *Unfiltered rectifiers*

	Half Wave	Full Wave	Bridge
Number of diodes	1	2	4
Peak output ideal	V_m	$\frac{V_m}{2}$ (center tapped)	V_m
Peak output (practical)	$V_m - 0.7$ V	$\frac{V_m}{2} - 0.7$ V	$V_m - 1.4$ V
DC output	$V_p(\text{out})/\pi$	$2\ V_p(\text{out})/\pi$	$\frac{2\ V_P(\text{out})}{\pi}$
Ripple frequency	f_{in}	$2f_{in}$	$2f_{in}$

V_m – peak secondary voltage

V_p(out) – peak output voltage

Voltage multiplier

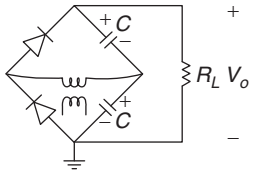
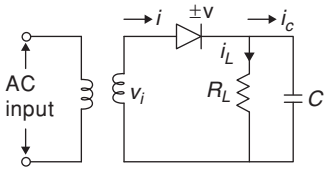


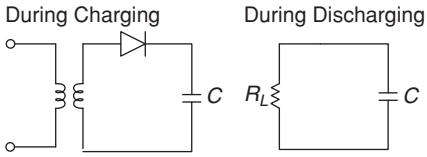
Figure 17 Bridge rectifier as voltage doubler

A common voltage doubling circuit which delivers a dc voltage approximately, equal to twice the transformer maximum voltage at no load. The circuit is operated by alternately charging each of two capacitors, to the transformer peak voltage V_m, current being continuously drained from capacitors through load. The capacitors also act to smooth out the ripple in the output.

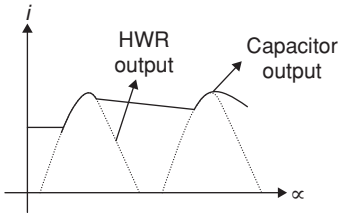
Capacitor filters



Capacitor stores the energy during the conduction period and delivers this energy to the load during non-conducting.



Diode having less forward resistance, and capacitor charges very fastly and the value of R_L is designed to be so high to get the small ripple at the output



Full wave rectifier with a capacitor filter is obtained by placing a capacitor 'C' across R_L

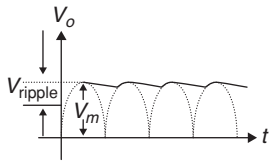


Figure 18 Approximate load voltage waveform for full wave rectifier with capacitive filter

The average value of the voltage is approximately

$$V_{DC} = V_m - \frac{V_{ripple}}{2}$$

$$V_{ripple} = \frac{I_{DC}}{2fC} \text{ (for full wave rectifier)}$$

$$V_{DC} = V_m - \frac{I_{DC}}{4fC}$$

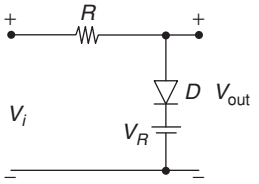
Clipping Circuits: Voltage Slicers

(i) An electronic circuit which cuts the given input wave form

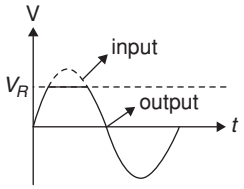
(ii) The amount of clipping is decided by the supply voltage by which the circuit is driven

Considering ideal diode

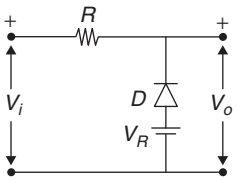
(i)

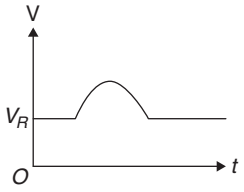


When $V_i > V_R$: Diode is forward biased $\Rightarrow V_o = V_R$
$V_i < V_R$: Diode is reverse biased $\Rightarrow V_o = V_i$



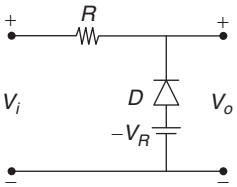
(ii)



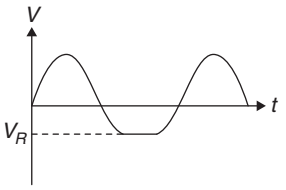


When $V_i > V_R$: Diode is reverse biased $\Rightarrow V_o = V_i$
$V_i < V_R$: Diode is forward biased $\Rightarrow V_o = V_R$

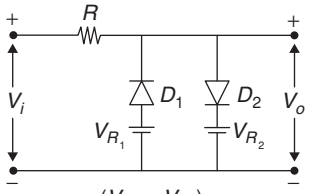
(iii)



When $V_i > (-V_R)$: Diode is reverse biased $\Rightarrow V_o = V_i$
$V_i < (-V_R)$: Diode is forward biased $\Rightarrow V_o = V_R$

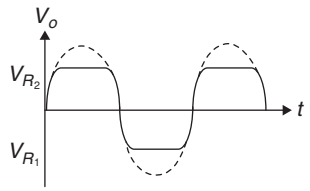


Clipping at two independent levels

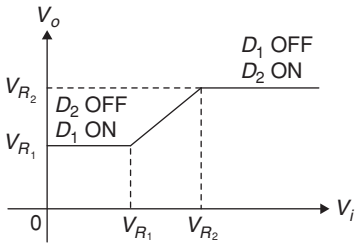


$(V_{R_2} > V_{R_1})$

Input	Diode state	V_0
$V_i \le V_{R_1}$	D_1 ON, D_2 OFF	V_{R_1}
$V_{R_1} \le V_i < V_{R_2}$	D_1 OFF, D_2 OFF	V_i
$V_i \ge V_{R_2}$	D_1 OFF, D_2 ON	V_{R_2}



Transfer Characteristics:



Clipping circuits are used to select for transmission, that part of a waveform which lies above or below same reference level.

Considering practical diode

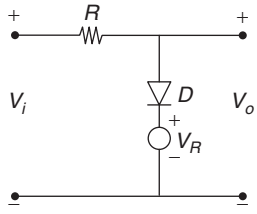


Figure 19 Diode clipping circuit

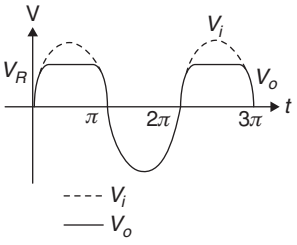


Figure 20 Input and output voltage waveforms

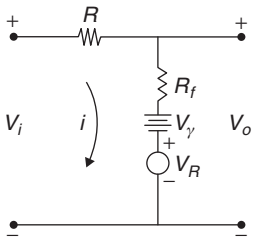


Figure 21 Model for circuit in forward bias $V_{in} > V_R + V_\gamma$

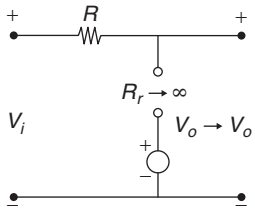


Figure 22 Model for circuit in reverse bias $V_{in} < V_R + V_\gamma$

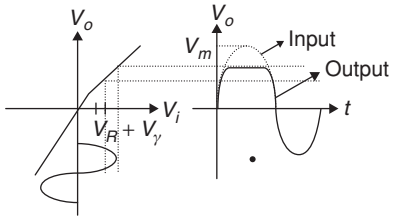
The diode will be ON during forward bias, as shown in Figure 21 by applying KVL

$$i = \frac{V_i - V_R - V_\gamma}{R_f + R}$$

$$V_0 = iR_f + V_\gamma + V_R = \frac{R_f}{R_f + R} V_i + \frac{R}{R + R_f}(V_R + V_\gamma)$$

The translation from OFF to ON occurs when V_i equals to $V_\gamma + V_R$.

This transition indicates abrupt change in slope in the plot of V_0 versus V_i, called transfer characteristic. The slope is unity, when diode is off, as $V_i = V_0$ (*see* Figure 22)



Transfer characteristic of clipping circuit

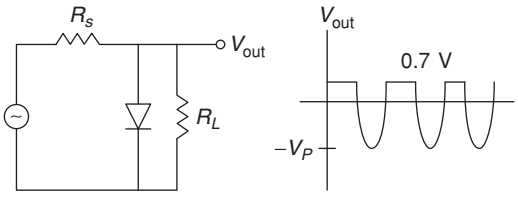


Figure 23 Positive clipper

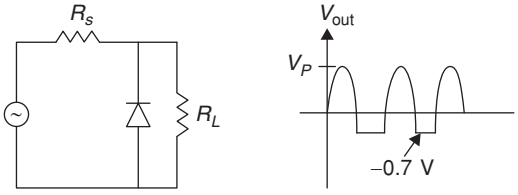


Figure 24 Negative clipper

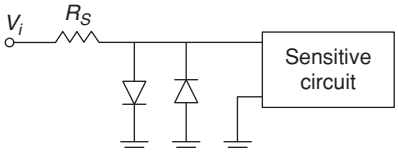


Figure 25 Limiter or diode clamp

If the input signal is so small (say 20 mV), none of the diode will be turned ON, so these small voltages will be not effected, but for higher voltages than ±0.7 V one of the diode will be forward biased and the input to sensitive circuit will never cross ±0.7 V.

A limiter on the input side of an op-amp will prevent excessive input voltage from being accidently applied.

Biased clippers

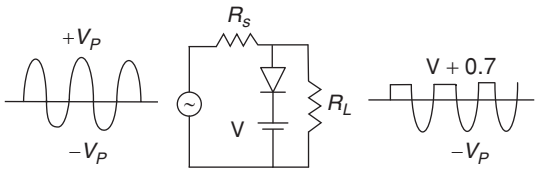


Figure 26 Positive clipper

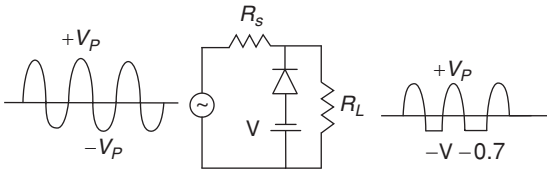


Figure 27 Negetive clipper

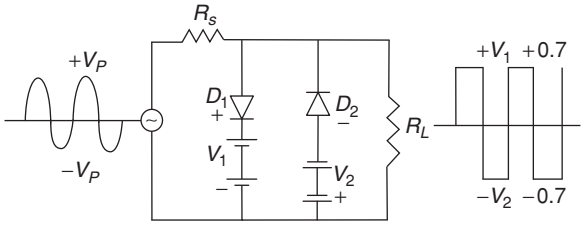


Figure 28 Biased positive-negetive clipper

In the above combination clipper diode D_1 clips off positive parts above positive bias level, and diode D_2 clips off parts below the negative bias level. If input voltage is very large compared to the bias levels, the output signal is a square wave

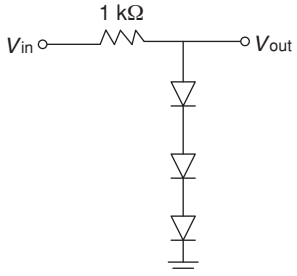


Figure 29 Clipper with three offset voltages

Since each diode has an offset of around 0.7 V, three diodes produce a clipping level of +2.1 V, we can use the same circuit (*see* Figure 29) as a diode clamp to protect sensitive circuit, that cannot tolerate more than 2.1 V input

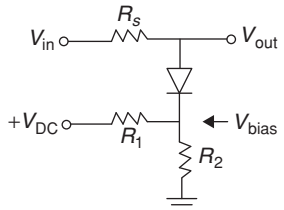


Figure 30 Voltage divider biases clipper

This is another way to bias a clipper without batteries. The voltage divider R_1 and R_2 to set the bias level, given by $V_{bias} = \frac{R_2}{R_1 + R_2} V_{DC}$.

In this case the output voltage is clipped or limited when the input is greater than $V_{bias} + 0.7$ V

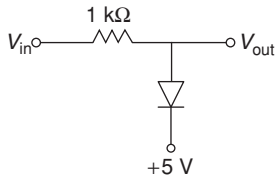


Figure 31 Diode clamp protects above 5.7 V

The circuit in Figure 31, it can be used to protect, sensitive circuits from excessive input voltages,

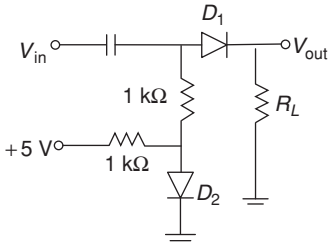


Figure 32 Circuit with zero off set

The circuit shown in Figure 32 is used to remove offset of limiting diode D_1.

The diode D_2 is biased slightly into forward conduction, so it has 0.7 V across it, this 0.7 V is applied to D_1 and R_L via 1 kΩ. This means that diode D_1 is on the verge of conduction. Therefore when a signal carried in, diode D_1 conducts near 0 V.

Clamper Circuits

When a positive clamper has a sine wave input, it adds a positive dc voltage, to the sine wave,

Stiff clamper: $R_L C > 100\ T$, T–time period.

The positive clamper shifts the AC reference level upto a DC level.

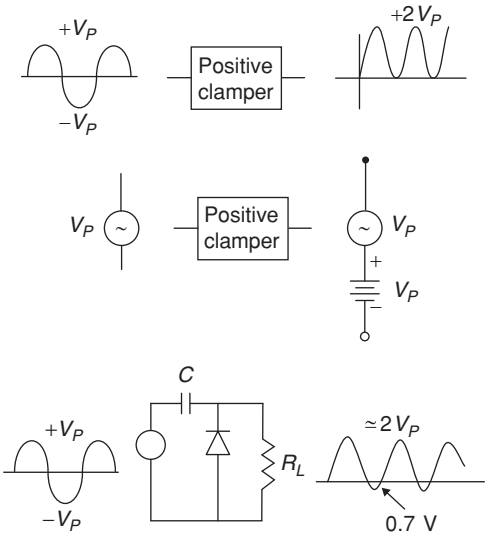


Figure 33 Practical clamper

The first quarter of the cycle charges the capacitor fully, then the capacitor retains almost all of its charge during subsequent cycles. The small charge that is lost between cycles, is replaced by diode conduction.

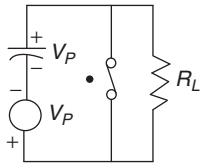


Figure 34 At positive peak

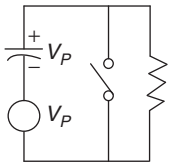


Figure 35 Beyond positive peak capacitor acts as battery

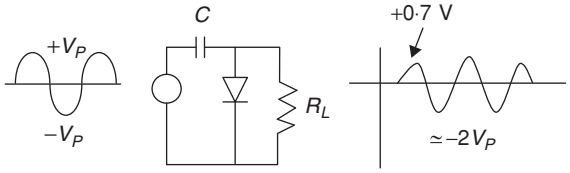


Figure 36 Negative clamper

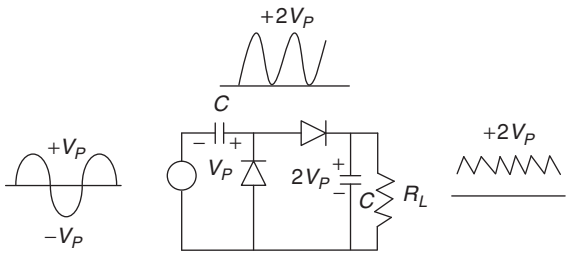


Figure 37 Peak to peak detector

A half wave rectifier with a capacitor input filter produces a DC output voltage approximately equal to the peak of input signal.

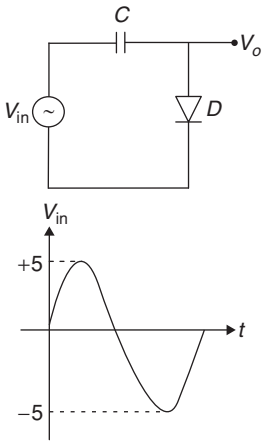
If we cascade a clamper and a peak detector, we will get a peak to peak detector.

The RC time constant should be much greater than the period of the signal (T) to get good clamping.

Steps to draw clamper output

(i) Find the voltage across the capacitor at input peak state.
(ii) Replace the diode with open circuit, and draw the output waveform.

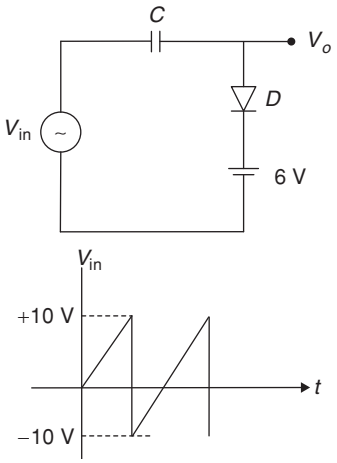
Example 4: Find the output of clamper circuit in the given figure



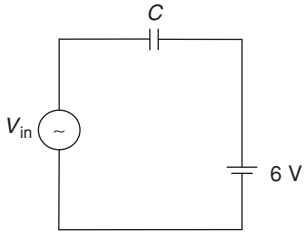
Solution: Under peak state V_c = +5 V KVL at input side gives $V_{in} - V_c - V_o = 0$

$$\boxed{V_o = V_{in} - 5}$$

Example 5: Find the output of clamper circuit in the given figure



Solution: At peak state condition, diode forward biased



$\therefore V_c$ (saturation) = 4 V

KVL at input gives, after replacing diode with short circuit

$V_o = V_{in} - 4$

Voltage Multipliers

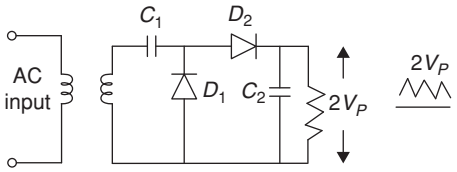


Figure 38 Voltage doubler with floating loads

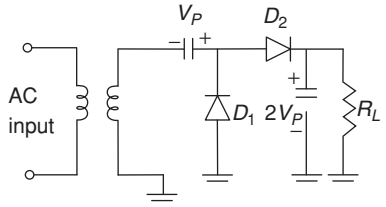


Figure 39 Voltage doubler with grounded loads

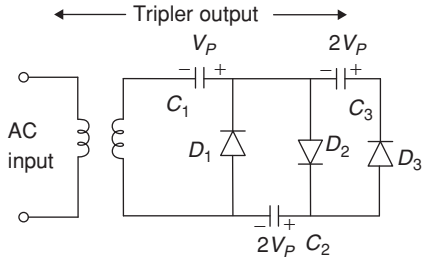


Figure 40 Voltage tripler with floating load

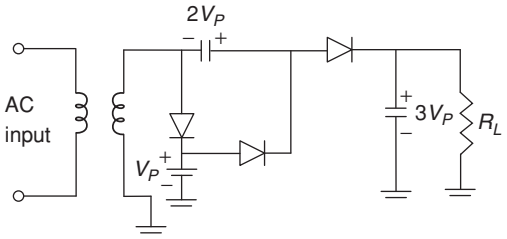


Figure 41 Voltage tripler with grounded load

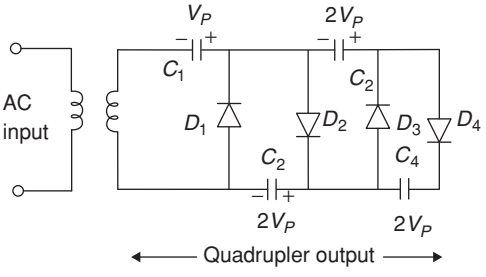


Figure 42 Voltage quadrupler with floating loads

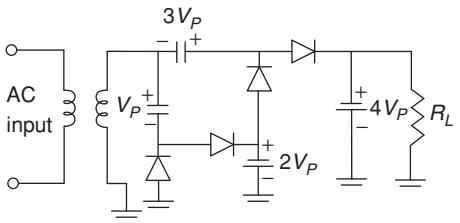


Figure 43 Voltage quadrupler with grounded load

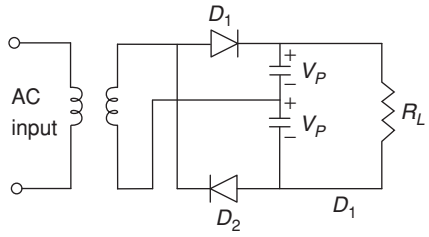


Figure 44 Voltage multiplier–full wave doubler

Voltage doubler is a redesign of the peak to peak detector, it uses rectifier diodes instead of small-signal diodes. It produces an output equal to the peak value of the rectified signal.

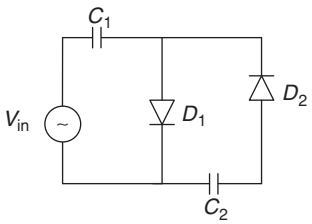
In voltage tripler, the first two sections act like a doubler, at peak of the negative half cycle, D_3 is forward biased. This charges C_2 to $2V_p$ with the polarity shown in figure 40. The tripler output appears across C_1 and C_3.

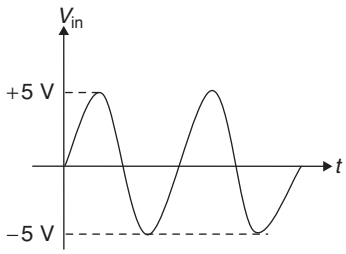
Voltage quadrupler with four sections in cascade, the first three sections are a tripler, and the fourth makes the overall circuit quadrupler. The quadrupler output is across the series connections C_2 and C_4.

Figure 44 shows a full wave voltage doubler, on the positive half cycle of the source, the upper capacitor charges to the peak voltage with the polarity shown, on the peak next half cycle, lower capacitor charges to the voltage, with indicated polarity, final output voltage is approximately $2V_p$.

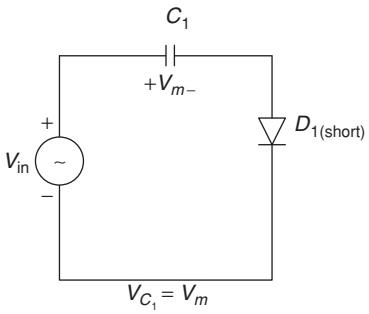
The voltage multipliers shown in figure 38–43 are half wave designs, i.e., the output ripple frequency is same as input frequency, but full wave voltage doubler, output ripple is double the input frequency. So its easy to filter the ripple and PIV is also V_p only.

Voltage doubler analysis

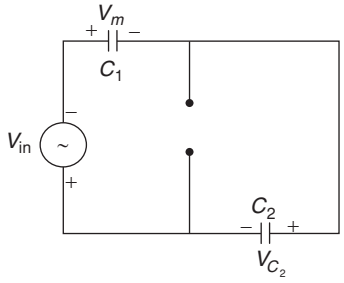




During positive half cycle D_1 is ON and hence 'C_1' is charged to $+V_m$

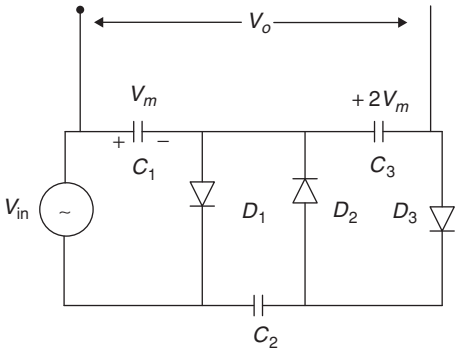


During negative cycle:



$$-V_{in} - V_m - V_{c_2} = 0$$
$$V_{c_2} = -V_{in} - V_m$$
$$V_{c_2} = -2V_m$$

Voltage tripler analysis



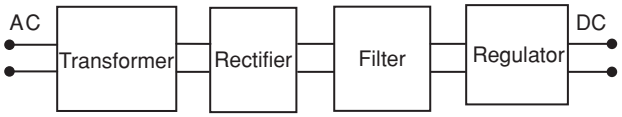
During first positive half cycle D_1 is ON. During next negative half cycle D_2 alone ON

$$\therefore\ V_o = V_{c1} + V_3 = 3V_m$$

During second positive half cycle D_1 and D_3 ON

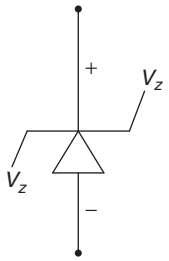
Power Supplies

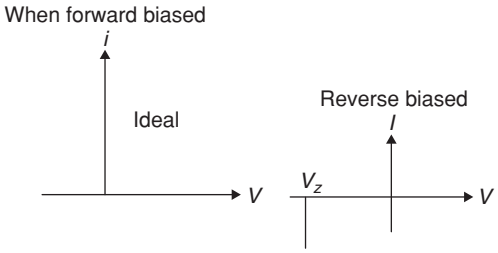
Transformers are used for step-up or step down the level of input ac voltage.



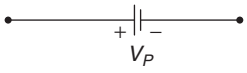
Voltage Regulators

Zener diode can be used as a voltage regulator.

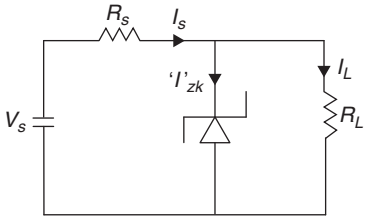




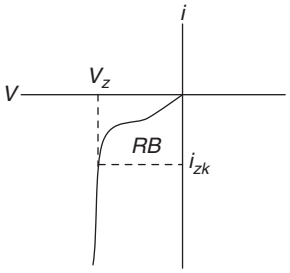
Zener reverse ON condition:



Zener voltage regulator



I-V Characteristics:



When source voltage varying:

$$I_s\,(\min) = I_z\,(\min) + I_L\,(\text{fix})$$
$$I_s\,(\max) = I_z\,(\max) + I_L\,(\text{fix})$$

When load is varying:

$$I_s\,(\text{fix}) = I_z\,(\min) + I_z\,(\max)$$
$$I_s\,(\text{fix}) = I_z\,(\max) + I_z\,(\min)$$

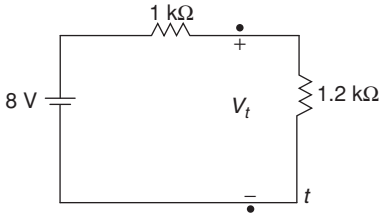
Zener diode withstand the current variations due to variations in source and load resistance when it is in reverse 'ON' condition and hence produces a constant output voltage across the Load.

Steps to determine the state of Zener Diode

(i) Replace the Zener diode with open circuit.
(ii) Find the terminal voltage across the diode from '*n*' to '*p*' side let say V_t.
(iii) If $V_t > V_z$ then Zener is reverse ON and diode is replaced with a voltage source of V_z.
(iv) If $V_t < V_z$, then Zener is OFF and hence replace it with open circuit.

Example 6: Check the status of Zener diode when
(i) $R_L = 1.2\ \text{k}\Omega$
(ii) $R_L = 3.6\ \text{k}\Omega$
[Given $V_z = 6$ V and $V_s = 8$ V, $R_s = 1\ \text{k}\Omega$ determine the output voltage]

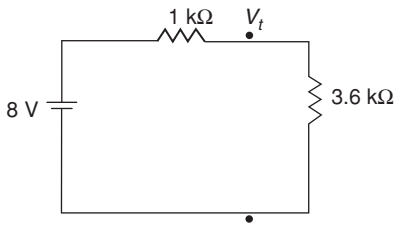
Solution: Case (1):
$R_L = 1.2\ \text{k}\Omega$



$$V_t = \frac{8(1.2)}{2.2} = 4.3\ \text{V}$$

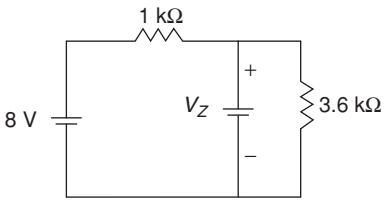
$V_t < V_z$, hence Zener diode is in OFF condition.
$\therefore\ V_o = 4.3$ V

Case (2): $R_L = 3.6\ \text{k}\Omega$



$$V_t = \frac{8(3.6)}{4.6} = 6.26087$$

$V_t < V_z$, hence Zener is in 'ON' condition.



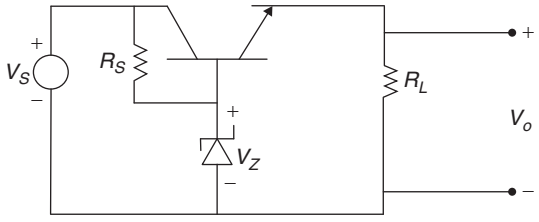
$\therefore\ V_o = V_z = 6\ \text{V}$

Note: The conventional direction of Zenar voltage is considered to be opposite to the direction of ordinary PN junction diode, because PN junction is used in FB direction, and Zenar diode is used in RB direction.

Characteristics of a ideal voltage regulator

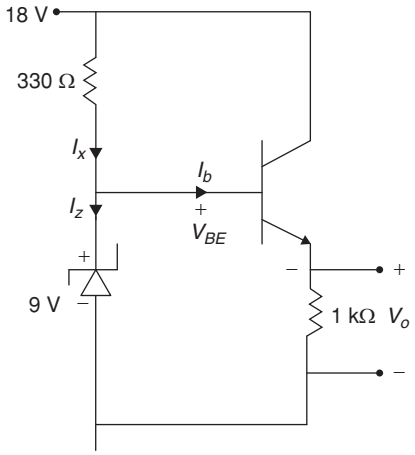
(i) It should maintain a constant output voltage irrespective of variations in input and the load.
(ii) Load current should depend on R_L.

Voltage regulator using BJT



as 'V_s' changes $\Rightarrow V_z$ constant
$\Rightarrow I_b$ is constant
$I_E = (\beta + 1)\, I_b$
Hence I_E = constant
$\therefore\ V_o = I_E R_L$ = constant irrespective of variations in the input circuit.

Example 7:



Find current I_z if $\beta = 99$ and $V_{BE} = 0.7$ V
(A) 0.0264 A (B) 0.352 A
(C) 0.476 A (D) 0.0123 A

Solution: (A)
KVL at input side gives $-9 + 0.7 + V_L = 0$
KCL $\Rightarrow I_x = I_b + I_z$
$I_z = I_x - I_b$

$$I_x = \frac{18 - 9}{330\ \Omega} = 0.02727\ \text{A}$$

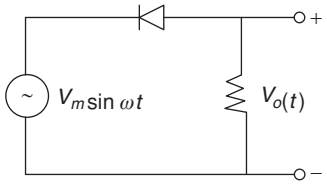
$$I_b = \frac{1}{100} \cdot \frac{V_L}{1\ \text{k}\Omega} = \frac{8.3}{100\ \text{k}\Omega} = 0.083\ \text{mA}$$

$\therefore\ I_z = 0.0264\ \text{A}$

Solved Examples

Directions for questions 1 to 8: Select the correct alternative from the given choices.

Example 1: The average value of $V_o(t)$ will be



(A) 0 (B) $\dfrac{-V_m}{\pi}$

(C) $\dfrac{-V_m}{\sqrt{2}}$ (D) $-V_m$

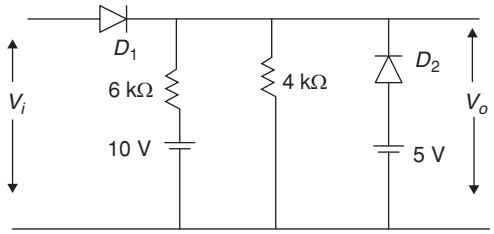
Solution: (B)

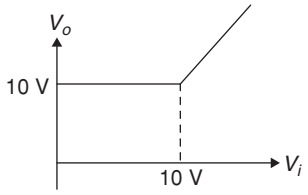
$$V_{av} = \frac{\int_{\pi}^{2\pi} V_m \sin \omega t d\omega t}{2\pi}$$

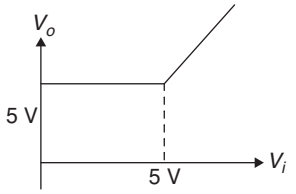
$$= \frac{V_m\left[-\cos\omega t\right]_{\pi}^{2\pi}}{2\pi} = \frac{-V_m}{\pi}$$

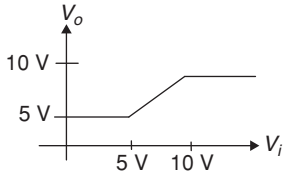
The diode will be ON for π to 2π.

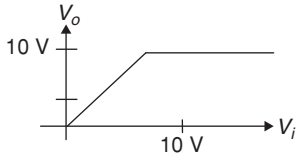
Example 2: If D_1, D_2 diode are ideal, which of the following represents transfer characteristics of circuit:



(A)


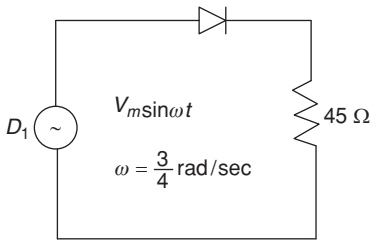
(B)


(C)


(D)


Solution: (B)
For $0 \le V_i \le 5$ V, D_2 is ON and $V_o = 5$ V
For $5 \le V_i \le 10$ V, D_2 is off and $V_o = V_i$
For $V_i > 10$ V, D_1 is ON, D_2 is OFF
$V_o = V_i$

Example 3: The forward resistance of the diode shown in circuit is 5 Ω, and the other parameters are the same as those of an ideal diode. Then what is the DC component of the source current?



(A) $\frac{-V_m}{50\pi}$ (B) $\frac{V_m}{45\pi\sqrt{2}}$

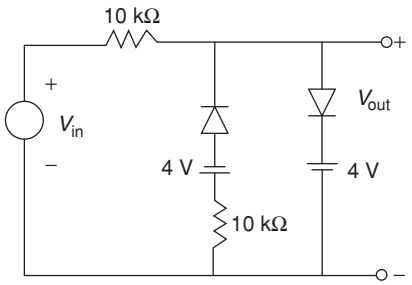
(C) $\frac{V_m}{100\pi\sqrt{2}}$ (D) $\frac{2V_m}{45}$

Solution: (A)

$$I_{DC} = \frac{\int_0^{\pi} V_m \sin\omega t \cdot d\omega t}{2\pi \cdot R} = \frac{\left|V_m \cdot \cos\omega t\right|_0^{\pi}}{2\pi \cdot R} = \frac{V_m}{50\pi}$$

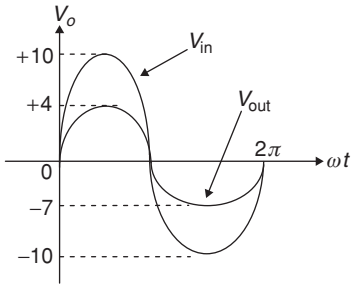
$R = R_{f(\text{diode})} + 45\ \Omega = 5 + 45\ \Omega = 50\ \Omega$

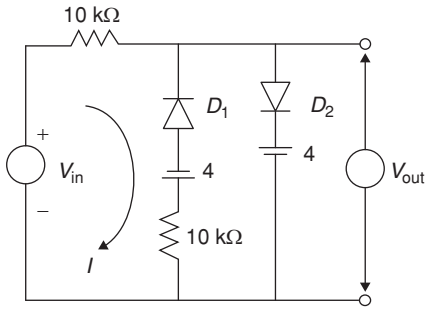
Example 4: A voltage signal 10 sinωt is applied to the circuit with ideal diode, as shown, The maximum and minimum. values of the output waveform V_{out} of the circuit are respectively



(A) +10 V and −10 V (B) +4 V and −4 V
(C) +7 V and −4 V (D) +4 V and −7 V

Solution: (D)
The voltage wavefrom at the output will be as shown below for +ve cycle





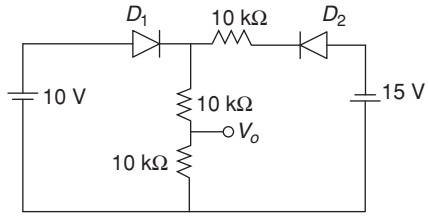
D_2 ON, D_1 OFF, $V_{out} = 4$ V,
for −ve cycle D_1 ON, D_2 OFF

$$I = \frac{V_{in} + 4}{20}$$

$$= \frac{-10 + 4}{20} = \frac{-3}{10} \text{mA}$$

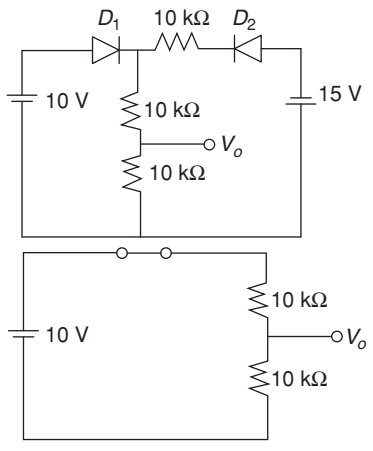
$V_{out} = I \cdot R - 4 = -7$ V

Example 5: Assuming that the diode in the given circuit is ideal, the voltage V_o is



(A) 4 V
(B) 5 V
(C) 7.5 V
(D) 12.1 V

Solution: (B)

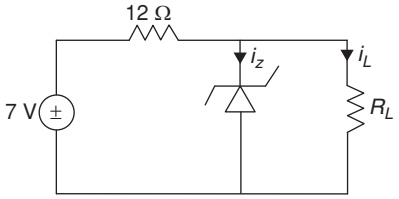


In this case the diode D_2 is always reverse biased

$$V_o = \frac{10K}{10K + 10K} \times 10 = 5 \text{ V}$$

Common Data for Questions 6 to 8:

Example 6:



In the voltage regulator circuit shown, the Zener diode current is to be limited to the range $5 \le i_z \le 100$ mA

$$V_z = 4 \text{ V}, R_z = 0\ \Omega$$

The range of posible load current is

(A) $5 \le i_L \le 130$ mA
(B) $25 \le i_L \le 250$ mA
(C) $10 \le i_L \le 110$ mA
(D) $150 \le i_L \le 245$ mA

Solution: (D)

Current through 12 Ω resistor $i = \frac{7-4}{12} = 250$ mA

$i_L = i - i_z$, and i_z is in range of 5 to 100 mA
So i_L will be 150 mA $\le i_L \le$ 245 mA.

Example 7: Range of possible load resistance is

(A) $60 \le R_L \le 375\ \Omega$
(B) $20 \le R_L \le 60\ \Omega$
(C) $40.1 \le R_L \le 92.1\ \Omega$
(D) $16.3 \le R_L \le 26.6\ \Omega$

Solution: (D)
$i_L R_L = 4$ V, 150 mA $\le i_L \le$ 245 mA
So R_L will be 16.3 Ω $\le R_L$ Ω 26.67

Example 8: The power rating required for load resistance is

(A) 576 mW
(B) 480 mW
(C) 360 mW
(D) 980 mW

Solution: (D)
The power rating required for output resistance is $i_L = 245$ mA,
$V_Z = 4$ V, $i_L V_Z = 0.98$ W

EXERCISES

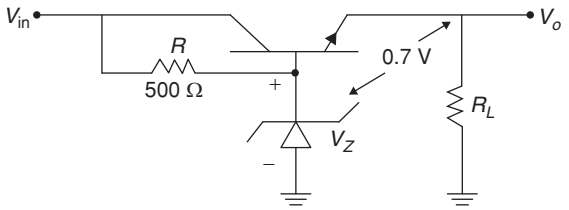
Practice Problems 1

Directions for questions 1 to 28: Select the correct alternative from the given choices.

1. A power supply having an output resistance 1.5 Ω supplies full load current of 500 mA to a 50 Ω load. What is the % voltage regulation of supply?
(A) 2% (B) 3%
(C) 6% (D) 1.5%

2. What is the no-load output voltage of the supply in above problem?
(A) 25 V (B) 24.25 V
(C) 25.75 V (D) 25.25 V

3. A transistor series voltage regulator is shown below. The input voltage V_{in} can carry from 18 V to 30 V. If $V_Z = 10$ V, $R_L = 1$ kΩ, β-very high



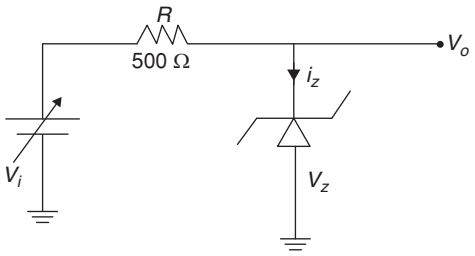
The minimum collector to emitter voltage (V_{CE}) of the transistor is,
(A) 8.7 V (B) 20.7 V
(C) 7.3 V (D) 19.3 V

4. The minimum power dissipated in the transistor is
(A) 0.192 W (B) 2.1 W
(C) 80.9 mW (D) None

5. The max current supplied by V_{in} is
(A) 40 mA (B) 9.3 mA
(C) 24.5 mA (D) 49.3 mA

6. A FWR is operated form 230 V, 50 Hz line has a capacitor filter across its output. Calculate the minimum value of capacitance required if the load is 1 kΩ and the ripple must not exceed 5%
(A) 57.8 μF (B) 10 μF
(C) 28.6 μF (D) 5.7 μF

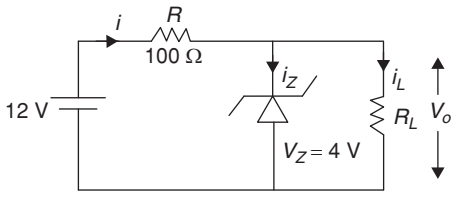
7. A Zener diode voltage regulator is given below with following specifications $I_{Z_{min}} = 4$ mA, $I_{z_{max}} = 24$ mA, $V_z = 4.8$ V, and $R_Z = 10$ Ω



Calculate minimum, and maximum input voltage that can be regulated by Zener diode
(A) 9.8 V and 25.04 V (B) 6.84 V and 25.04 V
(C) 9.8 V and 17.04 V (D) 6.84 V and 17.04 V

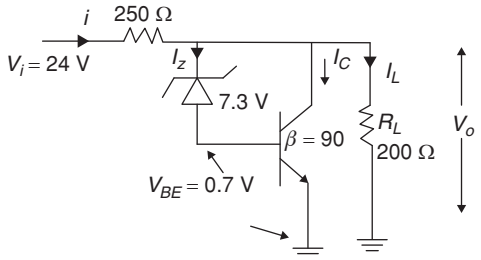
8. The line regulation is
(A) 1.3% (B) 2.5%
(C) 1.96% (D) 0%

9. In the circuit shown below, the knee current of ideal Zener diode is 8 mA. To maintain 4 V across R_L, the minimum value of R_L and minimum power rating of Zener diode in mW respectively are



(A) 55 Ω and 320 mW (B) 55 Ω and 288 mW
(C) 50 Ω and 320 mW (D) 50 Ω and 288 mW

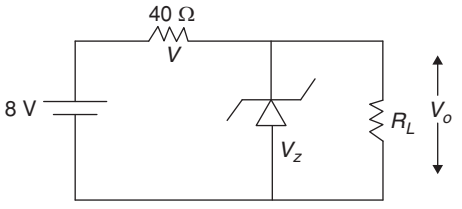
10. A transistor shunt voltage regulator is given below



The collector current i_c is
(A) 10 mA (B) 24 mA
(C) 64 mA (D) 40 mA

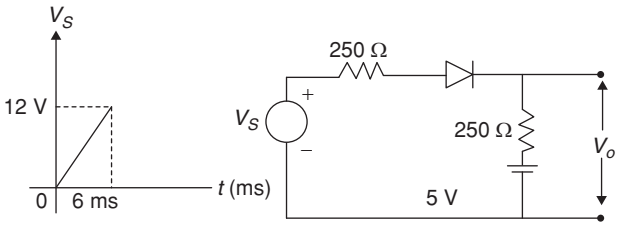
11. The power dissipation in Zener diode is
(A) 1.89 mW (B) 2.5 W
(C) 3.1 mW (D) None of these

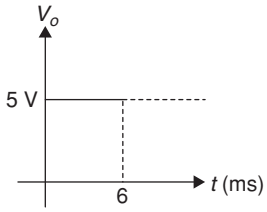
12. A Zener diode in the circuit shown below, has a knee current of 4 mA and power dissipation of 200 mW. What are the minimum and maximum load currents that can be drawn safely from the circuit, keeping the output $V_o = 5$ V.

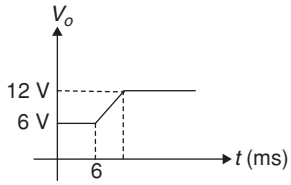


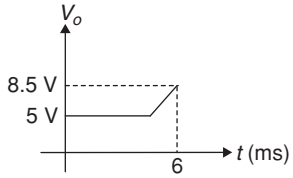
(A) 0 mA and 200 mA (B) 4 mA and 11 mA
(C) 35 mA and 71 mA (D) 35 mA and 200 mA

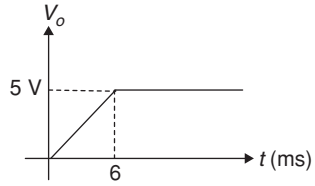
13. Sketch the output V_o of the circuit given for $0 \le t \le 6$ ms, assume the diode is ideal



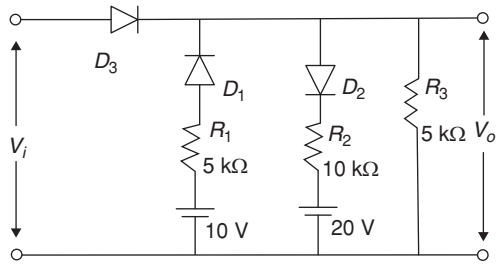
(A)


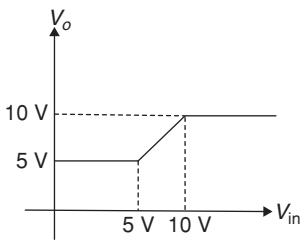
(B)


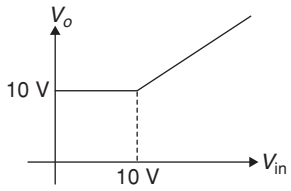
(C)


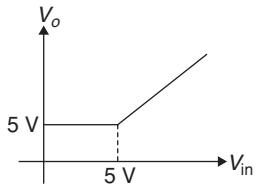
(D)


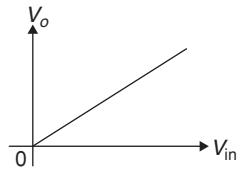
14. Draw the transfer characteristic (V_o V_s V_i) for the circuit given below.



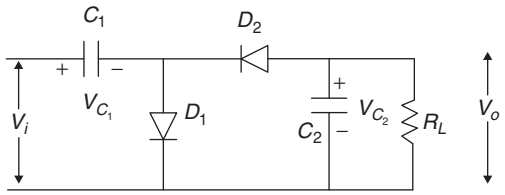
(A)


(B)


(C)


(D)


15. A voltage doubler is given below.



Where $V_i = 6 \sin \omega t$ V

The voltage across C_1 and C_2 are

(A) 6 V and 6 V (B) 6 V and –12 V

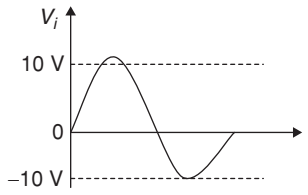
(C) 6 V and 12 V (D) –6 V and –12 V

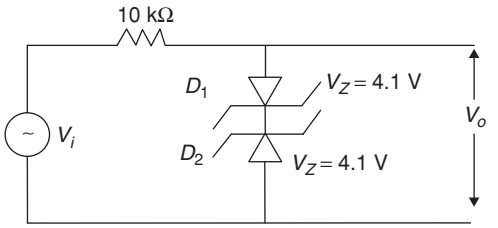
16. The PIV of D_1 and D_2 are

(A) 6 V and 6 V (B) 6 V and 12 V

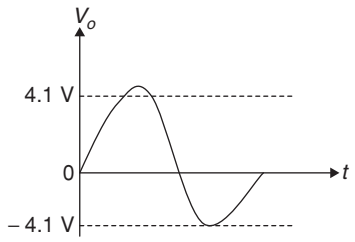
(C) 12 V and 12 V (D) 12 V and 6 V

17. Draw the output V_o of the figures given.

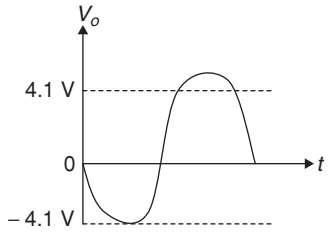




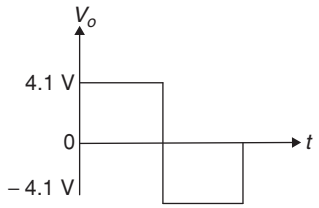
(A)



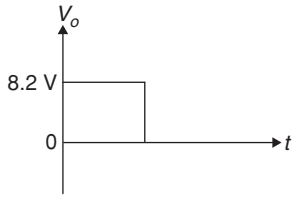
(B)



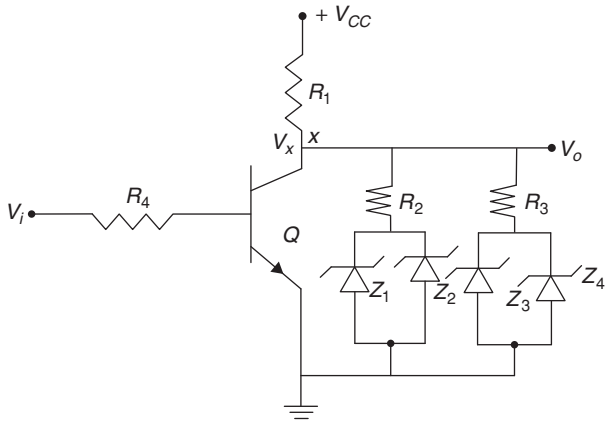
(C)



(D)

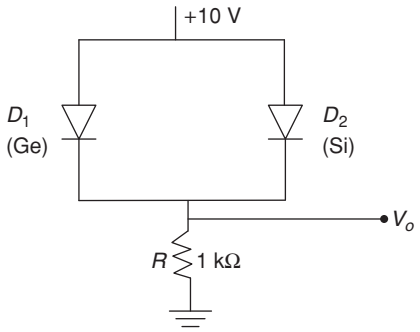


18. In the circuit given below, assume that V_{CC} = 12 V, Z_1, Z_2, Z_3 and Z_4 are identical Zener diodes with break down voltage of 4 V. $R_1 = R_4 = 4$ kΩ and $R_2 = R_3 = 8$ kΩ.
Calculate V_o when Q is OFF.



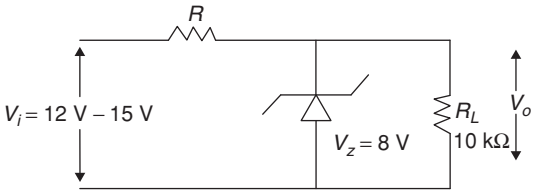
(A) 7.3 V (B) 8 V
(C) 10 V (D) 8.5 V

19. In the given circuit, two diodes made up of Ge (cut in voltage = 0.3 V) and Si (cut in voltage = 0.7 V) are used. What is the value of V_o.



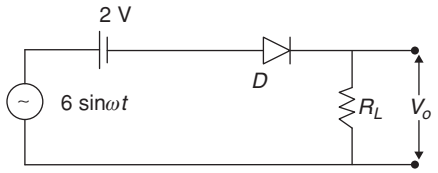
(A) 10 V (B) 9 V
(C) 9.7 V (D) 9.3 V

20. What is the minimum value of R to maintain $V_o = 8$ V constant and I_z is negligible.

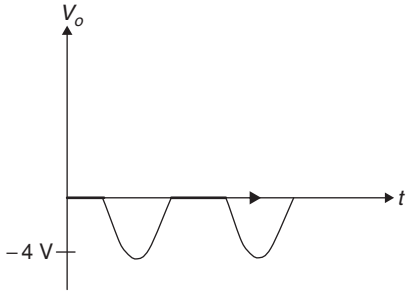


(A) $R \le 5$ kΩ (B) $R > 8.75$ kΩ
(C) 5 kΩ < R < 8.75 kΩ (D) None of these

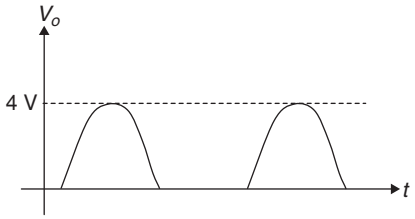
21. Draw the output V_o of the figure given below (assume diode is ideal)



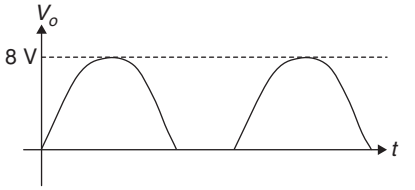
(A)



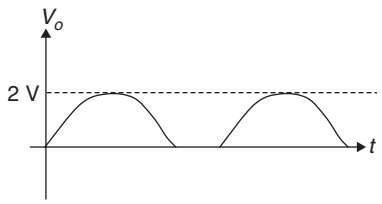
(B)



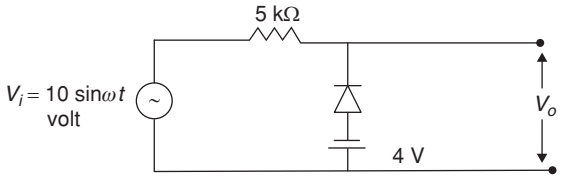
(C)



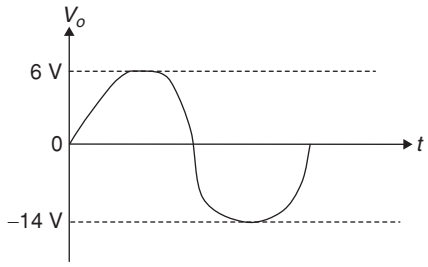
(D)



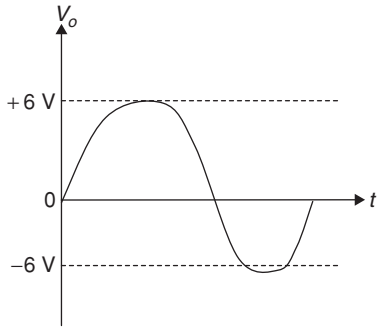
22. Draw the output V_o assume diode FB voltage drop is 0.6 V.



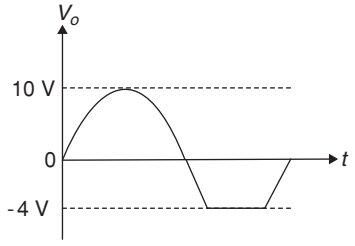
(A)



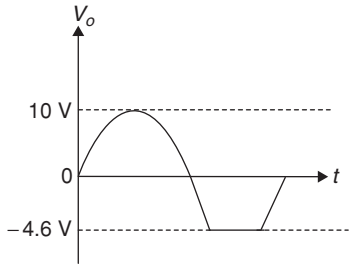
(B)



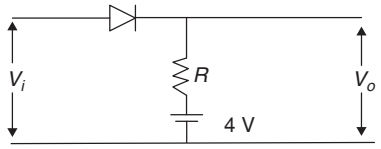
(C)



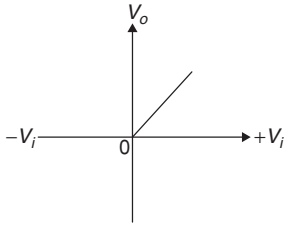
(D)



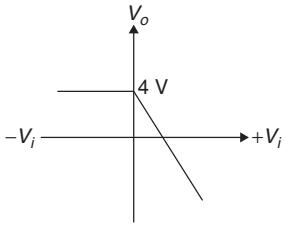
23. Draw the transfer characteristic of the figure given below (assume diode is ideal)



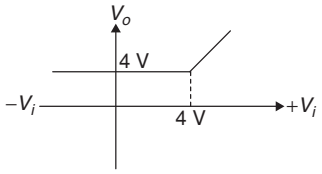
(A)



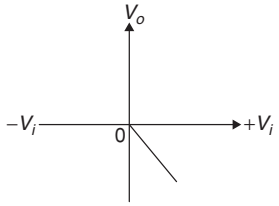
(B)



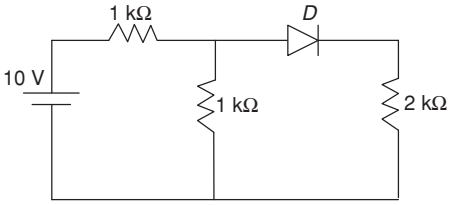
(C)



(D)

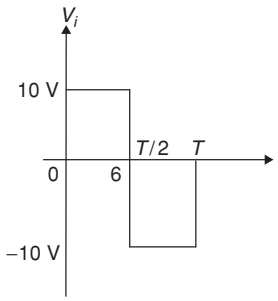


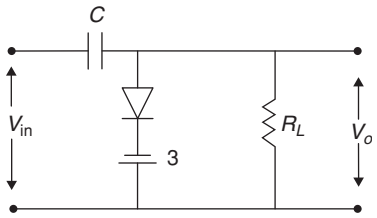
24. Calculate the voltage drop across 2 kΩ of the figure (assume the diode is ideal)



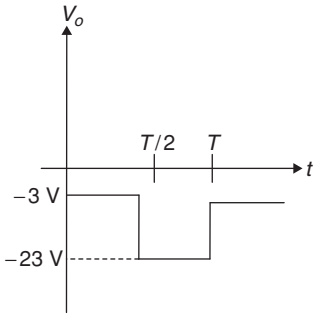
(A) 10 V (B) 2 V
(C) 4 V (D) 0 V

25. Sketch the output wave form for the circuit given.

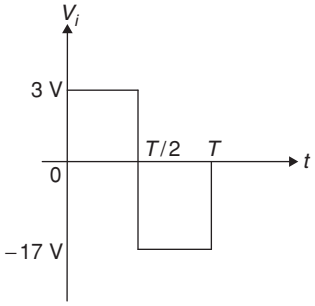




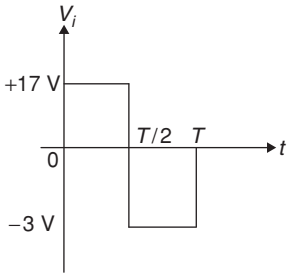
(A)



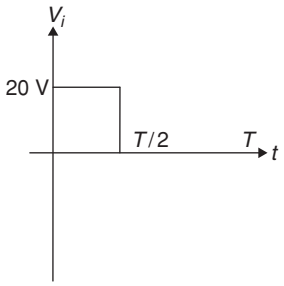
(B)



(C)



(D)



26. A HWR uses a transformer of turn ratio 2:1 and a load resistance of 500 Ω. If the rms primary voltage is 240 V. The DC output voltage is

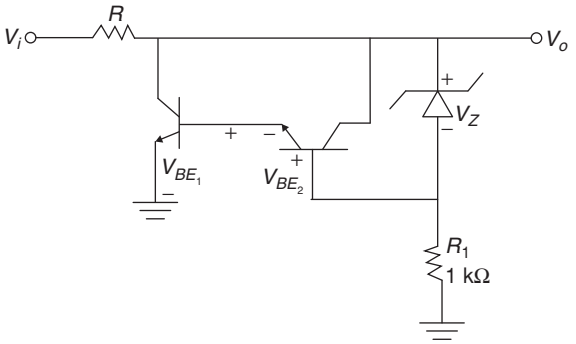
(A) 54 V (B) 38 V
(C) 76 V (D) None of these

27. The PIV of diode is

(A) 120 V (B) 240 V
(C) 360 V (D) 170 V

28. Find the output voltage of a shunt regulator given below with following specifications.

$V_{BE_1} = V_{BE_2} = 0.6$ V and $V_Z = 10$ V



(A) 10.6 V (B) 8.8 V
(C) 10 V (D) 11.2 V

Practice Problems 2

Directions for questions 1 to 26: Select the correct alternative from the given choices.

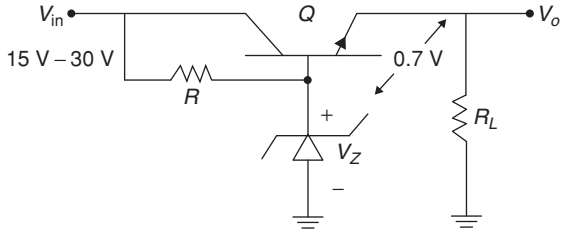
1. The DC output voltage is 45 V at full load and 47 V at no-load. The % regulation factor is

(A) 2.5% (B) 1%
(C) 4.25% (D) 4.44%

2. In a FWR, the rms output is 10 V. What is its DC output voltage.

(A) 10 V (B) 9 V
(C) 5 V (D) 6 V

3. A transistor series voltage regulator is given below.

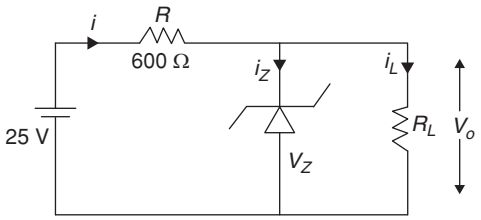


What break down voltage V_Z, should the Zener diode have if the load voltage is to be maintained at 12 V?

(A) 11.3 V (B) 12.7 V
(C) 12 V (D) None of these

4. If the Zener diode must conduct 12 mA to remain in the breakdown region, what minimum value should R have (in above problem)?
(A) 500 Ω (B) 1.4 kΩ
(C) 2 kΩ (D) 192 Ω

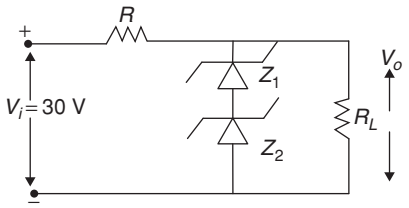
5. A Zener diode voltage regulator is given below with following specifications.
$V_Z = 4.1$ V, $I_{ZK} = 3$ mA, $I_{Z_{max}} = 28$ mA and $r_z = 0$



Calculate minimum and maximum load currents for which Zener diode will maintain regulation.
(A) 3 mA and 31.83 mA (B) 6.83 mA and 31.83 mA
(C) 6.83 mA and 28 mA (D) 3 mA and 28 mA

6. For above problem calculate the minimum value of R_L used
(A) 128 Ω (B) 85 Ω
(C) 600 Ω (D) 117 Ω

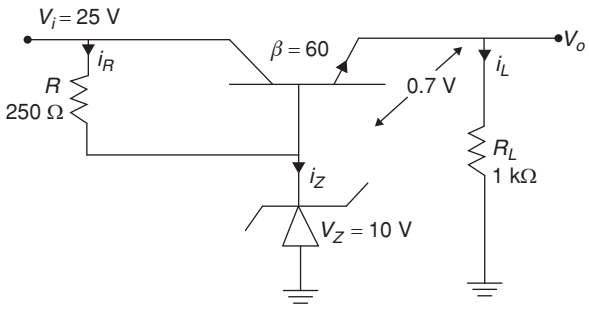
7. The circuit given below uses two Zener diodes, each is rated at 12 V, 100 mA and assume that load current is negligible.



The regulated output V_o is
(A) 24 V (B) 12 V
(C) 6 V (D) 0 V

8. For above problem the value of series resistance R is
(A) 180 Ω (B) 15 Ω
(C) 60 Ω (D) None of these

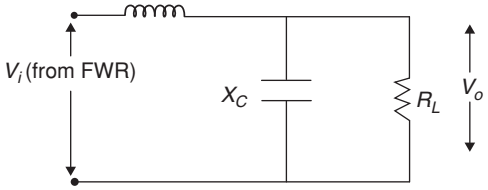
9. A transistor series voltage regulator is given below



The output voltage V_o is
(A) 9.3 V (B) 10.7 V
(C) 10 V (D) 0.7 V

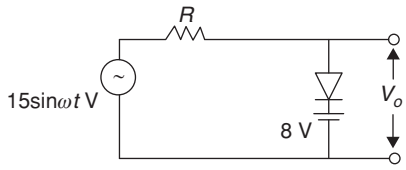
10. For above problem Zener current i_z is
(A) 60 mA (B) 0.15 mA
(C) 59.84 mA (D) 10 mA

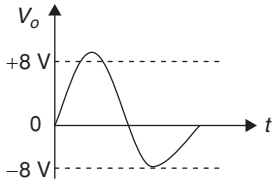
11. Calculate the ripple factor (γ) of L-section filter given below with $X_C = 10$ Ω, $X_L = 100$ Ω and $R_L = 1$ kΩ.

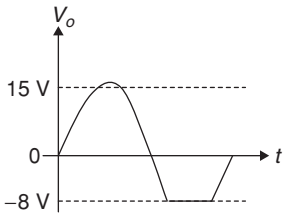


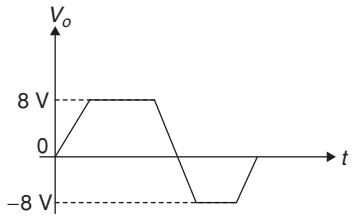
(A) 0.14 (B) 0.047
(C) 0.482 (D) 0.52

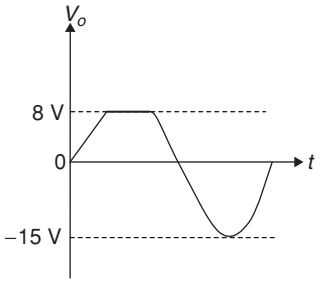
12. For the circuits given assume the diode is ideal sketch the output V_o.



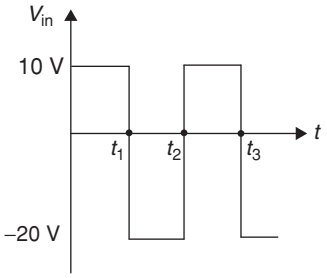
(A)


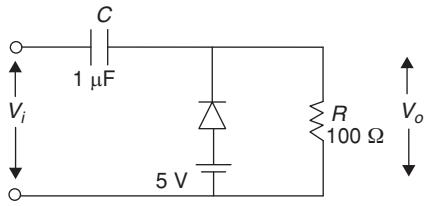
(B)


(C)


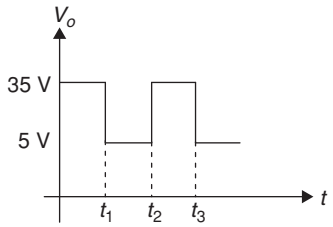
(D)


13. A DC restorer is shown below. Sketch the output V_o.

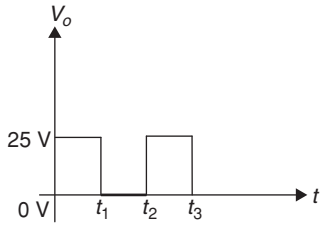




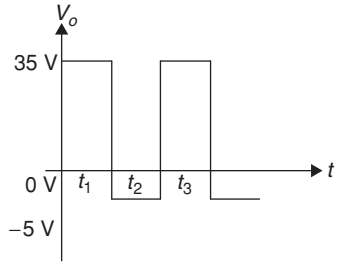
(A)



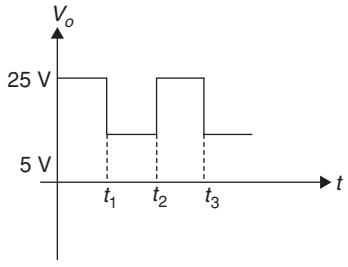
(B)



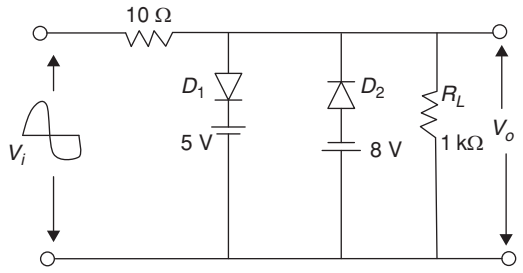
(C)



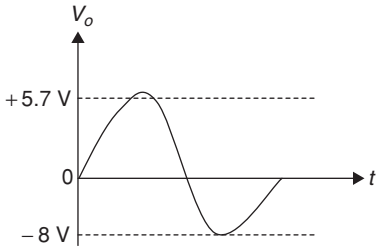
(D)



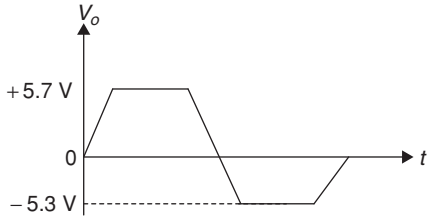
14. Draw the output V_o of the figure given. Assume forward voltage drop of each diode is 0.7 V.



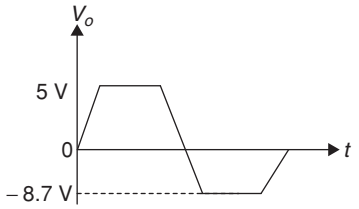
(A)



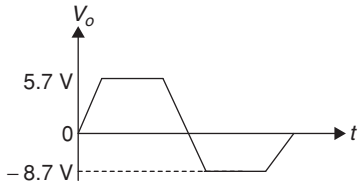
(B)



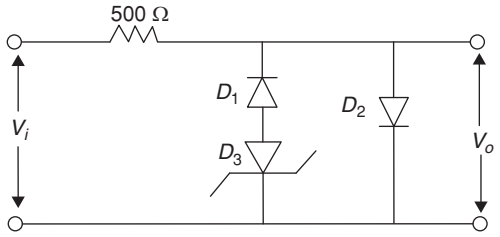
(C)



(D)

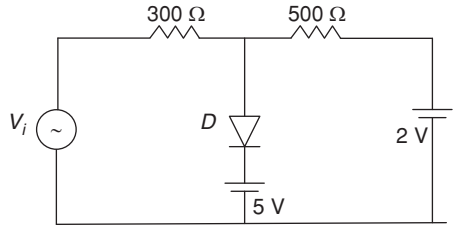


15. In the following limiter circuit, an input voltage $V_i = 12 \sin 100\pi t$ V is applied. Assume that the diode forward voltage drop is 0.6 V and Zener breakdown voltage is 4.2 V. What is the minimum and maximum value of output voltage?



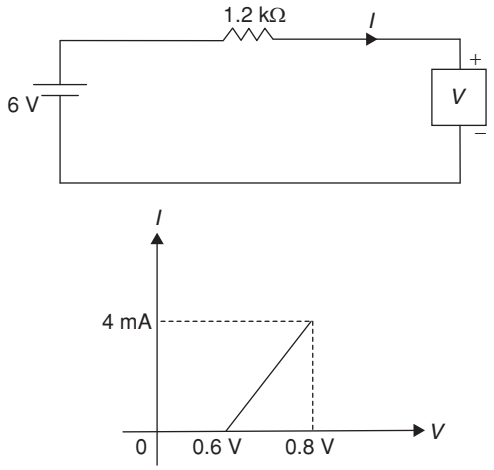
(A) –0.6 V and 4.8 V
(B) –4.8 V and 0.6 V
(C) 0.6 V and 4.2 V
(D) –4.2 V and –0.6 V

16. Find input V_i for the diode to be FB



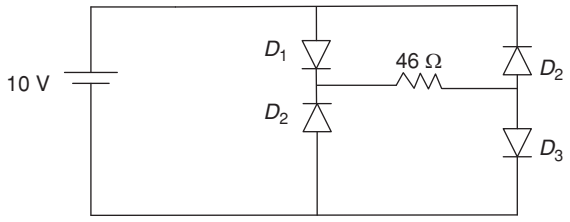
(A) 7 V (B) 6.8 V
(C) 3 V (D) 5 V

17. Find current I for the given characteristic



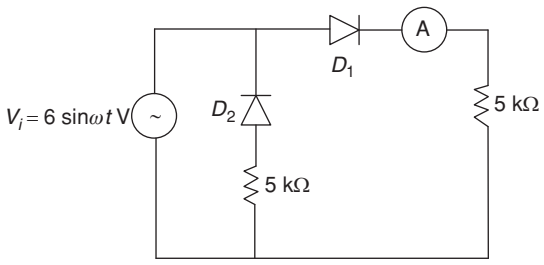
(A) 4.32 mA (B) 5 mA
(C) 4 mA (D) 0

18. If the cut-in voltage and forward resistance of each diode are 0.6 V and 2 Ω respectively. The current passing through 46 Ω is



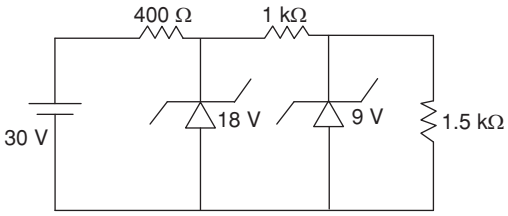
(A) 0.176 A (B) 0.16 A
(C) 0 A (D) 0.2 A

19. In the circuit shown, assume that diodes are ideal an ammeter is rms value indicating meter with zero internal resistance. The ammeter reading is



(A) 0.3 mA (B) 1.2 mA
(C) 0.6 mA (D) $\frac{1.2}{\pi}$ mA

20. What is the output voltage across 1.5 kΩ load in the circuit given below.



(A) 18 V (B) 10.8 V
(C) 9 V (D) 5.4 V

21. An AC supply of 200 V, 50 Hz is applied to bridge rectifier through a transformer turn ratio 4:1. The diode forward resistance is 200 Ω and load R_L = 1 kΩ. What is the DC output current?
(A) 31.8 mA (B) 26.54 mA
(C) 106 mA (D) None of these

22. What is the peak inverse voltage of diode?
(A) 50 V (B) 100 V
(C) 200 V (D) 400 V

23. Output ripple frequency is
(A) 100 Hz (B) 50 Hz
(C) 25 Hz (D) 0 Hz

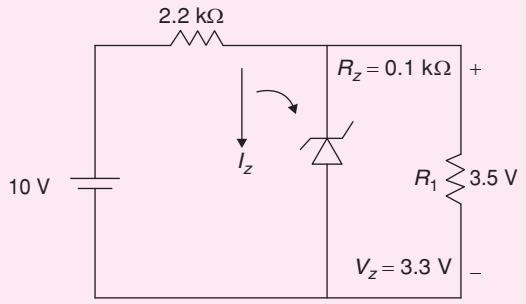
24. A power supply is delivering 100 W to a load of 10 kΩ. What is the AC (ripple) voltage across the load if the ripple factor is 0.1%.
(A) 100 V (B) 10 V
(C) 1 V (D) 0.1 V

25. A full wave rectifier uses a centre tapped transformer. The AC voltage from its centre tap to either end is 20sin314t V. The load resistance of the circuit is 80 Ω and diode resistance is 20 Ω. What is rms value of output current (I_{rms})?
(A) 0.2 A (B) 2 mA
(C) 20 mA (D) 0.14 A

26. Rectifier efficiency is
(A) 81.2% (B) 64.96%
(C) 58.7% (D) 40.6%

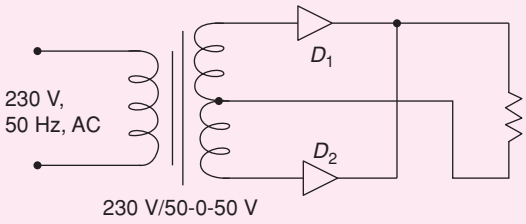
Previous Years' Questions

1. The current through the Zener diode in the figure is **[2004]**



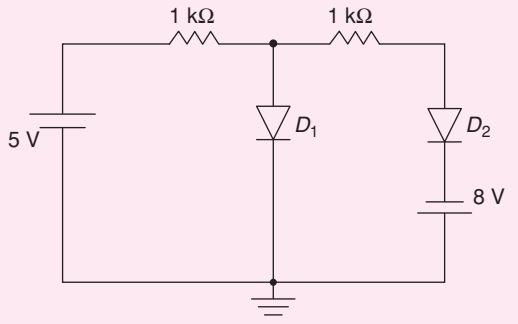
(A) 33 mA (B) 3.3 mA
(C) 2 mA (D) 0 mA

2. The circuit in figure shows a full wave rectifier. The input voltage is 230 V (rms) single-phase AC. The peak reverse voltage across the diodes D_1 and D_2 is **[2004]**



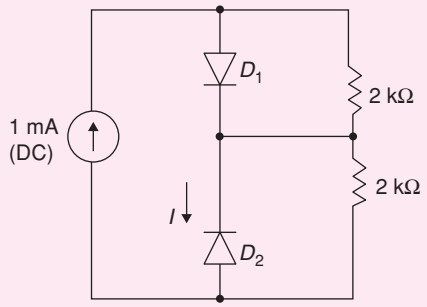
(A) $100\sqrt{2}$ V (B) 100 V
(C) $50\sqrt{2}$ V (D) 50 V

3. Assuming that the diodes are ideal in figure, the current in D_1 is **[2004]**



(A) 8 mA (B) 5 mA
(C) 0 mA (D) –3 mA

4. Assume that D_1 and D_2 in the figure are ideal diodes. The value of current I is: **[2005]**

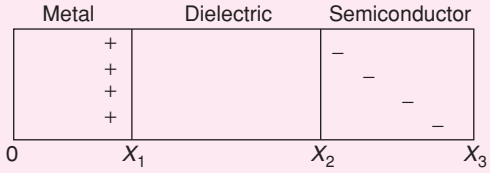


(A) 0 mA (B) 0.5 mA
(C) 1 mA (D) 2 mA

5. A three-phase diode bridge rectifier is fed from a 400 V rms, 50 Hz, three-phase AC source. If the load is purely resistive, the peak instantaneous output voltage is equal to **[2005]**

(A) 400 V (B) $400\sqrt{2}$ V
(C) $400\sqrt{\frac{2}{3}}$ V (D) $\frac{400}{\sqrt{3}}$ V

6. The charge distribution in a metal-dielectric-semiconductor specimen is shown in the figure. The negative charge density decreases linearly in the semiconductor as shown. The electric field distribution is as shown in **[2005]**



(A)



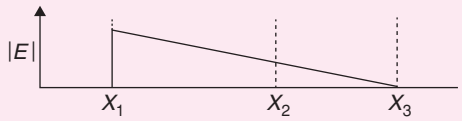
(B)



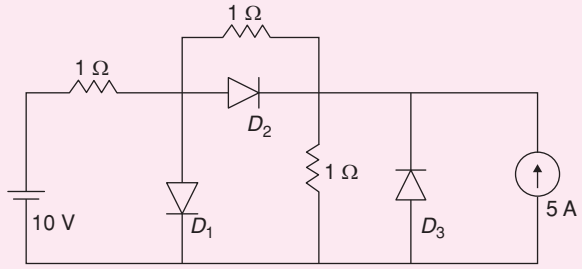
(C)



(D)

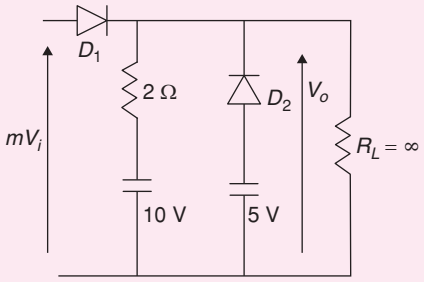


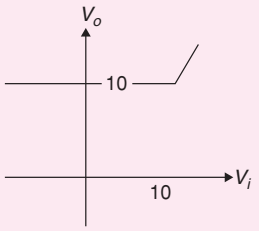
7. What are the states of the three ideal diodes of the circuit shown in figure? **[2006]**

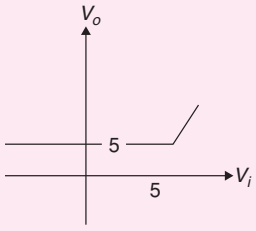


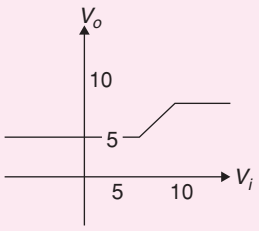
(A) D_1 ON, D_2 OFF and D_3 OFF
(B) D_1 OFF, D_2 ON and D_3 OFF
(C) D_1 ON, D_2 OFF and D_3 ON
(D) D_1 OFF, D_2 ON and D_3 ON

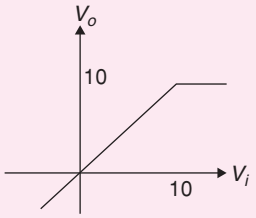
8. Assuming the diodes D_1 and D_2 of the circuit shown in the figure to be ideal ones, the transfer characteristics of the circuit will be **[2006]**



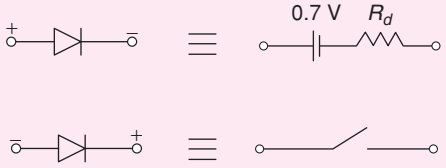
(A)


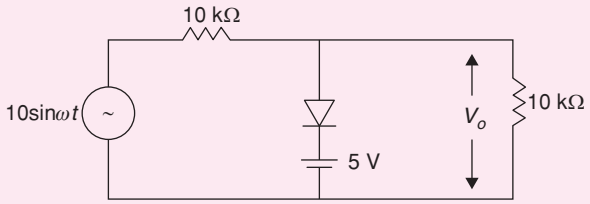
(B)


(C)


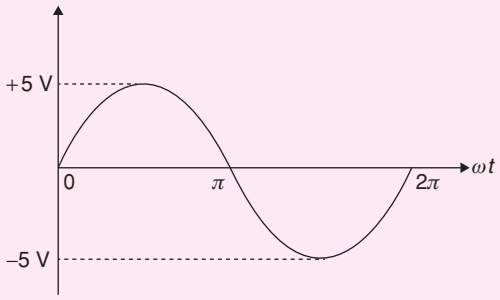
(D)


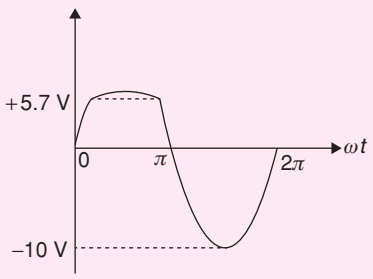
9. The equivalent circuits of a diode, during forward biased and reverse biased conditions, are shown in the figure.

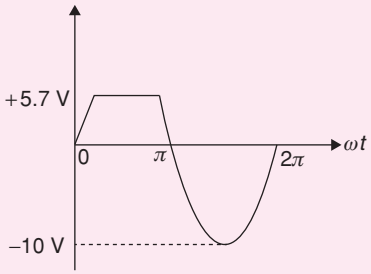


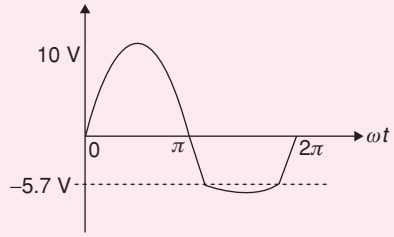


If such a diode is used in clipper circuit of figure given above, the output voltage (V_o) of the circuit will be **[2008]**

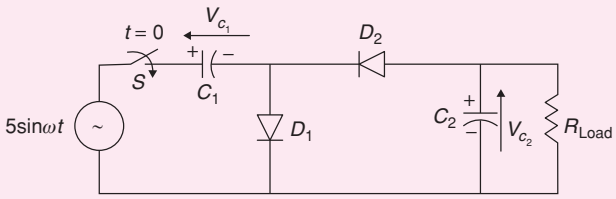
(A)


(B)


(C)


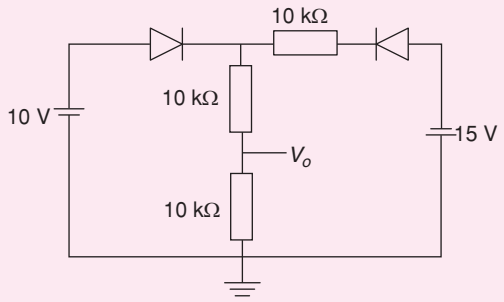
(D)


10. In the voltage doubler circuit shown in the figure, the switch '*S*' is closed at $t = 0$. Assuming diodes D_1 and D_2 to be ideal, load resistance to be infinite and initial capacitor voltages to be zero, the steady state voltage across capacitors C_1 and C_2 will be **[2008]**



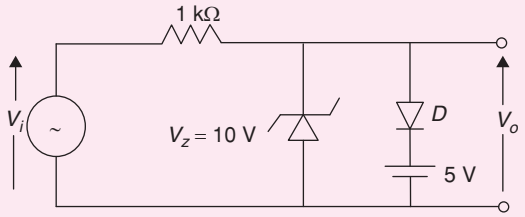
(A) $V_{c_1} = 10$ V, $V_{c_2} = 5$ V (B) $V_{c_1} = 10$ V, $V_{c_2} = -5$ V
(C) $V_{c_1} = 5$ V, $V_{c_2} = 10$ V (D) $V_{c_1} = 5$ V, $V_{c_2} = -10$ V

11. Assuming that the diodes in the given circuit are ideal, the voltage V_o is **[2010]**



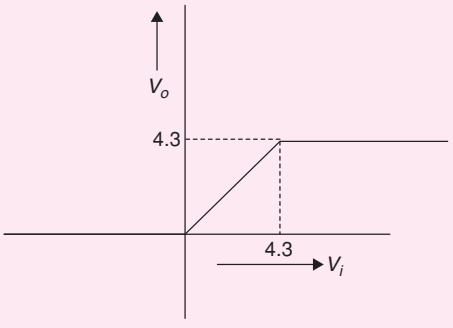
(A) 4 V (B) 5 V
(C) 7.5 V (D) 12.12 V

12. A clipper circuit is shown below.

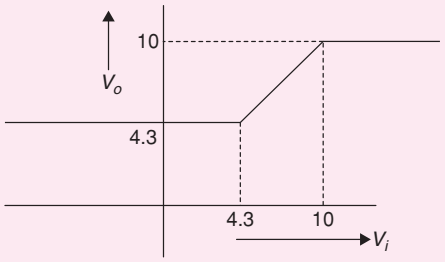


Assuming forward voltage drops of the diodes to be 0.7 V, the input-output transfer characteristic of the circuit is. **[2011]**

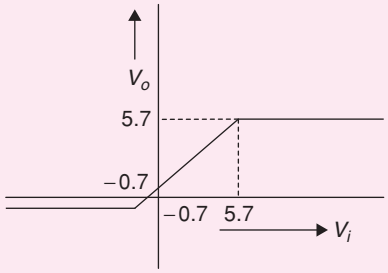
(A)



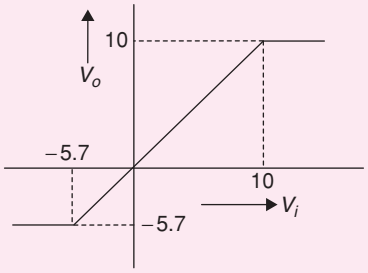
(B)



(C)



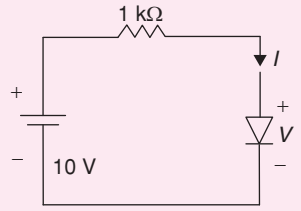
(D)



13. The I–V characteristics of the diode in the circuit given below are **[2012]**

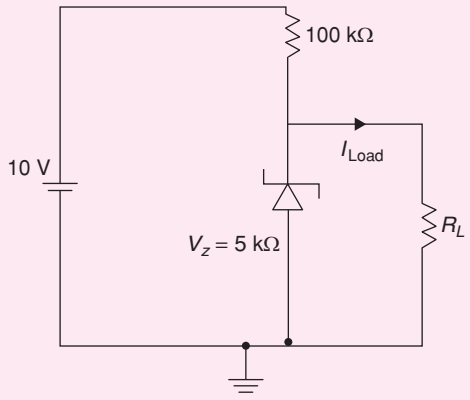
$$I = \begin{cases} \dfrac{V - 0.7}{500}\text{A}, V > 0.7\text{ V} \\ 0\text{A}, V < 0{\cdot}7\text{ V} \end{cases}$$

The current in the circuit is



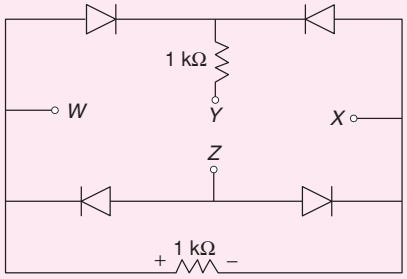
(A) 10 mA (B) 9.3 mA
(C) 6.67 mA (D) 6.2 mA

14. In the circuit shown below, the knee current of the ideal Zener diode is 10 mA. To maintain 5 V across R_L, the minimum value of R_L in Ω and the minimum power rating of the Zener diode in mW respectively are **[2013]**



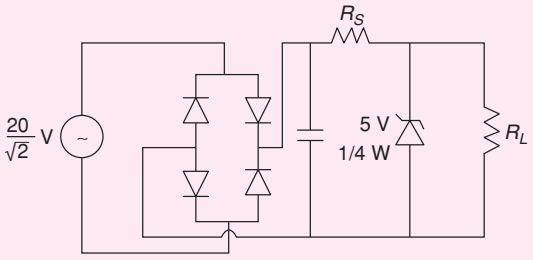
(A) 125 and 125 (B) 125 and 250
(C) 250 and 125 (D) 250 and 250

15. A voltage $1000\sin\omega t$ V is applied across YZ. Assuming ideal diodes, the voltage measured across WX in V is **[2013]**

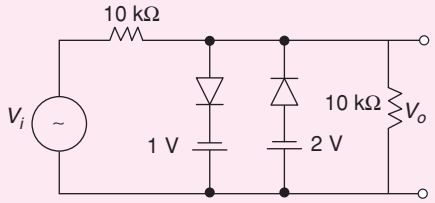


(A) $\sin\omega t$ (B) $(\sin\omega t + |\sin\omega t|)/2$
(C) $(\sin\omega t - |\sin\omega t|)/2$ (D) 0 for all t

16. A sinusoidal AC source in the figure has an rms value of $\frac{20}{\sqrt{2}}$ V. Considering all possible values of R_L, the minimum value of R_s in Ω to avoid burnout of the Zener diode is _____. **[2014]**



17. Assuming the diodes to be ideal in the figure, for the output to be clipped, the input voltage V_i must be outside the range **[2014]**



(A) −1 V to −2 V (B) −2 V to −4 V
(C) +1 V to −2 V (D) +2 V to −4 V

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** C	**3.** A	**4.** A	**5.** D	**6.** A	**7.** D	**8.** C	**9.** A	**10.** B
11. A	**12.** C	**13.** C	**14.** C	**15.** B	**16.** C	**17.** C	**18.** B	**19.** C	**20.** A
21. B	**22.** D	**23.** C	**24.** C	**25.** A	**26.** A	**27.** D	**28.** D		

Practice Problems 2

1. D	**2.** B	**3.** B	**4.** D	**5.** B	**6.** A	**7.** A	**8.** C	**9.** A	**10.** C
11. B	**12.** D	**13.** A	**14.** D	**15.** B	**16.** B	**17.** A	**18.** A	**19.** C	**20.** C
21. B	**22.** A	**23.** A	**24.** C	**25.** D	**26.** B				

Previous Years' Questions

1. C	**2.** A	**3.** C	**4.** A	**5.** B	**6.** A	**7.** A	**8.** A	**9.** A	**10.** D
11. B	**12.** C	**13.** D	**14.** B	**15.** D	**16.** 300	**17.** B			

Chapter 2

Bipolar Junction Transistors

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Transistor construction
- Transistor symbols
- Transistor current Components
- Transistor configurations
- Common-base configuration
- Operation modes of transistor
- Common emitter configuration
- Common collector configuration
- Thermal run away
- Power rating of transistor

Introduction

When a third doped element is added to a diode in such a way that two pn junctions are formed, the resulting device is known as a transistor. Transistors are smaller than vacuum tubes. Invented in 1948 by J. Barden and W.H. Brattain of Bell Laboratories, USA.

Transistor Construction

A transistor consists of two-pn-junctions formed by sandwiching either p-type or n-type semi-conductor between a pair of opposite types. Accordingly there are two types of transistors,

Namely, (i) *n-p-n* transistor;

(ii) *p-n-p* transistor;

An *n-p-n* transistor is composed of two *n*-type semiconductor separated by a thin section of *p*-type.

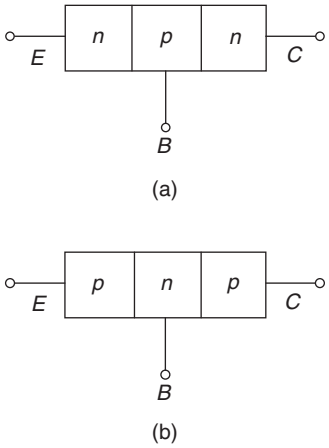


Figure 1 Types of transistors (a) *n-p-n*, and (b) *p-n-p*

- The emitter is heavily doped, the base is lightly doped, and the collector is moderately doped. According to area emitter is moderate, base is very thin and collector is large to dissipate heat.
- A transistor has two pn-junction, one junction is forward biased and the other junction is reverse biased. The forward junction has a low resistance path, whereas a reverse biased junction has a high resistance path. The weak signal is introduced at the low resistance circuit and output is taken from the high resistance circuit. Therefore, a transistor transfers a signal form low resistance to high resistance.

That is Transistor → Transfer + Resistor

A transistor has three sections of doped semiconductors. The section on one side is the emitter and the section on the opposite side is the collector. The middle section is called the base. It forms two junctions between the emitter and collector.

1. **Emitter:** The section on one side that supplies charge carriers (electrons or holes) is called the emitter. The emitter is always forward biased with refer to the base, so that it can supply a large number of majority carriers.
2. **Base:** The middle section which forms two pn-junctions between the emitter and collector is called the base. The base-emitter junction is forward biased andallowing low resistance for the emitter circuit. The base-collector junction is reverse biased. So it provides high resistance in the collector circuit.
3. **Collector:** The section on the other side that collects the charges is called the collector. The collector is always reverse biased.

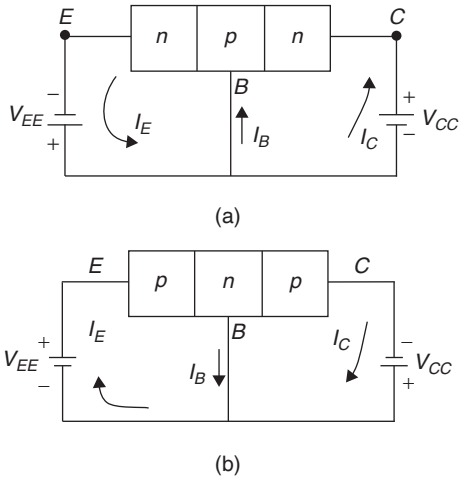


Figure 2 Types of transistors (a) *n-p-n*, (b) *p-n-p*

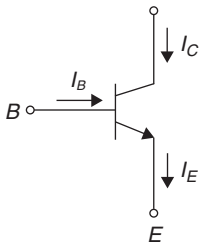
The transistor has two *pn* junctions i.e., it is like two diodes. The junction between emitter and base may be called emitter-base diode, or emitter diode. The junction between the base and collector may be called collector-base diode, or collector diode. The emitter diode is always forward biased and collector diode is always reverse biased.

The transistor offers Input Impedance very small and output Impedance high.

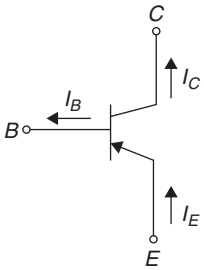
$$R_{in} \rightarrow \text{Small}$$
$$R_o \Rightarrow \text{High}$$

Transistor Symbols

(i) *n-p-n* transistor:



(ii) *p-n-p* transistor:



Transistor Current Components

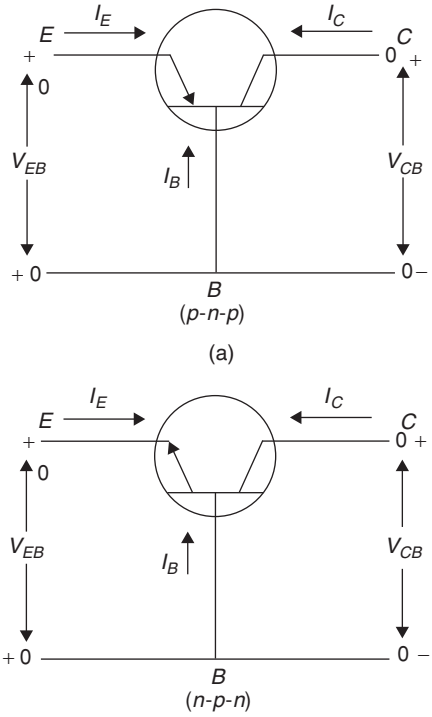
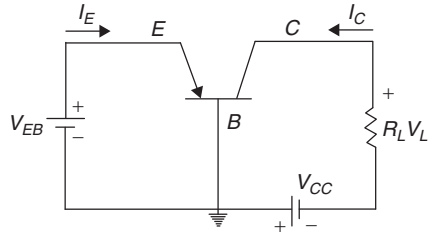


Figure 3 Circuitrepresentation of the two types of transistors (a) *p-n-p*, (b) *n-p-n*

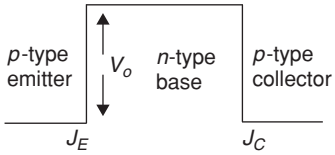
The Emitter, base and collector currents I_E, I_B, and I_C respectively are assumed to be positive when the current flows into the transistor.

(i) **For *p-n-p* transistor:**
I_E : positive (into)
I_B : negative (away)
I_C : negative (away)

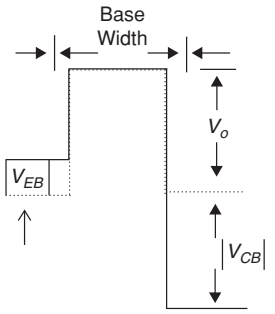


(ii) **For *n-p-n* transistor:**
I_E : negative (away)
I_B : positive (into)
I_C : positive (into)

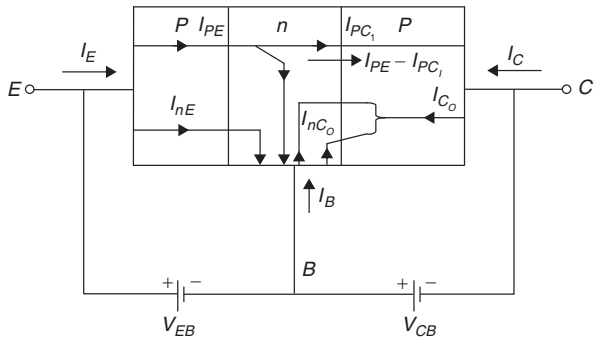
Un-biased Condition



Biased Condition



Transistor current components for p-n-p Transistor



KCL at the input junction gives:

$$I_E = I_{PE} + I_{nE}$$

Not all the holes crossing the emitter junction J_E reach the Collector junction, J_C, because some of them combine with the electrons in the *n*-type base.

$$-I_{C_O} = I_{nc_o} + I_{PC_O}$$

KVL at the Collector junction gives:

$$-I_C = I_{c_o} - I_{PC_l} = I_{C_O} - \alpha I_E$$

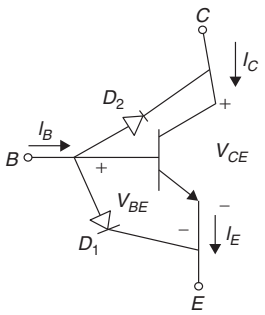
(i) Emitter efficiency $\gamma^* = \dfrac{I_{PE}}{I_E}$

(ii) Transport factor $\beta = \dfrac{I_{PC}}{I_{PE}}$

(iii) Current gain $\alpha = \dfrac{I_{PC}}{I_{IE}} = \dfrac{I_{PC}}{I_{PE}} \times \dfrac{I_{PE}}{I_E}$

$$\therefore \boxed{\alpha = \beta\gamma^*}$$

Operation Modes of Transistor



$$I_E = I_C + I_B$$
$$I_E \approx I_C.$$

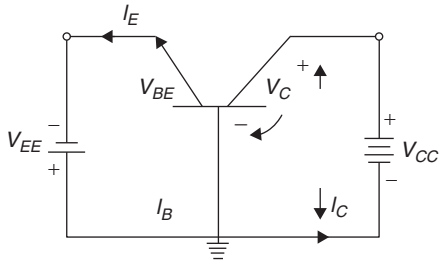
But $I_C < I_E$

1. **Active region:** Base-emitter junction forward biased and collector junction is reverse biased.
   i.e., $D_1 \rightarrow$ ON
   $D_2 \rightarrow$ OFF
   In this mode transistor works as an amplifier.
2. **Cut off region:** In this region emitter junction and collector junctions are both in reverse biased.
   $\therefore D_1 \rightarrow$ OFF
   $\Rightarrow D_2 \rightarrow$ OFF
   i.e., it is OFF switch.
3. **Saturation region:** In this region both junctions are in forward biased.
   i.e., $D_1 \rightarrow$ ON
   $\Rightarrow \; D_2 \rightarrow$ ON
   At $V_{CE} = 0$ V . ≈ 0.2 V for Si.

Transistor Configurations

Common-base Configuration

The base is common to both the input and output sides of the configuration.



The arrow in the transistor symbol represents the direction of the emitter current. In this circuit input is applied between emitter and base and output is taken from collector and base.

$$I_E = I_C + I_B$$

$$\alpha = \frac{\Delta I_C}{\Delta I_E} \text{ at } V_{CB} \text{ constant}$$

Where, $\alpha \rightarrow$ current amplification factor.

It is the ratio of change in output current (I_C) to change in input current (I_E) at constant V_{CB}.

$$\therefore \quad \alpha = \frac{\Delta I_C}{\Delta I_E} \text{ at } V_{CB} = \text{constant}$$

$$\alpha < 1$$

α, ranges from 0.9 to 0.99

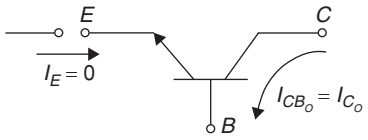


Figure 4 Reverse saturation current

Where $I_{CB_O} \Rightarrow$ collector to base emitter open circuit current.
i.e., Reverse saturation current

Total current $I_C = \alpha I_E + I_{CB_O}$

But $I_E = I_C + I_B$

$\therefore I_C = \alpha[I_C + I_B] + I_{CB_O}$

$I_C(1-\alpha) = \alpha I_B + I_{CB_O}$

$$I_C = \frac{\alpha}{1-\alpha} \cdot I_B + \frac{I_{CB_O}}{1-\alpha}.$$

Transfer characteristics

1. Input characteristics

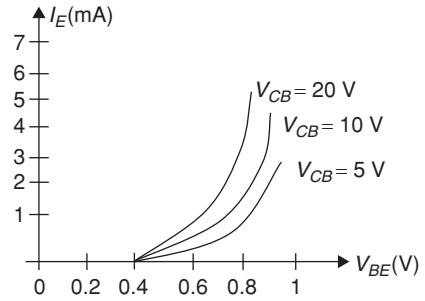


Figure 5 Input or driving point characteristics for a *CB* silicon transistor amplifier.

It is the curve between the I_E and V_{BE} at constant V_{CB}. V_{CB} increases I_E curves moves towards left.

2. Output characteristics

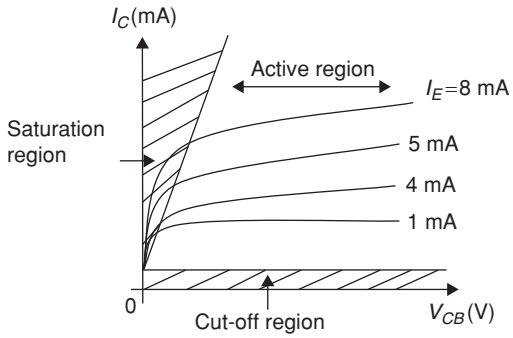


Figure 6 CB Output characteristics.

From the characteristics, it is seen that for a constant value of I_E, I_C is independent of V_{CB} and the curves are parallel to the axis of V_{CB}. I_C flows even when V_{CB} is equal to zero.

Early effect or base-width modulation

As the collector voltage V_{CB} is made to Increase the reverse bias, the space charge width between the collector and base is increased, with the result that the effective width of the base decreases, this effect is known as early effect.

Input resistance, $r_i = \dfrac{\Delta V_{BE}}{\Delta I_E}$ at constant V_{CB}

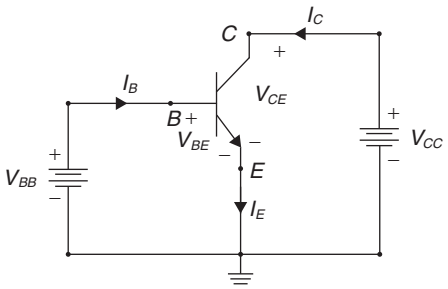
Output resistance $r_0 = \dfrac{\Delta V_{CB}}{\Delta I_C}$ at constant I_E r_i of *CB* circuit quite small, in the order of a few 'Ω's r_0 is very large, in the order of kΩ i.e., R_i small and R_0 high compare to others.

Example 1: The early effect in a bipolar junction transistor is caused by:
(A) Fast-turnoff
(B) Fast-turn-ON
(C) Large collector-base reverse bias
(D) Large V_{EB} forward bias

Solution: (C)

Common Emitter Configuration

1. *n-p-n*



2. *p-n-p*

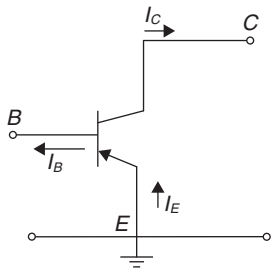


Figure 7 Notations and symbols used with the CE-configuration.

In this circuit, input is applied between base and emitter and output is taken form the collector and emitter.

Characteristic of a CE configuration

1. Input or base characteristics

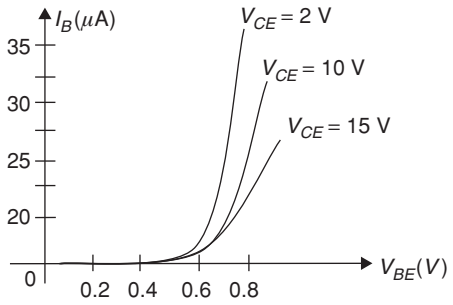


Figure 8 Si transistor input characteristics.

Input resistance

$$R_i = \frac{\Delta V_{BE}}{\Delta I_B} \text{ at const } V_{CE} \text{ and } T$$

The values of input resistance for a *CE* amplifier arein the order of a few hundreds of ohms.

2. Output or collector characteristics

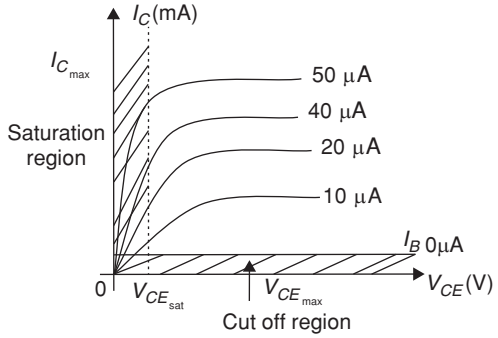
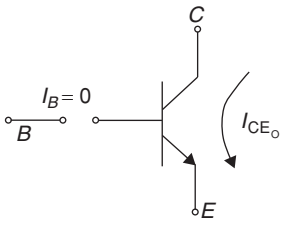


Figure 9 Collector characteristics

In the active region of a *CE* amplifier, the base emitter junction is forward-biased, where as the collector-base junction is reverse-biased.

Leakage current



Where, $I_{CE_O} \Rightarrow$ Base open circuit collector to emitter current in *CE* amplifier, a small current flows, even when the $I_B = 0$.

This is called collector cut off current and is denoted by I_{CE_O}.

$$(I_{CE_O} >> I_{CB_O})$$

Base current amplification factor (β) The ratio of change in collector current (ΔI_C) to the change in base current (ΔI_B) is known as current amplification factor.

$$\beta = \frac{\Delta I_C}{\Delta I_B}$$

β, ranges generally from 20 to 500.

If *DC* values are considered,

$$\beta = \frac{I_C}{I_B}$$

Expression for collector current We know,

$$I_E = I_B + I_C$$

$$I_C = |\alpha I_E| + I_{CB_O}$$

$$I_C = \alpha(I_B + I_C) + I_{CB_O}$$

$$\therefore \quad I_C = \frac{\alpha}{1-\alpha} \cdot I_B + \frac{I_{CBO}}{1-\alpha}$$

$$I_C = \beta I_B + I_{CEO}$$

From the above equation, we get

$$\beta = \frac{\alpha}{1-\alpha} \text{ and } I_{CE_o} = \frac{I_{CB_O}}{1-\alpha}$$

or $I_{CEO} = (1 + \beta) \cdot I_{CB_O}$

Common Collector Configuration

The CC-configuration is used for Impedance matching circuitsit has a high input impedance and low output impedance.

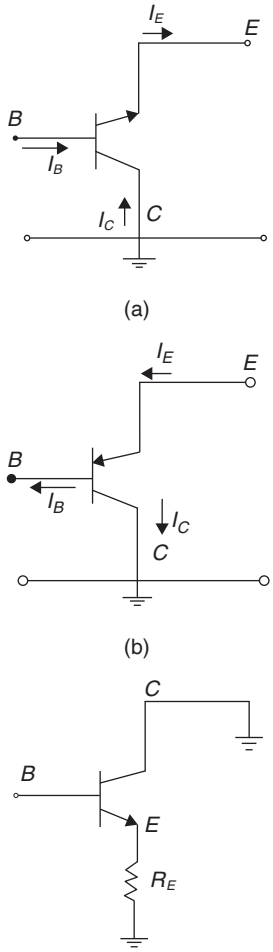


Figure 10 CC-configuration used for impedance matching purpose. (a) *n-p-n*, (b) *p-n-p*

The maximum power dissipation is equal to $P_{C(max)} = V_{CE} \cdot I_C$

1. Current amplification factor: The ratio of change in emitter current (ΔI_E), to the change in base current (ΔI_B) is known as current amplification factor in common collector.

$$\gamma_{ac} = \frac{\Delta I_E}{\Delta I_B}$$

The voltage gain of *CC* amplifier is always less than one.

2. Relation between α, β and γ:
For *DC*

$$\alpha = \frac{I_C}{I_E},\ \beta = \frac{I_C}{I_B}$$

$$\gamma = \frac{I_E}{I_B}$$

$$\alpha = \frac{\beta}{1+\beta};\ \beta = \frac{\alpha}{1-\alpha};\ \gamma = \frac{1}{1-\alpha}$$

$$\gamma = 1 + \beta$$

The currents are always following the below relation.

$$I_E : I_B : I_C = 1 : (1 - \alpha) : \alpha.$$

Comparison of Various Characteristics of Transistor

Sl. No.	Characteristic	CB	CE	CC
1.	R_i	Low (Ω's) < 100 Ω	Low (Ω's) < 100 Ω	Very high (kΩ's) < 100 kΩ
2.	R_0	Very high (about 500 kΩ)	High (about 50 kΩ)	Low (about 50 Ω)
3.	Voltage gain	Av ≈ 150	Av ≈ 500	Av < 1
4.	Current gain	A_i < 1 (α)	High (β)	High (γ)
5.	Applications	For high frequency applications	For audio frequency applications	Impedance matching

From the above analysis:

$R_i \rightarrow$ Low for CB configuration (compare to others);
$R_i \rightarrow$ Very high for CC configuration;

Output impedance (R_0):

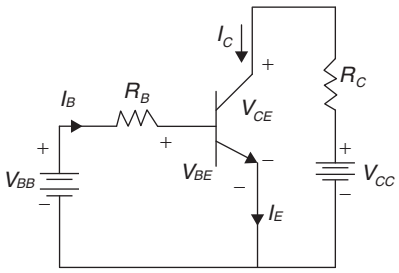
$R_0 \rightarrow$ Low for CC configuration;
$R_0 \rightarrow$ Very high for CB configuration;

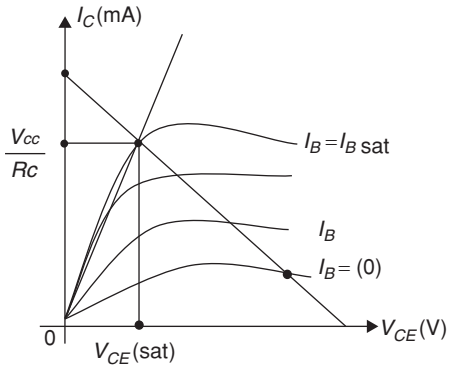
Out of the three transistor connections, the common emitter circuit is the most efficient, the main reason for this usage:

1. High current gain;
2. High voltage and power gain;
3. Moderate Output to Input ratio.
   That is R_0/R_i less compare to others.

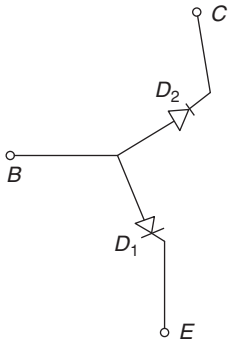
Transistor and Its Region of Operations

CE transistor circuit and the o/p characteristic along with the DC load line.

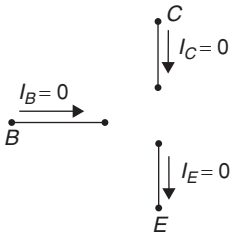




Cut-off



In this region emitter diode (D_1) and collector diode (D_2) both are in OFF state. i.e.,

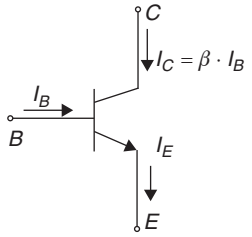


Transistor in cut-off mode: The point where the load line intersects the $I_B = 0$ curve is known as cut-off. At this point $I_B = 0$ and only small collector current or leakage current I_{CE_O} exists.

$\therefore\ V_{CE\ (\text{cut off})} = V_{CC}$.

Active region

The region between the cut off and saturation is known as active region. In this region collector-base junction remains reverse biased ($D_2 \rightarrow$ OFF) and base-emitter junction in forward biased ($D_1 \rightarrow$ ON).

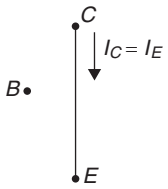


That is in active region emitter diode is ON and collector diode is OFF.

Note: We provide biasing to the transistor to it operates in the active region.

Saturation region

In this region both the junctions are in forward biased and normal transistor action is lost. i.e., Emitter diode and collector diodes are ON.



At $V_{CE} \approx 0$ V

$$\therefore I_{C(\text{sat})} = \frac{V_{CC}}{R_C}$$

$$V_{CE} = V_{CE(\text{sat})} = V_{\text{knee}}.$$

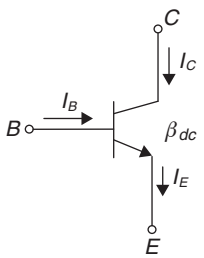
$V_{CE(\text{sat})}$ or V_{knee} can be neglected as compared to V_{CC}.

$\Rightarrow$ At room temperature V_{CE} drop of a silicon transistor at saturation is approximately $V_{CE(\text{sat})} = 0.3$ V

Example 2: If the transistor in the figure is in saturation, then

(A) $I_C = \beta_{dc} \cdot I_B$ (B) $I_C < b_{dc} \cdot I_B$

(C) $I_C \geq \beta_{dc} \cdot I_B$ (D) None of the above



Solution: (B)

In active region $I_C = \beta \cdot I_B$

Saturation region $I_{C(\text{Sat})} < \beta \cdot I_B$.

Example 3: If for a *Si npn* transistor, the $V_{BE} = 0.7$ V and $V_{CB} = 0.2$ V, then the transistor is operating in the

(A) Cut-off region

(B) Active region

(C) Saturation region

(D) Inverse active mode

Solution: (B) $V_{BE} = 0.7$ V

That is $V_B - V_E = 0.7$ V

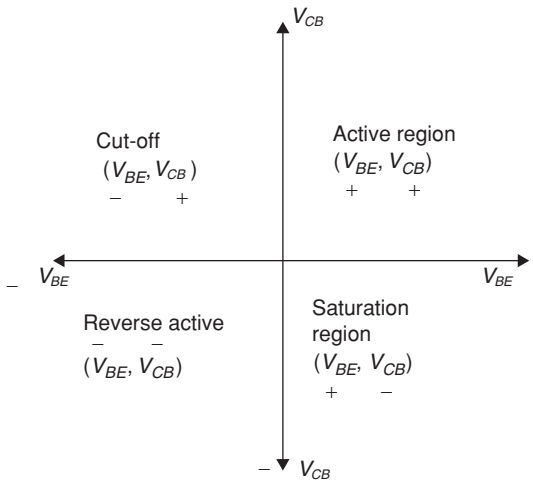
$\therefore V_B > V_E \Rightarrow$ emitter diode forward bias

$V_{CB} = 0.2$ V

$\Rightarrow V_C > V_B \Rightarrow$ collector junction is in reverse bias.

So, the transistor is in the normal active mode.

Modes of operation of BJT



Notes

1. For a silicon transistor generally considered:

$$V_{CE(\text{sat})} = 0.2 \text{ V}$$
$$V_{BE(\text{active})} = 0.7 \text{ V and}$$
$$V_{BE(\text{sat})} = 0.8 \text{ V.}$$

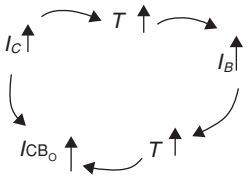
2. For Ge transistor:

$$V_{CE(\text{sat})} = 0.1 \text{ V}$$
$$V_{BE(\text{active})} = 0.2 \text{ V}$$
$$V_{BE(\text{sat})} = 0.3 \text{ V and}$$
$$V_{BE(\text{cut off})} = 0 \text{ V}$$

Thermal Run Away

The collector $I_C = \beta \cdot I_B + (1 + \beta) . I_{CB_O}$. The three variables in the equation β, I_B and I_{CB_O} increase with rise in temperature. The reverse saturation current I_{CO} is more sensitive with temperature, it doubles for every 10°C raise in temperature. As a result, I_C will increase still further. Which will further

rise the temperature at the collector-base junction. This process is called as 'Thermal run away'.



The collector is normally larger in size than the others because to help dissipate the heat developed at the collector junction.

Example 4: To avoid thermal run away in the design of an analog circuit, the operating point of the BJT should be such that it satisfies the condition.

(A) $V_{CE} = \frac{V_{CC}}{2}$ (B) $V_{CE} \geq \frac{V_{CC}}{2}$

(C) $V_{CE} < \frac{V_{CC}}{2}$ (D) None of the above

Solution: (C)

Power Rating of Transistor

The maximum power that a transistor can handle without destruction is known as power rating of the transistor.
The maximum power is dissipated by:

$$P_{max} = I_C \cdot V_{CE(max)}$$

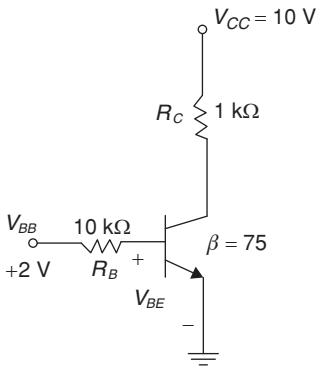
If $V_{CE} > V_{CE(max)}$, the transistor will be destroyed due to excessive heat.

Example 5: The maximum power dissipation of a transistor is 80 mΩ. If $V_{CE} = 10$ V, the maximum collector current that can be allowed without destruction of the transistor is:
(A) 5 mA. (B) 7 mA.
(C) 8 mA. (D) 10 mA.

Solution: (C)
We know, $P_{D(max)} = I_{C(max)} \cdot V_{CE}$

$$I_{CE(max)} = \frac{80 \text{ mW}}{10 \text{ V}} = 8 \text{ mA}.$$

Example 6: For the circuit shown in the Figure:



The power dissipation in the transistor is:
(A) 2 mW (B) –2.8 mW
(C) 2.43 mW (D) 3 mW

Solution: (C) Apply KVL to the input loop:

$$I_B = \frac{V_{BB} - V_{BE}}{R_B} = \frac{1.3}{10 \text{ k}\Omega} = 0.13 \text{ mA}$$

$$I_{C(active)} = \beta \cdot I_B = 75 \times 0.13 \text{ mA} = 9.75 \text{ mA}$$
$$V_{CE} = V_{CC} - I_C \cdot R_C = 10 - 9.75 = 0.25 \text{ V}$$
$$P_D = I_C \cdot V_{CE}$$
$$= 9.75 \times 0.25 \text{ mW} = 2.43 \text{ mW}.$$

Example 7: In a *CB* configuration current configuration factor is 0.9, if the emitter current is: 1.8 mA. The value of I_B is:
(A) 1.17 mA. (B) 0.18 mA.
(C) 13 μA. (D) 1.3 mA.

Solution: (B)

We know, $\alpha = \frac{I_C}{I_E}$

$$I_C = \alpha I_E = 0.9 \times 1.8 \text{ mA} = 1.62 \text{ mA}$$
$$I_B = I_E - I_C = 0.18 \text{ mA}.$$

Example 8: For a certain transistor, $I_B = 15$ μA, $I_C = 2$ mA and $\beta = 100$. The value of leakage current I_{CB_O} is:
(A) 4.95 μA. (B) 5 μA.
(C) 3 μA. (D) 4.95 mA.

Solution: (A)
We know, $I_C = \beta \cdot I_B + (1 + \beta) \cdot I_{CB_O}$

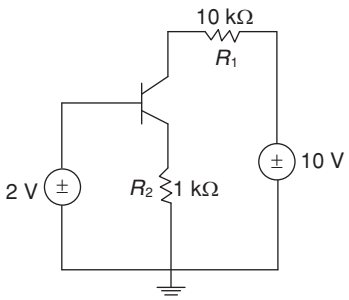
$$I_{CB_O} = \frac{0.5}{101} \text{mA} = 4.95 \text{ μA}.$$

Example 9: In a junction transistor, the collector cut off current I_{CB_O} reduces considerably by doping the:
(A) Emitter with low level of impurity.
(B) Emitter with high level of impurity.
(C) Collector with high level of impurity.
(D) Base with high level of impurity.

Solution: (B)
We know, $I_C = \alpha I_E + I_{CB_O}$

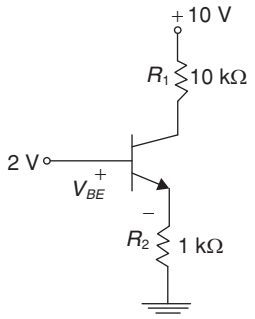
$$\alpha = \frac{I_C - I_{CB_O}}{I_E} \Rightarrow \alpha \uparrow \Rightarrow I_{CB_O} \downarrow.$$

Example 10: For a BJT circuit shown, assume that the 'β' of the transistor is very large of $V_{BE} = 0.7$ V. The mode of operation of the BJT is:



(A) Cut-off region (B) Saturation region
(C) Normal active (D) Reverse active

Solution: (B)
Let us assume transistor is in saturation region.



$$I_{C(sat)} = \frac{10}{11} \text{ mA}$$

Apply KVL in i/p loop

$$V_{BB} = V_{BE} + I_C \cdot R_2$$
$$V_{BE} = 2 - 0.909$$
$$= 1.09 \text{ V}$$

$\therefore V_{BE} > 0.7$. So transistor is in saturation mode.

Example 11: In a certain transistor, $I_C = 0.98$ mA and $I_B = 20$ mA. Determine (i) I_E, (ii) α and (iii) β

Solution:

(i) $I_E = I_B + I_C = 0.98 + 0.02$
$I_E = 1.0$ mA

(ii) $\alpha = \frac{I_C}{I_E} = \frac{0.98}{1.0} = 0.98$

(iii) $\beta = \frac{I_C}{I_B} = \frac{0.98}{0.02} = 49$

Example 12: A BJT has $I_B = 10$ μA, $\beta = 99$ and $I_{C_O} = 1$ μA. What is the collector current I_C?

Solution: $I_C = \beta I_B + (1 + \beta)\, I_{CB_O}$
$= 99 \times 10 \times 10^{-6} + (1 + 99)\, 1 \times 10^{-6}$
$I_C = 1.09$ mA

Example 13: Determine the emitter current. I_E, collector current I_C for a transistor with $\alpha_{DC} = 0.97$ and collector to base leakage current 10 μA, I_B is 50 μA.

Solution: $I_c = \beta \cdot I_B + (1 + \beta)\, I_{CB_O}$
$\beta = \frac{\alpha}{1-\alpha} = \frac{0.97}{1-0.97}$
$\beta = 32.33$
$I_C = 32.33 \times 50 \times 10^{-6} + (1 + 32.33) \times 10 \times 10^{-6}$
$I_C = 1.95$ mA
$I_E = I_B + I_C = 2$ mA

Example 14: In a particular transistor, the collector current is 5.6 mA and emitter current is 5.75 mA. Determine α_{DC}

Solution: $I_C = 5.6$ mA
$I_E = 5.75$ mA
$\alpha_{DC} = \frac{I_C}{I_E} = 0.974.$

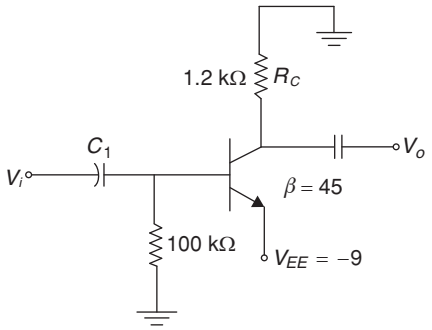
Example 15: A BJT has a base current of 200 μA and emitter current of 20 mA. Determine collector current and β

Solution: $I_B = 200$ μA
$I_E = 20$ mA
$I_c = I_E - I_B$
$= 20 \times 10^{-3} - 200 \times 10^{-6}$
$I_C = 19.8$ mA
$\beta = \frac{I_C}{I_B}$
$\beta = 99$

Example 16: A BJT has a collector current of 4 mA and base current of 20 μA. Determine its β.

Solution: $I_C = 4$ mA
$I_B = 20$ μA
$\beta = \frac{I_C}{I_B} = 200.$

Example 17: Determine V_C and V_B for the network.

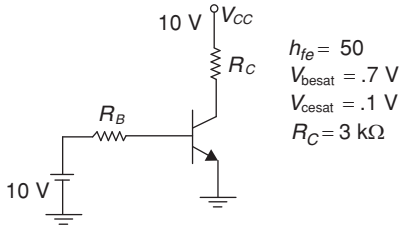


Solution: $-I_B R_B - V_{BE} + V_{EE} = 0$

$$I_B = \frac{V_{EE} - V_{BE}}{R_B} = \frac{9 - .7}{100 \text{ k}\Omega} = \frac{8.3}{100 \text{ k}\Omega} = 83 \text{ μA}$$

$I_C = \beta I_B = 45 \times 83$ μA $= 3.735$ mA
$V_C = -I_C R_C = -$ (3.735 mA) (1.2 kΩ)
$= -4.48$ V
$V_B = -I_B R_B$
$= -(83$ μA) (100 kΩ) $= -8.3$ V.

Example 18: For the given circuit find the value of R_B that would be just sufficient to drive the transistor to saturation?



Solution: The value of R_B required into drive the transistor to saturation.

$$I_C \le h_{F\in} \times \frac{V_{BB} - V_{B\text{sat}}}{R_B}$$

$$R_B \le 50 \times \frac{10 - .7}{I_C}$$

$$V_{CC} = I_C R_C$$

$$I_C = \frac{10}{3\ \text{k}\Omega} = 3.33\ \text{mA}$$

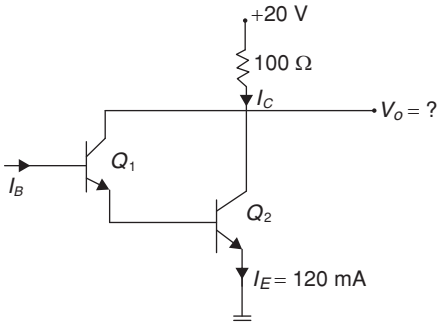
$$R_B \le \frac{50 \times 9.3}{3.33\ \text{mA}} = 139\ \text{k}\Omega \sim 140\ \text{k}\Omega$$

EXERCISES

Practice Problems I

Directions for questions 1 to 24: Select the correct alternative from the given choices.

Common Data for Questions 1 and 2:

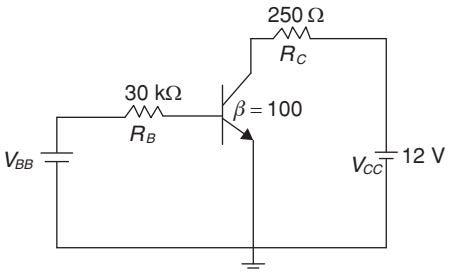


Assume both Transistors are in Active region and neglect reverse saturation currents. If $\alpha_1 = 0.99$ and $\alpha_2 = 0.98$

1. The value of V_o shown is:
(A) 6 V (B) 12 V
(C) 8 V (D) 10 V

2. The value of overall β is $\left(\frac{I_C}{I_B}\right)$ is:
(A) 5000 (B) 5001
(C) 4999 (D) 4998

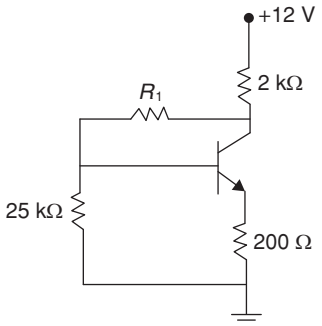
Common Data for Questions 3 and 4:



3. If $V_{CE} = 6$ V, the value of V_{BB} required is _____
(A) 7 V (B) 7.9 V
(C) 8 V (D) 7.8 V

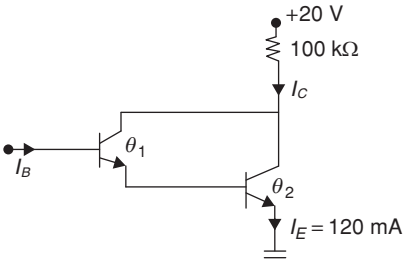
4. If V_{CC} is changed to 6 V in the given circuit, the value of R_C required to achieve the Q-point Q (2 V, 16 mA).
(A) 200 Ω (B) 250 Ω
(C) 2.5 kΩ (D) 2 kΩ

5.



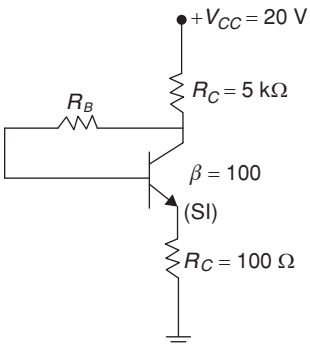
If $\alpha = 0.98$ and $V_{BE} = 0.7$ V, the value of resistor R_1 for an emitter current of 2 mA is _____
(A) 81.1 kΩ (B) 8.11 kΩ
(C) 44 kΩ (D) 19.6 kΩ

6.



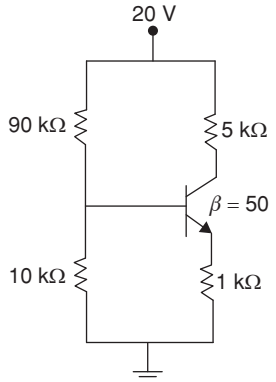
Assume both transistors are in Active regions. If $\alpha_1 = 0.99$ and $\alpha_2 = 0.98$, then the value of overall $\alpha\left(\frac{I_C}{I_E}\right)$ is _____
(A) 0.99 (B) 0.9998
(C) 0.998 (D) 0.98

7.



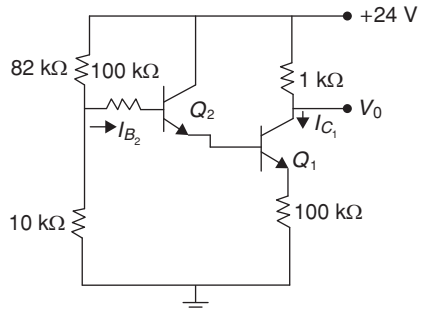
Assume the transistor is in active region. If V_{CE} = 4 V, find the value of R_B.
(A) 100 kΩ
(B) 106 kΩ
(C) 104 kΩ
(D) 98 kΩ

8.



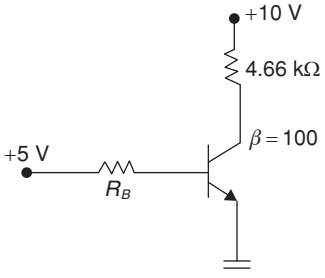
Assume the transistor is in Active region. The value of thecollector current. I_C is _____
(A) 2 mA
(B) 1.085 mA
(C) 1.85 mA
(D) 0.021 mA

9.



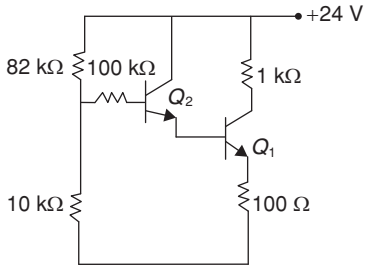
Assume, both Transistors are in Active region with V_{BE_1} = V_{BE2} = 0.7 V. β_1 = 100 and, β_2 = 50. The ratio of I_{C_1}/I_{B_2} is _____
(A) 5000
(B) 5100
(C) 4900
(D) 490

10.



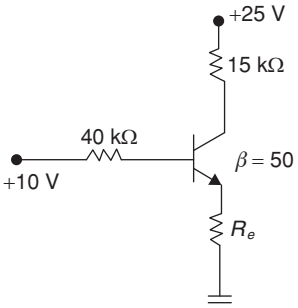
The Maximum value of R_B for which the Transistor remains at saturation is _____
(A) 20 kΩ (B) 2 kΩ
(C) 200 kΩ (D) 20 Ω

11.



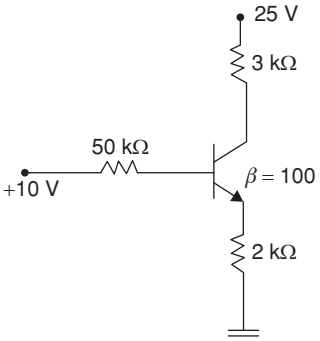
Assume, both Transistor are in Active region with β_1 = 100, β_2 = 50 and $V_{BE_1} = V_{BE_2}$ = 0.7 V. The value of V_{CE} of Q_1 is
(A) 14.1 V (B) 13.1 V
(C) 14.9 V (D) 13.9 V

12.



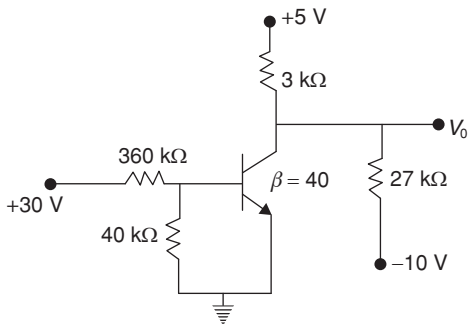
The value of R_e, for which the Transistor just comes out of saturation region.
(A) 742 Ω (B) 7.42 kΩ
(C) 472 Ω (D) 4.72 kΩ

13.



Find the region of operation of transistor shown.
(A) Cut-off (B) saturation
(C) Active (D) inverse Active

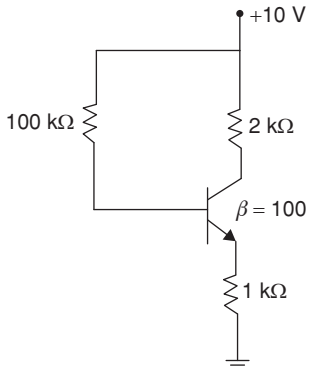
14.



Find the region of operation of the transistor, shown.

(A) Active (B) Saturation
(C) Cut-off (D) Reverse active

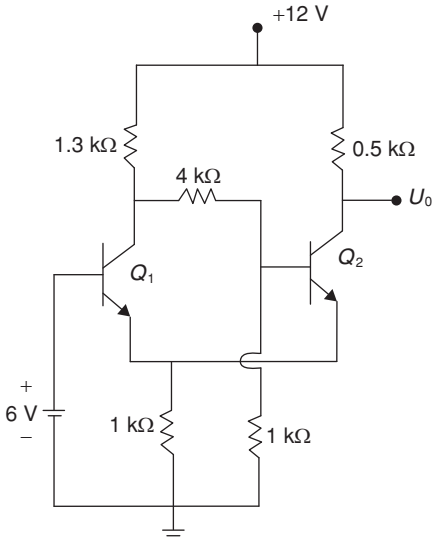
15.



Neglect the junction voltages. The transistor is operating in _____ region.

(A) Active (B) Saturation
(C) Reverse Saturation (D) Cut-off

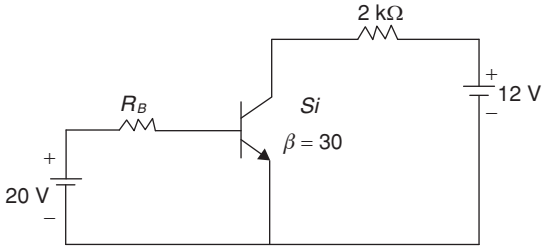
16.



Assume β of each transistor is 100. Find the value of V_o is:

(A) 8.5 V (B) 12 V
(C) 7.5 V (D) 9 V

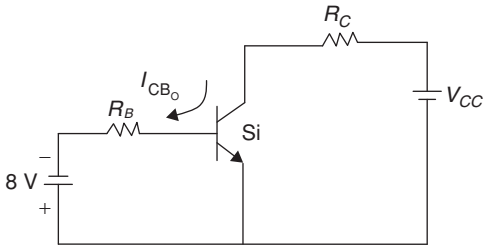
17.



For what values of R_B will the Transistor remain below cut off region if I_{CB_O} = 100 μA:

(A) $R_B \le 200$ kΩ (B) $R_B \ge 200$ kΩ
(C) $R_B \le 100$ kΩ (D) $R_B \ge 10$ kΩ

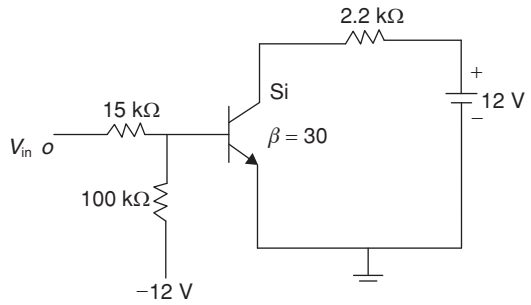
18.



If the reverse saturation current of Si Transistor is 10 nA at room temperature (25°C) and increases by a factor of 2 for each temperature increase of 10°C. The maximum allowable value for R_B if the transistor is to remain cut-off at a temperature of 185°C _____

(A) 122 kΩ (B) 12.2 kΩ
(C) 12.2 MΩ (D) 1.22 MΩ

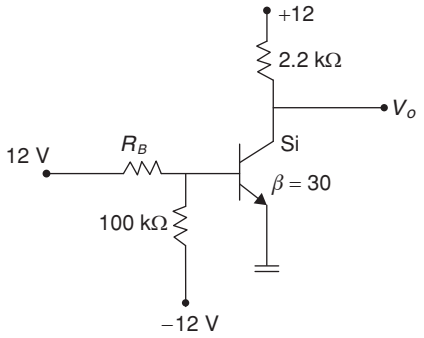
19.



Find V_{CE} if V_{in} = 12 V

(A) 8.8 V (B) 0.2 V
(C) 11.8 V (D) 3.8 V

20.



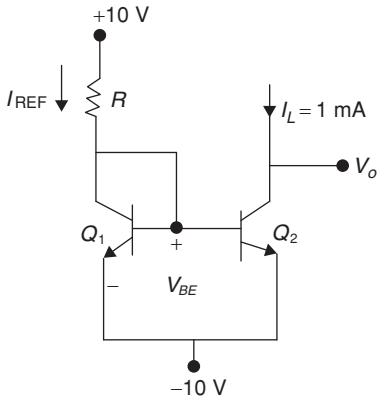
Find minimum value of R_1 for which the transistor is in the Active region.

(A) 17 kΩ (B) 27 kΩ
(C) 37 kΩ (D) 33 kΩ

21. Find the punch through voltage of a NPN silicon Transistor of alloy type, if the width of base region is 2 μm and resistivity of base is 1 Ω cm.

(A) 38 V
(B) 10 V
(C) 28 V
(D) 18 V

22.

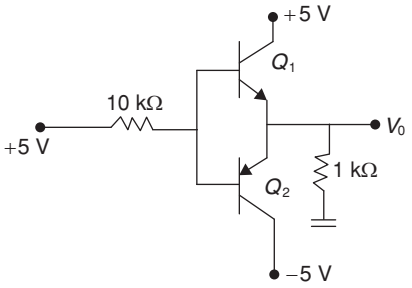


Find the value R, such that load current is equal to 1 mA.

(A) 10 kΩ (B) 9.3 kΩ

(C) 19.3 kΩ (D) 10.7 kΩ

23.



If β of each transistor is 100, find V_o.

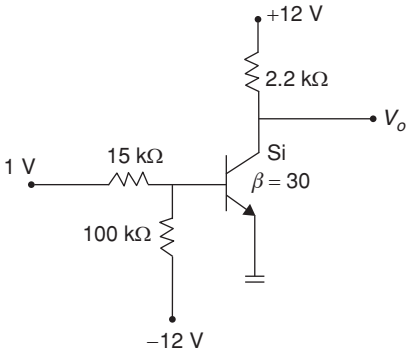
(A) +4 V

(B) +5 V

(C) –4 V

(D) –5 V

24.



Assume reverse saturation current I_{CB_O} = 10 nA at 25°C. Find the maximum temperature. At which transistor remains at cut-off.

(A) 129°C

(B) 149°C

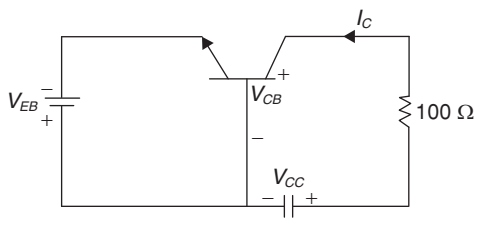
(C) 124°C

(D) 134°C

Practice Problems 2

Directions for questions 1 to 20: Select the correct alternative from the given choices.

1.



If I_C = 15 mA and V_{CB} = 3 V then the value of V_{CC} required is _____

(A) 4 V

(B) 4.5 V

(C) 3.15 V

(D) 18 V

2. Find the value of V_{CB}, if the supply voltage V_{CC} decreases by 1 V in part (i), and I_C remains the same,

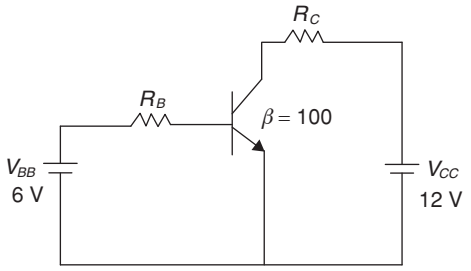
(A) 3 V

(B) 3.5 V

(C) 2 V

(D) 2.5 V

Common Data for Questions 3 and 4:



3. Assume the Transistor used is silicon with V_{BE} = 0.7 V, the values of R_C and R_B so that I_C = 12 mA and V_{CE} = 6 V.

(A) 0.5 kΩ, 44 kΩ

(B) 5 kΩ, 44 kΩ

(C) 4.4 kΩ, 50 kΩ

(D) 4 kΩ , 50 kΩ

4. The values of R_C and R_B if a 200 Ω emitter resistor is included so that I_C = 12 mA and V_{CE} = 6 V.

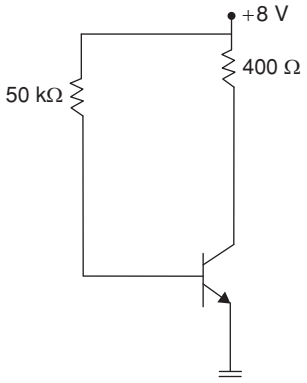
(A) 300 Ω, 24 kΩ

(B) 0.3 kΩ, 42 kΩ

(C) 24 kΩ, 42 kΩ

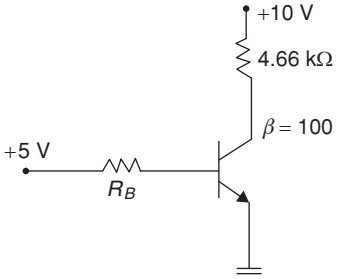
(D) 2.4 kΩ, 24 kΩ

5. Assume the transistor is in Active region. If $I_C = 19.6$ mA then the value of V_{CB} is _____



(A) 0.55 V (B) –0.55 V (C) 0.85 V (D) –0.85 V

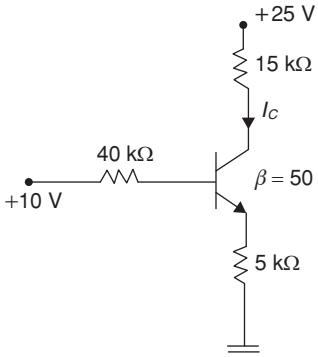
6.



The minimum value of R_B for which the transistor remains in Active region is _____

(A) 200 kΩ (B) 205 kΩ
(C) 20 kΩ (D) 21 kΩ

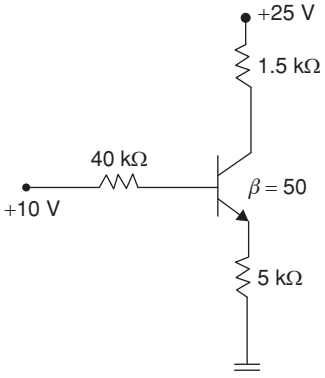
7.



The value of I_C is _____

(A) 1.57 mA (B) 3.15 mA
(C) 1.22 mA (D) 0.68 mA

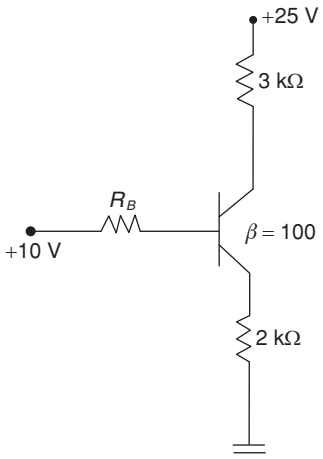
8.



Find the value of collector current, following through the circuit.

(A) 1.57 mA
(B) 3.15 mA
(C) 1.75 mA
(D) 3.51 mA

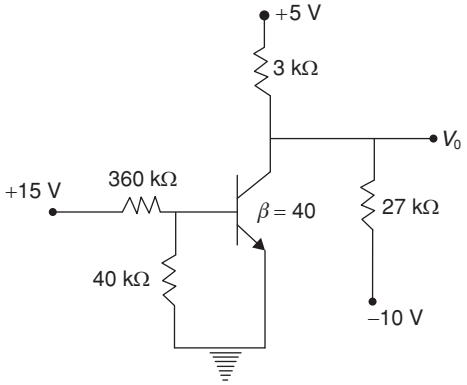
9.



The smallest value of R_B, such that the transistor is in active region.

(A) 24 kΩ (B) 2.4 kΩ
(C) 42 kΩ (D) 0 Ω (zero)

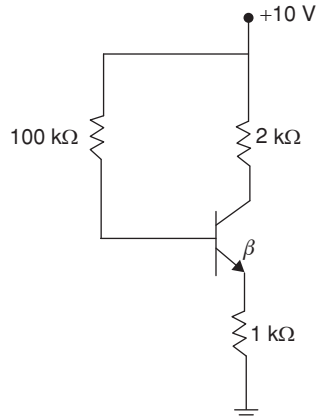
10.



Find the value of V_0.

(A) 3.5 V (B) +5 V
(C) –10 V (D) 1.1 V

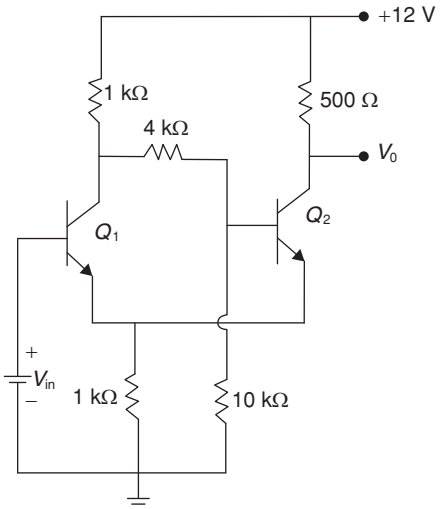
11.



Neglect junction voltages: Find the Minimum value of β, that will saturate the Transistor:

(A) 50 (B) 70
(C) 49 (D) 51

12.

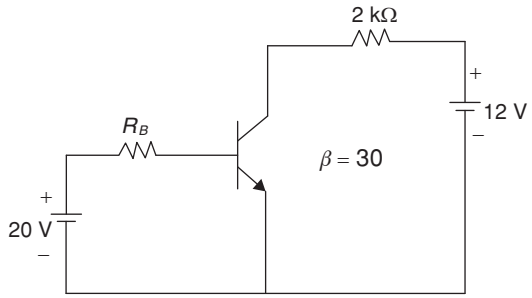


Neglect reverse saturation currents and assume each transistor has $\beta = 100$.

The value of V_o, if $V_{in} = 0$ V:

(A) 12 V (B) 9 V
(C) 8.5 V (D) 7.5 V

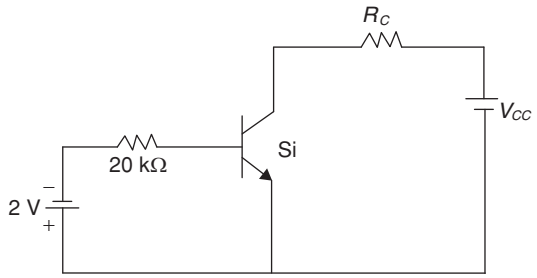
13.



The minimum value of R_B, which keeps the transistor in saturation region, is:

(A) 97 kΩ (B) 86 kΩ
(C) 125 kΩ (D) 68 kΩ

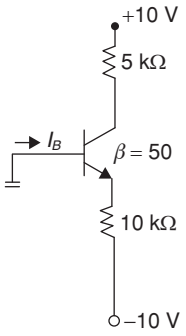
14.



If $I_{CB_O} = 10$ nA at 25°C, the maximum temperature that the transistor can with stand by keeping itself in cut-off region is:

(A) 148°C (B) 208°C
(C) 168°C (D) 188°C

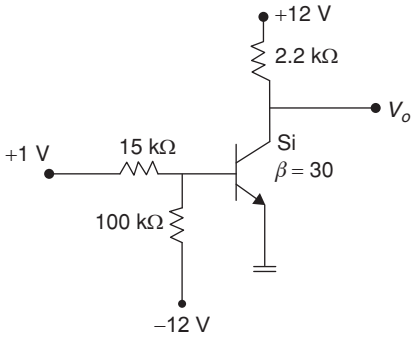
15.



If $V_{BE} = 0.7$ V, find V_{CE}.

(A) 5.45 V (B) 0.7 V
(C) 0.2 V (D) 6.15 V

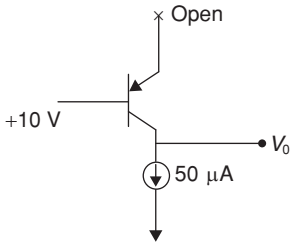
16.



Find the value of V_o

(A) 0.2 V (B) 12 V
(C) 7.6 V (D) 9.8 V

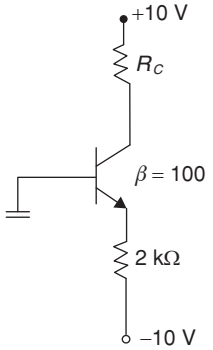
17.



Find the output voltage if the transistor has $BV_{CB_O} = 70$V:

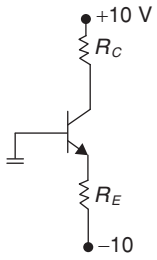
(A) –70 V (B) –10 V
(C) –10.7 V (D) –60 V

18. Find the largest value of R_C while maintaining the transistor in active mode:



(A) 2.28 kΩ (B) 2.42 kΩ
(C) 3.21 kΩ (D) 4.23 kΩ

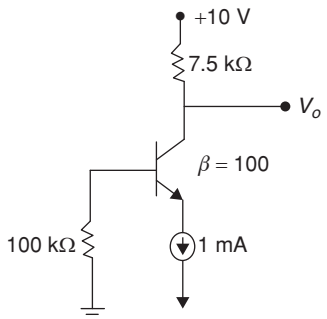
19.



Assume large value of β. Find the values of R_C and R_E, to achieve $I_C = 1$ mA and $V_{CB} = +4$ V.

(A) 9.3 K, 6 K
(B) 6 K, 10.7 K
(C) 3.9 K, 10.7 K
(D) 6 K, 3.9 K

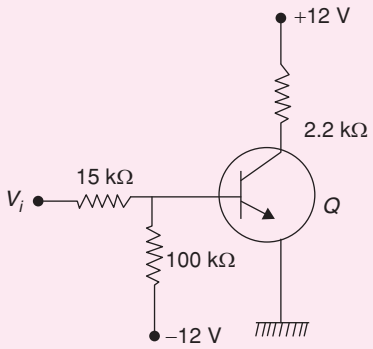
20.



The value of V_o for the given ckt is:
(A) 2.5 V
(B) 2.4 V
(C) 2.6 V
(D) 10 V

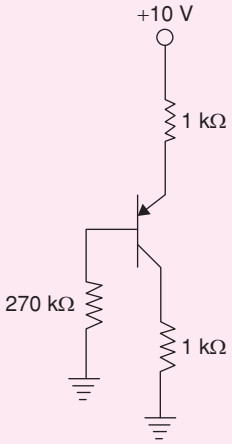
Previous Years' Questions

1. Consider the circuit shown in figure. If the β of the transistor is 30 and I_{CB_O} is 20 nA and the input voltage is +5 V, then transistor would be operating in: **[2006]**



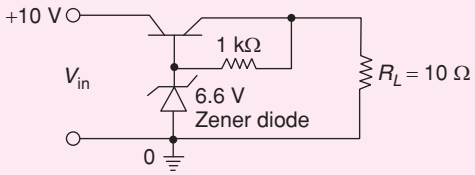
(A) Saturation region
(B) Active region
(C) Breakdown region
(D) Cut-off region

2. The common emitter forward current gain of the transistor shown is $\beta_F = 100$. **[2007]**



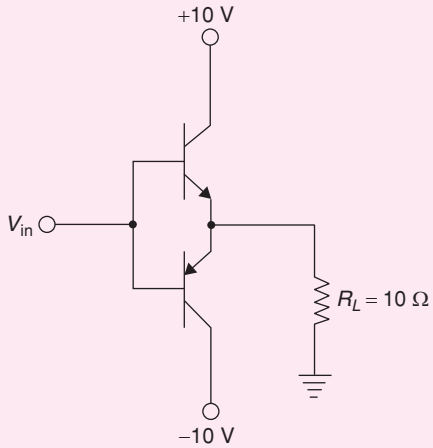
The transistor is operating in:
(A) Saturation region
(B) Cutoff region
(C) Reverse active region
(D) Forward active region

3. The three-terminal linear voltage regulator is connected to a 10 Ω load resistor as shown in the figure. If V_{in} is 10 V, what is the power dissipated in the transistor? **[2007]**



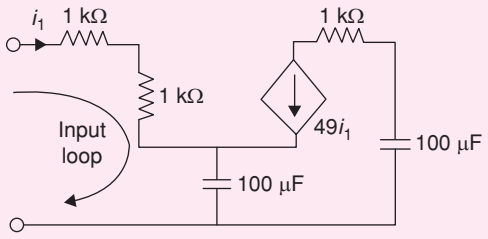
(A) 0.6 W
(B) 2.4 W
(C) 4.2 W
(D) 5.4 W

4. The input signal V_{in} shown in the figure is a 1 kHz square wave voltage that alternates between +7 V and –7 V with a 50% duty cycle. Both transistors have the same current gain, which is large. The circuit delivers power to the load resistor R_l. What is the efficiency of this circuit for the given input? Choose the closest answer. **[2007]**



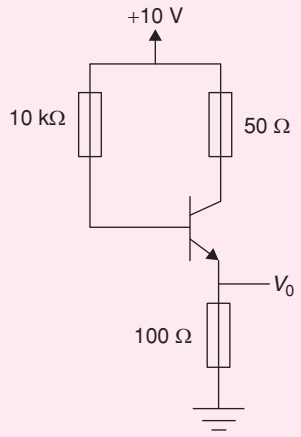
(A) 46%
(B) 55%
(C) 63%
(D) 92%

5. The equivalent capacitance of the input loop of the circuit shown is: **[2009]**



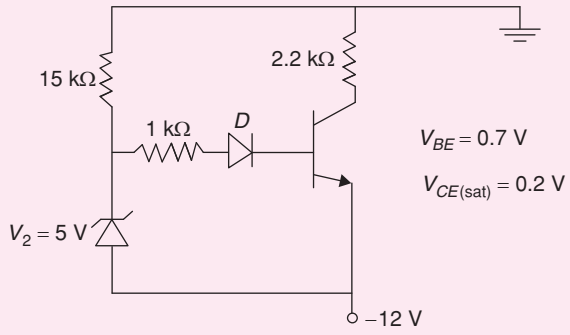
(A) 2 μF (B) 100 μF
(C) 200 μF (D) 4 μF

6. The transistor circuit shown uses a silicon transistor with $V_{BE} = 0.7$ V, $I_C \approx I_E$ and a DC current gain of 100. The value of V_o is: **[2010]**



(A) 4.65 V (B) 5 V
(C) 6.3 V (D) 7.23 V

7. The transistor used in the circuit shown below has a β of 30 and I_{CB_O} is negligible. **[2011]**

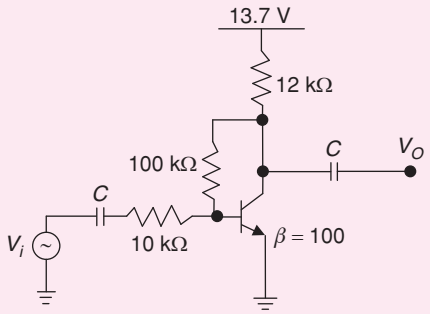


If the forward voltage drop of diode is 0.7 V, then the current through collector will be
(A) 168 mA (B) 108 mA
(C) 20.54 mA (D) 5.36 mA

8. Transformer and emitter follower can both be used for impedance matching at the output of an audio amplifier. The basic relationship between the input power P_{in} and output power P_{out} in both the cases is: **[2012]**

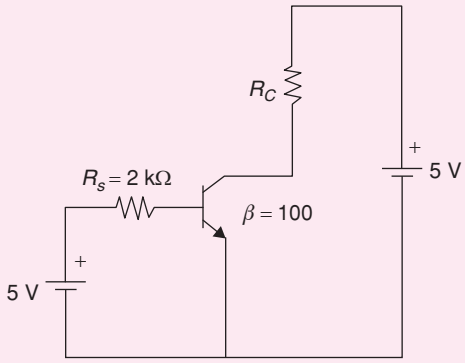
(A) $P_{in} = P_{out}$ for both transformer and emitter follower
(B) $P_{in} > P_{out}$ for both transformer and emitter follower
(C) $P_{in} < P_{out}$ for transformer and $P_{in} = P_{out}$ for emitter follower
(D) $P_{in} = P_{out}$ for transformer and $P_{in} < P_{out}$ for emitter follower

9. The voltage gain A_v of the circuit shown below is: **[2012]**

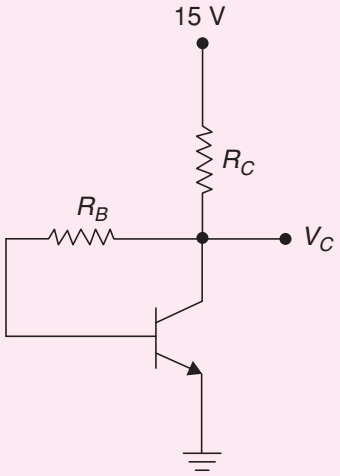


(A) $|A_v| \approx 200$ (B) $|A_v| \approx 100$
(C) $|A_v| \approx 20$ (D) $|A_v| \approx 10$

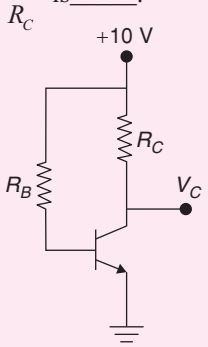
10. The transistor in the given circuit should always be in active region. Take $V_{CE(sat)} = 0.2$ V, $V_{BE} = 0.7$ V. The maximum value of R_c in Ω which can be used is _____. **[2014]**



11. In the given circuit, the silicon transistor has $\beta = 75$ and a collector voltage $V_C = 9$ V. Then the ratio of R_B and R_C is _____. **[2015]**



12. The following circuit, the transistor is in active mode and $V_C = 2$ V. To get $V_C = 4$ V, we replace R_C with R'_C. Then the ratio $\frac{R_C^1}{R_C}$ is_____. **[2015]**



13. When a bipolar junction transistor is operating in the saturation mode, which one of the following statements is TRUE about the state of its collector-base (CB) and the base-emitter (BE) junctions? **[2015]**

(A) The CB junction is forward biased and the BE junction is reverse biased.

(B) The CB junction is reverse biased and the BE junction is forward biased.

(C) Both the CB and BE junctions are forward biased.

(D) Both the CB and BE junctions are reverse baised.

14. A transistor circuit is given below. The Zener diode breakdown voltage is 5.3 V as shown. Take base to emitter voltage drop to be 0.6 V. The value of the current gain β is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. C	**2.** C	**3.** B	**4.** B	**5.** A	**6.** C	**7.** B	**8.** B	**9.** B	**10.** C
11. B	**12.** B	**13.** C	**14.** B	**15.** B	**16.** B	**17.** A	**18.** B	**19.** B	**20.** C
21. A	**22.** C	**23.** A	**24.** B						

Practice Problems 2

1. B	**2.** C	**3.** A	**4.** A	**5.** B	**6.** B	**7.** C	**8.** A	**9.** D	**10.** D
11. A	**12.** C	**13.** A	**14.** B	**15.** D	**16.** B	**17.** D	**18.** A	**19.** B	**20.** C

Previous Years' Questions

1. B	**2.** D	**3.** B	**4.** B	**5.** A	**6.** A	**7.** D	**8.** D	**9.** D
10. 22.32	**11.** 105.133		**12.** 0.74 to 0.76		**13.** C	**14.** 18.0 to 20.0		

Chapter 3

Field Effect Transistors

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- JFET
- Parameters of FET
- Relation between BJT and JFET
- Comparisons of JFET and BJT
- Introduction to MOSFET
- D–MOSFET
- E–MOSFET
- The body effect
- N–MOS transistor
- PMOS transistors
- Conditions for saturation region

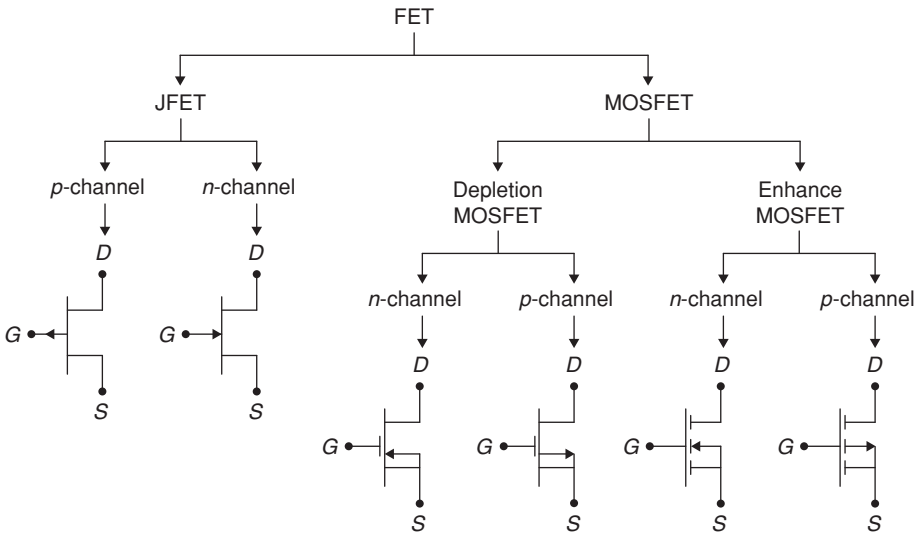
Introduction

The bipolar junction transistor relies on two types of charge carriers i.e., free electrons and holes. Another kind of transistor is called the field effect transistor (FET). This type of device is unipolar because its operation depends on only one type of charge, either free electrons or holes. In other words, an FET has majority carriers but not minority carriers.

The FET is generally much less noisy than the BJT. BJT is a current controlled device, and FET is a voltage controlled current device.

These are two types:

1. Junction field effect transistors (JFET).
2. Metal oxide semiconductor field effect transistor (MOSFET).



JFET

A junction field effect transistor (see Figure 1) is a three terminal semiconductor device in which current conduction is by majority types of carriers i.e., electrons or holes. JFET has high input impedance and low noise level.

Construction of JFET

A junction field effect transistor is a three terminal semiconductor device in which current conduction is by one type of carrier i.e., electrons or holes.

A JFET consists of a *p*-type or *n*-type silicon material, containing two *p*-*n* junctions at the sides. If the bar is *n*-type, it is called a *n*-channel JFET and if the bar is of *p*-type, it is called a *p*-channel JFET.

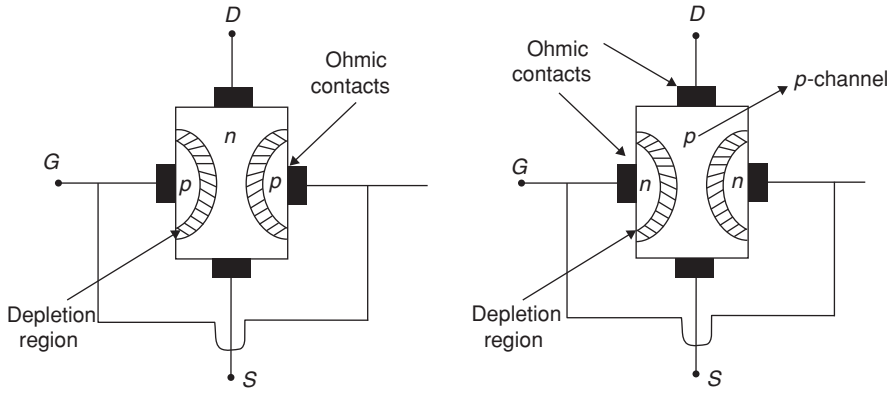


Figure 1 Junction field effect transistors.

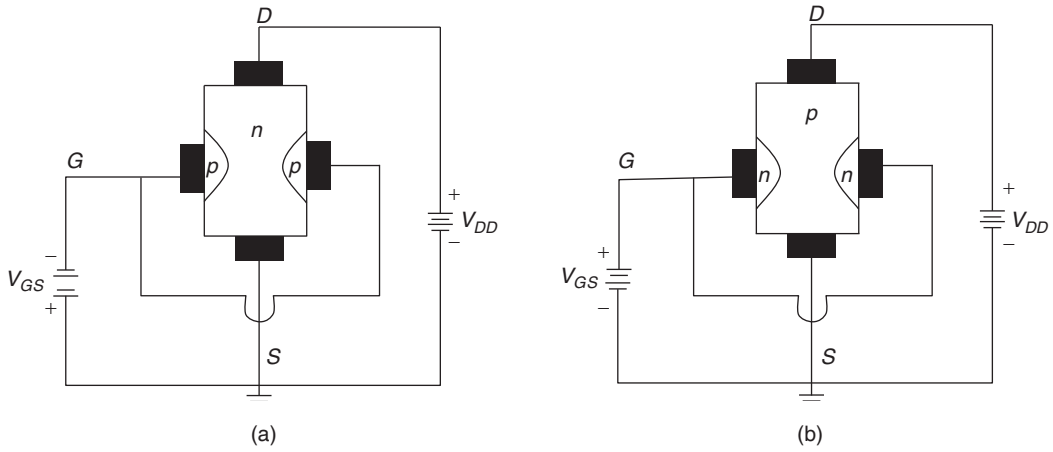


Figure 2 JFET polarities. (a) *n*-channel JFET, (b) *p*-channel JFET

The input circuit (gate to source) of a JFET is reverse biased and output circuit (drain to source) of a JFET is forward bias, i.e., the device has high input impedance and low output impedance.

Working Principle

The two *p*–*n* junctions at the sides form two depletion layers. The current conduction by charge carriers (majority carriers) is through the channel between the two depletion layers and out of the drain. The width of the channel, i.e., resistance of this channel can be controlled by changing the input voltage V_{GS}.

The reverse voltage V_{GS} increases, the wider will be the depletion layers and narrower will be the conduction channel i.e., channel width decreases means resistance of the channel increases, so drain to source current decreases. Otherwise the reverse will be happen when V_{GS} decreases.

Thus, JFET operates on the principle that width and resistance of the conducting channel can be varied by changing the reverse voltage V_{GS}.

***Case* (1):** For *n*-channel JFET

$$V_{GS} = 0 \text{ V}, V_{DS} = 0 \text{ V}$$

In this case drain current $I_D = 0$, because $V_{DS} = 0$ V.

***Case* (2):** When $V_{GS} = 0$ V and V_{DS} some +ve value

As the voltage V_{DS} is increased from 0 V to a few Volts, the current will increase as determined by ohm's law and the characteristics plot of $I_D V_S V_{DS}$ shown in Figure 4.

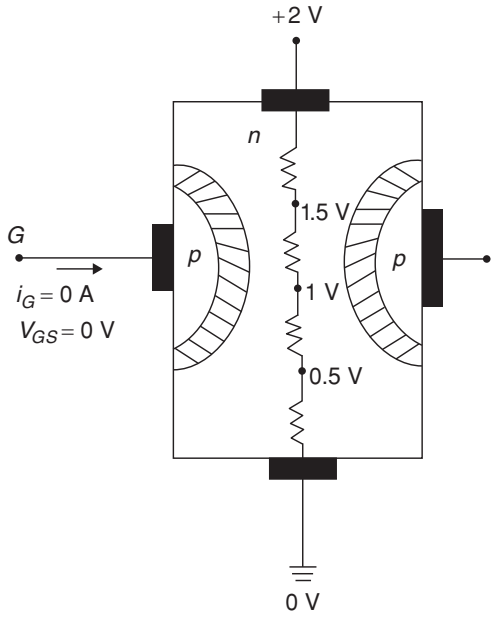


Figure 3 Varying reverse bias potential across the *p-n* junction of an *n*-channel JFET.

I–V Characteristics

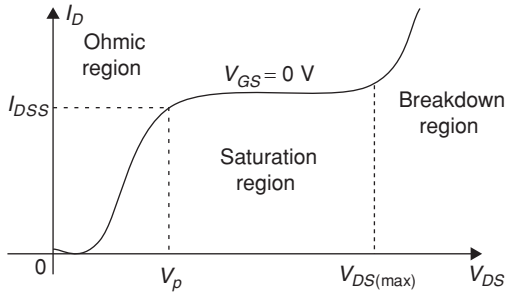


Figure 4 I_D vs. V_{DS} for V_{GS} = 0 V

The ohmic relationship between V_{DS} and I_D continues till V_{DS} reaches a certain critical value is called pinch off voltage (V_p).

If the drain voltage exceeds $V_{DS(max)}$, JFET enters into break down region.

The region between the V_P and $V_{DS(max)}$ is called constant current region because when V_{DS} equal to V_P, the channel is effectively closed and does not allow further increase in drain current shown in Figure 5.

***Case* (3):** If $V_{GS} < 0$ and V_{DS} = Constant

If the reverse voltage V_{GS} on the gate is continuously increased, a state is reached when the two depletion layers touch each other at $|V_{GS(off)}| = |V_P|$ and the channel is cut-off.

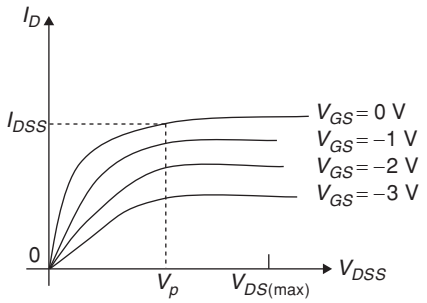
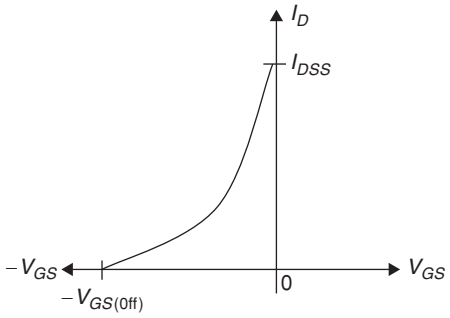


Figure 5 *n*-channel JFET characteristics.



***Shorted-gate drain current* (I_{DSS})** It is the drain current with source, short circuited to gate ($V_{GS} = 0$) and drain voltage (V_{DS}) equal to pinch off voltage. It is the maximum drain current (I_{DSS}). The region between V_P and $V_{DS(max)}$ (break down voltage) is called constant-current region or saturation region. JFET behaves as a constant-current device. For proper working of JFET, it must be operated in the saturation region or active mode.

***Pinch off Voltage* (V_P)** It is the minimum drain-source voltage at which the drain current becomes constant. For values of V_{DS} greater than V_P, the drain current is almost constant. Because when V_{DS} is equal to V_P, the channel is effectively closed and does not allow further increase in drain current. However, V_{DS} should not exceed $V_{DS(max)}$ otherwise JFET may break down.

***Gate-source cut-off Voltage* $V_{GS(off)}$** It is the gate-source voltage where the channel is completely cut off and the drain current becomes zero. As the reverse gate-source voltage is increased, the cross sectional area of the channel decreases. This in turn decreases the drain current. At some gate to source voltage, the depletion region is increased and touches each other. In this condition, the channel is cut off and the drain current reduces to zero.

$$V_P = |V_{GS(off)}|$$

Parameters of GFET

Drain current (I_D)

$$I_D = I_{DSS}\left[1 - \frac{V_{GS}}{V_P}\right]^2 \text{A}$$

Where

I_D = drain current at given V_{GS}

I_{DSS} = short circuit gate drain current

V_{GS} = gate-source voltage

V_P = gate-source cut-off voltage (or) pinch off voltage

Drain resistance (r_d)

$$r_d = \frac{\Delta V_{DS}}{\Delta I_D}; \text{ at constant } V_{GS}$$

The drain resistance of a JFET has a large value range from 10 kΩ to 1 MΩ.

Transconductance (g_m)

It is the ratio of change in drain current (ΔI_D) to the change in gate-source voltage (ΔV_{GS}) at constant drain-source voltage.

$$g_m = \frac{\Delta I_D}{\Delta V_{GS}} \text{ at constant } V_{DS}.$$

Amplification factor (μ)

It is the ratio of change in drain-source voltage (ΔV_{DS}) to the change in the gate to source voltage (ΔV_{GS})

$$\mu = \frac{\Delta V_{DS}}{\Delta V_{GS}}; \text{ at constant } I_D.$$

Relation among JFET parameters

$$\mu = \frac{\Delta V_{DS}}{\Delta V_{GS}} \times \frac{\Delta I_D}{\Delta I_D}$$

$$\mu = \frac{\Delta V_{DS}}{\Delta I_D} \times \frac{\Delta I_D}{\Delta V_{GS}} = r_d \times g_m$$

$\therefore \mu = g_m \cdot r_d$

$$g_m = g_{m_o}\left[1 - \frac{V_{GS}}{V_P}\right]$$

Where,

$$g_{m_o} = \frac{2I_{DSS}}{|V_P|} \text{ or } \frac{-2I_{DSS}}{V_P}$$

Or

$$g_m = g_{m_o} \cdot \sqrt{\frac{I_D}{I_{DSS}}}$$

DC drain resistance, $R_{DS} = \frac{V_{DS}}{I_D}$

voltage gain $A_V = -g_m \cdot R_D$

Solved Examples

Example 1: A JFET has $V_p = -4.5$ V, $I_{DSS} = 10$ mA and $I_D = 2.5$ mA. Determine the transconductance.

Solution: $g_m = \frac{-2I_{DSS}}{V_p}\left[1 - \frac{V_{GS}}{V_P}\right]$

$$V_{GS} = V_P\left[1 - \sqrt{\frac{I_D}{I_{DSS}}}\right]$$

$$= 2.22 \text{ mA/V}$$

Example 2: For an *n*-channel JFET $I_{DSS} = 8.7$ mA, $V_p = -3$ V, $V_{GS} = -1$ V. Find the values of

(A) I_D (B) g_{m_0} (C) g_m

Solution:

(A) $I_D = I_{DSS}\left[1 - \frac{V_{GS}}{V_P}\right]^2 = 3.8667$ mA

(B) $g_{m_0} = \frac{-2I_{DSS}}{V_P} = 5.8$ mA/V

(C) $g_m = g_{m_0}\left(1 - \frac{V_{GS}}{V_P}\right) = 3.867$ mA/V

Example 3: For a *n*-channel JFET, $I_{DSS} = 8$ mA, $V_p = -4$ V, $V_{GS} = -1$ V. Determine I_D, g_{m_o} and g_m.

Solution: $I_D = I_{DSS}\left(1 - \frac{V_{GS}}{V_P}\right)^2$

$$= 8 \times 10^{-3}\left(1 - \frac{(-1)}{(-4)}\right)^2$$

$I_D = 4.5$ mA

$g_{m_o} = \frac{-2I_{DSS}}{V_p}$

$= 4$ mA/V

$g_m = g_{m_o}\left(1 - \frac{V_{GS}}{V_p}\right) = 3$ mA/V.

Example 4: In a JFET, the drain current is changed by 0.25 mA when the gate to source voltage is changed by 0.25 V, keeping drain-source voltage constant. Calculate the transconductance of the JFET.

Solution: $g_m = \left.\frac{\Delta I_D}{\Delta V_{GS}}\right|_{V_{DS}=\text{constant}}$

$$= \frac{0.25 \times 10^{-3}}{0.125}$$

$g_m = 2$ mA/V.

Example 5: A JFET has $g_m = 10$ ms, $I_{DSS} = 10$ μA. Calculate $V_{GS(OFF)}$.

Solution: $g_m = g_{m_o} = \dfrac{-2I_{DSS}}{V_P} = \dfrac{-2I_{DSS}}{V_{GS(OFF)}}$

$$10\times10^{-3} = \frac{-2\times10\times10^{-6}}{V_{GS(OFF)}}$$

$$V_{GS(OFF)} = -2 \text{ mV}$$

Example 6: A FET has a drain current of 4 mA. If I_{DSS} = 6 mA and $V_{GS(OFF)} = -6$ V. Find the values of V_{GS} and V_p.

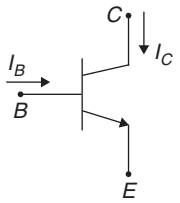
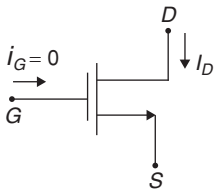
Solution: $I_D = 4$ mA; $I_{DSS} = 6$ mA

$$V_{GS} = V_{GS(OFF)}\left[1-\sqrt{\frac{I_D}{I_{DSS}}}\right]$$

$$= -6\left[1-\sqrt{\frac{4\text{ mA}}{6\text{ mA}}}\right] = -1.1 \text{ V}$$

Pinch-off voltage, $V_p = V_{GS(OFF)} = -6$ V

Table 1 *Relation between BJT and JFET*

S. No.	BJT	JFET
i.		
ii.	$I_C = \beta \cdot I_B$	$I_D = I_{DSS}\left(1-\dfrac{V_{GS}}{V_P}\right)^2$
iii.	$I_C \approx I_E$	$I_D = I_S$
iv.	$V_{BE} = 0.7$ V	$I_G \approx 0$ A
v.	Current controlled (I_B) current device (I_C)	voltage controlled (V_{GS}) current device (I_D)

Comparisons of JFET and BJT

1. FET is a unipolar device i.e., operation depends only on the flow of majority carriers. BJT operations depends on both minority and majority charge carriers.
2. FET is a voltage controlled device and BJT is a current controlled device.
3. FET is less noisy than BJT (minority carriers more noise)
4. FET is smaller than the BJT's
5. BJT's are faster than the FET's.
6. The input circuit of FET is reverse biased. (i.e., input impedance higher $R_{in} \approx 100$ MΩ). FET offers a larger input impedance and lower output impedance. So FET can act as an excellent buffer amplifier but the BJT has low input impedance because its input is forward biased.
7. FET does not suffer from minority charge carrier storage effects, it has 'higher switching speeds and cut-off frequency' BJT has lower switching speed and cut-off frequencies.
8. BJT's are cheaper than FET's.
9. FET amplifiers have low gain band width product due to the junction capacitance. (disadvantage)
10. FET has a negative (zero) temperature coefficient at high current levels, it prevents the FET thermal runaway. The BJT has a '+ve' temperature coefficient at high current levels which leads to thermal break down.
11. No current enters the gate of JFET i.e., $I_G = 0$ A. However, typical BJT base current I_B be a few μA.
12. In JFET, there are no junctions as in an ordinary transistor. The conduction is through an *n*-type or *p*-type semiconductor material.

Example 7: A JFET has I_{DSS} = 16 mA and $V_P = -4$ V. $V_{GS} = -2$ V, the value of drain current is

(A) 8 mA. (B) 4 mA.
(C) 16 mA. (D) 6 mA.

Solution: (B)

We know, $I_D = I_{DSS}\left[1-\dfrac{V_{GS}}{V_P}\right]^2$

$$= 16\left[1-\frac{(-2)}{(-4)}\right]^2 \times 10^{-3}$$

$$= 16 \times 10^{-3} \times 0.25 = 4 \text{ mA}.$$

Example 8: The transconductance of a JFET used as a voltage amplifier is 2500 $\mu\Omega^{-1}$ and drain resistance is 15 kΩ. The voltage gain of the amplifier is

(A) 25 (B) 30
(C) 37.5 (D) 40

Solution: (C)

Given $g_m = 2500 \times 10^{-6}\Omega^{-1}$

$R_D = 15 \times 10^3\Omega$

voltage gain $A_V = g_m \cdot R_D$

$= 2500 \times 10^{-6} \times 15 \times 10^3$

$= 37.5.$

Example 9: In a *p*-channel JFET, the charge carriers are

(A) Electrons. (B) Holes.
(C) Both electrons and holes. (D) Either e^-'s or holes.

Solution: (B)

Example 10: If the reverse bias voltage is increased, then the width of the conducting channel of JFET is

(A) Increased. (B) Decreased.
(C) Remains same. (D) None of the above

Solution: (B)

Example 11: The input control parameter of JFET is

(A) Source voltage. (B) Gate voltage.
(C) Drain voltage. (D) None of these

Solution: (B)

Example 12: The gate voltage in a JFET at which drain current becomes zero is called
(A) Break down (B) Active.
(C) Cut-off. (D) Saturation.

Solution: (C)

Example 13: In a common source JFET amplifier, the output voltage is
(A) In phase with the input. (B) 90° out of phase.
(C) 180°out of phase with input. (D) None of the above.

Solution: (C)

Example 14: The constant current region of a JFET lies between
(A) Cut-off and saturation.
(B) Cut-off and pinch off.
(C) Pinch off and break down regions.
(D) 0 and I_{DSS}.

Solution: (C)

Example 15: The trans-conductance of a JFET ranges from
(A) 0.5 to 30 mA/V. (B) 100 to 500 mA/V.
(C) 500 to 750 mA/V. (D) Above 750 mA/V.

Solution: (A)

Introduction to MOSFET

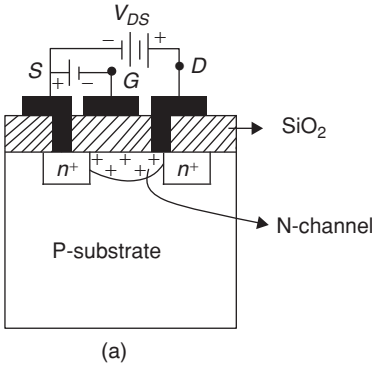
The main drawback of JFET is that its gate must be reverse biased for proper operations of the device i.e., it can only have negative gate operation for *n*-channel and positive gate operation for *p*-channel. The name MOSFET stands for "Metal Oxide Semiconductor Field Effect Transistor". The insulating layer between the gate and the channel has resulted in another name for the device, insulated gate FET or IGFET.

MOSFET's broadly classified into two categories. These are

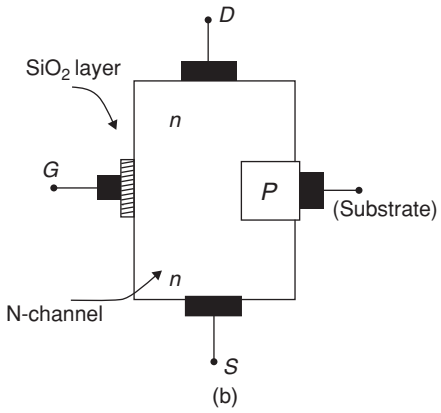
1. Depletion mode MOSFET or D-MOSFET
2. Enhancement mode MOSFET or E-MOSFET

D-MOSFET

Construction of the n-channel depletion mode MOSFET



(a)



(b)

Figure 6 (a) *N*-Channel, (b) D-MOSFET

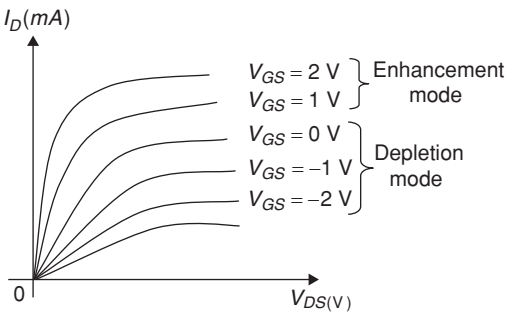
The D-MOSFET can be operated in both the depletion-mode and the enhancement-mode. The device consists of a *p*-type substrate into which are diffused two heavily doped n^+-type semiconductor blocks. In between the blocks, is a lightly doped *n*-type channel. The majority carriers e^-'s flowing from source to drain must pass through the narrow channel between the gate and the *p*-type region.

A thin layer of metal oxide (SiO_2) is deposited over a small portion of the channel. A metallic gate is deposited over the oxide layer. A SiO_2 is an insulator; therefore gate is insulated from the channel. Since the gate is insulated from the channel, we can apply either negative (or) positive voltage to the gate. Since, D-MOSFET can be operated in both depletion-mode and enhancement-mode JFET can be operated only In depletion-mode. If a '+ve' potential is applied to the drain, since electrons of the *n*-type drain block are attracted towards the +ve terminal. If a '–ve' voltage is applied to the gate, +ve charge carriers would be induced in the *n*-type channel. There tend to join with free e^-'s present in the channel, they creating a depletion region and increasing the channel resistance the MOSFET is termed as depletion mode MOSFET.

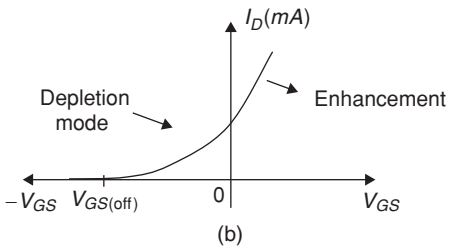
If the gate is given +ve bias with respective source, the number of charge carriers i.e., e^-'s in the *n*-channel further increases, with the result that the drain current increases. It means that a '+ve' gate bias the depletion mode MOSFET can be operated as enhancement mode MOSFET shown in Figure 6.

Transfer characteristics of D-MOSFET

The drain and transfer characteristics of a *n*-channel depletion–enhancement mode MOSFET.



(a)



Fifure 7 (a) Drain characteristic, (b) transfer characteristics

Enhancement Mode MOSFET or E-MOSFET

There are two types, enhancement *p*-channel and *n*-channel types.

p-channel E-MOSFET

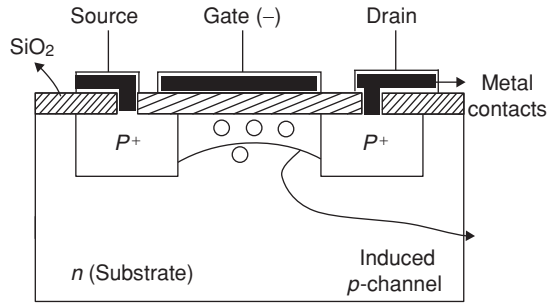


Figure 8 *p*-channel Enhancement type

A lightly doped *n*-type substrate into which two highly doped p^+ regions are diffused. The p^+ regions act as source and drain. A thin layer of SiO_2 is grown over the surface of the structure, and holes are cut into the oxide layer, allowing contact with the source and drain. The gate metal area is overlaid on the oxide, covering the entire channel region. Metal contacts are made to the drain and source. When the gate is given negative supply voltage, positive charges will be induced is the substrate between the source and drain, forms an inversion layer. Current flows from source to drain through the channel.

n-Channel E-MOSFET

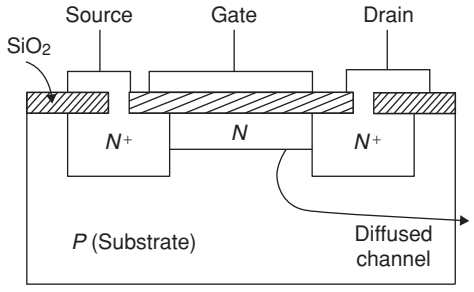


Figure 9 *n*-channel E-MOSFET

A *n*-channel is diffused between two *n*+ regions is a *p*-substrate. With gate potential is zero, maximum drain current flows. When gate is given negative supply, positive ions are induced into the channel, thus reduces the conductivity. The operation is similar to that of JFET.

$$I_D = I_{DSS} = \left(1 - \frac{V_{DS}}{V_T}\right)^2 \text{ for } |V_{DS}| \geq |V_T|$$

E-MOSFET operates only in the enhancement mode and has no depletion mode, and the E-MOSFET has no physical channel from source to drain because the substrate extends completely to the SiO_2 layer. The minimum value of V_{GS} of proper polarity that turn 'ON' the E-MOSFET is called 'Threshold voltage' [$V_{GS(Th)}$]. The *n*-channel device requires positive $V_{GS} \geq V_{GS(th)}$ and the *p*-channel device requires negative $V_{GS} (\geq V_{GS(th)})$.

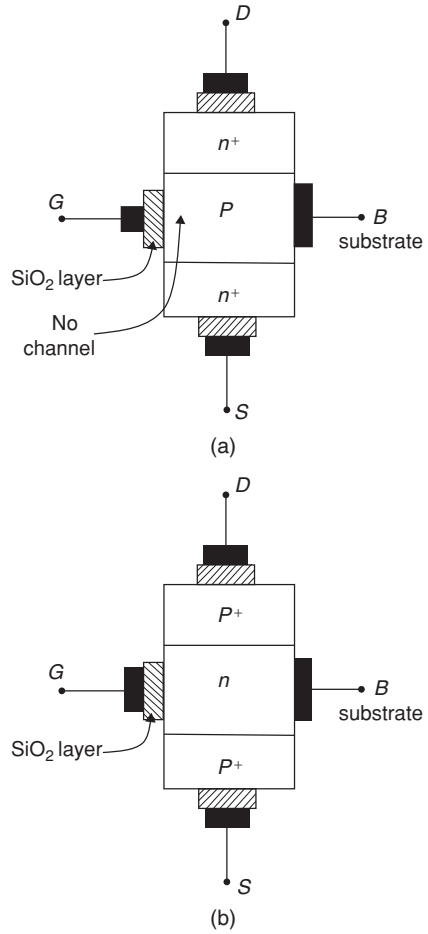


Figure 10 (a) *n*-channel E-MOSFET, (b) *p*-channel E-MOSFET

It does not conduct when gate-source voltage $V_{GS} = 0$. This is also called normally-off MOSFET. In these MOSFETs drain current I_D flows only when $V_{GS} \geq V_{GS(th)}$.

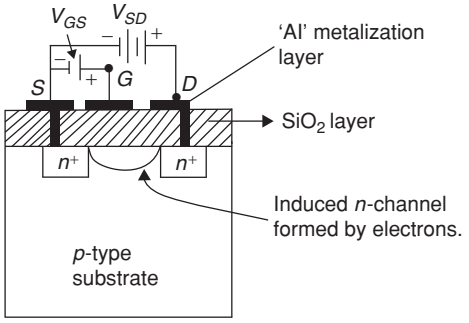


Figure 11 Operation of *n*-channel E MOSFET

Case (1): $V_{GS} = 0$ V and V_{DS} = +ve voltage

When V_{DS} = +ve voltage and no potential is applied to the gate, the two *n*-regions and one *p*-substrate form two *p*-n junctions connected back-back with a resistance of the *p*-substrate. Both the junctions are in reverse biased at the same time. So only a reverse leakage current flows.

The minimum value of gate-to-source voltage V_{GS} that is required to form the inversion layer (*N*-type) is termed the gate-to-source threshold voltage $V_{GS(th)}$.

If $V_{GS} < V_{GS(th)}$ then $I_D = 0$ A (OFF)

$V_{GS} \geq V_{GS(th)}$ (ON)

An *n*-type inversion layer connects the source to drain and the drain current I_D is large depending upon the device being used $V_{GS(th)}$ may vary from less than 1 V to more than 5 V.

JFET and D-MOSFET are classified as the depletion mode devices because their conductivity depends on the action of depletion layers. E-MOSFET is classified as an enhancement mode device because its conductivity depends on the action of the inversion layer. Depletion mode devices are normally ON when $V_{GS} = 0$ V also. Whereas enhancement-mode devices are normally OFF when $V_{GS} = 0$ V shown in Figure 12.

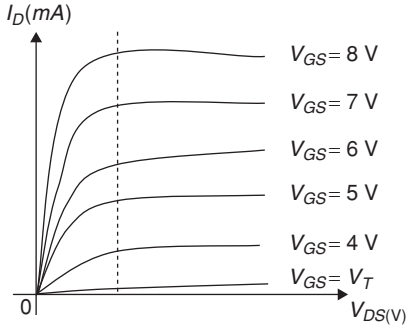


Figure 12 Drain characteristic of an *n*-channel enhancement MOSFET

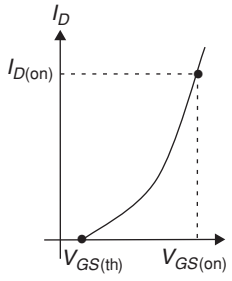


Figure 13 Transconductance curve for *n*-channel E-MOSFET

Drain current relations

$$I_D = K[V_{GS} - V_T]^2$$

Where, $V_T = V_{GS(th)}$ = Threshold voltage

$$K = \frac{I_{D(on)}}{[V_{GS(on)} - V_{GS(th)}]^2}$$

Increasing order of input impedance:

BJT < op-amp < JFET < MOSFET

Relationship between I_D and V_{GS}

MOSFET in triode region

$$V_{GS} \geq V_{th}$$

$$I_D = \mu_n C_{ox} \cdot \frac{W}{L}\left[[V_{GS} - V_T] \cdot V_{DS} - \frac{V_{DS}^2}{2}\right]$$

Where, $K'_n = \mu_n C_{ox}$

$\Rightarrow$ transconductance parameter

$\frac{W}{L}$ $\Rightarrow$Aspect ratio

$$C_{ox} = \frac{\varepsilon_{ox}}{t_{ox}}$$

$$\therefore I_D = K_n' \frac{W}{L}\left[(V_{GS} - V_T) \cdot V_{DS} - \frac{1}{2}V_{DS}^2\right]$$

$$I_D = K\left[(V_{GS} - V_T)V_{DS} - \frac{1}{2}V_{DS}^2\right]$$

Where, $K = \frac{\mu_n C_{ox} W}{L}$

Case (2): MOSFET in saturation region

$V_{GS} \geq V_T$ and $V_{DS} \geq (V_{GS} - V_T)$

Let $V_{DS(sat)} = V_{GS} - V_T$

$$\therefore \quad I_D = \frac{K_n'}{2}\frac{W}{L}[(V_{GS} - V_T)^2] = K(V_{GS} - V_T)^2$$

Where, $K = \frac{\mu_n C_{ox} W}{2L}$

For *n*-channel E-MOSFET

The Body Effect

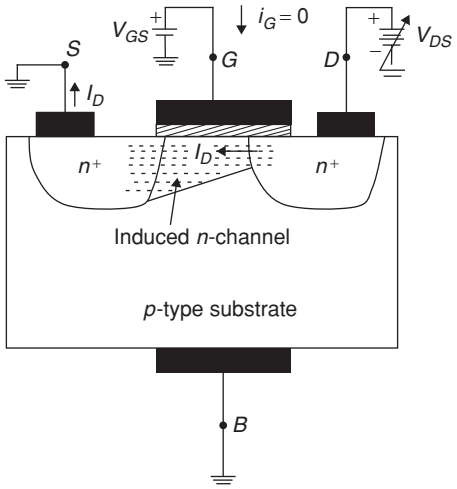


Figure 14 Enhancement *n*-MOS transistor as V_{DS} is increased.

In many applications the source terminal is connected to the substrate (or body) terminal B, which results in the p-n junction between the substrate and the induced channel having a constant zero (cut-off) bias. In such a case the substrate does not play any role in circuit operation and its existence can be ignored altogether shown in Figure 14.

In Integrated circuits, the substrate is usually common to many MOS transistors.

In order to maintain the cut off condition for all the substrate-to-channel junctions, the substrate is usually connected to the most negative power supply. In an N-MOS circuit, (the most positive in a P-MOS circuit). The resulting reverse-bias voltage between source and body (V_{SB} in an n-channel) will have an effect on device operation.

The reverse bias voltage will widen the depletion region; this in turn reduces the channel depth. To return the channel to its former state V_{GS} has to be increased the effect of V_{SB} on the channel can be represented as a change in the V_T.

$$V_T = V_{TO} + \gamma[\sqrt{2\phi_f + V_{SB}} - \sqrt{2\phi_f}]$$

Where,

$V_{T0} \Rightarrow$ Threshold voltage for $V_{SB} = 0$;

$\varphi_f \Rightarrow$ constant ($2\varphi_f \approx 0.6$ V)

$$\gamma = \frac{\sqrt{2qN_A\varepsilon_s}}{C_{ox}}$$

$V_T \rightarrow$ Threshold voltage

From the above equation it indicates that an incremental change in V_{SB} gives rise to an incremental change in V_T i.e., in an incremental change in I_D even through V_{GS} = constant.

Since the body voltage controls I_D, then the body acts as another gate for the MOSFET, this is known as the body effect.

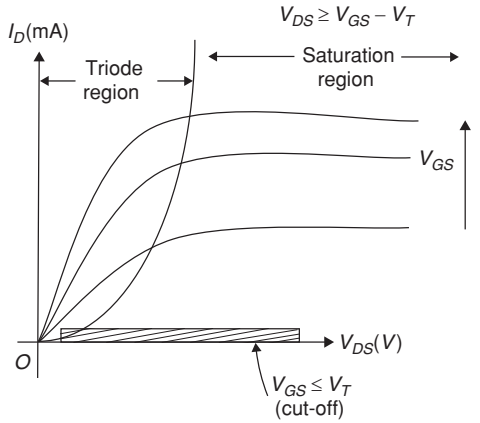
Note: Both V_T and K are temperature sensitive the magnitude V_T decreases by about 2 mV/°C rise in temperature and K' decrease with temperature.

$\therefore V_T \downarrow \Rightarrow I_D \uparrow K' \downarrow \downarrow$ (more decrement)

$\Rightarrow$ i.e., $I_D \downarrow$ (Overall effect)

$\therefore$ The overall observed effect of a temperature increases in a decrease in drain current.

Summary of the MOSFET Current – Voltage Characteristics



N-MOS Transistor

1. Triode region: conditions.

   $V_{GS} \geq V_T$ (MOSFET ON)

   $V_{DS} < V_{GS} - V_T$ (continuous channel)

$$C_{ox} = \frac{\varepsilon_{ox}}{t_{ox}}$$

I-V relationships

$$I_D = \mu_n \cdot C_{ox} \frac{W}{L}\left[(V_{GS} - V_T)V_{DS} - \frac{1}{2}V_{DS}^2\right]$$

For $V_{DS} << 2(V_{GS} - V_T)$

$$\therefore r_{ds} = \frac{V_{DS}}{I_{DS}} = \frac{1}{\left[\mu_n C_{ox} \cdot \frac{W}{L}(V_{GS} - V_T)\right]}$$

2. Operation in the saturation region: conditions.

   $V_{GS} \geq V_T$, $V_{DS} \geq V_{GS} - V_T$

   $\therefore V_{DS(sat)} = V_{GS} - V_T$

I-V relationships

$$I_D = \frac{1}{2}\mu_n C_{ox} \frac{W}{L}(V_{GS} - V_T)^2(1 + \lambda V_{DS})$$

If $\lambda = 0$

$$I_{Dsat} = \frac{1}{2}\mu_n C_{ox} \cdot \frac{W}{L}(V_{GS} - V_T)^2$$

Threshold voltage:

$$V_T = V_{TO} + \gamma(\sqrt{2\phi_f + |V_{SB}|} - \sqrt{2\phi_f})$$

Parameters

$$C_{ox} = \frac{\varepsilon_{ox}}{t_{ox}} \text{ (F/m}^2\text{)}$$

$$K'_n = \mu_n C_{ox} \text{ (A/V}^2\text{)}$$

$$\gamma = \frac{1}{V_A} (V^{-1})$$

$$\gamma = \frac{\sqrt{2qN_A\varepsilon_s}}{C_{OX}(V^{1/2})}$$

Constants

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

$\varepsilon_{ox} = 3.9\varepsilon_0$

$\varepsilon_s = 11.7\varepsilon_0$

$Q = 1.6 \times 10^{-19}$ C

PMOS Transistors

Conditions for triode region

$V_{GS} \leq V_T$ (ON State)

$V_{DS} \geq V_{GS} - V_T$ (continuous channel)

$$I_D = K'_p \frac{W}{L}\left[(V_{GS} - V_T)\cdot V_{DS} - \frac{1}{2}V_{DS}^2\right]$$

$$K'_p = \mu_P C_{ox}.$$

Conditions for saturation region

$V_{DS} \le V_{GS} - V_t$ (pinched-off channel)

$$I_D = \frac{1}{2}\cdot K'_P \frac{W}{L}[V_{GS} - V_T]^2(1 + \lambda \cdot V_{DS})$$

Where V_{GS}, V_T, λ and V_{DS} all are negative.
If $\lambda = 0$

$$I_D = \frac{1}{2}\mu_P \cdot C_{ox} \cdot \frac{W}{L}[V_{SG} - |V_T|]^2$$

Example 16: The threshold voltage of an *n*-channel MOSFET can be increased by
(A) Increasing the channel dopant concentration.
(B) Reducing channel dopant concentration.
(C) Reducing the gate-oxide-thickness.
(D) Reducing the channel length.

Solution: (C)

Example 17: A MOS capacitor made using *p*-type substrate is in the accumulation mode. The dominant charge in the channel is due to the presence of
(A) Holes. (B) Electrons.
(C) Positively charged ions. (D) Negatively charged ions.

Solution: (A)

Example 18: The drain of an *n*-channel MOSFET is shorted to the gate so that $V_{GS} = V_{DS}$. The threshold voltage (V_T) of MOSFET is 2 V. If the drain current I_D is 2.5 mA for $V_{GS} = 3$ V then for $V_{GS} = 5$ V I_D is
(A) 4 mA. (B) 9 mA.
(C) 15 mA. (D) 22.5 mA.

Solution: (D)
If $V_{GS} = V_{DS} \Rightarrow$ MOSFET in saturation region,

$$\therefore \quad I_D = \frac{k_n}{2}(V_{GS} - V_T)^2$$

$$I_D = 2.5 \text{ mA}$$

$$V_{GS} = 3 \text{ V}; \; V_T = 2 \text{ V}$$

$$2.5\times10^{-3} = \frac{k_n}{2}(3-2)^2$$

$$K_n = 5 \times 10^{-3} \text{ A/V}^2$$

$$K_n = 5 \text{ mA/V}^2$$

$$\therefore \quad I_D = \frac{K_n}{2}(V_{GS} - V_T)^2$$

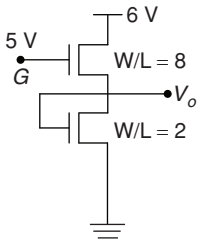
$V_{GS} = 5$ V then,

$$I_D = 2.5 \times 10^{-3}(5-2)^2$$

$$I_D = 22.5 \text{ mA}.$$

Example 19: In the circuit shown below for the MOS transistors. $\mu_n C_{ox}$ = 150 μA/V², and the threshold voltage V_T = 1 V. The voltage V_0 at the source of the upper transistor is
(A) 2 V. (B) 3 V.
(C) 4 V. (D) 5 V.

Solution: (B)



From the figure,

$$I_{DS_1} = I_{DS_2}$$

$$\therefore \quad \frac{K_{n_1}}{2}(V_{GS1} - V_T)^2 = \frac{K_{n_2}}{2}(V_{GS2} - V_T)^2$$

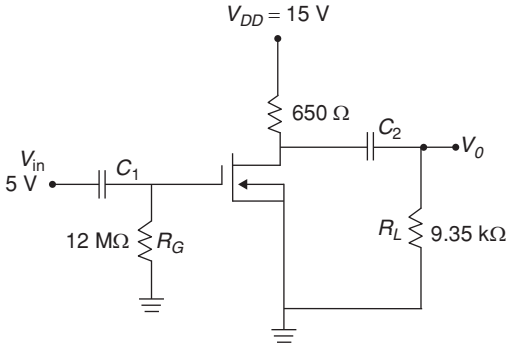
Where $V_{GS_1} = 5 - V_0$

$$\frac{1}{2}(150\times10^{-6})(8)(5 - V_o - 1)^2 = \frac{1}{2}\cdot(150\times10^{-6})(2)(V_0 - 1)^2$$

$$2(5 - V_o - 1) = (V_o - 1) \Rightarrow 8 - 2V_0 = V_o - 1$$

$$3V_o = 9 \Rightarrow V_o = 3 \text{ V}.$$

Common Data for Questions 20 and 21: The D-MOSFET used in the amplifier shown below, has an I_{DSS} = 10 mA and g_m = 3.5 mA/V



Example 20: The DC drain-to-source voltage V_{DS} is
(A) 5 V. (B) 7.5 V.
(C) 8.5 V. (D) 10 V.

Solution: (C)
The input signal is capacitive coupled to the gate.
For D.C value of $V_{GS} = 0$ V (C → open ciruit)

$$\therefore \; I_D = I_{DSS} = 10 \text{ mA}$$

$$V_{DS} = V_{DD} - I_D \cdot R_D$$

$$= 15 - 10 \times 10^{-3} \times 650 = 8.5 \text{ V}.$$

Example 21: The AC output voltage for V_{in} = 450 mA is
(A) 0.96 V. (B) 1.2 V.
(C) 1.5 V. (D) None of the above

Solution: (A)

$$R_{AC} = R_D \| R_L$$

$$= 650\ \Omega \| 9.35\ \text{k}\Omega$$

$$= \frac{6.077\times10^{6}}{10\times10^{3}}$$
$$= 608\ \Omega$$
$$V_{out} = A_v \cdot V_{in} = (g_m \cdot R_{AC}) \cdot V_{in}$$
$$= 3.5\times10^{-3}\times608\times450\times10^{-3}$$
$$= 0.96\ \text{V}.$$

Example 22: The MOSFET differs from a JFET mainly because
(A) The JFET has a pn junction.
(B) The MOSFET has two gates.
(C) Of power rating.
(D) None of the above

Solution: (A)

Example 23: The *p*-channel D-MOSFET with a negative V_{GS} is operating in
(A) The depletion mode. (B) The enhancement-mode.
(C) Cut-off region. (D) Saturation region.

Solution: (B)

Common Data for Questions 24 and 25: Consider a process technology for which $L_{min} = 0.5\ \mu m$, $t_{ox} = 10$ nm, $\mu_n = 500\ cm^2/V\text{-sec}$, and $V_T = 0.7$ V.

Example 24: Find C_{ox} and K_n
(A) $3.45\times10^{-3}\ F/m^2$, $172.5\ mA/V^2$
(B) $5.5\times10^{-3}\ F/m^2$, $172.5\ \mu A/V^2$
(C) $3.45\times10^{-3}\ F/m^2$, $172.5\ \mu A/V^2$
(D) $3.45\times10^{-6}\ F/m^2$, $172.5\ mA/V^2$

Solution: (C)

We know $C_{ox} = \frac{\varepsilon_{0x}}{t_{0x}} = \frac{3.95\cdot\varepsilon_0}{t_{0x}}$

$$= \frac{3.45\times10^{-11}}{10\times10^{-9}} = 3.45\times10^{-3}\ F/m^2.$$

$$K_n = \mu_n \cdot C_{ox}$$
$$= 500\times\frac{cm^2}{V-s}\times3.45\times10^{-3}\ F/m^2$$
$$= 500\times10^{-4}\ \frac{m^2}{V-sec}\times3.45\times10^{-3}\ F/m^2$$
$$= 500\times3.45\times10^{-7}\ A/V^2$$
$$K_n = 172.5\ \mu A/V^2.$$

Example 25: For a MOSFET with $\frac{W}{L} = 5$, calculate the values of V_{GS} and $V_{DS(min)}$ to operate the transistor in the saturation region with a DC current $I_D = 80\ \mu A$
(A) $V_{GS} = 1.13$, $V_{DS} = 0.43$ V
(B) $V_{GS} = 0.43$ V, $V_{DS} = 1.13$ V
(C) $V_{GS} = 2.25$ V, $V_{DS} = 1.55$ V
(D) None of the above

Solution: (A)

We know $I_D = \frac{K_n}{2}\cdot\frac{W}{L}(V_{GS}-V_T)^2$

$I_D = 80\ \mu A$

$$80\times10^{-6} = \frac{172.5}{2}\times10^{-6}\times5(V_{GS}-V_T)^2$$
$$(V_{GS}-V_T)^2 = 0.1855$$
$$V_{GS}-V_T = V_{DS(min)} = 0.43\ \text{V}$$
$$V_{GS} = 0.43 + 0.7 = 1.13\ \text{V}.$$

Example 26: The value of V_{GS} required causing the device to operate as a 1.25 kΩ resistor for very small V_{DS} is
(A) $V_{GS} = 1.2$ V (B) $V_{GS} = 1.8$ V
(C) $V_{GS} = 1.63$ V (D) $V_{GS} = 2$ V

Solution: (C)
For the MOSFET in the triode region V_{DS} very small

$$I_D = K_n \cdot \frac{W}{L}(V_{GS}-V_T)V_{DS}\left(\text{neglect}\ \frac{1}{2}V^2{}_{DS}\right)$$

We know $r_{DS} = \frac{V_{DS}}{i_D}$, small V_{DS}

$$r_{DS} = \frac{1}{K_n\left(\frac{W}{L}\right)(V_{GS}-V_T)}$$
$$1250 = \frac{1}{172.5\times10^{-6}(V_{GS}-V_T)\times5}$$
$$V_{GS}-V_T = 0.9275$$
$$V_{GS} = 1.63\ \text{V}.$$

Example 27: An *n*-channel JFET has $I_{DSS} = 1$ mA and $V_p = -4$ V. Its transconductance g_m (mA/V) for an applied gate to source voltage V_{GS} of –3 V is

Solution: $g_m = \frac{\partial I_D}{\partial V_{GS}} = \frac{2I_{DSS}}{|V_P|}\left(1-\frac{V_{GS}}{V_P}\right)$

$$= \frac{2\times1\times10^{-3}}{4}\left(1-\frac{-3}{-4}\right)$$
$$= \frac{2\times1\times10^{-3}}{4\times4} = 0.125\ \text{mA/V}.$$

Example 28: An ideal *p*-channel MOSFET has $C_{ox} = 8\times10^{-8}\ F/cm^2$. $\mu_p = 300\ cm^2/Vs$, $l = 2\ \mu m$, $t_{ox} = 250\ A°$, $V_{TP} = -0.8$ V, $w = 15\ \mu m$. If transistor is operating in non saturation region at $V_{SD} = 0.6$ V, value of g_m is

Solution: $I_D = \frac{\mu C_{ox} w}{2L}\left(2(V_{GS}+V_T)V_{SD} - V_{SD}^2\right)$

$$I_D = \frac{300\times8\times10^{-8}\times15\times10^{-4}}{2\times2\times10^{-4}}$$
$$\left[2(V_{SG}+V_T)V_{SD} - V_{SD}^2\right]$$
$$= 9\times10^{-5}\left[2(V_{SG}+V_T)V_{SD} - V_{SD}^2\right]$$
$$g_m = \frac{\partial I_D}{\partial V_{GS}} = 9\times10^{-5}\times2\times V_{SD}$$
$$= 0.6\times9\times10^{-5}\times2 = 0.108\ \text{ms}.$$

Example 29: For a given JFET, $I_{DSS} = 15$ mA, $V_P = -4$ V, $V_{DS} = 10$ V. Value of resistance R_S for a drain current of $I_{DS} = 7$ mA is

Solution: $I_D = I_{DSS}\left(1-\frac{V_{GS}}{V_P}\right)^2$

$$7\times10^{-3} = 15\times10^{-3}\left(1-\frac{(V_{GS})}{-4}\right)^2$$

$$0.6831 = 1-\frac{V_{GS}}{-4}$$

$$V_{GS} = 0.3168\times 4$$

$$= -1.2674$$

$$V_{GS} = -I_{DS}\cdot R_s$$

$$R_s = \frac{1.2674}{7\text{ mA}}$$

$$= 181.05\ \Omega.$$

Example 30: A JFET fixed bias configuration has an operating point defined by $V_{GSQ} = -2$ V and $I_{DQ} = 5.625$ mA with $I_{DSS} = 10$ mA and $V_P = -8$ V. $Y_{os} = 40$ µs. Determine the value of *Av* and *rd*.

Solution: $g_m = \frac{2I_{DSS}}{|V_p|} = \frac{2\times10\text{ mA}}{8\text{ V}} = 2.5$ ms

$$g_m = g_{m_o}\left(1-\frac{V_{gsQ}}{V_p}\right) = 2.5\text{ ms}$$

$$\left(1-\frac{-2}{-8}\right) = 1.88\text{ ms}$$

$$A_v = -g_m R_D = -1.88\text{ ms}\times 2\text{ k}\Omega = -3.76$$

$$R_d = \frac{1}{y_o} = \frac{1}{40\,\mu\text{s}} = 25\text{ k}\Omega.$$

Example 31: Consider an *n*-channel depletion mode MOSFET having the parameters $V_{TN} = -1.5$ V and $K_n = 1$ mA/V^2. If $V_{GS} = 0$ V and $V_{DS} = 0.3$ V then I_D is

Solution: $V_{DS(\text{sat})} = V_{GS} - V_{TN}$

$$= 0 - (-1.5) = 1.5\text{ V}.$$

$$V_{DS} < V_{DS(\text{sat})}$$

Biased in non-saturated region

$$I_D = kn\left[2(V_{GS}-V_{TN})V_{DS} - V_{DS}^2\right]$$

$$= 1[2(0-1.5)0.3-(0.3)^2]$$

$$= 1[2(1.5)0.3-(0.3)^2]$$

$$= [0.9-0.09] = 0.81\text{ mA}.$$

Common Data for Questions 32 and 33: Consider an *n*-channel Si JFET with the parameters $N_a = 5\times10^{15}$/cm^3, $N_d = 10^{18}$/cm^3. $a = 0.3$ µm, $L = 6$ µm, $w = 25$ µm, $\mu_n = 1000$ cm^2/VS. $V_{GS} = 0$

Example 32: The pinch off current I_{D1} is

Solution: $I_{D1} = \frac{\mu(eN_a)^2 wa^3}{6\varepsilon L}$

$$= \frac{1000\times(0.6\times10^{-19}\times5\times10^{15})^2\times25\times10^{-4}\times(0.3\times10^{-4})^3}{6\times(11.7\times8.85\times10^{-14}\times6\times10^{-4})}$$

$$= \frac{1000\times6.4\times10^{-7}\times 6.75\times10^{-17}}{3727.62\times10^{-18}}$$

$$= \frac{4.32\times10^{-20}}{3727.62\times10^{-18}} = 1.15\times10^{-5} = 0.015\text{ mA}.$$

Example 33: Maximum drain current $I_{DI(\text{sat})}$ is

Solution: $I_{D_1}(\max) = I_{D_1}\left[1-\frac{3Vbi}{Vpo}\left(1-\frac{2}{3}\sqrt{\frac{Vbi}{Vpo}}\right)\right]$

$$V_{po} = \frac{ea^2Na}{2\varepsilon_o}$$

$$= \frac{(1.6\times10^{-19})(5\times10^{15})\times(3\times10^{-4})^2}{2(11.7\times8.85\times10^{-14})}$$

$$= 0.347\text{ V}$$

$$V_{bi} = 0.0259\ln\left(\frac{(5\times10^{15}\times10^{18})}{(1.5\times10^{10})^2}\right)$$

$$= 0.0259\ln\left(\frac{5\times10^{33}}{2.25\times10^{20}}\right)$$

$$= 0.0259\ln(2.22\times10^{13}) = 0.79593\text{ V}$$

$$I_{Di\max} = 0.15\left(1-\left(3\times\frac{0.795}{0.347}\right)\right)\left(1-\frac{2}{3}\sqrt{\frac{0.796}{0.347}}\right]$$

$$= 3.2\text{ mA}.$$

Common Data for Questions 34 and 35: An *n*-channel JFET has a pinch of voltage of −3.5 V and $I_{DSS} = 6$ mA.

Example 34: At what value of V_{GS} will I_{DS} equal to 2 mA?

Solution: $I_D = I_{DSS}\left(1-\frac{V_{GS}}{V_P}\right)^2$

$$V_{GS} = -3.5\left(1-\sqrt{\frac{2\times10^{-3}}{6\times10^{-3}}}\right)$$

$$= -3.5\ (0.42264)$$

$$= -1.4792\text{ V}.$$

Example 35: Find the value of g_m at this I_{DSS}?

Solution: $g_m = \frac{-2I_{DSS}}{V_P}\times\left(1-\frac{V_{GS}}{V_P}\right)$

$$g_m = \frac{-2\times6\times10^{-3}}{-3.5}\left(1-\frac{1.4772}{3.5}\right]$$

$$= \frac{-2\times6\times10^{-3}\times0.57737}{-3.5}$$

$$= 1.97\text{ ms}.$$

EXERCISES

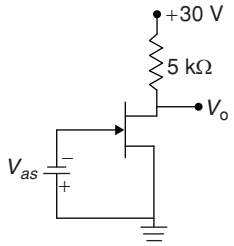
Practice Problems I

Directions for questions 1 to 27: Select the correct alternative from the given choices.

1. Find the resistivity of a *p*-channel Ge JFET with half channel height of 2 μm and pinch-off voltage of 3.94 V. (Consider $\mu_p = 1800$ cm²/V-sec. $\varepsilon_r = 16$)

(A) 0.02 Ω-cm (B) 0.2 Ω-cm
(C) 0.02 Ω-m (D) 0.2 Ω-m

2.



For the given *n*-channel JFET, $I_{Dss} = 6$ mA and $I_D = 2.5$ mA when $V_{GS} = -1$ V.

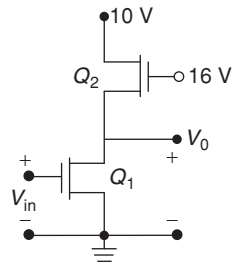
Calculate V_{GS} for a drain current of 3 mA is

(A) −0.71 V (B) −0.79 V
(C) −0.86 V (D) −0.69 V

3. Consider an *n*-channel silicon JFET with $a = 4$ μm, $W = 120$ μm and $L = 6$ μm. Find drain to source resistance for $V_{GS} = 0$ V if channel resistivity is 10 cm.

(A) 425 Ω (B) 625
(C) 475 (D) 675

4.



Assume threshold voltage of both transistors is 2 V and $\beta_1 = \beta_2$. Find V_o when $V_{in} = 6$ V.

(A) 5 V (B) 13.25 V
(C) 8.35 V (D) 3.25 V

5. Consider a MOSFET with $V_T = 0.7$ V, W = 8 μm, $L = 0.8$ μm, $K = 194$ μA/v².

The value of $V_{DS_{min}}$ need to operate the transistor in saturation region with a D.C current $I_D = 100$ μA.

(A) 1.02 V (B) 0.7 V
(C) 0.32 V (D) 1.72 V

6. Consider a MOSFET with $V_T = 0.7$ V, $W = 8$ μm, $L = 0.8$ μm, $K = 194$ μA/v².

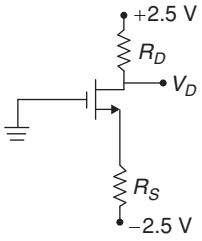
The value of V_{Gs} required causing the device to operate as a 100 resistor for very small V_{DS}.

(A) 1.02 V (B) 1.2 V
(C) 0.52 V (D) 1.52 V

7. An *n*-MOS enhancement transistor with $V_T = 0.7$ V conducts a current of $I_D = 100$ μA when $V_{GS} = V_{DS} = 1.2$ V. Find the value of I_D for $V_{Gs} = 1.5$ V and $V_{DS} = 3$ V.

(A) 128 μA (B) 100 μA
(C) 256 μA (D) 200 μA

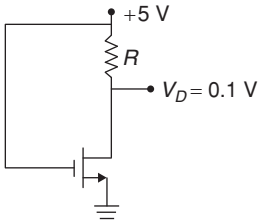
8.



Assume $V_T = 1$ V, $K = 60$ μA/V² and $\frac{W}{L} = \frac{120}{3}$. Find the value of R_S and R_D so that the transistor operates at $I_D = 0.3$ mA and $V_D = 0.4$ V.

(A) 7 k and 3.3 k
(B) 3.3 k and 7 k
(C) 5 k and 3.3 k
(D) 3.3 k and 5 k

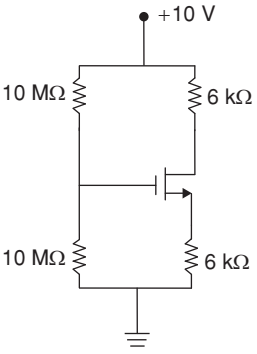
9.



Assume $V_T = 1$ V and $K \cdot \frac{W}{L} = 1$ mA/V². Find the value of *R*, to obtain a drain voltage of 0.1 V

(A) 11.4 k (B) 12.4 k
(C) 10.5 k (D) 9.6 k

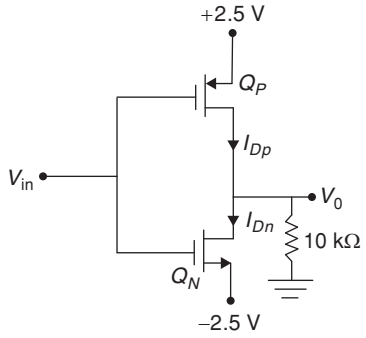
10.



Let $V_T = 1$ V, $K \cdot \frac{W}{L} = 1$ mA/V^2. Find the region of operation and drain current.

(A) Active, 0.89 mA (B) Saturation, 0.89 mA
(C) Active, 0.5 mA (D) Saturation, 0.5 mA

11.

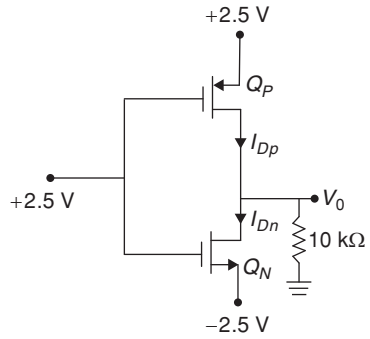


Assume both *n*-MOS and *p*-MOS has same $K \cdot \frac{W}{L} = 1$ mA/V^2 and $V_{t_n} = -V_{t_p} = 1$ V.

Find I_{DN} and I_{DP}, when $V_{in} = 0$ V

(A) 1.125 mA, 0 mA
(B) 0 mA, 1.125 mA
(C) 1.125 mA, 1.125 mA
(D) 0 mA, 0 mA

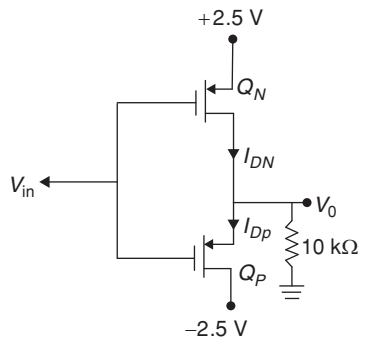
12.



Assume $V_T = 1$ V $K \cdot \frac{W}{L} = 1$ mA/V^2 for both. Find V_o when $V_{in} = +2.5$ V.

(A) –2.44 V (B) +2.44 V
(C) 5 V (D) 0 V

13.

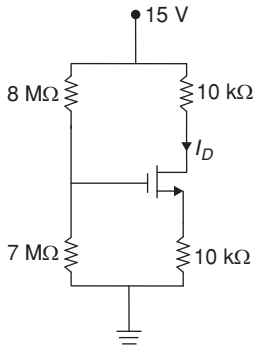


Assume both transistors are perfectly matched with $K \cdot \frac{W}{L} = 1$ mA/V^2 and $V_{t_n} = -V_{t_p} = 1$ V.

Find V_o when $V_{in} = +2.5$ V.

(A) –1.04 V (B) +1.04 V
(C) 0 V (D) +2.5 V

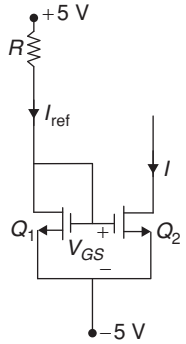
14.



Assume the MOSFET has V_T = 1 V and $K \cdot \frac{W}{L}$ = 1 mAV^2. If the current flowing is $I_D = 0.5$ mA initially, calculate the percent change in the value of I_D when the MOSFET is replaced with another having same $K \cdot \frac{W}{L}$ but $V_T = 1.5$ V.

(A) Decreases by 9% (B) Increases by 9%
(C) Remains same (D) None of the above

15.



Assume $V_T = 1$ V, $K \cdot \left(\frac{W}{L}\right)_1$, = 0.8 mA/V^2 and two Transistors Q_1 and Q_2 having equal length but widths related by $W_2 = 5W_1$. Find the value of *R*, to obtain *I* = 0.5 mA.

(A) 45 kΩ (B) 85 kΩ
(C) 65 kΩ (D) 55 kΩ

16. For a CMOS inverter with matched MOSFETS having $V_{t_n} = -V_{t_p} = 1$ V and $V_{DD} = 5$ V.

Find noise margin for HIGH level.

(A) 2.1 V
(B) 2.5 V
(C) 2.9 V
(D) 2 V

17. For a CMOS inverter with matched MOSFETS having $V_{t_n} = -V_{t_p} = 1$ V and $V_{DD} = 5$ V

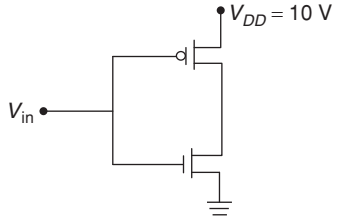
Find noise margin for LOW level.

(A) 2.1 V
(B) 2.9 V
(C) 2.5 V
(D) 2 V

18. A 1.2 μm CMOS inverter uses $L_n = L_p = 1.2$ μm, $W_n = 1.8$ μm, $k_n = 80$ μA/V², $k_p = 27$ μA/v², $V_{t_n} = 0.8$ V, $V_{DD} = 5$ V. Calculate the value of output resistance of the inverter when $V_o = V_{oL}$.

(A) 1 k (B) 1.5 k
(C) 2 k (D) 3 k

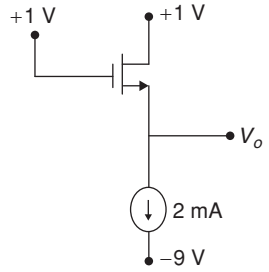
19.



Assume $V_{t_n} = -V_{t_p} = 2$ V, $\left(\frac{W}{L}\right)_n = 20$, $\left(\frac{W}{L}\right)_p = 40$ and $\mu_n C_{ox} = 2\mu_p$, $C_{ox} = 20$ μA/v². Find the peak current drawn from V_{DD} during switching.

(A) 1.2 mA
(B) 1.8 m A
(C) 0.8 mA
(D) 1.6 Ma

20.



Assume $V_t = +1$ V, $K \cdot \left(\frac{W}{L}\right) = 1$ mA/V². Find V_o for the circuit shown.

(A) +2 V (B) −2 V
(C) −9 V (D) +1 V

Common Data for Questions 21 and 22: For a depletion mode NMOS transistor with $V_t = -2$ V and $K\left(\frac{W}{L}\right) = 2$ mA/V².

21. Find the minimum V_{DS} required to operate in the saturation region when $V_{gs} = +1$ V

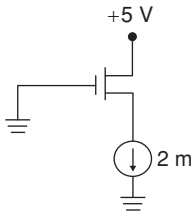
(A) −1 V (B) +1 V
(C) 2 V (D) 3 V

22. For a depletion mode NMOS transistor with $V_t = -2$ V and $K\left(\frac{W}{L}\right) = 2$ mA/V².

The value of drain current I_D, when $V_{gs} = +1$ V is _____

(A) 3 mA (B) 6 mA
(C) 7.5 mA (D) 9 mA

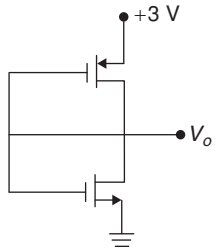
23.



For the depletion mode MOSFET, assume $K\left(\frac{W}{L}\right) = 4$ mA/V² and $V_t = -2$ V. Find the voltage across the source terminal.

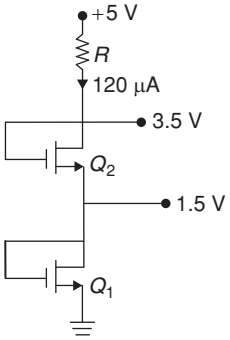
(A) −1 V (B) +1 V
(C) −3 V (D) +3 V

24. Assume $k_P = 2.5$ μA/V², $k_n = 10$ μA/V², $|V_t| = 1$ V, $L = 10$ μm, $W = 30$ μm. Find the output voltage V_o as shown.



(A) 4.1 V (B) 2.0 V
(C) 2.8 V (D) 4.2 V

25.

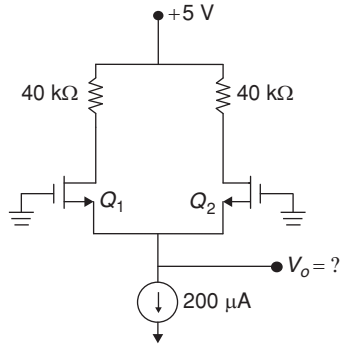


Assume each NMOS transistor has $V_t = 1$ V, $K = 120$ μA/V², $L_1 = L_2 = 1$ μm.

Find the value of gate width of Q_2.

(A) 8 μm (B) 4 μm
(C) 2 μm (D) 1 μm

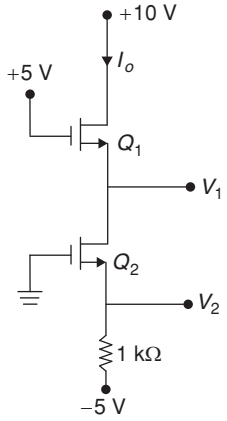
26.



Assume both Transistors have $V_t = 1$ V, $K = 100\ \mu A/V^2$, $\frac{W}{L} = 20$. Find the value of V_o indicated.

(A) 1.32 V
(B) 1.68 V
(C) −1.32 V
(D) −1.68 V

27. Assume each transistor has $V_t = 1$ V, and $K \cdot \frac{W}{L} = 2$ mA/V^2. Find the current flowing through drain of Q_1.



(A) 1.23 mA
(B) 2.44 mA
(C) 1.48 mA
(D) 2.65 mA

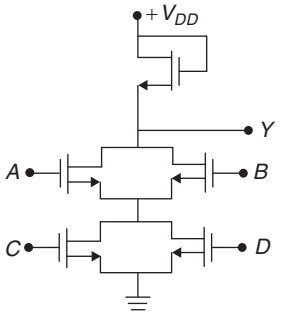
Practice Problems 2

Directions for questions 1 to 18: Select the correct alternative from the given choices.

1. Find the pinch off voltage of a silicon *n*-channel JFET with half channel height of 2 μm and donor concentration of 7×10^{14} atoms/cm^3

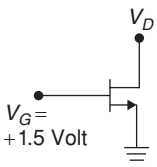
(A) 3.1 V
(B) 4.1 V
(C) 2.1 V
(D) 1.1 V

2. The logic function realized by the given circuit is _____



(A) $\overline{(A+C)(B+D)}$
(B) $\overline{(A+C)}+\overline{(B+D)}$
(C) $\overline{(A+B)}+\overline{(C+D)}$
(D) $\overline{A+B+C+D}$

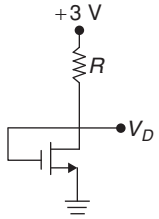
3.



For the given NMOS enhancement with $K = 100\ \mu A/v^2$, $W = 10$ μm, $L = 1$ μm and $V_T = 0.7$ V. Find the region of operation and drain current when $V_D = 0.9$ V.

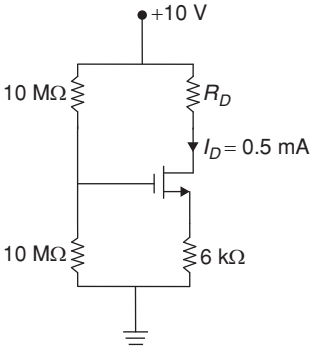
(A) Triode region, 275 μA
(B) Saturation, 320 μA
(C) Triode, 320 μA
(D) Saturation, 275 μA

4. Find the value of aspect ratio (W/L) and R, to obtain drain current of 160 μA and V_{DS} of 1 V.
Assume $V_T = 0.6$ V
$K = 200\ \mu A/V^2$



(A) 8 and 25 k
(B) 10 and 25 k
(C) 10 and 12.5 k
(D) 8 and 12.5 k

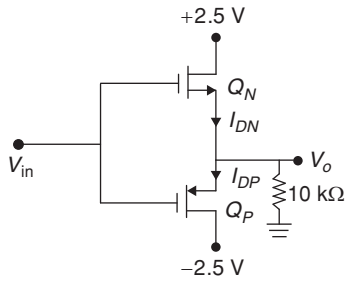
5.



Let $V_T = 1$ V and $K \cdot \frac{W}{L} = \frac{1\ mA}{V^2}$. Find the largest value that R_D can have, while the transistor remains in saturation with 0.5 mA of drain current.

(A) 8 k
(B) 10 k
(C) 12 k
(D) 16 k

6.

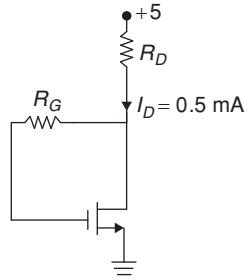


Assume both transistors are perfectly matched with $K \cdot \frac{W}{L} = 1 \text{ mA/V}^2$ and $V_{t_n} = V_{t_p} = 1 \text{ V}$

Find I_{D_N} and I_{D_P}, when $V_{\text{in}} = 0$ V.

(A) 0 mA, 0 mA (B) 0 mA, 0.104 mA
(C) 0.104 mA, 0 mA (D) 0.104 mA, 0.104 mA

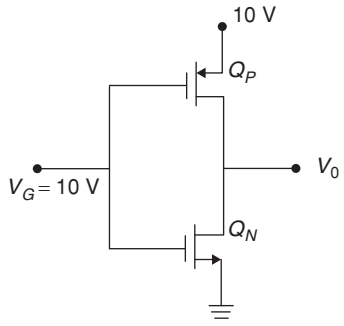
7.



Assume $V_T = 1 \text{ V}, K \cdot \frac{W}{L} = 1 \text{ mA/v}^2$. Find the value of R_D.

(A) 3 k (B) 6 k
(C) 9 k (D) 7.5 k

8.



The given CMOS inverter has $V_{t_n} = -V_{t_p} = 2 \text{ V}, \left(\frac{W}{L}\right)_p = 40, \left(\frac{W}{L}\right)_n = 20,$ $\mu_n C_{ox} = 20 \text{ } \mu \text{ A/V}^2$.

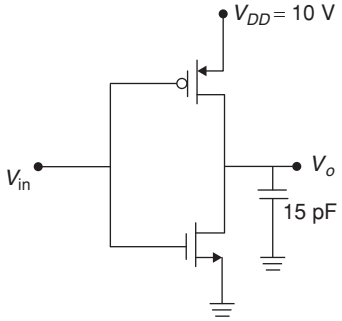
Find the maximum current that the inverter can sink while $V_O = 0.5$ V.

(A) 0 mA (B) 1.25 mA
(C) 1.55 mA (D) 1.75 mA

9. A 1.2 μm CMOS inverter uses $L_p = L_n = 1.2$ μm, $W_N = 1.8$ μm. Find the value of W_p that would result in Q_N and Q_P being matched. Assume $K_n = 80$ μA/V², $K_P = 27$ μA/V², $V_{t_n} = 0.8$ V and $V_{DD} = 5$ V.

(A) 1.8 μm (B) 3.6 μm
(C) 5.4 μm (D) 7.2 μm

Common Data for Questions 10 and 11:



Assume $V_{t_n} = -V_{t_p} = 2 \text{ V}, \left(\frac{W}{L}\right)_n = 20; \left(\frac{W}{L}\right)_p = 40, K_n = 2k_p = 20 \text{ μA/V}^2$.

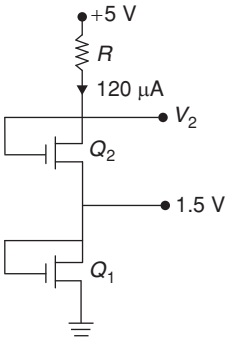
10. Find the dynamic power dissipation when the inverter is switched at a frequency of 2 MHz.

(A) 3 mW (B) 30 mW
(C) 6 mW (D) 15 mW

11. Average current drawn from the power supply is

(A) 0.3 mA (B) 0.6 mA
(C) 3 mA (D) 30 mA

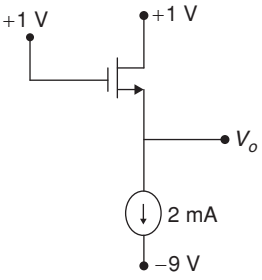
12.



Assume $V_t = 1$ V, $K = 120$ μA/V², $L_1 = L_2 = 1$ μm, $W_1 = 8$ μm, $W_2 = 2$ μm. Find the value of R indicated

(A) 25 k (B) 12.5 k
(C) 1.25 k (D) 2.5 k

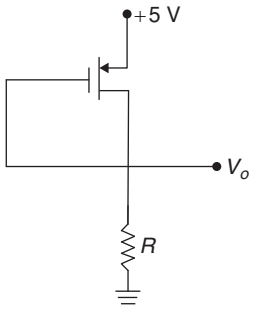
13.



Assume $V_t = +1 \text{ V}, K \cdot \left(\frac{W}{L}\right) = 1 \text{ mA/V}^2$. Find V_o for the circuit shown.

(A) +2 V (B) −2 V
(C) −9 V (D) +1 V

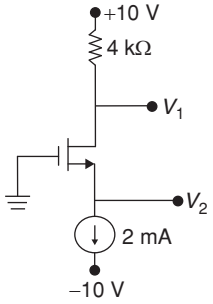
14.



For the PMOS transistor, assume V_t = –0.7 V, K_P = 60 μA/V², L = 0.8 μm. Find the value of W, in order to establish a drain current of 115 μA and a voltage, V_D of 3.5 V.

(A) 0.8 μm (B) 2.4 μm
(C) 4.8 μm (D) 3.6 μm

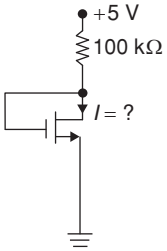
15. Assume $V_t = 2$ V, $K \cdot \frac{W}{L} = 1$ mA/V². Find V_1 and V_2 labeled as shown



(A) 2 V and 4 V (B) –2 V and –4 V
(C) 2 V and –4 V (D) –2 V and 4 V

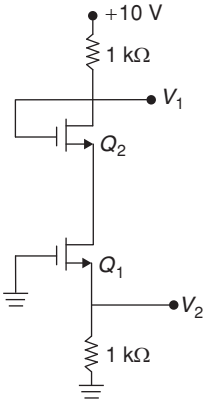
16. Assume $V_t = 1$ V, $K \cdot \left(\frac{W}{L}\right) = 0.4$ mA/V².

Find the value of drain current.



(A) 0.024 mA
(B) 0.036 mA
(C) 0.048 mA
(D) 0.072 mA

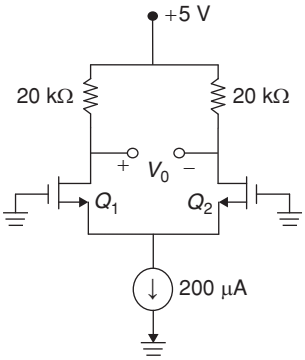
17.



Assume each NMOS has $V_t = 1$ V, and $K \cdot \frac{W}{L} = 2$ mA/V². Find the voltage V_1 shown in figure.

(A) 5.71 V
(B) 6.25 V
(C) 2.52 V
(D) 5.23 V

18.



Assume $V_t = 1$ Volt, $K = 100$ μA/V² and $W/L = 10$. Find the value of V_o as shown.

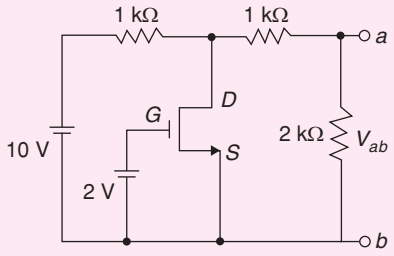
(A) 1 V (B) 3 V
(C) 2 V (D) 0 V

Previous Years' Questions

1. The value of R for which the PMOS transistor in figure will be biased in linear region is **[2004]**

(A) 220 Ω (B) 470 Ω
(C) 680 Ω (D) 1200 Ω

2. Assume that the n-channel MOSFET shown in figure is ideal, and that its threshold voltage is +1.0 V. The voltage V_{ab} between nodes a and b is: **[2005]**

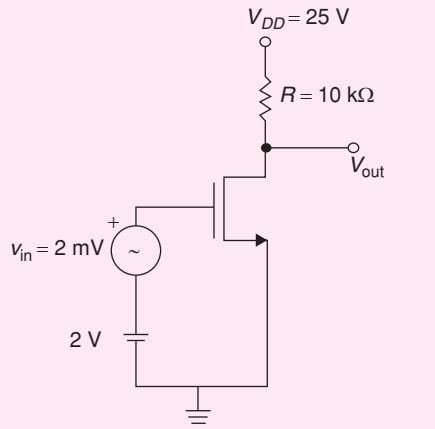


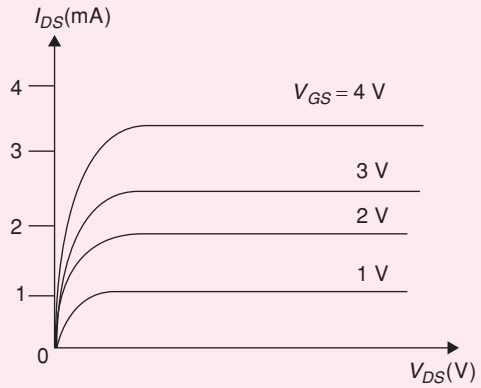
(A) 5 V (B) 2 V
(C) 1 V (D) 0 V

3. The conduction loss versus device current characteristic of a power MOSFET is best approximated by **[2005]**

(A) a parabola
(B) a straight line
(C) a rectangular hyperbola
(D) an exponentially decaying function

Common Data for Questions 4a and 4b: Assume that the threshold voltage fo the n-channel MOSFET shown in figure is +0.75 V. The output characteristics of the MOSFET are also shown. **[2005]**





4. (a) The transconductance of the MOSFET is:
(A) 0.75 ms (B) 1 ms
(C) 2 ms (D) 10 ms

(b) The voltage gain of the amplifier is:
(A) +5 (B) –7.5
(C) +10 (D) –10

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** A	**3.** B	**4.** C	**5.** C	**6.** B	**7.** C	**8.** B	**9.** B	**10.** D
11. C	**12.** A	**13.** B	**14.** A	**15.** B	**16.** A	**17.** A	**18.** C	**19.** B	**20.** B
21. D	**22.** D	**23.** B	**24.** B	**25.** C	**26.** C	**27.** B			

Practice Problems 2

1. C	**2.** C	**3.** B	**4.** C	**5.** C	**6.** A	**7.** B	**8.** C	**9.** C	**10.** A
11. A	**12.** B	**13.** B	**14.** C	**15.** C	**16.** B	**17.** B	**18.** D		

Previous Years' Questions

1. D **2.** D **3.** A **4.** (a) B (b) D

Chapter 4

Transistor Biasing

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

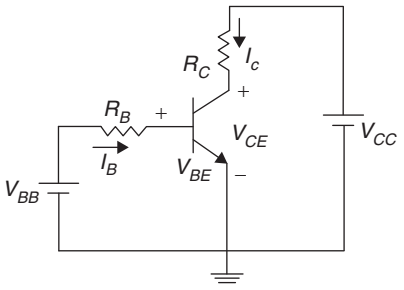
- The operating point
- Bias stability
- Stability factor
- 3 Biasing the BJT
- Bias compensation
- Compensation techniques
- Thermistor compensation
- Sensistor compensation
- Thermal run away
- Thermal resistance
- Biasing the FET
- Four resistor bias circuit

The Operating Point

A Transistor is a current controlled 3-terminal device having 2 junctions (i) Base-emitter junction and (ii) Base-collector junction.

Base-emitter Junction	Base-collector Junction	Application	Region of Operation
FB	FB	ON switch	Saturation
FB	RB	Amplifier	Active
RB	FB	–	Inverse active
RB	RB	OFF switch	Cut-off

Transistor Circuit Under DC Condition



Apply KVL at input side gives $V_{BB} - I_B R_B - V_{BE} = 0$

$$\Rightarrow V_{BE} = V_{BB} - I_B R_B$$

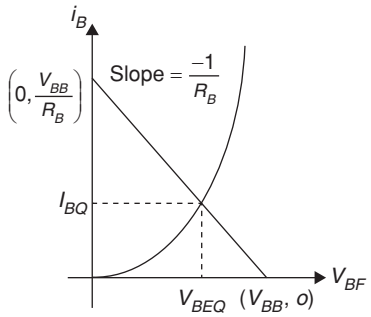
When $V_{BE} = 0 \quad \Rightarrow \quad I_B = \frac{V_{BB}}{R_B}$

$$I_B = 0 \Rightarrow V_{BE} = V_{BB}$$

DC load line is the line joining the two points

$$(V_{BB},\ 0) \quad \text{and} \quad \left(0,\ \frac{V_{BB}}{R_B}\right)$$

Input Characteristics of a CE Transistor



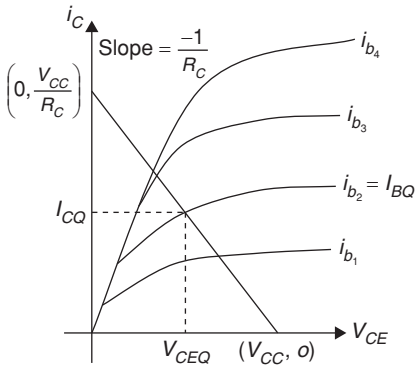
Intersection of the load line with the characteristics of the transistor gives the Q point (operating point). Apply KVL at the output:

$$V_{CC} - I_C R_C - V_{CE} = 0$$

$$V_{CE} = 0 \quad \Rightarrow \quad I_C = \frac{V_{CC}}{R_C}$$

$$I_C = 0 \quad \Rightarrow \quad V_{CE} = V_{CC}$$

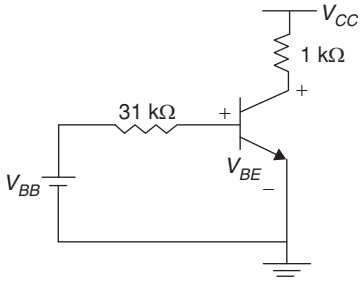
Output Characteristics of a CE Transistor



Intersection of the load line with the output characteristics for a particular value of I_B gives the Quiscent point/ Operating point.

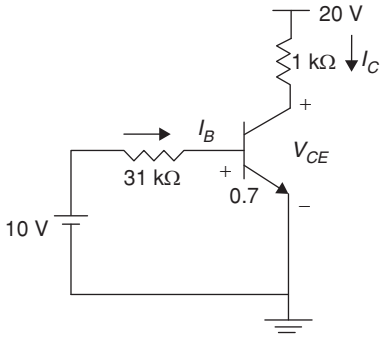
Hence 'Q' point is given by $Q(I_{CQ}, V_{CEQ})$

Example 1: Determine the 'Q' point for the following circuit if $\beta = 50$ and $V_{BB} = 10$ V, $V_{BE} = 0.7$ V, $V_{CC} = 20$ V.



(A) (0.3 mA and 5 V) (B) (15 mA and 5 V)
(C) (0.6 mA and 4 V) (D) (15 mA and 4 V)

Solution: (B)



Writing KVL for the base emitter loop gives
$10 - 31\, I_B - 0.7 = 0$
$\Rightarrow I_B = 0.3$ mA
$I_{CQ} = \beta\,(0.3\text{ mA}) = 15$ mA
Writing KVL for the collector loop gives
$20\text{ V} - I_C\,(1\text{ k}\Omega) - V_{CE} = 0$
$\Rightarrow V_{CE} = 20 - (15\text{ mA})\,(1\text{ k})$
$= 20 - 15$
$V_{CE} = 5$ V
$(I_C, V_{CEQ}) = (15\text{ mA}, 5\text{ V})$

Bias Stability

Operating point defines where the transistor will operate on its characteristics curve, under DC conditions. For linear (minimum distortion) amplification, the DC operating point should not be too close to the maximum power and voltage or current rating and should avoid the regions of saturation and cut off. To keep the operating point stable, we use external network to bias the transistor.

The stability of operating point depends on V_{BE}, β, I_{CO}. β is very sensitive to temperature, and V_{BE} decreases about 2.5 mV for each 1°C increase in temperature. The reverse saturation current typically doubles for every 10°C increase is temperature.

Stability Factor(S)

$$I_C = f(I_{CO}, V_{BE}\, \beta)$$

$$S = \frac{\partial I_C}{\partial I_{Co}},\ S^1 = \frac{\partial I_C}{\partial V_{BE}} \cdot S^{11} = \frac{\partial I_C}{\partial \beta}$$

$$I_C = \beta I_B + (1 + \beta)\, I_{CO}$$

Differentiating wrt I_C gives

$$1 = \beta \frac{\partial I_B}{\partial I_C} + (1+\beta)\frac{\partial I_{Co}}{\partial I_C}$$

$$\Rightarrow \quad 1 = \beta \frac{\partial I_B}{\partial I_C} + \frac{1+\beta}{S} \quad \Rightarrow \quad S = \frac{1+\beta}{1 - \beta \dfrac{\partial I_B}{\partial I_C}}$$

Where 'S' is the stability factor.

Smaller value of 'S' is desirable for a proper biasing of circuit to act as an amplifier.

The networks are the most stable and least sensitive to temperature changes have the smallest stability factors.

Biasing the BJT

Base Bias or Fixed Bias

Input loop

$$V_{CC} = I_B R_B + V_{BE}$$

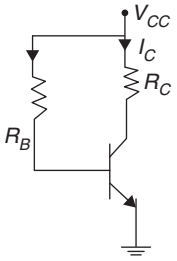
$$I_b = \frac{V_{CC} - V_{BE}}{R_B}$$

Output loop

$$V_{CC} - I_C R_C - V_{CE} = 0$$

$$V_{CE} = V_{CC} - I_C R_C$$

$$I_C = \beta I_b = \beta \left(\frac{V_{CC} - V_{BE}}{R_b} \right)$$



Voltage drop across R_C can never be greater than V_{CC}.

$$I_C R_C < V_{CC} \quad \Rightarrow \quad I_C < \frac{V_{CC}}{R_C}$$

If I_C becomes greater than this value, the operating point will lie in saturation region.

$$\text{Stability factor} = S = 1 + \beta \left(\because \frac{\partial I_B}{\partial I_C} = 0 \right)$$

It's a simple circuit with few parts.

Operating point can be fixed anywhere in active region, by simply varying R_B.

I_C depends on β. β in turn unstable wrt temperature, this biasing in useful is switching and digital applications.

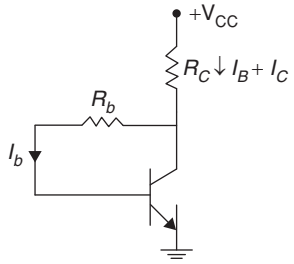
Collector to Base Bias Circuit

Input loop

$$V_{CC} - (I_B + I_C) R_C - R_b I_B - V_{BE} = 0$$

$$\Rightarrow \quad \frac{\partial I_B}{\partial I_C} = \frac{-R_C}{R_C + R_B}$$

$$I_b = \frac{I_C}{\beta}$$



$$\Rightarrow \quad I_B = \frac{V_{CC} - V_{BE}}{(\beta + 1) R_C + R_B}$$

$$\therefore I_C = \frac{\beta (V_{CC} - V_{BE})}{(\beta + 1) R_C + R_B}$$

If 'β' is very large ($I_B = 0$) then the approximate value of

$$I_C = \frac{V_{CC} - V_{BE}}{R_C}$$

KVL at output side:

$$V_{CC} - (I_C + I_B) R_C - V_{CE} = 0$$
$$\Rightarrow V_{CE} = V_{CC} - (I_C + I_B) R_C$$

If $(\beta + 1) R_C >> R_B$, then $I_C = \frac{V_{CC} - V_{BE}}{R_B}$ independent of 'β'.

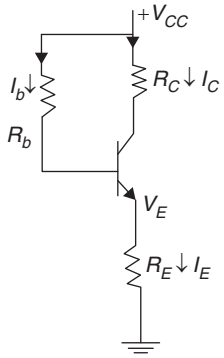
The feedback resistor R_B provides negative feedback for AC input consequently reduces the AC gain of the circuit.

$$\text{Stability factor } S = \frac{1 + \beta}{1 + \beta \left(\frac{R_C}{R_C + R_B} \right)}$$

Collector to base bias is having lesser stability factor than fixed bias. Thus it provides better stability.

Emitter Feedback Bias

$$I_E = I_C + I_B$$
$$V_E = I_E R_E = (I_C + I_B) R_E$$
$$V_B = V_{BE} + (I_C + I_B) R_E$$
$$I_B = \frac{V_{CC} - V_B}{R_b}$$



Output loop

$$(I_C + I_B) R_E + R_C I_C - V_{CC} + V_{CE} = 0$$

$$\text{Stability factor } S = \frac{1 + \beta}{1 + \beta \left(\frac{R_E}{R_E + R_b} \right)}$$

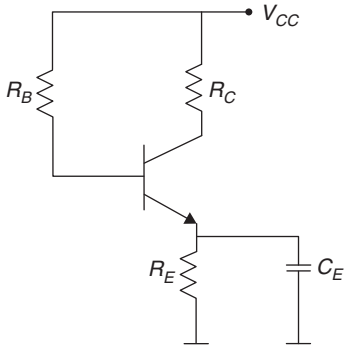
I_C can be made insensitive to β, if we choose $R_E >> R_b$.

Advantages

1. Simplest biasing scheme
2. Better stability over fixed bias if $R_E >> \frac{R_B}{(\beta + 1)}$

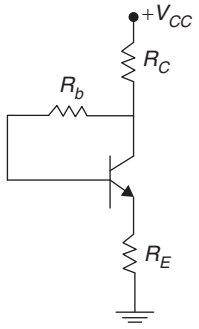
Disadvantages: The Resistor R_E provides a negative feed back for AC input consequently reduces the gain.

To increase the stability of the emitter feedback circuit, a bypass capacitor has been placed in parallel with R_E.



The purpose of bypass capacitor is to make $R_E = 0$ for AC input hence no gain reduction at the output of the circuit.

Collector–Emitter Feedback Bias



By applying KVL to the circuit.

Output loop

$$V_{CC} - (I_C + I_B)\, R_C - V_{CE} - R_E\,(I_C + I_B) = 0$$

Input loop

$$V_{CC} - (I_C + I_B)\, R_C - I_B R_b - V_{BE} - R_E\,(I_B + I_C) = 0$$

$$\text{Stability factor } S = \frac{1+\beta}{1+\beta(R_e + R_b)/(R_e + R_c + R_b)}$$

Self-bias, Emitter-bias, or Voltage–divide–bias

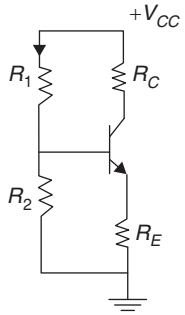


Figure 1 Self biasing circuit

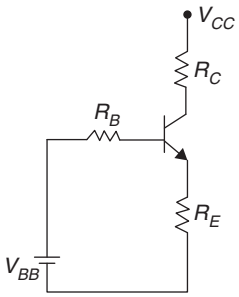


Figure 2 Thevenine's equivalent circuit

$$V_{BB} = \frac{R_2 \cdot V_{CC}}{R_2 + R_1},\quad R_B = \frac{R_2 R_1}{R_2 + R_1}$$

Input loop

$$V_{BB} = I_B R_B + V_{BE} + (I_b + I_c) R_e$$

Output loop

$$V_{cc} - R_C I_c + V_{CE} + (I_b + I_c) R_c$$

$$\text{Stability factor } S = \frac{1+\beta}{1+\beta\left(\dfrac{R_E}{R_E + R_B}\right)}$$

KVL at input side:

$$V_{BB} - I_B R_B - V_{BE} - (I_B + I_C)\, R_E = 0$$

$$I_C = \beta I_B$$

$$\Rightarrow I_B = \frac{V_{BB} - V_{BE}}{(\beta+1)\,R_E + R_B} \quad \Rightarrow \quad I_C = \frac{\beta(V_{BB} - V_{BE})}{(\beta+1)R_E + R_B}$$

KVL at out put side:

$$V_{cc} - I_c R_c - V_{CE} - (I_B + I_c)\, R_E = 0$$

$$\Rightarrow \quad V_{CE} = V_{cc} - I_c R_c - (\beta+1)\, I_B R_E$$

If 'β' is very large, then approximate value of

$$I_C \simeq \frac{V_{BE} - V_{BE}}{R_E}$$

Where $V_{BB} = \dfrac{V_{CC} \cdot R_2}{R_1 + R_2}$

$R_B = R_1 \| R_2$

If $(\beta + 1)\, R_E \gg R_B$, then I_c is Independent of 'β'

(i) Better stability over, all biasing schemes if

$$R_E \gg \frac{R_B}{(1+\beta)}$$

(ii) Most of the amplifiers uses self-biasing scheme.

Bias Compensation

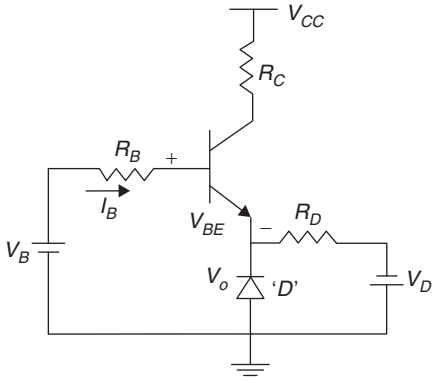
Compensation Techniques

$$I_c = f(I_{co}, V_{BE}, \beta)$$

I_{co} (reverse saturation current), V_{BE} and β are temperature dependent parameters. Hence to stabilize the circuit against the variation in temperature, i.e., (to make I_c independent of I_{co}, V_{BE}, β) compensation techniques are used.

1. Reverse saturation current doubles for every 10°C rise in temperature.
2. V_{BE} decreases for 7.5 mv for 1°C rise in temperature.

Compensation for V_{BE}

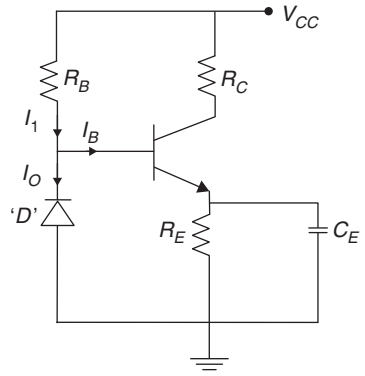


If the value of V_D and R_D are designed such that the voltage across the diode is V_{BE} from p-n side, I_B is $V_B - I_B R_B - V_{BE} + V_{BE} = 0$

$$I_B = \frac{V_B}{R_B} \text{ gives } I_C = \frac{\beta V_B}{R_B} \text{ independent of } V_{BE}$$

V_{BE} tracks V_0 with respect to temperature, so I_C will be insensitive to variations in V_{BE}.

Compensation for I_{co}



Let us assume that the transistor and the diode are made up of same material. Hence I_{co} is same for both

$$\therefore I_c = \beta I_B + (1 + \beta) I_{co}$$

$$I_c = \beta (I_1 - I_0) + (1 + \beta) I_{co} \quad (\therefore \text{ from the figure})$$

If 'β' is assumed to be so large and $I_0 = I_{co}$

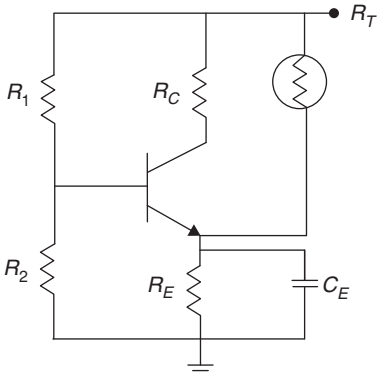
$$\Rightarrow \beta + 1 \approx \beta$$

$$I_c = \beta I_1 - \beta I_0 + \beta I_{co}$$

$I_c = \beta I_1$ whcih is independent of reverse saturation current.

If the diode and transistor are the same type of material, the reverse, saturation current of diode will increase with temperature at the same rate as the transistor collector saturation current I_{CO}.

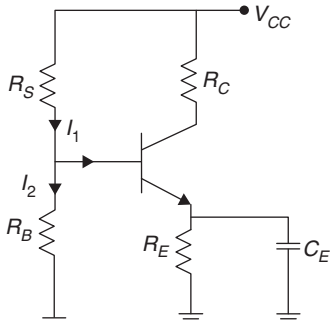
Thermistor compensation



The thermistor has a negative temperature coefficient. Its resistance decreases exponentially with increase in temperature.

As T rises, R_T decreases, and the current fed through R_T into R_e increase. Since the voltage drop across R_e is in the direction to reverse-bias the transistor. The temperature sensitivity of R_T acts so as to tend to compensate the increase in I_C due to temperature.

Sensistor compensation



R_s: sensistor

Sensistor is having a positive temperature coefficient of resistance whose resistance increases, as temperature increases, (as the collector current is increasing). Hence the current through sensistor decreases.

$$I_B = I_1 - I_2$$

I_1 decreases I_B also starts decreasing.

Since, $I_c = \beta I_B$ so the collector current is in control thus providing a better stability.

Thermal Run Away

Power dissipated at collector causes self heating, as a consequence junction temperature rises, and this in turn increase collector current, with a subsequent increase in power dissipation, if this phenomenon (referred to as thermal run away) continues, it may result in permanently damaging the transistor.

Thermal Resistance

It is found that steady state temperature rise at the collector junction is proportional to the power dissipated at the junction.

$\Delta T = T_J - T_A = \theta P_D$

T_J = Junction temperature

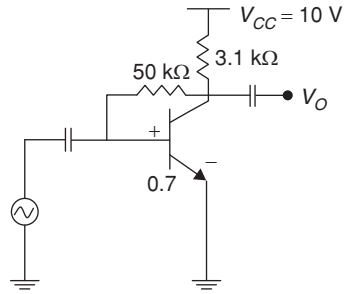
T_A = Ambient temperature

P_D = Power dissipated at collector junction

θ = Thermal resistance

The required condition for thermal stability is the rate at which heat is released must not exceed the rate at which the heat can be dissipated $\frac{\partial P_C}{\partial T_j} < \frac{\partial P_D}{\partial T_j} \Rightarrow \frac{\partial P_C}{\partial T_j} < \frac{1}{\theta}$ is the condition to avoid thermal runway.

Example 2: Find the DC collector current if V_{BE} = 0.7 V and 'β' is considered as very large



(A) 2 mA (B) 4 mA
(C) 5 mA (D) 3 mA

Solution: (D)

$$I_C = \frac{\beta(V_{CC} - V_{BE})}{(\beta+1)R_C + R_B}$$

As 'β' is large

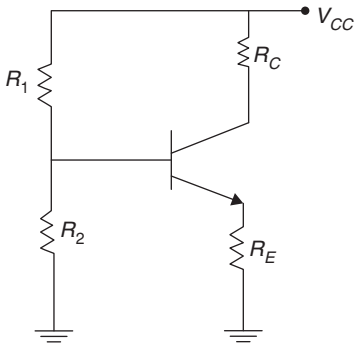
$$(\beta + 1)\, R_C \simeq \beta\, R_C \gg R_B$$

$$I_C \simeq \frac{V_{CC} - V_{BE}}{R_C}$$

$$= \frac{10 - 0.7}{3.1\ \text{k}\Omega}$$

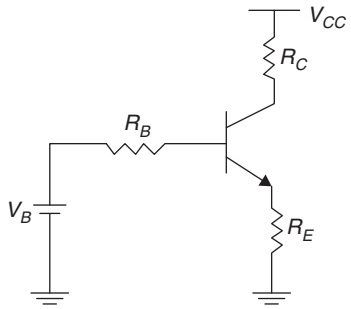
$$I_c = 3\ \text{mA}.$$

Example 3: A common emitter amplifier has a potential Divider bias using $V_{cc} = 20$ V, $R_1 = R_2 = 6.2$ kΩ and $V_{BE} = 0.7$ V, β is assumed to be so large then, the operating point is (given R_c as 1 kΩ and R_E as 2 kΩ)



(A) (4.65 mA and 6.05 V) (B) (3 mA and 12 V)
(C) (9 mA and 6 V) (D) (9 ma and 14 V)

Solution: (A)



$$V_{BB} = \frac{V_{CC} \cdot R_2}{R_1 + R_2} = 10\ \text{V}$$

$R_B = R_1 \| R_2 = 3.1\ \text{k}\Omega$

$$I_C = \frac{\beta(V_{BB} - V_{BE})}{(\beta+1)R_E + R_B}$$

As 'β' is very large, i.e., $(\beta + 1)\, R_E \gg R_B$

$$\therefore I_C \simeq \frac{V_{BB} - V_{BE}}{R_E}$$

$$I_C = \frac{10 - 0.7}{3.1\ \text{k}\Omega}$$

$$I_c = 4.65\ \text{mA}$$

Apply KVL at output side

$V_{CEQ} = V_{cc} - I_{CQ}R_c - I_E R_E$ as 'β' is large $I_E \simeq I_{CQ}$

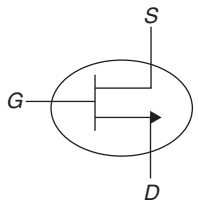
$\therefore\ V_{CEQ} = V_{cc} - I_{CQ}\,[R_c + R_E]$

$= 20 - (4.65\ \text{mA})\ (1 + 2\ \text{k}\Omega)$

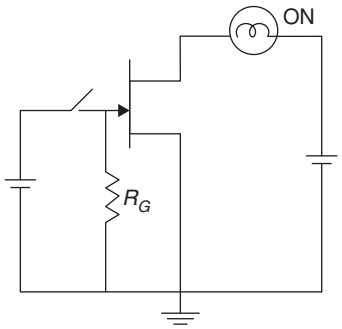
$V_{CEQ} = 6.05$ V.

Biasing the FET

Field effect transistor (FET) is a voltage controlled device having 3 terminals gate, source and drain.



FET is a symmetrical device (i.e., source and drain are interchangeable). FET acts as a better switch over BJT



For JFET and depletion mode MOSFET'S Shockley's equation is valid. R_G is called the bleeder resistor, FET is ON when input circuit is open

$\therefore\ I_G = 0$

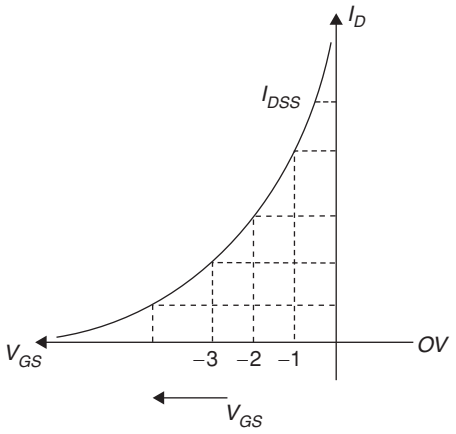


Figure 3 Transfer characteristics

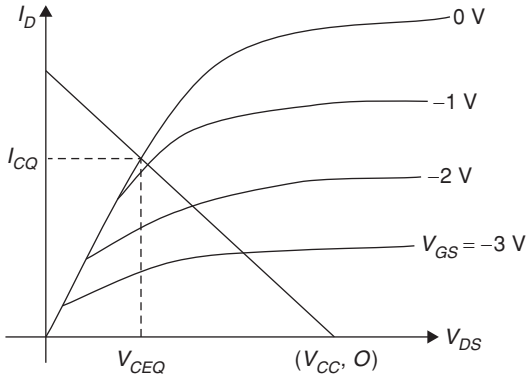


Figure 4 Output characteristics

Transfer characteristics are governed by the equation

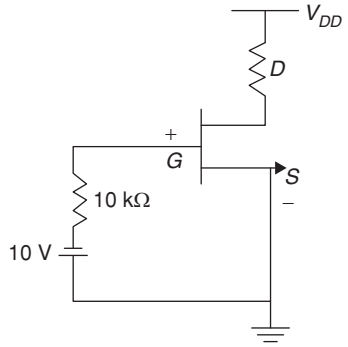
$$I_D = I_{DSS}\left[1-\frac{V_{GS}}{V_P}\right]^2$$

When I_D = drain current

$I_D = I_{DSS}$ when $V_{GS} = 0$, i.e., drain saturation current

V_P = pinch off voltage

Example 4: Find I_D, V_{GS} for the given circuit assuming V_{DD} = 30 V, V_P = – 5 V, I_{DSS} = 30 mA.



(A) 0.27 A, 10 V
(B) 0.27 A, 20 V
(C) 0.135 A, 10 V
(D) 0.27 A, – 10 V

Solution: (D)

$I_G = 0$ A

Apply KVL at input side gives

$-10 - V_{GS} - 0\,(10\text{ k}) = 0$

$$V_{GS} = -10 \text{ V}$$

$$I_D = I_{DSS}\left(1-\frac{V_{GS}}{V_P}\right)^2$$

$$= 30 \text{ mA}\left[1+\frac{10}{5}\right]^2$$

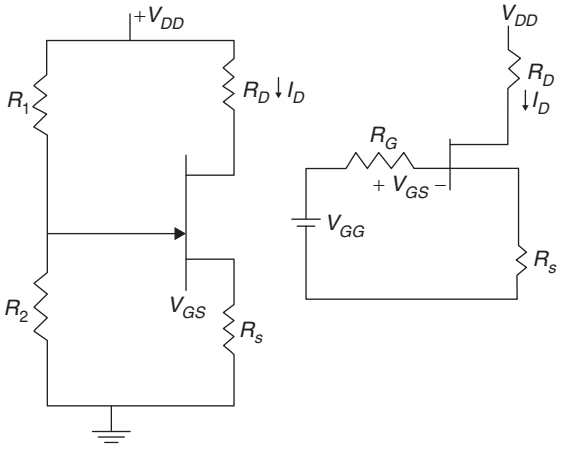
$$= 270 \text{ mA}$$

$$I_D = 0.27 \text{ A}.$$

In MOS circuits biasing schemes control deviations in the operating point caused by fabrication variations in the Threshold Voltage V_T and transconductance (processing) parameter k. Both IC and discrete-component JFET circuits are biased so that the effects of unit-to-unit variations in the pinch off voltage V_P and zero bias drain saturation current I_{DSS} are controlled.

BJT	FET
(i) Current-controleld device	(i) Voltage-controlled device
(ii) Apply KVL at Input gives I_B	(ii) $I_G = 0$ A
(iii) $I_c = \beta I_B$	(iii) $I_D = I_{DSS}\left[1-\frac{V_{GS}}{V_P}\right]^2$
(iv) Not symmetrical	(iv) Symmetrical device (source and drain are interchangable)

Four Resistor Bias Circuit



$$V_{GG} = \frac{R_2}{R_1 + R_2}\cdot V_{DD}$$

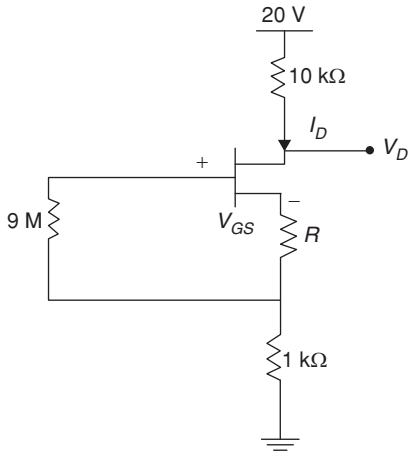
$$R_G = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2},\ I_s = I_D$$

By KVL for the gate loop $-V_{GG} + V_{GS} + I_D R_S = 0$, by KVL for drain and source loop

$$-V_{DD} + R_D I_D + V_{DS} + R_S I_D = 0$$

In this analysis $I_G = 0$ was assumed, however the small reverse saturation current I_{GSS} exits in the gate loop. The resistance R_G is selected to be as large as it is Feasible while maintaining the voltage drop ($I_{GSS} R_G << V_{GG}$) at a negligible value.

Example 5: Find the value of resistor 'R' if the drain to ground voltage = 0 given $I_{DSS} = 50$ mA, $V_p = -6$ V.



(A) 1.2 kΩ (B) 1.4 kΩ
(C) 1.3 kΩ (D) 1.5 kΩ

Solution: (B)

$$I_G = 0 \text{ A}$$

$$I_D = \frac{20 - V_D}{10 \text{ k}}$$

$$I_D = \frac{20 - 0}{10 \text{ k}} \simeq 2 \text{ mA}$$

$$I_D = I_S = 2 \text{ mA}$$

$$I_D = I_{DSS}\left[1 - \frac{V_{GS}}{V_P}\right]^2$$

$$2 \text{ mA} = 50 \text{ mA}\left[1 + \frac{V_{GS}}{6}\right]^2$$

$$\Rightarrow \quad 1 + \frac{V_{GS}}{6} = \frac{1}{5}$$

$$\Rightarrow V_{GS} = -4.8 \text{ V}$$

KVL at input gives

$$-V_{GS} - I_D[R + 1 \text{ k}\Omega] = 0$$

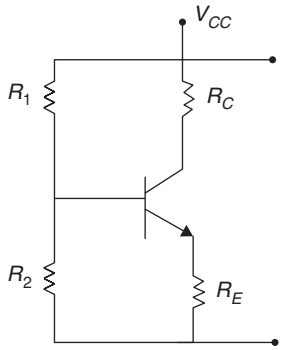
$$\Rightarrow 4.8 = I_D\,[R + 1 \text{ k}\Omega]$$

$$\Rightarrow 4.8 = 2 \text{ mA}\,[R + 1 \text{ k}\Omega]$$

$$\Rightarrow R = 1.4 \text{ k}\Omega.$$

Solved Examples

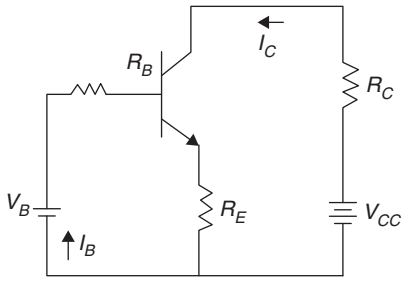
Example 1: A common emitter amplifier has a potential divider bias using $V_{CC} = 12$ V. If $R_C = 4$ kΩ, $R_E = 1$ kΩ, $R_1 = 20$ kΩ, $R_2 = 5$ kΩ, $V_{BE} = 0.6$ V, $\beta = 100$, then, the operating point is _____.



(A) 5 V and 1 mA (B) 4 V and 1.4 mA
(C) 5.14 V and 1.2 mA (D) 3.41 V and 1.71 mA

Solution (D)

$$V_B = \frac{R_2}{R_1 + R_2} \times V_{CC} = \frac{5}{20 + 5} \times 12 = 2.4 \text{ V}$$



Writing KVL for the base emitter loop

$$V_B = I_B \cdot R_B + V_{BE} + (I_B + I_C) R_E$$

$$R_B = \frac{R_1 \cdot R_2}{R_1 + R_2} = \frac{20 \times 5}{20 + 5} = 4 \text{ k}\Omega, I_B = \frac{I_C}{\beta}$$

$$2.4 = \frac{I_C}{100} \times 4 + 0.6 + \left(\frac{I_C}{100} + I_C\right) \times 1$$

$$1.8 = \frac{21}{20} \cdot I_C \Rightarrow I_C = 1.714 \text{ mA}$$

Writing KVL for the collector loop

$$V_{CC} = I_C \cdot R_C + V_{CE} + (I_C + I_B) R_E$$

$$12 = 1.714 \times 4 + V_{CE} + \left(1.714 + \frac{1.714}{100}\right) \times 1$$

$$V_{CE} = 3.41 \text{ V}$$

Example 2: When β is doubled, what are the operating point and stability factor in the previous problem?

(A) 1.756 mA, 3.21 V and 4.9
(B) 1.71 mA, 3.14 V and 4.9
(C) 1.71 mA, 3.21 V and 4.81
(D) 1.756 mA, 3.21 V and 4.81

Solution: (A)

$$V_B = 2.4 \text{ V},\ R_B = 4 \text{ k}\Omega,\ I_B = \frac{I_C}{\beta} = \frac{I_C}{200}$$

Writing KVL for the base emitter loop

$$V_B = I_B \cdot R_B + V_{BE} + (I_B + I_C)R_E$$

$$2.4 = \frac{I_C}{200} \times 4 + 0.6 + \left(\frac{I_C}{200} + I_C\right) \times 1$$

$$= \frac{5}{200} I_C + I_C + 0.6$$

$1.8 = 1.025 I_C \Rightarrow I_C = 1.756$ mA

Writing KVL for the collector loop

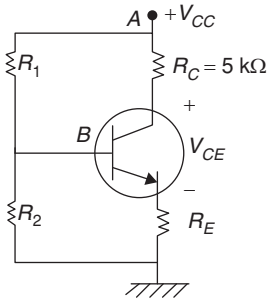
$$V_{CC} = I_C + R_C + V_{CE} + (I_C + I_B)R_E$$

$$12 = 1.756 \times 4 \cdot V_{CE} + \left(\frac{1.756}{200} + I_C\right) \times 1$$

$V_{CE} = 3.21$ V

$$S_1 = (\beta + 1)\frac{1'}{1 + \frac{\beta R_E}{R_B + R_E}} = \frac{201}{1 + \frac{200}{4+1}} = 4.9024.$$

Example 3: A silicon transistor with $\beta = 100$ is to be used in the self biasing circuit, shown in figure such that the quiescent point corresponds to $V_{CE} = 10$ V and $I_C = 2$ mA, find 'R'$_E$ if $V_{CC} = 24$ V, $R_C = 5$ kΩ?



(A) 2 kΩ (B) 1.98 kΩ (C) 980 Ω (D) $R_1 \| R_2$

Solution: (B)

By Applying KVL between V_{CC} and ground

$$V_{CC} = V_{RC} + V_{CE} + V_{RE}$$

$$= I_C \cdot R_c + V_{CE} + (I_B + I_c) R_E$$

$$24 \text{ V} = 2 \times 10^{-3} \times 5 \times 10^3 + 10 + \left(\frac{2 \times 10^{-3}}{100} + 2 \times 10^{-3}\right) \cdot R_E$$

$$\Rightarrow R_E = \frac{4}{2 \times 10^{-3}(1.01)} = 1.98 \text{ k}\Omega$$

Example 4: In the above question, it is desired to have $S \leq 3$, find the limiting value of R_1 and R_2 assuming $V_{BE\ (\text{active})} = 0.65$ V.

(A) 41.39 kΩ and 10.16 kΩ
(B) 20.69 kΩ and 10.16 kΩ
(C) 20.69 kΩ and 5.08 kΩ
(D) 41.39 kΩ and 5.08 kΩ

Solution: (C)

$$S = \frac{(\beta + 1)(R_B + R_E)}{R_B + R_E(\beta + 1)} = \frac{R_B + R_E}{R_E + \frac{R_B}{\beta + 1}}$$

$$3 = \frac{101(R_B + 1.98)}{R_B + 1.98(101)}$$

$$3R_B + 599.94 = 101R_B + 199.98$$

$$98\ R_B = 3999.96 \Rightarrow R_B = 4.08 \text{ k}\Omega$$

$$R_B < 4.08 \text{ k}\Omega \text{ for } S \leq 3$$

$$R_B = \frac{R_1 R_2}{R_1 + R_2}, V_B = V_{CC} \cdot \frac{R_2}{R_1 + R_2}$$

By solving

$$R_1 = \frac{V_{CC} \cdot R_B}{V_B},\ R_2 = \frac{R_1 V_B}{V_{CC} - V_B}$$

$$V_B = I_B \cdot R_B + V_{BE} + V_{RE}$$

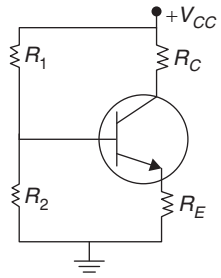
$$= 0.02 \times 10^{-3} \times 4.08 \times 10^3 + 0.65 + 4$$

$$= 4.7312 \text{ V}$$

$$R_1 = \frac{24 \times 4.08}{4.7312} = 20.69 \text{ k}\Omega$$

$$R_2 = \frac{20.69 \times 4.7312}{24 - 4.73} = 5.08 \text{ k}\Omega$$

Example 5: Consider the transistor circuit of figure where $V_{cc} = 24$ V, $R_C = 5.6$ kΩ, $R_E = 1$ kΩ, $R_2 = 10$ kΩ, and $R_1 = 80$ kΩ, $\beta = 60$, $V_{BE} = 0.6$ V, the transistor operates in the active region. The operating point of the transistor is



(A) 1.78 mA and 13.254 V
(B) 1.78 mA and 12.28 V
(C) 1.39 mA and 13.25 V
(D) 1.39 mA and 12.28 V

Solution: (B)

$$R_B = \frac{R_1 \cdot R_2}{R_1 + R_2} = \frac{80 \times 10}{80 + 10} = 8.89 \text{ k}\Omega$$

The operating point can be found from the forward characteristics by writing the bias equation

$$I_B = \frac{I_C}{\beta} = \frac{I_C}{60}$$

$$V_B = I_B R_B + V_{BE} + I_E R_E$$

$$\frac{V_{CC} \cdot R_2}{R_1 + R_2} = I_B R_B + V_{BE} + (I_C + I_B) R_E$$

$$\frac{24 \times 10}{80 + 10} = I_B \times 8.89 + 0.6 + (60 + 1) I_B \times 1$$

$$2.67 = I_B (69.89) + 0.6$$

$$I_B = 0.0296 \text{ mA} \Rightarrow 29.6 \text{ mA}$$

$$I_C = 60 \times 29.6 \ \mu\text{A} = 1.78 \text{ mA}$$

$$V_{CE} = V_{CC} - I_C R_C - (I_B + I_C) \cdot R_E = 12.23 \text{ V}$$

$$(I_{CQ},\ V_{CEQ}) = (1.78 \text{ mA},\ 12.23 \text{ V})$$

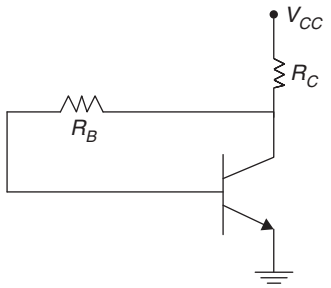
Example 6: The stability factor of the above transistor is
(A) 1.99
(B) 3.13
(C) 6.31
(D) 8.63

Solution: (D)

$$S = \frac{R_B + R_E}{R_E + \frac{R_B}{\beta + 1}} = \frac{8.89 + 1}{1 + \frac{8.89}{61}} = 8.63$$

Example 7: The fixed bias circuit as shown is used as collector bias circuit, the value of R_B is?

Quiescent point (9.2 mA, 4.4 V), the transistor has DC current gain 115. $V_{BE} = 0.7$ V, $V_{CC} = 9$ V, $R_C = 500\ \Omega$.



(A) 500 Ω
(B) 45.75 kΩ
(C) 75.42 kΩ
(D) 125.63 kΩ

Solution: (B)

$$V_{CC} = (I_B + I_C) R_C + I_B \cdot R_B + V_{BE}$$

$$9 = \left(\frac{9.2}{115} + 9.2\right) \times 0.5 + \frac{9.2}{115} \times R_B + 0.7$$

$$R_B = \frac{3.66}{0.08} = 45.75 \ \text{k}\Omega$$

Example 8: The stability factor and the thermal stability factor for the above circuit are
(A) 62.81 and 0.525
(B) 51.71 and 0.446
(C) 42.36 and 0.345
(D) 33.81 and 0.824

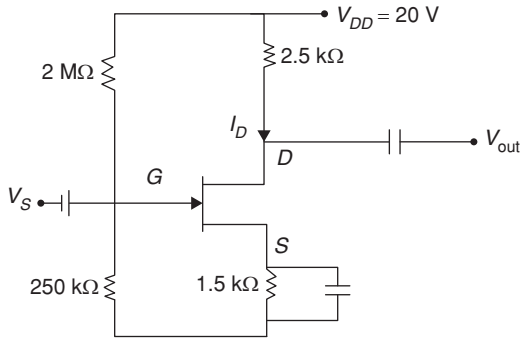
Solution: (B)

$$S = \frac{1 + h_{FE}}{1 + \frac{h_{FE} \cdot R_C}{R_C + R_B}} = \frac{1 + 115}{1 + \frac{115 \times 0.500}{0.5 + 45.75}} = 51.711$$

Thermal stability factor $K = \dfrac{S}{1 + h_{FE}}$

$$= \frac{51.7111}{116} = 0.446$$

Example 9: FET voltage divider bias is shown in given figure if $I_D = 4$ mA, then V_{GS} and V_{DS} will be respectively



(A) −3.78 V and 4 V
(B) 4 V and −3.78 V
(C) −3.78 V and −4 V
(D) −4 V and −3.78 V

Solution: (A)

$$V_G = \frac{20 \times 0.25}{2 + 0.25} = \frac{20}{9}$$

$$V_{GS} = V_G - I_D \cdot R_S = \frac{20}{9} - 4 \times 1.5 = -3.78 \text{ V}$$

$$V_{DS} = V_{DD} - I_D (R_S + R_D)$$

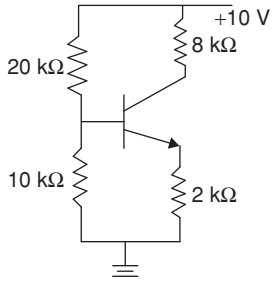
$$= 20 - 4 \times (1.5 + 2.5) = 4 \text{ V}$$

EXERCISES

Practice Problems I

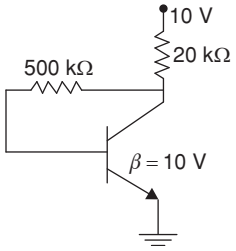
Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. For the circuit showing in figure assume $\beta = 100$ for BJT, the transistor will be in



(A) Cutoff region
(B) Active region
(C) Inverse active region
(D) Saturation region.

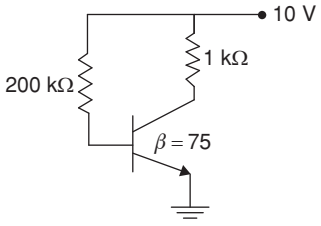
2. For the given biasing network, the stability factor '*S*' is



(A) 101 (B) 17.14
(C) 20.84 (D) 34

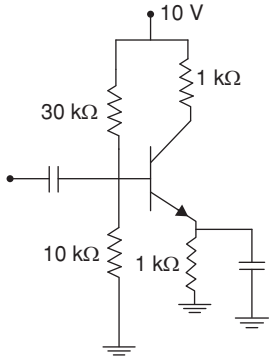
3. In a fixed bias circuit, $S = 51$ for $I_c = 2$ mA. Given $V_{cc} = 10$ V. The value of R_B is_____
(A) 250 kΩ
(B) 250 Ω
(C) 25 kΩ
(D) 2.5 kΩ

4. In the given fixed bias circuit, $V_{CE(sat)} = 0$ V, the minimum value of I_B to saturate the transistor is _____



(A) 10 mA (B) 0.25 mA
(C) 0.133 mA (D) 0.16 mA

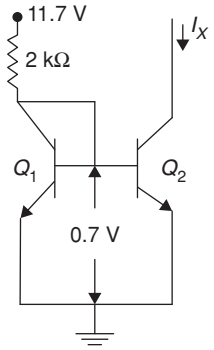
5. For the given biasing network, β is taken as very large. Assume $V_{BE} = 0.2$ V, the value of I_E and V_{CE} are _____



(A) 2.3 mA and 0 V (B) '0' mA and 10 V
(C) 10 mA and 5.4 V (D) 2.3 mA and 5.4 V

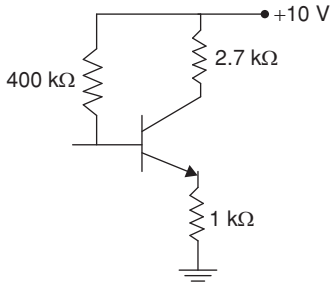
6. A GE transistor is used in the self bias arrangement with $V_{CC} = 20$ V, $R_C = 2$ kΩ, $R_E = 500$ Ω. The Q-point is selected at $V_{CE} = 10$ V, $I_C = 5$ mA and $S = 12$. If $V_{BE} = 0.3$ V and $\beta = 60$, then the values of R_1 and R_2 are _____.
(A) 40.65 kΩ and 8.24 kΩ
(B) 8.24 kΩ and 40.65 kΩ
(C) 40.65 kΩ and 18.6 kΩ
(D) 18.6 kΩ and 4.65 kΩ

7. In the BJT current mirror shown below, assume that the emitter area of Q_1 is double that of Q_2. What is the value of I_x?



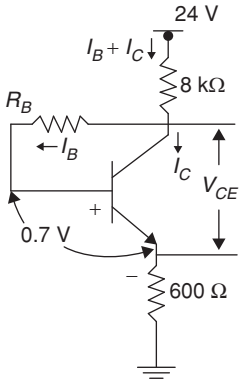
(A) 5.5 mA (B) 2.75 mA
(C) 6.2 mA (D) 11 mA

8. The transistor has following parameters. $\beta = 50$ and $V_{BE} = 0.6$ V. If β is increased by 5% what is the change in V_{CE}?



(A) Increased by 4.57% (B) Increased by 2.97%
(C) Decreased by 4.57% (D) Decreased by 2.97%

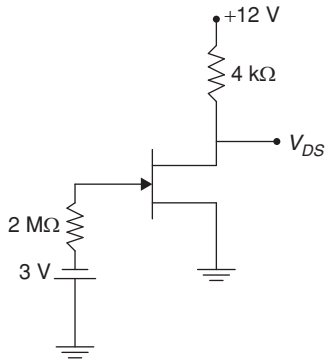
9. A Si transistor with $\beta = 50$ at 50°C is used and it is desired that $V_{CE} = 8$ V. Consider I_{CO} varies from 0.5 to 10 μA when temperature changes from 50°C to 100°C.



The values of R_B, stability factor, and I_c at 100°C are respectively _____

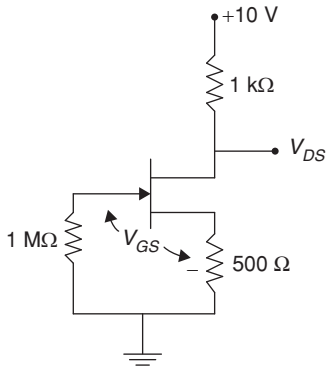
(A) 200 kΩ, 51 and 2.5 mA
(B) 20.5 kΩ, 50 and 1 mA
(C) 200.5 kΩ, 16.7 and 1.98 mA
(D) 2 kΩ, 10.4 and 1.75 mA

10. Given $I_{DSS} = 6$ mA and $V_P = -4$ V for the given FET biasing Network, the value of V_{DS} is _____



(A) 12 V (B) 0 V
(C) 10.5 V (D) 13.5 V

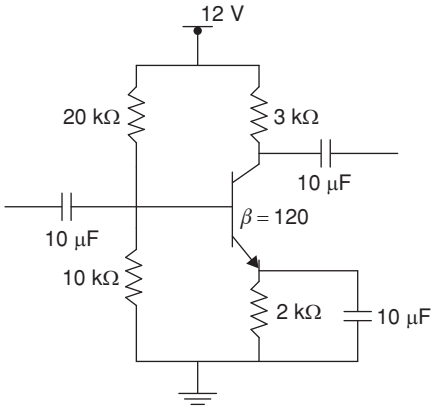
11. A self bias network with FET is given below; has $I_D = 5$ mA



What are the values of V_{GS} and V_{DS}?

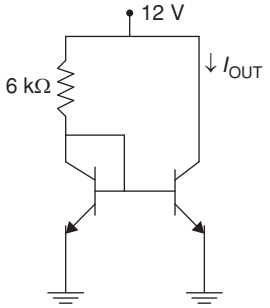
(A) –2.5 V and –2.5 (B) –2.5 V and 2.5 V
(C) –5 V and 0 V (D) 0 V and –5 V

12. For the given biasing network, calculate I_B and V_{CE} (assume $V_{BE} = 0.7$ V)



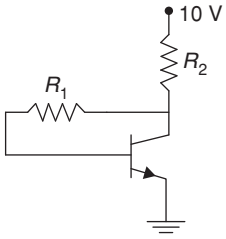
(A) 13.4 μA and 3.95 V (B) 13.4 μA and 6 V
(C) 1.34 μA and 3.95 V (D) 1.34 μA and 6 V

13. A constant current source using two matched npn transistors with $\beta = 100$ and $V_{BE} = 0.6$ V is shown in the figure below. Calculate I_{out}.



(A) 1.5 mA (B) 0.2 mA
(C) 1.86 mA (D) 2 mA

14. A transistor with $V_{BE} = 0.7$ V is shown. The values of R_1 and R_2 are such that the transistor is operating at $V_{CE} = 5$ V and $I_C = 2$ mA where $\beta = 100$. What is the operating point values for a transistor with $\beta = 200$.

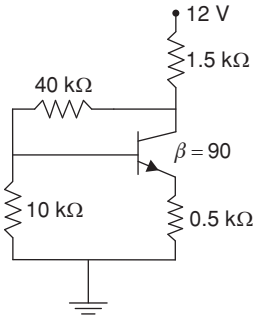


(A) 3.78 V and 3.1 mA (B) 3.5 V and 2.6 mA
(C) 5 V and 2.5 mA (D) 3.5 V and 1.25 mA

15. A transistor with $\beta = 100$ is used in CE mode. It is connected in collector to base bias mode with $\frac{R_B}{R_C} = 9$. Calculate the stability factor?

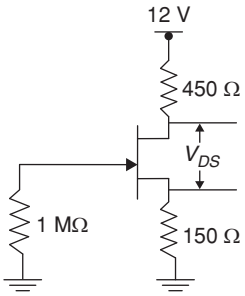
(A) 99 (B) 19.1
(C) 20.86 (D) 9.18

16. Calculate the stability factor for the given biasing network.



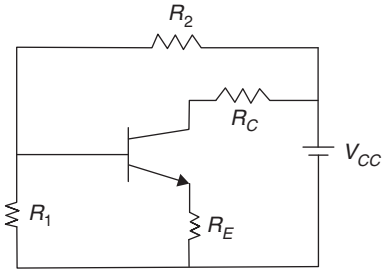
(A) 10 (B) 6.2
(C) 4.8 (D) 1

17. For a given FET biasing network, $V_{DS} = \frac{1}{2}V_{DD}$, what are the values of I_D and V_{GS}?



(A) 1 mA and −1 V (B) 10 mA and −1 V
(C) 1 mA and −1.5 V (D) 10 mA and −1.5 V

18. The values of resistors R_C, R_E respectively, for I_C = 5 mA, V_{CE} = 8 V, V_E = 6 V and $S(I_{co})$ = 10, h_{fe} = 200 and V_{CC} = 20 V, are?



(A) 1.194 kΩ and 1.2 kΩ
(B) 1.2 kΩ and 1.194 kΩ
(C) 10.3 kΩ and 17.43 kΩ
(D) 2.1 kΩ and 4.12 kΩ

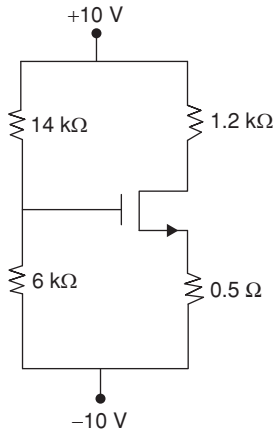
19. The values of resistors R_1 and R_2 in the above circuit are
(A) 17.2 kΩ and 32.86 kΩ
(B) 12.8 kΩ and 23.6 kΩ
(C) 12.8 kΩ and 22.86 kΩ
(D) 17.2 kΩ, 32.86 kΩ

20. A pnp transistor uses potential divider bias V_{BE} = −0.7 V, and β = 100, the operating point is to be at I_C = −1 mA, and V_{CE} = −5 V, V_{CC} = −20 V, voltage drop across R_E is −3 V, for SC(I_{co}) = 5, the value of R_C and R_E are
(A) 12 kΩ and 3 kΩ
(B) 11.7 kΩ and 2.97 kΩ
(C) 11.7 kΩ and 3 kΩ
(D) 12 kΩ and 2.97 kΩ

21. The values of R_1 and R_2 (bias resistances) in the above problem are?
(A) 4.2 kΩ and 12.4 kΩ
(B) 64.4 kΩ and 15.35 kΩ
(C) 42 kΩ and 12 kΩ
(D) 6.3 kΩ and 24 kΩ

Common Data for Questions for 22 to 24

22. In the current circuit, if the transistor parameters are V_{th} = 2 V, $k_n = 60\mu$ A/V^2, $\frac{W}{L} = 60$, and the transistor is in saturation, the value of V_{GS} is



(A) 3.62 V (B) −3.62 V
(C) 0.74 V (D) −0.74 V

23. The value of I_D is
(A) 12.5 mA (B) 9.2 mA
(C) 6.3 mA (D) 4.7 mA

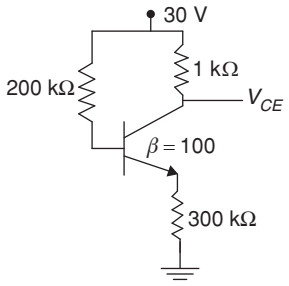
24. The value of V_{DS} is
(A) 12.95 V (B) 11.9 V
(C) 9.3 V (D) 7.5 V

25. A BJT for which the manufacturer specifies $P_{D(max)}$ = 125 MW at 25°C free air temperature and max junction temperature is 150°C. The thermal resistance of BJT is _____.
(A) 5°C/W (B) 1°C/MW
(C) 1°C/W (D) 10°C/MW

Practice Problems 2

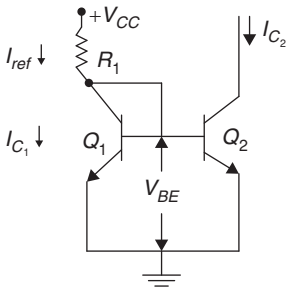
Directions for questions 1 to 18: Select the correct alternative from the given choices.

1. For the given biasing network, the collector to emitter voltage V_{CE} is _____ Volt.



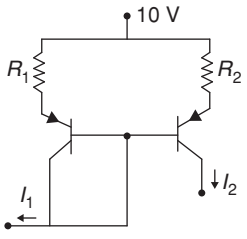
(A) 0 V (B) 30 V
(C) 15.2 V (D) 13.1 V

2. A current mirror shown below, provide a 1.5 mA current with $V_{CC} = 12$ V. Assume $\beta = 150$ and $V_{BE} = 0.7$ V. What is R_1.



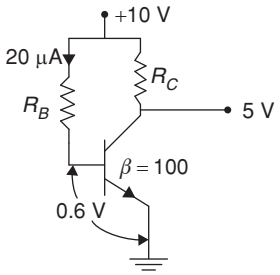
(A) 9.25 kΩ
(B) 7.43 kΩ
(C) 7.53 kΩ
(D) 8.5 kΩ

3. In the figure given $I_1 = 10$ μA and $I_2 = 40$ μA. What is the ratio R_1/R_2.



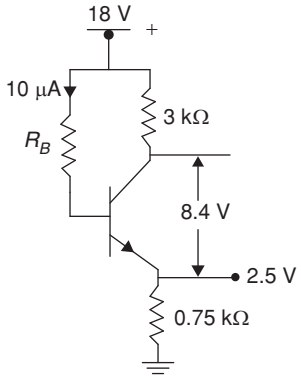
(A) 1 (B) 2
(C) $\frac{1}{2}$ (D) 4

4. In the given biasing network, the value of R_B and R_C are_____



(A) 470 kΩ and 2.5 kΩ
(B) 2.5 kΩ and 470 kΩ
(C) 4.7 kΩ and 250 Ω
(D) 470 kΩ and 25 kΩ

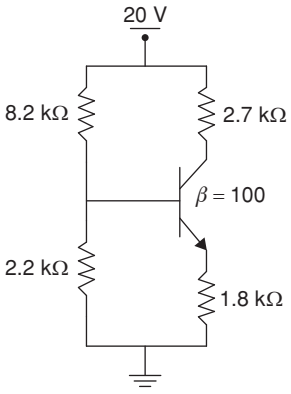
5. A transistor with emitter to base bias is given below.



The values β of transistor, the base resistance R_B are _____

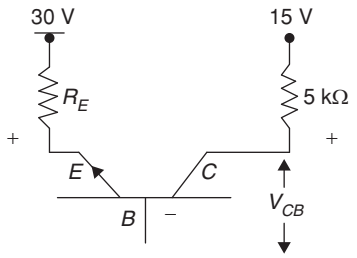
(A) 100 and 155 kΩ (B) 166 and 1.75 MΩ
(C) 333 and 1.55 MΩ (D) 158 and 1.8 MΩ

6. The self bias Network is given below. Its stability factor (S) is_____



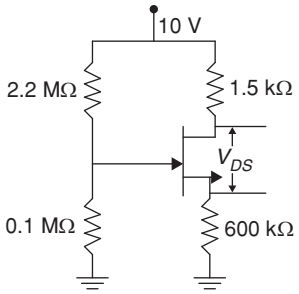
(A) 101 (B) 9.5
(C) 58 (D) 2

7. The value of R_E which will saturate the transistor is nearly _____ kΩ.



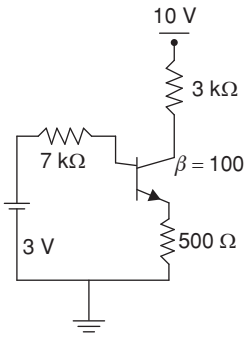
(A) 5 (B) 25
(C) 10 (D) 20

8. Calculate I_D and V_{GS} for $V_{DS} = 2$ V



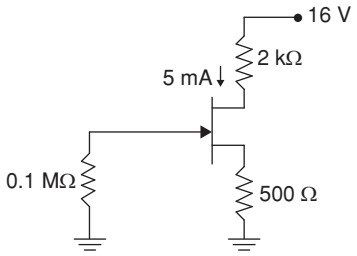
(A) 4.6 mA and −1.85 V (B) 3.8 mA and −0.63 V
(C) 4.6 mA and −0.63 V (D) 3.8 mA and −1.85 V

9. Find the region of operation of transistor.



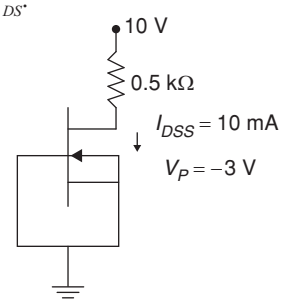
(A) Saturation (B) Active
(C) Cut-off (D) None

10. A FET biasing network is used with following specifications.
$V_P = 5$ V, $I_{DSS} = 8$ mA
Calculate g_m.



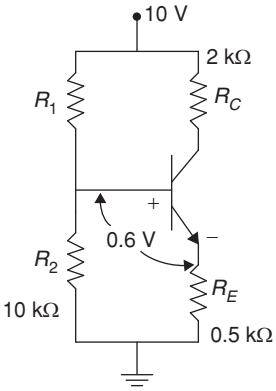
(A) 1.6 mS (B) 2 mS
(C) 10 mS (D) 2.3 mS

11. The figure below shows a MOSFET biasing network. Calculate V_{DS}.



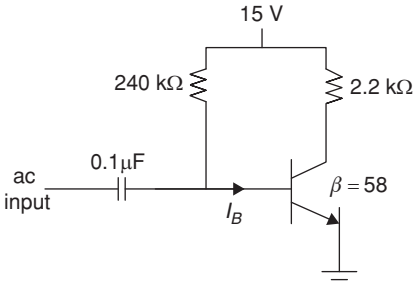
(A) 5 V (B) 10 V
(C) 0 V (D) −5 V

12. A transistor with β is assumed to be very large is shown below. What is the value of R_1 required to make $I_C = 2.5$ mA.



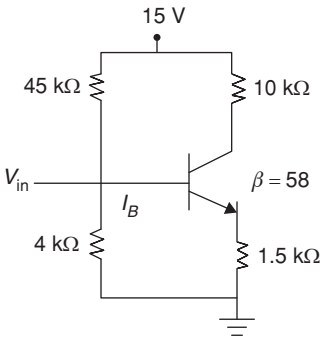
(A) 2 kΩ (B) 10 kΩ
(C) 44 kΩ (D) 54 kΩ

13. I_B for a given circuit is



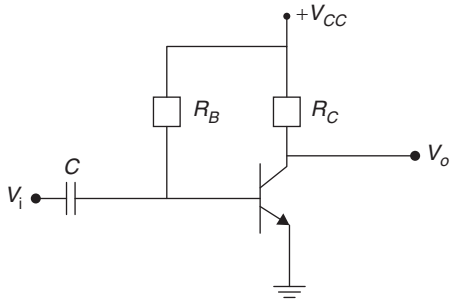
(A) 59.6 μA (B) 49 μA
(C) 62.5 μA (D) 68.1 μA

14. The equivalent base resistance of the given circuit is



(A) 49 kΩ (B) 45 kΩ
(C) 4 kΩ (D) 3.67 kΩ

15. The circuit shown in figure is that of a fixed bias circuit. Estimate the values of the collector load resistor R_c, and bias resistor R_B, if the quiescent collector current and voltage values are 10 mA and 5 V respectively. The transistor has a DC current gain of 110. V_{BE} is 0.7 V, V_{CC} = 12 V.

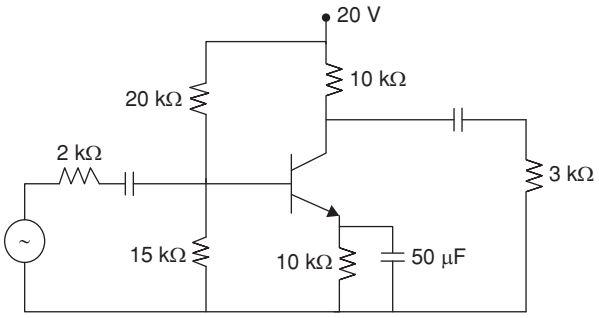


(A) 500 Ω and 103.7 kΩ
(B) 500 Ω and 124.3 kΩ
(C) 700 Ω and 124.3 kΩ
(D) 700 Ω and 103.7 kΩ

16. Calculate the thermal stability factor for the above circuit?

(A) 111 (B) 1
(C) 12 (D) 18

17. The circuit gives details for one stage of a transistor amplifier. Calculate the quiescent values of I_E, V_{CE}, h_{fe} = 100, V_{BE} = 0.



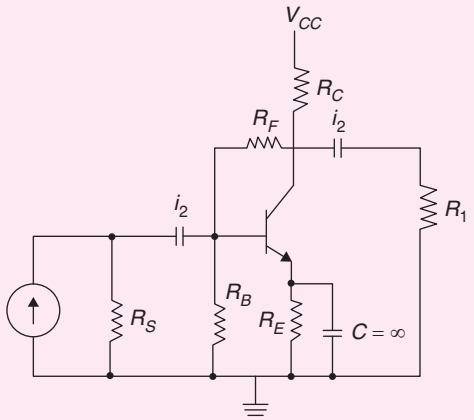
(A) 0.857 mA and 2.959 V
(B) 1.23 mA and 5.67 V
(C) 1.11 mA and 7.69 V
(D) –0.857 mA and 2.95 V

18. The thermal resistance of power transistor is 20°C/W. The ambient temperature is 40°C and junction temperature is 160°C. If V_{CE} = 4.2 V, the maximum the collector can carry without destruction is _____.

(A) 1.2 A
(B) 2.5 A
(C) 1.66 A
(D) 2 A

Previous Years' Questions

1. The feedback used in the circuit shown in figure can be classified as **[2004]**

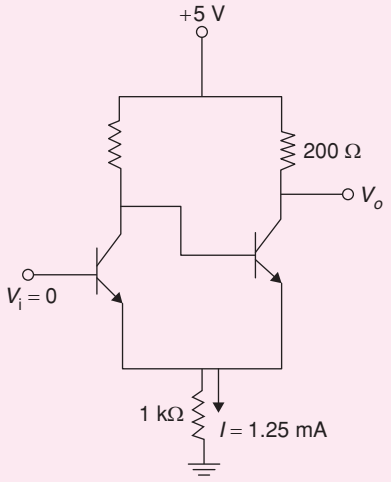


(A) shunt-series feedback
(B) shunt-shunt feedback
(C) series-shunt feedback
(D) series-series feedback

2. A bipolar junction transistor (BJT) is used as a power control switch by biasing it in the cut-off region (OFF state) or in the saturation region (ON state). In the ON state, for the BJT **[2004]**

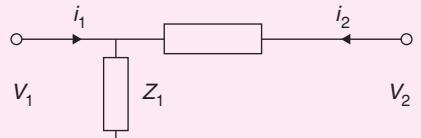
(A) Both the base-emitter and base-collector junctions are reverse biased.
(B) The base-emitter junctions is reverse biased, and the base-collector junction is forward biased.
(C) The base-emitter junction is forward biased, and the base-collector junction is reverse biased.
(D) Both the base-emitter and base-collector junctions are forward biased.

3. In the Schmitt trigger circuit shown in figure, if $V_{CE(sat)}$ = 0.1 V, the output logic low level (V_{OL}) is **[2004]**



(A) 1.25 V (B) 1.35 V
(C) 2.50 V (D) 5.00 V

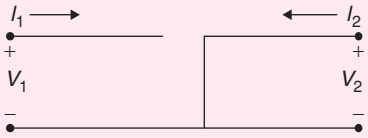
4. For the two-port network shown in the Figure the Z-matrix given by **[2005]**



(A) $\begin{bmatrix} Z_1 & Z_1+Z_2 \\ Z_1+Z_2 & Z_2 \end{bmatrix}$ (B) $\begin{bmatrix} Z_1 & Z_1 \\ Z_1+Z_2 & Z_2 \end{bmatrix}$

(C) $\begin{bmatrix} Z_1 & Z_2 \\ Z_2 & Z_1+Z_2 \end{bmatrix}$ (D) $\begin{bmatrix} Z_1 & Z_1 \\ Z_2 & Z_1+Z_2 \end{bmatrix}$

5. The parameter type and the matrix representation of the relevant two port parameters that describe the circuit shown are **[2006]**



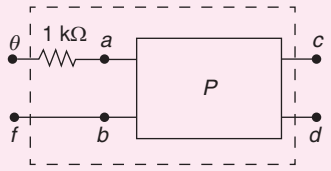
(A) z parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(B) h parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) h parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(D) z parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. The two-port network P shown in the figure has ports 1 and 2, denoted by terminals (a, b) and (c, d), respectively. It has an impedance matrix Z with parameters denoted by z_{ij}. A 1 Ω resistor is connected in series with the network at port 1 as shown in the figure. The impedance matrix of the modified two-port network (shown as a dashed box) is **[2010]**



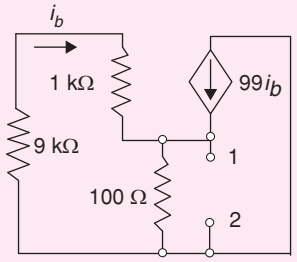
(A) $\begin{pmatrix} z_{11}+1 & z_{12}+1 \\ z_{21} & z_{22}+1 \end{pmatrix}$

(B) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21} & z_{22}+1 \end{pmatrix}$

(C) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$

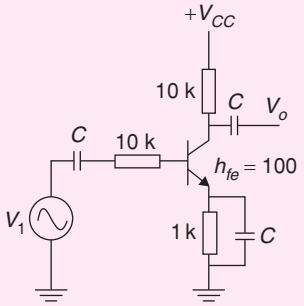
(D) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21}+1 & z_{22} \end{pmatrix}$

7. The impedance looking into nodes 1 and 2 in the given circuit is **[2012]**



(A) 50 Ω (B) 100 Ω
(C) 5 kΩ (D) 10.1 kΩ

8. The magnitude of the mid-point voltage gain of the circuit shown in figure is (assuming h_{fe} of the transistor to be 100) **[2014]**



(A) 1
(B) 10
(C) 20
(D) 100

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** C	**3.** B	**4.** C	**5.** D	**6.** A	**7.** B	**8.** D	**9.** C	**10.** C
11. B	**12.** A	**13.** C	**14.** B	**15.** D	**16.** C	**17.** D	**18.** B	**19.** D	**20.** D
21. B	**22.** A	**23.** D	**24.** B	**25.** B					

Practice Problems 2

1. D	**2.** B	**3.** D	**4.** A	**5.** C	**6.** D	**7.** C	**8.** D	**9.** A	**10.** A
11. A	**12.** C	**13.** A	**14.** D	**15.** C	**16.** A	**17.** A	**18.** C		

Previous Years' Questions

1. B	**2.** D	**3.** B	**4.** D	**5.** C	**6.** C	**7.** A	**8.** D

Chapter 5

Amplifiers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- The re transistor model
- BJT AC analysis
- Emitter follower configuration
- Common base configuration
- 3 π – model
- The hybrid equivalent model
- Hybrid h – parameter model for common emitter
- Two port systems approach
- Frequency response analysis of amplifiers
- Miller's theorem
- Multi stage amplifiers
- RC coupled BJT amplifier
- Cascode connection
- Darlington connection
- Frequency response
- FET amplifier stages
- AC equivalent model of MOS FET
- AC equivalent Circuit
- Amplifier AC equivalent model

Introduction

The transistors can be employed as an amplifying device, i.e., the output AC power is greater than the input power. There is an exchange of DC power to AC domain, that power permits establishing a higher output AC power.

The superposition theorem is applicable for the analysis and design of the DC and AC components of a BJT network, permitting the separation of the analysis of the DC and AC responses of the system.

There are three models commonly used in the small signal AC analysis of transistor network

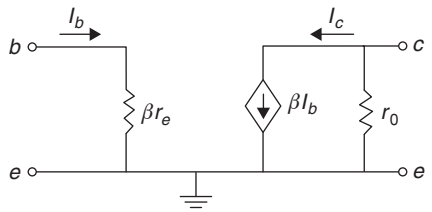
1. r_e model,
2. Hybrid π model
3. Hybrid equivalent model.

The AC equivalent of a network is obtained by

1. Setting all DC sources to zero, and replacing them by a short circuit equivalent.
2. Replacing all capacitors by a short circuit equivalent.
3. Removing all elements by passed by short circuit equivalents, introduced by steps 1 and 2.
4. Redrawing the network in more convenient and logical form.

The r_e Transistor Model

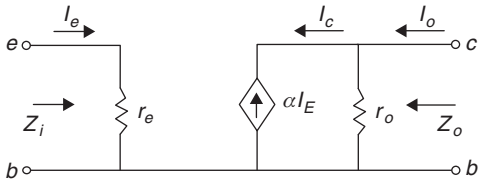
Common Emitter Configuration



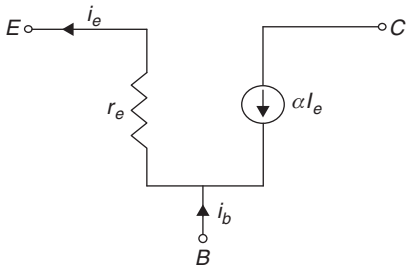
$$r_e = \frac{26\text{ mV}}{I_E},\ Z_{e=}(\beta+1)\, r_e \simeq \beta r_e$$

$$r_o = \frac{\Delta V_{CE}}{\Delta I_C}$$

Common Base Configuration



$$r_e = \frac{26\text{ mV}}{I_E}$$



For common collector configuration, the equivalent is same as common Emitter configuration.

For p-n-p transistor, the direction of currents are reversed

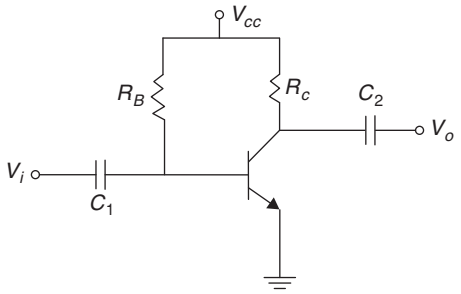
r_e = dynamic emitter resistance $= \dfrac{26\text{ mV}}{I_E}$

α = current gain of CB

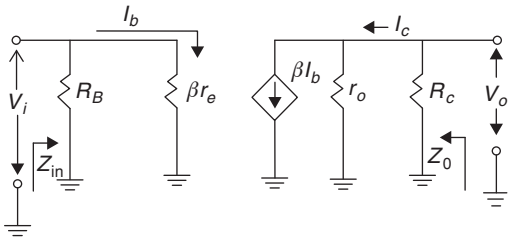
β = current gain of CE

BJT AC Analysis

Common Emitter Fixed Bias Configuration



(1) Common Emitter fixed bias configuration.



$Z_i = R_B \parallel \beta r_e\ \Omega$ input resistance
$Z_i \simeq \beta r_e$ if $R_B \geq 10\ \beta r_e$
Output impedance $Z_o = R_C \parallel r_o$
If $r_o \geq 10R_c$ $Z_o \cong R_c$

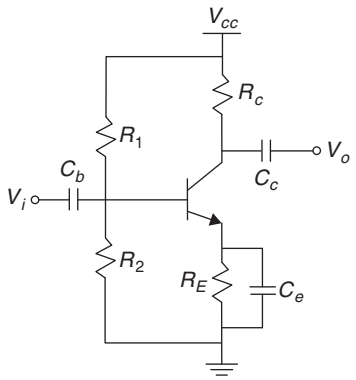
$$V_o = -\beta I_b (R_1 \parallel r_o),\ I_b = \frac{V_i}{\beta r_e}$$

$$\text{So}\quad V_o = -\beta\left(\frac{V_i}{\beta r_e}\right)(R_c \parallel r_o)$$

$$A_v = \frac{V_o}{V_i} = \frac{-(R_c \parallel r_o)}{r_e} = \frac{-R_c}{r_e};\ r_o \geq 10R_c,\ A_i = \beta$$

The negative sign in the equation A_v, reveals that a 180° phase shift occurs between the input and output signals.

Voltage Divider Bias



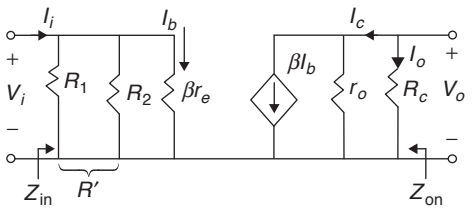


Figure 1 AC equivalent circuit

$$\text{Say}\ \ R' = \frac{R_1 R_2}{R_1 + R_2} = R_1 \parallel R_2$$

Z_i Input impedance $= R' \parallel \beta r_e$
Z_o Output impedance $= R_c \parallel r_o$

$$A_v = \frac{V_o}{V_1} = \frac{-R_c \parallel r_o}{r_e}$$

$$\text{if } r_o \geq 10\ R_c \text{ then } A_v = \frac{-R_c}{r_e}$$

Negative sign reveals a 180° phase shift between V_o and V_i.

Common Emitter, Emitter-bias Configuration

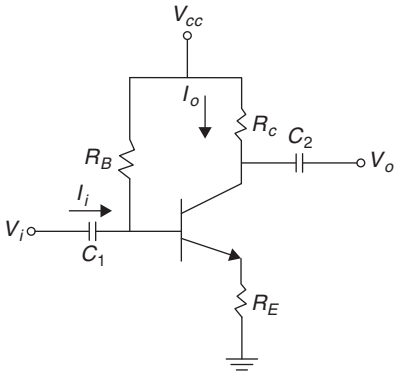


Figure 2 CE, emitter bias configuration

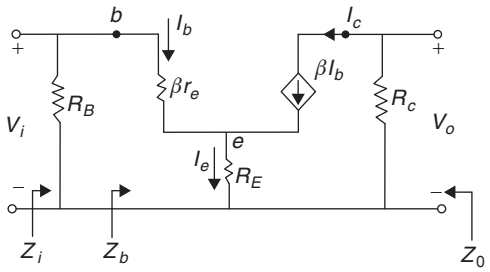


Figure 3 AC equivalent circuit

Applying Kirchhoff's voltage law to the input,

$$V_i = I_B \beta r_e + I_e R_E = I_B \beta r_e + (\beta + 1)\, I_B R_E$$

$$Z_b = \frac{V_i}{I_B} = \beta r_e + (\beta + 1) R_E$$

$$\simeq \beta r_e + \beta R_E = \beta(r_e + R_E)$$

$$\simeq \beta R_E \text{ as } R_E >> r_e$$

Z_i Input impedance $(Z_i) = Z_b \parallel R_B = Z_b \parallel R_B$
With V_i set to zero, $I_b = 0$, and βI_b can be replaced by an open circuit equivalent,
Out put impedance $Z_o = R_c$
Gain A_v,

$$I_b = \frac{V_i}{Z_b}$$

$$V_o = -I_o R_c = -\beta I_b R_c$$

$$= -\beta \left(\frac{V_i}{Z_b} \right) R_c$$

$$A_v = \frac{V_o}{V_i} = \frac{-\beta R_c}{Z_b}$$

$Z_b = b(r_e + R_E)$ gives

$$A_v = \frac{V_o}{V_i} = \frac{-R_c}{r_e + R_E} \simeq \frac{-R_c}{R_E}$$

Effect of r_o

$$Z_b = \beta r_e + (\beta + 1) R_E \simeq \beta (r_e + R_E)$$

$$r_o \geq 10(R_c + R_E)$$

$$Z_i = Z_b \parallel R_B,\ Z_o = R_c,\ A_v = \frac{\beta R_c}{Z_b} \quad r_o \geq 10 R_c$$

If R_E is by passed by a capacitor, the AC circuit will be same as of fixed bias configuration.

Emitter Follower Configuration

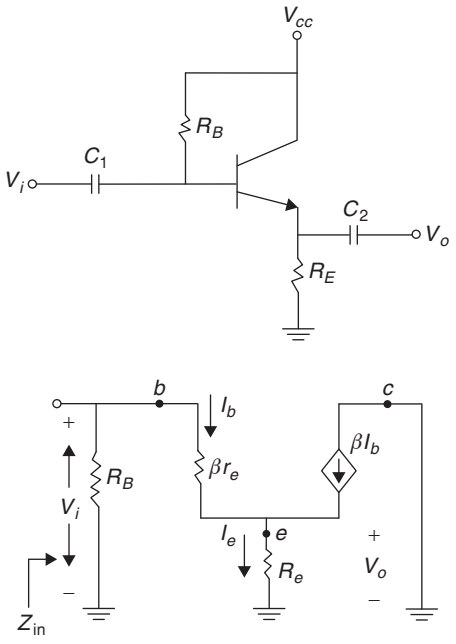


Figure 4 AC equivalent circuit

$$Z_i = R_B \parallel Z_b$$

$$Z_b = \beta r_e + (\beta + 1)\, R_E$$

$$Z_b = \beta R_E \qquad (R_E >> r_e)$$

$$Z_o = R_E \parallel r_e$$

$$Z_o = r_e \quad (R_E >> r_e)$$

$$A_v = \frac{V_o}{V_i} = \frac{R_E}{R_E + r_e} \simeq \frac{R_E}{R_E} = 1$$

V_o and V_i are in phase for Emitter-follower configuration.

Common Base Configuration

The common base configuration is characterized as having relatively low input and a high output impedance and a current gain less than 1. The voltage gain can be quite large.

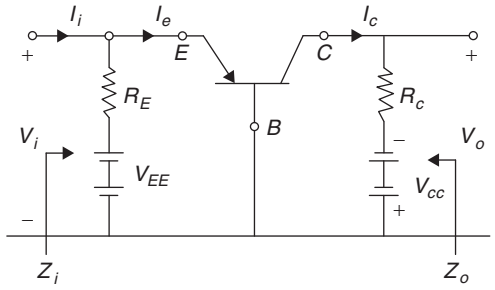
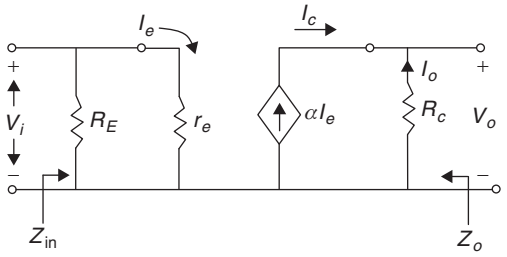


Figure 5 Common base configuration (p-n-p)



$$Z_i = R_E \parallel r_e$$

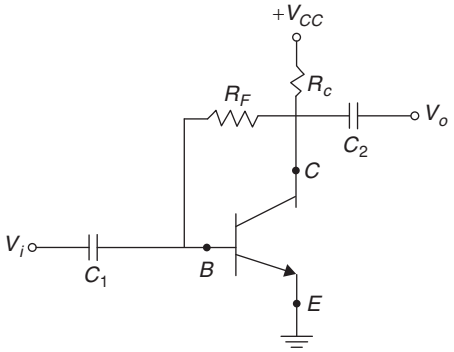
$$Z_o = R_c$$

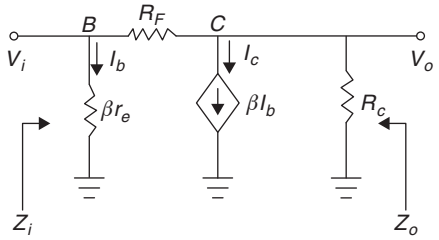
$$A_v = \frac{V_o}{V_i} = \frac{\alpha R_c}{r_e} \simeq \frac{R_c}{r_e}$$

$$A_i = \frac{I_o}{I_i} = -\alpha \simeq -1$$

The fact that A_v is a positive number shows that V_o and V_i are in phase for the common base configuration.

Collector Feedback Configuration





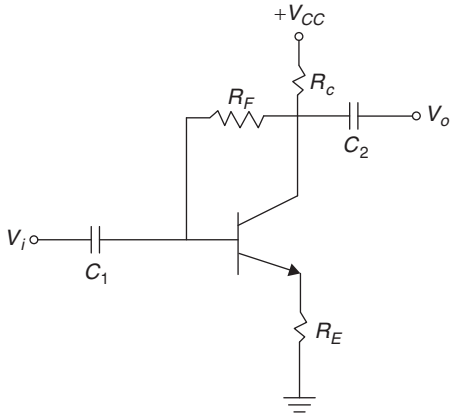
Substituting r_e equivalent circuit into the AC network,

$$Z_i = \frac{r_e}{\frac{1}{\beta} + \frac{R_c}{R_F}}$$

$$Z_o = R_c \parallel R_F$$

$$A_v = \frac{V_o}{V_i} = \frac{-R_c}{r_e},\ A_i = \frac{R_F}{R_c}$$

The negative sign in A_v equation shows 180° phase shift between input V_i and out put V_o.



Collector feedback with R_E

$$Z_i = \frac{R_E}{\left[\frac{1}{\beta} + \frac{R_E + R_c}{R_F}\right]}$$

$$Z_o = R_c \parallel R_F$$

$$A_v = \frac{-R_c}{R_E}$$

Determining the Current Gain

$$A_{iL} = -A_{vL}\frac{Z_i}{R_L}$$

Effect of R_L and R_s

The loaded voltage gain of an amplifier is always less than the no-load gain.

The gain obtained with a source resistance in place will always be less than that obtained under loaded or unloaded conditions.

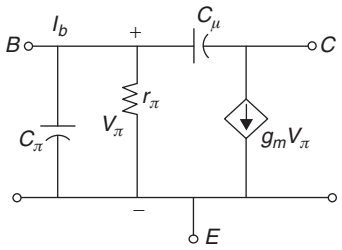
For the same configuration

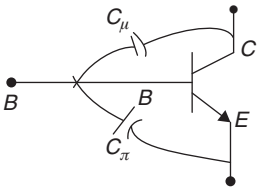
$$A_{v\ \text{no load}} > A_{v\ \text{load}} > A_{v\ \text{load and source}}$$

for a particular design, the larger the level of R_L the greater is the level of AC gain.

For a particular amplifier, the smaller the internal resistance of the signal source, the greater is the overall gain.

π–Model





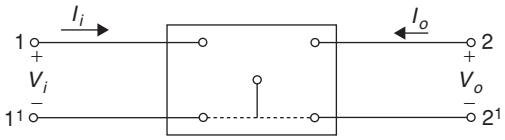
C_π = Forward biased diffusion capacitance

C_μ = Reverse biased diffusion capacitance

$r_\pi = \beta r_e$

$$g_m = \frac{1}{r_e}$$

The Hybrid Equivalent Model



$$V_i = h_{11}I_i + h_{12}V_o$$

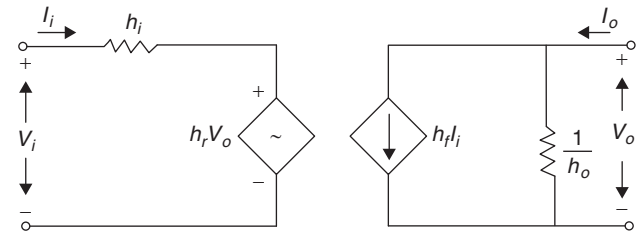
$$I_o = h_{21}I_i + h_{22}V_o$$

h_{11} – input resistance – $h_i = \left.\frac{V_i}{I_i}\right|_{V_o=0}^{\text{ohm}}$

h_{12} – reverse transfer voltage ratio – $h_r = \left.\frac{V_i}{I_i}\right|_{V_0=0}$

h_{21} – forward transfer current ratio – $h_f = \left.\frac{I_o}{I_i}\right|_{V_0=0}$

h_{22} – output conductance – $h_o = \left.\frac{I_o}{V_o}\right|_{I_i=0}^{\text{Siemens}}$



For common emitter circuit $h_{ie} = \beta r_e$, $h_{fe} = \beta_{ac}$

For common base circuit $h_{ib} = r_e$, $h_{fb} = -\alpha \sim -1$

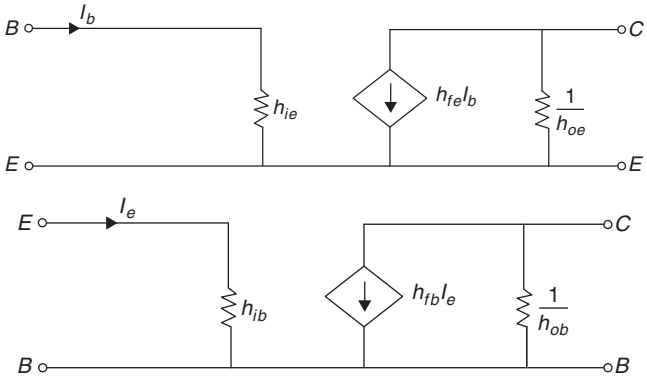
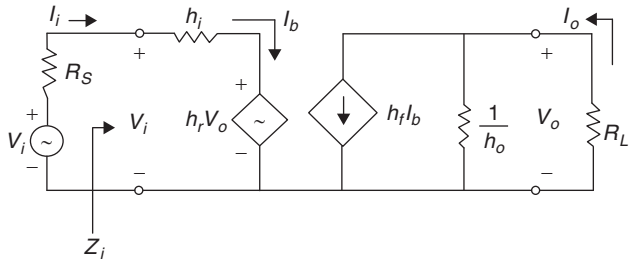


Figure 6 Approximate common base hybrid equivalent circuit

H–parameter Model with R_s and R_L

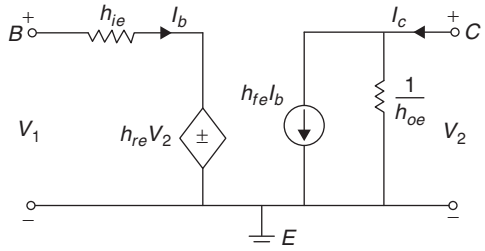


Current gain $A_i = \dfrac{I_o}{I_i} = \dfrac{h_f}{1 + h_o R_L}$

Voltage gain $A_v = \dfrac{V_o}{V_i} = \dfrac{-h_f R_L}{h_i + (h_i h_o - h_f h_r) R_L}$

$$Z_i = h_i - \frac{h_f h_r R_L}{1 + h_o R_L}, \quad Z_o = \frac{1}{h_o - [h_f h_r / (h_i + R_S)]}$$

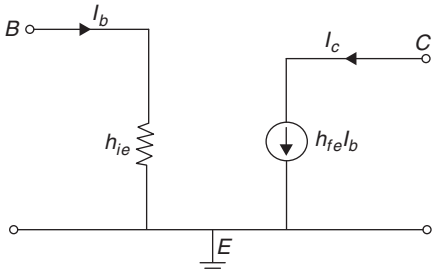
Hybrid H–parameter Model for Common Emitter



Typical values of *h*-parameters for CE configuration:

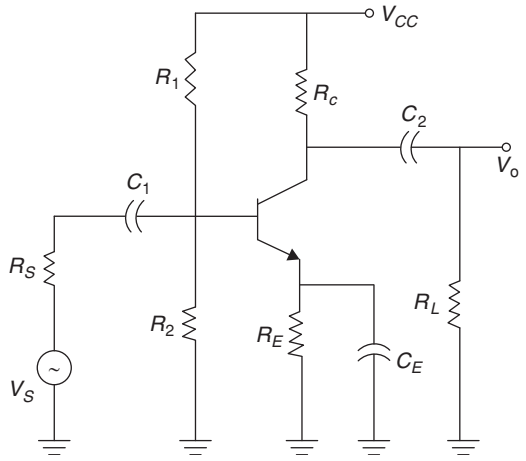
$$h_{ie} = 3.5 \text{ k}\Omega$$
$$h_{re} = 1.3 \times 10^{-4}$$
$$h_{fe} = 120$$
$$h_{oe} = 8.5 \text{ } \mu\text{S}$$
$$\Rightarrow \quad \frac{1}{h_{0e}} = 0.12 \text{ m}\Omega$$

Simplified '*h*' parameters model: (CE *n-p-n* configuration)



For *p-n-p* transistor, the direction of current is reversed in the equivalent circuit.

RC Coupled CE Amplifier



Small signal analysis using simplified *h*-parameter model

1. Make all the DC sources to zero.
2. Replace the coupling capacitors with short.

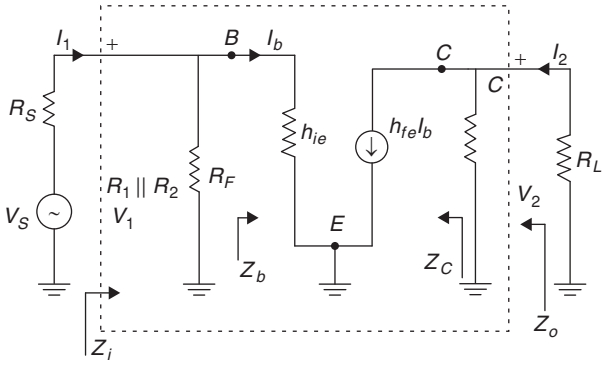
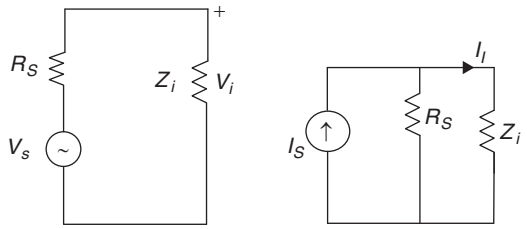


Table 1 *Equivalent circuit without considering R_E*

Without R_E	Considering R_E
$Z_b = h_{ie}$	$Z_b = h_{ie} + (1 + h_{fe})\, R_E$
$Z_i = h_{ie} \| R_1 \| R_2$	$Z_i = Z_b \| R_1 \| R_2$
$Z_C = \frac{1}{h_{oe}}$	$Z_C = \frac{1}{h_{0e}}$
$Z_0 = \left(\frac{1}{h_{oe}}\right)$	$Z_0 = \left(\frac{1}{h_{oe}}\right) \| R_c \| R_L$
$A_v = \frac{-h_{ie} R_C}{h_{ie}}$	$A_v = \frac{(R_C \| R_2)}{r_e}$

Voltage and current gain by taking source into consideration:

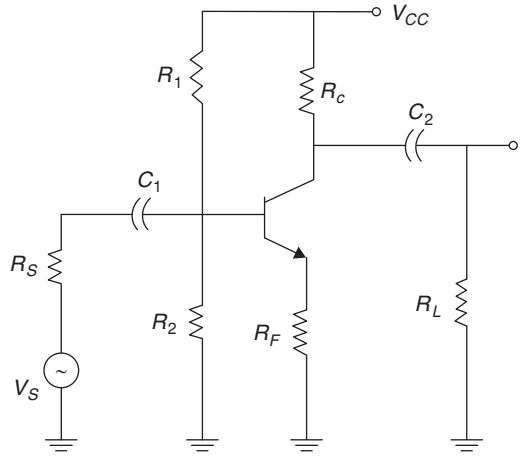


$$A_{VS} = \frac{V_0}{V_S} = \frac{V_0}{V_1} \times \frac{V_1}{V_S} \qquad A_{IS} = \frac{I_0}{I_S} = \frac{I_0}{I_1} \times \frac{I_1}{I_S}$$

$$A_{VS} = \frac{A_V Z_i}{R_S + Z_i} \qquad A_{IS} = \frac{A_I R_S}{R_S + Z_i}$$

Example 1: Determine A_V, A_I, A_{VS}, A_{IS}, Z_i, Z_o for the following amplifier circuit

$C_1 = 10\ \mu F$ $R_1 = 40\ k\Omega$ $R_L = 2.2\ k\Omega$
$C_E = 20\ \mu F$ $R_2 = 10 k\Omega$ $\beta = 100$
$C_C = 1\ \mu F$ $R_E = 2\ k\Omega$ $C_{CC} = 20\ V$
$R_2 = 1\ k\Omega$ $R_C = 4\ k\Omega$



Solution:

(i) $A_V = \frac{-(R_C \| R_L)}{r_e}$

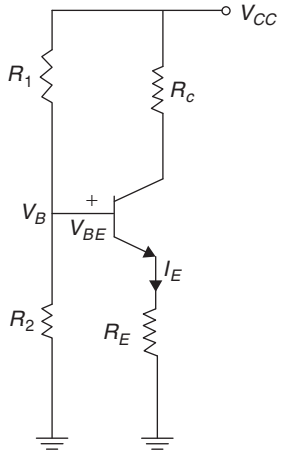
Determining 'r_e':

$$r_e = \frac{26\ mV}{I_E}$$

So DC equivalent circuit is needed to calculate I_E.

DC Equivalent Circuit

1. Make AC sources short.
2. Replace coupling capacitors with open.



Apply KVL at input side gives

$$V_B - V_{BE} - I_E R_E = 0$$

$$I_E = \frac{V_B - V_{BE}}{R_E}$$

$$V_B = \frac{4 - 0.7}{2\,k\Omega} = 1.65\ mA$$

$$\therefore \quad r_e = \frac{26\ mv}{I_E} = 15.76\ ohms$$

$$\therefore \quad A_V = -\frac{(R_C \| R_L)}{r_e} = -90$$

3. $A_I = \beta = 100$
4. $A_{VS} = \frac{A_V \cdot Z_i}{R_S + Z_i}$

   $Z_i = Z_b \| R_1 \| R_2$

   $Z_b = h_{ie} = \beta r_e = 15.76\ k\Omega$

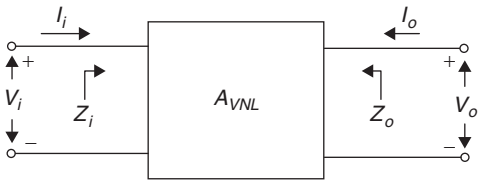
   $Z_i = 1.32\ k\Omega$

   $$A_{VS} = \frac{-90(1.32\ k\Omega)}{(2.32\ k\Omega)} = -51.21$$

5. $A_{IS} = \frac{A_I \cdot R_S}{R_S + Z_i} = \frac{100 \times 1\,k\Omega}{2.32\,k\Omega}$

   $A_{IS} = 43.103$
6. $Z_0 \times R_C = 4\ k\Omega$

Two Port Systems Approach

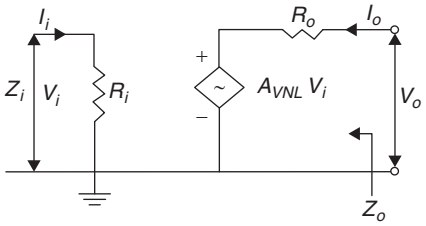


If we take a Thevenin look at the output terminals we find V_i, set to zero.

$$Z_{th} = Z_o = R_o$$

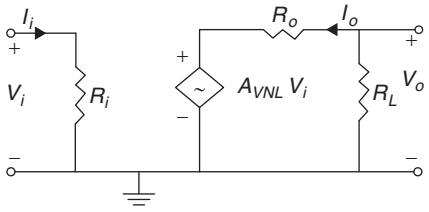
E_{th} is the open circuit voltage between the output terminals identified as V_o.

$$A_{VNL} = \frac{V_o}{V_i}, V_o = A_{VNL} V_i = E_{th}$$



Substituting the internal elements for 2 port network.

Applying a Load to the Two Port Network System

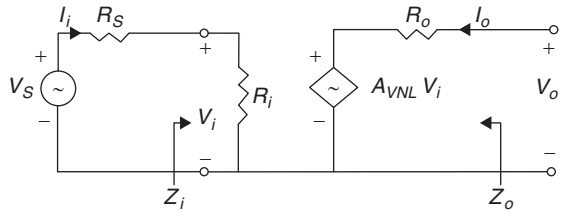


$$V_o = \frac{R_L}{R_L + R_o} \cdot A_{VNL} \cdot V_i$$

$$A_{VL} = \frac{V_o}{V_i} = \frac{R_L}{R_L + R_o} \cdot A_{VNL}$$

$$A_{iL} = -A_{VL} \cdot \frac{Z_i}{R_L}$$

Effects of Source Resistance R_s



The parameters Z_i and A_{VNL} of a two port system are unaffected by the internal resistance of the applied source, the output impedance may be affected by the magnitude of R_S.

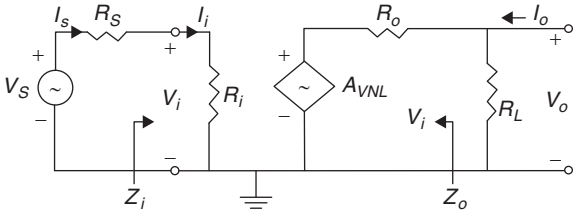
$$V_i = \frac{R_i}{R_i + R_S} \cdot V_S$$

$$V_o = A_{VNL} \cdot V_i$$

$$= A_{VNL} \frac{R_i}{R_i + R_s} \cdot V_S$$

$$A_{VS} = \frac{V_o}{V_S} = \frac{R_i}{R_i + R_S} \cdot A_{VNL}$$

The Effects of R_s and R_L



$$V_i = \frac{R_i}{R_i + R_S} \cdot V_S, \; V_o = \frac{R_L}{R_L + R_o} \cdot A_{VNL} \cdot V_i$$

$$A_{VL} = \frac{V_o}{V_i} = \frac{R_L \cdot A_{VNL}}{R_L + R_o}$$

$$A_{VS} = \frac{V_o}{V_S} = \frac{V_o}{V_i} \cdot \frac{V_i}{V_S} = \frac{R_i}{R_i + R_S} \cdot \frac{R_L}{R_L + R_o} \cdot A_{VNL}$$

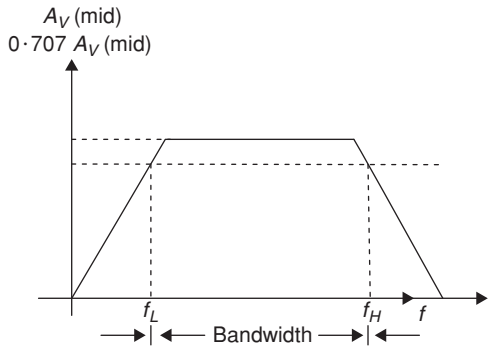
$$A_{iL} = -A_{VL} \cdot \frac{R_i}{R_L} \text{ as } I_i = \frac{V_i}{R_i}$$

$$I_S = \frac{V_S}{R_S + R_i}$$

$$A_{iS} = -A_{VS} \frac{R_S + R_i}{R_L}$$

Frequency Response Analysis of Amplifiers

$$|A_v| = \left| \frac{V_0}{V_i} \right|$$

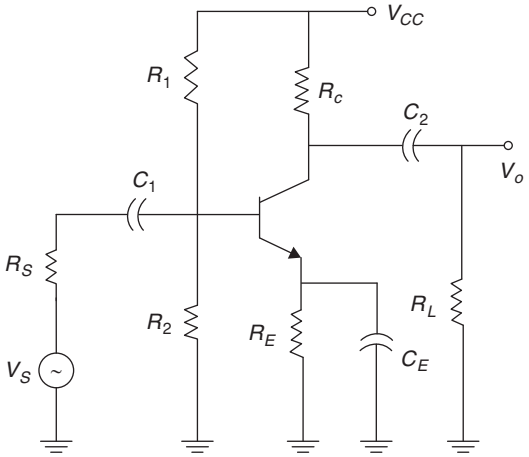


Frequency Rrange	Coupling and by Pass Capacitor	Parasitic Capacitance
low	Consider	Open
mid	Short	Open
high	Short	Consider

$A_{V(\text{mid})}$ = mid band gain of the amplifier

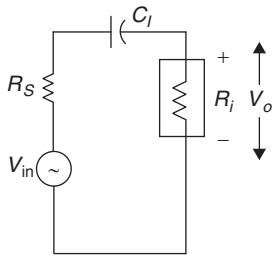
Bandwidth = $f_H - f_L$

Low Frequency Response of an RC Coupled CE Amplifier

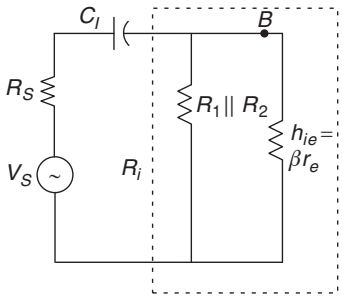


Low frequency Response depends on C_1, C_2, and C_E

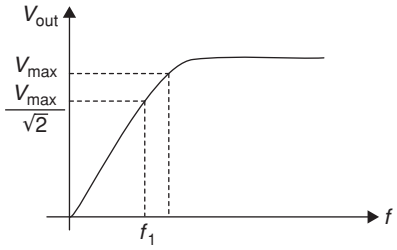
Due to C_I



AC equivalent network



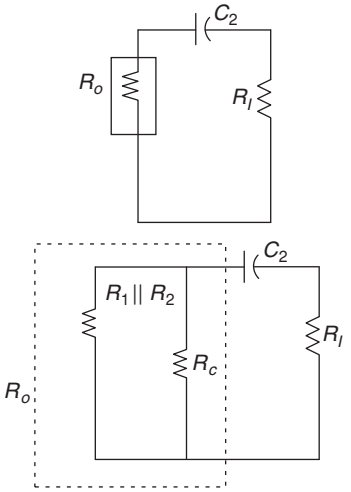
Frequency response due to C_I



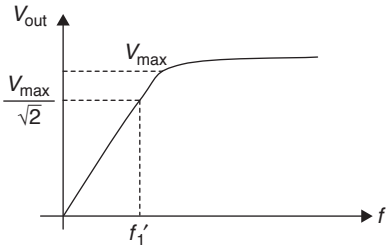
$$f_1 = \frac{1}{2\pi(R_S + R_i)C_1}$$

Where $R_i = R_1 \| R_2 \| \beta r_e$

Due to C_2



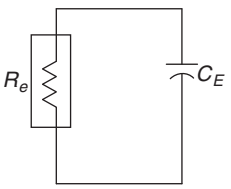
Frequency response due to C_2



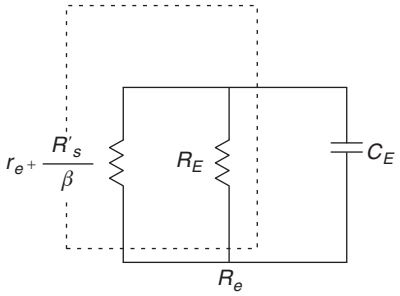
$$f_1' = \frac{1}{2\pi(R_S + R_i)C_2}$$

Where $R_0 = r_0 \alpha R_C$

Due to C_E



AC equivalent network

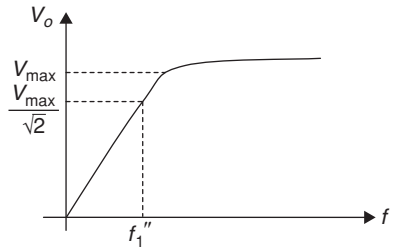


$$f_1'' = \frac{1}{2\pi R_e C_E}$$

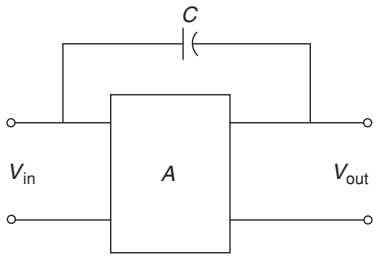
Where $R_e = R_E \| \left(r_e + \frac{R'S}{\beta} \right)$,

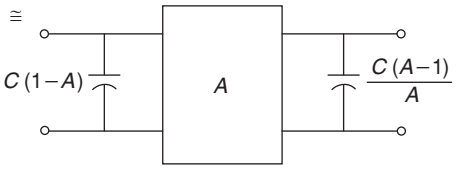
$R_S' = R_S \| R_1 \| R_2$

Frequency response



Miller's theorem





C_{iw}, C_{ow} = Wiring (stray) capacitance

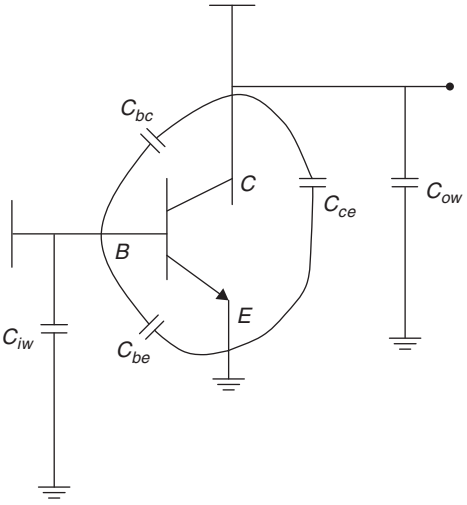
C_{bc} = miller capacitance

C_{be}, C_{ce} = junction capacitance/parasitic capacitance

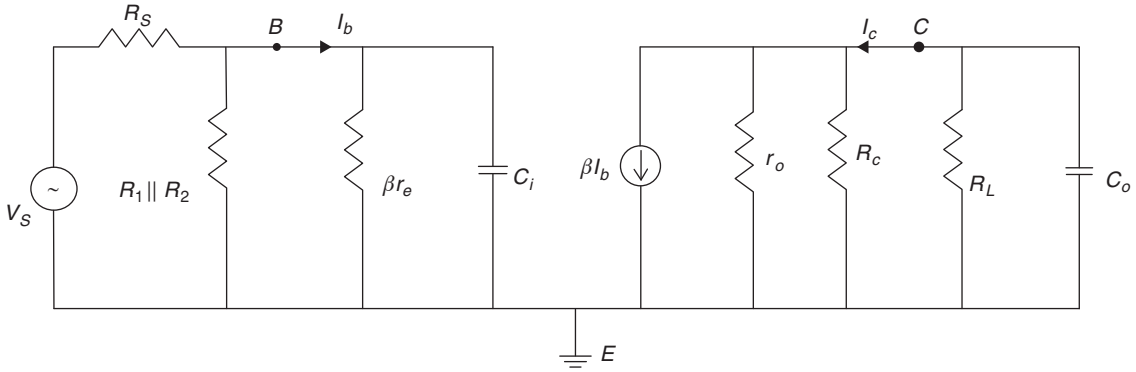
C_{in} (miller) = $C(1 - A)$

$$C_{out}(\text{miller}) = \frac{C(A-1)}{A}$$

High Frequency Response of an RC Coupled CE Amplifier

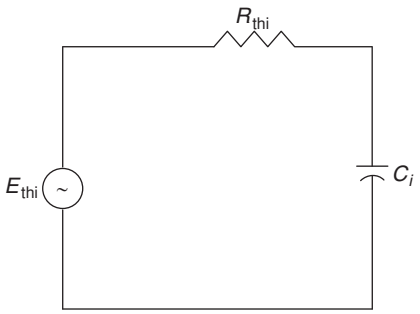


AC equivalent network



Thevenin's circuit

Input side

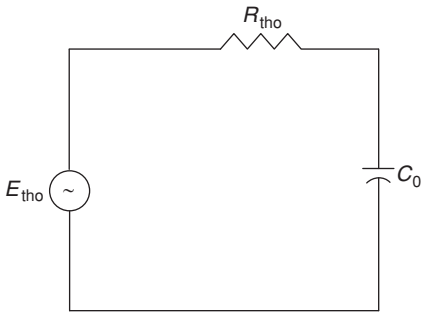


$$f_i = \frac{1}{2\pi R_{thi} C_i}$$

$R_{thi} = R_S \,||\, R_1 \,||\, R_2 \,||\, \beta r_e$

$C_i = C_{wirei} + C_{be} + (1 - A)\, C_{bc}$

Output side

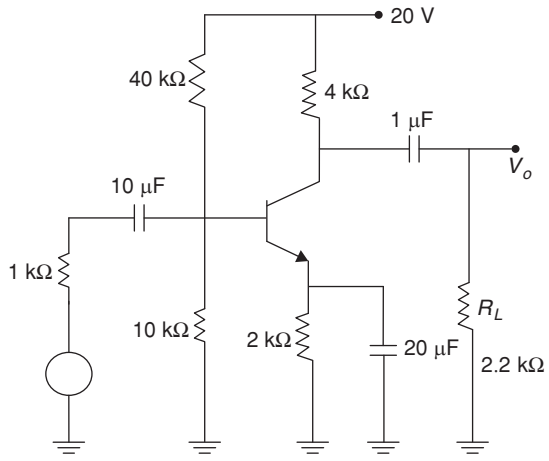


$$F_0 = \frac{1}{2\pi R_{tho} C_0}$$

$R_{tho} = r_0 \,||\, R_C \,||\, R_L$

$$C_0 = C_{wireo} + C_{ce} + C_{bc}\frac{(A-1)}{A}$$

Example 2: Determine the lower cut off frequency for the circuit given $\beta = 100$, $r_0 = \infty$ ohms



(A) 6.86 Hz (B) 25.68 Hz
(C) 327 Hz (D) 350 Hz

Solution: (C)
Determining 'r_e'

$$r_e = \frac{26 \text{ mv}}{I_E}$$

$R_e = 15.76$ ohms (refer GE04557)
$\beta r_e = 1.576$ kΩ

Mid band gain $A_V = \dfrac{-(R_C \,||\, R_L)}{r_e}$

$$A_V = -90$$

$$R_i = R_1 \,||\, R_2 \,||\, \beta r_e = 1.32 \text{ kΩ}$$

$$R_s' = R_s \,||\, R_1 \,||\, R_2 = 0.889 \text{ kΩ}$$

$$R_e = R_E \,||\left(\frac{Rs'}{\beta} + r_e\right) = 24.35\ \Omega$$

$$f_1 = \frac{1}{2\pi(R_S + R_i)C_1} = 6.86 \text{ Hz}$$

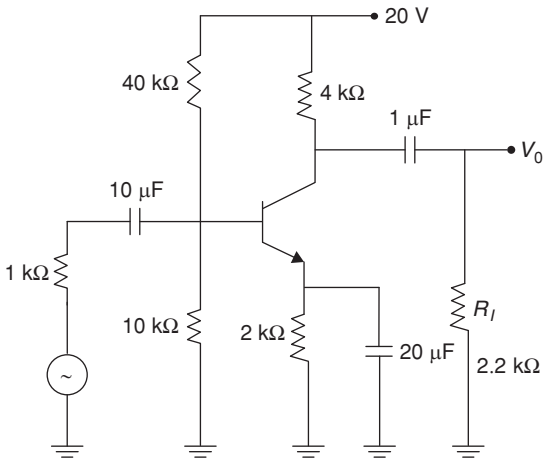
$$f_1' = \frac{1}{2\pi(R_C + R_L)C_2} = 25.86 \text{ Hz}$$

$$f_1^{\alpha} = \frac{1}{2\pi R_E C_E} = 327 \text{ Hz}$$

$\therefore$ Dominant lower cut-off frequency
= 327 Hz (highest).

Example 3: Determine the higher cut off frequency for the given circuit given $\beta = 100$, $r_0 = \infty$ ohms, $V_{cc} = 20$ V, $C_{be} = 36$ pF, $C_{bc} = 4$ pF

$C_{ce} = 1$ pF, $C_{wi} = 6$ pF, $C_{wo} = 8$ pF



(A) 738.24 kHz (B) 8.6 MHZ
(C) 10 MHZ (D) 800 kHz

Solution: (A)

$$R_{thi} = R_s \parallel R_1 \parallel R_2 \parallel \beta r_e$$
$$r_e = 1.32\ \text{k}\Omega$$
$$R_{thi} = 0.531\ \text{k}\Omega$$
$$C_i = C_{wirei} + C_{be} + (1 - A)\, C_{bc}$$
$$A = -90 \text{ (refer GE04557)}$$
$$C_i = 406\ \text{pF}$$
$$R_{tho} = R_c \parallel R_L = 1.419\ \text{k}\Omega$$

$$C_0 = C_{wo} + C_{ce} + C_{bc}\frac{(A-1)}{A} = 13.04\ \text{pF}$$

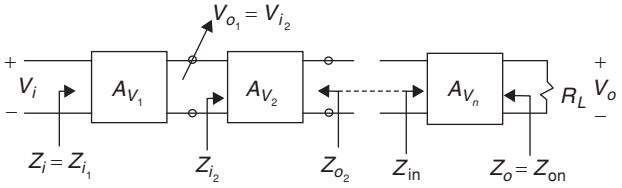
$$f_i = \frac{1}{2\pi R_{thi} RC_i} = 738.24\ \text{kHz}$$

$$f_0 = \frac{1}{2\pi R_{tho} RC_0} = 8.6\ \text{MHZ}$$

$\therefore$ Higher cut-off frequency is 738.24 kHz (lowest).

Multi Stage Amplifiers

Cascaded Systems



The two port systems approach is partially useful for cascaded systems such as that appearing in fig. where Av_1, Av_2, …, Av_n are voltage gains of each stage, under loaded conditions, that is Av_1 is determined with the input impedance to Av_2 acting as load on Av_1.

The total gain of the system is product of individual gains.

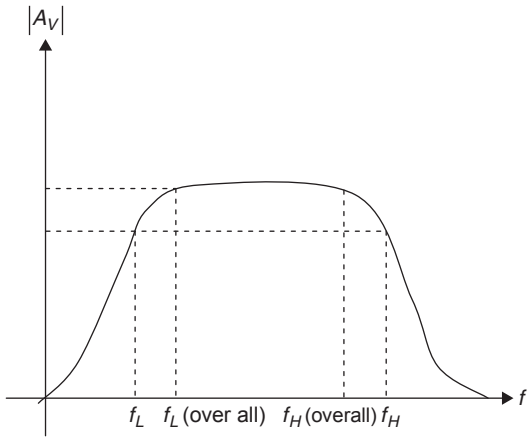
$Av = Av_1 \cdot Av_2 \cdot Av_3$, …, Av_n and total current gain is

$$A_i = -Av\frac{Z_i}{R_L}$$

Effect of cascading on frequency response:

1. Gain is multiplied.
2. Bandwidth is reduced.

If n' amplifiers having lower cutoff frequency of f_L and highest cut-off frequency of f_H, then the frequency response is given below



$$f_l(\text{overall}) = \frac{f_L}{\sqrt{2^{\frac{1}{n}} - 1}}$$

$$f_h(\text{overall}) = f_H\sqrt{2^{\frac{1}{n}} - 1}$$

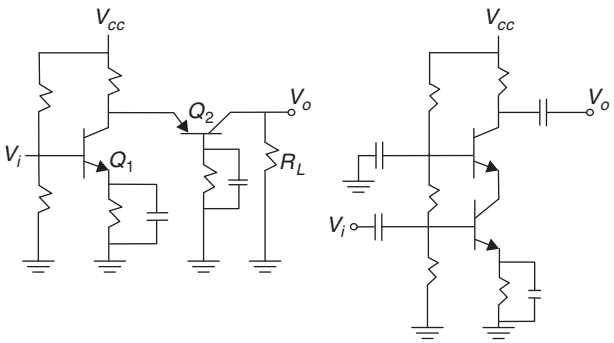
Where f_L and f_H are the lower and higher cut off frequencies of the individual amplifier stages.

RC coupled BJT amplifier

The name is derived from the Capacitive coupling (Capacitor (C_C)) and the fact that the load on first stage is on RC combination. The Coupling capacitor isolates the two stages from a DC point of view, but acts as short circuit equivalent for the AC response. The input impedance of second stage acts as load for the first stage.

Cascode connection (CE–CB)

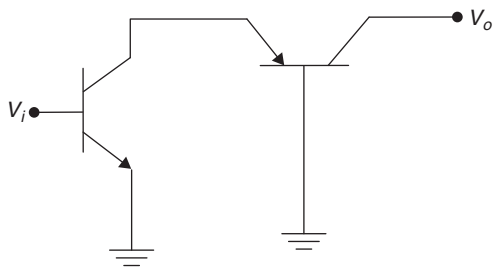
If the collector of the loading transistor is connected to the emitter of the following transistor, it is called as cascode connection. The possible configurations are



Advantages:

1. Larger output impedance
2. Wider band width

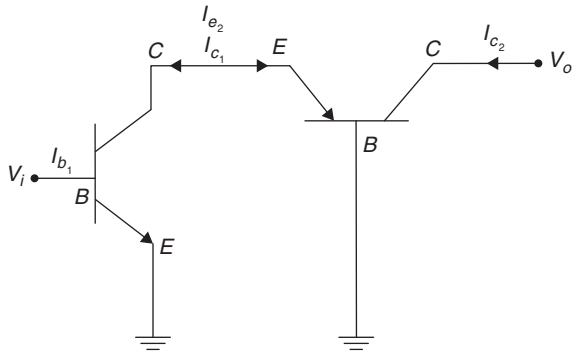
Example 4: For the connection shown below, find the overall transconductance of the circuit. Given g_{m_1}, is the transconductance of CE amplifier and g_{m_2} is the transconductance of CB-amplifier



(A) g_{m_2} (2) $g_{m_1} \| g_{m_2}$

(C) g_{m_1} (D) $g_{m_1} + g_{m_2}$

Solution: (C)

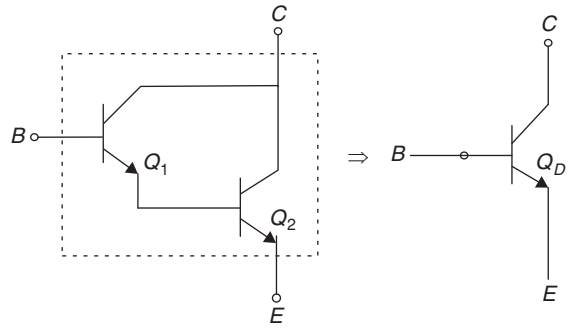


Overall transconductance $= \dfrac{I_{c_2}}{V_i}$

$$I_{C_2} = I_{e_2} = I_{C_1}$$

$$\therefore \quad g_m(\text{overall}) = \frac{I_{c_1}}{V_i} = g_{m_1}.$$

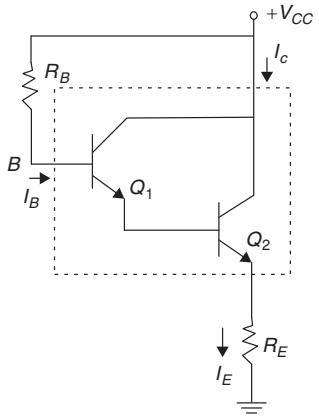
Darlington connection



The main feature of the Darlington connection is that the composite transistor acts as a single unit with a current gain that is product of the current gains of the individual transistors.

$$\beta_D = \beta_1 \cdot \beta_2$$

A darling ton transistor connection provides a transistor having a very large current gain typically a few thousand, and high input resistance



DC bias

$$I_B = \frac{V_{CC} - V_{BE}}{R_B + \beta_D R_E}$$

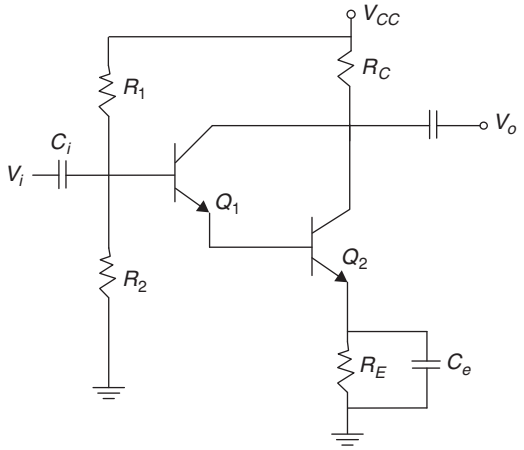
$$I_E = (\beta_D + 1)\, I_B$$

AC Analysis

$$Z_i = R_B \| \beta_1 \beta_2 R_E, \quad A_i = \frac{\beta_D R_E}{R_B + \beta_D R_E}$$

$$A_v \cong 1, \; Z_o = \frac{r_{e_1}}{\beta_2} + r_{e_2}$$

Where R_E is by passed and output taken at collector



$$Z_i = R_1 \| R_2 \| \beta_1 (r_{e_1} + \beta_D r_{e_2})$$

$$A_i = \frac{\beta_D (R_1 \| R_2)}{R_1 \| R_2 + Z_i'}$$

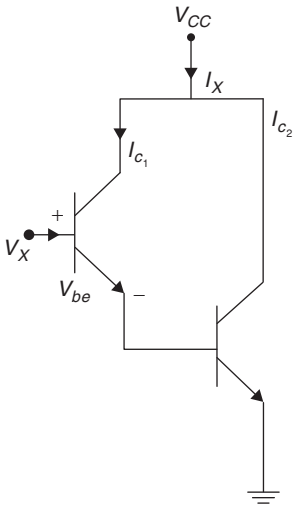
Where $Z_i' = \beta (r_{e_1} + \beta_2 r_{e_2})$

$$Z_o = RC \| r_{o_2}$$

$$A_v = \frac{V_o}{V_i} = \frac{\beta_D R_C}{Z_i^1}$$

Example 5: Given the transconductance of Q_1 is g_{m_1} and Q_2 is g_{m_2}.

Find the overall transconductance $\frac{I_X}{V_X}$ (β is large)



(A) g_{m_1} (B) $0.5\ g_{m2}$
(C) $0.5\ g_{m_1}$ (D) g_{m_2}

Solution: (B)

$$I_X = I_{C_1} + I_{C_2}$$

$$I_{C_2} \simeq I_{E_2} = (\beta+1) I_{E_1}$$

$$I_{C_2} \simeq (\beta+1) I_{C_1}$$

$$I_{C_2} \simeq (\beta+1)^2 I_{b_1}$$

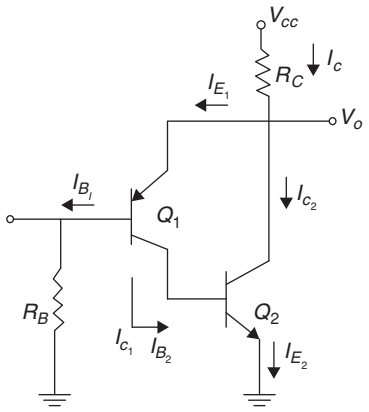
as 'β' is large $I_{C_2} >> I_{C_1}$

$$\therefore \quad I_X \simeq I_{C_2}$$

$V_x = 2V_{be}$ (KVL at input)

$$\frac{I_X}{V_X} = \frac{I_{C_2}}{2V_{be}} = 0.5 g_{m_2}.$$

Feedback pair



DC Analysis

$$I_{B_1} = \frac{V_{cc} - V_{BE_1}}{R_B + \beta_D R_C}$$

$$I_c \simeq I_{c_2}$$

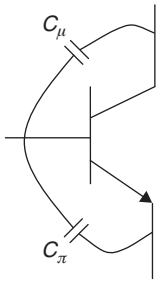
AC Analysis

$$Z_i' = \beta_1 \beta_2 R_c, \; Z_i = R_B \,||\, Z_i'$$

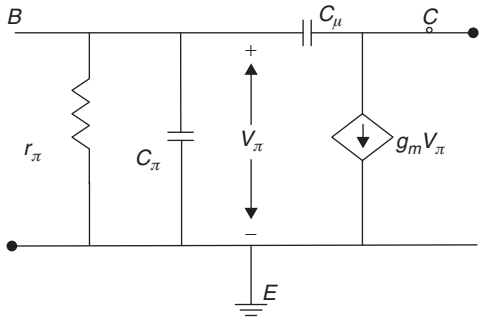
$$A_i = \frac{I_o}{I_i} = \frac{-\beta_1 \beta_2 R_B}{R_B + \beta_1 \beta_2 R_c}$$

$$A_v = \frac{\beta_2 R_c}{r_{e_1} + \beta_2 R_c} \simeq 1, \; Z_o = \frac{r_{e_1}}{\beta_2}$$

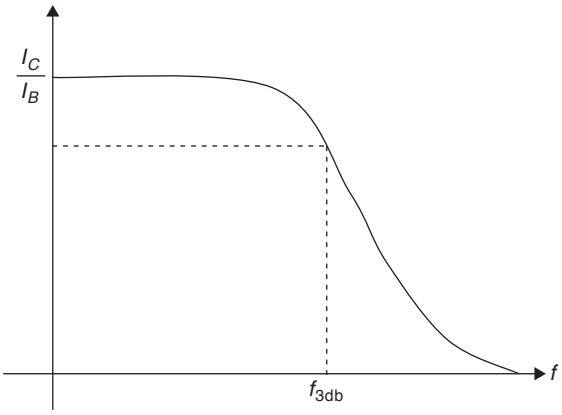
High Frequency Response of an Amplifier (π model)



AC Equivalent Circuit



Frequency Response



f_T = unity gain frequency

f_{3dB} = 3 – dB frequency

$$f_{3dB} = \frac{1}{2\pi r_\pi (C_\pi + C\mu)}$$

$$f_T = \frac{g_m}{2\pi (C_\pi + C\mu)}$$

$$\frac{f_T}{f_{3dB}} = g_m r_\pi = h_{fe}$$

Example 6: An amplifier has a gain of 60 dB with a band width of 10 MHz. Find the unity gain frequency?

(A) 10 GHz (B) 100 GHz

(C) 10 MHz (D) 100 MHz

Solution: (A)

Gain = A_{in} dB = 60

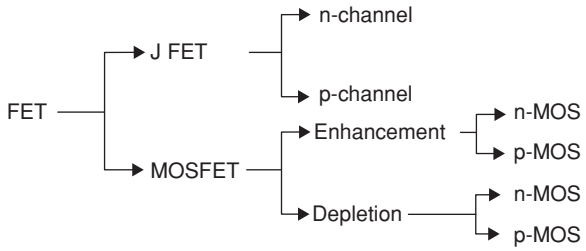
20 log A = 60

$\Rightarrow$ A = 10^3 = 1000

$f_T = (1000)\ (10 \times 10^6)$

f_T = 10 GHz.

FET Amplifier Stages



For JFET and depletion mode MOSFET analysis, Shockley's equation holds good.

$$\text{i.e., } I_D = I_{DSS}\left[1 - \frac{V_{GS}}{V_P}\right]^2 = \text{Source Current}$$

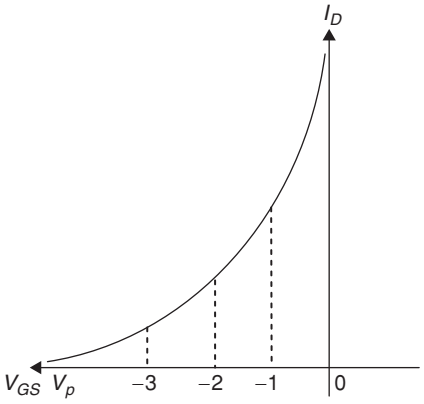
I_D =Drain current = I_S

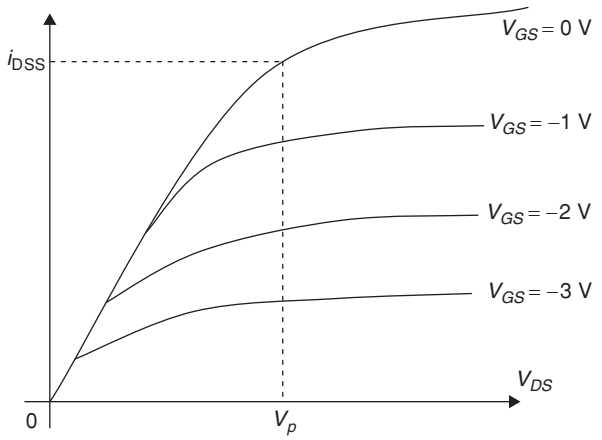
I_{DSS} = Drain saturation current

V_P = Pinch off voltage where I_D = 0 A

JFET Transfer Characteristics

Output characteristics:





Enhancement Mode MOSFET Analysis

$i_D = K(V_{GS} - V_T)^2$

i_D = Instantaneous drain current

V_{GS} = Instantaneous gate-source voltage

V_T = Threshold voltage

Symbols used to represent DC, AC and instantaneous values:

I_d = AC drain current

I_D = DC drain current

i_D = Instantaneous drain current = $I_D + i_d$

Region of Operation:

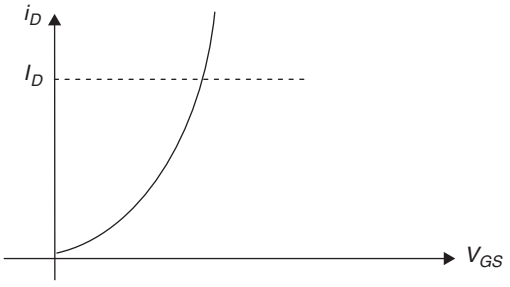
Case (i): $V_{GS} < V_T$: cut-off region

Case (ii): $V_{GS} > V_T$ and $V_{DS} < (V_{GS} - V_T)$: linear/triode region

Case (iii): $V_{GS} > V_T$ and $V_{DS} \geq (V_{GS} - V_T)$: saturation region

Case (iv): $V_{GS} > V_T$ and $V_{DS} \geq 2(V_{GS} - V_T)$: break down region

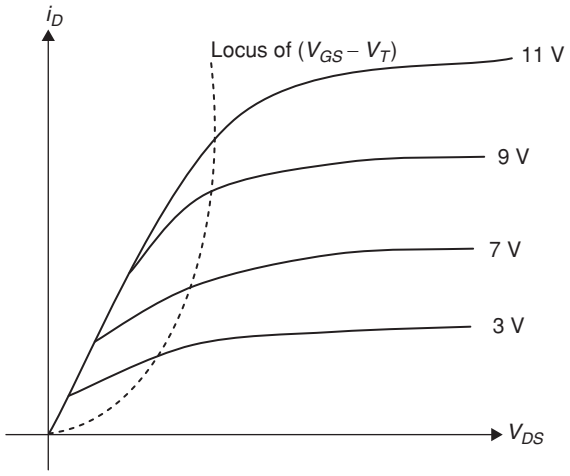
Transfer Characteristics:



$$\text{Slope} = g_m = \frac{\Delta i_d}{\Delta V_{gs}}\Big|_{V_{GSQ}}$$

g_m : Transconductance

Output characteristics:



$$\text{Slope} = r_D = \left(\frac{di_D}{dv_{DS}}\right)^{-1} \Big|_{V_{GS} = V_{GSQ}}$$

r_D = Drain resistance

Transconductance (g_m)

1. $i_D = k\,(V_{GS} - V_T)^2$
   $i_D = I_D + i_d$
   $v_{GS} = V_{GS} + V_{gs}$
   $i_d = 2k_n\,(V_{GS} - V_T)$ at V_{GSQ}

   Where $k_n = \mu_n C_{ox}\left(\frac{W}{L}\right)$ represents MOSFET

Fabrication constant

$$g_m = 2k_n\sqrt{\frac{i_D}{k_n}}$$

$$g_m = 2\sqrt{k_n I_{DQ}}$$

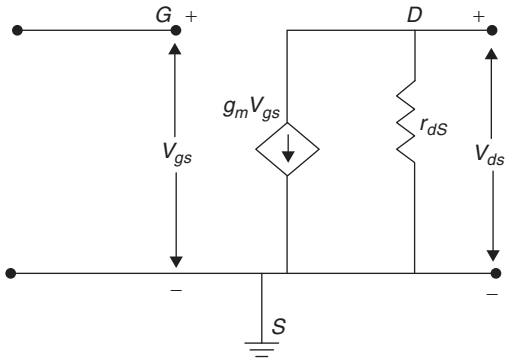
AC Equivalent Model of MOSFET

$$i_d = f(V_{gs}, V_{ds})$$

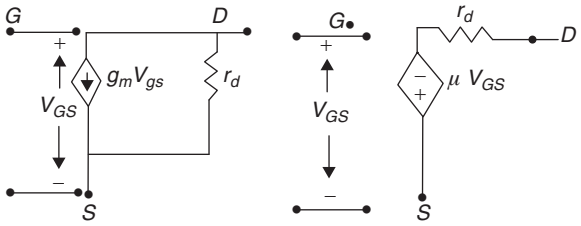
By super position theorem

$$i_d = g_m V_{gs} + \frac{V_{ds}}{r_{ds}}$$

Where g_m = Trans conductance
r_{ds} = Drain resistance



Small signal equivalent circuits



$$\mu = g_m r_d$$

$$g_m = \frac{2}{|V_P|}\sqrt{I_{DQ} I_{DSS}} \quad \text{for JFET}_S$$

$$g_m = 2\sqrt{k\frac{W}{L} I_{DQ}} \quad \text{for MOSFET}_S$$

$$r_d = \frac{1}{\lambda I_{DQ}} = \frac{V_A}{I_{DQ}}$$

The values of both g_m and r_d are bias-dependent. The output resistance r_d is usually not sufficiently large, so it may not be neglected (as r_o is often neglected in the BJT model).

The quantity μ is called amplification factor $\mu = g_m r_d$. The open circuit between gate and source in the model makes $I_g = 0$, so we consider $A_V R_o$ only (A_I and R_i are very High).

Equations for FET stages

Common source stage

$$A_v = -\frac{\mu R_D}{r_d + R_D} = \frac{-g_m R_D}{1 + R_D/r_d}$$

$$R_o = r_d, R_o' = R_D \| r_d$$

Common source stage with source resistance

$$A_v = \frac{\mu R_D}{r_d + R_o + (1+\mu)R_S} = \frac{-g_m\,(r_o \| R_D)}{1 + g_m R_S R_L / R_D}$$

$$R_o = r_d + R_S\,(1+\mu) = r_d\,(1 + g_m R_S)$$

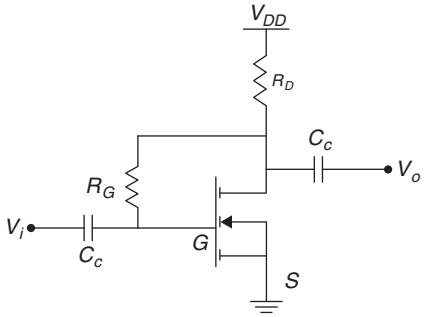
$$R_o' = R_o \| R_D$$

Common drain configuration

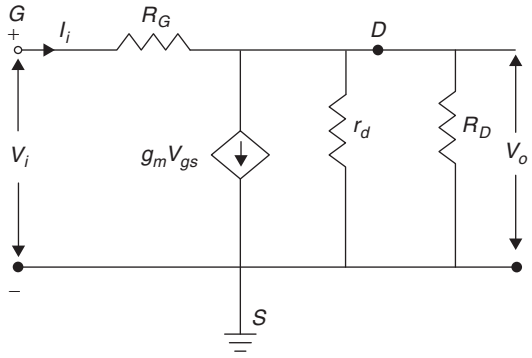
$$A_v = \frac{\mu R_S}{r_d + R_S(1+\mu)} = \frac{g_m R_S}{1 + g_m R_S},\ R_o = \frac{r_d}{1+\mu} = \frac{1}{g_m}$$

$$R_o' = R_s \| R_o$$

Enhancement Mode MOSFET Drain Feedback Configuration



AC equivalent model



$r_d = r_{ds}$ = Drain resistance

Input-impedance:

$$z_i = \frac{R_F + (r_d \parallel R_D)}{[1 + (r_d \parallel R_D g_m)]}$$

$$\text{if } R_F \gg (r_d \parallel R_D) \Rightarrow z_i = \frac{R_F}{1 + g_m (r_d 11 R_D)}$$

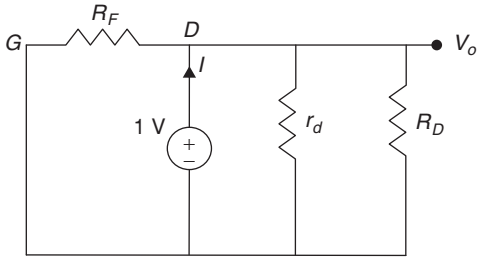
$$\text{if } R_F \gg (r_d \parallel R_D) \text{ and } \Rightarrow z_i = \frac{R_F}{1 + g_m R_D}$$

$$r_d \geq 10 R_D$$

Output-impedance:

1. Make input voltage = 0
2. Connect a voltage source at the output and find $\frac{V}{I}$

AC equivalent circuit



$z_0 = R_F \parallel r_d \parallel R_D$

if $R_F >> (r_d \parallel R_D)$ and $r_d \geq 10 R_D$

$z_0 \simeq R_D$

Voltage gain:

$$A_v = \frac{\left(\frac{1}{R_F}\right) - g_m}{\left(\frac{1}{R_F}\right) + \frac{1}{(r_d \parallel R_D)}}$$

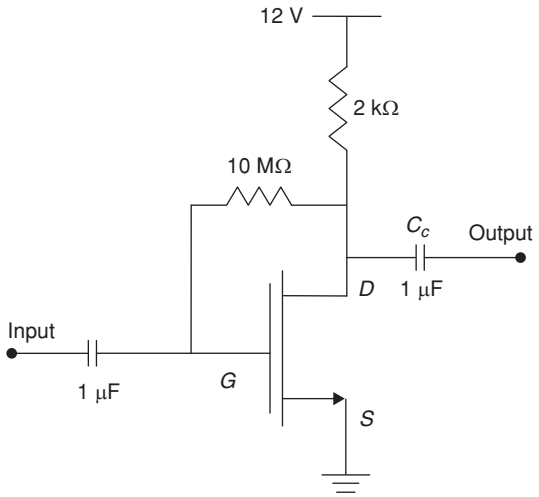
Approximations:

1. If $R_F \gg (r_d \parallel R_D)$
   $A_V \simeq -g_m (r_d \parallel R_D)$
2. if $R_F \gg (r_d \parallel R_D)$ and $r_d \geq 10 R_D$
   $A_V \simeq -g_m R_D$
3. $g_m \gg \frac{1}{R_F}$ then $A_V = -g_m (r_d \parallel R_D \parallel R_F)$

Summary

Without Approximation	With Approximation
$A_V = -g_m (r_d \text{II} R_D)$	$A_V = -g_m R_D$
$z_i = \frac{R_F}{1 + g_m (r_d \parallel R_D)}$	$z_i = \frac{R_F}{1 + g_m R_D}$
$z_o = r_d \text{II} R_D$	$z_0 = R_D$

Example 6:



Find g_m, r_d, z_i, z_0, A_v?
K = 0.2×10^{-3} A/V^2
$V_T = 3$ V
$Y_d = 20$ μs
$V_{GSQ} = 6.4$ V
$I_{DQ} = 2.75$ mA

Solution:

(i) $g_m = 2\text{ k}(V_{GSQ} - V_T)$
$= 1.632$ mA/V
$g_m = 1.632$ ms

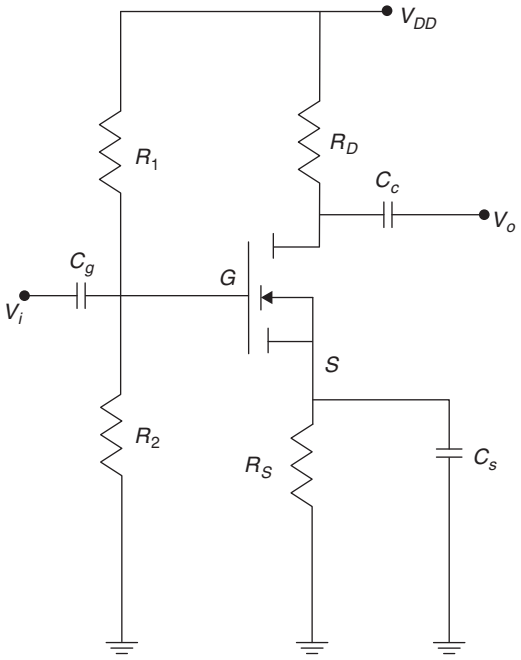
(ii) $r_d = \frac{1}{y_d} = 50$ kΩ

(iii) $z_i = \frac{R_F}{1 + g_m (r_d \parallel R_D)} = 2.42$ mΩ

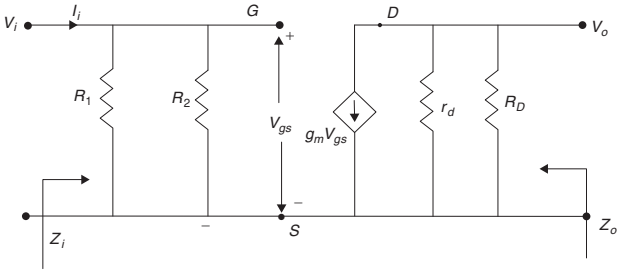
(iv) $z_0 = R_F \parallel r_d \parallel R_D = 1.923$ kΩ

(v) $A_v = -g_m (r_d \parallel R_D \parallel R_F) = -3.14$

Common Source Amplifier (Voltage Divider-bias)



Amplifier AC equivalent model

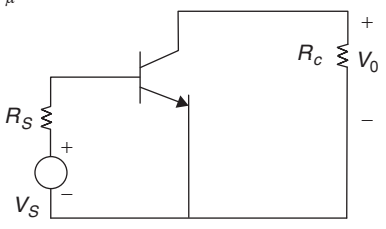


1. $z_i = R_1 \| R_2$
2. $z_0 = r_d \| R_D$
3. $A_v = -g_m(r_d \| R_D)$
   If $r_d \gg 10\, R_D$ then $A_v \simeq -g_m R_D$

Solved Examples

Directions for questions 1 to 7: Select the correct alternative from the given choices.

Example 1: The amplifier circuit with $R_S = 500\ \Omega$, $R_C = 1.5$ kΩ operates at $I_C = 1$ mA, the parameters are $\beta_0 = 125, f_T = 300$ MHz, $C_\mu = 0.5$ pF. The mid band voltage gain is



(A) −51.72 (B) −32.42
(C) −28.34 (D) −13.50

Solution: (A)

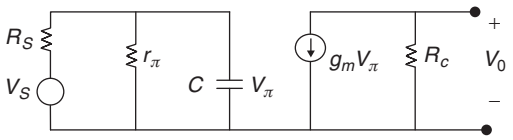
$$g_m = \frac{I_{C(\text{mA})}}{25} = \frac{1}{25}$$

$$\beta_o = g_m \cdot r_\pi \quad \Rightarrow \quad r_\pi = \frac{\beta_0}{g_m} = \frac{125}{1/25} = 3125\ \Omega$$

$$\omega_T = \frac{g_m}{C_\pi + C_\mu} = 2\pi \times 300 \times 10^6$$

$$C_\pi + C_\mu = \frac{g_m}{\omega_T} = \frac{1}{25 \times 2\pi \times 300 \times 10^6} = 21\ \text{pF}$$

The AC equivalent circuit using unilateral hybrid π.



$$\frac{V_o}{V_1} = g_m R_C$$

$$C = C_x + C_\mu(1 + g_m R_C)$$

$$= C_\pi + C_\mu + C_\mu g_m R_C$$

$$V_\pi = \frac{V_S r_\pi}{R_S + r_\pi} \cdot \frac{1}{1 + \frac{S}{\omega_H}} \text{ where } \omega_H = \frac{1}{RC}$$

$$R = R_S \| r_\pi$$

$$\frac{V_o}{V_S} = \frac{-\beta_o R_C}{R_S + r_\pi} \cdot \frac{1}{1 + \frac{s}{\omega_H}}$$

Mid band gain $A_0 = \dfrac{-\beta_o R_C}{R_S + r_\pi} = \dfrac{125 \times 1.5}{0.5 + 3.125} = -51.72$

Example 2: The value of $\omega_H = 2\pi f_H$ which is the upper cut off frequency in the above question?
(A) 75.7 M radians/second (B) 54.6 M radians/second
(C) 39.2 M radians/second (D) 32.9 M radians/second

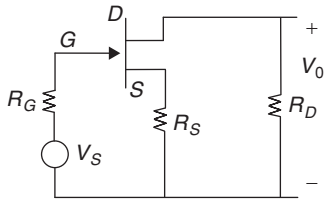
Solution: (C)
$C = (C_\pi + C_\mu) + C_\mu g_m R_C$

$$= 21 + 0.5 \times \frac{1}{25} \times 1500 = 21 + 30 = 51\ \text{pF}$$

$$\omega_H = \frac{1}{RC} = \frac{1}{500 \times 51\ \text{pF}}$$

$$= 39.2\ \text{M radians/second}$$

Example 3: For the common source amplifier with R_S, find the values of $\frac{V_O}{V_S}$ and R_O (output impedance), given R_D = 16 kΩ, R_S = 1 kΩ, FET parameters are r_d = 32 kΩ, μ = 60.



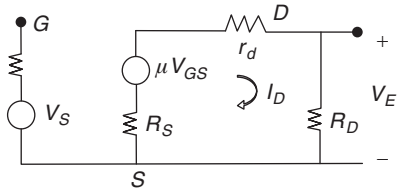
(A) 8.8, 93 kΩ
(B) −8.8, 93 kΩ
(C) −4.2, 56 kΩ
(D) +4.2, 56 kΩ

Solution: (B)
The equivalent current in the loop equation from KVL

$I_d (R_S + R_D , r_d) = \mu V_{gs}$

$V_{gs} = V_s - R_s I_d$

$V_0 = -I_d R_D$



By solving $A_V = \frac{V_O}{V_S} = \frac{-\mu R_D}{r_d + R_S(1+\mu) + R_D}$

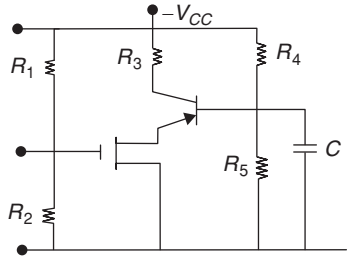
$R_0 = r_d + R_S (1 + \mu)$

R_S = 1 kΩ, R_D = 16 kΩ, r_d = 32 kΩ, μ = 60

$A_V = -8.8$

$R_O = 32 + 1 \times 61 = 93$ kΩ

Example 4: The given figure shows a composite transistor consisting of a MOSFET and a bipolar transistor in cascade. The MOSFET has trans conductance g_m of $\frac{3 \text{ mA}}{\text{V}}$ and bipolar transistor has β of 99. The overall transconductance of the composite transistor is? (C is very large)



(A) 1.98 mA/V
(B) 19.8 mA/V
(C) 29.7 mA/V
(D) 2.97 mA/V

Solution: (C)

$$g_m \cdot V_{\text{in}} = I_E (BJT), \ \alpha = \frac{I_C}{I_E} = (BJT)$$

$$I_c = \alpha I_E = \alpha \cdot g_m \cdot V_{\text{in}},$$

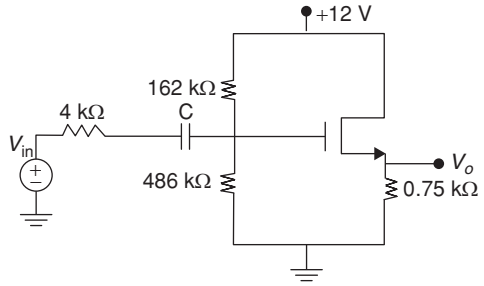
$$\alpha = \frac{\beta}{\beta + 1} = 0.99$$

$$g_m (\text{over all}) = \frac{I_c}{V_{\text{in}}} = g_m \cdot a$$

$$= 3 \times 10^{-3} \times 0.99 = 2.97 \times 10^{-3} \text{ Mho}$$

Common Data for Questions 5 to 7:

Example 5: The source follower amplifier circuit shown in following figure transistor parameters are V_{th} = 1.2 V, k_n = 4 mA/V², λ = 0.01 V⁻¹ small signal trans conductance and I_D are

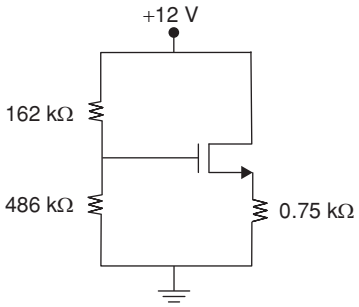


(A) 8.47 mA and 14.8 mA/V
(B) 6.47 mA and 14.8 mA/V
(C) 6.47 mA and 11.6 mA/V
(D) 8.47 mA and 11.6 mA/V

Solution: (D)
DC analysis of the circuit

$$\frac{V_G - 12}{162} + \frac{V_G - 0}{486} = 0 \quad \Rightarrow \quad V_G = 9 \text{ V}$$

$$I_D = \frac{V_G - V_{GS}}{(0.75 \text{ kΩ})} = k_n (V_{GS} - V_{\text{th}})^2$$



$$(9 - V_{GS}) = (0.75)(4)(V_{GS} - 1.2)^2$$

$$(9 - V_{GS}) = 3(V_{GS} - 1.2)^2$$

$$9 - V_{GS} = 3V_{GS}^2 - 7.2V_{GS} + 4.32$$

$$3V_{GS}^2 - 6.2V_{GS} - 4.68 = 0$$

$V_{GS} = 2.65$ or -0.58 (by solving)

V_{GS} should be positive $V_{GS} = 2.65$ V

$$I_D = \frac{V_G - V_{GS}}{0.75\ \text{k}\Omega} = \frac{6.35}{0.75} = 8.47\ \text{mA}$$

Small signal trans conductance is

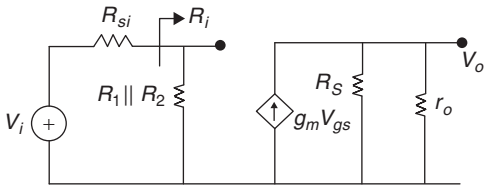
$$g_m = \frac{dI_D}{dV_{GS}} = 2\,k_n(V_{GS} - V_{\text{th}})$$

$$= 2 \times 4 \times (2.65 - 1.2) = 11.6\ \text{mA/V}$$

Example 6: Small signal voltage gain $A_v = \frac{V_o}{V_i}$ is is

(A) 1.2 (B) 1.13
(C) 0.98 (D) 0.86

Solution: (D)
The small signal equivalent circuit of the given amplifier is



$R_{si} = 4$ kΩ, $R_1 = 162$ kΩ, $R_2 = 486$ kΩ, $R_s = 0.75$ kΩ

$$r_o \approx [\lambda I_D]^{-1} = 1/0.01 \times 8.47 = 11.8\ \text{k}\Omega$$

$$R_i = R_1 \| R_2 = 121.5\ \text{k}\Omega$$

Voltage gain $A_g = \frac{g_m(R_S \| r_o)}{1 + g_m(R_S \| r_o)} \cdot \frac{R_i}{R_i + R_{si}}$

$$= \frac{11.6(0.75 \| 11.8)}{1 + 11.6(0.75 \| 11.8)} \times \frac{121.5}{121.5 + 4} = 0.86$$

Example 7. The output resistance of the amplifier circuit is
(A) 87.8 Ω (B) 82.3 Ω (C) 79.3 Ω (D) 76.6 Ω

Solution: (D)
By small signal equivalent circuit, output independence is

$$R_0 = \frac{1}{gm} \| R_s \| r_0$$

$$= 0.086 \| 0.75 \| 11.8\ \text{k}\Omega$$

$$0.076\ \text{k}\Omega \Rightarrow 76.6\ \Omega$$

EXERCISES

Practice Problems I

Directions for questions 1 to 30: Select the correct alternative from the given choices.

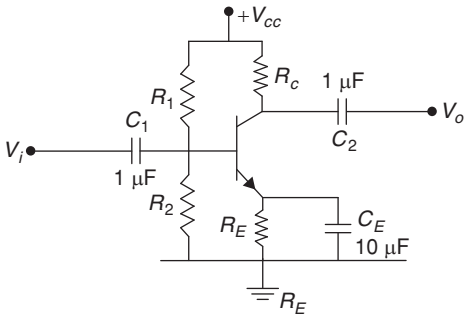
Common Data for Questions 1 and 2: A BJT has $h_{fe} = 220$, $g_m = 40 \times 10^{-3}\Omega$, $C_{cb} = 10$ pF and $C_{be} = 50$ pF.

1. What is unity gain frequency f_T?
(A) 80.6 MHz (B) 106.6 MHz
(C) 20 MHz (D) 100 kHz

2. What is β cutoff frequency f_β?
(A) 4.84 kHz (B) 0.359 MHz
(C) 3.59 kHz (D) 0.484 MHz

Common Data for Questions 3 and 4: A CE amplifier is shown below with the following specifications.

$R_1 = 40$ kΩ, $R_2 = 4.7$ kΩ
$R_C = 4$ kΩ, $R_E = 1.2$ kΩ, $\beta = 100$
And $V_T = 26$ mV and $V_{CC} = 16$ V, $V_{BE} = 0.7$ V.

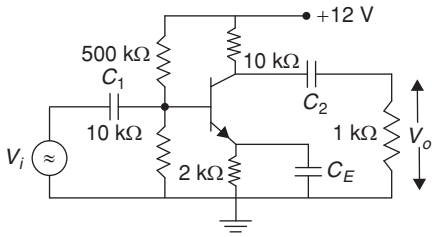


3. What is the voltage gain A_V of the amplifier?
(A) −132.3 (B) −10
(C) −121.5 (D) −30.6

4. If the bypass capacitor C_E is removed, the voltage gain is _____
(A) −100 (B) −3.33
(C) −2.73 (D) −121.5

5. The transistor shown below has $r_e = \frac{V_T}{I_E} = 2.6\ \Omega$.

The mid band voltage gain of the amplifier is _____



(A) −350 (B) −177
(C) −212 (D) −182

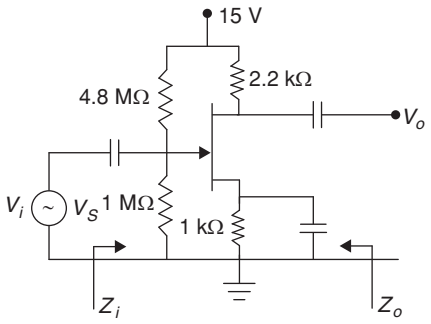
6. A multi stage amplifier is to be constructed using 4 identical stages, each of which has a lower cut-off frequency 60 Hz and upper cut-off frequency 1MHz. The band width of multi stage amplifier is _____
(A) 0.39 MHz (B) 4 MHz
(C) 0.435 MHz (D) 0.51 MHz

7. A multistage amplifier has 3 stages. The voltage gain of 3 stages are 30, 50 and 60 respectively. The overall gain in db is _____
(A) 60 dB (B) 105.2 dB
(C) 140 dB (D) 99.06 dB

8. A multistage amplifier has 2 identical stages. If each stage has $R_{in} = 2\ k\Omega$, $\beta = 80$ and $R_C = 3.5\ k\Omega$. The overall voltage gain is _____
(A) 2090 (B) 19600
(C) 280 (D) 70

9. The following parameters were measures on transistor biased at $I_C = 2$ mA
$h_{fe} = 80$ $h_{re} = 0.5 \times 10^{-4}$
$h_{oe} = 1.4 \times 10^{-6}$ A/V
Calculate g_m and r_π?
(A) $g_m = 0.769$ ℧, $r_\pi = 10.4\ k\Omega$
(B) $g_m = 76.9$ ℧, $r_\pi = 1.04\ k\Omega$
(C) $g_m = 76.9 \times 10^{-3}$ ℧, $r_\pi = 1.04\ k\Omega$
(D) $g_m = 76.9 \times 10^{-3}$ ℧, $r_\pi = 10\ k\Omega$

Common Data for Question 10 to 12: A JFET common source amplifier is given below. The specifications are $r_d = 100\ k\Omega$, $g_m = 3 \times 10^{-3}$S



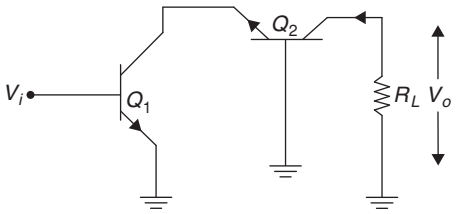
10. The input impedance Z_i is
(A) 827 kΩ (B) 82.7 MΩ
(C) 1 MΩ (D) 5 MΩ

11. The output impedance Z_o is
(A) 220 Ω (B) 3.2 kΩ
(C) 2.2 kΩ (D) 2.152 kΩ

12. The voltage gain A_V is
(A) −6.6 (B) −3
(C) −300 (D) −6.456

13. In a single stage amplifier,
$R_C = 10\ k\Omega$, $R_{in} = 2\ k\Omega$, $\beta = 50$ and $R_L = 10\ k\Omega$. A small signal voltage $V_i(t) = 0.5 \sin \omega t$ mV is applied, the output voltage V_o is _____
(A) $-62.5 \sin\omega t$ mV (B) $-125 \sin\omega t$, mV
(C) $-31.5 \sin\omega t$, mV (D) $62.5 \sin\omega t$, volt

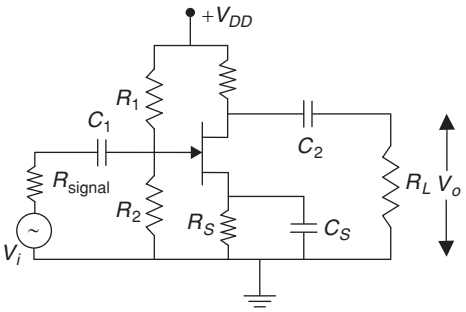
14. In the cascode amplifier shown in figure, if the common emitter stage Q_1 has $r_{e_1} = 26\ \Omega$ and common base Q_2 has $r_{e_2} = 12\ \Omega$, then the output current i_o for $V_i = 0.2$ V is _____



(A) 16 mA (B) 7.7 mA
(C) 6.2 mA (D) None

15. A CE amplifier has a unity gain frequency of 305 MHz with a gain of 92 dB. The 3 dB bandwidth is _____
(A) 16.2 MHz
(B) 305 MHz
(C) 4 MHz
(D) 7.66 kHz

Common Data for Questions 16 and 17: A FET amplifier is shown with the following specifications.
$R_1 = 1\ M\Omega$, $R_2 = 200\ k\Omega$, $R_S = 1\ k\Omega$, $R_D = 2\ k\Omega$, $R_L = 1.8\ k\Omega$, $R_{signal} = 5\ k\Omega$, $C_1 = C_2 = 0.1\ \mu F$ and $V_{DD} = 20$ V. The other parameters of FET are $r_d = 100\ k\Omega$, $C_{gs} = 4$ PF, $C_{ds} = 0.5$ PF, $C_{gd} = 1.2$ PF and $g_m = 4 \times 10^{-3}$ ℧

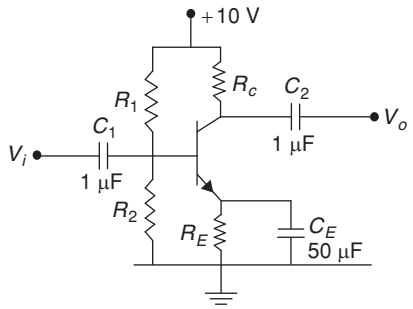


16. The approximate lower cutoff frequency is _____
(A) 9.35 Hz
(B) 60.5 Hz
(C) 21.5 Hz
(D) 425.5 Hz

17. The approximate upper cutoff frequency is _____
(A) 84.6 MHz
(B) 10 MHz
(C) 5.7 MHz
(D) 210 MHz

18. An amplifier circuit with $R_C = 2\ k\Omega$ operating at $I_C = 1.2$ mA. If the gain bandwidth product of the amplifier is 125 kHz then the bandwidth of the same amplifier, assuming $V_T = 26$ mv, is _____
(A) 2.7 kHz (B) 62.5 kHz
(C) 1.35 kHz (D) None of these

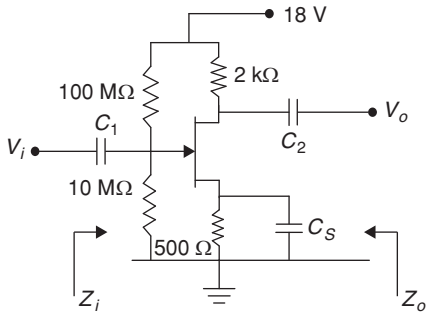
19. A RC coupled CE amplifier is shown below with following specifications.
$R_1 \| R_2 = 4\ k\Omega$, $\beta = 50$, $h_{ie} = 1\ k\Omega$, $R_C = 2\ k\Omega$ and $R_E = 0.5\ k\Omega$. What is the loss in gain if the bypass capacitor C_E is removed?



(A) 4%
(B) 0%
(C) 0.4%
(D) 96%

Common Data for Questions 20 to 22: A AFET amplifier is given below with following specifications.

$I_{DSS} = 5$ mA, $V_P = -2.5$ V and $r_d = 80$ kΩ



20. The input impedance Z_i is
(A) 100 MΩ
(B) 9.09 MΩ
(C) 110 MΩ
(D) 10 MΩ

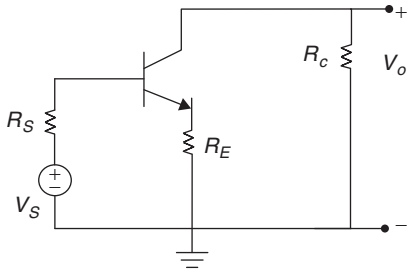
21. The output impedance Z_o is
(A) 1.95 kΩ
(B) 80 kΩ
(C) 10 kΩ
(D) 2.5 kΩ

22. The output V_o for $V_i = 30$ mV is
(A) −0.95 V
(B) −150 mV
(C) −40.5 mV
(D) −81.432 mV

23. The emitter follower using a p-n-p transistor with $\beta_0 = 150$ is biased at $I_C = 0.25$ mA, the voltage signal source has $R_S = 3$ kΩ. In order to make the overall $R_0 = 110$ Ω, the value of R_E is
(A) 1.42 Ω
(B) 2.04 Ω
(C) 0.71 kΩ
(D) 1.42 kΩ

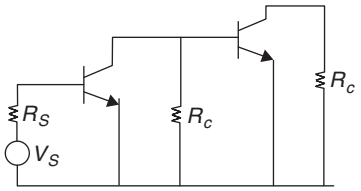
24. For the above value of R_E, the values of A_V, R_i (input resistance) are
(A) 1, 230 kΩ
(B) 2.8, 330 kΩ
(C) 0.92, 230 kΩ
(D) 0.92, 330 kΩ

25. In the circuit shown, the transistor has $\beta_0 = 125$, and operated $I_C = 0.3$ mA, the element values are $R_S = 2$ kΩ, $R_G = 5$ kΩ, $R_E = 100$ Ω, the value of V_O/V_S is?



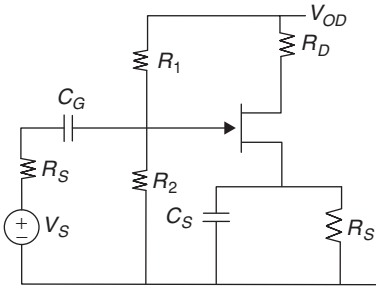
(A) −18.35
(B) 20.46
(C) −24.98
(D) 32.46

26. Both the transistors have $\beta_0 = 120$, and operated at $I_C = 1$ mA, $R_S = 0.5$ kΩ, $R_C = 1.5$ kΩ, overall gain is?



(A) 91.4
(B) 1028
(C) 2056
(D) 3084

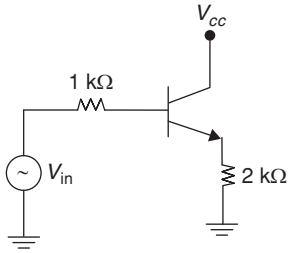
27.



A FET amplifier shown in the above figure has parameters $g_m = 2$ m Mho, $r_d = 20$ kΩ, what is the value of load R_D for mid band gain of −20? Assume that R_S, C_S bias is short $R_S = 300$ Ω, $R_1 = 160$ kΩ, $R_2 = 40$ kΩ
(A) 20 kΩ
(B) 30 kΩ
(C) 40 kΩ
(D) 0 Ω

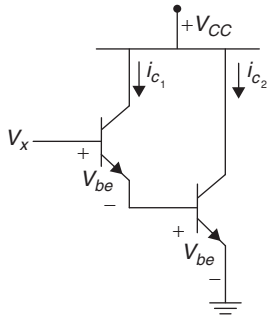
28. The value of input coupling capacitor C_G for lower cut-off frequency $f_L = 300$ Hz in the above circuit is.
(A) 0.025 μF
(B) 0.052 μF
(C) 0.016 μF
(D) 0.043 μF

29. An amplifier circuit is shown in the figure given below; assume that the transistor works in active region. The low frequency small signal parameters for the BJT are $g_m = 20$ mS, $\beta = 50$, $r_o = \infty$, $r_b = 0$, $\frac{V_0}{V_{in}}$ of the amplifier is



(A) 0.967 (B) 0.976
(C) 0.983 (D) 0.998

30. The Darlington pair stage is shown in the following figure. If the transconductance of Q_1 is 8×10^{-3} ℧ and Q_2 is 6×10^{-3} ℧, the overall transconductance g_m is _____.

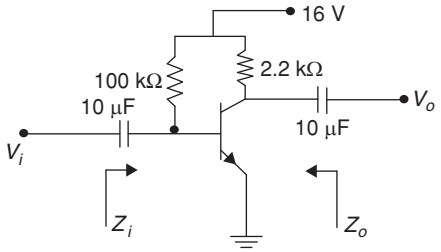


(A) 14×10^{-3} ℧
(B) 2×10^{-3} ℧
(C) 3×10^{-3} ℧
(D) 6×10^{-3} ℧

Practice Problems 2

Directions for questions 1 to 22: Select the correct alternative from the given choices.

Common Data for Questions 1 to 3: The h-parameters of the transistor are h_{ie} = 400 Ω, h_{fe} = 80 and the transistor amplifier is given below



1. What is the input impedance Z_i as seen from source?
(A) 400 Ω (B) 100 kΩ
(C) 500 kΩ (D) 398 Ω

2. What is the output impedance Z_o?
(A) 2.2 kΩ (B) 10 kΩ
(C) 6.2 kΩ (D) 6 kΩ

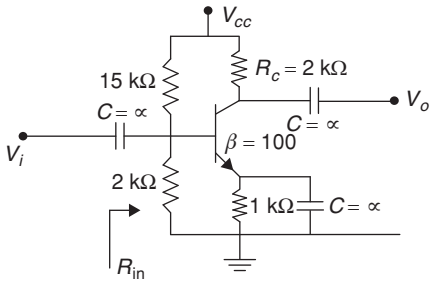
3. What is the voltage gain of the amplifier?
(A) –80 (B) –440
(C) –612 (D) 1

Common Data for Questions 4 and 5: The typical parameters of the hybrid-π model of transistor at room temperature and for I_C = 1.8 mA are
C_π = 100 pF and f_T = 80 MHz

4. The value of g_m in mA/V is _____
(A) 6.9 (B) 0.69
(C) 0.069 (D) 69

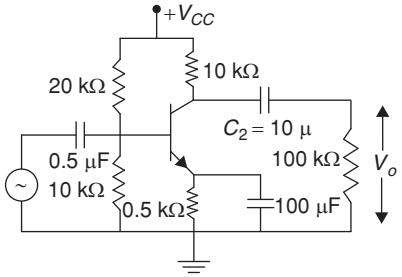
5. The capacitance C_μ is _____
(A) 38 pF (B) 2 pF
(C) 3.8 pF (D) 0.38 pF

6. The Transconductance g_m of the transistor shown is 8 mS. The value of input resistance R_{in} seen from source is _____.



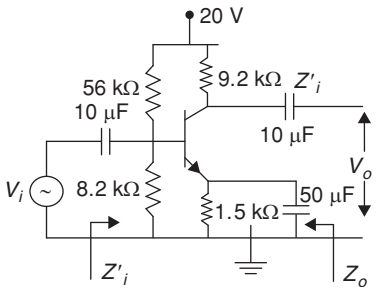
(A) 1.54 kΩ
(B) 12.5 kΩ
(C) 1.764 kΩ
(D) 2.5 kΩ

7. A RC coupled CE amplifier is shown in the figure. The lower cut-off frequency f_1 due to coupling capacitor C_2 is _____



(A) 29 Hz
(B) 145 Hz
(C) 50 HZ
(D) 32 Hz

Common Data for Questions 8 and 9: A CE transistor amplifier is shown below with

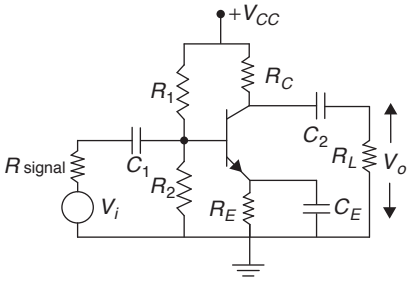


$h_{ie} = 2.12$ kΩ, $h_{re} = 0$, $h_{fe} = 50$ and $h_{oe} = 0.2 \times 10^{-3}$ ℧

8. The input impedance Z_i' is _____
(A) 7.15 kΩ (B) 8.78 kΩ
(C) 2.12 kΩ (D) 1.63 kΩ

9. The output impedance Z_o is _____
(A) 7.77 kΩ (B) 9.2 kΩ
(C) 43.2 kΩ (D) 50 kΩ

Common Data for Questions 10 to 12: A RC coupled CE amplifier is shown below.



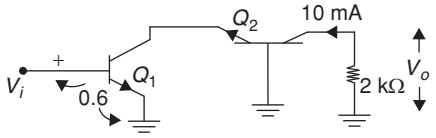
The specifications are
$R_{signal} = 1$ kΩ, $R_1 = 20$ kΩ, $R_2 = 5$ kΩ, $R_C = 1$ kΩ, $R_E = 1$ kΩ, $R_L = 1$ kΩ, $C_1 = 0.5$ μF, $C_2 = 1$ μF and $C_E = 100$ μF.

10. The lower cut-off frequency due to C_1 is
(A) 23.4 Hz (B) 64 Hz
(C) 32 Hz (D) 128 Hz

11. The lower cut-off frequency due to C_2 is
(A) 0 Hz (B) 100 Hz
(C) 64 Hz (D) 80 Hz

12. What is the value of C_2 required to have a lower cut-off frequency of 100 Hz.
(A) 80 μF (B) 8 pF
(C) 0.8 μF (D) 8 μF

13. A cascode amplifier is given below. The voltage gain $A_v = \frac{V_o}{V_i}$ is _____



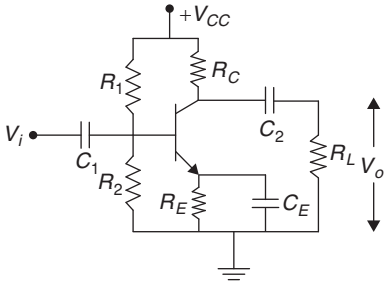
(A) −67 (B) −33.3
(C) −16.6 (D) −1

14. The 3 amplifier stages are cascoded to provide overall gain of 12000. The first two stages have a gain of 45 db and 25.12 respectively, the gain of 3rd stage in dB is _____
(A) 3 dB
(B) 28 dB
(C) 80 dB
(D) 8.58 dB

15. The rise time of the amplifier to the input is 70 n sec. The bandwidth of the amplifier is _____
(A) 1 MHz
(B) 5 MHz
(C) 10 MHz
(D) 0.5 MHz

16. The mid frequency gain of RC coupled CE amplifier is 200. The lower and upper cut-off frequencies are 50 Hz and 80 kHz.
The frequencies at which gain falls to 150 are _____ and _____
(A) 50 Hz and 80 kHz
(B) 56.69 Hz and 70.55 kHz
(C) 25 Hz and 40 kHz
(D) 10 Hz and 90 kHz

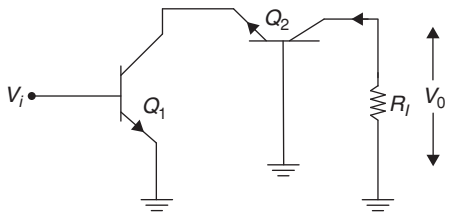
17. A CE transistor amplifier shown below specified with $h_{fe} = 50$, $h_{ie} = 2$ kΩ and $R_L = 10$ kΩ. The value of R_C required to maintain a mid band gain $A_V = -120$ is _____ kΩ.



(A) 10
(B) 5
(C) 2.75
(D) 9.23

18. The mid band gain of the amplifier is 62 dB. The gain at lower (or) upper 3 dB is _____
(A) 59 dB
(B) 31 dB
(C) $\frac{62}{\sqrt{2}}$ dB
(D) 65 dB

19. A cascode amplifier is shown below. The transistor Q_1 has $g_{m_1} = 5$ mA/V and
Q_2 has $g_{m_2} = 1.25$ mA/V.
The overall transconductance g_m of the cascode amplifier is _____



(A) 5 mA/V
(B) 1.25 mA/V
(C) 6.25 mA/V
(D) 4 mA/V

Common Data for Questions 20 to 22: A multi stage amplifier has 4 identical stages. Each stage is specified with gain $A = 60$, lower cutoff $f_1 = 30$ Hz and upper cutoff $f_2 = 90$ kHz.

20. The overall gain A' in dB is
(A) 47.6 dB
(B) 32.55 dB
(C) 142.25 dB
(D) 60 dB

21. The overall lower cutoff frequency f_1' is _____
(A) 120 Hz (B) 69 Hz
(C) 7.5 Hz (D) 21.2 Hz

22. The overall upper cutoff frequency f_2' is _____
(A) 22.5 kHz
(B) 45 kHz
(C) 39.13 kHz
(D) 360 kHz

Previous Years' Questions

1. The typical frequency response of a two-stage direct coupled voltage amplifier, as shown in the below figures, is. **[2005]**

(A) 

(B) 

(C) 

(D) 

Answers Keys

Exercises

Practice Problems 1

1. B	**2.** D	**3.** C	**4.** B	**5.** A	**6.** C	**7.** D	**8.** B	**9.** C	**10.** A
11. D	**12.** D	**13.** A	**14.** B	**15.** D	**16.** D	**17.** C	**18.** C	**19.** D	**20.** B
21. A	**22.** D	**23.** D	**24.** C	**25.** C	**26.** C	**27.** A	**28.** C	**29.** A	**30.** C

Practice Problems 2

1. D	**2.** A	**3.** B	**4.** C	**5.** A	**6.** A	**7.** B	**8.** D	**9.** A	**10.** B
11. D	**12.** C	**13.** B	**14.** D	**15.** B	**16.** B	**17.** D	**18.** A	**19.** A	**20.** C
21. B	**22.** C								

Previous Years' Questions

1. B

Chapter 6

Differential and Feedback Amplifiers

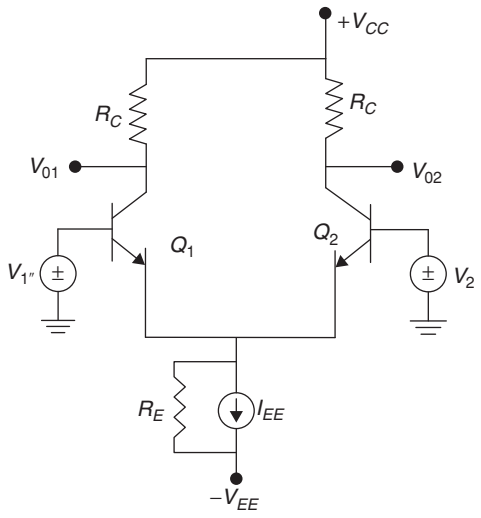
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- The differential amplifier
- FET differential amplifiers
- Feed back amplifiers
- Power amplifiers
- Amplifier efficiency
- Maximum theoretical efficiency
- Dissipation
- Class 'AB' amplifier
- Class 'C' amplifier
- Resonant frequency
- 555 timer
- Schmitt trigger
- Voltage controlled oscillator

The Differential Amplifier

The emitter coupled differential amplifier is an essential building block in modern IC amplifiers.



Differential mode voltage gain $A_{DM} = -g_m R_C$

Common mode voltage gain $A_{CM} = \dfrac{-R_C}{2R_E}$

Common mode rejection ratio

$$\text{CMRR} = \frac{A_{DM}}{A_{CM}} = 1 + 2g_m R_E$$

Output for arbitrary input signals: If V_1 and V_2 are inputs applied to transistors Q_1 and Q_2

$$V_{DM} = \frac{V_1 - V_2}{2}, V_{CM} = \frac{V_1 + V_2}{2}$$

$$V_{01} = A_{DM} V_{DM} + A_{CM} V_{CM}$$

$$= A_{DM} = \left(V_{DM} + \frac{V_{CM}}{\text{CMRR}} \right)$$

$$V_{02} = -A_{DM} V_{DM} + A_{CM} V_{CM}$$

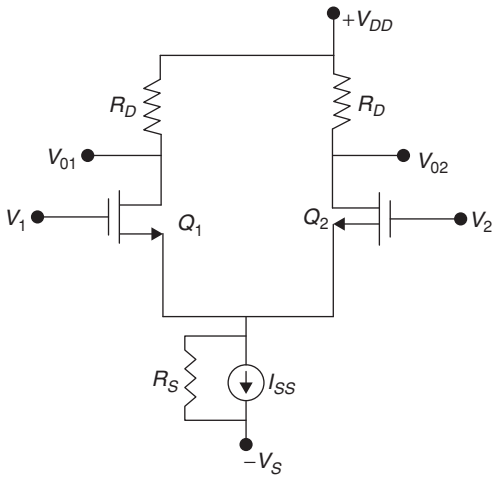
$$= -A_{DM} \left(V_{DM} - \frac{V_{CM}}{\text{CMRR}} \right)$$

Input and output resistances: Differential mode output resistance

$$R'_{O(DM)} = R_c \| r_o \cong R_C$$

Differential mode input resistance $R_{i(DM)} = 2r_e$

FET differential Amplifiers: The source-coupled pair differential amplifier with MOSFETS is shown in the figure



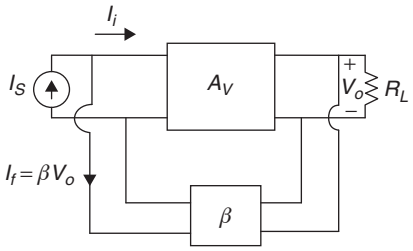
$$\text{CMRR} = 1 + \frac{2R_S\ (1+\mu)}{r_d + R_D}$$

For $r_d \gg R_D$ and $\mu \gg 1$

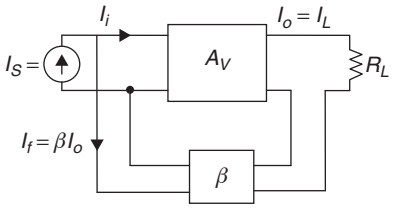
$$\text{CMRR} = 1 + 2g_m R_S \cong 2g_m R_S$$

Feedback Amplifiers

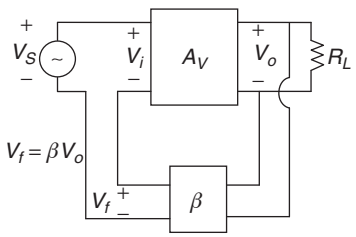
Feedback Amplifier Topologies



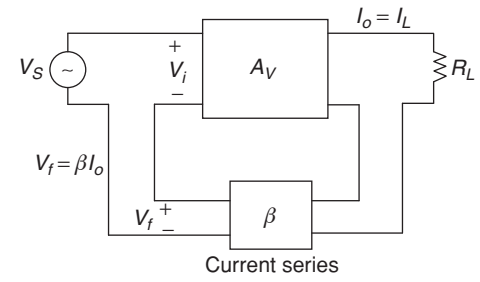
Voltage shunt



Current shunt



Voltage series



Current series

Classification of Amplifiers

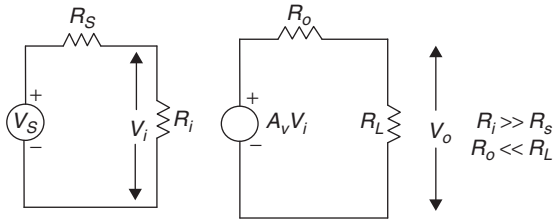


Figure 1 Voltage amplifier

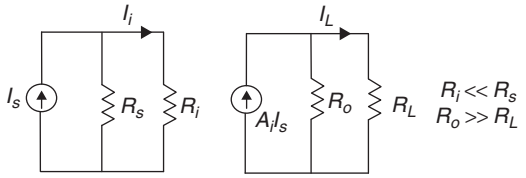


Figure 2 Current amplifier

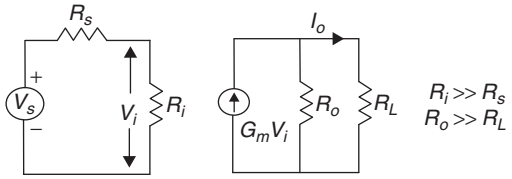


Figure 3 Transconductance amplifier

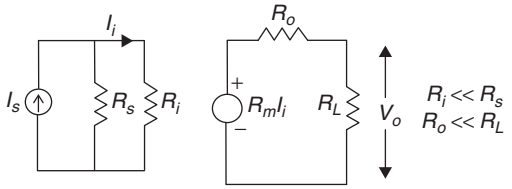


Figure 4 Transresistance amplifier

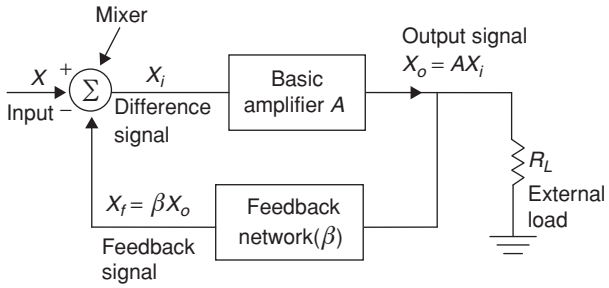


Figure 5 Schematic representation of a single loop feedback amplifier

Advantages of negativefeedback

High input resistance of a voltage amplifier can be made higher, and its lower output resistance can be lowered. The transfer gain A_f of the amplifier with feedback can be stabilized against variations of h-parameters of transistor. It improves frequency response.

Stability of gain

$$A_f = \frac{A}{1 + A\beta}$$

if $A\beta \gg 1$, $A_f = \frac{A}{A\beta} = \frac{1}{\beta}$

$$\frac{\partial A_f}{A_f} = \frac{1}{(1 + A\beta)} \frac{\partial A}{A}$$

$\frac{\partial A_f}{A_f}$ = fractional change in amplifier voltage gain with feedback

$\frac{\partial A}{A}$ = fractional change in amplifier voltage gain without feedback

Sensitivity $= \frac{1}{1 + A\beta}$

Desensitivity = $1 + A\beta$

Increase in input impedance, $Z_{if} = Z_i (1 + A\beta)$

Decrease in output impedance, $Z_{of} = \frac{Z_0}{1 + A\beta}$

Reduction in distortion and noise

$$D_f = \frac{D}{1 + A\beta}, N_f = \frac{N}{1 + \beta}$$

Increase in bandwidth with negative feedback lower cut-off frequency

$$f_L' = \frac{f_L}{1 + A_v\beta}$$

Upper cut-off frequency $f_H' = f_H (1 + A_v\beta)$

$\therefore$ Overall $B.W_f = BW (1 + A\beta)$.

Table 1 *Effect of negative feedback on amplifier*

	Voltage series	Current series	Current shunt	Voltage shunt
R_{output}	Decreases	Increases	Increases	Decreases
R_{input}	Increases	Increases	Decreases	Decreases
Improves char acteristics of	Voltage amplifier	Trans conductance amplifier	Current amplifier	Transresistance amplifier
Desentisizes	A_{vf}	G_{mf}	A_{If}	R_{mf}
Bandwaidth	Increases	Increases	Increases	Increases
Non linear distortion	Decreases	Decreases	Decreases	Decreases

Example 1: The gain of an amplifier with feedback is to be nominally 20, and a variation of 5% is permissible. If the magnitude of loop gain must be at least 1000 (so that $A\beta \gg 1$), then maximum permissible variation in open loop gain is ____

(A) 20 (B) 50
(C) 40 (D) 30

Solution: (B)

Given $A_f = 20$

If $A\beta >> 1 \Rightarrow A_f = \frac{1}{\beta} \quad \therefore \quad \beta = \frac{1}{20} = 0.05$

$$\left|\frac{dA_f}{A_f}\right| = \frac{1}{|1 + A\beta|} \left|\frac{dA}{A}\right|$$

$$\Rightarrow \quad \left|\frac{dA}{A}\right| = 50$$

Example 2: If the input impedance and voltage gain of a open loop voltage series feedback amplifier are 3 kΩ and 100, and the feedback factor is $\frac{1}{50}$, then the input impedance of closed loop configuration is _____

(A) 9 kΩ (B) 6 kΩ
(C) 3 kΩ (D) 12 kΩ

Solution: (A)

$$A = 100, R_{in} = 3\ \text{k}\Omega$$

$$R_{inf} = R_{in} (1 + A\beta)$$

$$= 3\ \text{k}\Omega \left(1 + 100 \frac{1}{50}\right)$$

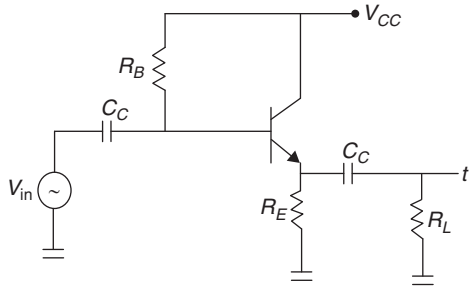
$$R_{inf} = 9\ \text{k}\Omega$$

Note: Voltage-series feedback

(i) $R_{inf} = R_{in} (1 + A\beta)$

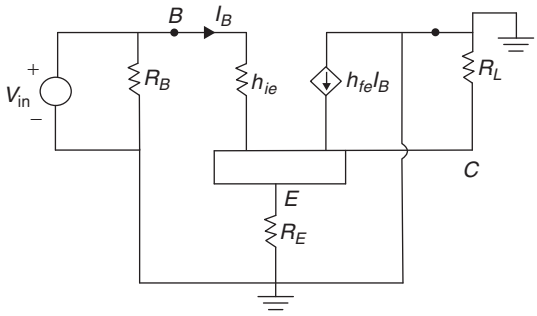
(ii) $R_{of} = \frac{R_o}{(1 + A\beta)}$

Example 3: Find the type of the feedback in the given circuit



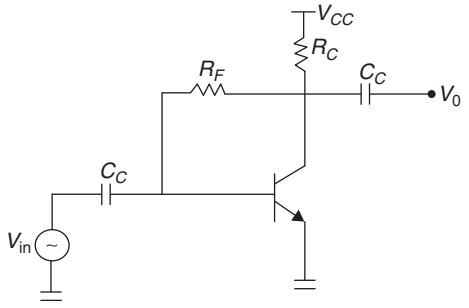
(A) Current series
(B) Voltage shunt
(C) Voltage series
(D) Current shunt

Solution: (C)
AC equivalent circuit:



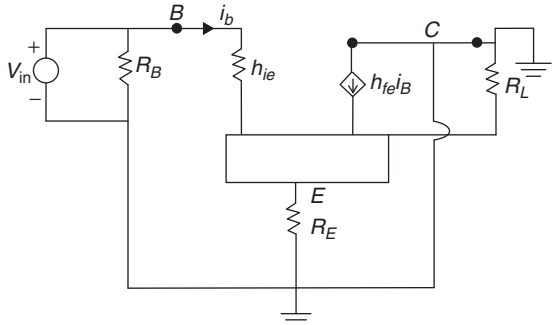
Input-series connection
Output-shunt connection
∴ Voltage series feedback.

Example 4: Find the type of the feedback in the given circuit.



(A) Voltage-series
(B) Current-series
(C) Current-shunt
(D) Voltage-shunt

Solution: (D)
AC equivalent model:



Input-shunt connection
Output-shunt connection
∴ Voltage-shunt feedback

Power Amplifiers

Classification

Class A

Class A amplifier is one in which the operating point and the input signal are such that the current in the output circuit flows all the times. Normally class A amplifier operates essentially over a linear portion of its characteristic.

Class B

A class B amplifier is one in which the operating point is at an extreme end of its characteristic, so that the quiescent power is very small, hence either the quiescent current or voltage is approximately zero. If the input signal is sinusoidal, amplification takes place for only one half of the cycle.

Class AB

A class AB, amplifier is one operating between the two extremes defined for class A and class B, hence the output signal is zero, for part but less than one half of an input sinusoidal signal cycle.

Class C

A class C amplifier is one in which the operating point is chosen so that the output current (voltage) is zero, for more than one half of an input sinusoidal signal cycle.

Efficiency of Class a Amplifier

$$\eta = \frac{\text{signal power delivered to load}}{\text{DC power supplied to output circuit}} \times 100$$

$$\eta = \frac{\frac{1}{2} \cdot V_m \cdot I_m}{V_{cc} \cdot I_{CQ}} \times 100$$

Where V_m (I_m) represents the peak sinusoidal voltage (current) swing.

Class of Operation	Conduction Angle	Efficiency
Class 'A'	360°	25–50%
Class 'AB'	108° to 360°	50–75%
Class 'B'	180°	78.5%
Class 'C'	Less than 180°	80–90%
Class 'D'	Pulse operation	>90%

(i) Class A → Series-fed
Class A → Transformer Coupled

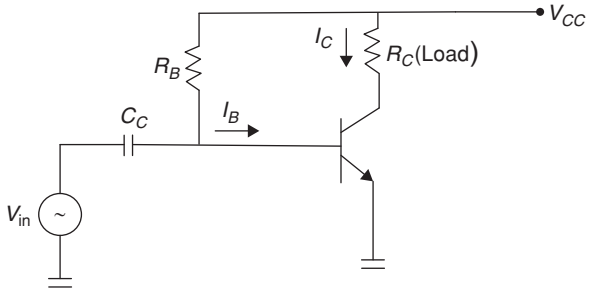
(ii) Class B → Push-Pull
Class B → Complementary symmetry

(iii) Class C circuits are used in tuned circuits, such as radio communications.

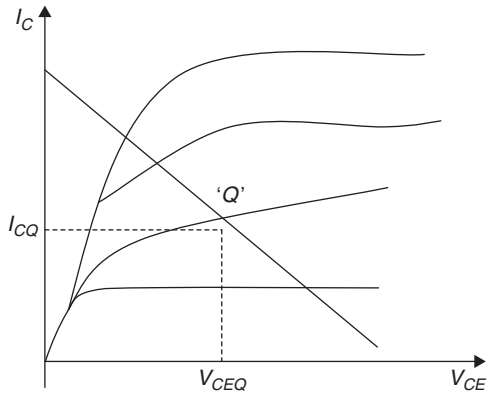
Amplifier efficiency (η):

$$\eta = \frac{\text{AC power delivered to the load}}{\text{DC input power}}$$

Series Fed Class A Amplifier



To have maximum output swing, 'Q' point should be located at the middle of the DC load line



Efficiency calculations

$$\eta = \frac{P_O(\text{AC})}{P_i(\text{DC})}$$

$$P_i(\text{DC}) = V_{cc} \cdot I_{CQ}$$

$$P_o(\text{AC}) = \frac{V_{CE}(P-P) I_C(P-P)}{8}$$

To have maximum output swing,

$$V_{CE}(P-P) = V_{cc}$$

$$I_C(P-P) = \frac{V_{CC}}{R_C},\ P_O(\text{AC}) = \frac{V^2_{CC}}{8R_C}$$

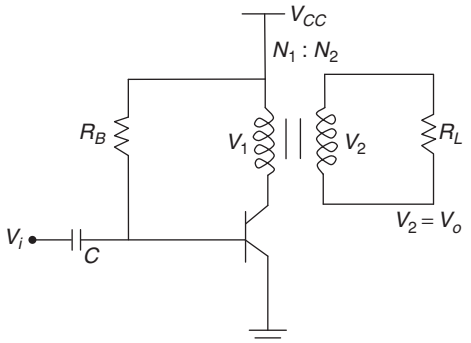
$$I_{CQ} = \frac{V_{cc}}{2R_C}$$

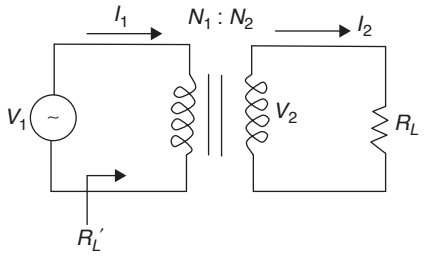
$$P_i(\text{DC}) = \frac{V_{CC}^{\ 2}}{2R_C}$$

$\therefore\ \eta = 25\%$

$\therefore$ Maximum efficiency of class A amplifier is 25%.

Transformer coupled class A power amplifies





$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_L' = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Example 5: Calculate the effective resistance seen looking into the primary of a 15 : 1 transformer connected to an 8 Ω load

(A) 2 kΩ (B) 1.8 kΩ
(C) 3 kΩ (D) 4 kΩ

Solution: (B)

$$R_L' = \left(\frac{N_1}{N_2}\right)^2 R_L = (15)^2 8$$

$$R_L' = 1.8\ \text{k}\Omega$$

Maximum theoretical efficiency:

$$\eta = 50\left(\frac{V_{\text{CE}_{\max}} - V_{\text{CE}_{\min}}}{V_{\text{CE}_{\max}} + V_{\text{CE}_{\min}}}\right)^2 \%$$

Class B Push-Pull Amplifier

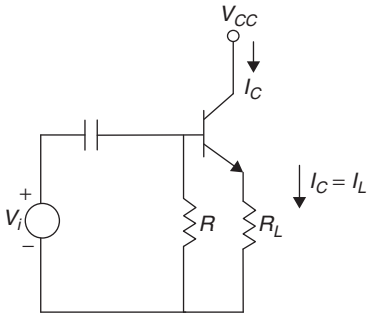


Figure 6 Emitter follower with zero bias operating as class B amplifier

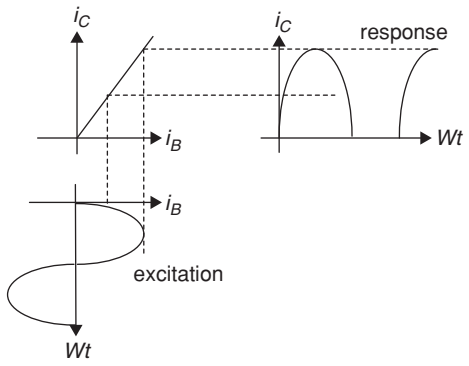
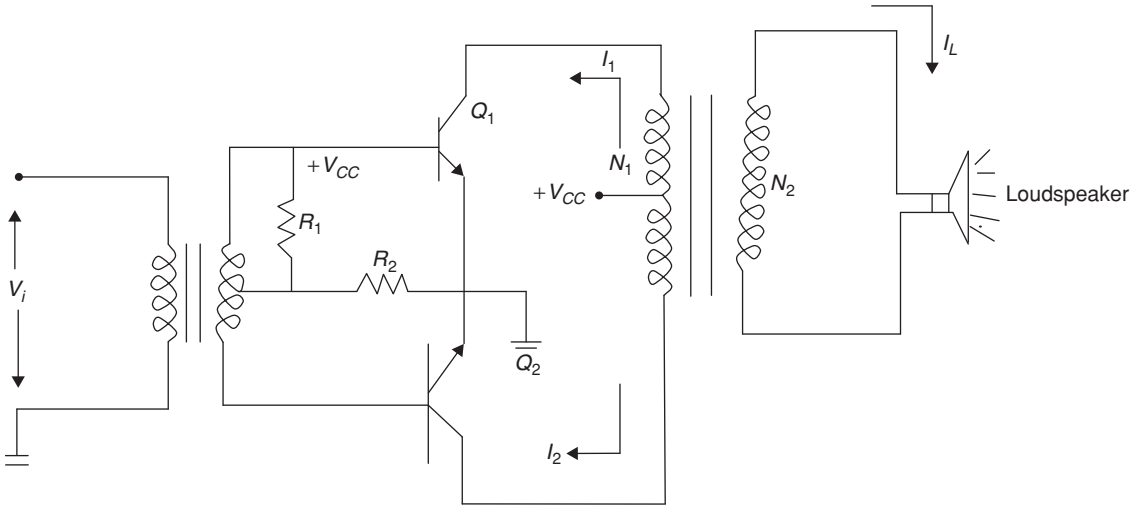


Figure 7 Dynamic transfer characteristic

The emitter follower operates in class B. Let us assume that the transistor output characteristics are equally spaced for equal intervals of excitation for such an idealized transistor the dynamic transfer curve (i_c vs i_B) is a straight line, passing through the origin. The graphical construction from which to determine the collector current wave shape is indicated.

Transformer coupled push-pull amplifier



During positive half cycle: Q_1 ON and Q_2 OFF
Negative half cycle: Q_1 OFF and Q_2 ON

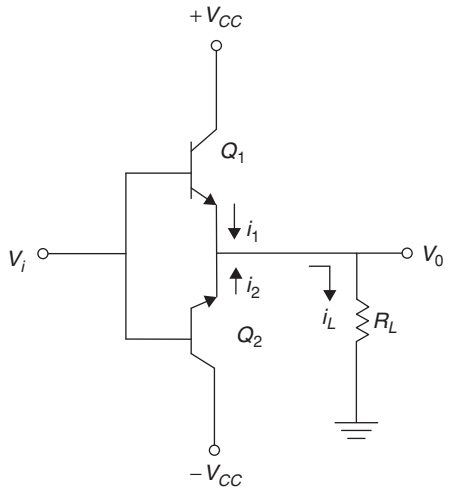


Figure 8 A complementary emitter follower

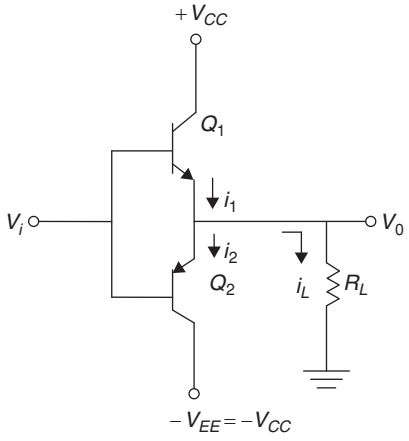
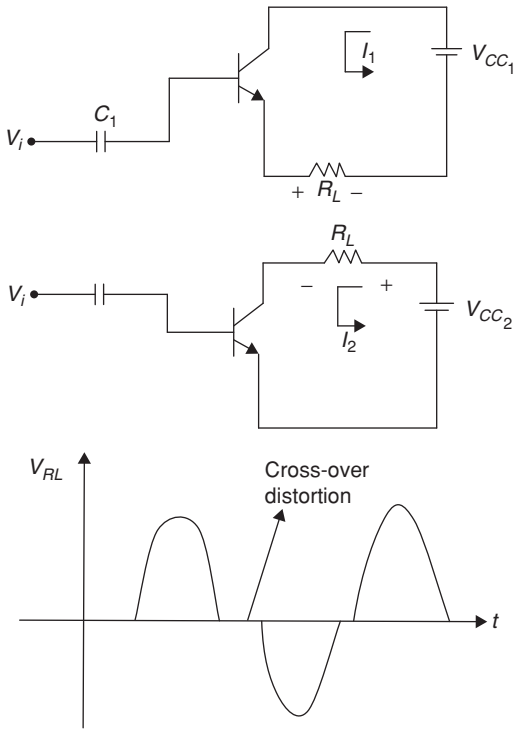


Figure 9 A complementary common-emitter push-pull amplifier

Transformer coupled push-pull amplifier



For positive values of sinusoidal input (V_1) Q_1 conducts and Q_2 is OFF ($i_2 = 0$) so that i_1 is the positive half sine wave.

For negative values of V_i, Q_1 is non conducting ($i_1 = 0$) and Q_2 conducts, resulting in a positive half sinusoid for i_2 which is 180° out of phase with that i_1

Since load current is difference between the two transistor emitter currents $i_L = i_1 - i_2$

Consequently for idealized transfer characteristics the load current is a perfect sinusoid.

The advantage of class B as compared with class A operating are:

It is possible to obtain greater power output, and efficiency is higher, and there is negligible power loss at no signal.

The disadvantages are higher harmonic distortion and power supply voltages must have good regulation.

Efficiency: If peak load voltage $V_m = I_m R_L$

$$\eta = \frac{P}{P_i} \times 100 = \frac{\pi}{4}\left[1 - \frac{V_{\min}}{V_{cc}}\right] \times 100\%$$

Dissipation

$$P_c = P_i - P = \frac{2}{\pi} \cdot \frac{V_{cc} \cdot V_m}{R_L} - \frac{V_m^2}{2R_L}$$

$$P_c(\max) = \frac{2V_{cc}^2}{\pi^2 R_L} = \frac{4}{\pi^2} \cdot P_{\max}$$

P_c (max) = maximum power dissipation

$P_{\max}$ = maximum power which can be delivered

Example 6: Calculate the efficiency of a class B amplifier for a supply voltage of 24 V with peak voltage of 8 V.

(A) 26.18% (B) 35.62%

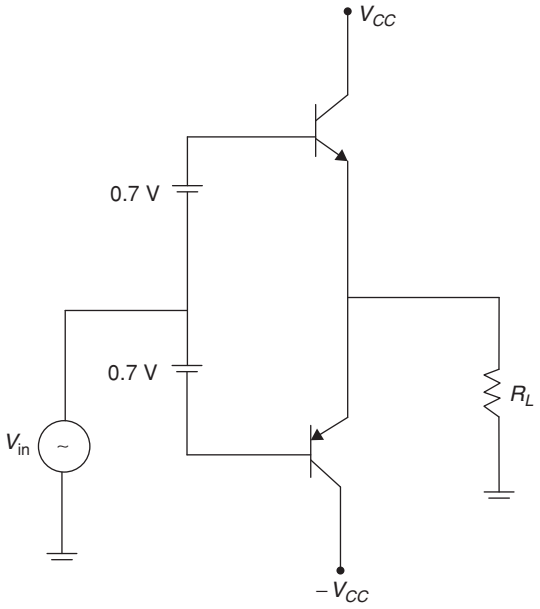
(C) 40.25% (D) 52.36%

Solution: (A)

$$\eta = 78.54 \frac{V_L(p)}{V_{CC}}\% = 78.54 \times \frac{8}{24}$$

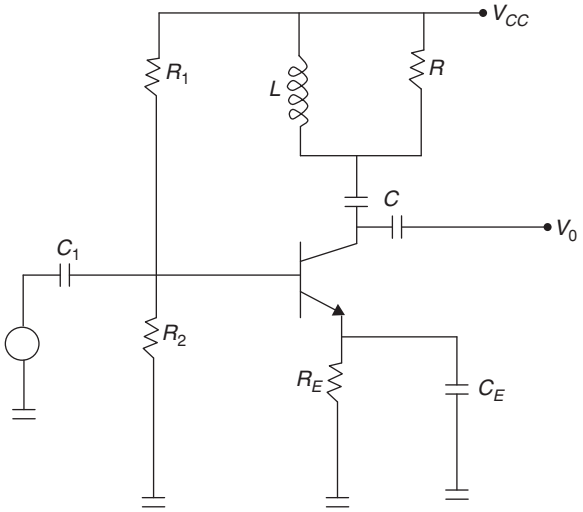
$$= 26.18 \text{ V}$$

Class AB Amplifier



Cross over distortion is eliminated by adding two diodes whose cut in voltages were +0.7 V and –0.7 V.

Class C Amplifier



Class C amplifier is used to generate a pulse wave form whose conduction angle is less than 180°. Efficiency of class C amplifier is 90%.

Resonant frequency

$$Z_L = (R \,||\, SL) + \frac{1}{sC}$$

$$R \,||\, SL = \frac{(R)(sL)}{(R)+(sL)}$$

$$Z_L = \frac{(R)(SL)}{(R)+(SL)} + \frac{1}{SC}$$

$$Z_L = \frac{s^2LCR + R + sL}{sC(R+sL)}$$

$$Y_L = \frac{sC(R+sL)}{s^2LCR + R + sL}$$

Put $S = j\omega$

$$Y_L = \frac{j\omega RC - \omega^2 LC}{-\omega^2 LCR + R + j\omega L}$$

$$Y_L = \frac{j\omega RC - \omega^2 LC}{[R-\omega^2 LCR]+(j\omega L)} \frac{[R-\omega^2 LCR]-j\omega L}{[R-\omega^2 LCR]-(j\omega L)}$$

To find resonant frequency, make imaginary part = 0

i.e., $j_\omega RC\,[R - \omega^2 LC] + j_{\omega^3} L^2C = 0$

$${}_\omega R^2C - \omega^2 RLC^2 + \omega^3 L^3 C = 0$$

$$R^2C = \omega^2[RLC^2 - L^2C]$$

$$\omega^2 = \frac{R^2C}{RLC^2 - L^2C}$$

$$\omega = \sqrt{\frac{R^2C}{RLC^2 - L^2C}}$$

555 Timer

The 555 timer is a highly stable device for generating accurate time delay or oscillation.

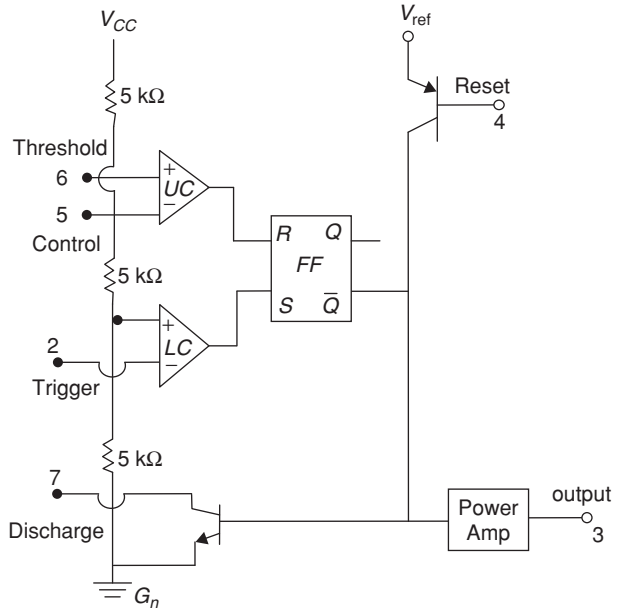


Figure 10 Functional diagram of 555 timer

Three 5 kΩ internal resistors act as voltage divider, providing bias voltage of $\frac{2}{3}V_{CC}$ to upper comparator (UC) and $\frac{1}{3}V_{CC}$ to lower comparator (LC). Since these two voltages fix the necessary comparator threshold voltage, they also aid in determining the timing interval. It is possible to vary time by applying a modulating voltage to control input terminal.

In applications where no such modulation is intended, it is recommended to connect a capacitor between control and ground.

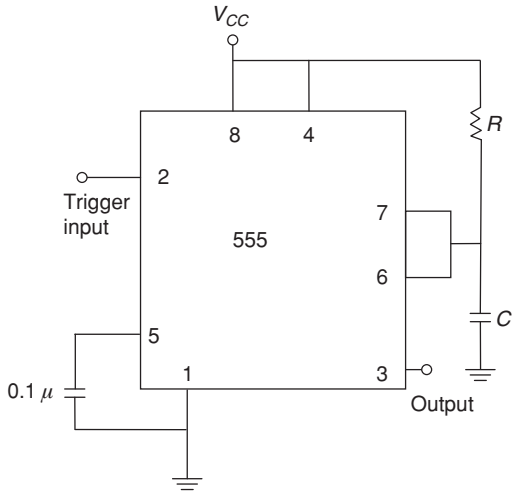
The reset pin provides a mechanism to reset *FF*, when reset is not used, it is returned to V_{cc}.

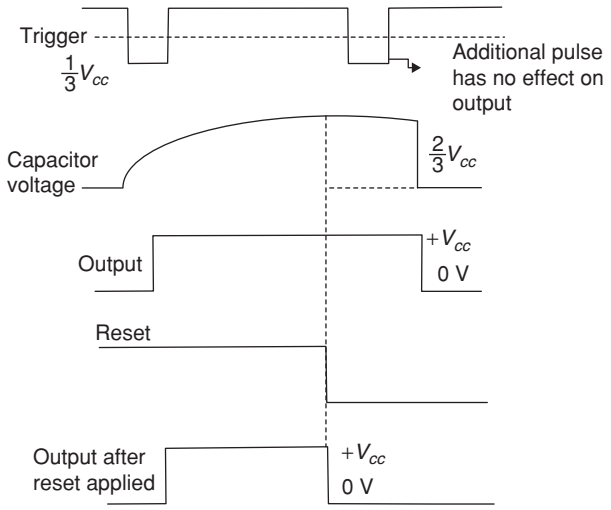
Operation:

1. V_{cc} is typically 5 V, The *R*'*s* are 5 kΩ each, and act as a voltage divider, to create voltages of value $\frac{V_{CC}}{3}, \frac{2V_{CC}}{3}$
2. Comparator 1 compares the voltage applied at the 'threshold' terminal with $\frac{2V_{CC}}{3}$, and comparator 2 compares the voltage at the 'trigger' terminal with $\frac{V_{CC}}{3}$.
3. External connection to the 'control' terminal will override the $\frac{2V_{CC}}{3}$ existing at that node, and allows the user added flexibility.
4. When the comparator 2 output is high, it 'sets' the *RS* flip-flop whose output (Q) goes high. This turns on the discharge transistor Q_1, and causes the 'discharge' terminal to be discharged to the ground.
5. When the comparator 1 output is high, it resets the *Rs*-flipflop, whose output goes low, which turns Q_1 OFF.

Monostable Operation

$T = 1.1\ RC$





Operation

Condition	R	S	Q	Output = Q	Transistor status
$V_{cc} > \frac{2}{3}V_{cc}$	1	0	1	0	ON, capacitor starts discharging
$V_{cc} = 0$ V	0	0	1	0	No change in state
Apply Trigger	0	1	0	1	OFF, capacitor starts charging
$V_{cc} > \frac{1}{3}V_{cc}$	0	0	0	1	No change in state
$V_{cc} > \frac{2}{3}V_{cc}$	1	0	1	0	ON, capacitor starts discharging
$V_{cc} = 0$ V	0	0	1	0	no change in state

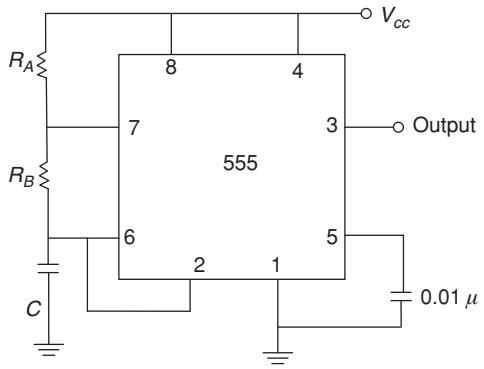
$$V_C(t) = V_{\text{final}} + (V_{\text{initial}} - V_{\text{final}})e^{\frac{-t}{RC}}$$

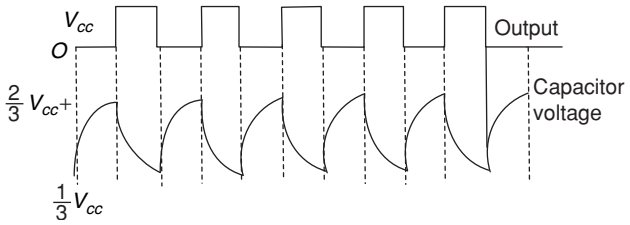
$$V_C(t) = V_{CC} + (0 - V_{CC})e^{\frac{-t}{RC}}$$

At $t = T, V_c(t) = \frac{2}{3}V_{cc}$, where '$T$' represents duration which 'Q' output goes high

$$\therefore \quad \frac{2}{3}V_{cc} = V_{cc}\left[1 - e^{\frac{-T}{RC}}\right] \quad \Rightarrow \quad T = 1.1\,\text{RC}$$

Astable Operation





Assume capacitor discharges, when power is switch ON, $V_{\text{trigger}} = V_{\text{threshold}} = 0$

$\Rightarrow$ comparator 2 will have a high output

Comparator 1 will have a low output.

$\Rightarrow R = 0, S = 1$ and $Q = 1$

So capacitor will charge through $(R_A + R_B)$

$V_C(t) > \frac{V_{cc}}{3}$, comparator 2 output will go low $\Rightarrow S = 0$

$V_C(t) > \frac{2V_{cc}}{3}$, comparator 2 output will go high $\Rightarrow Q = 0$

$$V_C(t) = \frac{2V_{cc}}{3}\exp\left(\frac{-t}{R_B \cdot C}\right)$$

$$V_C(T_1) = \frac{V_{cc}}{3} = \frac{2V_{CC}}{3}\exp\left(\frac{-T_1}{R_B C}\right)$$

$$T_1 = R_B C \ln^{(2)} = 0.69\, R_B\, C$$

Similarly, during the time period T_2

$$V_C(t) = V_{CC} - \left(V_{CC} - \frac{V_{CC}}{3}\right)\exp\left[\frac{-t}{(R_A + R_B)C}\right]$$

$$T_2 = (R_A + R_B)\ln(2) = 0.69\,(R_A + R_B)\,c$$

$$T = T_1 + T_2 = 0.69\,(R_A + 2R_B)\text{C}$$

$$\text{Duty cycle} = \frac{T_1}{T_1 + T_2} = \frac{R_A + R_B}{R_A + 2R_B}$$

$$t_{\text{high}} = 0.69(R_A + R_B)C$$

$$t_{\text{low}} = 0.69R_B C$$

$$T = t_{\text{high}} + t_{\text{low}}$$

$$= 0.69(R_A + 2R_B)C$$

$$f = \frac{1}{T} = \frac{1.45}{(R_A + 2R_B)C}$$

$$\text{Duty Cycle} = \frac{t_{\text{high}}}{T} \times 100 = \frac{R_A + R_B}{R_A + 2R_B} \times 100$$

When R_A is much smaller than R_B duty cycle approaches to 50%. If R_A is much greater than R_B duty cycle approaches to 100%. The circuit can be modified to enabled the duty cycle less than 50%. By placing a diode across R_B (anode at pin 7).

The capacitor will effectively charge through R_A and diode. The capacitor will discharge through R_B, $D = \frac{R_A}{R_A + R_B}$

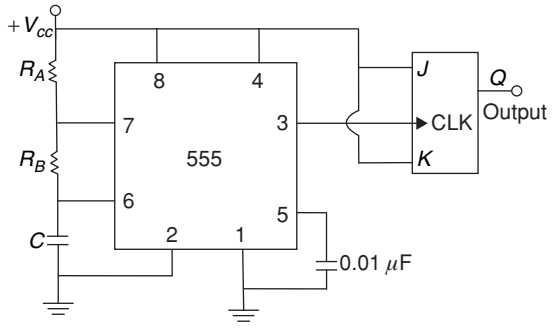
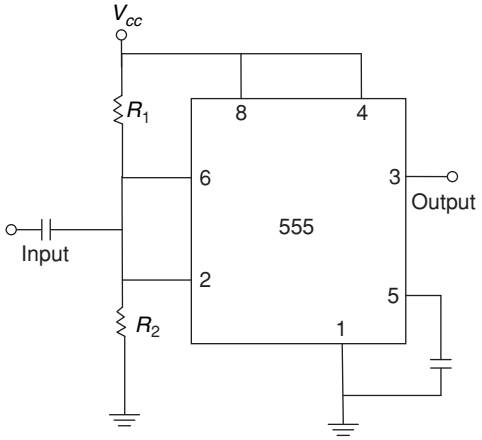


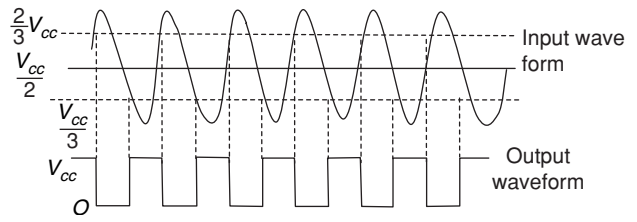
Figure 11 Symmetrical wave form generator.

The clocked flip-flop acts as binary divider to the timer output. The output frequency in this case will be one half that of the timer, the advantage of this circuit is of having output of 50% duty cycle, without any restriction on the choice of R_A and R_B.

Schmitt Trigger



The use of 555 timer as a Schmitt trigger input is shown in figure. Here the two internal comparators are tied together and externally biased at $\frac{V_{cc}}{2}$ by R_1 and R_2 resistors. The threshold levels of (UC) $= \frac{2}{3}V_{CC}$ and LC $= \frac{1}{3}V_{CC}$ and the bias provided by R_1 and R_2 determine the output wave form. Thus sine wave of sufficient amplitude, $\left(> \frac{V_{CC}}{6}, \text{i.e.,} \frac{2}{3}V_{CC} - \frac{V_{CC}}{2}\right)$ to exceed the reference levels causes the internal flip-flop to alternately set and reset, providing a square wave output



Unlike multivibrators, no frequency division is taking place and frequency of square wave remains same as that of input signal.

Voltage Controlled Oscillator

Voltage controlled oscillator (VCO) or voltage to frequency converter can be found in Applications like frequency modulation, tone generators, and frequency shift keying (FSK) etc, where frequency needs to be controlled by input voltage.

A typical example for VCO is IC 566, which provides simultaneous square wave and triangular wave outputs as a function of input voltage.

The frequency of osculation is determined by an external resistor R_1, capacitor C_1, and the voltage V_c applied.

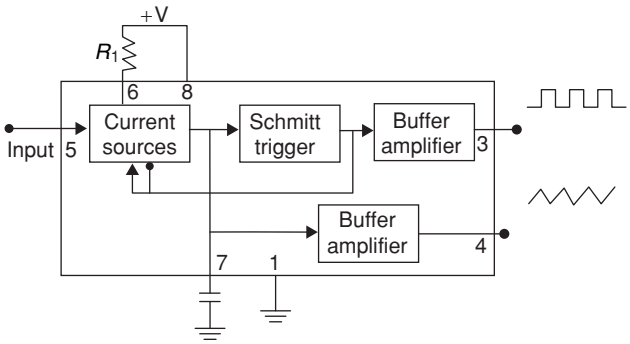


Figure 12 Block diagram. of 566

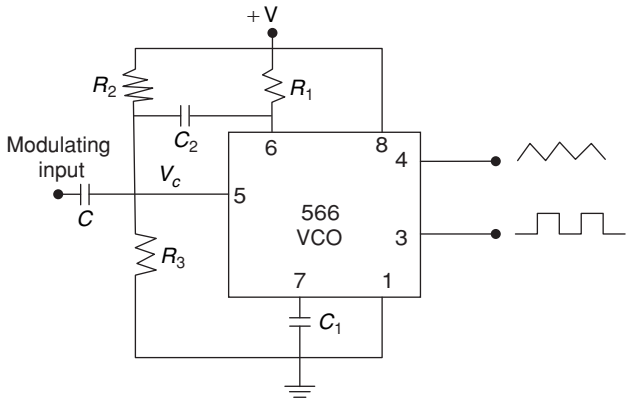


Figure 13 Typical connection diagram

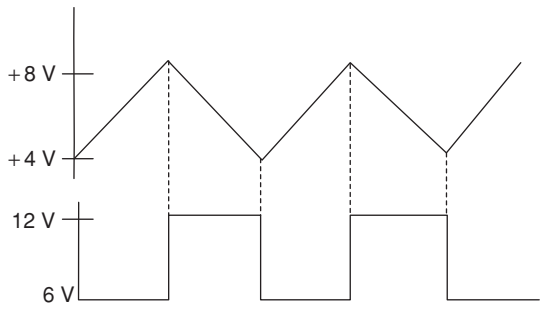


Figure 14 Output waveforms

$$f_o = \frac{2(V - V_c)}{R_1 C_1 (+V)},$$

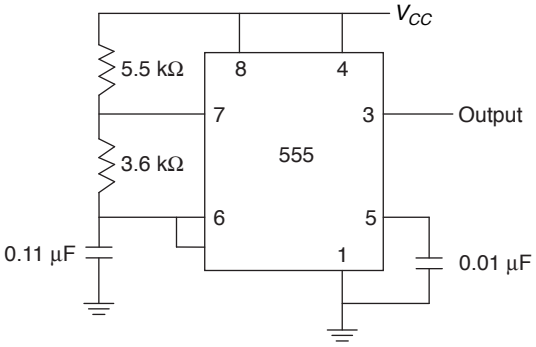
Where $\frac{3}{4}(+V) \le V_C \le (+V)$ and 2 kΩ < R, < 20 kΩ

VCO is commonly used in converting low frequency signal such as electroencephalograms (EEG) or electrocardiograms (ECG) into an audio frequency range.

Solved Examples

Directions for questions 7 to 11: Select the correct alternative from the given choices.

Example 7: The 555 timer circuit to generate a rectangular waveform is shown in figure. The frequency and duty cycle of the waveform are



(A) 1 kHz, 0.43 (B) 2 kHz, 0.43
(C) 2 kHz, 0.28 (D) 1 kHz, 0.28

Solution: (D)

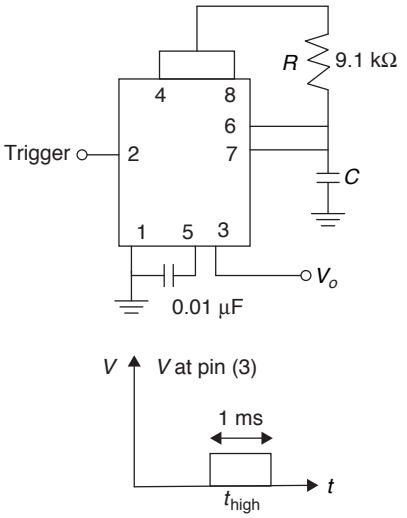
$$f = \frac{1.45}{(R_A + 2R_B)C} = \frac{1.45}{(5.5 + 2\times 3.6)\times 0.11\times 10^{-3}}$$

$$= 1.04 \text{ kHz} \approx 1 \text{ kHz}$$

$$\text{Duty cycle} = \frac{R_B}{R_A + 2R_B} = \frac{3.6}{5.5 + 2\times 3.6}$$

$$= 0.28$$

Example 8: A monostable multivibrator circuit is shown in the figure. The value of C would be nearly



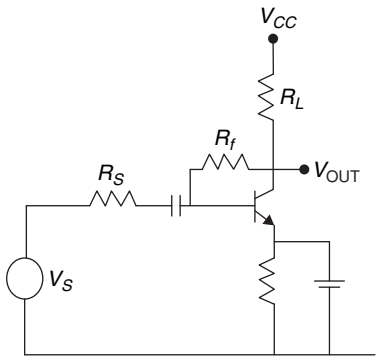
(A) 0.01 μF (B) 0.001 μF
(C) 0.1 μF (D) 1 μF

Solution: (C)

$T_{high} = 1{\cdot}1\ R.C$

$$\Rightarrow \quad C = \frac{1.0\times 10^{-3}}{1.1\times 9.1\times 10^{3}} = 0.1\ \mu\text{F}$$

Common Data for Questions 9 to10: For the circuit shown in the figure, the transistor parameters are h_{fe} = 100, h_{ie} = 1 kΩ, R_s = 2 kΩ, R_L = 20 kΩ, R_f = 200 kΩ.



Example 9: The type of feedback used in this circuit is

(A) Voltage shunt. (B) Current shunt.
(C) Voltage series. (D) Current series.

Solution: (A)

R_f is across the output and samples the output voltage V_0. At the input it is in shunt, resulting current feedback, the feedback in voltage shunt is

$$\beta = \frac{I_f}{V_0} = -\frac{1}{R_f} = \frac{1}{200\times 10^3} \text{ mho}$$

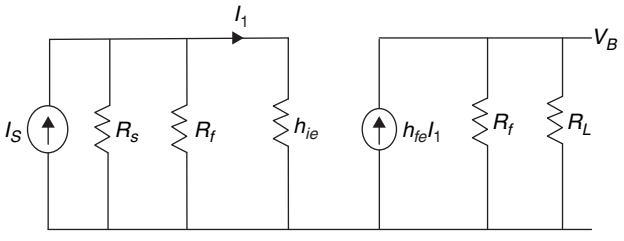
This feedback amplifier is a current controlled voltage source, hence both input and output impedances are reduced by feedback.

Example 10: The feedback factor is

(A) 5.543 (B) 4.636
(C) 3.423 (D) 2.942

Solution: (B)

The equivalent current of the amplifier without feedback will have R_f across input, and R_f across load resistance R_L



$$A_Z = \frac{V_0}{I_s} = \frac{V_0}{I_i} \times \frac{I_i}{I_s} \text{ (Total gain without feedback)}$$

Here $R_s = 2$ K, $h_{ie} = 1$ K, $R_f = 200$ K >> h_{ie}
Input resistance $R_i = 0.66$ kΩ

$$\frac{I_i}{I_s} = \frac{0.66}{1+0.66} = 0.397$$

$$R_f \| R_L = 200 \text{ K} \| 20 \text{ K} = 18.18 \text{ k}\Omega$$

$$\frac{V_0}{I_i} = -h_{ie} \times R_f \| R_L = -100 \times 18.18 \text{ k}\Omega$$

$$A_Z = \frac{V_0}{I_i} \times \frac{I_i}{I_s} = 72.72 \times 10^4$$

$$D = 1 + A_z\beta$$

$$= 1 + 72.72 \times 10^4 \times \frac{1}{200 \times 10^3} = 4.636$$

Example 11: The input and output impedance values in mid-band are
(A) 142 Ω, 3.92 kΩ
(B) 3.92 kΩ, 142 Ω
(C) 0.66 kΩ
(D) 18.18 kΩ, 0.66 kΩ

Solution: (A)

$$R_{\text{in}} = \text{Input resistance} = \frac{0.66 \text{ k}\Omega}{4.636} 0.142 \text{ k}\Omega$$

$$R_{\text{out}} = \text{Output resistance} = \frac{18.182}{4.636}$$

$$= 3.92 \text{ k}\Omega$$

EXERCISES

Practice Problems I

Directions for questions 1 to 43: Select the correct alternative from the given choices.

1. An amplifier has an open loop gain of 100 with 10% harmonic distortion at output. If 40 dB of negative feedback is applied, the distortion with feedback is _____.
(A) 0.1% (B) 10%
(C) 100% (D) None

2. The gain of amplifier is 1000 and $\beta = 0.1$. Due to temperature variations, gain is changed by 15%, the change in gain with −ve feedback is _____.
(A) 0% (B) 1.5%
(C) 0.15% (D) 1%

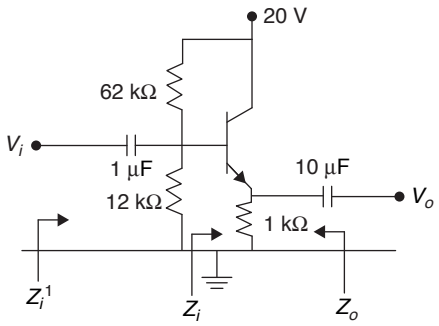
3. An amplifier has an open loop gain of 200, an input impedance of 5 kΩ and output impedance of 600 Ω. A feedback factor of $\beta = 20\%$ is connected to the amplifier in a voltage shunt feedback mode, the new input and output impedances are _____.
(A) $Z_{if} = 205$ kΩ $Z_{of} = 14.63$ Ω
(B) $Z_{if} = 122$ Ω $Z_{of} = 24.6$ kΩ
(C) $Z_{if} = 205$ kΩ $Z_{of} = 24.6$ kΩ
(D) $Z_{if} = 122$ Ω $Z_{of} = 14.63$ Ω

4. The mid-band gain of an amplifier is 1000. Its lower and upper cut-off frequencies are 60 Hz and 80 kHz respectively. A feedback Network with $\beta = 0.05$ is taken, the new lower and upper cut-off frequencies are _____.
(A) $f_L' = 3.1$ kHz $f_H' = 1.568$ kHz
(B) $f_L' = 3.1$ kHz $f_H' = 4.08$ MHz
(C) $f_L' = 1.176$ Hz $f_H' = 4.08$ MHz
(D) $f_L' = 1.176$ Hz $f_H' = 1.568$ kHz

5. The gain-band width product of amplifier without feedback is 6 MHz. with negative feedback, the bandwidth is 1 MHz, the closed loop gain is _____.
(A) 6 (B) 60
(C) 100 (D) 10

6. An amplifier has a voltage gain 200 with $Z_i = 10$ kΩ and $Z_o = 1$ kΩ. A negative feedback with $\beta = 0.04$ in current shunt feedback is applied, the result input and output impedance are _____.
(A) $Z_{if} = 1.11$ kΩ $Z_{of} = 0.11$ kΩ
(B) $Z_{if} = 1.11$ kΩ $Z_{of} = 9$ kΩ
(C) $Z_{if} = 90$ kΩ $Z_{of} = 0.11$ kΩ
(D) $Z_{if} = 90$ kΩ $Z_{of} = 9$ kΩ

Common Data for Questions 7 to 9: For the given emitter follower, the specifications are $\beta = 100$, $h_{ie} = 2$ kΩ

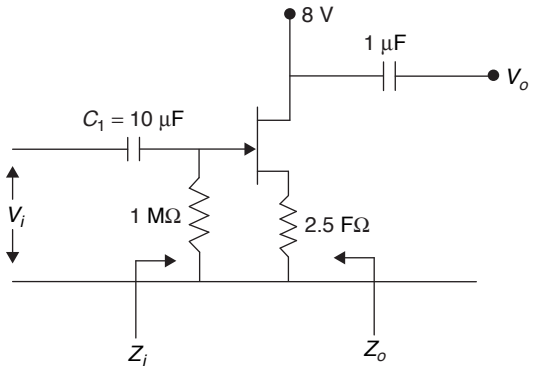


7. The input impedance Z_i is
(A) 1 kΩ (B) 101 kΩ
(C) 103 kΩ (D) 10.05 kΩ

8. The input impedance Z_i' as seen from source is _____.
(A) 9.156 kΩ (B) 10.05 kΩ
(C) 12 kΩ (D) 62 kΩ

9. The output impedance Z_o is _____.
(A) 1 kΩ (B) 20 Ω
(C) 1.02 kΩ (D) ∞

Common Data for Questions 10 to 12: A source follower is shown below with following specifications $I_{DSS} = 12$ mA, $V_P = 4$ V, $g_m = 2.28$ ms and $r_d = \infty$

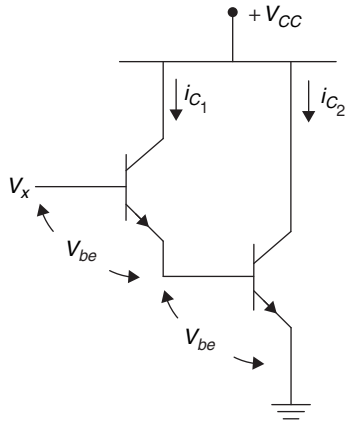


10. The input impedance Z_i is
(A) ∞ (B) 0
(C) 1 MΩ (D) None

11. The output impedance Z_o is
(A) 2.5 kΩ (B) 37 Ω
(C) 3.7 kΩ (D) 372 Ω

12. Voltage gain A_V is
(A) 0.85 (B) 1
(C) 99 (D) None

13. The Darlington pair stage is shown in the following figure. If the transconductance of Q_1 is 8×10^{-3} ℧ and Q_2 is 6×10^{-3} ℧, the overall transconductance g_m is _____.



(A) 14×10^{-3} ℧
(B) 2×10^{-3} ℧
(C) 3×10^{-3} ℧
(D) 6×10^{-3} ℧

14. The feedback N/w of Hartley oscillator have $L_1 = 10$ mH, $L_2 = 5$ mH and $C = 100$ PF. The mutual inductance b/w L_1 and L_2 is 2.5 mH. The frequency of oscillation is _____.
(A) 0.13 MHz (B) 11.3 kHz
(C) 0.113 MHz (D) None

15. The expression for output collector current in a power amplifier is given by
$I_C = 4 \sin \omega t + 1.5 \sin 2\,\omega t + 0.6 \sin 3\,\omega t$ amp
The % increase in power due to harmonic distortion is _____.
(A) 16.2% (B) 30.5%
(C) 12% (D) 1.6%

Common Data for Questions 16 to 18: A power amplifier working in class A operation has a zero signal collector current of 120 mA at $V_{CC} = 10$ volt

16. The maximum ac Output power is _____.
(A) 0.3 W (B) 0.15 W
(C) 0.6 W (D) 1.2 W

17. The power rating of transistor is _____.
(A) ≥ 1.2 W (B) ≥ 0.6 W
(C) ≤ 1.2 W (D) ≥ 0.3 W

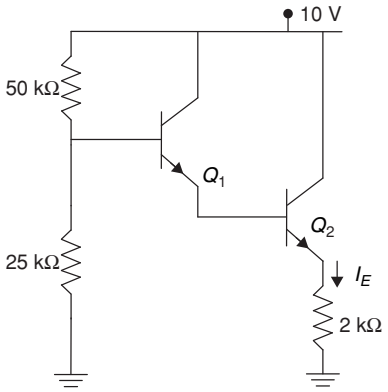
18. The power amplifier maximum collector efficiency is _____.
(A) 50% (B) 25%
(C) 78.5% (D) 18%

Common Data for Questions 19 and 20: A class B amplifier works on 20 V DC supply. The output impedance is 10 Ω and turn ratio is 5. The peak voltage across the load is 15 V.

19. The collector efficiency is _____.
(A) 28% (B) 50%
(C) 78.5% (D) 58.8%

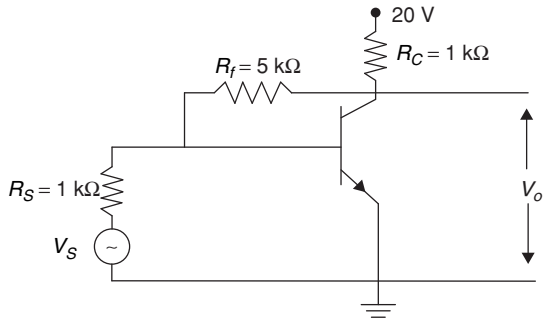
20. The power rating of transistor is _____.
(A) 0.45 W (B) 0.225 W
(C) 4.5 W (D) 2.25 W

21. What is the emitter current I_E if $\beta = 100$ and $V_{BE} = 0.7$ V for each transistor of the Darlington pair given below



(A) 1.5 mA (B) 0.964 mA
(C) 0.48 mA (D) None

Common Data for Questions 22 to 24: The feedback amplifier shown below has a open loop gain 1000.

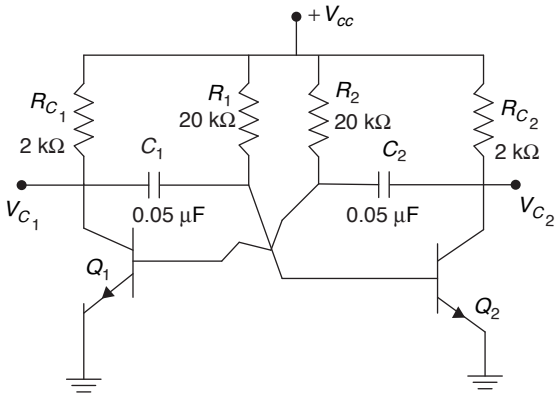


22. What is the type of feedback employed?
(A) series-series feedback
(B) series-shunt feedback
(C) shunt-series feedback
(D) shunt-shunt feedback

23. The feedback ratio β is _____.
(A) 1.6×10^{-4} (B) 1×10^{-3}
(C) 2×10^{-4} (D) 0.2×10^{-4}

24. The closed loop gain A_f is _____.
(A) 833.3 (B) 100
(C) 1000 (D) None

Common Data for Questions 25 and 26: An astable multi vibrator is shown below



25. The frequency of square wave Output is _____.
(A) 7.24 kHz (B) 5 kHz
(C) 0.5 kHz (D) 0.724 kHz

26. The minimum value of h_{fe} of transistor to ensure oscillations is _____.
(A) 1 (B) 0.2
(C) 10 (D) 8

27. An astable multi vibrator using Si transistor is designed to generate a square wave of amplitude 12 V and frequency 1 kHz. The transistor has $h_{fe_{mim}} = 40$, $I_{C_{sat}} = 6$ mA and $V_{CE_{sat}} = 0.2$ V. Assume $R_1 = R_2 = R$, $C_1 = C_2 = C$, calculate the value of R and C.
(A) $R = 1.96$ kΩ $C = 9$ nF
(B) $R = 1.96$ kΩ $C = 0.9$ μF
(C) $R = 196$ Ω $C = 9$ nF
(D) $R = 196$ Ω $C = 9$ μF

28. The AC power output of a class A amplifier is 4 W. If the collector efficiency is 45%, what is the power rating of transistor?
(A) ≥ 8.9 W
(B) < 8.9 W
(C) ≥ 4 W
(D) < 4 W

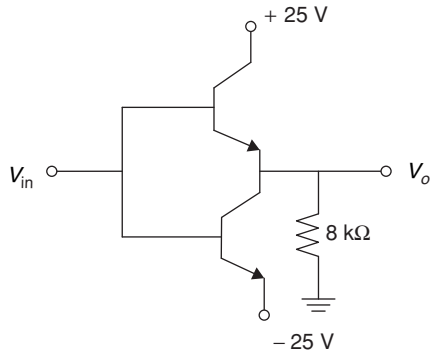
29. A class B push-pull amplifier uses 15 V DC supply, with sinusoidal input; a max peak to peak of 24 V is desired across a load of 100 Ω. What is the power dissipated by each transistor?
(A) 426 mW (B) 213 mW
(C) 1.146 W (D) 0.72 W

30. Match List I and List II:

List I	List II
P Class A amplifier	**1** Hifidelity
Q Class B amplifier	**2** Tuned amplifier
R Class C amplifier	**3** Power amplifier
S Class AB amplifier	**4** Low distortion power amplifier

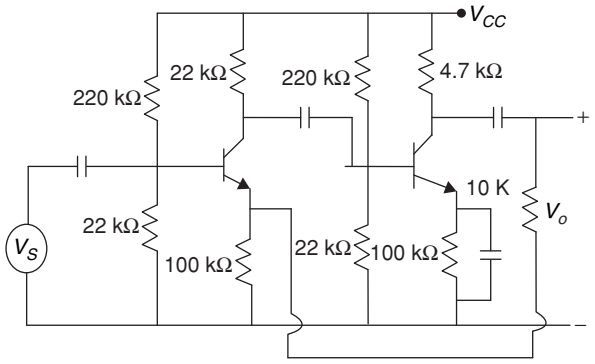
	P	Q	R	S
(A)	4	3	2	1
(B)	1	2	3	4
(C)	4	2	3	1
(D)	1	3	2	4

31. The circuit of a class B push pull amplifier is shown in the figure. If the peak output voltage V_o is 16 V. The power drawn from the DC source would be



(A) 10 W (B) 15 W
(C) 27 W (D) 32 W

Common Data for Questions 32 to 34: For the two stage feed back amplifier shown in figure $h_{ie} = 1.1$ kΩ, $h_{fe} = 50$



32. The nature of over-all feedback is?
(A) Voltage shunt
(B) Voltage series
(C) Current series
(D) Current shunt

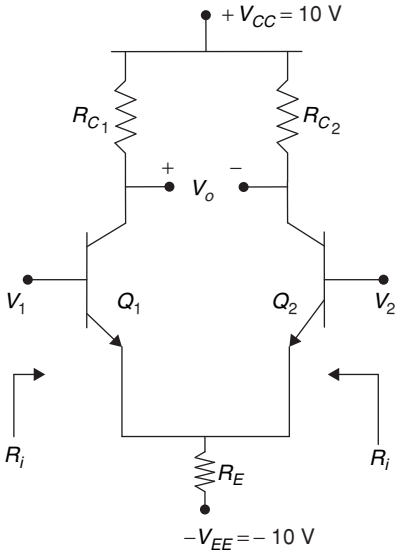
33. The values of over-all gain without feedback, and feedback factor are
(A) 600 and 10 (B) 1163 and 10
(C) 600 and 12.5 (D) 1163 and 12.5

34. The values of overall input and output resistance with feedback are
(A) 6.2 kΩ and 3.3 kΩ
(B) 77.63 kΩ and 3.3 kΩ
(C) 77.63 kΩ and 0.25 kΩ
(D) 6.2 kΩ and 0.25 kΩ

Common Data for Questions 35 to 39: A differential amplifier is shown below with following specifications.

$R_{C_1} = R_{C_2} = 2.2$ kΩ, $R_E = 4.5$ kΩ, $\beta = 100$, $V_{BE} = 0.7$ V

And $r_e = 28$ Ω



35. The input resistance R_i seen from each source is
(A) 5.6 kΩ (B) 28 Ω
(C) 1.4 kΩ (D) 2.8 kΩ

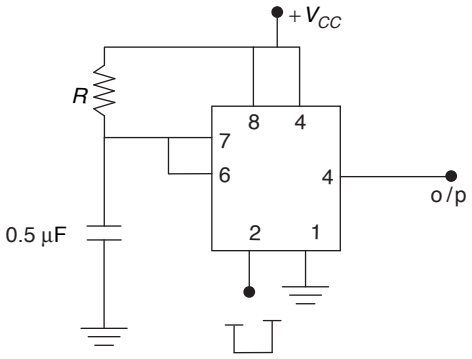
36. The output resistance R_O is
(A) 6.9 kΩ (B) 4.7 kΩ
(C) 2.2 kΩ (D) None

37. The differential mode gain is
(A) 39.3 (B) 78.5
(C) 0.48 (D) ∞

38. The common mode gain A_C is
(A) 0.48 (B) 0.24
(C) 0.97 (D) 0

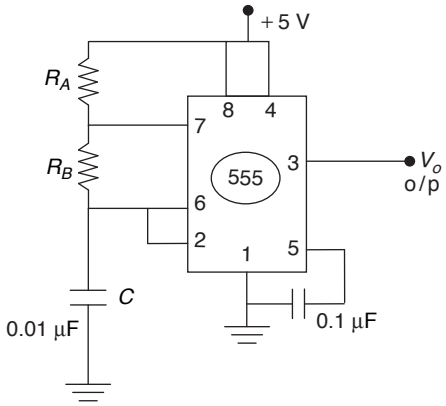
39. CMRR in dB is
(A) 50.29 dB (B) 25.14 dB
(C) 2.51 dB (D) 44.27 dB

40. A monostable multi vibrator is shown below. What is the value of R required for output rectangular frequency of 500 Hz.



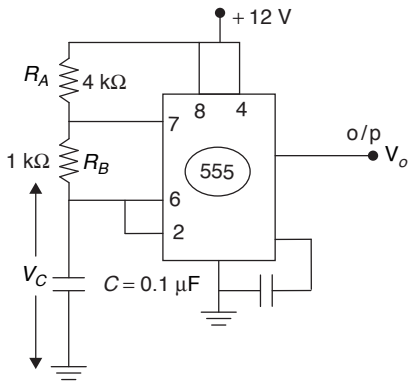
(A) 5.8 kΩ (B) 3.63 kΩ
(C) 6.2 kΩ (D) 9.1 kΩ

41. An astable multivibrator using *IC* 555 timer is given below; generate a square pulse of frequency 1 kHz with duty cycle 60%, what is the Relation between R_A and R_B?



(A) $R_B = R_A$ (B) $R_B = \frac{R_A}{2}$
(C) $R_B = 2R_A$ (D) $R_B = \frac{R_A}{3}$

42. An astable multivibrator using *IC* 555 is given below



The capacitor voltage V_c limits are

(A) 12 V and −12 V
(B) 4 V and −4 V
(C) 4 V and 8 V
(D) 8 V and −8 V

43. The frequency of output is

(A) 2. 4 kHz
(B) 1 kHz
(C) 1.2 kHz
(D) None of these

Practice Problems 2

Directions for questions 1 to 27: Select the correct alternative from the given choices.

1. An amplifier with −ve feedback has a voltage gain of 80, without feedback, an input signal of 40 mV is required to produce a given output where as with feedback; the input signal must be 0.5 V for the same output. The feedback ratio β is _____

(A) 1.5% (B) 1.15%
(C) 10% (D) 6.25%

2. An amplifier gives 1000 mV output with input of 1 mV without feedback. The desensitivity factor with negative feedback is 10. The feedback ratio β is _____.

(A) 2×10^{-3} (B) 9.9×10^{-2}
(C) 1×10^{-2} (D) 9×10^{-3}

3. The open loop gain of amplifier is 1000 ± 200 and feedback ratio is 4.2%, the % change in negative feedback gain is _____.

(A) 2.1% (B) 4.2%
(C) 0.46% (D) None

4. An amplifier has $A = 48000$ and $A_f = 6000$. The amounts of feedback in dB and feedback factor β are _____ respectively.

(A) 18.06 dB, 1.45%
(B) 18.06 dB, 0.145×10^{-3}
(C) 8 dB, 0.145×10^{-3}
(D) 8 dB, 1.45%

Common Data for Questions 5 and 6: The total harmonic distortion of an amplifier is reduced from 20% to 8% when 5% negative feedback is used.

5. The voltage gain without feedback is _____.

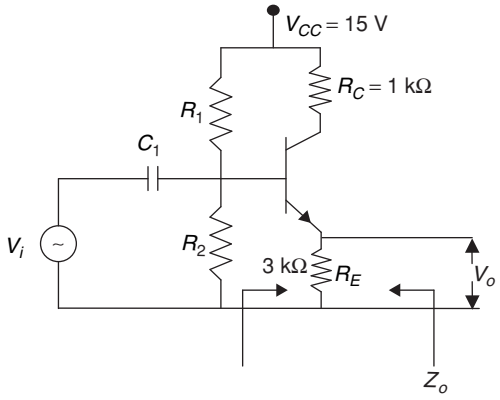
(A) 12 (B) 3
(C) 18 (D) 30

6. The voltage gain with negative feedback is _____.

(A) 1.2 (B) 30
(C) 8 (D) 12

Common Data for Questions 7 to 10: An emitter follower has following specifications.

$h_{ie} = 1000\ \Omega$, $h_{fe} = 100$ and $R_1 \parallel R_2 = 10\ \text{k}\Omega$



7. The voltage gain without feedback is _____.

(A) 1 (B) 300
(C) 101 (D) ∞

8. The voltage gain with feedback is _____.

(A) 0.99
(B) 0.85
(C) 30
(D) None

9. The input and output impedance are _____.

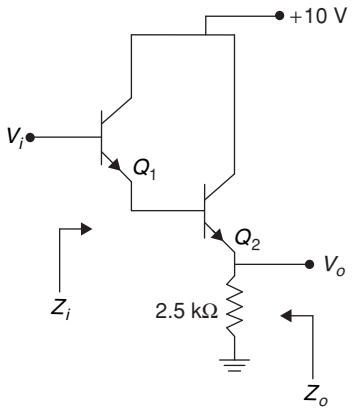
(A) $Z_i = 304\ \text{k}\Omega$ and $Z_o = 10\ \Omega$
(B) $Z_i = 10\ \Omega$ and $Z_o = 304\ \text{k}\Omega$
(C) $Z_i = 1.3\ \text{k}\Omega$ and $Z_o = 3\ \text{k}\Omega$
(D) $Z_i = 1\ \text{k}\Omega$ and $Z_o = 3\ \text{k}\Omega$

10. Current gain A_i is _____.

(A) 25 (B) 101
(C) 0.98 (D) 3

Common Data for Questions 11 to 14: Each transistor in the Darlington pair has following specifications

$h_{fe} = 99$, $h_{ie} = 1\ \text{k}\Omega$



11. The input impedance Z_i is _____.
(A) 2.5 kΩ (B) 25 MΩ
(C) 2.5 MΩ (D) 25 Ω

12. The output impedance Z_o is _____.
(A) 2.5 kΩ (B) 1 kΩ
(B) 100 Ω (D) 10.1 Ω

13. Current gain A_i is _____.
(A) 10^2 (B) 10^4
(C) 1 (D) 10

14. Voltage gain A_V is
(A) 0.8 (B) 10
(C) 1 (D) 12

15. An amplifier has a gain of 60 dB without feedback. What is the % loss in gain if $\frac{1}{50}$ of Output is feedback to input in out of phase?
(A) 4.76% (B) 8.3%
(C) 1% (D) 15%

16. In a certain amplifier, an output of 40 V is obtained when the input signal is 0.4 V. If 20% of the output is feedback to input in out of phase, by what value is the input signal to be changed so that the Output voltage remains constant?
(A) 10 V (B) 0.4 V
(C) 6 V (D) 8.4 V

17. In an astable multi vibrator each transistor is cutoff for 1m sec. The frequency of square wave output is _____.
(A) 500 Hz (B) 1 kHz
(C) 2 kHz (D) None

18. A transformer coupled class A amplifier has a turn ratio of 6:1 and the load is 25 Ω. If the zero signal collectors current is 80 mA, the maximum AC Output power is _____.
(A) 1.4 W (B) 2.88 W
(C) 5 W (D) 0.8 W

Common Data for Questions 19 and 20: An amplifier has voltage gain with feedback 50. If the gain without feedback change by 10% and the gain with feedback should not vary more than 1%.

19. The value of open loop gain A is _____.
(A) 500 (B) 250
(C) 1000 (D) 125

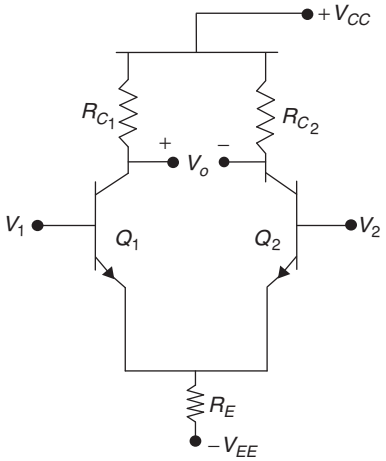
20. The feedback ratio β is _____.
(A) 0.9% (B) 0.45%
(C) 2.2% (D) 1.8%

21. An amplifier has 10% non linear distortion generated in its output stage. The amplifier gain without feedback is 200. If the distortions are reduced to 2% with negative feedback, the feedback ratio is _____.
(A) 0.5 (B) 0.2
(C) 0.02 (D) 1

22. Calculate the value of the capacitors to be used in astable multi vibrator to provide a train of pulse 4 μs wide at a repetition rate of 80 kHz if $R_1 = R_2 = 10$ kΩ.
(A) $C_1 = 0.58$ nF and $C_2 = 0.58$ nF
(B) $C_1 = 1.23$ nF, and $C_2 = 1.23$ nF
(C) $C_1 = 0.58$ nF, and $C_2 = 1.23$ nF
(D) $C_1 = 1.23$ nF, and $C_2 = 0.58$ nF

23. A power transistor used in class A amplifier is transformer coupled to a load of 10 Ω. If the signal has peak to peak swing of 200 mA and transformer turn ratio of 10, the AC Output power is _____.
(A) 5 W (B) 50 mW
(C) 10 W (D) 2.5 W

Common Data for Questions 24 to 26: A differential amplifier is shown below, has following specifications.
$R_{C_1} = R_{C_2} = 1.8$ kΩ, $R_E = 4$ kΩ $V_{CC} = 10$ V, $-V_{FE} = -10$ V, $\beta = 100$ and $V_{BE} = 0.7$ V



24. The emitter current I_E is
(A) 2.3 mA (B) 1.16 mA
(C) 1.7 mA (D) 0.98 mA

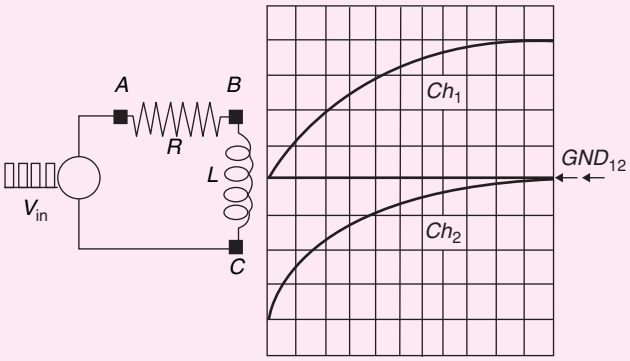
25. The differential mode voltage gain A_d is
(A) 10 (B) 8
(C) 80 (D) 40

26. What is the output V_O if $V_1 = 30$ mV and $V_2 = 40$ mV
(A) 0.8 V (B) 5.6 V
(C) 2.8 V (D) 0 V

27. In the mono stable multivibrator using IC555 timer, the external circuit elements connected is $C = 0.01$ μF and $R = 2.7$ kΩ. Calculate the duration of output pulse width.
(A) 18.6 μs (B) 29.7 μs
(C) 37.2 μs (D) None

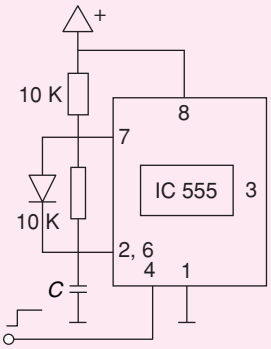
Previous Years' Questions

1. The probes of a non-isolated, two-channel oscilloscope are clipped to points A, B and C in the circuit of the adjacent figure. V_{in} is a square wave of a suitable low frequency. The display on Ch_1 and Ch_2 are as shown on the right. Then the 'Signal' and 'Ground' probes S_1, G_1 and S_2, G_2 of Ch_1 and Ch_2 respectively are connected to points. **[2007]**

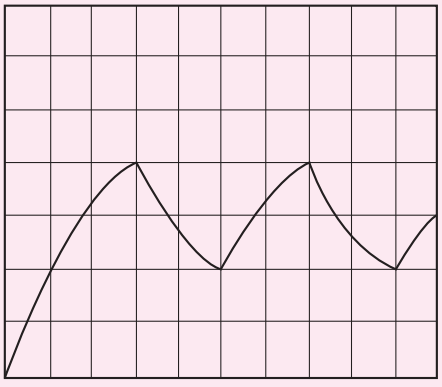


(A) A, B, C and A (B) A, B, C and B
(C) C, B, A and B (D) B, A, B and C

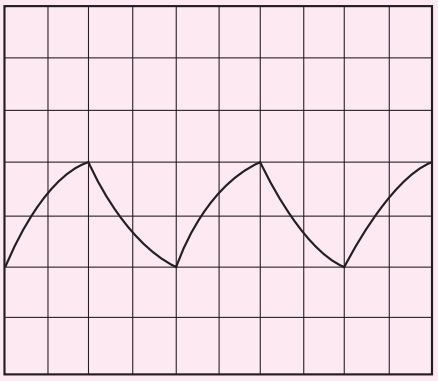
2. IC 555 in the adjacent figure is configured as an astable multi-vibrator. It is enabled to oscillate at $t = 0$ by applying a high input to pin 4. The pin description is: 1 and 8–supply: 2–trigger; 4–reset; 6–threshold; 7–discharge. The waveform appearing across the capacitor starting from $t = 0$, as observed on a storage CRO is **[2007]**



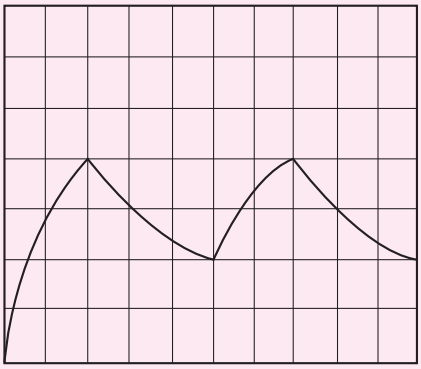
(A)



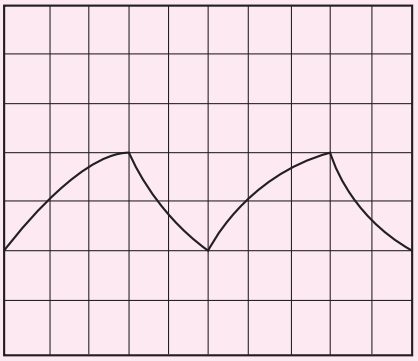
(B)



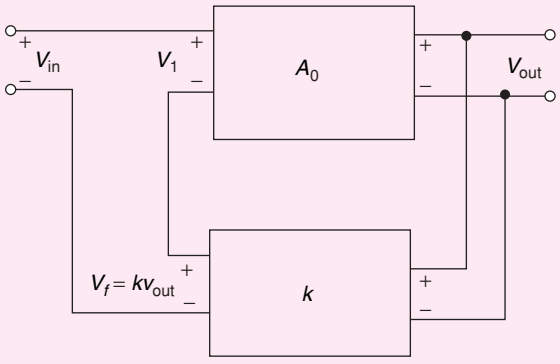
(C)



(D)



3. In the feedback network shown below, if the feedback factor k is increased, then the **[2013]**



(A) Input impedance increases and output impedance decreases.
(B) Input impedance increases and output impedance also increases
(C) Input impedance decreases and output impedance also decreases.
(D) Input impedance decreases and output impedance increases.

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** C	**3.** D	**4.** C	**5.** A	**6.** B	**7.** C	**8.** A	**9.** B	**10.** C
11. D	**12.** A	**13.** C	**14.** C	**15.** A	**16.** C	**17.** A	**18.** A	**19.** D	**20.** B
21. B	**22.** D	**23.** C	**24.** A	**25.** D	**26.** C	**27.** A	**28.** A	**29.** B	**30.** D
31. D	**32.** B	**33.** D	**34.** C	**35.** C	**36.** C	**37.** B	**38.** B	**39.** A	**40.** B
41. C	**42.** C	**43.** A							

Practice Problems 2

1. B	**2.** D	**3.** C	**4.** B	**5.** D	**6.** D	**7.** B	**8.** A	**9.** A	**10.** B
11. B	**12.** D	**13.** B	**14.** C	**15.** A	**16.** D	**17.** A	**18.** B	**19.** A	**20.** D
21. C	**22.** C	**23.** A	**24.** B	**25.** C	**26.** A	**27.** B			

Previous Years' Questions

1. B	**2.** A	**3.** A

Chapter 7

Operational Amplifiers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Operational Amplifiers
- Single–ended operation
- CMRR (common mode rejection ratio)
- Level–shifter
- 2 electrical parameters of op-Am
- Open loop op-Amp configurations
- The non-inverting amplifier
- Closed loop op-Amp configurations
- Filters
- Oscillators
- Theory of sinusoidal oscillators
- Waveform generators and wave shaping circuits
- Square wave generation from sinusoid
- Monostable multivibrator
- Power supplies

Operational Amplifiers

An operational amplifier is a direct, coupled, high gain amplifier consisting of one or more differential amplifiers, usually followed by a level translator and an output stage. The output stage is generally a push-pull or push-pull complementary-symmetry pair

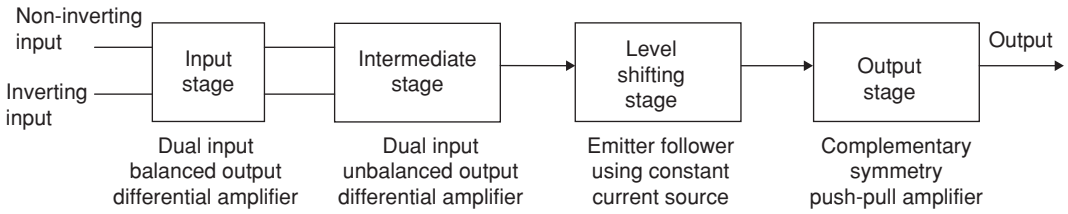
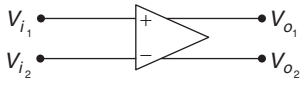


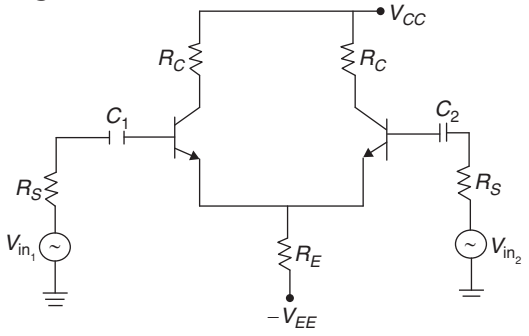
Figure 1 Block diagram of a typical op-amp

Differential Amplifier

Symbol:



Circuit diagram:



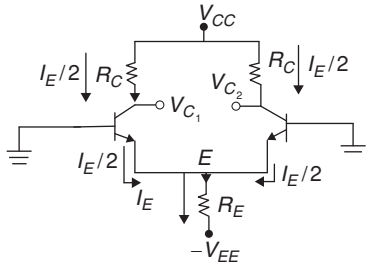
Modes of operation:

(i) Single-ended: Either of the input is grounded
(ii) Double-ended: Opposite polarity signals are applied at two inputs.
(iii) Common mode: Two similar input signals are applied at both inputs.

DC analysis:

(i) Make the AC sources ground
(ii) Replace the coupling capacitors open

Equivalent Circuit:



KVL at input side gives

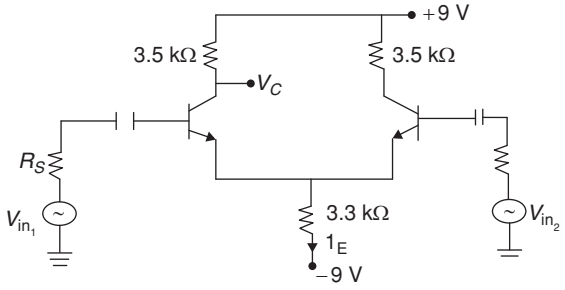
$0 - V_{BE} - V_E = 0$

$V_E = -0.7$ V

$V_E - I_E R_E + V_{EE} = 0$

$$I_E = \frac{V_{EE} - 0.7}{R_E}$$

Example 1: Determine I_E and V_C for the given circuit



(A) 4.1 V and 5 mA (B) 5 V and 2.5 mA

(C) 5 V and 5 mA (D) 4.6 V and 2.5 mA

Solution: (D)

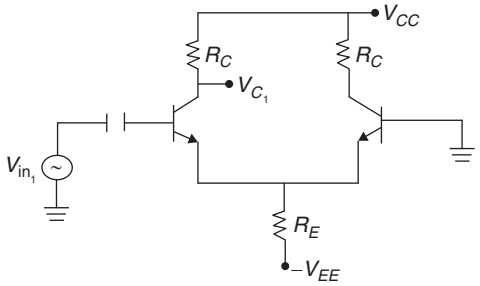
$$I_E = \frac{V_{EE} - 0.7}{R_E}$$

$$I_E = \frac{9 - 0.7}{3.3\text{ k}} = 2.5\text{ mA}$$

$$V_C = V_{CC} - \frac{I_E}{2} R_C$$

$V_C = 4.6$ V.

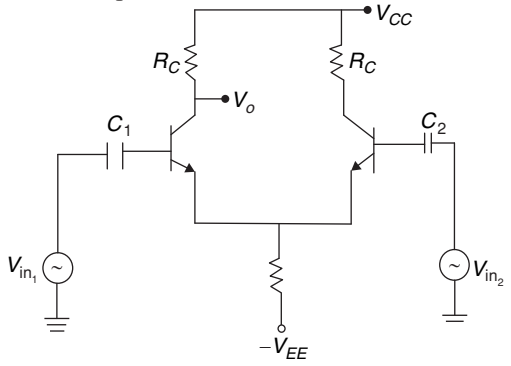
Single-ended operation:



Voltage gain $A_V = \frac{R_C}{2r_e} = \frac{V_{C_1}}{V_{in_1}}$

Input impedance $Z_i = 2\beta r_e$

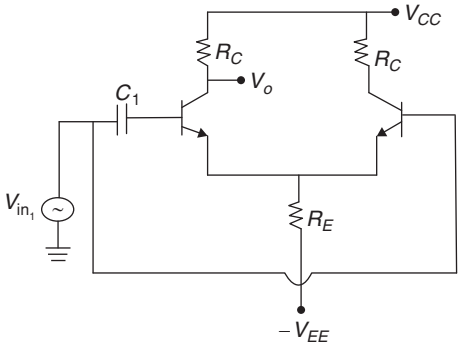
Double-ended Operation:



$$V_d = V_{in_1} - V_{in_2}$$

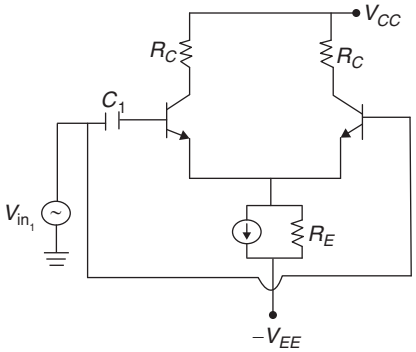
$$A_V = \frac{V_0}{V_d} = \frac{R_c}{2r_e}$$

Common-mode Operation:



$$A_C = \frac{V_0}{V_{in}} = \frac{\beta R_c}{\beta r_e + 2(\beta + 1)R_E}$$

In all IC design application, 'A_c' (common-mode gain) must be as small as possible. Hence R_E has to be so larger. So R_E is replaced by a constant current source to achieve smaller 'A_c'.



CMRR (Common-mode rejection ratio):

$$\text{CMRR} = \frac{A_d}{A_c}$$

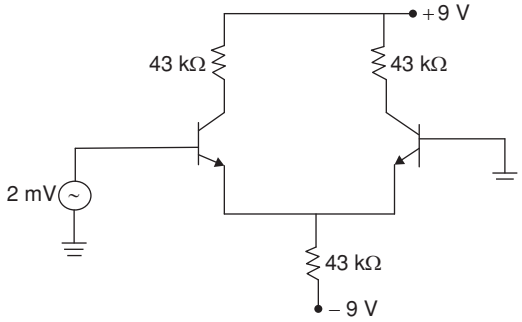
$$\text{CMRR in dB} = 20\log\left(\frac{A_d}{A_c}\right)$$

Where, A_d = Differential gain

A_c = common mode gain

In all practical applications, CMRR should be as high as possible. Ideal value of CMRR = ∞

Example 2: Calculate the single-ended voltage gain for the given circuit?



(A) 100 (B) 91.2
(C) 87.4 (D) 83

Solution: (C)

$$A_v = \frac{R_c}{2r_e}$$

$$r_e = \frac{26}{I_{E_1}}\text{ mV} = \frac{26}{I_{C_1}}\text{ mV}$$

$$I_E = \frac{V_{EE} - 0.7\text{ V}}{R_E} = 193\ \mu\text{A}$$

$$I_{C1} = \frac{I_E}{2} = 96.5\ \mu\text{A}$$

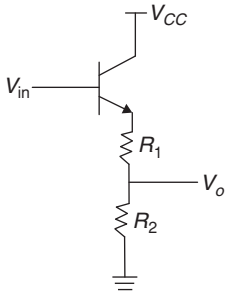
$$\therefore\quad r_e = \frac{26\text{ mv}}{96.5\ \mu\text{A}} = 269\ \Omega$$

$$A_v = \frac{R_C}{2r_e} = 87.4$$

Level-Shifter

The purpose of the coupling capacitor in any amplifier is to block DC and allows AC. In the absence of a capacitor, level-shifter circuit is used to block DC

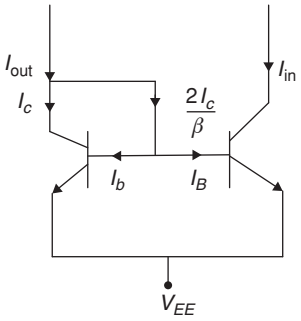
Simple Level Shifter:



$$V_0 = V_{\text{in}}\frac{R_2}{R_1 + R_2}$$

Resistor R_2 not only level-shifts DC by an amount $\frac{R_2}{R_1 + R_2}$ but also attenuates the AC signal. Remedy is to replace R_2 with current mirror circuit.

Current-mirror Circuit



If both the transistors are assumed to be symmetrical,

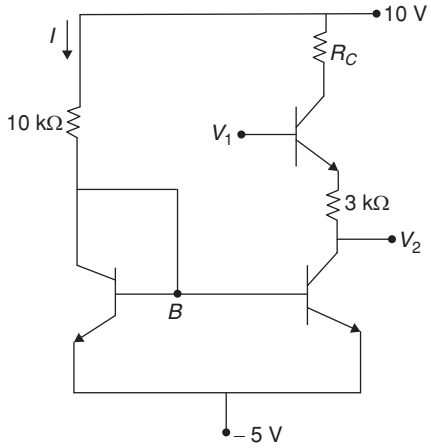
$$I_{\text{out}} = I_C + \frac{2I_c}{\beta}$$
$$= I_C[1 + 2/\beta]$$

If 'β' is designed to be so large, then

$$I_{\text{out}} \approx I_C \approx I_{\text{in}}$$

Advantage: I_C depends only on base current but not on the collector–emitter voltage.

Example 3: Find V_1–V_2 level shift in the given circuit if V_{BE} = 0.7 V and β is assumed to be so large?



(A) 4.52 (B) 5.23
(C) 4.99 (D) 6.25

Solution: (C)

$V_B - 0.7 + 5 = 0$

$V_B = -4.3$ V

$$I = \frac{10 - V_B}{10\text{ K}} = 1.43\text{ mA}$$

∴ Current through 3 kΩ resistor is 1.43 mA

$V_1 - V_2 = 0.7 + I(3\ \text{k}\Omega)$

$V_1 - V_2 = 4.99$

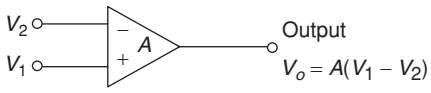
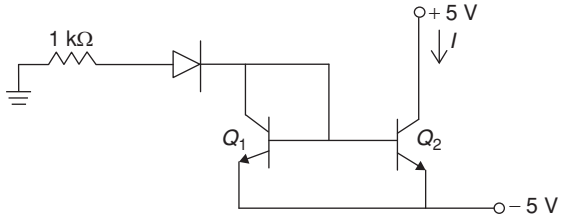


Figure 2 Schematic symbol for op-amp

Example 4: Two perfectly matched silicon transistors are connected as shown in the figure. Assuming β of the transistors to be very high and forward voltage drop in diode will be 0.7 V, the value of current I is



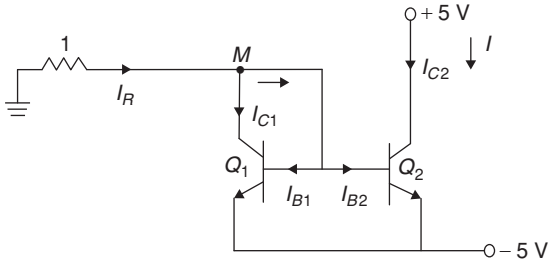
(A) 0 mA (B) 3.6 mA
(C) 4.3 mA (D) 5.7 mA

Solution: (B)

Since both are perfectly matched

$$V_{BE_1} = V_{BE_2}$$

$$\frac{I_{C_1}}{I_{C_2}} = \exp\left[\frac{V_{BE_1} - V_{BE_2}}{V_t}\right] = e^0 = 1$$



Since β is the same for both transistors, $I_{B_1} = I_{B_2} = I_B$

By KVL in loop at Q_1, $I_R = \dfrac{0 - 0.7 - 0.7(-5)}{1\ \text{k}\Omega} = 3.6\ \text{mA}$

KCL at point M,

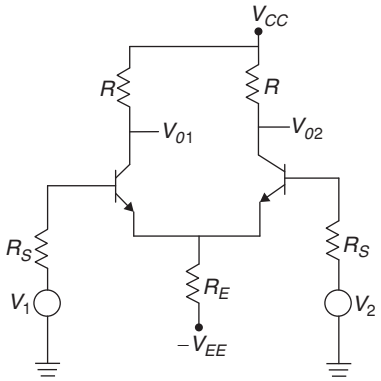
$$I_R = I_{C_1} + 2I_B = I_{C_1} + 2\frac{I_{C_1}}{\beta}$$

$$I_{C_1} = \frac{\beta}{\beta + 2} \cdot I_R$$

for large β

$$I_{C_1} \approx I_R = 3.6\ \text{mA}$$

Common Data for Questions 5 to 7: The differential amplifier, of shown in the figure has element values R_C = 50 kΩ, r_π = 1 MΩ, R_s = 2 kΩ, R_E = 200 kΩ, $\beta_0 = 2 \times 10^3$



Example 5: Differential mode voltage gain is

(A) –300 (B) –200

(C) –100 (D) 50

Solution: (C)

$$h_{fe} = \beta = 2 \times 10^3,\ r_\pi = 1\ \text{M}\Omega$$

$$R_C = R_L = 50\ \text{k}\Omega,\ R_E = 200\ \text{k}\Omega,\ R_S = 2\ \text{k}\Omega$$

$$A_{DM} = -\frac{h_{fe}R_L}{r_\pi} = -\frac{2\times10^3 \times 50\times10^3}{1\times10^6} = -100$$

Example 6: Common mode voltage gain is

(A) –0.5 (B) –0.25

(C) –0.125 (D) 0

Solution: (C)

$$A_{CM} = -\frac{R_L}{2R_E} \text{ when } r_\pi > R_S$$

$$= -\frac{50\ \text{k}\Omega}{2\times200\ \text{k}\Omega} = -\frac{1}{8} = -0.125$$

Example 7: The value of CMRR in dB is

(A) 800 (B) 60

(C) 58 (D) ∞

Solution: (C)

$$\text{CMRR} = \frac{A_{DM}}{A_{CM}} = \frac{100}{0.125} = 800$$

CMRR in dB = 20 log (800) = 58 dB

Electrical Parameters of Op-amp

Input offset voltage is the voltage that must be applied between the two input terminals of an op-amp to null the output.

The algebraic difference between the currents into the inverting and non-inverting terminals is referred to as input offset current $I_{io} = |I_{B_1} - I_{B_2}|$

Input bias current I_B is the average of the currents that flow into the inverting and non-inverting terminals of op-amp

$$I_B = \frac{I_{B_1} + I_{B_2}}{2}$$

Common mode rejection ratio is the ratio of differential voltage gain A_d, to the common mode voltage gain A_{CM}

$$\text{CMRR} = \frac{A_d}{A_{CM}}$$

The higher the value of CMRR, the better is the matching between the input terminals.

The change in an op-amp input offset voltage V_{io} caused by variations in supply voltages is called the supply voltage rejection ratio (SVRR), PSRR.

Slew rate is defined as the maximum rate of change of output voltage per unit of time and is expressed in volt per microseconds

$$\text{SR} = \left.\frac{dV_0}{dt}\right|_{\text{maximum}} \text{V/μs}$$

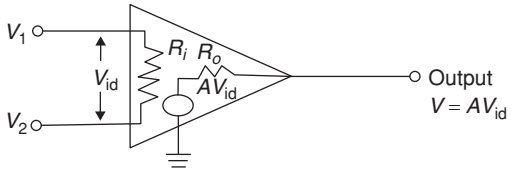
Slew rate indicates how rapidly the output of an op-amp can change in response to the changes in the input frequency, slew rate changes with change in voltage gain, and is normally specified at (+1) unity gain

The ideal op-amp characteristics:

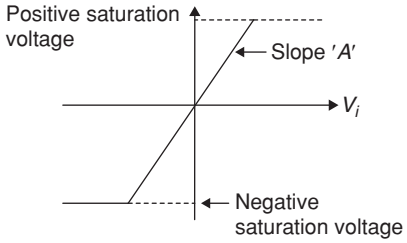
1. Infinite voltage gain (A)
2. Infinite input resistance (R_i)

3. Zero output resistance (R_0)
4. Zero output voltage when input voltage is zero
5. Infinite bandwidth
6. Infinite CMRR
7. Infinite slew rate

Equivalent Circuit of an Op-Amp

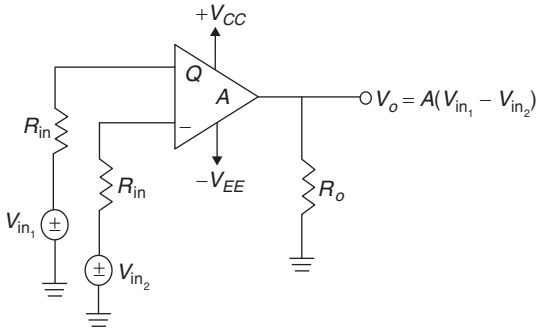


$$V_0 = AV_{id} = A(V_1 - V_2)$$

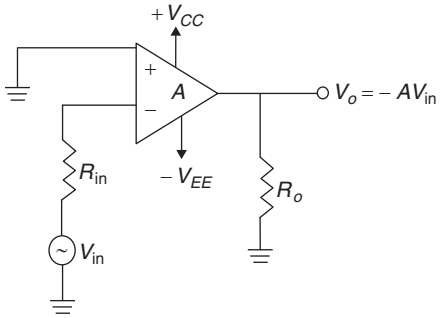


Open Loop op-amp Configurations

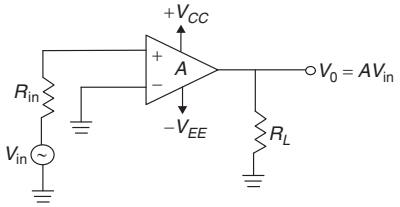
Differential Amplifier



The Inverting Amplifier



The Non-Inverting Amplifier

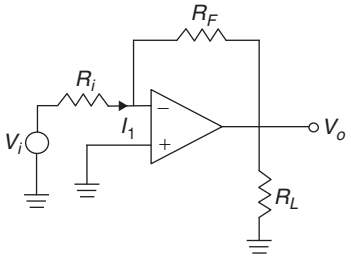


Very small values of input voltages can drive the open op-amp into saturation, as gain is very high, this prevents the use of open loop configurations of op-amps in linear applications.

Very high gain of open loop op-amp as well as its variation with temperature, power supply, and production yield makes the open loop op-amp configuration, unsuitable for linear applications.

Closed Loop Op-amp Configurations

Inverting Amplifier

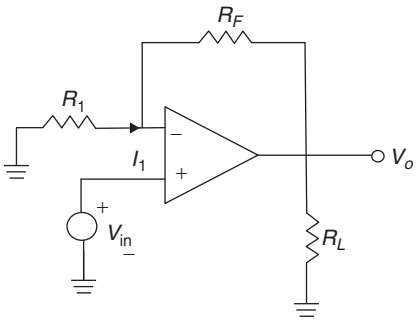


$$I_1 = \frac{V_i}{R_i}, V_0 = -I_1 R_f = \frac{-R_f}{R_1} V_i$$

$$A_v = \frac{-R_f}{R_i}$$

This is voltage shunt feedback circuit

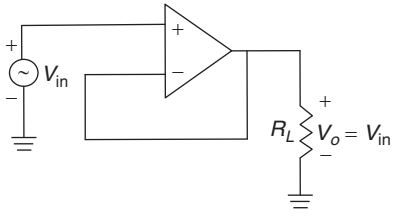
Non-inverting Amplifier



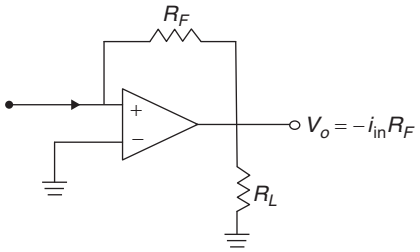
$$A_v = 1 + \frac{R_f}{R_i}$$

This is voltage series negative feedback configuration

Voltage follower



Current to Voltage Converter



The introduction of negative feedback stabilizes the gain; however, it is also smaller than the open loop gain. On the contrary, positive feedback is necessary in oscillator circuits.

The use of negative feedback in a non-inverting amplifier increases the input impedance, and bandwidth, and decreases output resistance, total output offset voltage.

The input resistance of the inverting amplifier is relatively smaller because it depends on the resistance connected in series with the input signal source.

A special case of non-inverting amplifier is voltage follower, while the current to voltage converter is the special case of inverting amplifier. The voltage converter produces output voltage proportional to input current.

Differential Amplifiers

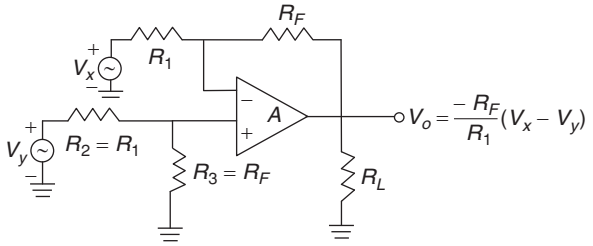


Figure 3 Differential amplifier with one op-amp

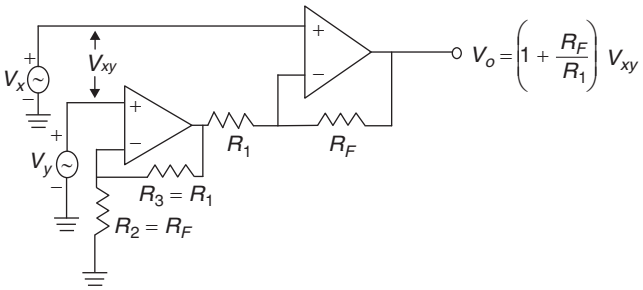
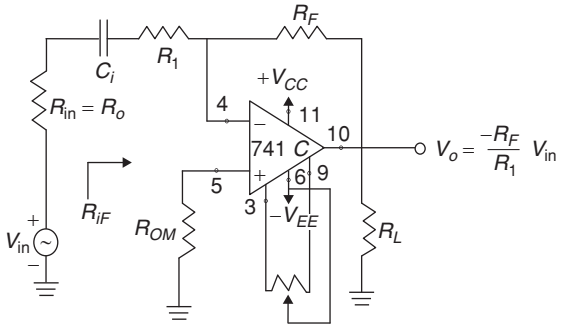


Figure 4 Differential amplifier with two op-amps

Negative feedback is also used in differential amplifier. The differential amplifier with one op-amp has the same characteristics as inverting amplifier, while the amplifier with two op-amps has the same characteristics as the non-inverting amplifier.

AC Amplifier–Inverting Amplifier



The coupling capacitor not only blocks the DC voltage but also sets the low frequency cutoff limit

$$f_L = \frac{1}{2\pi C_i (R_{iF} + R_o)}$$

The high frequency cutoff f_H or high end of the bandwidth depends on the closed loop gain of the amplifier

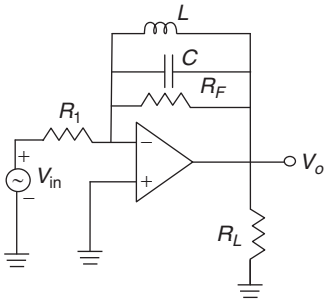
$$f_F = \frac{\text{(unity gain bandwidth)}}{A_F} \times k$$

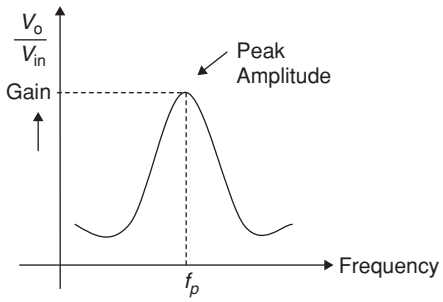
and $k = \dfrac{R_F}{R_1 + R_F}$,

$$A_F = \frac{-R_F}{R_i} \text{ (inverting amplifier)}$$

$$= 1 + \frac{R_F}{R_i} \text{ (non inverting amplifier)}$$

The Peaking Amplifier





The peaking response, that is the frequency response that peaks at a certain frequency, can be obtained by using a parallel LC network with the op-amp

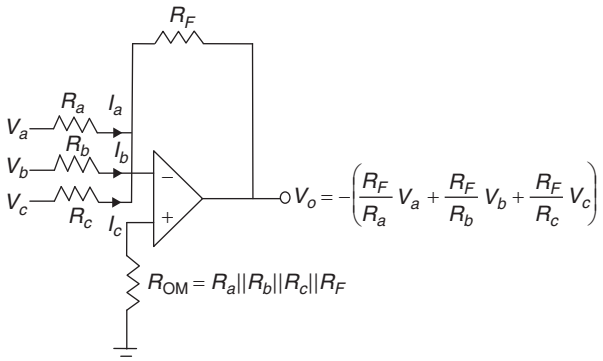
$$f_P = \frac{1}{2\pi\sqrt{LC}}, \text{ if } Q_{\text{coil}} \geq 10$$

The gain of amplifier at resonance is maximum and is given by $A_F = \dfrac{-R_F \parallel R_P}{R_1}$

R_P = equivalent resistance of the tank current = $Q^2_{\text{coil}}\, R$,
R = internal resistance of the coil

$$BW = \frac{f_p}{Q_p},\ Q_p = R_F \parallel \frac{R_p}{X_L}$$

Weighted Amplifier

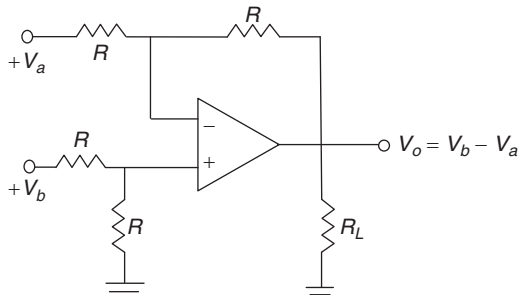


Summing amplifier:

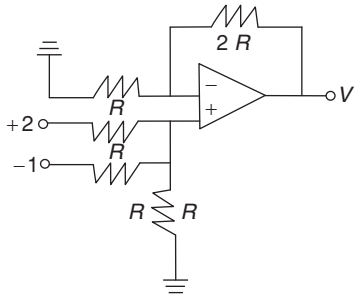
$$R_a = R_b = R_c = R,\ \text{then } V_0 = \frac{-R_F}{R}(V_a + V_b + V_c)$$

Averaging Amplifier: If $\dfrac{R_F}{R} = \dfrac{1}{n}$ for n inputs, then output is average of input voltages

Subtractor:



Example 8: The output voltage of the circuit



(A) 1.0 V
(B) 1.5 V
(C) 2.0 V
(D) 3 V

Solution: (B)

$$V_o = \left(1 + \frac{2R}{R}\right)\left[\frac{R}{R+R} \times 2 + \frac{R}{R+R} \times (-1)\right]$$
$$= 3\left[\frac{1}{2} \times 2 - \frac{1}{2}\right] = 1.5 \text{ V}$$

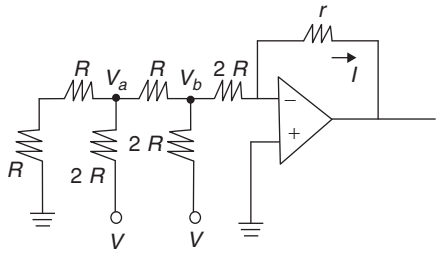
Example 9: An op-amp has a differential gain of 10^3 and a CMRR of 100, the ouptut voltage of the op-amp with inputs 120 μV and 80 μV will be

(A) 26 mV (B) 41 mV
(C) 100 mV (D) 200 mV

Solution: (B)

$$\text{CMRR} = \left(\frac{A_d}{A_c}\right) = 100 \quad \Rightarrow \quad A_c = \frac{A_d}{100} = 10$$
$$V_d = V_1 - V_2 = 40\ \mu\text{V}$$
$$V_c = \frac{V_1 + V_2}{2} = 100\ \mu\text{V}$$
$$V_o = A_d \cdot V_d + A_c \cdot V_c$$
$$= 10^3 \times 40 \times 10^{-6} + 10 \times 100 \times 10^{-6}$$
$$= 41 \text{ mV}$$

Example 10: The current I through resistance r in the circuit is



(A) $\dfrac{-V}{4R}$ (B) $\dfrac{V}{6R}$
(C) $\dfrac{V}{12R}$ (D) $\dfrac{V}{3R}$

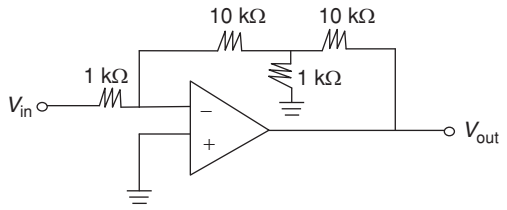
Solution: (C)

$$\frac{-V_a}{2R} = \frac{V_a - V}{2R} + \frac{V_a - V_b}{R}$$
$$4V_a = 2V_b + V \qquad (1)$$
$$\frac{V_b - V_a}{R} + \frac{V_b - V}{2R} + \frac{V_b}{2R} = 0 \qquad (2)$$

So we get $-V_a = V_b = \dfrac{V}{6}$

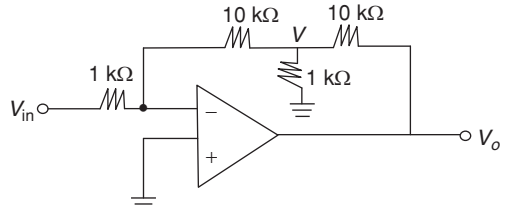
$$I = \frac{V}{12R}.$$

Example 11: Assuming the op-amp to be ideal, the gain $V_{\text{out}}/V_{\text{in}}$ for the circuit shown is?



(A) –1 (B) –20
(C) –100 (D) –120

Solution: (D)
Nodal equation at 'V'



$$\frac{V-0}{10}+\frac{V}{1}+\frac{V-V_{out}}{10}=0$$

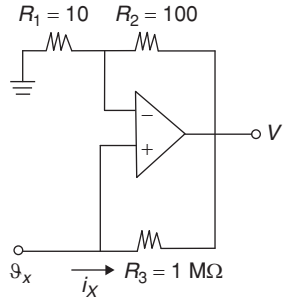
$$12\ \text{V} = V_{out}$$

Nodal equation at inverting node, we get

$$\frac{V_{in}-0}{1}=\frac{0-V}{10} \Rightarrow V_{in}=\frac{-V}{10}$$

$$\frac{V_{out}}{V_{in}}=\frac{12}{-1/10}=-120$$

Example 12: Input Resistance $R_{in} = (\vartheta_x/i_x)$ of the circuit is



(A) +100 kΩ (B) –100 kΩ
(C) +1 MΩ (D) –1 MΩ

Solution: (B)
Using KCL at inverting terminal, we have

$$\frac{v_x-V}{R_2}+\frac{V_x-0}{R_1}=0$$

$$\frac{v_x}{V}=\frac{R_1}{R_1+R_2}$$

Using same at Non-inverting terminal

$$i_x=\frac{v_x-V}{R_3}=\frac{v_x-v_x\left(\frac{R_1+R_2}{R_1}\right)}{R_3}$$

$$i_x=\frac{-v_xR_2}{R_1R_3}$$

$$R_{in}=\frac{v_x}{i_x}=\frac{-R_1R_3}{R_2}=-100\ \text{k}\Omega.$$

The Integrator

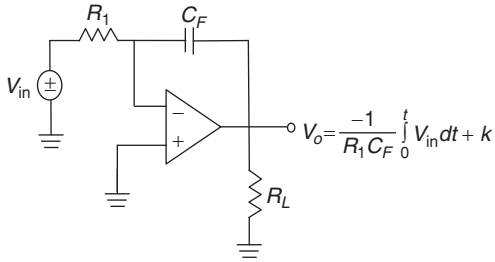


Figure 5 Basic Integrator

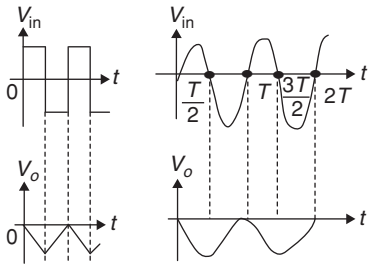
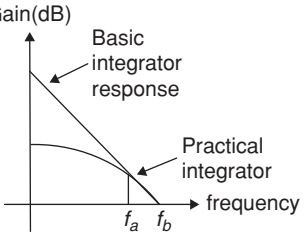
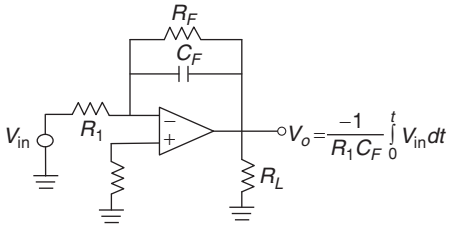


Figure 6 Input–Output wave forms



$$f_b=\frac{1}{2\pi R_1C_F}$$

The stability and low frequency roll-off problem can be corrected by the addition of a resistor R_F as shown in the practical integrator



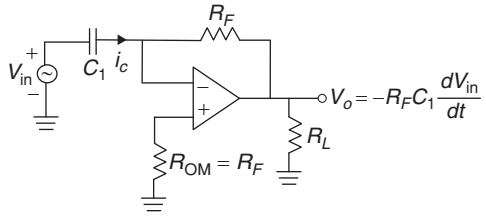
The gain limiting frequency $f_a=\frac{1}{2\pi R_FC_F}\cdot R_FC_F$ and R_1C_F values should be selected such that $f_a < f_b$.

The input signal will be integrated properly if the time period 'T' of the signal is larger than or equal to R_FC_F

$$T \geq R_F C_F$$

where, $R_F C_F = \dfrac{1}{2\pi f_a}$

The Differentiator



The output V_o is equal to $R_F C_1$ times the negative instantaneous rate of change of input voltage V_{in} with time

Differentiator performs the reverse of the integrator's function, a cosine wave input will produce a sine wave output, or a triangular input will produce a square wave output.

The frequency at which gain is zero is

$$f_a = \frac{1}{2\pi R_F C_1}$$

The gain $\dfrac{R_F}{X_{C_1}}$ of the circuit increases with increase in frequency at a rate of 20 dB/decade. This makes the circuit unstable, also the input impedance X_{C_1} decreases with increases in frequencies.

Both the stability and the high frequency noise problems can be corrected by the addition of two components R_1 and C_F

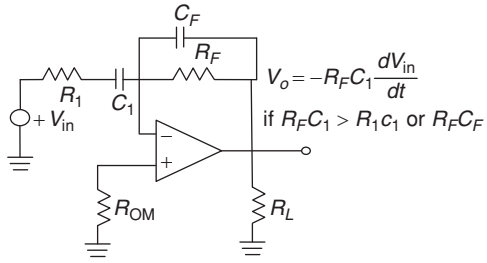


Figure 7 Practical Differentiator

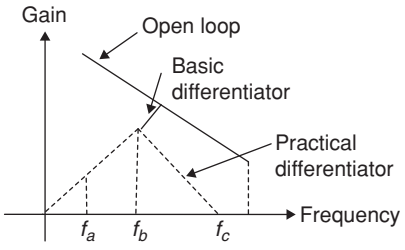


Figure 8 Frequency response

The gain limiting frequency f_b is given by

$$f_b = \frac{1}{2\pi R_1 C_1}$$

Generally the values of f_a, f_b, f_c are

$$f_a = \frac{1}{2\pi R_F C_1},\ f_b = \frac{1}{2\pi R_1 C_1},\ f_c$$

$$f_c = \frac{1}{2\pi R_F C_F}$$

f_c = unity gain bandwidth

The input signal will be differentiated properly if time period (T) of input signal $T \geq R_F C_1$

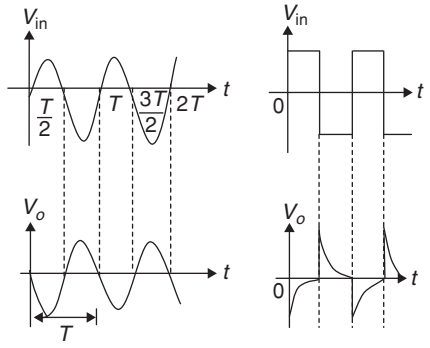
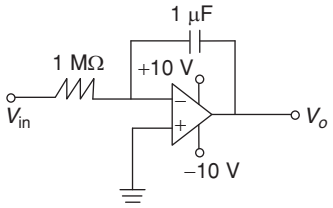


Figure 9 Input–output wave forms

Example 13: A unit positive step is applied at the input of the circuit shown in figure. After 20 seconds the output V_o will be



(A) +20 V (B) + 10 V
(C) –10 V (D) –20 V

Solution: (C)

$$V_o \frac{-1}{sCR} V_i = \frac{-1}{sCR} \cdot \frac{1}{s} (V_i \text{ is unit step})$$

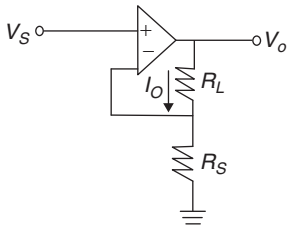
$$= \left[\frac{-1}{s^2 RC}\right]$$

$$V_o = \frac{-1}{RC} t \text{ after 20 seconds}$$

$$V_o = \frac{-1}{1 \times 1} \times 20 = -20 \text{ V}$$

But $\pm V_{sat} = \pm 10$ V so output is –10 V.

Example 14: The current I_o is

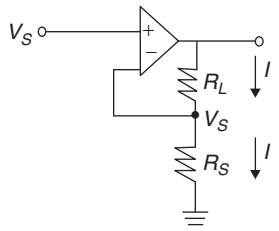


(A) $V_S \cdot \dfrac{R_L}{R_S(R_L + R_S)}$ (B) $\dfrac{V_S}{R_S}$

(C) $\dfrac{V_S}{R_L}$ (D) $V_S\left(\dfrac{1}{R_S} + \dfrac{1}{R_L}\right)$

Solution: (B)

No current flows into input $I_o = \frac{V_S}{R_S}$.



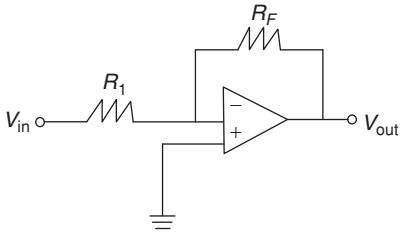
Example 15: The open loop voltage gain of an operational amplifier is 240. The noise level in the output without feedback is 100 mV; if negative feedback with $\beta = \frac{1}{60}$ is used, the noise level in the output will be

(A) 1.66 mV (B) 2.4 mV
(C) 4.0 mV (D) 20 mV

Solution: (D)

$$N_f = \frac{N}{1+\beta A} = \frac{100}{1+\frac{1}{60}\times 240} = \frac{100\text{ mV}}{5} = 20\text{ mV}$$

Example 16: For the operational amplifier circuit shown in the figure. What is the maximum possible value of R_1 if the voltage gain required is between –10 and –25? Upper limit of $R_F = 1\text{ M}\Omega$.



(A) ∞ (B) 1 MΩ
(C) 100 kΩ (D) 40 kΩ

Solution: (C)

$$A_v = \frac{V_o}{V_{in}} = \frac{-R_F}{R_1}$$

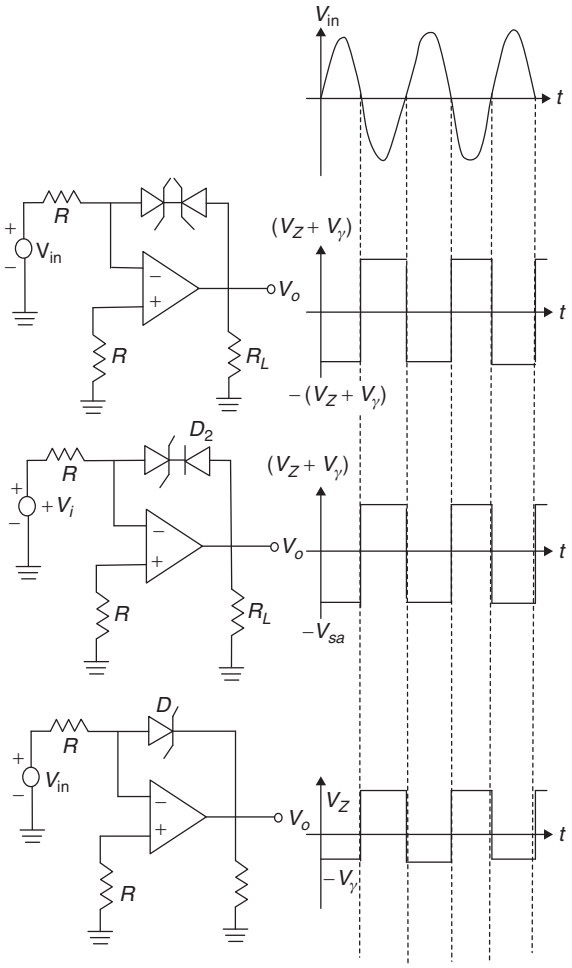
for $A_n = -10 = \frac{-R_F}{R_1} \Rightarrow R_1 = 100\text{ k}\Omega$

for $A_n = -25 = \frac{-R_F}{R_1} \Rightarrow R_1 = 40\text{ k}\Omega$

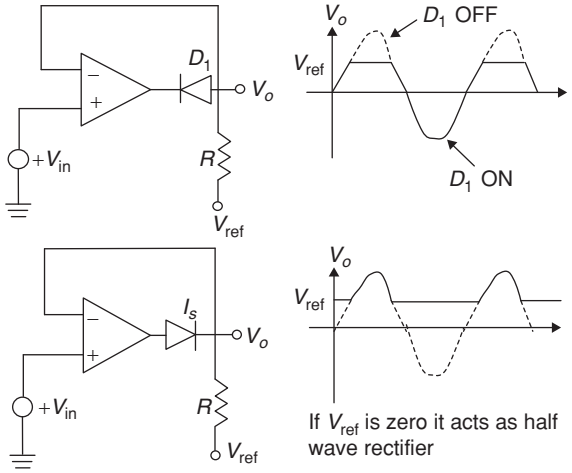
The maximum possible value of

$$R_1 = 100\text{ k}\Omega$$

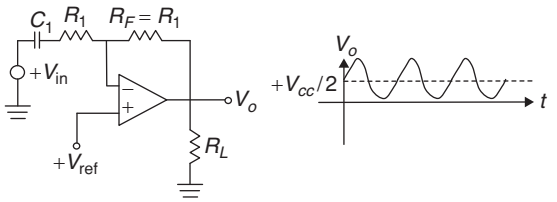
Voltage Limiters



Clippers



Clamper



Filters

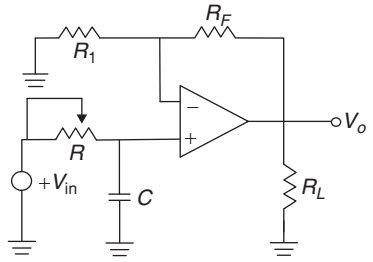
An electric filter is a frequency-selective circuit that passes a specified band of frequencies and blocks or attenuates signals of frequencies outside this band.

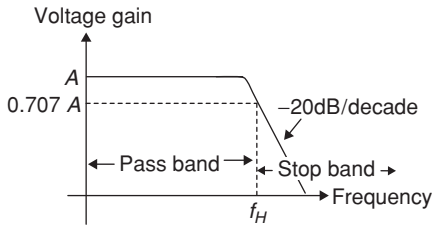
Filters can be classified in a number of ways

1. Analog or digital: Analog filters are designed to process analog signals, while digital filters process analog signals using digital techniques.
2. Passive or active: Elements used in passive filters are resistors, capacitors and inductors. Active filters on other hand employ transistors or op-amps in addition to passive components.
3. Audio or radio frequency filters: The type of element used dictates the operating frequency range of the filter, RC filters are used for audio or low frequency operation, where as LC or crystal filters are employed at RF or high frequencies, especially because of their high Q value.

Most Commonly Used Filters

First order low pass butterworth filter





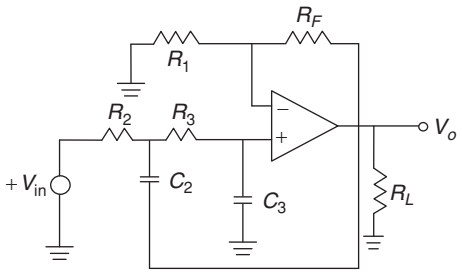
$$V_o = \left(1 + \frac{R_F}{R_1}\right) \frac{V_{in}}{1 + j\left(\frac{f}{f_H}\right)}$$

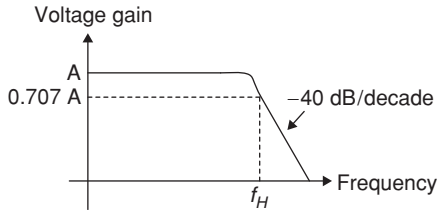
f = frequency of input signal

$$f_H = \frac{1}{2\pi RC} = \text{high cutoff frequency of filter}$$

$$\left|\frac{V_0}{V_{in}}\right| = \frac{A_F}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}}, \quad \phi = \tan^{-1}\left(\frac{f}{f_H}\right)$$

Second order low pass butterworth filter



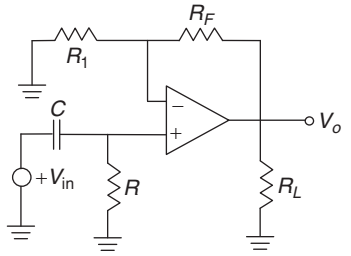


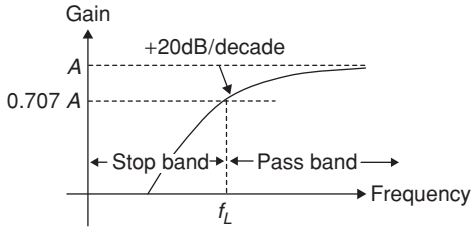
$$f_H = \frac{1}{2\pi\sqrt{R_2 R_3 C_2 C_3}}$$

$$\left|\frac{V_o}{V_{in}}\right| = \left|\frac{A_F}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}}\right|$$

$$A_F = 1 + \frac{R_F}{R_1} \text{ pass band gain}$$

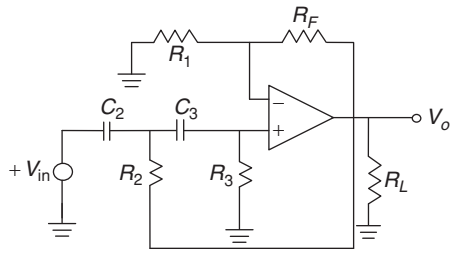
First order high pass butterworth filter

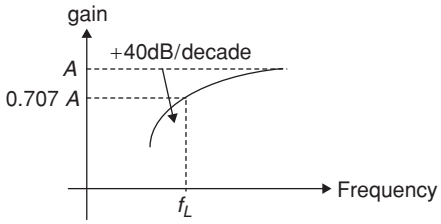




$$\left|\frac{V_0}{V_{in}}\right| = \frac{A_F\left(\frac{f}{f_L}\right)}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}}, \quad f_L = \frac{1}{2\pi RC}$$

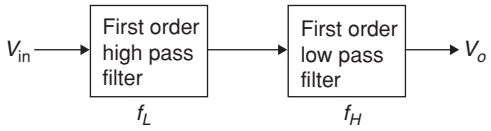
Second order high pass butterworth filter

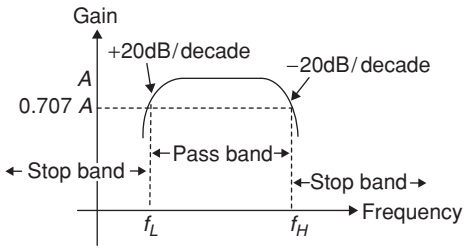




$$\left|\frac{V_o}{V_{in}}\right| = \frac{A_F}{\sqrt{1+\left(\frac{f_L}{f}\right)^4}}, \quad f_L = \frac{1}{2\pi\sqrt{R_2R_3C_2C_3}}$$

Wide band pass filter





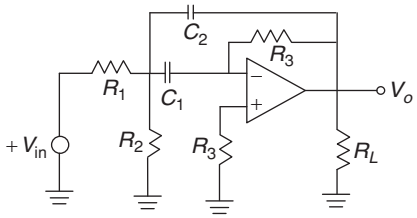
If figure of merit or Q factor $Q < 10$, then the filter is called as wide band pass filter

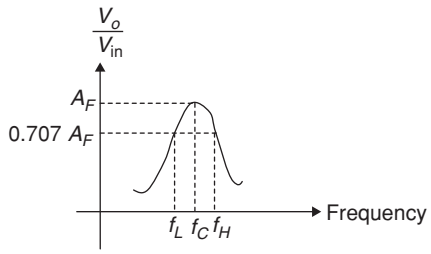
$$Q = \frac{f_0}{BW} = \frac{f_0}{f_H - f_L}$$

$$f_0 = \sqrt{f_H f_L}$$

f_H = high cutoff frequency
f_L = low cutoff frequency
f_0 = resonance frequency

Narrow band pass filter





if $C_1 = C_2 = C$

$$R_1 = \frac{Q}{2\pi f_c C A_F}, \; R_2 = \frac{Q}{2\pi f_c C(2Q^2 - A_F)},$$

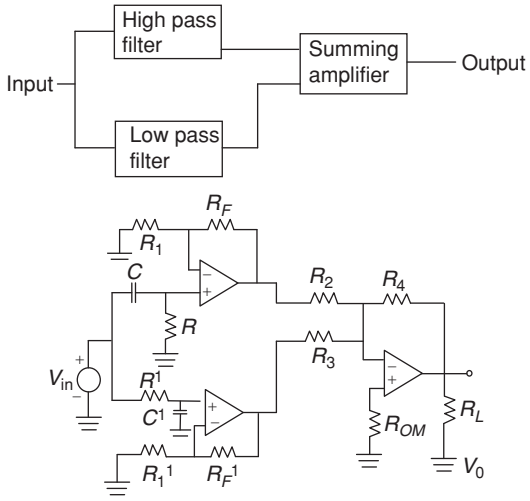
$$R_3 = \frac{Q}{\pi f_c}, \; A_F = \frac{R_3}{2R_1}, \; A_F = \text{gain at } f_c$$

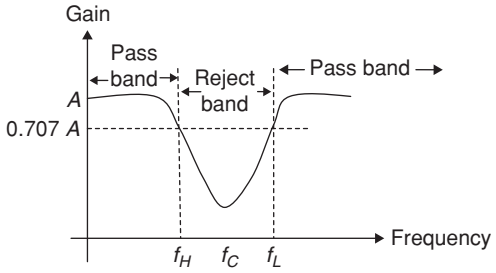
$$A_F < 2Q^2$$

Another advantage of the multiple feedback filter is, its centre frequency f_c, can be changed, to a new frequency f_c', without changing the gain or bandwidth, this is accomplished by changing R_2 to R_2' so that

$$R_2' = R_2\left(\frac{f_c}{f_c'}\right)^2$$

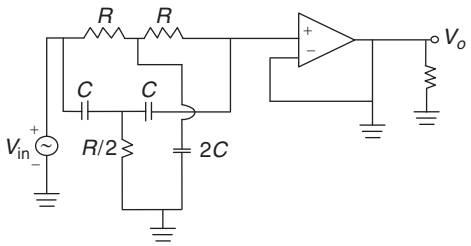
Wide band reject filter



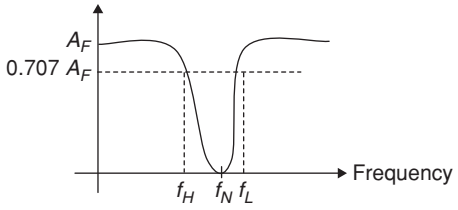


Wide band stop or wide band elimination filter, have lower ($Q < 10$); narrow band reject filter or notch filter will have higher Q.

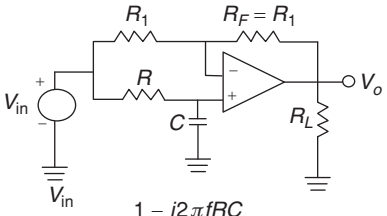
Narrow band reject filter



$$f_N = \frac{1}{2\pi RC}$$



All pass filter



$$V_0 = \frac{1 - j2\pi fRC}{1 + j2\pi fRC}$$

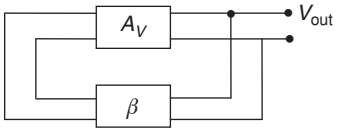
The phase shift between V_0 and V_{in} is a function of input frequency 'f'

$$\phi = -2\tan^{-1}\left(\frac{2\pi fRC}{1}\right)$$

OSCILLATORS

Theory of Sinusoidal Oscillators

To build a sinusoidal oscillator, we need to use an amplifier with positive feedback, to use the feedback signal in place of the input signal. If the feedback signal is large enough and has the correct phase, there will be an output signal even though there was no external input signal.



In any oscillator, the loop gain $A_v\beta$ is greater than 1, when the power is first turned on. A small starting voltage (thermal noise) is applied to the input terminals and output voltage builds up, after the output voltage reaches a certain level, $A_v\beta$ automatically decreases to 1, and the peak to peak output becomes constant.

Gain without feedback = A_v

Gain with feedback $A_f = \dfrac{A_v}{A_V\beta}$

If $A_v\beta = 1 \Rightarrow A_f = \infty$, which means we can get the output with no input. However, oscillator works on circuit transients.

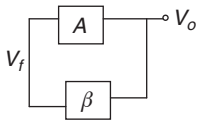
Bark–Hausen's criterion:

1. Magnitude of loop gain is unity
2. Phase angle of loop gain is 0° or 360°

Steps to solve an Oscillator problem:

1. Find β, $\beta = \dfrac{V_f}{V_0}$

   $\beta = \beta_{real} + \beta_{imaginary}$



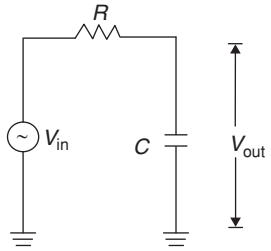
2. Find $A = \dfrac{V_0}{V_f}$

   Equate = $A\beta = 1$. Equate real parts and imaginary parts to get the frequency of oscillations and conditions for sustained oscillation.

The Wien Bridge Oscillator

The Wien bridge Oscillator is standard circuit for low to moderate frequencies in the range of 5 Hz to about 1 MHz.

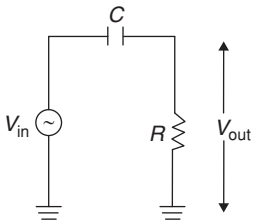
Lag circuit



$$\frac{V_{out}}{V_{in}} = \frac{X_c}{\sqrt{R^2 + X_c^2}}$$

The phase angle $\phi = -\tan^{-1}\left(\dfrac{R}{X_c}\right)$ negative sign in phase angle represents the output voltage lags the input voltage.

Lead circuit



$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + X_c^2}}$$

$$\phi = \tan^{-1}\left(\frac{X_c}{R}\right)$$

The phase angle is positive, means the output voltage leads the input voltage.

Lead-lag circuit

Wien bridge oscillator uses resonant feedback circuit called a lead-lag circuit.

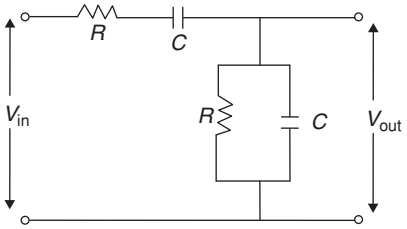


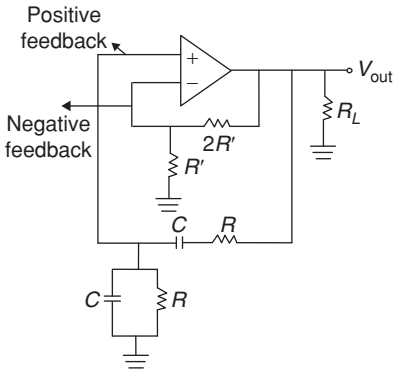
Figure 10 Lead-lag network

At very low frequencies the series capacitor appears, open to the input signal, and there is no output signal, at very high frequencies, the shunt capacitor looks shorted, and there is no output. In between these extremes, the output reaches a maximum value at resonant frequency f_r. at this frequency the feed back fraction B reaches a maximum value of 1/3.

Resonant frequency $f_r = \dfrac{1}{2\pi RC}$

$$\beta(\text{gain}) = \frac{1}{\sqrt{9 - \left(\frac{X_c}{R} - \frac{R}{X_C}\right)^2}},$$

$$\phi = \tan^{-1}\left(\frac{X_c/R - R/X_c}{3}\right)$$



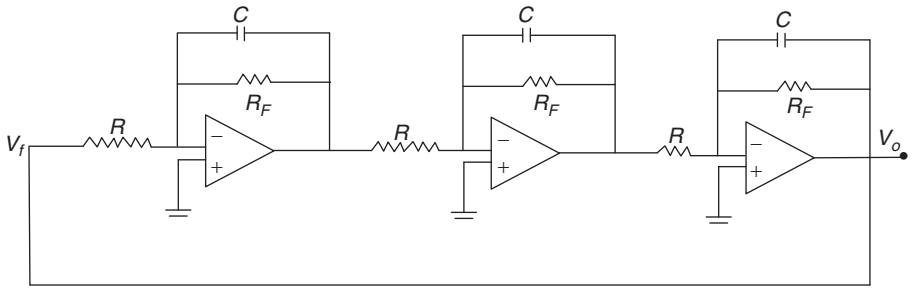
$$f_r = \frac{1}{2\pi RC}$$

Wien bridge oscillator uses both positive and negative feedback. The gain of the amplifier $(A_V) = 1 + \dfrac{2R'}{R'} = 3.$ at resonance ($X_c = R$), the gain of lead-lag network $\beta = \dfrac{1}{3}$. The loop gain $= A_V \beta = 1$.

The Wien bridge acts like a notch filter, a circuit with zero output at one particular frequencies, for a Wien bridge, the notch frequency equals to, $f_r = \dfrac{1}{2\pi RC}$.

Although it is superb at low frequencies, the Wien-bridge oscillator is not suited to high frequencies (well, above 1MH$_Z$). The main problem is the limited band width of op-amp.

RC–Phase Oscillator



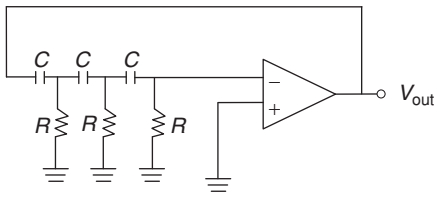
Properties:

1. Each section gives 120° phase shift
2. Frequency of oscillations required is

$$F = \frac{\sqrt{3}}{2\pi R_F C}$$

3. Condition for sustained oscillations: $R_F = 2R$
4. Minimum gain required is $|A| \geq 2$

Phase-Shift Oscillator



Phase-shift oscillator with three lead circuit in the feedback path, at some frequency the total phase shift of the three

lead circuit is 180°. The amplifier has an additional 180° of phase shift, because the signal drives the inverting input. As a result, the phase shift around the loop will be 360° equal to 0°.

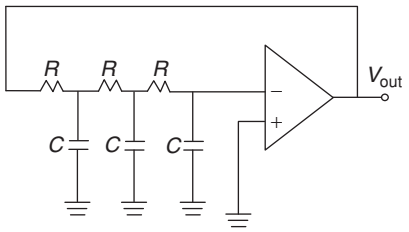
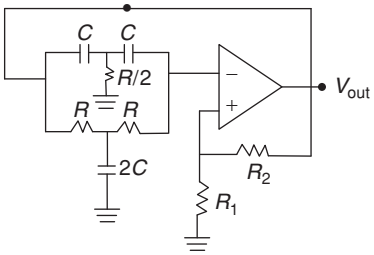


Figure 11 Phase shift oscillator with lag circuits

Properties:

1. Each RC section doesn't produce 60° phase shift due to loading effect. However 3 RC sections together produces a phase shift of 180°.
2. Frequency of oscillations $f = \dfrac{1}{2\pi RC\sqrt{6}}$.
3. Condition for sustained oscillations was $R_F = 29R_1$.
4. Minimum gain required was $|A| \geq 29$

Twin-T Oscillator



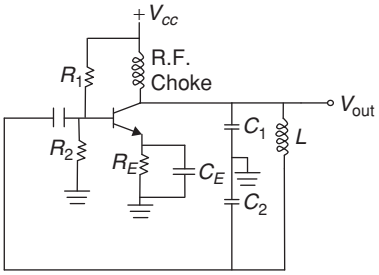
$$f_r = \frac{1}{2\pi RC}$$

The positive feedback to the non-inverting input is through a voltage divider. The negative feedback is through the twin T-filter.

To ensure that the oscillation frequency is close to the notch frequency, the voltage divider should have R_2 much larger than R_1, This forces the oscillator to operate at a frequency near the notch frequency.

The limitation is it works only at one frequency, it cannot be easily adjusted over a large frequency range.

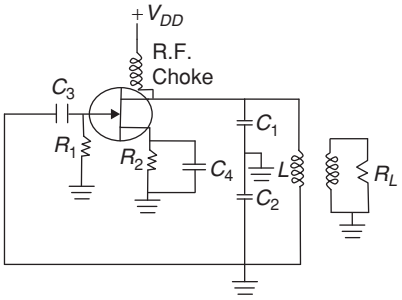
The Colpitts Oscillator



CE connection, Colpitts oscillator resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$C = \frac{C_1C_2}{C_1 + C_2},$$

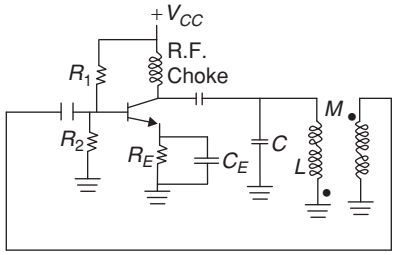
for $A_V B \geq 1$, $B = \dfrac{C_1}{C_2}$, $A_V = \dfrac{C_2}{C_1}$



JFET Oscillator has less loading effect on tank circuit. Resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$C = \frac{C_1C_2}{C_1 + C_2},\ B = \frac{C_1}{C_2},\ A_{V(\min)} = \frac{C_2}{C_1}$$

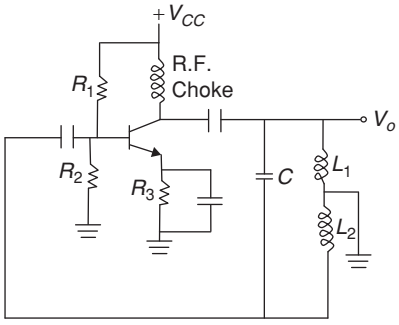
Armstrong Oscillator



$$f_r = \frac{1}{2\pi\sqrt{LC}},\ B = \frac{M}{L},\ A_{V(\text{mid})} = \frac{L}{M}$$

M-mutual inductance at output

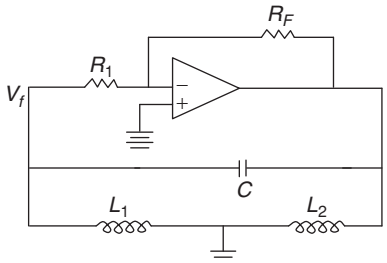
Hartley Oscillator



Hartley oscillator

$$f_r = \frac{1}{2\pi\sqrt{LC}},\ L = L_1 + L_2,\ B = \frac{L_2}{L_1},\ A_{V(\min)} = \frac{L_1}{L_2}$$

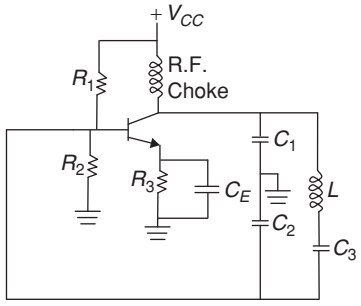
Hartley oscillator by using op-amp:



Frequency of Oscillation $f = \dfrac{1}{2\pi\sqrt{L_{eq} \cdot C}}$

$L_{eq} = L_1 + L_2 + 2M$ (series aiding)
$= L_1 + L_2 - 2M$ (series opposing)

Clapp oscillator

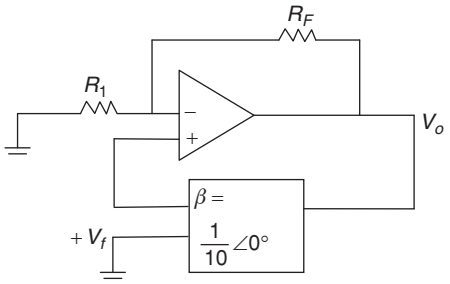


$$f_r = \frac{1}{2\pi\sqrt{LC}},$$

$$C = \frac{1}{1/c_1 + 1/c_2 + 1/c_3},\ B = \frac{C_1}{C_2}$$

$$A_{V(\text{mid})} = C_2/C_1$$

Example 16: Relation between R_1 and R_2 for sustained Oscillations if $\beta = \dfrac{1}{10}\angle 0°$ for the following circuits.



(A) $R_F = 4R_1$ (B) $R_F = 10R_1$
(C) $R_F = 2R_1$ (D) $R_F = 9R_1$

Solution: (D)

$$A = 1 + \frac{R_F}{R_1}$$

$$A\beta = 1$$

$$\Rightarrow \left[1 + \frac{R_F}{R_1}\right]\frac{1}{10}\angle 0° = 1\angle 0°$$

$$1 + \frac{R_F}{R_1} = 10$$

$$R_F = 9R_1$$

Example 17: A Colpitts oscillator has a coil with an inductance of 50 μH and is tuned by a capacitor 400 pF across amplifier input and 200 pF across the output. Then the frequency of oscillation and the minimum gain for maintaining oscillations?

(A) 1.95 MHz, 3 (B) 1.95 MHz, 2
(C) 2.9 MHz, 3 (D) 2.9 MHz, 2

Solution: (B)
For Colpitts oscillator

$$f_0 = \frac{1}{2\pi\sqrt{LC_{eq}}},\ C_{eq} = \frac{C_1C_2}{C_1 + C_2}$$

$$C_{eq} = \frac{400 \times 200}{600} = 133.33 \text{ pF}$$

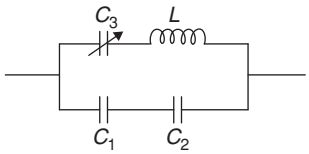
$$f_0 = \frac{1}{2\pi\sqrt{50 \times 133.33 \times 10^{18}}} = 1.95 \text{ MHz}$$

For maintaining oscillations, $A\beta \Rightarrow 1$

$$A_{\text{loop}} = A_{v0} \cdot \frac{C_2}{C_1} = 1$$

$$A_{v0} = \frac{C_1}{C_2} = \frac{400}{200} = 2$$

Example 18: A clapp oscillator has the following circuit components C_1 = 5000 pF, C_2 = 500 pF, L = 50 μH and C_3 is a 50 to 250 pF variable capacitor. Find out the tuning frequency range?



(A) 1.98 – 10.34 MHz (B) 2.34 – 11.56 MHz
(C) 3.35 – 12.5 MHz (D) 5.34 – 15.4 MHz

Solution: (C)

$$f_0 = \frac{1}{2\pi\sqrt{LC_{eq}}},\ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_{eq} = \frac{5000 \times 500 \times 50}{5000 \times 500 + 5000 \times 50 + 500 \times 50}$$

$$= 45.045 \text{ pF for } C_3 = 50 \text{ pF}$$

$$f_0 = \frac{1}{2\pi\sqrt{50 \times 45.045 \times 10^{-18}}} = 3.35 \text{ MHz}$$

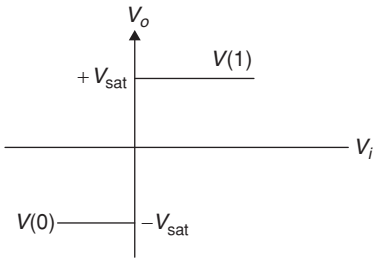
$$C_{eq} = 161.29 \text{ pF for } C_3 = 250 \text{ pF}$$

Then $f_0 = 12.5$ MHz

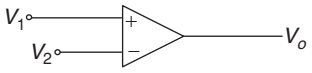
Waveform Generators and Wave Shaping Circuits

Comparators

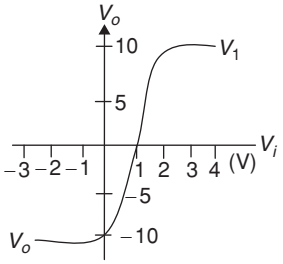
An analog comparator or simply comparator has two input voltages V_1 and V_2 and one output voltage V_o. Often one of the Input (V_2) is a constant reference voltage V_R, and the other is a time varying signal.



1. Transfer characteristics of ideal comparator

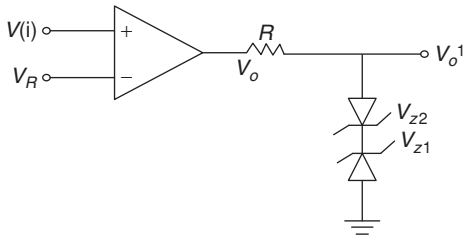


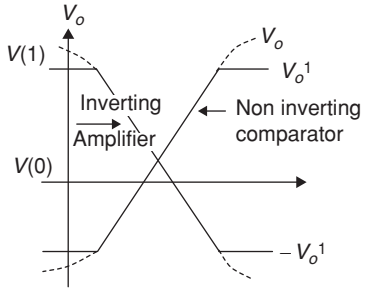
2. Op-amp operated in open loop becomes a comparator



3. Transfer characteristics of practical comparator.

   The Input is compared with the reference and the output is digitized into one of two states: $V_o = V(0)$ when $v(i) < V_R$ and $V_o = V(1)$ when $V_i > V_R$.





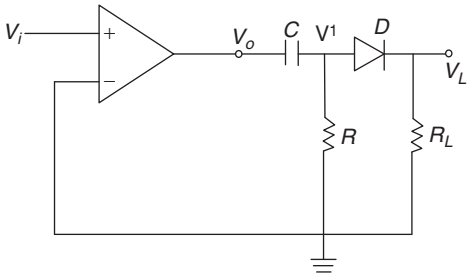
Practical comparator and transfer characteristics:

To obtain limiting output voltages which are independent of the power supply voltages, a Resistor R_1 and two back to back Zener diodes are added to clamp the output of comparator.

One more advantage of adding the Zener diodes is that the limiting may be much sharper for V_0^1 than for V_0 and disadvantage is poor transient response of the avalanche diode.

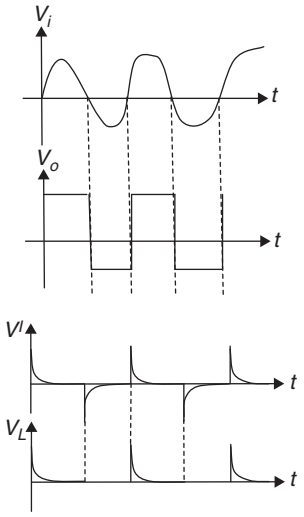
Square wave generation from sinusoid

The comparator performs highly non-linear wave shaping because the output bears no resemblance to the input wave form.



A zero crossing detector converts the sinusoid input (V_i) into a square wave, (V_0)

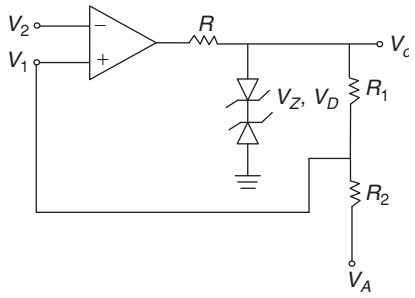
The pulse wave forms V' and V_L result from V_0 being fed into a short time-constant RC Circuit in cascade with a diode clipper.



Here a sinusoid has been converted into either a square wave or a pulse train.

Spurious positive and negative voltage spikes called noise super imposed on the input signal in the neighbour hood of the amplitude (V_R) may cause the output to chatter (change from one binary voltage to another) several times before settling down to the correct level. This difficulty can be avoided if positive feedback or regeneration is added to a comparator.

Regenerative Comparator (Schmitt Trigger)



Schmitt trigger

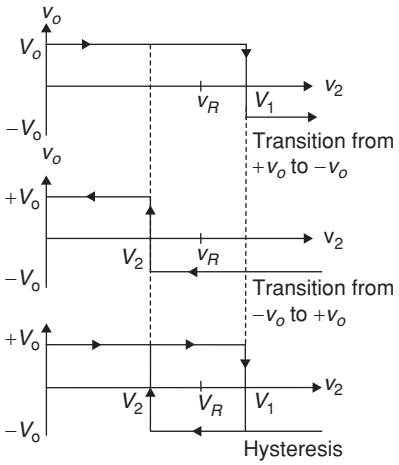
$$V_o = V_z + V_D,$$

assume $v_2 < v_1$, so that $V_o = +V_o$, and the voltage at the non inverting terminal (v_1) is

$$V_1 = V_A + \frac{R_2}{R_1 + R_2}(V_o - V_A)$$

If v_2 is now increased, the V_o remains constant at V_o, and $v_1 = V_1$ = constant until $v_2 = V_1$, at this threshold critical, or triggering voltage the output regeneratively switches to $v_o = -V_0$, and remains at this value as long as $v_2 > V_1$.

The voltage at the non inverting terminal (v_1) for $v_2 > V_1$ is $v_1 = V_A - \frac{R_2}{R_1 + R_2}(V_o + V_A) = V_2$.



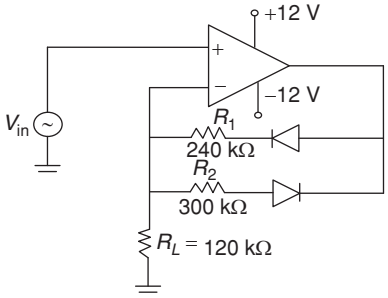
If we decrease v_2, the output remains at $-V_0$, until v_2 equals the voltage at positive terminal, or until $v_2 = V_2$. At this stage, regenerative transition takes place and the output returns to $+V_0$ almost instantly. One of the most important uses made of the Schmitt trigger is to convert a slowly varying input voltage into an output wave from displaying an abrut, almost discontinuous change.

Note: $V_2 < V_1$, and the difference between these two values is called hysteresis

$$V_H = V_1 - V_2 = \frac{2R_2V_0}{R_1 + R_2}$$

UTP: (V_1) Output changes form $+V_{sat}$ to $-V_{sat}$
LTP: (V_2) Output changes form $-V_{sat}$ to $+V_{sat}$

Example 20: Find UTP, LTP and Hysteresis width for the following circuit assuming diodes are ideal.



Solution:

UTP: $V_0 = +V_{sat}$; Upper diode is 'ON'.

$$V_{UTP} = +V_{sat} \cdot \frac{R_L}{R_1 + R_L}$$

$$V_{UTP} = \frac{12(120\text{ K})}{(360\text{ K})} = +4\text{ V}$$

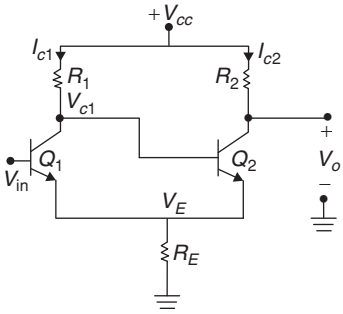
LTP: $V_0 = -V_{sat}$; Lower diode is 'ON'

$$V_{LTP} = \frac{V_{sat} * R_L}{R_L + R_2}$$

$$V_{LTP} = \frac{-12(120\text{ K})}{(420\text{ K})} = -3.4\text{ V}$$

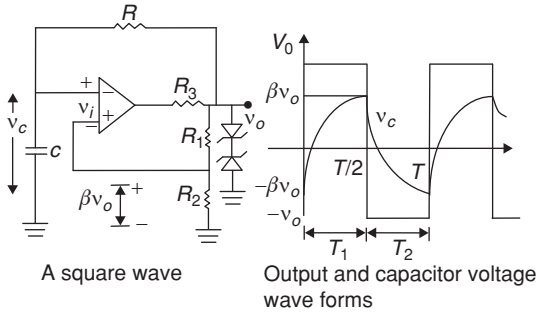
Hysteresis width = $V_{UTP} - V_{LTP} = 7.4$ V

Emitter coupled schmitt trigger



The basic Emitter coupled pair can be converted into a regenerative comparator. The resistances R_1 and R_2 are unequal ($R_1 > R_2$) and hence Q_1 and Q_2 have different currents when saturated. This difference results in the hysteresis as different input voltages are required to saturate and cutoff Q_1 and Q_2.

Square Wave Generator (Astable Multivibrator)

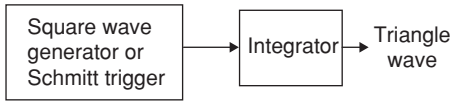


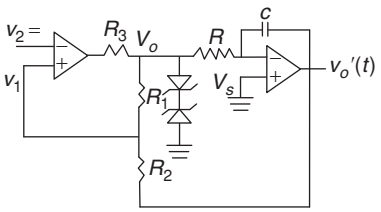
The inverting Schmitt trigger can be used to obtain a free running square wave generator by connecting an *RC* network between output and the inverting input.

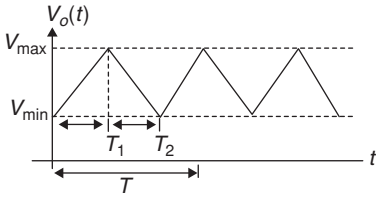
$$V_i = v_c - \beta v_0 = v_c - \frac{R_2}{R_1 + R_2} v_0$$

$$T = 2RC\ell_n\left(1+\frac{2R_1}{R_2}\right),\ f_0 = \frac{1}{2RC} \text{ for } R_2 = 1.16R_1$$

Triangle Wave Generator







Output wave form, $T_1 = T_2$ if $V_s = 0$,

$$V_{max} = V_R\left(\frac{R_1 + R_2}{R_1}\right) + V_0\frac{R_2}{R_1}$$

$$V_{min} = V_R\left(\frac{R_1 + R_2}{R_1}\right) - V_0\frac{R_2}{R_1}$$

Peak to peak swing

$$V_{max} - V_{min} = 2V_0\frac{R_2}{R_1}$$

$$T = \frac{4R_2RC}{R_1},\ f = \frac{1}{T}$$

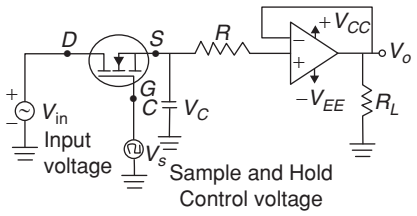
The frequency is independent of v_0. The maximum frequency is limited by either the slew rate of the integrator or its maximum output current, which determines the charging rate of C.

If unequal sweep intervals $T_1 \neq T_2$ is required apply $V_s \neq 0$ for integrator.

Then $\frac{T_1}{T_2} = \frac{V_0 - V_s}{V_0 + V_s}$ duty cycle $\frac{T_1}{T_0} = \delta = \frac{1}{2}\left(1 - \frac{V_s}{V_0}\right)$

Sample and Hold Circuit

The sample and hold circuit as it name implies, samples an input signal and holds on to its last sampled value until the input is sample again.



The Analog signal V_{in} to be sampled is applied to the drain, and sample and hold central voltage (V_s) is applied to the gate of E-MOSFET.

During positive portion of V_s, the MOSFET conducts and acts as a closed switch. This allows the input voltage to charge capacitor *C*, i.e., input voltage appears across *C* and in turn at the output.

When V_s is zero, the E-MOSFET is off (non-conductive) and acts as an open switch, the only discharge path for *C* is through op-amp. However the input resistance of op-amp is very high, so voltage across *C* is retained.

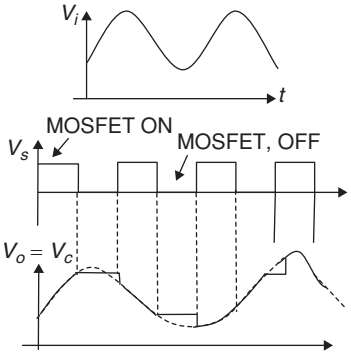
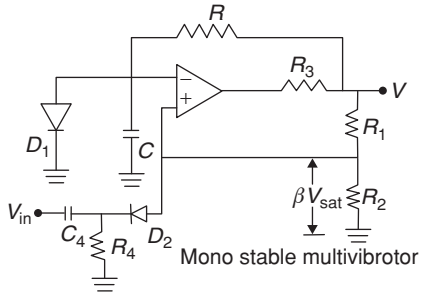


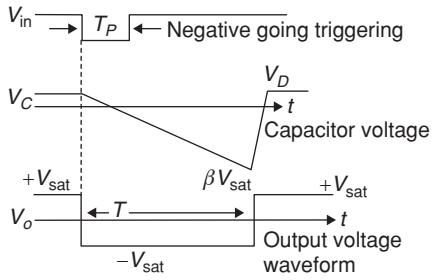
Figure 12 Input and output waveforms

To obtain close approximation of the input wave form, the frequency of the sample and hold control voltage must be significantly higher than that of the input. The sample and hold circuit is commonly used in digital interfacing and communications.

Monostable Multivibrator

Monostable multivibrator has one stable state and other is quasi-stable state. The circuit is useful for generating single output pulse of adjustable time duration, in response to a triggering signal.





V_D – diode forward voltage

$$T = RC\ell n\frac{(1+V_D/V_{sat})}{1-\beta}$$

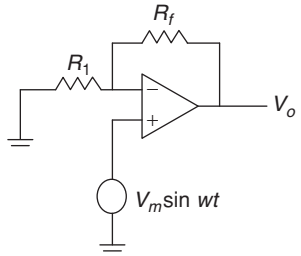
$$\beta = \frac{R_2}{R_1+R_2}$$

if $V_{sat} >> V_D$, $R_1 = R_2$ So that $\beta = 0.5$, then $T = 0.69\ RC$.

Let us assume that in the stable state, the output is at $+V_{sat}$. The diode D_1 conducts and V_c the voltage across capacitor gets clamped to 0.7 V.

The voltage at (+) input terminal through R_1, R_2 potentiometer divider is $+\beta V_{sat}$. Now if a negative trigger of magnitude V_1 is applied to the (+) terminal. So that the effective signal at this terminal is less than 0.7 V. i.e., ($[\beta\ V_{sat} + (-V_1)$ < 0.7 V). The output of op-amp will switch from $+V_{sat}$ to $-V_{sat}$ through the resistance R. The voltage at the (+) input terminal is now $-\beta V_{sat}$, when capacitor C voltage becomes just slightly more negative than this voltage. The output of op-amp switches back to $+V_{sat}$. The capacitor C now starts charging to $+V_{sat}$ through R until V_c clamped to 0.7 V.

Example 21:



The op-amp has a slew rate of 1 V/μ second. The unity gain frequency is 1 MHz and the output saturation levels are of ±14 V. What is the maximum output frequency at which undistorted output can be obtained?

(A) 15.4 kHz (B) 13.8 kHz
(C) 12.39 kHz (D) 11.36 kHz

Solution: (D)

$$\text{Slew rate} = \left.\frac{dV}{dt}\right|_{max} = \omega V_m$$

$$2\pi f_{max} V_m = \text{Slew rate}$$

$$f_{max} = \frac{\text{slew rate}}{2\pi V_m}$$

Here slew rate = 1 V/μs

$V_m = 14$ V

$$f_{max} = \frac{1/10^{-6}}{2\pi \times 14} = 11.36 \text{ kHz}$$

Example 22: The values of R_f, R_1 for a non inverting amplifier to provide maximum low frequency gain for a bandwidth of 100 kHz are

(A) 9 kΩ, 1 kΩ
(B) 19 kΩ, 1 kΩ
(C) 29 kΩ, 1 kΩ
(D) Cannot be determined

Solution: (A)

For 1 MHz bandwidth, gain = 1 (unity)

$$\text{For 0.1 MHz, gain } = \frac{1}{0.1} = 10$$

$$\text{Closed loop gain } = 10 = 1 + \frac{R_f}{R_1}$$

$\Rightarrow R_1 = 1$ kΩ, $R_f = 9$ kΩ.

Example 23: The maximum peak to peak amplitude of the input signal of frequency 100 kHz for undistorted output?

(A) 15.9 V (B) 3.18 V
(C) 2.12 V (D) Not possible

Solution: (B)

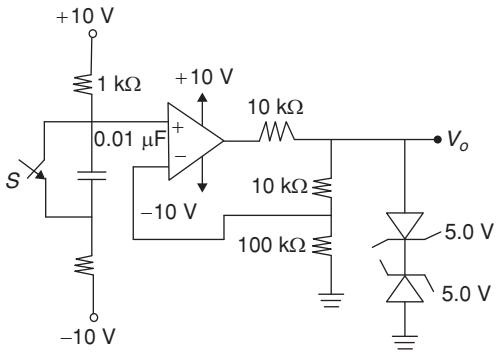
Slew rate = $2\pi f_{max} \cdot V_m$

$$\text{Peak amplitude of input } = \frac{\text{Slew rate}}{2\pi f_{max}}$$

$$= \frac{1/10^{-6}}{2\pi \times 100 \times 10^3} = 1.59 \text{ V}$$

Peak to peak (P–P) = 2 × 1.59 V = 3.18 V

Example 24: The switch S in the circuit was initially closed and opened at time t = 0, you may neglect the zener diode forward voltage drops. What is the behaviour of V_{out} for $t > 0$?



(A) It makes a transition from –10 V to +10 V at t = 12.98 μS.
(B) It makes a transition from –5 V to +5 V at t = 2.57 μS.

(C) It makes a transition from + 5 V to –5 V at $t = 2.57$ μS.
(D) It makes a transition from + 10 V to –10 V at $t = 12.95$ μS.

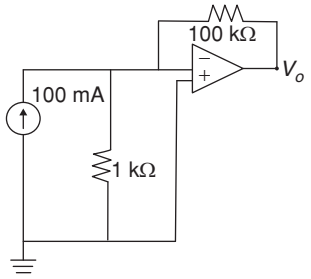
Solution: (C)
It is a limiter circuit.
It makes a transition from +5 V to –5 V

$$20(1-e^{-t/Rc}) = 5\times\frac{100}{110}$$

= Voltage Across 100 kΩ

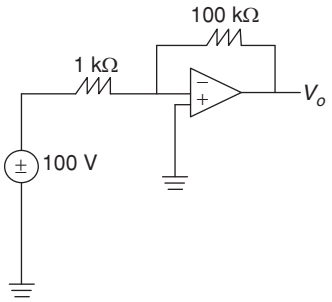
$RC = 0.01 \times 10^{-6} \times 10^{3}$
$t = 2.57$ μS

Example 25: In the circuit shown, the output voltage is



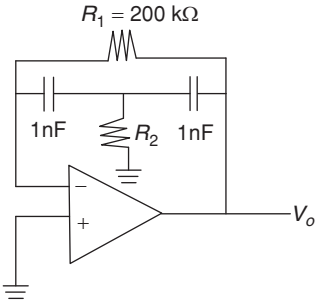
(A) – 9 V (B) – 10 V
(C) + 10 V (D) + 9 V

Solution: (B)
We can observe the current source 100 mA, in parallel with resistance 1 kΩ as



So $V_0 = \frac{-100}{1000}\times 100 = -10$ V

Example 26: In the active filter circuit, if $Q = 1$, a pair of poles will be realized with ω_o, equal to



(A) 1000 rad/s (B) 100 rad/s
(C) 10 rad/s (D) 1 rad/s

Solution: (A)

$$\frac{V_o(s)}{V_i} = \frac{\frac{s}{R_2C_1}}{s^2 + \frac{C_1+C_2}{R_1C_1C_2}s + \frac{1}{R_1R_2C_1C_2}}$$

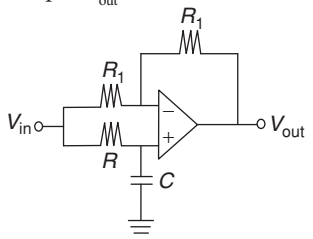
$$A_v(s) = \frac{(\omega_o/Q)A_o s}{s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2}$$

By comparing, we have $\frac{\omega_o}{Q} = \frac{C_1+C_2}{R_1C_1C_2}$

$$\omega_o = Q\left(\frac{C_1+C_2}{R_1C_1C_2}\right) = 1\times\frac{(1+1)\times10^{-9}}{200\times10^{3}\times10^{-18}}$$

$= 1000$ rad/s.

Example 27: For the circuit with an ideal op-amp, the maximum phase shift of the output V_{out} with reference to input V_{in} is



(A) –90° (B) 0°
(C) +90° (D) ±180°

Solution: (D)

$$V_+ = \frac{V_{in}}{1+j\omega RC}$$

$V_- = V_+$ (as its ideal op-amp)

$$\frac{V_{in}-V_-}{R_1} = \frac{V_- - V_o}{R_1}$$

$$V_o = 2V_- - V_{in} = 2V_+ - V_{in} = \left[\frac{2}{1+j\omega RC} - 1\right]V_{in} = \frac{1-j\omega RC}{1+j\omega RC}\cdot V_{in}$$

$\angle(V_o/V_i) = -2\tan^{-1}\omega RC$
for $-90° \le 0 \le +90°$, phase shift $\angle(V_o/V_i) = \pm 180°$

Power Supplies

The function of voltage regulator is to provide, astable DC voltage for powering other electronic circuits.

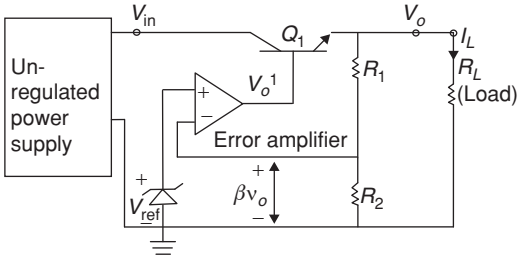


Figure 13 A regulated power supply

The circuit consists of four parts

1. Reference voltage Circuit
2. Error amplifier
3. Series pass transistor(Q_1)
4. Feedback network.

The power transistor Q_1 is in series with the unregulated DC voltage V_{in} and regulated output V_0. So it absorbs the difference between these two voltages, for fluctuations in voltages. Q_1, is connected as emitter follower and it provides sufficient current gain to drive load. The output voltage is sampled by $R_1 - R_2$ divider and feedback to the (–) input terminal, of the error-amplifier, the sampled voltage is compared with the reference voltage V_{ref} (obtained by a zener diode). The output of error amplifier drives the Q_1 transistor.

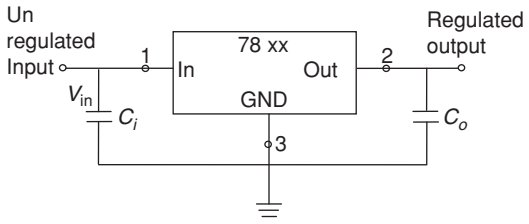
$$\beta = \frac{R_2}{R_1 + R_2}$$

This in turn, reduces the output voltage V_o, of the diff amp $(v_0^1 = \beta v_0)$, so V_0 follows v_o^1, so V_o also reduces. Hence, increase in V_0 is nullified, and reduction in output voltage also regulated.

IC Voltage Regulators

78XX series are three terminal, positive fixed voltage regulator available in 05, 06, 08, 12, 15, 18 and 24 V.

The last (XX) two numbers indicate output voltages.



The three terminal regulators have the following limitations:

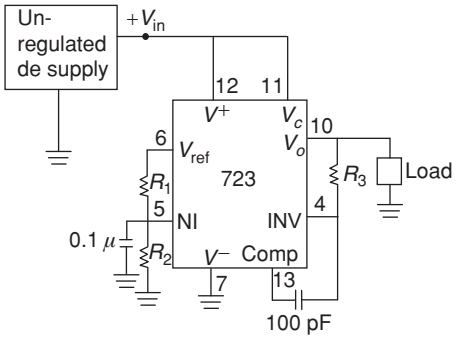
1. No short circuit protection
2. Output voltage (positive or negative) is fixed

These limitations have been overcome in 723 general purpose regulator.

Advantages: Adjustable to wide range of both + ve and –ve voltages, it can be boosted to provide 5A or more current

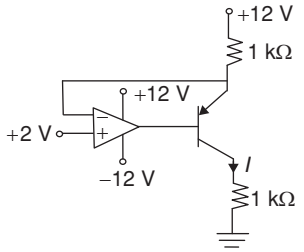
Limitations:

- No built in thermal protection
- No short circuit current limits.



Example 28: An op-amp circuit is shown in the figure, the current 'I' is

$$V_o = V_{ref}\frac{R_2}{R_1 + R_2}$$



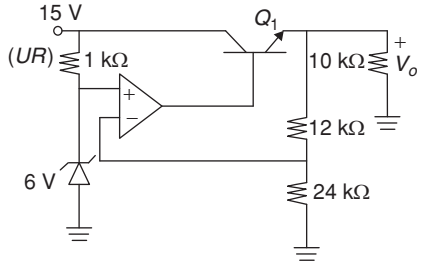
(A) 0 (B) 5 mA
(C) 10 mA (D) 12 mA

Solution: (C)

The current through emitter $I_E = \frac{(12-2)\text{ V}}{1\text{ k}\Omega} = 10\text{ mA}$

The current through collector = $I_C \approx I_E \approx 10$ mA

Example 29: A regulated power supply, shown in the figure below, has an unregulated input (UR) of 15 V, and generates a regulated output V_{out}. Use the component values shown, the power dissipation across the transistor Q_1 in the figure is



(A) 4.8 W (B) 5.0 W
(C) 5.4 W (D) 6.0 W

Solution: (C)

The zener is in break down region

$$V_+ = V_Z = +6\text{ V} = V_{in}$$

$$V_o = V_{in}\left(1 + \frac{RF}{R_1}\right)$$

$$V_{out} = V_o = 6\left(1 + \frac{12\text{ k}}{24\text{ k}}\right) = 9\text{ V}.$$

The current in 10 Ω resistor is very high when compared with current in 12 kΩ + 24 kΩ resistor, so it is negligible

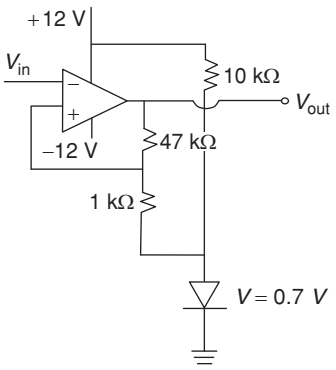
$$I_C \approx I_E \approx \frac{V_{out}}{R_L} = \frac{9}{10} = 0.9\text{A}$$

$$V_{CE} = 15 - 9 = 6\text{ V}$$

The power dissipated in transistor is P

$$= V_{CE} \times I_C = 6 \times 0.9 = 5.4\text{ W}$$

Example 30: The Schmitt trigger circuit is shown in the circuit, if $\pm V_{sat} = \pm 10$ V, the tripping point for the increasing input voltage will be?



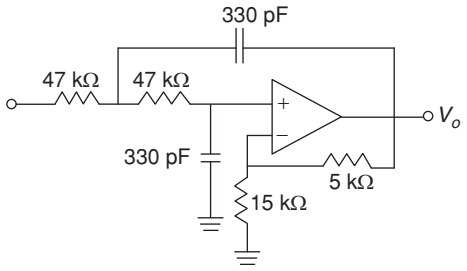
(A) 1 V
(B) 0 · 893 V
(C) 0 · 477 V
(D) 0 · 416 V

Solution: (B)

$$\text{Tripping point} = \frac{R_1}{R_1+R_2}\cdot V_R + \frac{R_2}{R_1+R_2}\cdot V_0$$

$$= \frac{1}{48}\times 10.7 + \frac{47}{48}\times 0.7 = 0.893 \text{ V}$$

Example 31: The pole frequency and Q of the filter shown in figure is



(A) 0.6 and 10.3 KHz
(B) 0.5 and 10.3 KHz
(C) 0.5 and 5.15 KHz
(D) 0.6 and 5.15 KHz

Solution: (A)

$$A_v = \frac{5}{15}+1 = 1.33$$

$$Q = \frac{1}{3-A_v} = \frac{1}{3-1.33} = 0.6 \text{ (Same for LPF and HPF)}$$

$$f_P = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}} = \frac{1}{2\pi RC} = 10.3 \text{ KHz for Butterworth filter.}$$

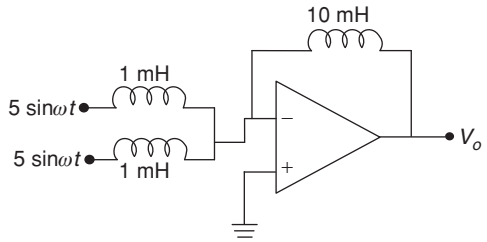
Pole frequency is same as cutoff.

EXERCISES

Practice Problems I

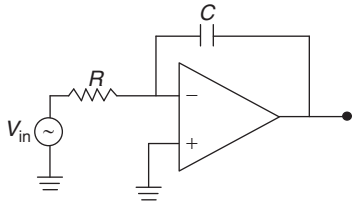
Directions for questions 1 to 41: Select the correct alternative from the given choices.

1. Calculate the output V_o of the figure given



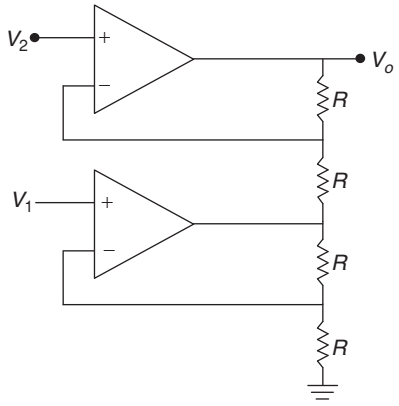
(A) 7 sinωt V
(B) 3 sinω V
(C) −70 sinωt V
(D) −10 sinωt V

2. An op-amp integrator with R = 10 MΩ and C = 0.1 μF is shown below. An input of 4sin100t V is applied. Calculate the time in which output V_o reach 20 mV.



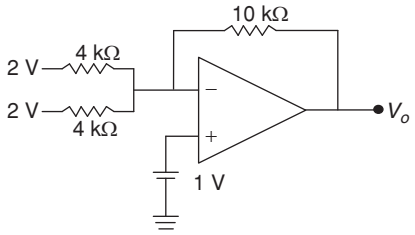
(A) 1.047 sec
(B) 10.47 sec
(C) 10.47μsec
(D) 10.47msec

3. What is the output V_o in the given figure?



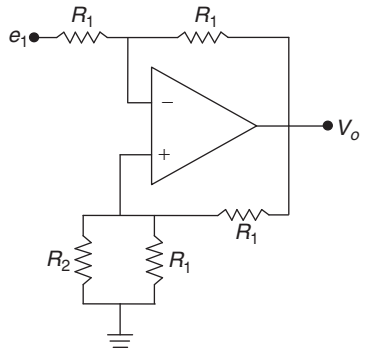
(A) $V_1 - V_2$
(B) $V_1 + V_2$
(C) $2(V_1 - V_2)$
(D) $2(V_2 - V_1)$

4. What is the output V_o of the given figure?



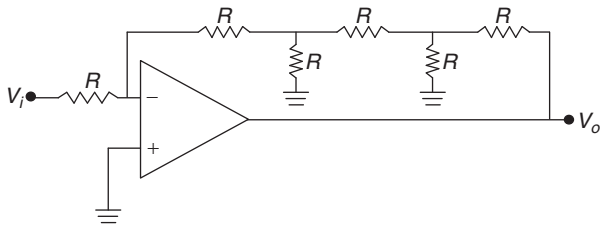
(A) 10 V
(B) –6 V
(C) –4 V
(D) –10 V

5. What is the current in R_2, given that $R_1 = 1$ kΩ, $R_2 = 2$ kΩ and $e_1 = 1$ volt?



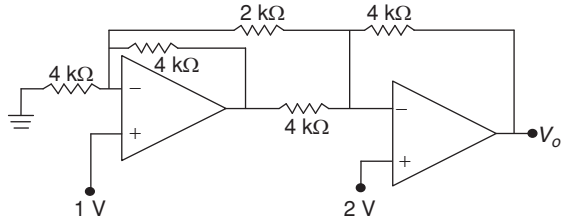
(A) –0.5 mA (B) –1 mA
(C) –2 mA (D) None

6. Calculate $\frac{V_o}{V_i}$ of the given figure.



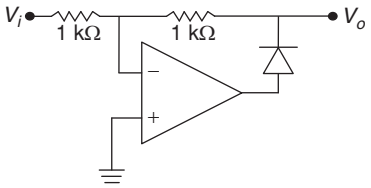
(A) 4 (B) –8
(C) –4 (D) None

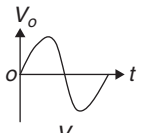
7. What is the output V_o?

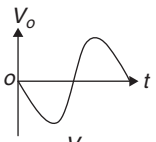


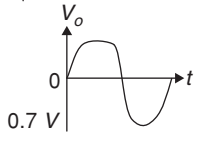
(A) 2 V (B) 4 V
(C) 6 V (D) 0 V

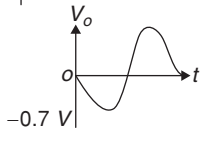
8. What is the output wave form for the input sinusoidal for the circuit given assuming the forward voltage drop of diode is 0.7 V?



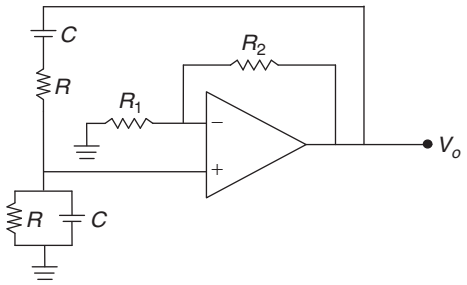
(A) 

(B) 

(C) 

(D) 

9. A Wien bridge oscillator is shown below with $R = 2$ kΩ, $C = 0.16$ μF.



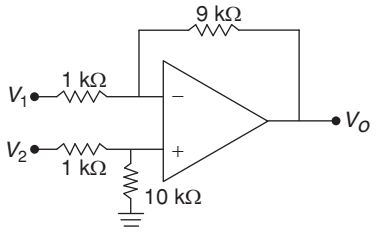
If $R_1 = 0.5$ kΩ, the value of R_2 to sustained oscillations is

(A) 0.5 kΩ (B) 1 kΩ
(C) 2 kΩ (D) 1.5 kΩ

10. The frequency of output is

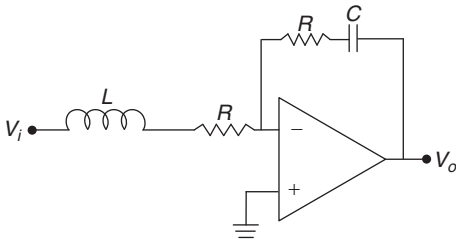
(A) 50 Hz (B) 1 KHz
(C) 5 KHz (D) 500 Hz

11. The CMRR of the given figure is



(A) 900 (B) 9
(C) 90 (D) Infinite

12. Given op-amp is ideal $R = \sqrt{\frac{L}{C}}$. What is the phase angle between V_o and V_i at $\omega = \frac{1}{\sqrt{LC}}$?

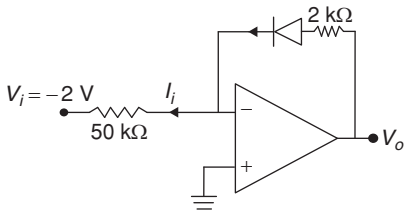


(A) 180° (B) 270°
(C) 0° (D) 90°

13. The op-amp shown is ideal. The V–I characteristic of diode is given by

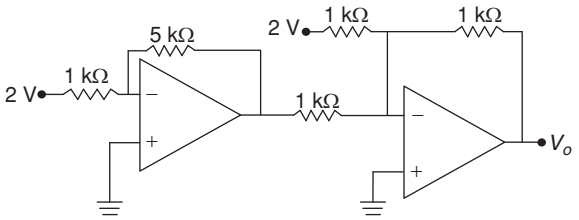
$$I = I_o\left[e^{\frac{V_d}{nV_T}} - 1\right] \text{ where}$$

$V_T = 26$ mV, $I_o = 10$ μA and V_d = Voltage across diode. The output V_o for $V_i = -2$ V is _____



(A) 0.122 V (B) 0.6 V
(C) 1.2 V (D) None

14. What is the output V_o in the given figure?



(A) 8 V (B) −8 V
(C) 10 V (D) −10 V

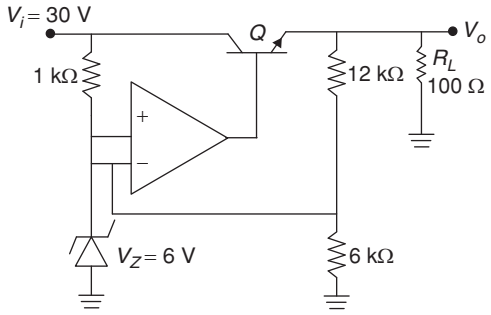
15. An op-amp has open loop gain 'A' of 2×10^3, the input impedance Z_i = 100 kΩ and output impedance Z_o = 2 kΩ. The op-amp is used in non-inverting mode with closed loop gain 100. The effective input and output impedances are _____

(A) $Z_{if} = 2.1$ MΩ and $Z_{of} = 42$ kΩ
(B) $Z_{if} = 4.8$ kΩ and $Z_{of} = 95$ Ω
(C) $Z_{if} = 2.1$ MΩ and $Z_{of} = 95$ Ω
(D) $Z_{if} = 4.8$ kΩ and $Z_{of} = 42$ kΩ

16. A differential amplifier has a typical common mode gain 30 db and CMRR of 70 dB. What is the output voltage V_o when input voltages are 0.14 mV and 0.16 mV

(A) 2.047 V (B) 1.99 V
(C) 4.74 V (D) 4.4 V

17. An op-amp voltage regulator is given below



The regulated output V_o is

(A) 10 V (B) 20 V
(C) 9 V (D) 18 V

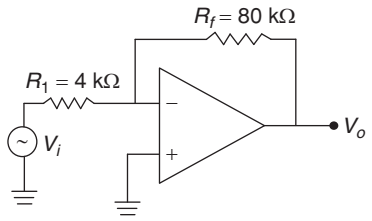
18. The power dissipation by transistor is

(A) 1.08 W (B) 2.16 W
(C) 4 W (D) 3.2 W

19. For a non-inverting amplifier R_1 = 20 kΩ, R_f = 100 kΩ, is used. A peak to peak voltage of 2 V is applied at input. Given slew rate = 0.5 V/μs, what is maximum operating frequency

(A) 6.66 kHz (B) 13.33 kHz
(C) 16 kHz (D) 8 kHz

20. An op-amp circuit given below



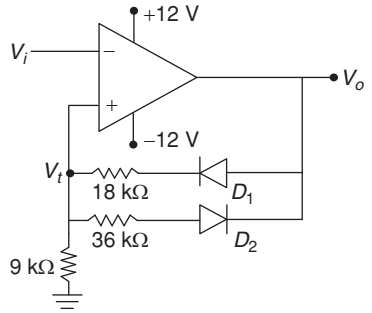
The feedback implemented is

(A) voltage series
(B) voltage shunt
(C) current series
(D) current shunt

21. Feedback ratio β is

(A) 95.2% (B) 5.2%
(C) 5% (D) 4.7%

22. Given that diodes and op-amp are ideal, calculate UTP and LTP.

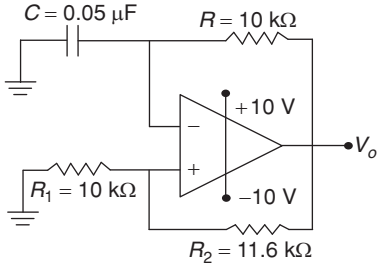


(A) $V_{UTP} = 8$ V $V_{LTP} = -8$ V
(B) $V_{UTP} = 9.6$ V $V_{LTP} = -9.6$ V
(C) $V_{UTP} = 4$ V $V_{LTP} = -2.4$ V
(D) $V_{UTP} = -4$ V $V_{LTP} = 2.4$ V

23. An op-amp has unity gain frequency of 400 MHz with band width of 60 kHz. Calculate gain in dB

(A) 56.5 dB (B) 20 dB
(C) 76.48 dB (D) None

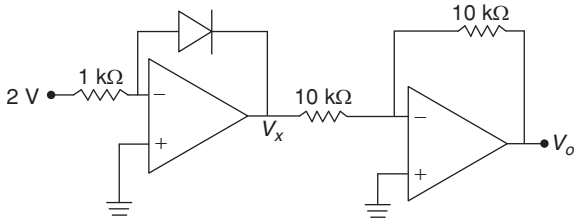
Common Data for Questions 24 and 25: A square wave generator (astable) using op-amp is shown below.



24. The frequency of the square wave output is
(A) 500 Hz (B) 2.5 kHz
(C) 1 kHz (D) None

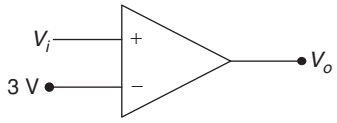
25. The feedback ratio β of the circuit is
(A) 0.463 (B) 1
(C) 0.72 (D) 0.537

26. Calculate the output V_o of the given figure. Given V_T = 25 mV



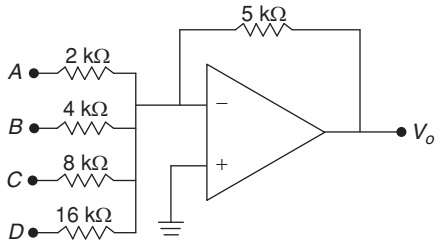
(A) –17.3 mV (B) 17.3 mV
(C) 0.69 mV (D) –0.69 mV

27. A comparator is shown below. A sinusoidal input of peak value 6 V is applied. The length (in degrees) for which the output high is



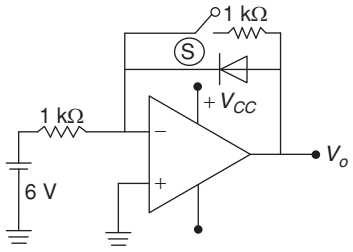
(A) 180° (B) 60°
(C) 120° (D) 150°

28. Calculate the output voltage of the figure given below if the digital input 0101 =



Digital input is 0 V (or) +5 V for 0 or 1 respectively
(A) –7.81 V (B) –4.87 V
(C) –5.87 V (D) –3.87 V

29. Consider the op-amp and diode are ideal

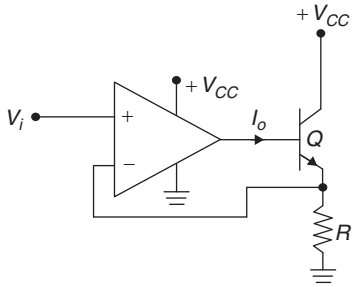


What is the output V_o when switch 'S' is open
(A) 0.7 V (B) $+V_{cc}$
(C) 0 V (D) $-V_{EE}$

30. What is the output V_o when switch S is closed
(A) –60 V (B) 0 V
(C) –6 V (D) $-V_{EE}$

31. In an inverting op-amp, the input bias current is –2μA. The input and feedback resistance are both 50 kΩ. Calculate the output voltage for input voltage of 1.5 V.
(A) –1.6 V (B) –2 V
(C) –1.5 V (D) –3 V

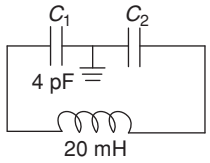
32. Assume that V_i = 3 V, R = 20 Ω, β of a transistor is 80, V_{CC} = 12 V. Calculate I_o



(A) 3.8 mA (B) 1.8 mA
(C) 0.15 A (D) 4 mA

33. The feedback network of Hartley oscillator have L_1 = 10 mH, L_2 = 5 mH and C = 100 PF. The mutual inductance between L_1 and L_2 is 2.5 mH. The frequency of oscillation is _____.
(A) 0.13 MHz (B) 11.3 kHz
(C) 0.113 MHz (D) None

Common Data for Questions 34 and 35: The feedback network used in Colpitts oscillator is given below.

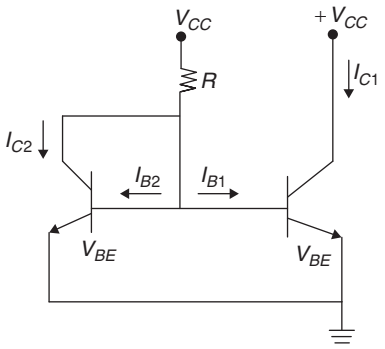


34. The value of C_2 to have frequency of oscillation 1 MHz is _____.
(A) 0.8 pF (B) 4 pF
(C) 1.8 pF (D) 2.2 pF

35. Feedback ratio β is _____.
(A) 0.45 (B) 0.2
(C) 0.9 (D) None

36. The RC network of Wien bridge oscillator has $R_1 = R_2$ = 80 kΩ, let us assume $C_1 = C_2 = C$ then what is the value of C required for the frequency of oscillation 5 kHz?
(A) 200 pF (B) 400 pF
(C) 50 pF (D) 1 pF

37. The current mirror in the figure is designed to provide I_C = 0.5 mA, V_{CC} = 10 V, β = 100, V_{BE} = 0.7 V. The value of R is



(A) 18.23 kΩ (B) 8.5 kΩ
(C) 12.9 kΩ (D) 15.3 kΩ

38. If V_{BE} changes by $\frac{-2.2\text{ mV}}{°\text{C}}$ what is the permissible temperature range if I_{C1} is to remain within 1% of its normal design value of the above problem?
(A) 21°C (B) 42°C
(C) 93°C (D) 12°C

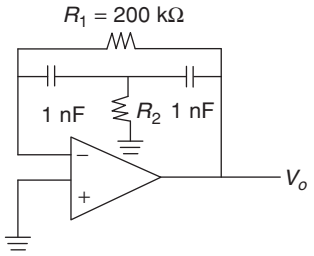
39. Match List I and List II

LIST I	LIST II
P. Wien bridge oscillator	**1.** Low output Impedance
Q. Voltage shunt feedback Amplifier	**2.** RF frequency range
R. Crystal Oscillator	**3.** Audio Frequency range
S. Current Shunt feedback Amplifier	**4.** High Input Impedance
	5. High Output Impedance

	P	Q	R	S
(A)	2	1	3	5
(B)	5	4	2	1
(C)	3	4	2	1
(D)	3	1	2	5

40. The slew rate of an op-amp is 0.5 V/micro seconds. The maximum frequency of a sinusoidal input of $2V_{(rms)}$ that can be handled without excessive distortion is
(A) 3 kHz
(B) 30 kHz
(C) 200 kHz
(D) 2 MHz

41. In the active filter circuit, if $Q = 2$, a pair of poles will be realized with ω_o, equal to
(A) 2000 rad/s
(B) 200 rad/s
(C) 20 rad/s
(D) 2 rad/s

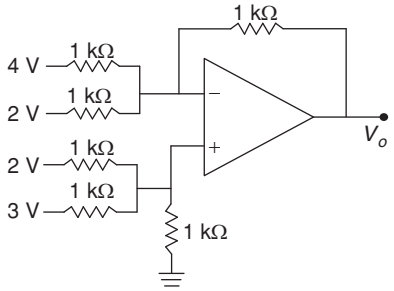


Practice Problems 2

Directions for questions 1 to 39: Select the correct alternative from the given choices.

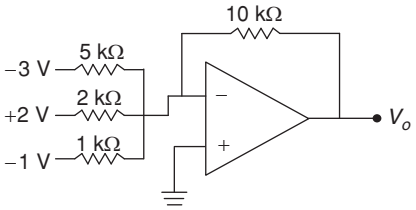
1. An op-amp has an open loop gain 60 dB and common mode gain is 1.2. Calculate CMRR in dB
(A) 50 dB (B) 58.42 dB
(C) 72 dB (D) 58.8 dB

2. What is the output V_o of the given figure?



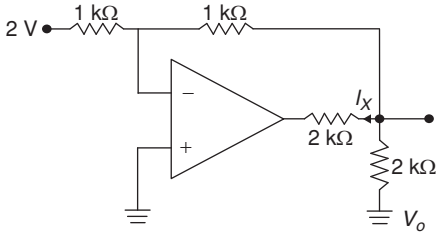
(A) –1 V (B) +1 V
(C) 0 V (D) 2 V

3. What is the output V_o of the given figure?



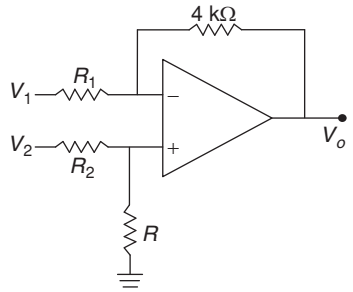
(A) 0 V (B) 6 V
(C) –6 V (D) –14 V

4. What is the current I_x of the given figure?



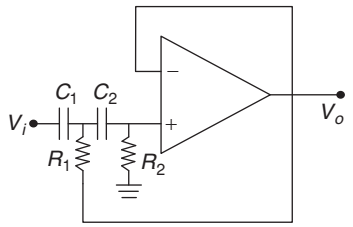
(A) 3 mA (B) 2 mA
(C) 1.5 mA (D) 0

5. What is the value of R as if the magnitude of gains in inverting and non-inverting modes is equal. ($R_1 = R_2$)



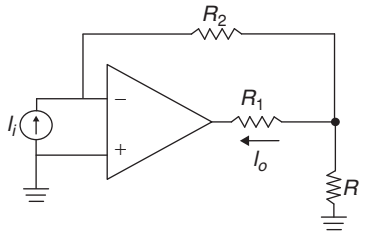
(A) 4 kΩ (B) 2 kΩ
(C) 1 kΩ (D) 8 kΩ

6. What is the cutoff frequency of the given figure.
$C_1 = c_2 = 0.08$ μF
$R_1 = 7.07$ kΩ
$R_2 = 14.14$ kΩ



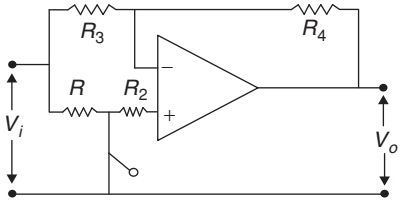
(A) 2 kHz (B) 20 kHz
(C) 0.2 kHz (D) None

7. In the circuit given below, I_o/I_i is



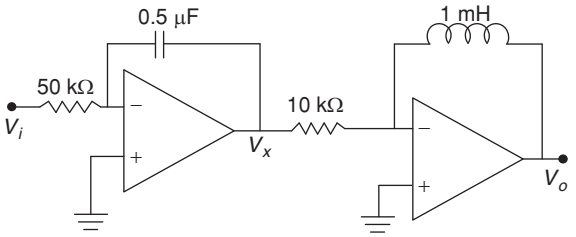
(A) $1+\frac{R_2}{R}$ (B) $1+\frac{R_1}{R}$
(C) $\frac{R_2}{R}$ (D) $\frac{R_1}{R}$

8. What is the ratio of gain of the circuit given below when switch open to switch close condition



(A) $\frac{-R_3}{R_4}$ (B) $R_1+\frac{R_3}{R_4}$
(C) $\frac{-R_4}{R_3}$ (D) $R_2+\frac{R_4}{R_3}$

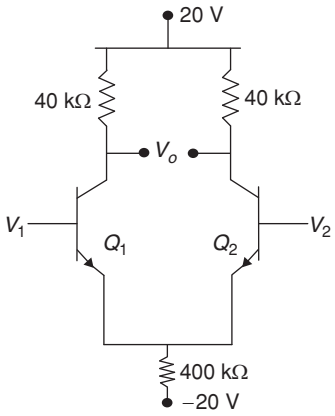
9. Calculate the output V_o of the figure given below.



Where $V_i = 10\cos 100t$ Volt
(A) $0.04\cos 100t$ mV (B) $0.2\sin 100t$ V
(C) $-4\cos 100t$ mV (D) $-4\sin 100t$ V

Common Data for Questions 10 to 12: A differential amplifier is given below that has following specifications:

$r_\pi = 2$ MΩ, $\beta = 4 \times 10^3$

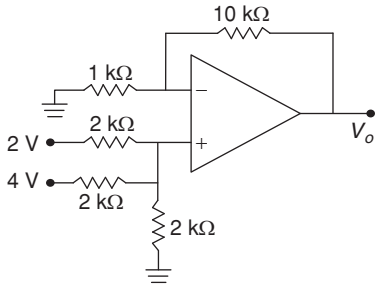


10. The differential mode gain is
(A) 100 (B) 80
(C) 90 (D) None

11. Common mode gain is
(A) 0.125 (B) 0.05
(C) 1 (D) 0

12. CMRR in dB is
(A) 64.08 dB (B) 21.8 dB
(C) 41.3 dB (D) None

13. What is the output V_o of the figure given below?

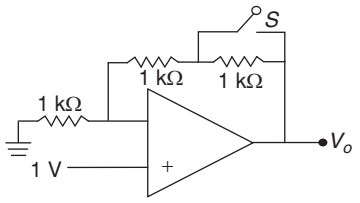


(A) 11 V (B) 33 V
(C) 22 V (D) 5.5 V

14. The CMRR of a differential amplifier is 60 dB. If the differential mode gain is 1400, what is the common mode gain A_C?

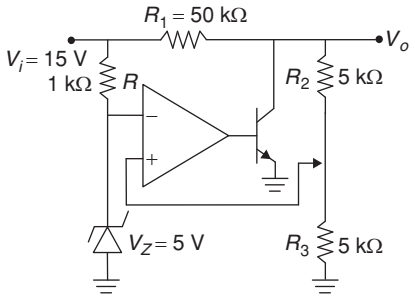
(A) 1.4 (B) 2.1
(C) 0.8 (D) None

15. An op-amp shown below give output is x when the switch S is open. What is the output when switch S is closed?



(A) x (B) $2x$
(C) $\frac{1}{2}x$ (D) $\frac{2}{3}x$

16. Calculate the power rating of R_1 if the maximum input voltage is 15 V.



(A) 4.5 W (B) 3.1 W
(C) 2.7 W (D) None

17. The change in the op-amp input offset voltage V_{io} caused by variation in supply voltage is 60 μv. Calculate the change in supply voltage. Given SVRR = 104 dB

(A) 0 V (B) 9.5 V
(C) 2 V (D) None

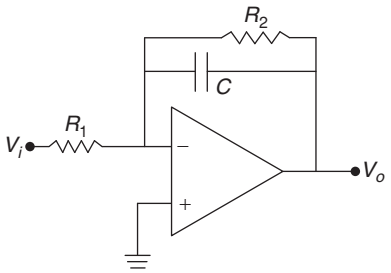
18. For non-inverting amplifier how much is the feedback resistance required to produce an output of 10 V for input of 1.5 V, given that input resistance is 2.5 kΩ?

(A) 2.26 kΩ
(B) 0.37 kΩ
(C) 14.2 kΩ
(D) 0.37 kΩ

19. An amplifier using op-amp with a slew rate 0.5 V/μsec has a gain 20 dB. If this amplifier has to faithfully amplify sinusoidal signals from DC to 30 kHz without any distortion, what is the limit of the input signal level?

(A) 266 mV (B) 2.6 V
(C) 26.6 mV (D) None

20. Recognize the type of filter shown below

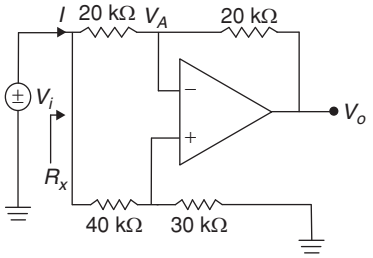


(A) Low pass filter (B) High pass filter
(C) Band pass filter (D) None

21. If R_1 = 10 kΩ, R_2 = 1 kΩ and C = 0.01μf, the cutoff frequency f_o is_____

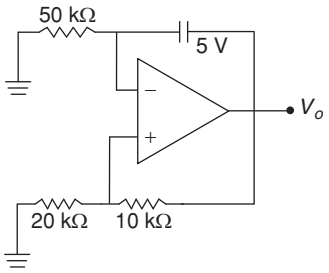
(A) 1.6 kHz (B) 1.45 kHz
(C) 160 Hz (D) None

22. Calculate R_x of the figure given below



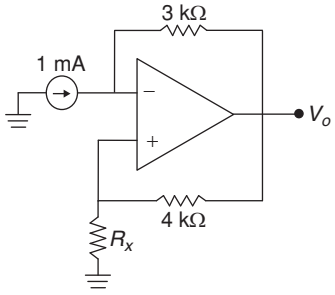
(A) 20 kΩ (B) 60 kΩ
(C) 23.3 kΩ (D) $\frac{40}{3}$ kΩ

23. Calculate V_o of the given circuit.



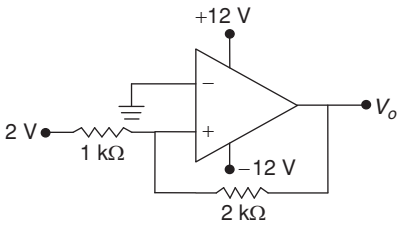
(A) 15 V (B) 0 V
(C) −5 V (D) 5 V

24. What is the value of R_x to get V_o = −4.5 V?



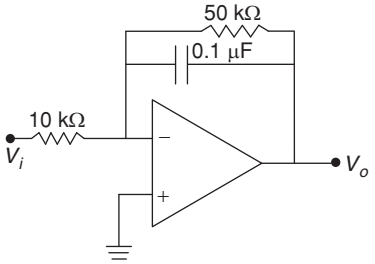
(A) 2 kΩ (B) 1 kΩ
(C) 1.5 kΩ (D) 4 kΩ

25. What is the output V_o of the given figure?



(A) +12 V (B) ±12 V
(C) −12 V (D) 6 V

26. What is the very low frequency gain of low pass filter shown in the figure given below?

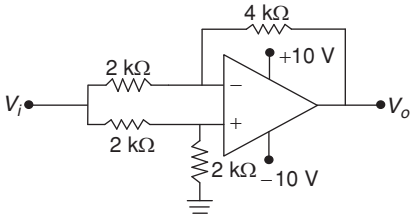


(A) ∞ (B) 6
(C) 5 (D) 0

27. An op-amp based non-inverting amplifier has a gain of 20 and bandwidth of 200 kHz. If the gain of the amplifier is reduced to unity, the band width will changes to _____

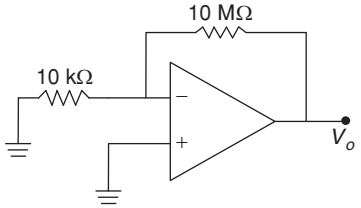
(A) 200 kHz (B) 4 MHz
(C) 1 MHz (D) 10 MHz

28. In the op-amp circuit shown, the voltage ratio V_o/V_i is



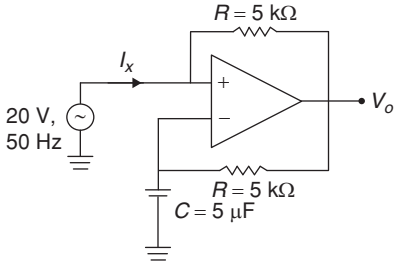
(A) –0.5 (B) –1
(C) 0.5 (D) 4

29. An op-amp has offset voltage of 1.2 mV and is ideal in all other respects. If this op-amp is used in the circuit shown in the figure what is the output voltage



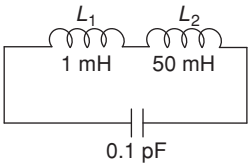
(A) 1.2 mV (B) ±1.2 V
(C) 0.6 V (D) 0 V

30. In the figure given, calculate I_x from the source



(A) 31.4 mA lead by 90° (B) 31.4 mA lag by 90°
(C) 62.8 mA lead by 90° (D) 62.8 mA lag by 90°

31. The feedback network of Hartley oscillator is shown below. The minimum gain of a transistor required for oscillation is _____.

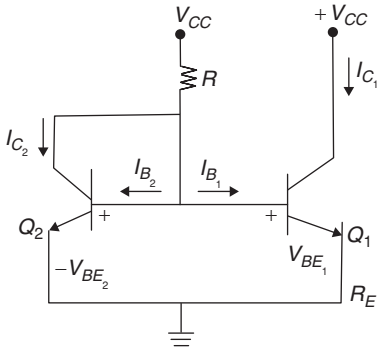


(A) 0.02 (B) 1
(C) 50 (D) None

32. The resonant circuit oscillator has a frequency of 5 MHz. If the capacitance is increased by 50%, the new frequency of oscillation is _____

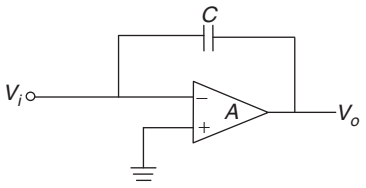
(A) 5 MHz (B) 10 MHz
(C) 2.5 MHz (D) 4.08 MHz

33. For low collector current I_{C_1} = 50 μA, this circuit is used. Determine the value of R_E, given I_{C_2} = 0.5 mA, $\beta = 100$, $V_T = 25$ mV.



(A) 1.139 kΩ
(B) 4.34 kΩ
(C) 7.57 kΩ
(D) Its not possible to calculate

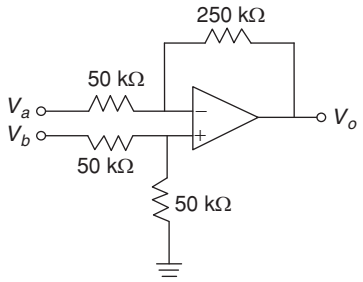
34. An amplifier of gain A is bridged by a capacitor C as shown. The effective input capacitance is



(A) $\frac{C}{A}$ (B) $C(1-A)$

(C) $C(1+A)$ (D) CA

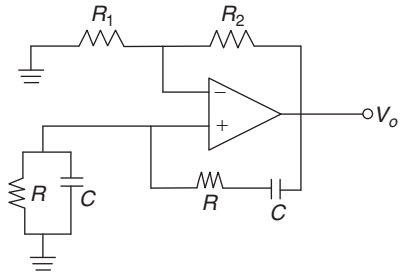
35. What is the output voltage V_o of the given circuit?



(A) $-5V_a + 2.5V_b$ (B) $-5V_a + 3V_b$

(C) $-2.5V_a + 2.5V_b$ (D) $-2.5V_a + 5V_b$

36. Consider the following statements for the circuit shown in figure?



(1) For $R = 1\ \text{k}\Omega, C = \left(\frac{1}{2\pi}\right)\mu\text{F}, f = 1\ \text{kHz}$

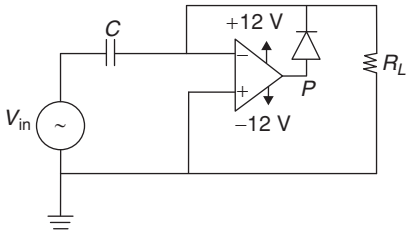
(2) for $R = 3\ \text{k}\Omega, C = \left(\frac{1}{18\pi}\right)\mu\text{F}, f = 3\ \text{kHz}$

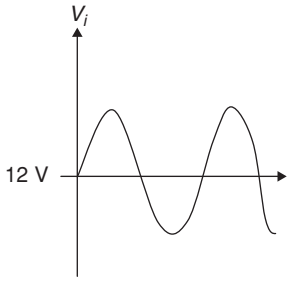
Which of the statements given are true?

(A) (1) only (B) (2) only

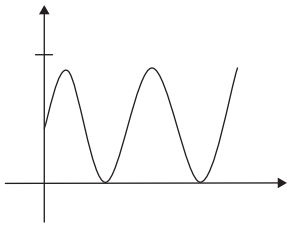
(C) both (D) Neither

37. For the given sinusoidal input voltage, the voltage waveform at point P of the clamper circuit shown in figure is.

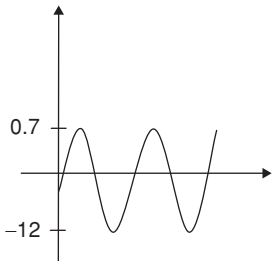




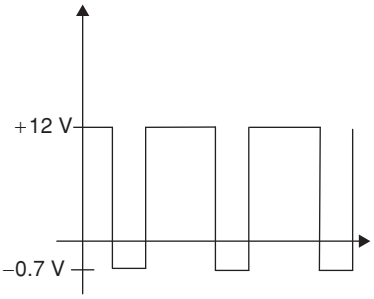
(A)



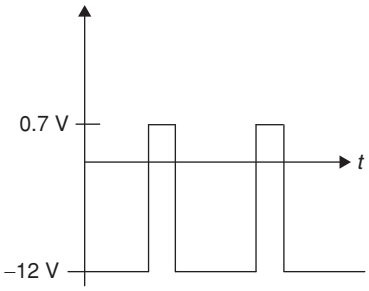
(B)



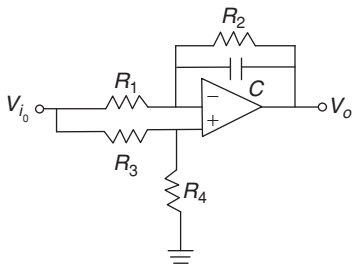
(C)



(D)



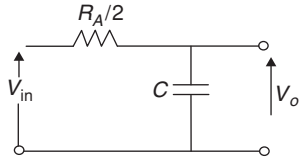
38. A general filter circuit is shown in the figure if $R_1 = R_2 = R_A$, and $R_3 = R_4 = R_B$ then the circuit acts as



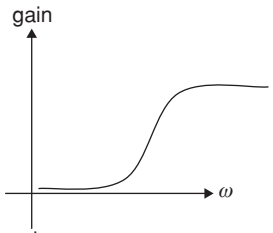
(A) All pass filter (B) Band pass filter

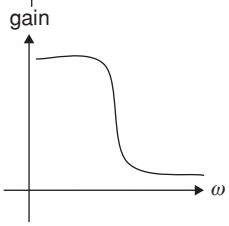
(C) High pass filter (D) Low pass filter

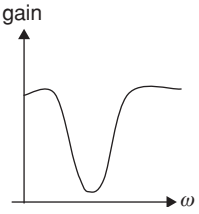
39. The output of the above filter is given to input of the circuit shown in this figure.

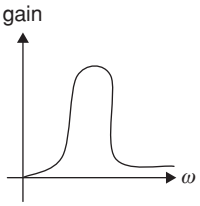


The gain V_s frequency characteristic of the output V_o will be

(A) 

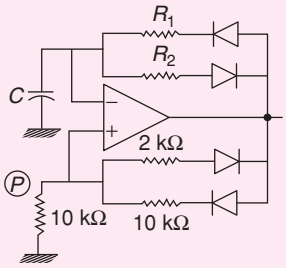
(B) 

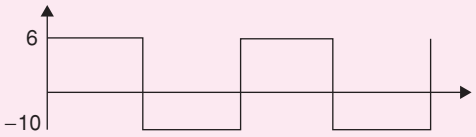
(C) 

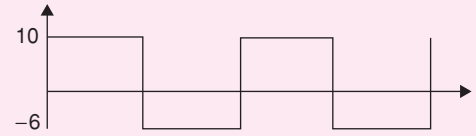
(D) 

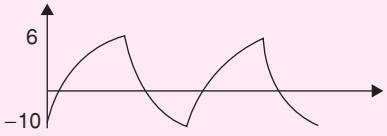
Previous Years' Questions

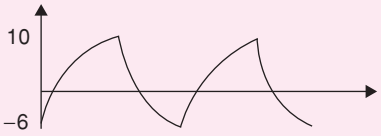
1. A relaxation oscillator is made using op-amp as shown in figure. The supply voltages of the op-amp are ±12 V. The voltage waveform at point P will be **[2006]**



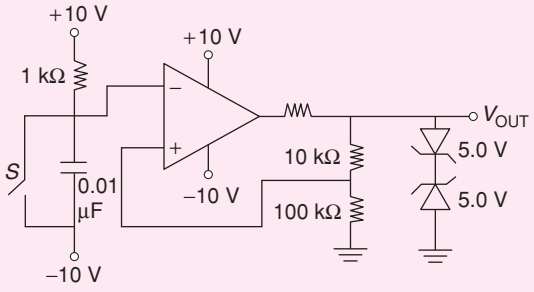
(A) 

(B) 

(C) 

(D) 

2. The switch S in the circuit of the figure is initially closed. It is opened at time $t = 0$. **[2007]**



You may neglect the Zener diode forward voltage drops. What is the behaviour of V_{OUT} for $t > 0$?

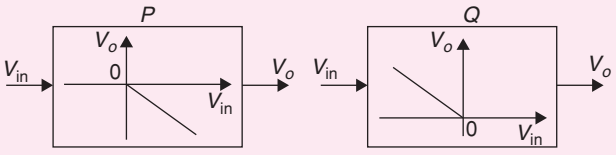
(A) It makes a transition from –5V to +5V at t = 12.98 μs

(B) It makes a transition from –5V to +5V at t = 2.57 μs

(C) It makes a transition from –5V to +5V at t = 12.98 μs

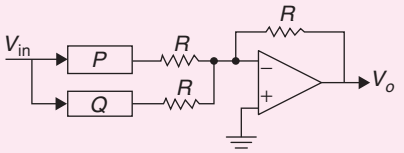
(D) It makes a transition from + 5V to –5V at t = 2.57 μs

3. The block diagrams of two types of half wave rectifiers are shown in the figure. The transfer characteristics of the rectifiers are also shown within the block.

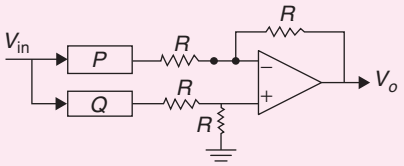


It is desired to make full-wave rectifier using above two half-wave rectifiers. The resultant circuit will be **[2008]**

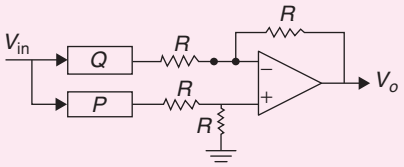
(A)



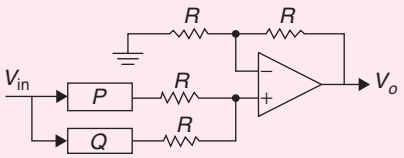
(B)



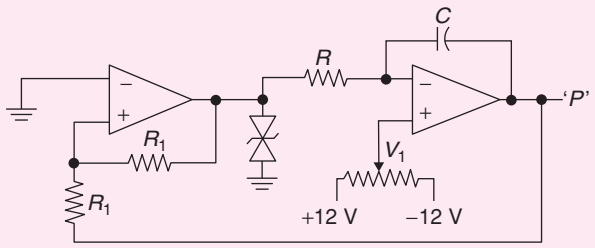
(C)



(D)

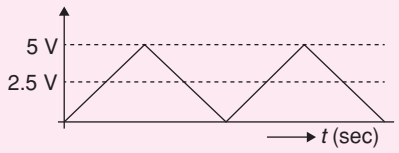


4. A waveform generator circuit using op-amps is shown in the figure. It produces a triangular wave at point '*P*' with a peak to peak voltage of 5 V for $V_i = 0$ V. **[2008]**

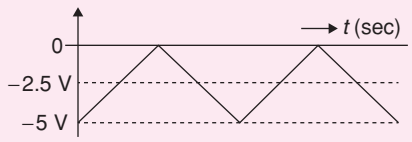


If the voltage *v* is made +2.5 V, the voltage waveform at point '*P*' will become

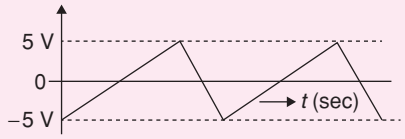
(A)



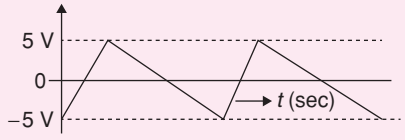
(B)



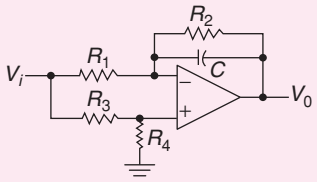
(C)



(D)



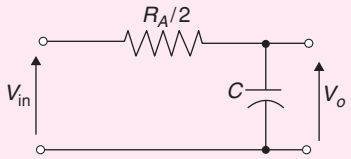
Common Data for Questions 5 and 6: A general filter circuit is shown in the figure



5. If $R_1 = R_2 = R_A$ and $R_3 = R_4 = R_s$, the circuit acts as a **[2008]**

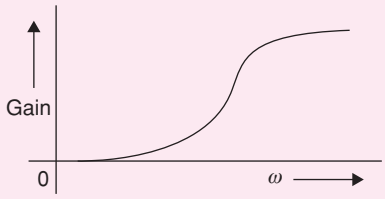
(A) all pass filter (B) band pass filter
(C) high pass filter (D) low pass filter

6. The output of the filter in question 12 is given to the circuit shown in figure **[2008]**

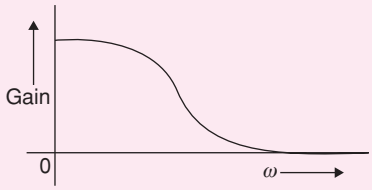


The given vs. frequency characteristic of the output (V_0) will be

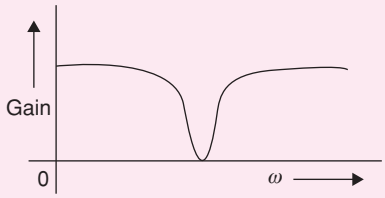
(A)



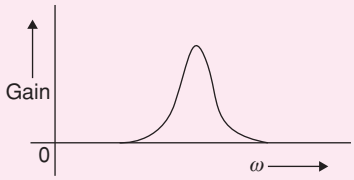
(B)



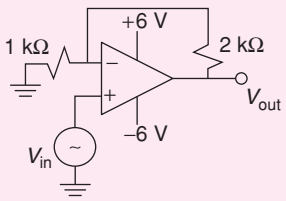
(C)



(D)

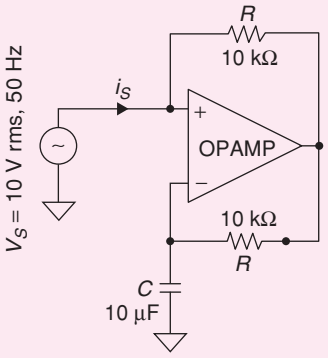


7. The nature of feedback in the op-amp circuit shown is **[2009]**



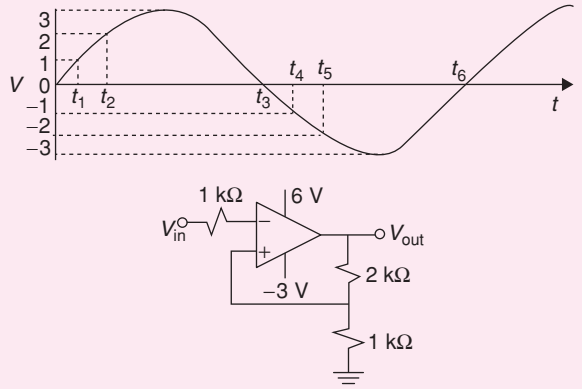
(A) Current–Current feedback
(B) Voltage–Voltage feedback
(C) Current–Voltage feedback
(D) Voltage–Current feedback

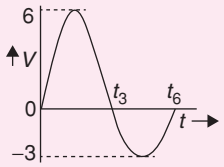
8. The following circuit has R = 10 kΩ, C = 10 μF. The input voltage is a sinusoid at 50 Hz with an rms value of 10 V. Under ideal conditions, the current i_s from the source is **[2009]**

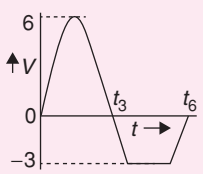


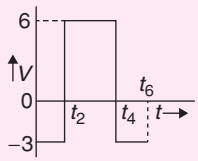
(A) 10π mA leading by 90°
(B) 20π mA leading by 90°
(C) 10 mA leading by 90°
(D) 10π mA lagging by 90°

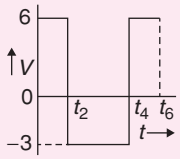
9. An ideal op-amp circuit and its input waveform are shown in the figures. The output waveform of this circuit will be **[2009]**



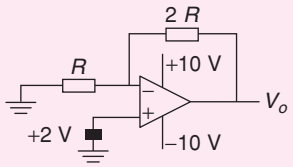
(A)


(B)


(C)


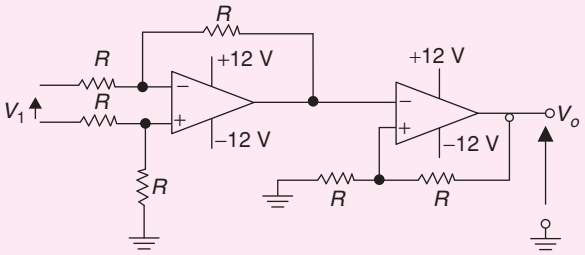
(D)


10. Given that the op-amp is ideal, the output voltage V_0 is **[2010]**

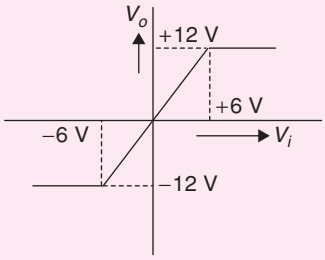


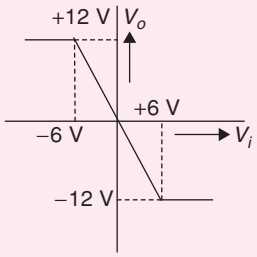
(A) 4 V
(B) 6 V
(C) 7.5 V
(D) 12.12 V

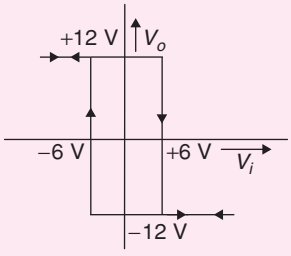
11. For the circuit shown below. **[2011]**



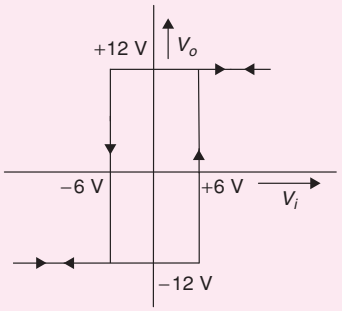
the correct transfer characteristic is

(A)


(B)


(C)


(D)



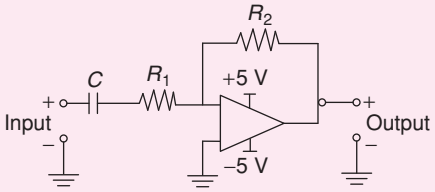
12. A dual trace oscilloscope is set to operate in the alternate mode. The control input of the multiplexer used in the y-circuit is fed with a signal having a frequency equal to: **[2011]**

(A) the highest frequency that the multiplexer can operate properly.
(B) twice the frequency of the time base (sweep) oscillator.
(C) the frequency of the time base (sweep) oscillator.
(D) half the frequency of the time base (sweep) oscillator.

13. A low-pass filter with a cutoff frequency of 30 Hz is cascaded with a high-pass filter with a cutoff frequency of 20 Hz. The resultant system of filters will function as. **[2011]**

(A) an all-pass filter
(B) an all-stop filter
(C) a band-stop (band-reject) filter
(D) a band-pass filter

14. The circuit shown is a **[2012]**



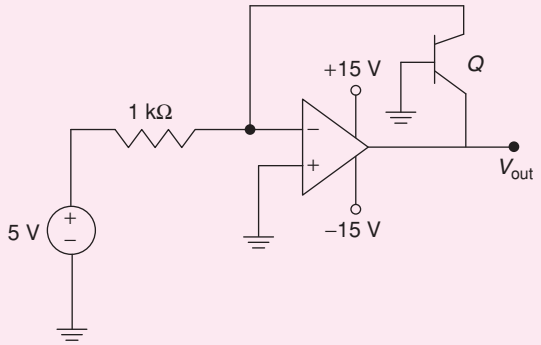
(A) low pass filter with $f_{3dB} = \dfrac{1}{(R_1 + R_2)C}$ rad/s

(B) high pass filter with $f_{3dB} = \dfrac{1}{R_1 C}$ rad/s

(C) low pass filter with $f_{3dB} = \dfrac{1}{R_1 C}$ rad/s

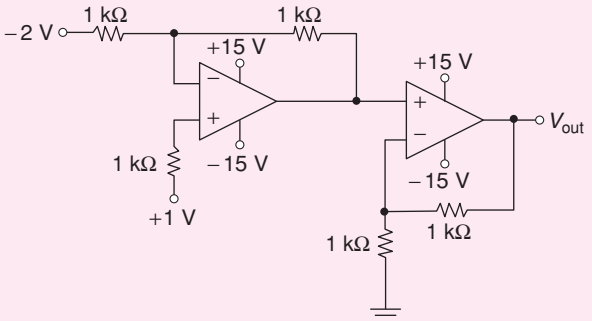
(D) high pass filter with $f_{3dB} = \dfrac{1}{(R_1 + R_2)C}$ rad/s

15. In the circuit shown below, what is the output voltage (V_{out}) in V if a silicon transistor Q and an ideal op-amp are used? **[2013]**



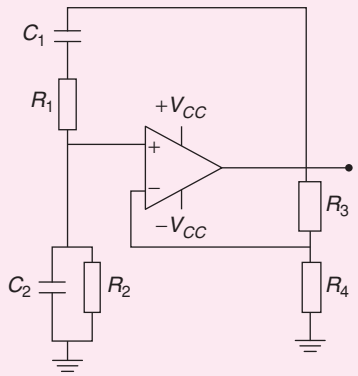
(A) −15 (B) −0.7
(C) +0.7 (D) +15

16. In the circuit shown below the op-amps are ideal. Then V_{out} in V is **[2013]**



(A) 4 (B) 6
(C) 8 (D) 10

17. In the Wien bridge oscillator circuit shown in figure, the bridge is balanced when **[2013]**



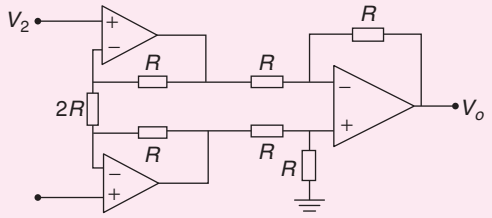
(A) $\dfrac{R_3}{R_4} = \dfrac{R_1}{R_2}, \omega = \dfrac{1}{\sqrt{R_1 C_1 R_2 C_2}}$

(B) $\dfrac{R_2}{R_1} = \dfrac{C_2}{C_1}, \omega = \dfrac{1}{R_1 C_1 R_2 C_2}$

(C) $\frac{R_3}{R_4}=\frac{R_1}{R_2}+\frac{C_2}{C_1}, \omega=\frac{1}{\sqrt{R_1C_1R_2C_2}}$

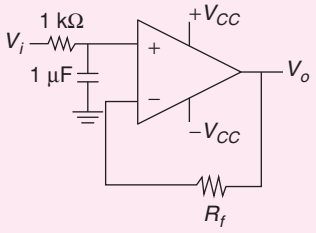
(D) $\frac{R_3}{R_4}+\frac{R_1}{R_2}=\frac{C_2}{C_1}, \omega=\frac{1}{R_1C_1R_2C_2}$

18. Given that the op-amps in the figure are ideal, the output voltage V_o is **[2014]**

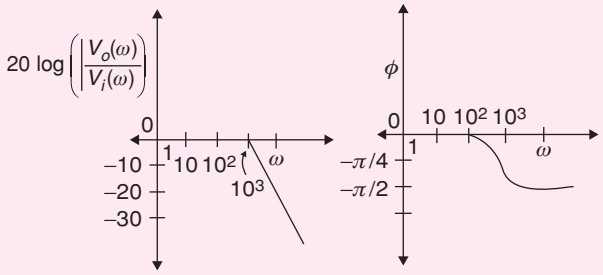


(A) (V_1-V_2) (B) $2(V_1-V_2)$
(C) $(V_1-V_2)/2$ (D) (V_1+V_2)

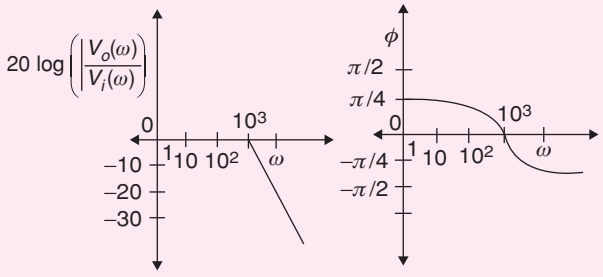
19. In the figure shown, assume the op-amp to be ideal. Which of the alternatives gives the correct Bode plots for the transfer function $\frac{V_o(\omega)}{V_t(\omega)}$? **[2014]**



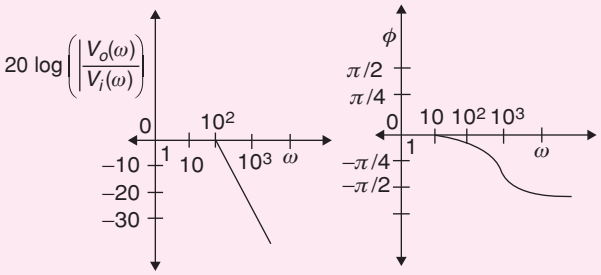
(A)



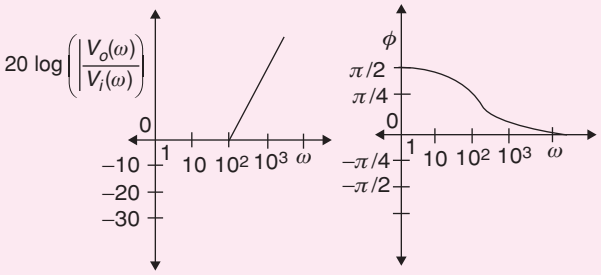
(B)



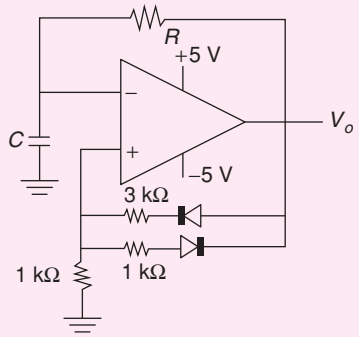
(C)



(D)

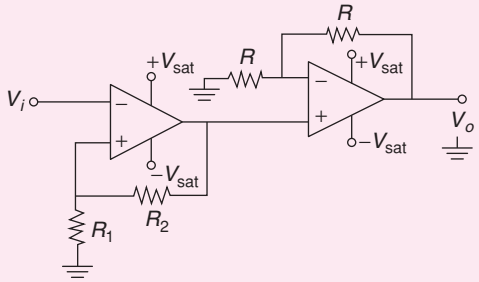


20. An oscillator circuit using ideal op-amp and diodes is shown in the figure. **[2014]**



The time duration for +ve part of the cycle is Δt_1 and for –ve part is Δt_2. The value of $e^{\frac{\Delta t_1-\Delta t_2}{RC}}$ will be _____.

21. An operational-amplifier circuit is shown in the figure. **[2014]**

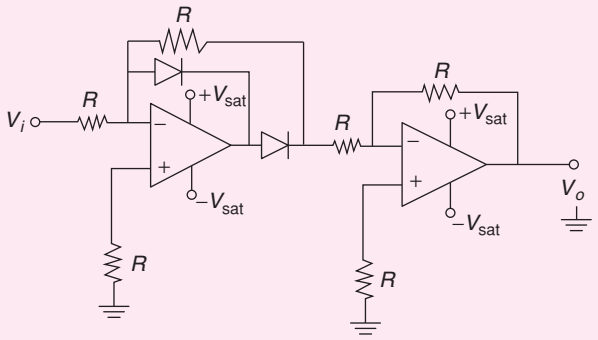


The output of the circuit for a given input v_1 is

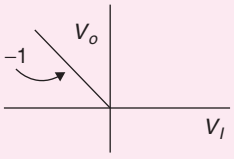
(A) $-\left(\frac{R_2}{R_1}\right)v_i$ (B) $-\left(1+\frac{R_2}{R_1}\right)v_i$

(C) $\left(1+\frac{R_2}{R_1}\right)v_i$ (D) $+V_{sat}$ or $-V_{sat}$

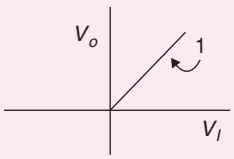
22. The transfer characteristic of the op-amp circuit shown in figure is **[2014]**



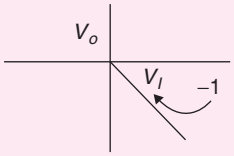
(A)



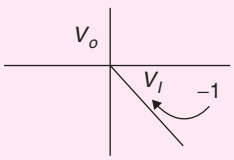
(B)



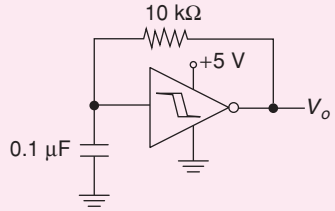
(C)



(D)

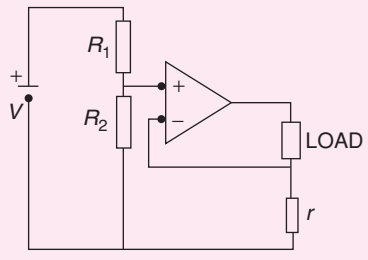


23. A hysteresis type TTL inverter is used to realize an oscillator in the circuit shown in the figure. **[2014]**



If the lower and upper trigger level voltages are 0.9 V and 1.7 V, the period (in ms), for which output is low, is _____.

24. The circuit shown in the figure is **[2007]**



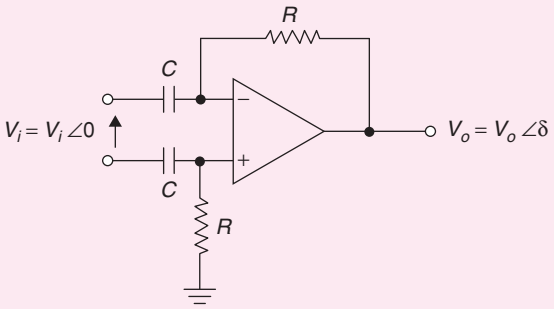
(A) a voltage source with voltage $\frac{rV}{R_1 \| R_2}$

(B) a voltage source with voltage $\frac{r \| R_2}{R_1} V$

(C) a current source with current $\frac{r \| R_2}{R_1 + R_2} \cdot \frac{V}{r}$

(D) a current source with current $\frac{R_2}{R_1 + R_2} \cdot \frac{V}{r}$

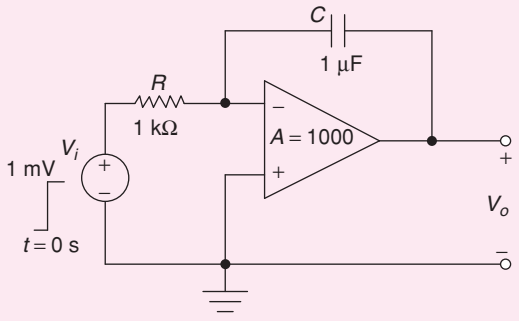
25. Consider the circuit shown in the figure. In this circuit $R = 1\ k\ \Omega$, and $C = 1\ \mu F$. The input voltage is sinusoidal with a frequency of 50 Hz, represented as a phasor with magnitude V_i and phase angle 0 radian as shown in the figure. The output voltage is represented as a phasor with magnitude V_o and phase angle δ radian. What is the value of the output phase angle δ(in radian) relative to the phase angle of the input voltage? **[2015]**



(A) 0 (B) π

(C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{2}$

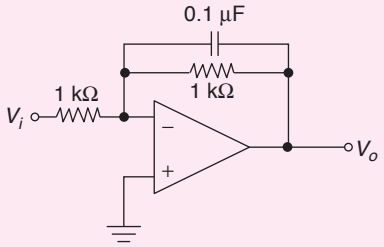
26. The op-amp shown in the figure has finite gain $A = 1000$ and an infinite input resistance. A step-voltage $V_i = 1$ mV is applied at the input at time $t = 0$ as shown. Assuming that the operational amplifier is not saturated, the time constant (in millisecond) of the output voltage V_o is **[2015]**



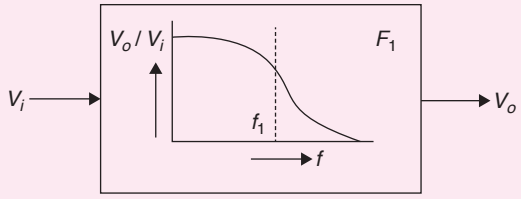
(A) 1001 (B) 101

(C) 11 (D) 1

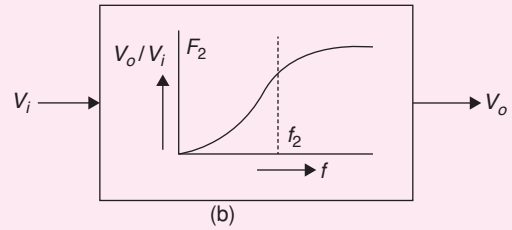
27. The operational amplifier shown in the figure is ideal. The input voltage (in Volt) is $V_i = 2\sin(2\pi \times 2000t)$. The amplitude of the output voltage V_o (in Volt) is _____. **[2015]**



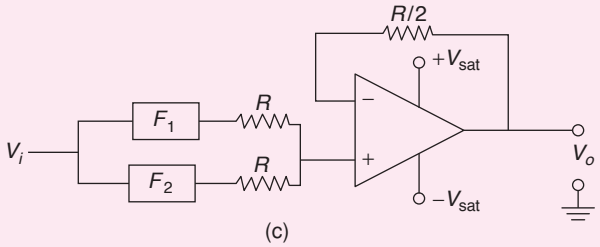
28. The filters $F1$ and $F2$ having characteristics as shown in Figures (a) and (b) are connected as shown in Figure (c). **[2015]**



(a)



(b)

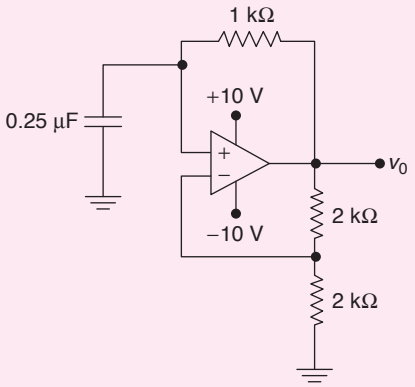


(c)

The cut-off frequencies of $F1$ and $F2$ are f_1 and f_2 respectively. If $f_1 < f_2$, the resultant circuit exhibits the characteristics of a

(A) Band-pass filter (B) Band-stop filter
(C) All pass filter (D) High-Q filter

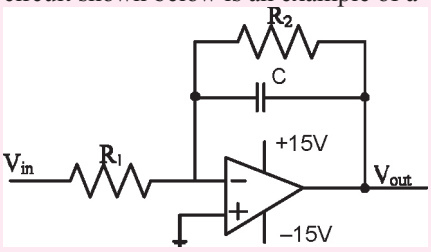
29.



The saturation voltage of the ideal op-amp shown below is ±10 V. The output voltage v_0 of the following circuit in the steady-state is **[2015]**

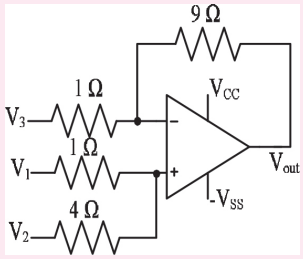
(A) square wave of period 0.55 ms.
(B) triangular wave of period 0.55 ms.
(C) square wave of period 0.25 ms.
(D) triangular wave of period 0.25 ms.

30. The circuit shown below is an example of a **[2016]**



(A) Low pass filter (B) Band pass filter
(C) high pass filter (D) Notch filter

31. For the circuit shown below, taking the opamp as ideal, the output voltage V_{out} in terms of the input voltages V_1, V_2, and V_3 is _____. **[2016]**



(A) $1.8V_1 + 7.2V_2 - V_3$ (B) $2V_1 + 8V_2 - 9V_3$
(C) $7.2V_1 + 1.8V_2 - V_3$ (D) $8V_1 + 2V_2 - 9V_3$

Answer Keys

Exercises

Practice Problems 1

1. C	**2.** D	**3.** D	**4.** C	**5.** B	**6.** B	**7.** C	**8.** D	**9.** B	**10.** D
11. C	**12.** D	**13.** A	**14.** A	**15.** C	**16.** A	**17.** D	**18.** B	**19.** B	**20.** B
21. D	**22.** C	**23.** C	**24.** C	**25.** A	**26.** B	**27.** C	**28.** A	**29.** D	**30.** C
31. A	**32.** B	**33.** C	**34.** C	**35.** A	**36.** C	**37.** A	**38.** B	**39.** D	**40.** B
41. A									

Practice Problems 2

1. B	**2.** A	**3.** B	**4.** A	**5.** A	**6.** C	**7.** A	**8.** A	**9.** A	**10.** B
11. B	**12.** A	**13.** B	**14.** A	**15.** D	**16.** A	**17.** B	**18.** C	**19.** A	**20.** A
21. A	**22.** C	**23.** A	**24.** A	**25.** A	**26.** C	**27.** B	**28.** A	**29.** B	**30.** B
31. C	**32.** D	**33.** A	**34.** B	**35.** B	**36.** C	**37.** D	**38.** C	**39.** D	

Previous Years' Questions

1. A	**2.** D	**3.** B	**4.** A	**5.** C	**6.** D	**7.** B	**8.** D	**9.** D	**10.** B
11. D	**12.** D	**13.** D	**14.** B	**15.** B	**16.** C	**17.** C	**18.** B	**19.** A	
20. 0.7977		**21.** D	**22.** C	**23.** 0.635	**24.** D	**25.** D	**26.** A	**27.** 1.1 to 1.4	
28. B	**29.** A	**30.** A	**31.** D						

Chapter 8

Boolean Algebra and Minimization of Functions

LEARNING OBJECTIVES

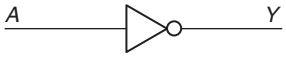
After reading this chapter, you will be able to understand:

- Logic gates
- Boolean algebra
- AXIOMS and Laws of boolean algebra
- Properties of boolean Algebra
- Boolean functions, min terms and max terms
- Minterm and maxterm
- Minimization of boolean functions
- Implementation of function by using NAND/NOR gates
- Digital IC families
- Bipolar logic families
- Unipolar logic families
- Classification of digital ICs

Logic Gates

Inverter or NOT Gate (7404 IC)

The inverter performs a basic logic operation called inversion or complementation. The purpose of the inverter is to change one logic level to the opposite level. In terms of digital circuits, it converts 1 to 0 and 0 to 1.



Symbol

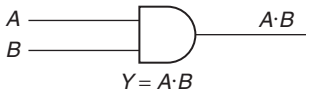
$Y = \overline{A}$

Truth Table

Input	Output
A	***Y***
0	1
1	0

AND Gate (Logical Multiplier 7408 IC)

The AND gate performs logical multiplication more commonly know as AND function. The AND gate is composed of 2 or more inputs and a single output



$Y = A \cdot B$

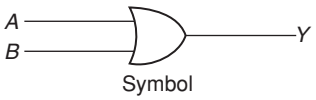
Figure 1 2 input AND gate

Truth Table

Input		Output
A	***B***	***Y***
0	0	0
0	1	0
1	0	0
1	1	1

OR Gate (Logical Adder 7432 IC)

The OR gate performs logical, addition commonly known as OR function.



Symbol

$Y = A + B$

Figure 2 2 input OR gate

Truth Table

Input		Output
A	***B***	***Y***
0	0	0
0	1	0
1	0	0
1	1	1

NAND Gate (7400 IC)

The NAND gate's function is basically AND + NOT function.

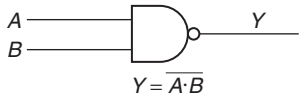


Figure 3 2 input NAND gate

Truth Table			
Input			Output
A	*B*	$A \cdot B$	$\overline{A + B}$ (*Y*)
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

NOR Gate (7402 IC)

The NOR gate is basically OR + NOT function.

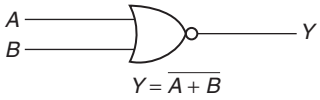


Figure 4 2 input NOR gate

Truth Table			
Input			Output
A	*B*	$A + B$	$\overline{A + B}$ (*Y*)
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

Exclusive OR Gate X-OR (7486 IC)

X-OR is a gate in which unequal inputs create a high logic level output; and if both inputs are equal, the output, will be low. Other name for EX-OR gate is unequivalent gate.

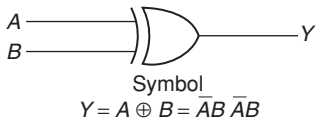


Figure 5 2 input X-OR gate

Truth Table		
A	*B*	*Y*
0	0	0
0	1	1
1	0	1
1	1	0

7. **Exclusive NOR gate (X-NOR):** X-NOR is a gate in which equal inputs create a high logic level output; and if both inputs are unequal, then the output will be low. Other name for EX-NOR gate is equivalent gate.

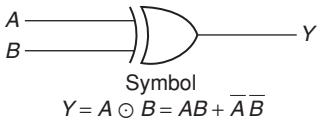


Figure 6 2 input X-NOR gate

Truth Table		
A	*B*	*Y*
0	0	1
0	1	0
1	0	0
1	1	1

EX-NOR gate is complement of EX-OR gate.

Boolean Algebra

Boolean algebra is a system of mathematical logic. It is an algebraic system consisting of the set of elements (0, 1), two binary operators OR and AND and one unary operator NOT. The Boolean algebra is governed by certain well-developed rules and laws.

Axioms and Laws of Boolean Algebra

Axioms

1. **AND operation**
   - (i) $0 \cdot 0 = 0$
   - (ii) $0 \cdot 1 = 0$
   - (iii) $1 \cdot 0 = 0$
   - (iv) $1 \cdot 1 = 1$
2. **OR operation**
   - (v) $0 + 0 = 0$
   - (vi) $0 + 1 = 1$
   - (vii) $1 + 0 = 1$
   - (viii) $1 + 1 = 1$
3. **NOT operation**
   - (ix) $\overline{1} = 0$
   - (x) $\overline{0} = 1$

Laws

1. **Complementation law**
   - (i) $\overline{0} = 1$
   - (ii) $\overline{1} = 0$
   - (iii) If $A = 0$, then $\overline{A} = 1$
   - (iv) If $A = 1$, then $\overline{A} = 0$
   - (v) $\overline{\overline{A}} = A$
2. **AND laws**
   - (i) $A \cdot 0 = 0$ (NULL law)
   - (ii) $A \cdot 1 = A$ (Identity law)
   - (iii) $A \cdot A = A$
   - (iv) $A \cdot \overline{A} = 0$

3. **OR laws**
   (i) $A + 0 = A$ (NULL law)
   (ii) $A + 1 = 1$ (Identity law)
   (iii) $A + A = A$
   (iv) $A + \overline{A} = 1$
4. **Commutative laws**
   (i) $A + B = B + A$
   (ii) $A \cdot B = B \cdot A$
5. **Associative laws**
   (i) $(A + B) + C = A + (B + C)$
   (ii) $(A \cdot B)C = A(B \cdot C)$
6. **Distributive laws**
   (i) $A(B + C) = AB + AC$
   (ii) $A + BC = (A + B)(A + C)$
7. **Redundant literal rule (RLR)**
   (i) $A + \overline{A}B = A + B$
   (ii) $A(\overline{A} + B) = AB$
8. **Idempotence laws**
   (i) $A \cdot A = A$
   (ii) $A + A = A$
9. **Absorption laws**
   (i) $A + A \cdot B = A$
   (ii) $A(A + B) = A$

Theorems

1. **Consensus theorem**

Theorem 1: $AB + \overline{A}C + BC = AB + \overline{A}C$

Proof:

$$\begin{aligned}\text{LHS} &= AB + \overline{A}C + BC \\ &= AB + \overline{A}C + BC(A + \overline{A}) \\ &= AB + \overline{A}C + BCA + BC\overline{A} \\ &= AB(1 + C) + \overline{A}C(1 + B) \\ &= AB(1) + \overline{A}C(1) \\ &= AB + \overline{A}C \\ &= \text{RHS}\end{aligned}$$

Theorem 2: $(A + B)(\overline{A} + C)(B + C) = (A + B)(\overline{A} + C)$

Proof:

$$\begin{aligned}\text{LHS} &= (A + B)(\overline{A} + C)(B + C) \\ &= (A\overline{A} + AC + B\overline{A} + BC)(B + C) \\ &= (AC + BC + \overline{A}B)(B + C) \\ &= ABC + BC + \overline{A}B + AC + BC + \overline{A}BC \\ &= AC + BC + \overline{A}B\end{aligned}$$

$$\begin{aligned}\text{RHS} &= (A + B)(\overline{A} + C) \\ &= A\overline{A} + AC + BC + \overline{A}B \\ &= AC + BC + \overline{A}B \\ &= \text{LHS}\end{aligned}$$

2. **Transposition theorem**

$$AB + \overline{A}C = (A + C)(\overline{A} + B)$$

Proof:

$$\begin{aligned}\text{RHS} &= (A + C)(\overline{A} + B) \\ &= A\overline{A} + C\overline{A} + AB + CB \\ &= 0 + \overline{A}C + AB + BC \\ &= \overline{A}C + AB + BC(A + \overline{A}) \\ &= AB + ABC + \overline{A}C + \overline{A}BC \\ &= AB + \overline{A}C \\ &= \text{LHS}\end{aligned}$$

3. **De Morgan's theorem**

Law 1: $\overline{A + B} = \overline{A} \times \overline{B}$

This law states that the complement of a sum of variable is equal to the product of their individual complements.

Law 2: $\overline{AB} = \overline{A} + \overline{B}$

This law states that the complement of the product of variables is equal to the sum of their individual complements.

Example 1: Simplify the Boolean function $Y = A(A + \overline{B})$

$$Y = A \cdot A + A \cdot \overline{B}$$

Solution:

$$\begin{aligned}Y &= A + A\overline{B} \\ &= A(1 + \overline{B}) \\ &= A\end{aligned}$$

Example 2: Simplify the Boolean function $Y = A + \overline{A}B$

Solution:

$$\begin{aligned}Y &= A \cdot (B + 1) + \overline{A} \cdot B \\ &= A \cdot B + A + \overline{A}B \\ &= B(A + \overline{A}) + A \\ &= A + B\end{aligned}$$

Example 3: Simplify the Boolean function

$$Y = A(A + B) + B(\overline{A} + B)$$

Solution:

$$\begin{aligned}Y &= A \cdot A + A \cdot B + B \cdot \overline{A} + B \cdot B \\ &= A + B(A + \overline{A}) + B \\ &= A + B \cdot 1 + B \\ &= A + B + B \\ &= A + B\end{aligned}$$

Example 4: Simplify the Boolean function

$$Y = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}B\overline{C} + AB\overline{C} + A\overline{B}\,\overline{C}$$

Solution:

$$\begin{aligned}Y &= \overline{A}\,\overline{C}(\overline{B} + B) + A\overline{C}(B + \overline{B}) \\ &= \overline{A}\,\overline{C} + A\overline{C} \\ &= \overline{C}(\overline{A} + A) \\ &= \overline{C}\end{aligned}$$

Example 5: Simplify the Boolean function

$$Y = \overline{\overline{A}BC + \overline{A}B\overline{C} + \overline{A}\overline{B}C}$$

Solution:

$$\begin{aligned}&= \overline{\overline{A}C(B + \overline{B}) + \overline{A}B\overline{C}} \\ &= \overline{\overline{A}C + \overline{A}B\overline{C}} \\ &= \overline{\overline{A}(C + B\overline{C})} \\ &= \overline{\overline{A}(C + B)} \\ &= A + \overline{C}\,\overline{B}\end{aligned}$$

Example 6: Simplify the Boolean function

$$Y = AB + C\overline{B} + CA + ABD$$

Solution: $Y = AB(1 + D) + C\overline{B} + CA$

$= AB + C\overline{B} + CA$

$= AB + C\overline{B}$

Properties of Boolean Algebra

With n variables, maximum possible distinct functions $= 2^{2^n}$.

Duality

consider the distributive law

1. $x(y + z) = xy + xz$
2. $x + yz = (x + y)(x + z)$

The second one can be obtained from the law 1, if the binary operators and the identity elements are interchanged. This important property of Boolean algebra is called the *duality principle*.

The dual of an algebraic expression can be written by interchanging OR and AND operators, 1's by 0, and 0's by 1's.

Example 7: $x + x' = 1 \xleftrightarrow{\text{Dual}} x \cdot x' = 0$

Solution: $xy + xy' = x \xleftrightarrow{\text{Dual}} (x + y)(x + y') = x$

$x + x'y = x + y \xleftrightarrow{\text{Dual}} x(x' + y) = xy$

Example 8: The dual of $F = xy + xz + yz$ is?

Solution: Dual of $F = (x + y)(x + z)(y + z)$

$= (x + xz + xy + yz)(y + z) = xy + yz + xz$

So dual of $xy + xz + yz$ is same as the function itself, for N variables maximum possible self-dual functions $= 2^{2^{n-1}} = 2^{(2^n/2)}$

Example 9: Which of the following statements is/are true

S_1: The dual of NAND function is NOR

S_2: The dual of X-OR function is X-NOR

(A) S_1 ND S_2 are true

(B) S_1 is true

(C) S_2 is true

(D) None of these

Solution: (A)

NAND $= (xy)' = x' + y'$

Dual of NAND $= (x + y)' = x'y'$

X-OR $= xy' + x'y$

Dual of X-OR $= (x + y')(x' + y) = xy + x'y' =$ X-NOR

Both S_1 and S_2 are true

Operator Precedence

The operator precedence for evaluating Boolean expression is

1. Parentheses
2. NOT
3. AND
4. OR

So the expression inside the parentheses must be evaluated before all the operations. The next operation to be performed is the complement and then follows AND and finally the OR.

Complement of Function

The complement of a function F is F' is obtained from an interchange of 0, s for 1, s and 1, s for 0, s in the value of F. The complements of a function may be derived algebraically through De Morgan's theorems.

$$(x_1 \cdot x_2 \cdot x_3 \ldots x_n)' = x'_1 + x'_2 + x_3' + \cdots + x_n'$$
$$(x_1 + x_2 + x_3 + \cdots + x_n)' = x'_1 \cdot x_2' \cdot x'_3 \cdot x'_4 \ldots x_n'$$

Example 10: The complement of function $F = a(b'c + bc')$ is?

Solution: $(F)' = [a(b'c + bc')]'$

$= a' + (b'c + bc')'$

$= a' + (b'c)' \cdot (bc')'$

$= a' + (b + c)'(b' + c)$

$F' = a' + bc + b'c'$

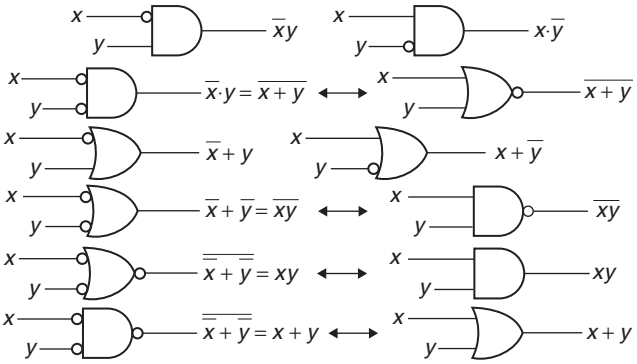


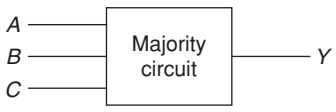
Figure 7 Gates with inverted inputs

Boolean Functions, Min Terms and Max Terms

The starting point for designing most logic circuits is the truth table, which can be derived from the statement of problem. The truth table is then converted into a Boolean expression and finally the assembly of logic gates accordingly.

Let us consider the example of majority circuit. This circuit takes three inputs (A, B, C) and have one output (Y) which will give the majority of the inputs, i.e., if A, B, C are having more number of zeros. $Y = 0$ else if A, B, C are having more number of 1, s, $Y = 1$.

So from the statements we can derive the truth table as follows:



As we are using three Boolean variables A, B and C, total number of combinations in truth table are $2^3 = 8$.

Similarly for n variables, the truth table will have total of 2^n combinations, for a Boolean function.

	Input			Output	
Sl. No.	A	B	C	Y	
1	0	0	0	0	⟶ Y = 0, If inputs are having more zeros.
2	0	0	1	0	
3	0	1	0	0	
4	0	1	1	1	⟶ Y = 1, If inputs are having more 1's
5	1	0	0	0	
6	1	0	1	1	
7	1	1	0	1	
8	1	1	1	1	

For some combinations, output $Y = 1$, and for others $Y = 0$. The input combinations for which output $Y = 1$ are called as *min terms*.

Similarly the input combinations for which output $Y = 0$ are called as *max terms*.

Min terms are expressed as product terms. Similarly, max terms are expressed as sum terms.

The output $Y = 1$, only in rows 4, 6, 7, 8.

So the min terms combinations are 011, 101, 110, 111 in Boolean Algebra, 1 input will be written as A, B, C and 0 input will be written as $\bar{A}, \bar{B}, \bar{C}$ in complement form, we express these min terms as product terms, $\bar{A}BC$, $A\bar{B}C$, $AB\bar{C}$, ABC.

To express Y as Boolean expression, we can write it as sum of the min terms.

$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

We know that AND operation is a product while OR is sum. So the above equation is a sum of the products (SOP), (or) min terms expression.

The other way of expressing Y is $Y = \Sigma m\ (3, 5, 6, 7)$.

$$Y = m_3 + m_5 + m_6 + m_7.$$

The min term numbers are the decimal equivalent of input binary combinations.

Similar to SOP, we can have product of sums (POS) Boolean expression.

The output $Y = 0$ for the input combinations 000, 001, 010, 100. For max terms, one input will be indicated as $\bar{A}, \bar{B}, \bar{C}$ in complement form, 0 input will be indicated as A, B, C and max terms are expressed as sum terms.

$$A + B + C,\ A + B + \bar{C},\ A + \bar{B} + C, \bar{A} + B + C$$

Any function can be expressed as product of max terms.

So $Y = (A + B + C)(A + B + \bar{C})(A + \bar{B} + C)(\bar{A} + B + C)$

The above equation is a product of sum expression (POS) or max terms expression.

In other way $Y = \pi M\ (0, 1, 2, 4)$

$$= M_0 \cdot M_1 \cdot M_2 \cdot M_4$$

The max term numbers are decimal equivalents of corresponding input binary combinations.

Min Term and Max Term

All the Boolean expressions can be expressed in a standard sum of product (SOP) form or in a standard product of sum (POS) form.

- A standard SOP form is one in which a number of product terms, each contains all the variables of the function either in complement or non-complement form are summed together.
- A standard POS form is one in which a number of sum terms, each one of which contain all the variable of the function either in complemented or non-complemented form are multiplied together.
- Each of the product term in standard SOP form is called a *min term*.
- Each of the sum term in the standard POS form is called a *max term*.

Conversion from min terms to max terms representation

$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

$$Y' = (\bar{A}BC + A\bar{B}C + AB\bar{C} + ABC)'$$

$$= (\bar{A}BC)'(A\bar{B}C)'(AB\bar{C})'(ABC)'$$

$$(Y')' = [(A + \bar{B} + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C})]'$$

$$= [\pi(3, 5, 6, 7)]' = \pi(0, 1, 2, 4)$$

$$Y = (A + B + C)(A + B + \bar{C})(A + \bar{B} + C)(\bar{A} + B + C)$$

or $Y = \Sigma(3, 5, 6, 7) = \pi(0, 1, 2, 4)$

Conversion from normal SOP/POS form to canonical SOP/POS

Let us consider $f(A, B, C) = A + BC + \bar{A}C$

The above function is in normal (minimized) SOP form, to convert this function to standard SOP or canonical SOP form, include missing variable in each and every term, to make it complete. First term A, missing literals are B, C. Consider $A\ X\ X$, so possible combinations are $\bar{A}BC$, $A\bar{B}C$, $AB\bar{C}$, ABC or we can write

$$A = A(B + \bar{B})(C + \bar{C}) = ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C}$$

Second term BC = missing literal is A. Consider $XBC \Rightarrow$ so possible combinations are ABC, $\bar{A}BC$ or we can write

$$BC = (A + \bar{A})BC$$

$$= ABC + \bar{A}BC$$

Third term $\bar{A}C$ = missing literal is B. Consider $\bar{A}XC \rightarrow$ so possible combinations are $\bar{A}BC$, $\bar{A}$, $\bar{B}C$ or we can write

$$\bar{A}C = \bar{A}(B + \bar{B})C$$

$$= \bar{A}BC + \bar{A}\bar{B}C$$

Now, $f(A, B, C) = ABC + AB\overline{C} + A\overline{B}C + \overline{A}BC + \overline{A}\overline{B}C$, after removing the redundant terms.

Now consider

$$f(A, B, C) = (A + B)(\overline{A} + C)$$

To convert this expression to canonical form or standard POS form we can write

$$f(A, B, C) = (A + B + C \cdot \overline{C})(\overline{A} + B \cdot \overline{B} + C)$$

Here, the C variables is absent from first term and B from second term. So add $C \cdot \overline{C}$ (= 0) to first, and $B \cdot \overline{B}$ to second, and using distributive law arrive at the result.

$$f(A, B, C) = (A + B + C)(A + B + \overline{C})(\overline{A} + B + C)(\overline{A} + \overline{B} + C)$$

Minimization of Boolean Functions

Simplification Procedure

- Obtain truth table, and write output in canonical form or standard form
- Generate K-map!
- Determine prime implicants
- Find minimal set of prime implicants.

Karnaugh Map (K-map) Method

Karnaugh map method is a systematic method of simplifying the Boolean expression. K-map is a chart or a graph composed of an arrangement of adjacent cell, each representing a particular combination of variable in sum or product form. (i.e., min term or max term).

Two-variable K-map

x	y	F
0	0	m_0
0	1	m_1
1	0	m_2
1	1	m_3

m_0	m_1
m_2	m_3

x \ y	0	1
0	x′y′	x′y
1	xy′	xy

Three-variable K-map

A three-variable map will have eight min terms (for three variables $2^3 = 8$) represented by 8 squares

x	y	z	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

x \ yz	00	01	11	10
0	x′y′z	x′y′z	x′yz	x′′
1	xy′z′	xy′z	xyz	xyz′

3-variable K-map

Four-variable K-maps

The K-map for four variables is shown here, 16 min terms are assigned to 16 squares.

wx \ yz	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

The map is considered to lie on a surface with the top and bottom edges as well as the right and left edge touching each other to form adjacent squares.

- One square → a min term of four literals
- Two adjacent square → a term of three literals
- Four adjacent square → a term of two literals
- Eight adjacent square → a term of one literal
- Sixteen adjacent square → the constant one

Don't-care Combinations

It can often occurs that for certain input combinations, the value of the output is unspecified either because the input combination are invalid or because the precise value of the output is of no consequence. The combination for which the values of the expression are not specified are called don't-care combinations. During the process of design using an SOP, K-map, each don't-care is treated as a 1 if it is helpful in map reduction, otherwise it is treated as a 0 and left alone.

During the process of design using a POS K-map, each Don't-care is treated as a 0 if it is useful in map reduction, otherwise it is treated as a 1 and left alone.

Example 11: Find the minimal expression
$\Sigma m\,(1, 5, 6, 12, 13, 14) + d(2, 4)$

Solution:

AB \ CD	00	01	11	10
00		1		×
01	×	1		1
11	1	1		1
10				

$\therefore \quad F = B\overline{C} + B\overline{D} + \overline{A}\overline{C}D$

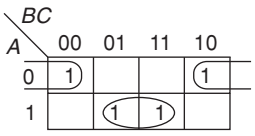
Pairs, Quads and Octets

A \ BC	00	01	11	10
0		1	1	
1				

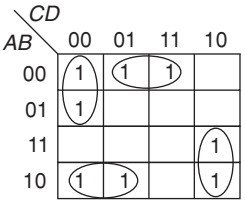
The map contains a pair of 1, *s* those are horizontally adjacent. Those cells represent $\overline{A}\overline{B}C$, $\overline{A}BC$.

For these two min terms, there is change in the form of variable *B*. By combining these two cells we can form a pair, which is equal to $\overline{A}\overline{B}C + \overline{A}BC = \overline{A}C(\overline{B}+B) = \overline{A}C$.

If more than one pair exists on K-map, OR the simplified products to get the Boolean function.



$F = \overline{A}\overline{C} + AC$

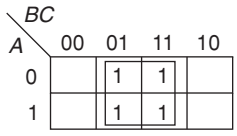


$F = \overline{A}C\overline{D} + \overline{A}B\overline{D} + A\overline{B}\overline{C} + AC\overline{D}$

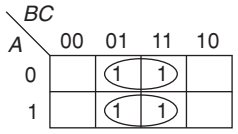
So pair eliminates one variable by minimization.

Quad

Quad is a group of four 1, *s* those are horizontally or vertically adjacent.

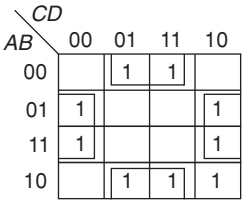


$F = C$



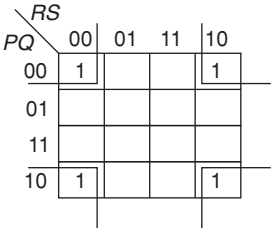
$F = \overline{A}C + AC = (\overline{A}+A)C = C$

By considering two pairs, also it will be simplified to *C*. Quad eliminates two variables from the function



$F = B\overline{D} + \overline{B}D$

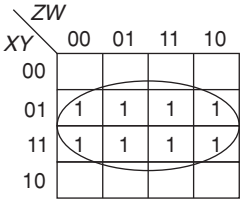
Corner min terms can from a quad:



$F = \overline{Q}\overline{S}$

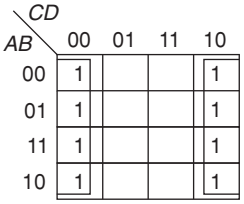
Octet

The group of eight cells will form one octet.



$F = Y$

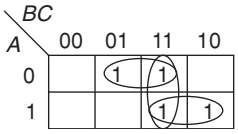
Other variable *X*, *Z*, *W* changes their form in octet. Octet can eliminate three variables and their complements.



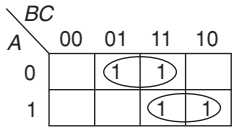
$F = \overline{D}$

Other variable *A*, *B*, *C* are vanished.

Eliminating Redundant Groups

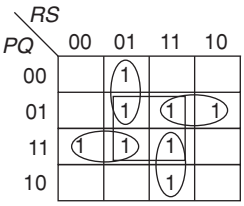


$AB + \overline{A}C + BC$

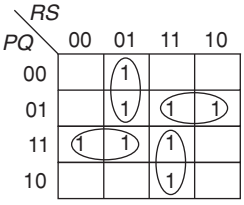


$AB + \overline{A}C$

Here *BC* is redundant pair, which covers already covered min terms of AB, $\overline{A}C$.



This K-map gives fourpairs and one quad.

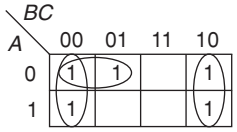


But only four pairs are enough to cover all the min times, quad is not necessary.

$\overline{P}\overline{R}S + \overline{P}QR + PQ\overline{R} + PRS$ is minimized function.

Prime Implicant

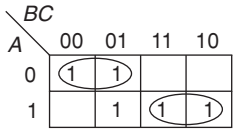
The group of adjacent min terms is called a prime implicant, i.e., all possible pairs, quads, octets, etc.



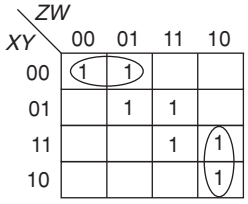
Prime implicants are $\overline{B}\overline{C}$, $B\overline{C}$, $\overline{C}$, $\overline{A}\overline{B}$. Minimized function is $\overline{C} + \overline{A}\overline{B}$.

Essential Prime Implicant

The prime implicant which contains at least one min term which cannot be covered by any other prime implicant is called essential prime implicant.



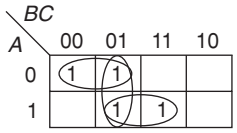
Min term 0, 6 can be grouped with only one pair each. The total possible prime implicants are $\overline{A}\overline{B}$, $\overline{B}C$, AC, AB but min term 0, 6 can be covered with $\overline{A}\overline{B}$, AB . So we call them as essential prime implicants. Min term 5 can be paired with any of 1 or 7 min term.



The essential prime implicants are $XZ\overline{W}$, $\overline{X}\overline{Y}\overline{Z}$

Redundant Prime Implicant

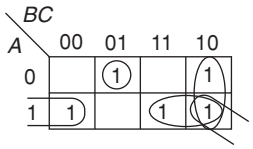
The prime implicant whose min terms are already covered by at least one min term is called redundant prime implicants.



Here prime implicants are $\overline{A}\overline{B}$, AC, $\overline{B}C$. But $\overline{B}C$ is already covered by other min terms. So $\overline{B}C$ is redundant prime implicant.

Example 12: Find the minimal expression for $\Sigma m(1, 2, 4, 6, 7)$ and implement it using Basic gates.

Solution: K-map is



$$F = A\overline{C} + AB + B\overline{C} + \overline{B}C + \overline{A}\,\overline{B}C$$

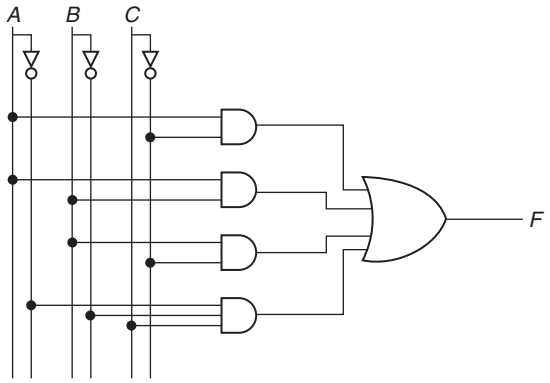
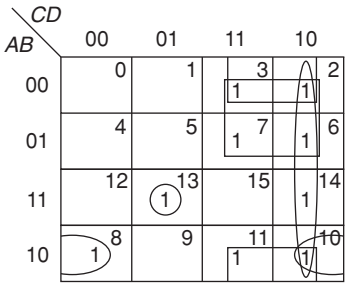


Figure 8 Logic diagram

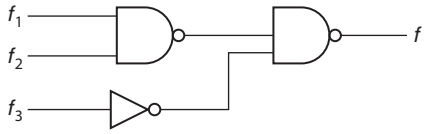
Example 13: Find the minimal expression for Σm (2, 3, 6, 7, 8, 10, 11, 13, 14)

Solution: K-map is:



$$\therefore \quad F(A, B, C, D) = AB\overline{C}D + A\overline{B}\overline{D} + \overline{A}C + \overline{B}C + C\overline{D}$$

Example 14: Three Boolean functions are defined as below $f_1 = \Sigma m(0, 1, 3, 5, 6)$, $f_2 = \Sigma m(4, 6, 7)$ $f_3 = \Sigma m(1, 4, 5, 7)$, then find f.



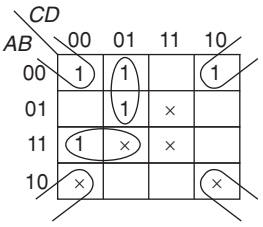
Solution: When two Boolean functions are ANDed, the resultant will contain the common min terms of both of the functions (e.g., intersection of min terms). If two Boolean functions are ORed, then resultant is the combination of all the min terms of the functions (like union of min terms)

Here $f = \overline{\overline{f_1 f_2} \cdot \overline{f_3}} = f_1 f_2 + f_3$

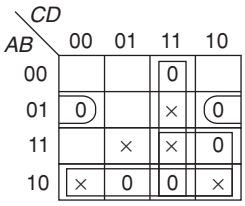
Here $f_1 \cdot f_2$ = Common min terms in f_1 and $f_2 = \Sigma m(6)$
$f_1 \cdot f_2 + f_3$ = Combination of min terms of $f_1 \cdot f_2$ and f_3
$= \Sigma(1, 4, 5, 6, 7)$

Example 15: What is the literal count for the minimized SOP, and minimized POS form for the following function?
$f(A, B, C, D) = \Sigma m(0, 1, 2, 5, 12) + \phi d(7, 8, 10, 13, 15)$

Solution: $f(A, B, C, D) = \Sigma m(0, 1, 2, 5, 12) + \phi(7, 8, 10, 13, 15)$



f = 1 quad + 2 pairs
Literal count = $1 \times 2 + 2 \times 3 = 8$
$f(A, B, C, D) = \pi M(3, 4, 6, 9, 11, 14) + \phi(7, 8, 10, 13, 15)$



f will consists of three quads + one pair = $3 \times 2 + 1 \times 3 = 6 + 3 = 9$

Implementation of Function by Using NAND–NOR Gates

NAND or NOR gates are called as universal gates, because any function can be implemented by using only NAND gates or only using NOR gates.

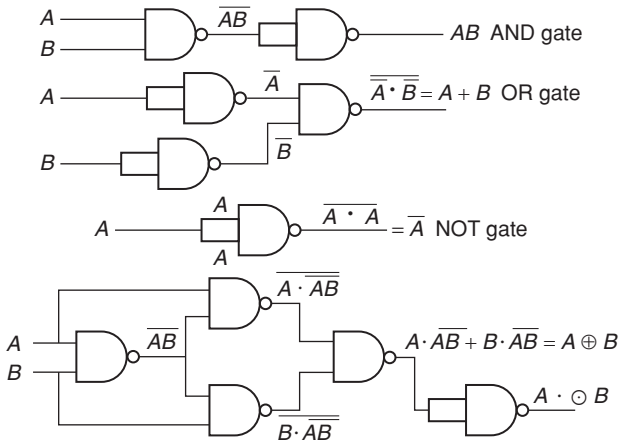


Figure 9 Implement of basics gates by using NAND gates

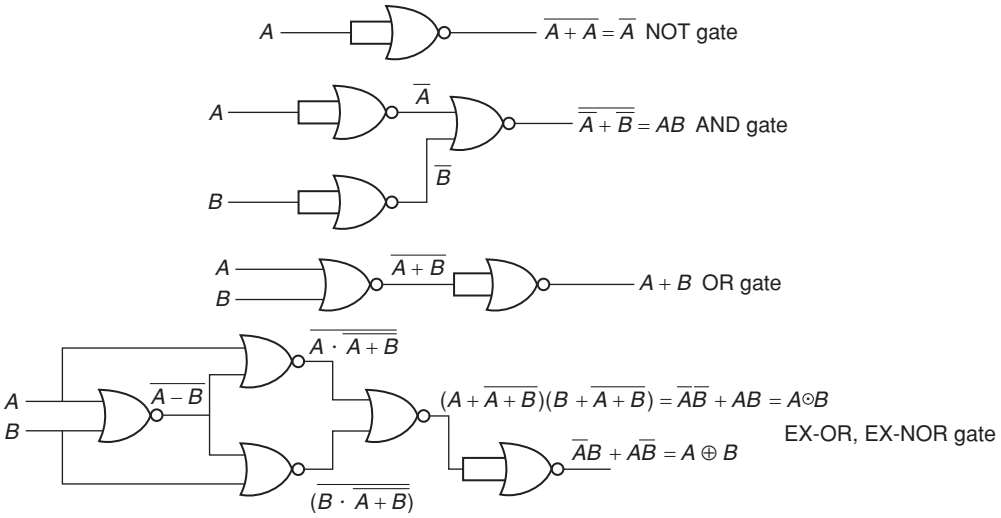
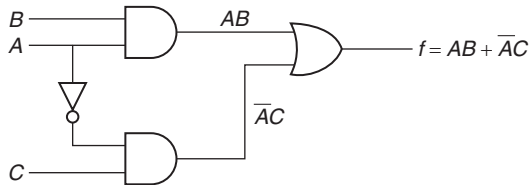


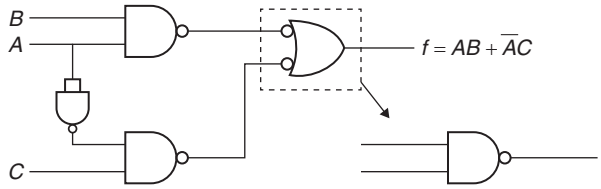
Figure 10 Implementation of basic gates by using NOR gates

Any function which is in the SOP form can be implemented by using AND-OR gates, which is also equivalent to NAND–NAND gates.

$F = AB + \overline{A}C$



By considering bubble at AND gate output and OR gate input, and by changing NOT gates to NAND gates, the circuit becomes as,



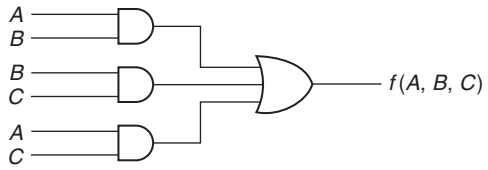
Now the circuit is in completely in NAND – NAND form

So the functions expressed in SOP form can be implemented by using AND – OR, (or) NAND–NAND gates.

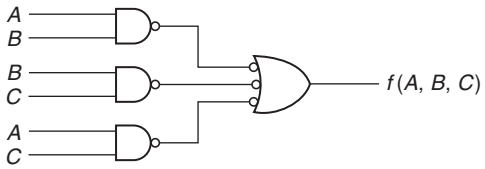
Any function in POS from can be implemented by using OR–AND gates, which is similar to NOR–NOR gate.

Example 16: How many number of NAND gates are required to implement $f(A, B, C) = AB + BC + AC$
(A) 3 (B) 4 (C) 5 (D) 6

Solution:



By considering bubbles at output of AND gate and input of OR gate.

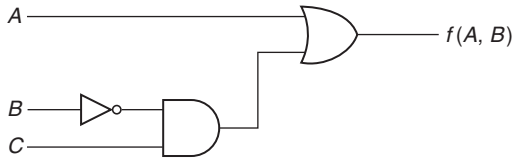


So four NAND gates are required.

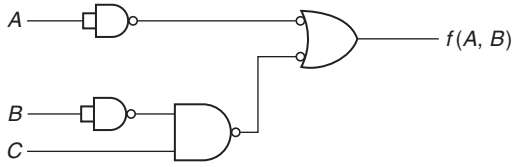
Example 17: Number of NAND gates required for implementation of $f(A, B) = A + \overline{B}C$ is

(A) 3 (B) 4 (C) 5 (D) 6

Solution:



To convert the all gates into NAND gates, place bubble output of AND, and inputs of OR gates. Now, the circuit can be drawn as



Four NAND gates are required.

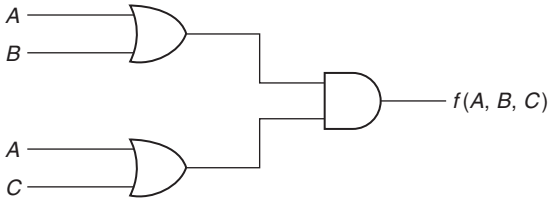
Example 18: $f = A + BC$, the number of NOR gates required to implement f, are?

(A) 3 (B) 4 (C) 5 (D) 2

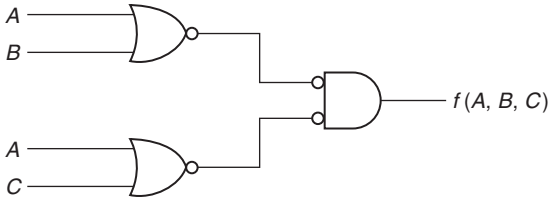
Solution: $A + BC$ is in SOP form.

To implement this function by using NOR gates, we can write $f(A, B, C) = A + BC = (A + B)(A + C)$

Which is in the form of POS?



By including bubbles at output of OR gate, and input of AND gate, the circuit becomes



Now the circuit consists of all NOR gates. So three NOR Gates are required.

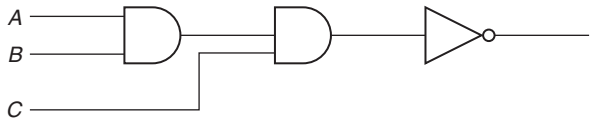
Example 19: How many number of two-input NAND–NOR gates are required to implement three-input NAND–NOR gates respectively?

(A) 2, 2 (B) 2, 3

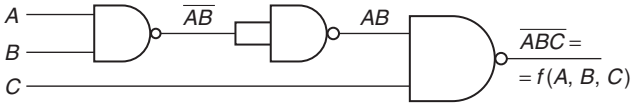
(C) 3, 2 (D) 3, 3

Solution: $f(A, B, C) = \overline{ABC} = \overline{AB} + \overline{C}$

(1) Implement above function by using two-inputs gates:

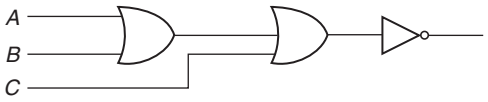


Now convert each gate to NAND gate:

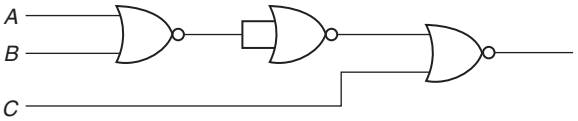


Three two-input NAND gates are required.

(2) $G(A, B, C) = \overline{A + B + C}$ – Implement it by using two-input gates



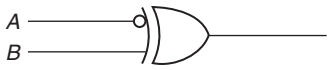
Now convert each gate to NOR gate:



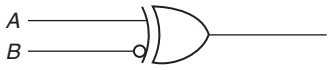
Three two-input, NOR gates are required.

EX-OR, EX-NOR Gates

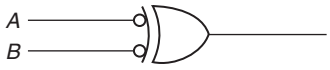
Inverted inputs for EX-OR, EX-NOR gates



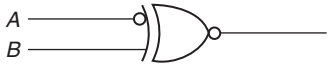
$$\overline{A} \oplus B = \overline{\overline{A}B} + \overline{A}\,\overline{B} = AB + \overline{A}\,\overline{B} = A \odot B$$



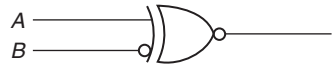
$$A \oplus \overline{B} = A\overline{\overline{B}} + \overline{A}\,\overline{B} = AB + \overline{A}\,\overline{B} = A \odot B$$



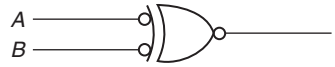
$$\overline{A} \oplus \overline{B} = \overline{A}\,\overline{\overline{B}} + \overline{\overline{A}}\,\overline{B} = \overline{A}B + A\overline{B} = A \oplus B$$



$$\overline{A} \odot B = \overline{A}B + \overline{\overline{A}}\,\overline{B} = \overline{A}B + A\overline{B} = A \oplus B$$



$$A \odot \overline{B} = A\overline{B} + \overline{A}\overline{\overline{B}} = A\overline{B} + \overline{A}B = A \oplus B$$

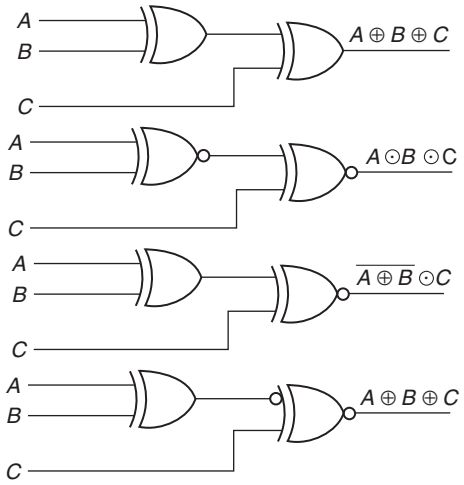


$$\overline{A} \odot \overline{B} = \overline{A}\overline{B} + \overline{\overline{A}}\overline{\overline{B}} = \overline{A}\overline{B} + AB = A \odot B$$

From the above discussions, we can conclude that inverted input EX-OR gate is EX-NOR gate.

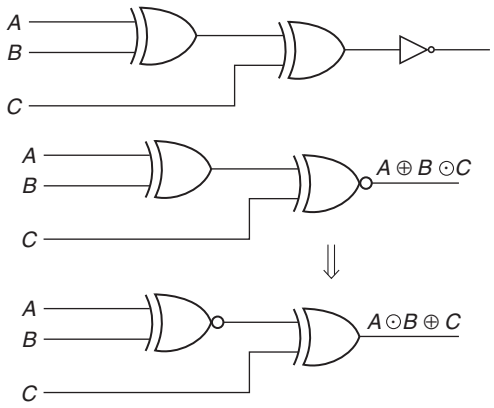
Similarly inverted input EX-NOR gate is EX-OR gate. If both inputs are inverted, the EX-OR/EX-NOR will remain as it is.

Consider a three-inputs X-OR gates by using two-input X-OR gates.



So we can conclude that $A \oplus B \oplus C = A \odot B \odot C$

$$\overline{A \oplus B \oplus C} = \overline{A \odot B \odot C}$$



$\overline{A \oplus B \oplus C} = \overline{A \odot B \odot C} = A \oplus B \odot C$

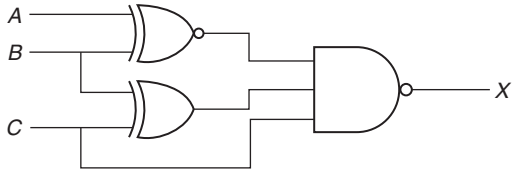
$= A \odot B \oplus C$

$A \oplus B \oplus C \oplus D = A \oplus B \odot C \odot D = A \odot B \odot C \oplus D = A \odot B \oplus C \odot D$

$A \odot B \odot C \odot D = \overline{A \oplus B \oplus C \oplus D}$

$A \odot B \odot C \odot D = A \oplus B \oplus C \odot D = A \oplus B \odot C \oplus D$

Example 20: For the logic circuit shown in figure, the required input condition (A, B, C) to make the output $X = 0$ is?



(A) 1, 1, 1 (B) 1, 0, 1
(C) 0, 1, 1 (D) 0, 0, 1

Solution: (D)

To get output $X = 0$, all inputs to the NAND gate should be 1, so $C = 1$. When $C = 1$, the output of X-OR gate $B \oplus C = 1$ only when $B = 0$. If $B = 0$, the output of X-NOR gate $A \odot B = 1$.

Only when $A = 0$.

So $X = 1$, only when $(A, B, C) = (0, 0, 1)$.

Example 21: The minimized expression of

$(A + \overline{B})(A\overline{B} + AC)(\overline{A}\overline{C} + \overline{B})$ is

Solution: $(A + \overline{B})(A\overline{B} + AC)(\overline{A}\overline{C} + \overline{B})$

$= (A + \overline{B})(A\overline{B} \cdot \overline{A}\overline{C} + A\overline{B} \cdot \overline{B} + AC \cdot \overline{A}\overline{C} + AC \cdot \overline{B})$

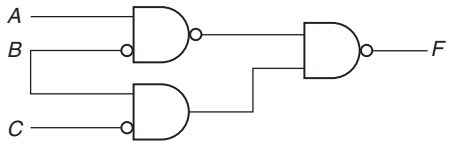
$= (A + \overline{B})(A\overline{B} + A\overline{B}C) = (A + \overline{B})A\overline{B}(1 + C)$

$= A\overline{B} + A\overline{B} = A\overline{B}$

Example 22: The Boolean function f is independent of

(A) a (B) b
(C) c (D) None of these

Solution: (A)

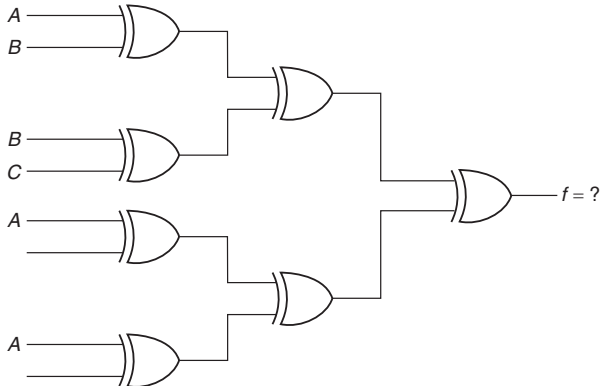


$F = \overline{\overline{ab} \cdot \overline{bc}}$

$= ab + \overline{\overline{bc}} = ab + b + \overline{c}$

$= b + \overline{c}$ is independent of 'a'.

Example 23:



Solution: $f = \{A \oplus B \oplus B \oplus C\} \oplus \{A \oplus C \oplus B \oplus A\}$

$= \{A \oplus 0 \oplus C\} \oplus \{0 \oplus C \oplus B\}$

$= A \oplus C \oplus C \oplus B = A \oplus 0 \oplus B = A \oplus B$

Digital IC Families

Digital logic gates will be implemented by using digital ICs. Logic family is the way to describe the internal circuitry of the IC. Each logic family is characterized by a circuit configuration, semiconductor technology and set of desirable properties.

Bipolar Logic Families

There are two types of operations in bipolar logic families:

1. Saturated
2. Non-saturated

In saturated logic, the transistors in the IC are driven to saturation whereas in the case of non-saturated logic, the transistors are not driven into saturation.

The saturated logic families are

- Diode transistor logic (DTL) transistor–transistor logic (TTL), etc.
- The non-saturated logic families are schottky TTL, emitter coupled logic (ECL), etc.

Unipolar Logic Families

MOS devices are unipolar devices and only MOSFETS are employed in MOS logic circuits,

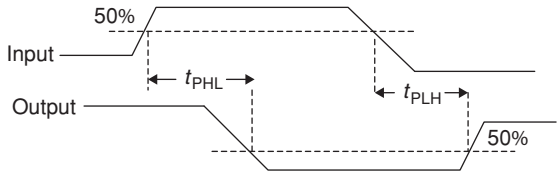
MOS logic families are PMOS, NMOS, CMOS, etc.

Table 1 *Classification of digital ICs*

IC classification	Equivalent individual gates per chip	Number of components
Small-scale integration (SSI)	Less than 12	Up to 99
Medium-scale integration (MSI)	12–99	100–999
Large-scale integration (LSI)	100–99	1000–9,.999
Very large-scale integration VLSI)	Above 1000	Above 10,000
Ultra large-scale integration (ULSI)	—	10^6–10^7
Giant-scale integration (GSI)	—	Above 10^7

Characteristics of Digital ICs

Speed of Operations The speed of digital circuits is specified in terms of propagation delay time:



Input and output wave form to define propagation delay times

The propagation delay time of logic gates is taken as average of these two delay times.

Power Dissipation It is determined by the current I_{CC} that is drawn from V_{CC} supply and is given as $V_{CC} \times I_{CC}$. I_{CC} is the average of $I_{CC}(0)$ and $I_{CC}(1)$. It is known as static power dissipation as it is power consumed by current when input signals are not changing.

Figure of Merit It is a product of speed and power.

Figure of merit = propagation delay (ns) × power (mW)

Fan out It is the number of similar gates which can be driven by a gate, where high fan out is advantageous.

Current and Voltage Parameters

High-level input voltage V_{IH}: minimum input voltage which is recognized by the gate as logic 1.

Low-level input voltage V_{IL}: maximum input voltage which is recognized by gate as logic 0.

High-level output voltage V_{OH}: minimum voltage available at the output corresponding to logic 1.

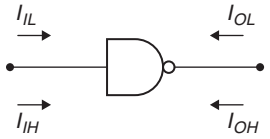
Low-level output voltage V_{OL}: maximum voltage available at the output corresponding to logic 0.

High-level input current I_{IH}: Minimum current which must be supplied by a driving source corresponding to logic 1 level voltage.

Low-level input current I_{IL}: minimum current which must be supplied by a driving source corresponding to logic 0 level voltage.

High-level output current I_{OH}: maximum current which the gate can sink in 1 level.

Low-level output current I_{OL}: This is the maximum current which the gate can sink in 0 level.



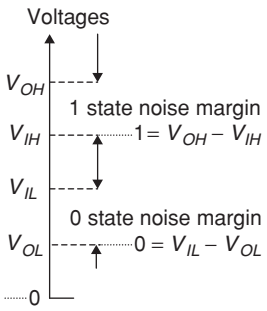
A gate with current directions marked:

$I_{CC}(1)$: High-level supply current when output is at logic 1.

$I_{CC}(0)$: Low-level supply current, when output of gate is at logic 0.

Noise immunity

Stray electric and magnetic fields may induce unwanted voltage, known as noise, on connecting wires between logic circuits.



This may cause the voltage at the input to a logic circuit to drop below V_{IH} or rise above V_{IL} and may produce undesired operation.

The circuit's ability to tolerate noise signals is referred to as noise immunity, a quantitative measure of which is called noise margin.

Note:

Noise margin: High-state noise margin $V_{NH} = V_{OH(\min)} - V_{IH(\min)}$

Low-state noise margin $V_{NL} = V_{IL(\max)} - V_{OL(\max)}$

$\therefore$ Overall noise margin $NM = \min \{V_{NH}, V_{NL}\}$

HTL logic family has highest noise margin among all logic families.

Fan out: Fan out (High) $= F_{OH} = \dfrac{I_{OH(\max)}}{I_{IH(\max)}}$

Fan out (Low) $= F_{OL} = \dfrac{I_{OL(\max)}}{I_{IL(\max)}}$

$\therefore$ Overall fan out = min $\{F_{OH}, F_{OL}\}$

CMOS logic family has highest fan out among all families.

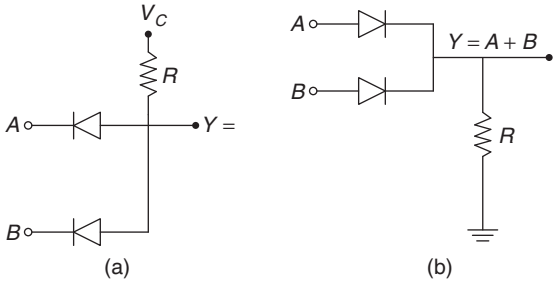
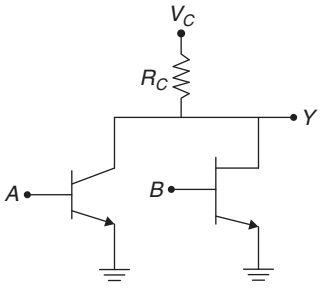


Figure 11 Basic gate and it diode circuit diagram (a) AND gate, (b) OR gate

Direct coupled transistor logic (DCTL)

In the RTL logic family, if the base resistor is disconnected, we obtain DCTL.

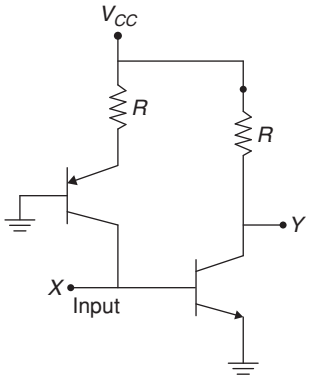


It is suffering from current hogging problem.

- The noise margin of the DCTL is very poor.
- The basic gate is a NOR gate.

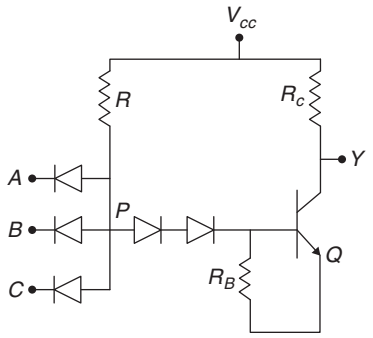
Integrated ilogic Integrated Injection logic (I^2L) is the modified version of DCTL. In direct coupled transistor logic (DCTL) suffers from the current hogging problem. It is also called merged-transistor logic (MTL).

In I^2L logic family, PNP and NPN transistors are integrated; so, it uses very small silicon chip area.



- I^2L available with multi-collector output. So its fan out is higher than RTL logic family.
- The speed of operation of I^2L depends upon the charging current and the propagation delay time is inversely proportional to the charging current.
- Figure of merit in the range of 0.1 to 0.7 pJ.
- I^2L has the highest figure of merit among all logic families.

Diode transistor logic (DTL)

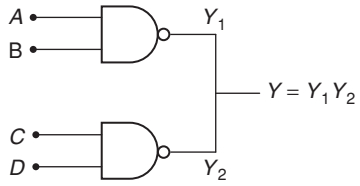


If any of the input is low, the input diode will be forward biased and the voltage at P drives transistor Q to cut off, so output Y is V_{CC}. If all the inputs are HIGH, then none of the input diode will be forward biased and voltage V_{CC} through 'R' and P will drive transistor Q into saturation and output Y will be V_{CE}, sat, which is (logic 0).

If we consider voltage corresponding to logic 1 and 0 as V_{CC} and V_{CE}, respectively, this circuit performs NAND operation.

The propagation delay time of commercially available DTL gates are of the order of 30 to 80 ns.

If outputs of gates are connected together, additional logic is preformed without additional hardware. This type of connection is referred to as wired logic (or) wired AND,



$$Y = Y_1 Y_2 = \overline{AB} \cdot \overline{CD} = \overline{AB + CD}$$

Transistor–transistor logic (TTL)

It is the most successful bipolar logic. TTL logic families use transistors, both to perform logic functions and to provide high-output drive capability.

The NAND gate is the basic TTL logic circuit. It uses multiple-emitter transistor Q_1.

The number of inputs (fan in) to the gate is same as the number of emitters fabricated during its manufacturing.

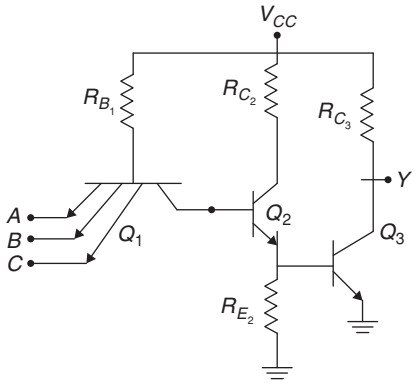


Figure 12 3-Input TTL NAND gate

If at least one input is low, the corresponding emitter base junction is forward biased, so the base voltage of Q_1 will not able to make base collector junction of Q_1 to forward bias, and for Q_2 and Q_3 to be conductions hence Q_2 and Q_3 are OFF and $Y = V_{CC} = V(1)$.

If all inputs are HIGH, the emitter base junctions of Q_1 are reverse biased. If we assume that Q_2 and Q_3 are ON, therefore $Y = V(0) = 0.2$ V.

The fan out (N) of a gate depends upon the maximum collector current rating of the transistor Q_3. The total collector current $I_{C_3} = I_{C_3,\,\text{sat}} = N \cdot I_L$, when the output is in the low state.

If the output of gates are connected together, wired AND logic is performed.

The output voltage is pulled up by the time constant $T = R_{C_3} \cdot C_0$ resistance R_{C_3} is known as pull-up resistor.

It is possible in TTL gates to hasten the charging of output capacitance without corresponding increase in power dissipation with the help of an output circuit arrangement referred to as on active pull up or totem pole output.

Wired–AND connection must not be used for totem–pole output circuits because of the current spike problem.

Open collector output: A circuit with open collector output is the same as the circuit except for the collector – resistor R_{C3} of Q_3 which is missing. The collector terminal is available outside the IC, and the passive pull up is to be connected externally.

If any input of a TTL gate is left disconnected, (open or floating), the corresponding $E - B$ junction of Q_1 will not be forward biased. Hence it acts exactly in the same way as if a logic 1 is applied to that input. Therefore, in TTL ICs, all unconnected inputs are treated as logic 1's.

Schottky TTL

The speed limitation of TTL is mainly due to the turn-off time delays involved in transistors while making transistors from saturation to cut off. This can be eliminated by replacing the transistors of TTL gate by Schottky transistors.

Emitter-coupled logic (ECL)

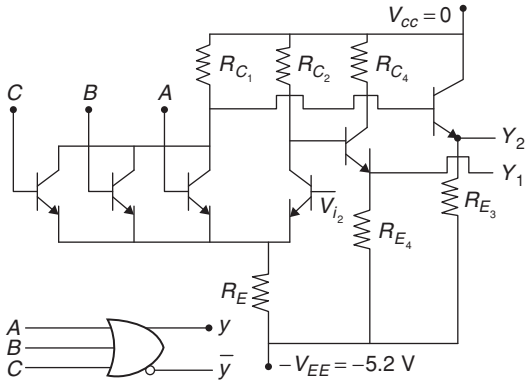


Figure 13 3-input ECL OR / NOR gate

ECL is the fastest of all logic families, and therefore is used in applications where very high speed is essential. High speeds have become possible in ECL because the transistors are used in different amplifier configurations, in which they are never driven into saturation and thereby the storage time is eliminated. Propagation delays of less than 1 ns per gate have become possible in ECL.

ECL is realized using difference amplifier in which the emitters if the two transistors are connected and hence it is referred to as emitter coupled logic. Two outputs, Y_1 and

Y_2 which are complementary are available at ECL output, Y_1 corresponds to OR output, and Y_2 corresponds to NOR output.

In ECL, the positive end of the supply is connected to ground in contrast to other logic families. Because of the low output impedance and high input impedance, the fan out of ECL is large.

Wired OR logic: The outputs of two or more ECL gates can be connected to obtain wired OR logic without using additional hardware.

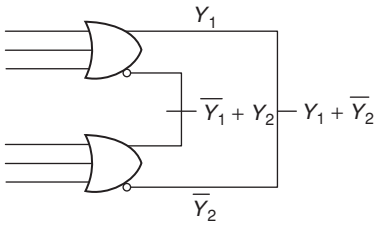


Figure 14 Wired–OR connection of ECL

If any input of an ECL gate is left unconnected, the corresponding *E–B* junction of input transistor will not be conducting, hence it acts as if a logic 0 level voltage is applied to that input.

MOS logic MOSFETS have become very popular for logic circuits due to high density of fabrication and low power dissipation. When MOS devices are used in logic circuits, there can be circuits in which either only p-channel (PMOS) or n-channel (NMOS) devices are used.

It is also possible to fabricate enhancement mode P-channel and n-channel MOS devices (CMOS) on the same chip.

The power dissipation is extremely small for CMOS and hence CMOS logic has became very popular.

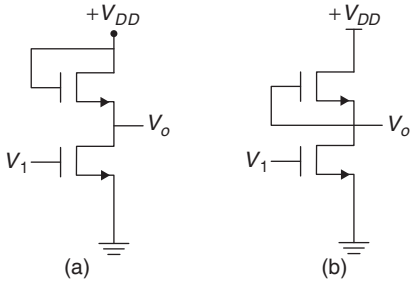


Figure 15 The basic MOS gate is an inverter

A MOS inverter with

(a) Enhancement load
(b) Depletion load

The logic levels of MOS Circuits are

$$V(0) = 0$$
$$V(1) = V_{DD}$$

MOSFET NAND and NOR gates

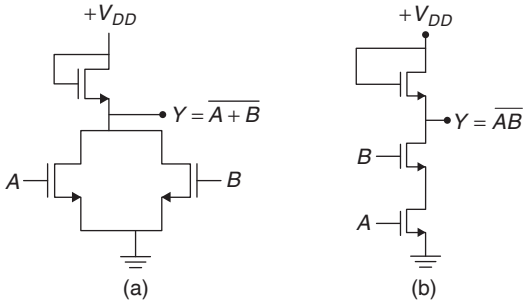


Figure 16 (a) 2-input NMOS NOR gate, (b) 2-Input NMOS NAND gate

NOR gates can be obtained by using multiple drives in parallel, whereas for NAND gates the drives are to be connected in series.

If the input is 0, the corresponding transistor is OFF, else it will be in ON state.

Since MOS devices have very high input impedance, therefore the fan out is large, but driving a large number of MOS gates increases the capacitance at the output of the driving gate, which in turn reduces the speed of MOS gates.

The propagation delay of MOS devices is large because of large capacitance present at the input and output of these devices.

MOS devices have very high input impedance and even a small static charge flowing into this high impedance can develop a dangerously high voltage. Therefore MOS ICs inputs must not be left unconnected.

CMOS logic A complementary MOSFET (CMOS) is obtained by connecting a p- and n-channel MOSFET in series, with drains tied together and the output is taken at the common drain.

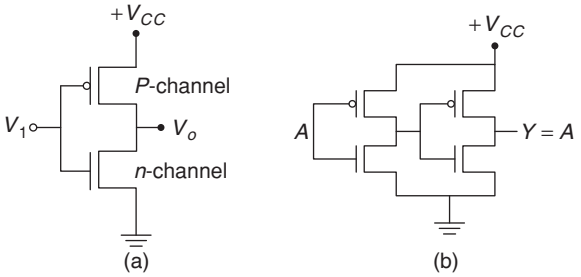


Figure 17 (a) A CMOS inverter, (b) CMOS non-inverting Buffer

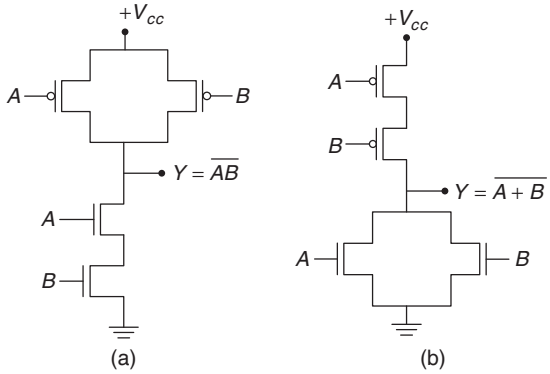
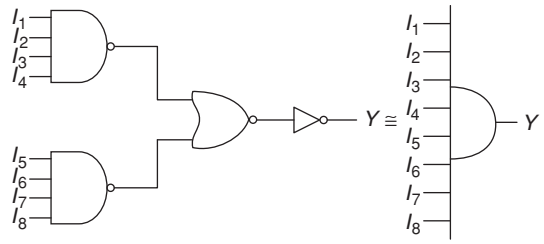


Figure 18 (a) 2-input CMOS NAND gate, (b) 2-input CMOS NOR gate

In CMOS logic, gates with larger number of inputs can be made by cascading gates with fewer inputs which are faster and smaller than the gates obtained by extending series parallel transistors.



Internal structure and equivalent logic for eight input CMOS NAND.

In the given logical structure of eight input CMOS NAND gate, the total propagates delay of this circuit is less than the delay of one-level eight input NAND circuit.

CMOS logic levels are functions of supply voltage, and are approximately

$$V_{OH} = V_{CC} - 0.1 \text{ V}$$

$$V_{IH} = 0.7V_{CC}, V_{IL} = 0.3V_{CC}, V_{OL} = 0.1 \text{ V}$$

The DC margins are

$$\Delta 1 = V_{OH} - V_{IH} = V_{CC} - 0.1 \text{ V} - 0.7 V_{CC}$$

$$\Delta 0 = V_{IL} - V_{OL} = 0.3 V_{CC} - 0.1 \text{ V}$$

The CMOS noise margins are much higher than the TTL noise margins.

For HC series, CMOS current ratings are

$$I_{IH} = 1 \text{ μA}, I_{IL} = -1 \text{ μA}, I_{OH} = 0.02 \text{ mA}, I_{OL} = 0.2 \text{ mA}$$

$$\text{The low-sate fan out} = \frac{I_{OL}}{I_{IL}} = \frac{0.02}{0.001} = 20$$

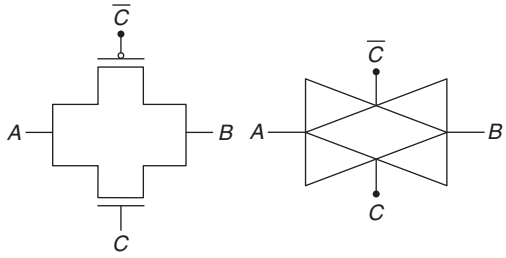
$$\text{The HIGH-state fan out} = \frac{I_{OH}}{I_{IH}} = \frac{0.02}{0.001} = 20$$

An unused NAND or AND input should be tied to logic 1 level, voltage through a pull up resistor and an unused NOR/OR, input should be tied to logic 0 level voltage through a pull down resistor.

Due to the presence of parasitic transistors between V_{CC} and ground of any CMOS device for high input voltages, they cause a virtual short circuit, between V_{CC} and ground, resulting in a very high dissipation enough to destroy the device.

This condition is known as latch up, i.e., stay ON permanently to prevent latch up condition; modern CMOS logic circuits are fabrication with special structures like zener diodes at inputs, for protection.

CMOS transmission of a gate It is controlled by gate voltages C and $\overline{C}$, if $C = 1$ then signal transmitted from A to B, if $C = 0$ transmission is not possible.



Solved Examples

Example 1: Simplify the Boolean function, $x y + x'z + y z$

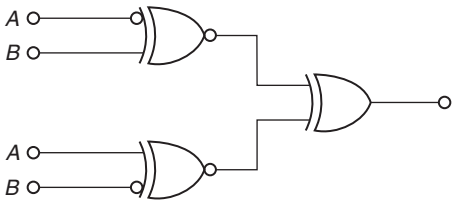
Solution: $x y + x' z + y z$

By using consensus property

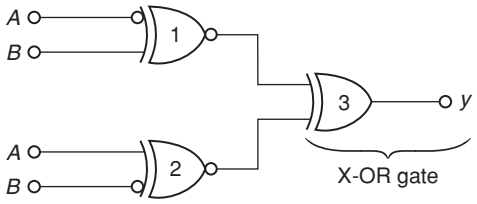
$xy + x'z + yz = xy + x'z$

$Y = xy + x'z$

Example 2: The output of the given circuit is equal to



Solution: $\overline{A} \odot B = \overline{A}B + A\overline{B}$

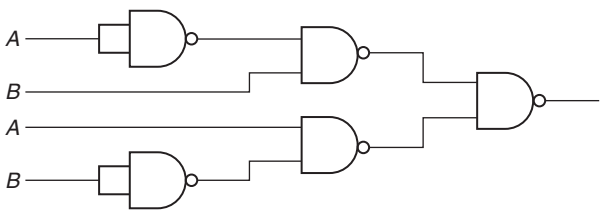


$$A \odot \overline{B} = \overline{A}B + A\overline{B}$$

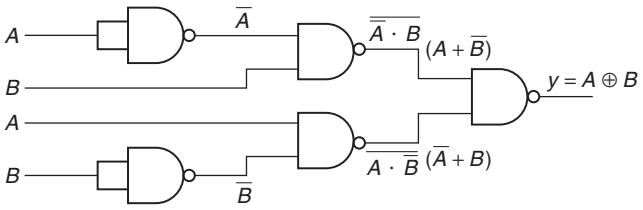
So the output of above circuit is '0'. As two inputs are same at third gate. Output of X-OR gate with two equal inputs is zero.

$\therefore y = 0$

Example 3: The circuit shown in the figure is functionally equivalent to



Solution:



$$Y = \overline{(\overline{\overline{A}\cdot B}\cdot\overline{A\overline{B}})} = \overline{(A+\overline{B})(\overline{A}+B)} \quad \because \quad (\overline{A\cdot B} = \overline{A}+\overline{B})$$

$$= \overline{(A+\overline{B})} + \overline{(\overline{A}+B)} = \overline{A}\cdot\overline{\overline{B}} + \overline{\overline{A}}\cdot\overline{B}$$

$$= \overline{A}\cdot B + A\cdot\overline{B} = A \oplus B$$

Example 4: Simplify the Boolean function $A \oplus \overline{A}B \oplus \overline{A}$

Solution: $A \oplus \overline{A}B \oplus \overline{A}$

Associativity

$$= 1 \oplus \overline{A}B = \overline{\overline{A}B}$$

$$= A + \overline{B} \quad (\because \text{ De morgan's})$$

Example 5:

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	×	×	1
11	×	×	1	×
10	1	0	1	1

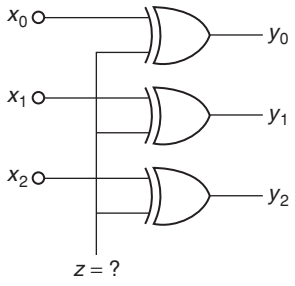
The minimized expression for the given K-map is

Solution:

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	×	×	1
11	×	×	1	×
10	1	0	1	1

$$= A + \overline{B}C$$

Example 6: In the figure shown, y_2, y_1, y_0 will be 1's complement of $x_2\, x_1\, x_0$ if z = ?

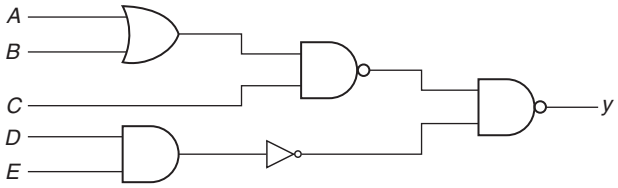


Solution: We are using X-OR gate

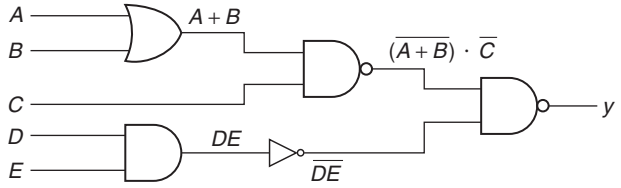
$\therefore$ X-OR output is complement of input only when other input is high.

$\therefore Z = 1$

Example 7: The output y of the circuit shown is the figure is



Solution:



$$y = \overline{\overline{(A+B)\cdot C}\cdot\overline{DE}} = \overline{\overline{(A+B)\cdot C}} + \overline{\overline{DE}}$$

$$= (A+B)C + DE(\overline{x\cdot y} = \overline{x} + \overline{y})$$

Example 8: Simplify the following function

$$f = \overline{\overline{A}(\overline{AB})\cdot\overline{B}(\overline{AB})}$$

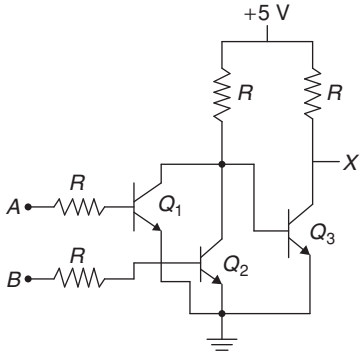
Solution: $\overline{\overline{A}(\overline{AB})\cdot\overline{B}(\overline{A\cdot B})}$

$$\overline{[A+(AB)]\cdot[B+(AB)]} = \overline{A+(AB)} + \overline{(B+AB)}$$

$$= \overline{A}\cdot\overline{(AB)} + \overline{B}\cdot\overline{(AB)} = \overline{A}\cdot(\overline{A}+\overline{B}) + \overline{B}(\overline{A}+\overline{B})$$

$$= \overline{A}\cdot\overline{A} + \overline{A}\overline{B} + \overline{B}\overline{A} + \overline{B}\cdot\overline{B} = \overline{A} + \overline{B} = \overline{AB}$$

Example 9: Which gate is represented by the circuit as shown in the figure?



(A) NAND (B) NOR

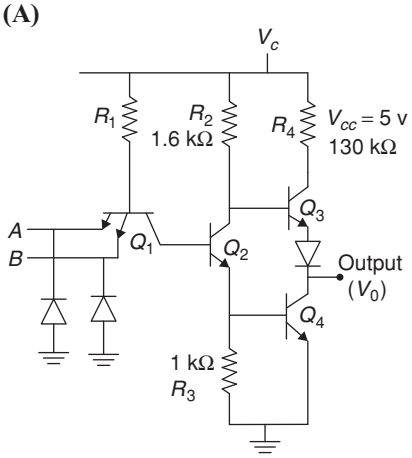
(C) OR (D) AND

Solution: (C)

Transistors with A, B inputs form RTL–NOR gate, other transistor is inverter, so the circuit is OR gate. If any input is 1, either Q_1 or Q_2 will be in saturation, so Q_3 will be in cut off, $x \approx v_{cc}$ (logic 1), hence it is OR gate.

Example 10: In a 7400 TTL NAND gate, V_{cc} = +5 V and a 5 kΩ load is connected to its output. Find the output voltage
(i) when both the inputs are +5 V and
(ii) when both the inputs are 0 V
(A) .0.1 V, 4.09 V (B) 4.8 V, 2.6 V
(C) 4.09 V, 2.26 V (D) .1 V, 3 V

solution: (A)



When both the inputs are high, i.e., +5 V, Q_2 and Q_4 are in saturation. When Q_4 is in saturation, its output is $V_{ce(sat)}$ = low = 0.1 V.
When both inputs are low 0 off the output is high.
(Q_3 is ON but Q_2 and Q_4 are off)

$$\therefore V_{OH} = V_{cc} - I_L R_4 - V_{ce(sat)} - V_D$$
$$= 5 - (I_{OH} \times 130) - .1 - .7$$
$$= (5 - 0.8 - 130 I_L)\ \text{V}$$

$$I_{OH} = \frac{V_{OH}}{5\ \text{k}\Omega}$$

$$V_{OH}\ 4.2 - 130 \times \frac{V_{OH}}{5000}\ \text{V}$$

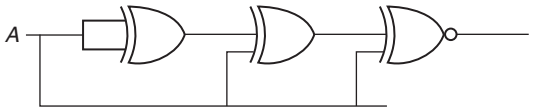
$$V_{OH} = \frac{4.2}{1 + \frac{13}{500}}\ \text{V} = 4.09\ \text{V}.$$

EXERCISES

Practice Problems I

Directions for questions 1 to 33: Select the correct alternative from the given choices.

1. The output of the following circuit is

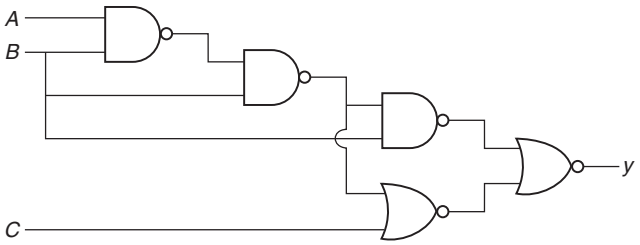


(A) 0 (B) 1
(C) A (D) A'

2. The circuit which will work as OR gate in positive level will work as ___ gate in negative-level logic
(A) NOR gate
(B) NAND gate
(C) Both NAND and NOR gate
(D) AND gate

3. Four logical expressions are given below:
(a) $\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D}\overline{E} \cdot \overline{F} \cdot \overline{G} \cdot \overline{H}$
(b) $\overline{AB} \cdot \overline{CD} \cdot \overline{EF} \cdot \overline{GH}$
(c) $\overline{A} + \overline{B} + \overline{C} + \overline{D} + \overline{E} + \overline{F} + \overline{G} + \overline{H}$
(d) $(\overline{A} + \overline{B})(\overline{C} + \overline{D})(\overline{E} + \overline{F})(\overline{G} + \overline{H})$
Two of these expression are equal. They are
(A) c and d (B) b and d
(C) a and b (D) a and c

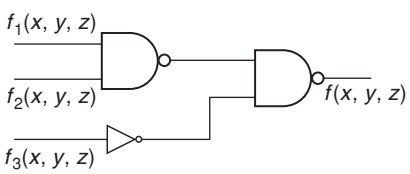
4. For the logic circuit shown in figure, the simplified Boolean expression for the out put y is



(A) $A + B + C$ (B) A
(C) $AB\overline{C}$ (D) $B\overline{C}$

5. In a digital system, there are three inputs A, B and C. The output should be high when at least two inputs are high. The minimized Boolean expression for the out put is
(A) $AB + BC + AC$
(B) $ABC + AB\overline{C} + \overline{A}BC + A\overline{B}C$
(C) $AB\overline{C} + A\overline{B}C + \overline{A}BC$
(D) $A\overline{B} + B\overline{C} + \overline{A}C$

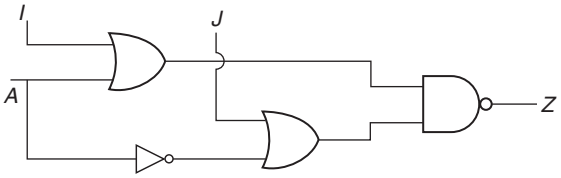
6. Consider the following logic circuit whose inputs are functions f_1, f_2, f_3 and output is f.



Given that $f_1(x, y, z) = \Sigma(0, 1, 3, 5)$ $f_2(x, y, z) = \Sigma(6, 7)$ and $f(x, y, z) = \Sigma(1, 4, 5)$, then f_3 is

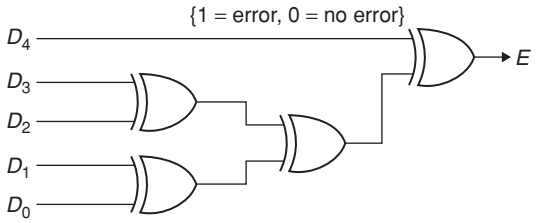
(A) $\Sigma(1, 4, 5)$ (B) $\Sigma(6, 7)$
(C) $\Sigma(0, 1, 3, 5)$ (D) None of these

7. The circuit shown above is to be used to implement the function $z = f(A, B) = \overline{A} + B$ what values are to be selected for I and J?



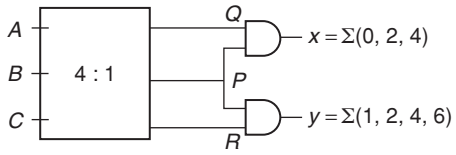
(A) $I = 0, J = B$ (B) $I = 1, J = B$
(C) $I = B, J = 1$ (D) $I = B, J = 0$

8. Parity checker output from the below figure, if input is 11111 ($D_4\ D_3\ D_2\ D_1\ D_0$) and 10000 ($D_4\ D_3\ D_2\ D_1\ D_0$).



(A) error, error (B) error, no error
(C) no error, error (D) no error, no error

9. For the given combinational network with three inputs A, B and C, three intermediate outputs P, Q and R, and two final outputs $X = P \cdot Q = \Sigma(0, 2, 4)$ and $Y = P \cdot R = \Sigma(1, 2, 4, 6)$ as shown below. Find the smallest function P(containing minimum number of min terms that can produce the output x and y).



(A) $\Sigma(2, 4)$ (B) $\Sigma(0, 1, 2, 4, 6)$
(C) $\Sigma(3, 5, 7)$ (D) $\Sigma(1, 2, 6)$

10. The standard form of expression $AB + ACD + \overline{A}C$ is

(A) $AB\overline{C}\overline{D} + ABC\overline{D} + AB\overline{C}D + ABCD + A\overline{B}CD + \overline{A}BCD + \overline{A}BC\overline{D} + \overline{A}B\overline{C}D + \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}C\overline{D}$

(B) $AB + ACD + \overline{A}C$

(C) $AB\overline{C} + ABC + ABCD + \overline{A}CB + \overline{A}C\overline{B}\,\overline{D}$

(D) $\overline{A}\,\overline{B}\overline{C}\overline{D} + ABCD + \overline{A}BC + AB\overline{D} + ABC$

11. Factorize $\overline{A}\,\overline{B}\overline{C}\overline{D} + \overline{A}\overline{B}\overline{C}D + A\overline{B}\overline{C}\overline{D} + A\overline{B}\overline{C}D$

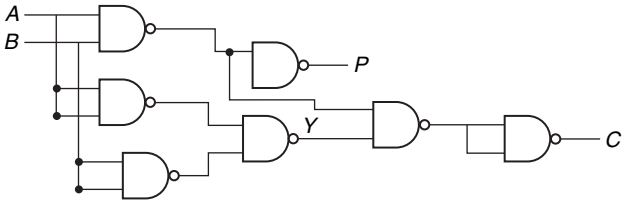
(A) $B + C$ (B) $AB + CD$
(C) $\overline{B}\overline{C}$ (D) AC

12. The K-map of a function is as shown. Find the function.

wx \ yz	00	01	11	10
00	1			1
01	1	1	1	1
11	1			1
10	1			1

(A) wx (B) $\overline{z}$
(C) $\overline{w}(z + \overline{z}) + \overline{z}w$ (D) $\overline{w}x + \overline{z}$

13. The Boolean expression for P is



(A) AB (B) $\overline{AB}$
(C) $\overline{A} + \overline{B}$ (D) $A + B$

14. The Boolean expression for the truth table is

A	B	C	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

(A) $B(A + C)(\overline{A} + \overline{C})$ (B) $\overline{B}(A + \overline{C})(\overline{A} + \overline{C})$
(C) $B(A + \overline{C})(\overline{A} + C)$ (D) $\overline{B}(A + C)(\overline{A} + \overline{C})$

15. Simplify (d represents don't-care)

C \ AB	00	01	11	10
0	1	d		1
1	1	d		1

(A) $\overline{B}$ (B) $\overline{B} + C$
(C) $\overline{B} + \overline{A}$ (D) $A + \overline{C}$

16. Simplify $\overline{(AB + \overline{C})(\overline{A + B} + C)}$

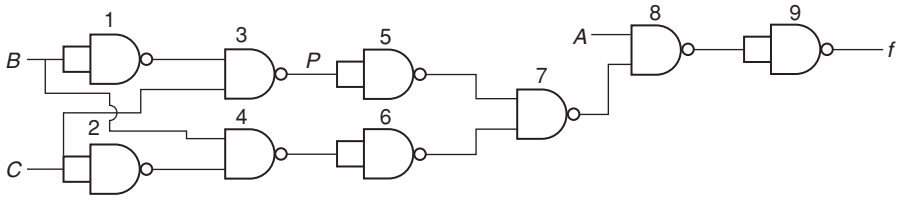
(A) $(\overline{A} + \overline{B} + \overline{C}) \cdot (A + B + C)$

(B) $(\overline{A} + B + C) \cdot (A + \overline{B} + \overline{C})$

(C) $(\overline{A} + \overline{B}) \cdot (A + B + C)$

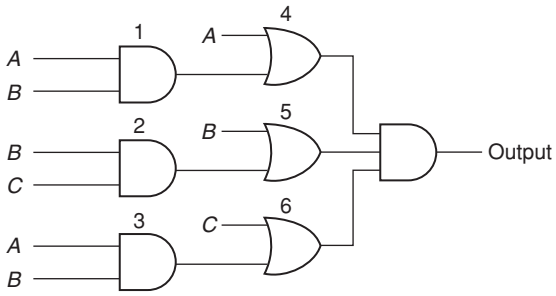
(D) None of these

17. The point P in the figure is stuck at 1. The output f will be



(A) $\overline{ABC}$ (B) $\bar{A}$ (C) $AB\bar{C}$ (D) A

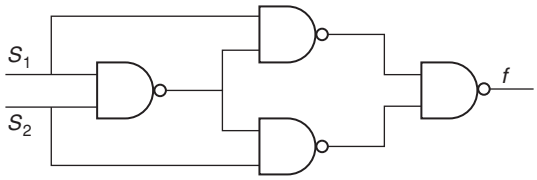
18. Find the function represented by the figure



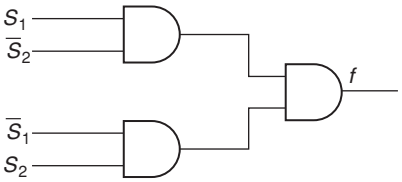
(A) $A + B + C$ (B) AB
(C) $AB + C$ (D) $B + C$

19. A staircase light is controlled by two switches, one is at the top of the stairs and other at the bottom of stairs. Realization of this function using NAND logic results in which of the following circuits? (Assume S_1 and S_2 are the switches)

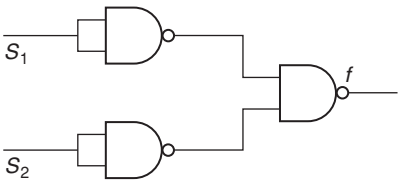
(A)



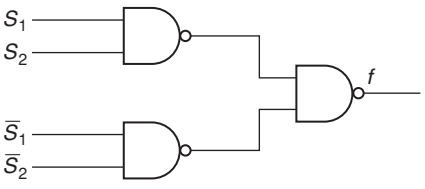
(B)



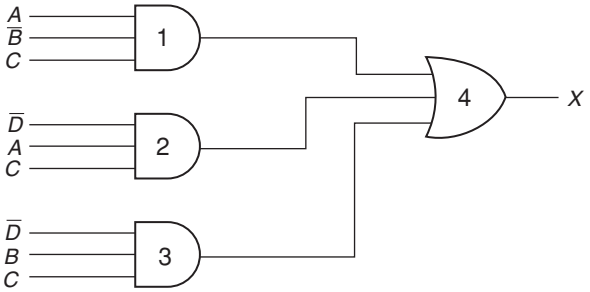
(C)



(D)



20. For the given figure simplify the expression and find which is the redundant gate?

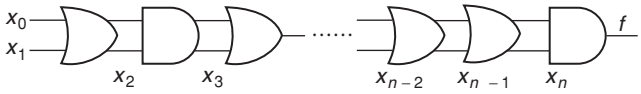


(A) $ABC + DBC, 4$ (B) $A\bar{B}C + \bar{D}AC, 3$
(C) $\bar{D}AC + \bar{D}BC, 1$ (D) $A\bar{B}C + \bar{D}BC, 2$

21. The function $f = A \oplus B \oplus C \oplus D$ is represented as

(A) $f(A, B, C, D) = \sum(2, 6, 10, 11, 12, 13, 14)$
(B) $f(A, B, C, D) = \sum(3, 5, 7, 10, 11, 12, 13, 14)$
(C) $f(A, B, C, D) = \sum(1, 2, 6, 8, 10, 12, 13, 14)$
(D) $f(A, B, C, D) = \sum(1, 2, 4, 7, 8, 11, 13, 14)$

22. Find the function represented



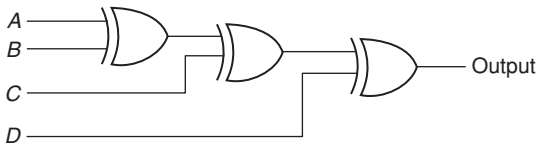
(A) $(x_0 + x_1)(x_2 + x_3)(x_4 + x_5) \ldots (x_{n-1} + x_n)$
(B) $x_0 + x_1 + x_2 + x_3 + \cdots + x_n$
(C) $x_0x_2x_4 \ldots x_n + x_1x_2 \cdots x_n + x_{n-1}x_n$
(D) $x_0x_1 + x_2x_3 + \cdots + x_{n-1}x_n$

23. The minimum number of NAND gates required to implement $A \oplus B \oplus C$ is

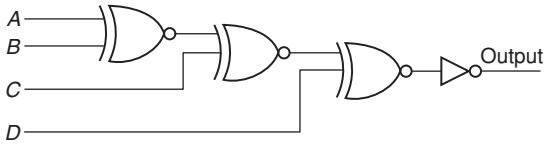
(A) 8 (B) 10
(C) 9 (D) 6

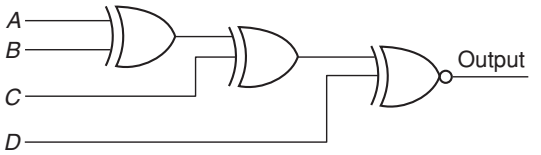
24. Which of the following circuit will generate an odd parity for a 4-bit input? (Assume ABCD as input)

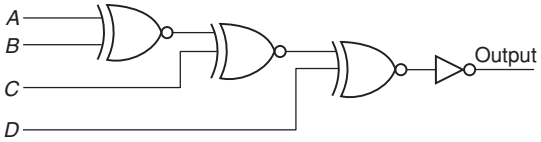
(A)



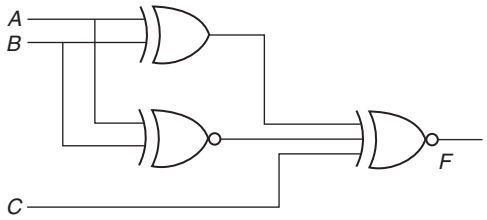
(B)



(C)


(D)


25. For the output F to be 1 in the circuit, the input combination should be



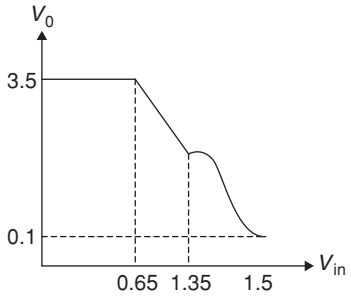
(A) $A = 1, B = 1, C = 0$
(B) $A = 1, B = 0, C = 0$
(C) $A = 0, B = 1, C = 0$
(D) $A = 0, B = 0, C = 1$

26. The highest noise margin is offered by
(A) TTL (B) ECL
(C) I^2L (D) CMOS

27. A LSI IC Package contains _____ logic gates.
(A) 10 (B) 50
(C) less than 50 (D) more than 100

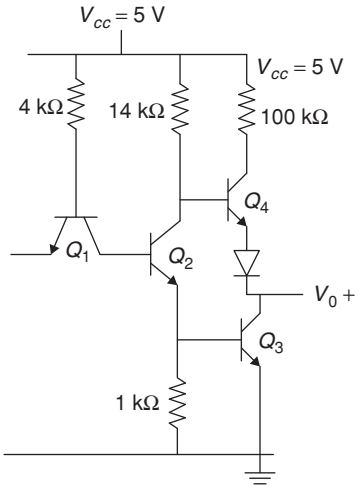
28. For interfacing two logic gates, the condition is
(A) $V_{OH} > V_{IH}, V_{IL} > V_{OL}$
(B) $V_{OH} < V_{IH}, V_{IL} < V_{OL}$
(C) $V_{OH} > V_{IH}, V_{IL} < V_{OL}$
(D) $V_{OH} = V_{IH}, V_{IL} = V_{OL}$

29. The I/O characteristics of which logic family is specified in the figure.



(A) TTL (B) CMOS
(C) I^2L (D) DTL

30. The circuit diagram of a standard TTL NOT gate is shown. If input is 2.5 V, the modes of operation of transistor is



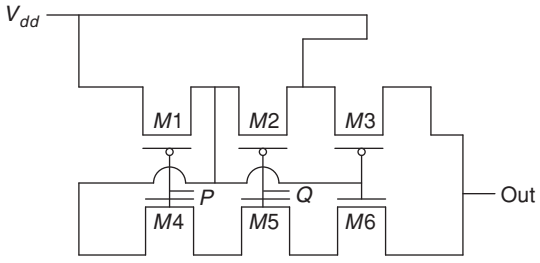
(A) Q_1: reverse active
Q_2: normal active
Q_3: saturation
Q_4: cut-off

(B) Q_1: reverse active
Q_2: saturation
Q_3: saturation
Q_4: cut-off

(C) Q_1: normal active
Q_2: cut-off
Q_3: cut-off
Q_4: saturation

(D) Q_1: saturation
Q_2: saturation
Q_3: saturation
Q_4: normal active

31. The logic function implemented by the following circuit at output terminal is



(A) P NOR Q (B) P NAND Q
(C) P OR Q (D) P AND Q

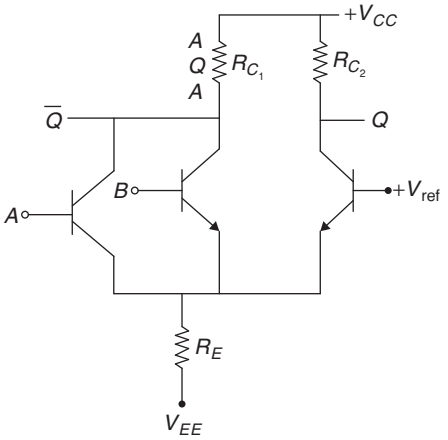
32. Match the options in column 1 with column 2

Column 1	Column 2
(a) TTL	1. Superfast computers
(b) NMOS	2. MSI
(c) CMOS	3. LSI
(d) ECL	4. VLSI

(A) a – 2, b – 3, c – 4, d – 1
(B) a – 3, b – 2, c – 4, d – 1

(C) a – 2, b – 3, c – 4, d – 1
(D) a – 1, b – 3, c – 1, d – 2

33. Which gate is represented by the circuit?



(A) NAND (B) NOR
(C) OR (D) AND

Practice Problems 2

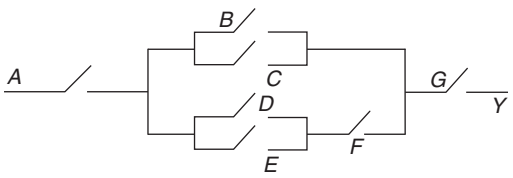
Directions for questions 1 to 33: Select the correct alternative from the given choices.

1. An OR Gate has six inputs. How many input words are there in its truth table?
(A) 6 (B) 36
(C) 32 (D) 64

2. Sum of product form can be implemented by using
(A) AND–OR (B) NAND–NAND
(C) NOR–NOR (D) Both A and B

3. Which one of the following is equivalent to the Boolean expression?
$Y = AB + BC + CA$
(A) $\overline{AB + BC + CA}$
(B) $(\overline{A} + \overline{B})(\overline{B} + \overline{C})(\overline{A} + \overline{C})$
(C) $\overline{(A + B)(B + C)(A + C)}$
(D) $\overline{(\overline{A} + \overline{B})(\overline{B} + \overline{C})(\overline{C} + \overline{A})}$

4. What Boolean function does the following circuit represents?



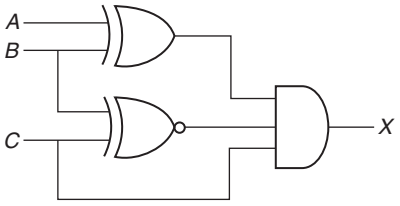
(A) $A\,[F + (B + C)\cdot(D + E)]\,G$
(B) $A + BC + DEF + G$
(C) $A\,[(B + C) + F\,(D + E)]\,G$
(D) $ABG + ABC + F(D + E)$

5. The minimum number of two input NOR gates are required to implement the simplified value of the following equation
$f(w, x, y, z) = \Sigma m(0, 1, 2, 3, 8, 9, 10, 11)$
(A) One (B) Two
(C) Three (D) Four

6. The output of a logic gate is '1' when all inputs are at logic '0'. Then the gate is either
(1) NAND or X-OR gate
(2) NOR or X-OR gate
(3) NOR or X-NOR gate
(4) NAND or X-NOR gate
(A) 1 and 2 (B) 2 and 3
(C) 3 and 4 (D) 4 and 1

7. If the functions w, x, y, and z are as follows.
$w = R + \overline{P}Q + \overline{R}S$
$x = PQ\overline{R}\,\overline{S} + \overline{P}\,\overline{Q}R\overline{S} + P\overline{Q}\,\overline{R}\,\overline{S}$
$y = RS + PR + P\overline{Q} + \overline{P}\cdot\overline{Q}$
$z = R + S + \overline{PQ + \overline{P}\cdot\overline{Q}\cdot\overline{R} + P\cdot\overline{Q}\cdot\overline{S}}$
(A) $w = z\,,\ x = y$ (B) $w = z,\ x = \overline{z}$
(C) $w = y$ (D) $w = y = z$

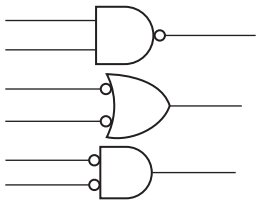
8. For the logic circuit shown in the figure, the required in put condition (A, B, C) to make the output (x) = 1 is



(A) 0, 0, 1 (B) 1, 0, 1
(C) 1, 1, 1 (D) 0, 1, 1

9. Which of the following is a basic gate?
(A) AND (B) X-OR
(C) X-NOR (D) NAND

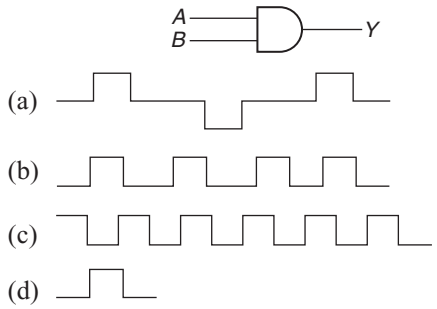
10. Which of the following represent the NAND gate?
(a)
(b)
(c)



(A) a only (B) a, b, c
(C) b, a (D) a, c

11. The universal gates are
(A) NAND and NOR (B) AND, OR, NOT
(C) X-OR and X-NOR (D) All of these

12. In the circuit the value of input A goes from 0 to 1 and part of B goes from 1 to 0. Which of the following represent output under a static hazard condition?



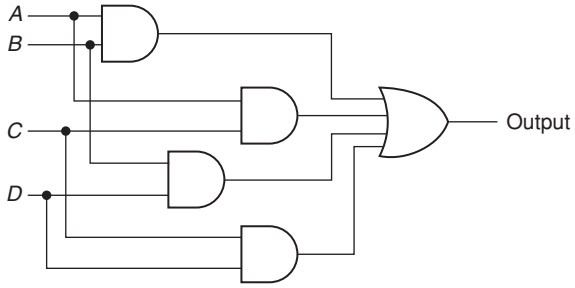
(A) Output a (B) Output b
(C) Output c (D) Output d

13. The consensus theorem states that
(A) $A + \bar{A}B = A + B$
(B) $A + AB = A$
(C) $AB + \bar{A}C + BC = AB + \bar{A}C$
(D) $(A + B)\cdot(A + \bar{B}) = A$

14. The dual form of expression
$AB + \bar{A}C + BC = AB + \bar{A}C$ is
(A) $(A + B)(\bar{A} + C)(B + C) = (A + B)(\bar{A} + C)$
(B) $(A + B)(\bar{A} + C)(B + C) = (\bar{A} + \bar{B})(A + \bar{C})$
(C) $(\bar{A} + \bar{B})(\bar{A} + \bar{C})(\bar{B} + \bar{C}) = (\bar{A} + \bar{B})(A + \bar{C})$
(D) $\bar{A}\bar{B} + A\bar{C} + \bar{B}\bar{C} = \bar{A}\bar{B} + A\bar{C}$

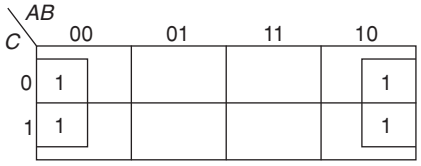
15. The max term corresponding to decimal 12 is
(A) $\bar{A} + \bar{B} + C + D$ (B) $A + B + \bar{C} + \bar{D}$
(C) $\bar{A}\bar{B}CD$ (D) $AB\bar{C}\bar{D}$

16. The given circuit is equivalent to



(A) $(A + C)(B + D)$ (B) $AC + BD$
(C) $(A + D)(B + C)$ (D) $(\bar{A} + \bar{B})(\bar{C} + \bar{D})$

17. Minimized expression for Karnaugh map is



(A) $AB + C$ (B) $\bar{A}B + C$
(C) $\bar{B}$ (D) $\bar{B} + C$

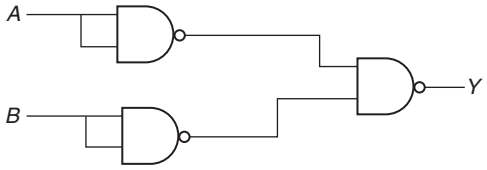
18. An X-OR gate will act as _____ when one of its input is one and as _____ when one of its input is zero.
(A) buffer, buffer (B) buffer, inverter
(C) inverter, buffer (D) inverter, inverter

19. The minimum number of two-input NAND gates required to implement $A \odot B$ if only A and B are available
(A) 6 (B) 3
(C) 5 (D) 4

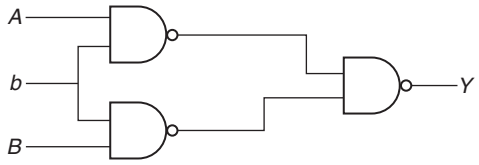
20. Negative logic in a logic circuit is one in which
(A) logic 0 and 1 are represented by GND and positive voltage respectively.
(B) logic 0 and 1 are represented by negative and positive voltage.
(C) logic 0 voltage level is lower than logic 1 voltage level.
(D) logic 0 voltage level is higher than logic 1 voltage level.

21. If the input to a gate is eight in number, then its truth table contains _____ input words.
(A) 128 (B) 8
(C) 64 (D) 256

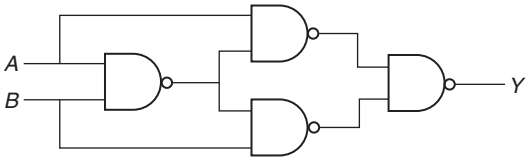
22. The X-OR gate implementation using NAND gate is
(A)



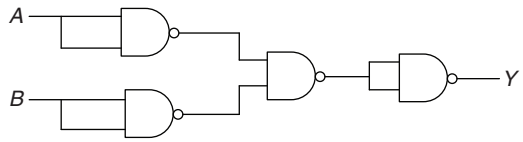
(B)



(C)



(D)



23. The equivalent of AND–OR logic circuit is
(A) NAND–NOR (B) NOR–AND
(C) NAND–NAND (D) NAND–OR

24. The X-OR is equivalent to
(A)



(B)



(C)

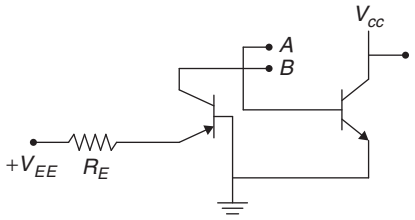


(D)



25. Simplify $A\bar{B}C + B + B\bar{D} + AB\bar{D} + \bar{A}C$
(A) B (B) $B + C$
(C) $C + A$ (D) $\bar{A} + B$

26. The gate represented by the circuit is



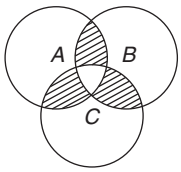
(A) NAND (B) NOR
(C) OR (D) AND

27. The full form of I^2L and ECL is
(A) Inter-integrated logic and emitter-coupled logic
(B) Inter-injection logic and emitter-combined logic
(C) Integrated injection logic and emitter-coupled logic
(D) Integrated Injection Logic and Emission-coupled logic

28. The most popular TTL family member among digital designers
(A) High speed TTL
(B) Standard TTL
(C) Low power TTL
(D) Low power schottky TTL

29. The totempole in standard TTL refers to
(A) Output buffer
(B) input emitter stage
(C) open collector output state
(D) phase splitter

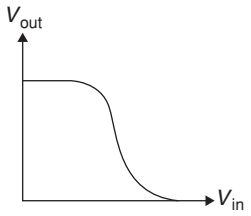
30. Boolean expression for shaded portion is



(A) $ABC + \overline{ABC}$
(B) $AB + BC + CA$
(C) $AB\overline{C} + A\overline{B}C + \overline{A}BC$
(D) $AB\overline{C} + \overline{A}B\overline{C} + \overline{ABC}$

31. The output of the 74 series of TTL gates is taken from a BJT in
(A) totempole and common collector configuration
(B) Either totempole or open collector configuration
(C) common base configuration
(D) common collector configuration

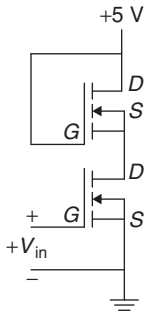
32. The voltage transfer characteristics shown in that of



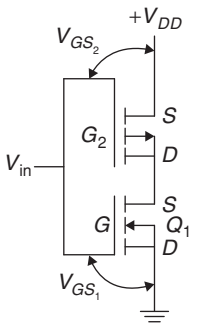
(A) an NMOS inverter with enhancement mode transistor as load.
(B) an NMOS inverter with depletion mode transistor as load
(C) a CMOS inverter
(D) a BJT inverter

33. Which of the following represent NMOS inverter?

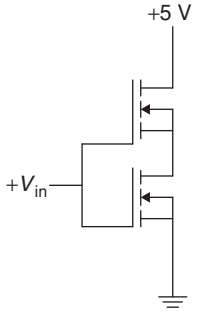
(A)



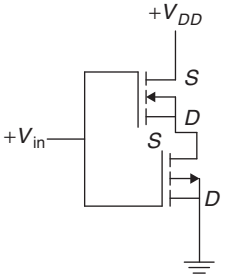
(B)



(C)

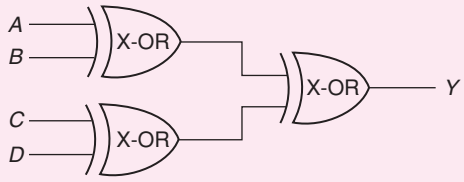


(D)



Previous Years' Questions

1. A, B, C and D are input bits, and Y is the output bit in the X-OR gate circuit of the figure below. Which of the following statements about the sum S of A, B, C, D and Y is correct? **[2007]**

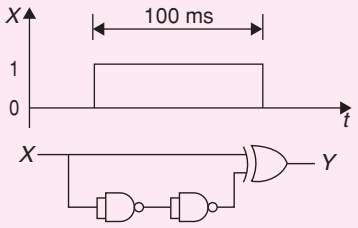


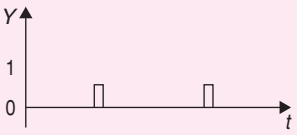
(A) S is always either zero or odd
(B) S is always either zero or even
(C) $S = 1$ only if the sum of A, B, C and D is even
(D) $S = 1$ only if the sum of A, B, C and D is odd

2. The octal equivalent of the HEX number $AB.CD$ is **[2007]**

(A) 253.314 (B) 253.632
(C) 526.314 (D) 526.632

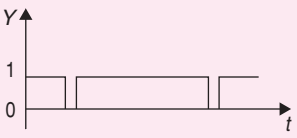
3. The TTL circuit shown in the figure is fed with the waveform X (also shown). All gates have equal propagation delay of 10 ns. The output Y of the circuit is **[2010]**



(A) 

(B) 

(C) 

(D) 

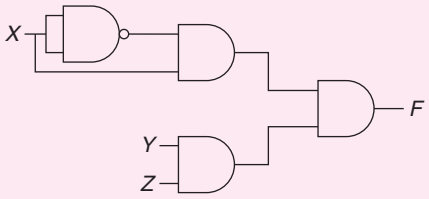
Common Data for Questions 4 and 5: The following karnaugh map represents a function F.

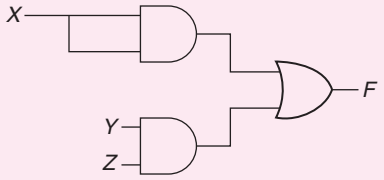
F \ YZ	00	01	11	10
X = 0	1	1	1	0
X = 1	0	0	1	0

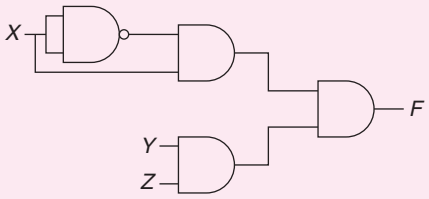
4. A minimized form of the function F is **[2010]**

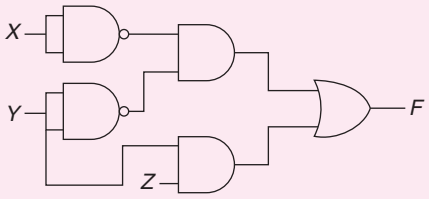
(A) $F = \overline{X}Y + YZ$ (B) $F = \overline{X}\overline{Y} + YZ$
(C) $F = \overline{X}\overline{Y} + Y\overline{Z}$ (D) $F = \overline{X}\overline{Y} + \overline{Y}Z$

5. Which of the following circuits is a realization of the above function F? **[2010]**

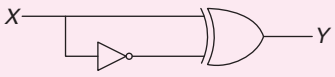
(A) 

(B) 

(C) 

(D) 

6. The output Y of the logic circuit given below is **[2011]**



(A) 1 (B) 0
(C) X (D) $\overline{X}$

7. In the sum of products function $f(X, Y, Z) = \sum(2, 3, 4, 5)$, the prime implicants are **[2012]**

(A) $\overline{X}Y, X\overline{Y}$

(B) $\overline{X}Y, X\overline{Y}\,\overline{Z}, X\overline{Y}Z$

(C) $\overline{X}Y\overline{Z}, \overline{X}YZ, X\overline{Y}$

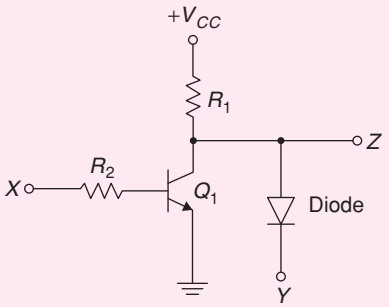
(D) $\overline{X}Y\overline{Z}, \overline{X}YZ, X\overline{Y}\,\overline{Z}, X\overline{Y}Z$

8. A bulb in a staircase has two switches, one switch being at the ground floor and the other one at the first floor. The bulb can be turned ON and also can be turned OFF by any one of the switches irrespective of the state of the other switch. The logic of switching of the bulb resembles **[2013]**

(A) an AND gate (B) an OR gate

(C) an X-OR gate (D) a NAND gate

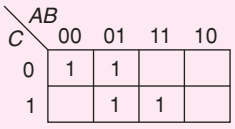
9. In the circuit shown below, Q_1 has negligible collector -to-emitter saturation voltage and the diode drops negligible voltage across it under forward bias. If V_{cc} is +5 V, X and Y are digital signals with 0 V as logic 0 and V_{cc} as logic 1, then the Boolean expression for Z is **[2013]**



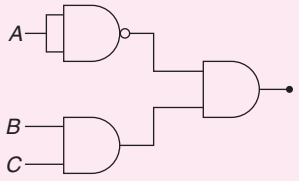
(A) XY (B) $\overline{XY}$

(C) $X\overline{Y}$ (D) $\overline{XY}$

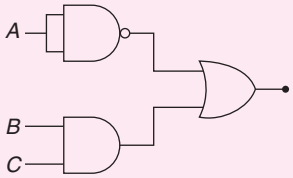
10. Which of the following logic circuits is a realization of the function F whose karnaugh map is shown in figure **[2014]**



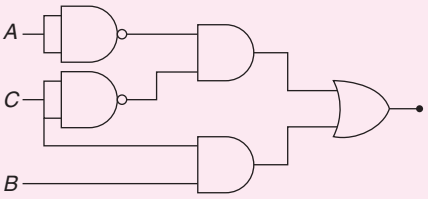
(A)



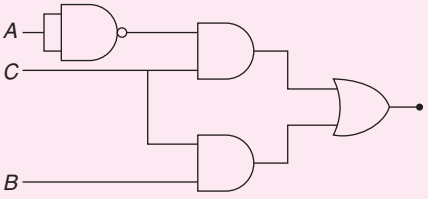
(B)



(C)



(D)



11. Which of the following is an invalid state in an 8-4-2-1 Binary-Coded Decimal counter **[2014]**

(A) 1 0 0 0 (B) 1 0 0 1

(C) 0 0 1 1 (D) 1 1 0 0

12. The SOP (sum of products) form of a Boolean function is $\Sigma(0, 1, 3, 7, 11)$, where inputs are A, B, C, D (A is MSB, and D is LSB). The equivalent minimized expression of the function is **[2014]**

13. $f(A, B, C, D) = \prod M(0, 1, 3, 4, 5, 7, 9, 11, 12, 13, 14, 150$ is a maxterm representation of a Boolean function $f(A, B, C, D)$ where A is the MSB and D is the LSB. The equivalent minimized representation of this function is **[2015]**

(A) $(A + \overline{C} + D)(\overline{A} + B + D)$

(B) $A\overline{C}D + \overline{A}BD$

(C) $\overline{A}C\overline{D} + A\overline{B}C\overline{D} + A\overline{B}\,\overline{C}\,\overline{D}$

(D) $(B + \overline{C} + D)(A + \overline{B} + \overline{C} + D)(\overline{A} + B + C + D)$

14. Consider the following Sum of Products expression, F. **[2015]**

$F = ABC + \overline{A}\,\overline{B}C + A\overline{B}C + \overline{A}BC + \overline{A}\,\overline{B}\,\overline{C}$

The equivalent Product of Sums expression is

(A) $F = (A + \overline{B} + C)(\overline{A} + B + C)(\overline{A} + \overline{B} + C)$

(B) $F = (A + \overline{B} + \overline{C})(A + B + C)(\overline{A} + \overline{B} + \overline{C})$

(C) $F = (\overline{A} + \overline{B} + \overline{C})(A + \overline{B} + \overline{C})(A + B + C)$

(D) $F = (\overline{A} + \overline{B} + C)(A + B + \overline{C})(A + B + C)$

(A) $(\overline{B}+C)(\overline{A}+C)(\overline{A}+\overline{B})(\overline{C}+D)$

(B) $(\overline{B}+C)(\overline{A}+C)(\overline{A}+\overline{C})(\overline{C}+B)$

(C) $(\overline{B}+C)(\overline{A}+C)(\overline{A}+\overline{C})(\overline{C}+\overline{D})$

(D) $(\overline{B}+C)(A+\overline{B})(\overline{A}+\overline{B})(\overline{C}+D)$

15. The output expression for the Karnaugh map shown below is **[2016]**

A \ BC	00	01	11	10
0	1	0	0	1
1	1	1	1	1

(A) $A+\overline{B}$ (B) $A+\overline{C}$
(C) $\overline{A}+\overline{C}$ (D) $\overline{A}+C$

16. The Boolean expression $\overline{\left(a+\overline{b}+c+\overline{d}\right)+\left(b+\overline{c}\right)}$ simplifies to **[2016]**
(A) 1 (B) $\overline{a.b}$
(C) $a.b$ (D) 0

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** D	**3.** B	**4.** C	**5.** A	**6.** A	**7.** B	**8.** A	**9.** B	**10.** A
11. C	**12.** D	**13.** A	**14.** A	**15.** A	**16.** A	**17.** D	**18.** B	**19.** A	**20.** D
21. D	**22.** C	**23.** A	**24.** C	**25.** D	**26.** D	**27.** D	**28.** A	**29.** A	**30.** B
31. D	**32.** C	**33.** B							

Practice Problems 2

1. D	**2.** D	**3.** D	**4.** C	**5.** A	**6.** C	**7.** B	**8.** D	**9.** A	**10.** C
11. A	**12.** D	**13.** A	**14.** A	**15.** A	**16.** C	**17.** C	**18.** C	**19.** C	**20.** D
21. D	**22.** C	**23.** C	**24.** B	**25.** B	**26.** A	**27.** C	**28.** D	**29.** A	**30.** C
31. B	**32.** C	**33.** A							

Previous Years' Questions

1. D	**2.** B	**3.** A	**4.** B	**5.** D	**6.** A	**7.** A	**8.** C	**9.** B	**10.** C
11. D	**12.** A	**13.** C	**14.** A	**15.** B	**16.** D				

Chapter 9

Combinational Circuits

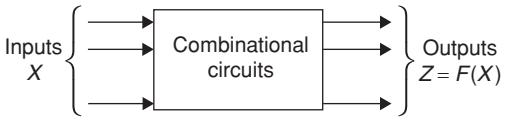
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Combinational logic design
- Arithmetic circuits
- Code converters
- Decoder
- Combinational logic implementation
- Encoders
- Multiplexer
- Basic gates by using MUX
- De-multiplexer
- Memory and programmable logic
- Random access memory

Introduction

Combinational logic is a type of logic circuit whose output is a function of the present input only.



Combinational Logic Design

The design of combinational circuit starts from the problem, statement and ends with a gate–level circuit diagram.

The design procedure involves the following steps:

1. Determining the number of input variables and output variables required, from the specifications.
2. Assigning the letter symbols for input and output.
3. Deriving the truth table that defines the required relationship between input and output.
4. Obtain the simplified Boolean function for each output by using K-map or algebraic relations.
5. Drawing the logic diagram for simplified expressions.

We will discuss combinational circuits under the following categories:

- Arithmetic circuits
- Code converters
- Data processing circuits

Arithmetic Circuits

Arithmetic circuits are the circuits that perform arithmetic operation. The most basic arithmetic operation is addition.

Half Adder

Addition is an arithmetic operation, and here to implement addition in digital circuits we have to implement by logical gates. So the addition of binary numbers will be represented by the logical expressions. Half adder is an arithmetic circuit which performs the addition of two binary bits, and the result is viewed in two output—sum and carry.

The sum 'S' is the X-OR of 'A' and 'B' where A and B are inputs.

$$\therefore \quad S = A\overline{B} + B\overline{A} = A \oplus B$$

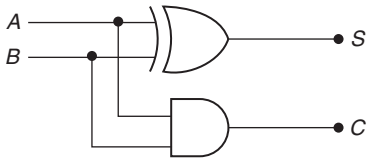
The carry 'C' is the AND of A and B.

$$\therefore C = AB$$

Truth Table

Inputs		Outputs	
A	B	S	C
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

So, half adder can be realized by using one X-OR gate and one AND gate.



Half adder can also be realized by universal logic, such as only NAND gate or only NOR gate as given below.

NAND logic

$$S = A\bar{B} + \bar{A}B$$
$$= A\bar{B} + A\bar{A} + \bar{A}B + B\bar{B}$$
$$= A(\bar{A} + \bar{B}) + B(\bar{A} + \bar{B})$$
$$= \overline{\overline{A\overline{AB}} \cdot \overline{B\overline{AB}}}$$
$$C = AB = \overline{\overline{A \cdot B}}$$

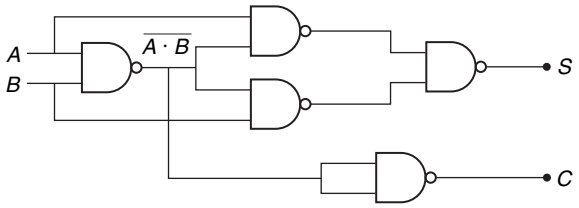


Figure 1 Half adder using NAND logic

NOR logic

$$S = A \cdot \bar{B} + \bar{A}B$$
$$= A\bar{B} + A\bar{A} + \bar{A}B + B\bar{B}$$
$$= A(\bar{A} + \bar{B}) + B(\bar{A} + \bar{B})$$
$$= (A + B)(\bar{A} + \bar{B})$$
$$= \overline{\overline{A + B} + (\overline{\bar{A} + \bar{B}})}$$
$$C = A \cdot B = \overline{\overline{A \cdot B}} = \overline{\bar{A} + \bar{B}}$$

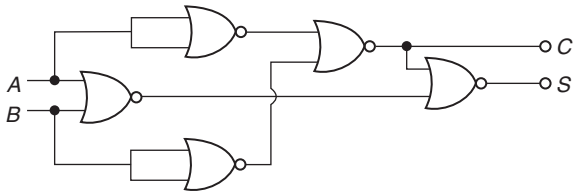


Figure 2 Half adder using NOR logic

Full Adder

Full adder is an arithmetic circuit that performs addition of two bits with carry input. The result of full adder is given by two outputs—sum and carry. The full adder circuit is used in parallel adder circuit as well as in serial adder circuit.

For full adder, if total number of 1's is odd at input lines, the sum output is equal to logic 1, and if total number of 1's at input lines are more than or equal to 2, then the carry output is logic 1.

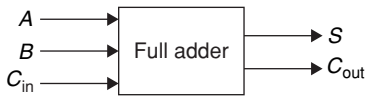


Figure 3 Block diagram

Truth Table

A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$S = \bar{A}\bar{B}C_{in} + \bar{A}B\overline{C_{in}} + A\bar{B}\overline{C_{in}} + ABC_{in}$$
$$= A \oplus B \oplus C_{in}$$
$$C_{out} = \bar{A}BC_{in} + A\bar{B}C_{in} + AB\overline{C_{in}} + ABC_{in}$$
$$= AB + (A \oplus B)\, C_{in}$$
$$= AB + A\, C_{in} + B\, C_{in}$$

Full adder can also be realized using universal logic gates, i.e., either only NAND gates or only NOR gates as explained below.

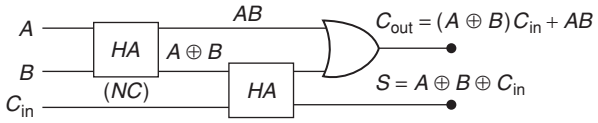


Figure 4 Block diagram of full adder, by using half adder

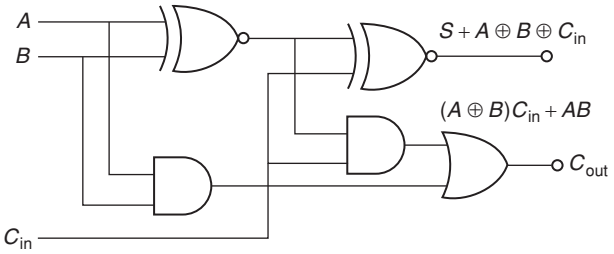


Figure 5 Logic diagram of full adder

NAND logic

$$A \oplus B = \overline{\overline{A\overline{AB}}\ \ \overline{B\overline{AB}}}$$

So $A \oplus B \oplus C_{in}$

Let $A \oplus B = x$ then $s = \overline{\overline{X \cdot \overline{XC_{in}}}\ \ \overline{C_{in}\overline{X \cdot C_{in}}}}$

$$= X \oplus C_{in}$$

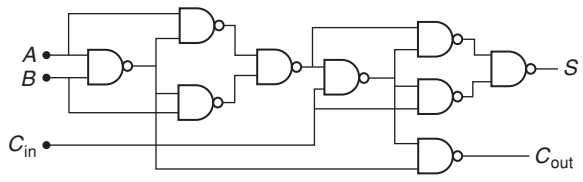


Figure 6 Logic diagram of a full adder using only 2-input NAND gates

NOR logic

Full subtractor outputs.

Sum = $a \oplus b \oplus c$, carry = $ab + bc + ac$ are self dual functions.

[$\because$ A function is called as self dual if its dual is same as the function itself $f^D = f$].

For self dual functions, the number of NAND gates are same as number of NOR gates.

By taking the dual for above NAND gate implementation, all gates will become NOR gates, and the output is dual of the sum and carry, but they are self dual ($f^D = f$).

So, output remain same, and only 9 NOR gates are required for full adder, structure similar to NAND gate circuit.

Half Subtractor

Half subtractor is an arithmetic circuit which performs subtraction of one bit (subtrahend) from other bit (minuend), and the result gives difference and borrow each of one bit. The borrow output is logic 1 only if there is any subtraction of 1 from 0.

When a bit 'B' is subtracted from another bit 'A', a difference bit (d) and a borrow bit (b) result according to the rule given below.

Truth Table

A	B	d	b
0	0	0	0
1	0	1	0
1	1	0	0
0	1	1	1

$$d = A\bar{B} + B\bar{A}$$
$$= A \oplus B$$
$$b = \bar{A}B$$

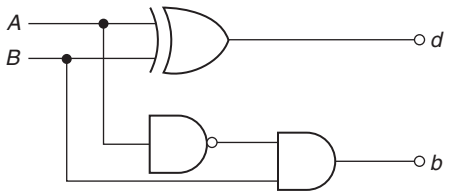


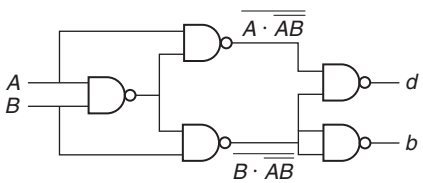
Figure 7 Logic diagram of a half subtractor

A half subtractor can also realized using universal logic either using only NAND gates or only NOR gates as explained below.

NAND logic

$$d = A \oplus B$$
$$= \overline{\overline{A\overline{AB}} \cdot \overline{B\overline{AB}}}$$

$$b = \bar{A}B = B(\bar{A} + \bar{B}) = B(\overline{AB}) = \overline{\overline{B \cdot \overline{AB}}}$$



NOR logic

$$d = A \oplus B$$
$$= A\bar{B} + \bar{A}B$$
$$= A\bar{B} + B\bar{B} + \bar{A}B + A\bar{A}$$
$$= \bar{B}(A + B) + \bar{A}(A + B)$$
$$= \overline{B + \overline{A + B}} + \overline{A + \overline{A + B}}$$
$$b = \bar{A}B$$
$$= \bar{A}(A + B)$$
$$= \overline{\overline{\bar{A}(A + B)}}$$
$$= \overline{A + \overline{(A + B)}}$$

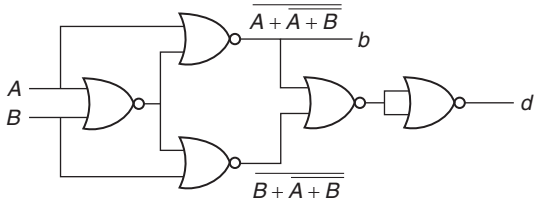


Figure 8 Logic diagram of half subtractor using NOR gate

Full Subtractor

Full subtractor is an arithmetic circuit similar to half subtractor but it performs subtraction with borrow, it involves subtraction of 3-bits—minuend, subtrahend and borrow-in, and two outputs—difference and borrow. The subtraction of 1 from 0 results in borrow to become logic 1. The presence of odd number of 1's at input lines make difference as logic 1.

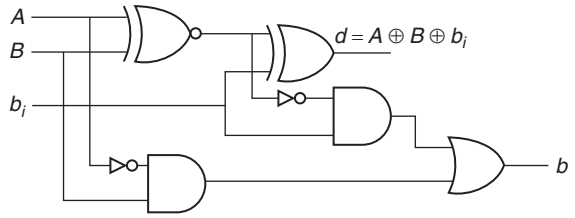
Truth Table

A	B	b_i	d	b
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	1
1	0	0	1	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

$$d = \bar{A}\bar{B}b_i + \bar{A}B\bar{b_i} + A\bar{B}\bar{b_i} + ABb_i$$
$$= b_i(AB + \bar{A}\bar{B}) + \bar{b_i}(A\bar{B} + \bar{A}B)$$
$$= b_i(\overline{A \oplus B}) + \bar{b_i}(A \oplus B)$$
$$= A \oplus B \oplus b_i$$

and

$$b = \bar{A}\bar{B}b_i + \bar{A}B\bar{b_i} + \bar{A}Bb_i$$
$$= \bar{A}B + (\overline{A \oplus B})b_i$$



NAND logic

$$d = A \oplus B \oplus b_i$$
$$= \overline{\overline{(A \oplus B)(\overline{A \oplus B})b_i}\,\overline{b_i(A \oplus B)b_i}},$$
$$b = \bar{A}B + b_i\overline{(A \oplus B)}$$
$$= \overline{\overline{\bar{A}B + b_i(\overline{A \oplus B})}}$$
$$= \overline{\bar{A}Bb_i\,\overline{(A \oplus B)}}$$
$$= \overline{\overline{B(\bar{A} + \bar{B})b_i}\left[\overline{\bar{b}_i + (A \oplus B)}\right]}$$

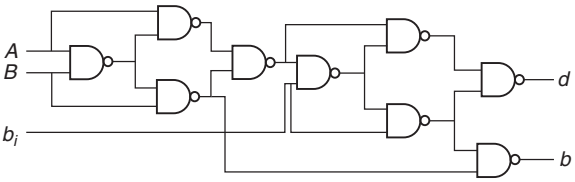


Figure 9 Logic diagram of a full subtractor using NAND logic

NOR logic

Outputs of full subtractor are also self dual in nature. Therefore, same circuit with all NAND gates replaced by NOR gates gives the NOR gate full subtractor. For this 9 NOR gates required.

Example 1: How many NAND gates are required for implementation of full adder and full subtractor?
(A) 11, 10 (B) 11, 11 (C) 9, 9 (D) 9, 10

Solution: (C)
From the circuit diagrams in the previous discussion, full adder requires 9 NAND gates, and full subtractor requires 9 NAND gates.

Binary Adder

A binary adder is a digital circuit that produces the arithmetic sum of two binary numbers.

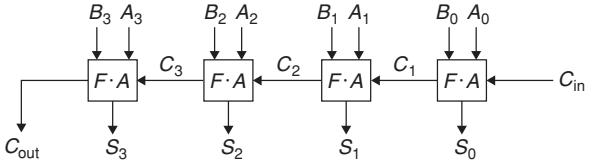


Figure 10 4-bit parallel adder

The output carry from each full adder is connected to the input carry of next full adder.

The bits are added with full adders, starting from the LSB, position, to form the sum bit and carry bit.

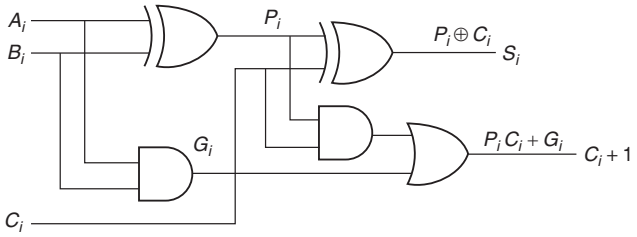
The longest propagation delay time in parallel adder is the time it takes the carry to propagate through the full adders.

For n-bit parallel adders consider t_{pds} is the propagation delay for sum of each full adder, and t_{pdc} is the propagation delay of carry.

The total time required to add all n-bits at the nth full adder is

$$T_S = t_{pds} + (n-1)t_{pdc}$$

So propagation delay increases with number of bits. To overcome this difficulty, we use look ahead carry adder. Look ahead carry adder is the fastest carry adder.



Consider the full adder circuit for ith stage, in parallel adder, with two binary variables A_i, B_i, input carry C_i are:

Carry propagate (P_i) and carry generate (G_i)

$$P_i = A_i \oplus B_i$$
$$G_i = A_i \cdot B_i$$

The output sum and carry can be expressed as

$$S_i = P_i \oplus C_i$$
$$C_{i+1} = P_i C_i + G_i$$

Now, the Boolean functions for each stage can be calculated as substitute $i = 0$

C_0 is input carry

$$C_1 = G_0 + P_0 C_0$$

Substitute $i = 1, 2 \ldots$

$$C_2 = G_1 + P_1C_1 = G_1 + P_1 (G_0 + P_0C_0)$$
$$= G_1 + P_1G_0 + P_1 P_0 C_0$$
$$C_3 = G_2 + P_2C_2 = G_2 + P_2 (G_1 + P_1G_0 + P_1 P_0C_0)$$
$$= G_2 + P_2G_1 + P_2P_1G_0 + P_2P_1P_0C_0$$

Since the Boolean function for each output carry is expressed in SOP form, each function can be implemented with AND–OR form or two level NAND gates.

From the above equations, we can conclude that this circuit can perform addition in less time as C_3 does not have to wait for C_2 and C_1 to propagate: C_3, C_2, C_1 can have equal time delays.

The gain in speed of operation is achieved at the expense of additional complexity (hardware).

n-bit Comparator

The comparison of two numbers is an operation that determines whether one number is greater than, less than, or equal to the other number.

A magnitude comparator is a combinational circuit that compares two input numbers A and B, and specifies the output with three variables:

$$A > B, A = B, A < B$$

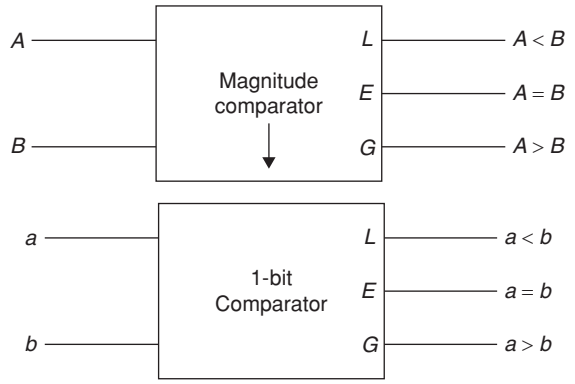


Figure 11 1-bit comparator will have only 1-bit input such as *a*, *b*.

a	b	a < b	a = b	a > b
0	0	0	1	0
0	1	1	0	0
1	0	0	0	1
1	1	0	1	0

By considering minterms for each output.

$$(a < b) = a'b$$
$$(a = b) = a'b' + ab = a \odot b$$
$$(a > b) = ab'$$

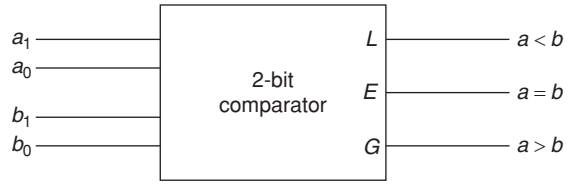
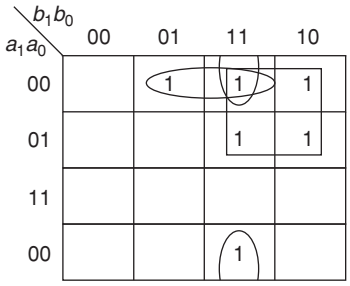


Figure 12 2-bit comparator will have 2-bit inputs, such as a_1 a_0 and b_1 b_0.

a_1	a_0	b_1	b_0	L a < b	E a = b	G a > b
0	0	0	0	0	1	0
0	0	0	1	1	0	0
0	0	1	0	1	0	0
0	0	1	1	1	0	0
0	1	0	0	0	0	1
0	1	0	1	0	1	0
0	1	1	0	1	0	0
0	1	1	1	1	0	0
1	0	0	0	0	0	1
1	0	0	1	0	0	1
1	0	1	0	0	1	0
1	0	1	1	1	0	0
1	1	0	0	0	0	1
1	1	0	1	0	0	1
1	1	1	0	0	0	1
1	1	1	1	0	1	0

$$(a < b) = \Sigma(1, 2, 3, 6, 7, 11)$$
$$(a > b) = \Sigma(4, 8, 9, 12, 13, 14)$$
$$(a = b) = \Sigma(0, 5, 10, 15)$$



$$a < b = a_1'a_0'b_0 + a_0'b_1b_0 + a_1'b_1$$
$$L = \overline{a_1}b_1 + (a_1 \odot b_1)\overline{a_0}b_0$$

Similarly, $a > b = a_0b_1'b_0' + a_1a_0b_0' + a_1b_1'$

$$G = a_1\overline{b_1} + (a_1 \odot b_1)a_0\overline{b_0}$$

$a = b$ is possible when $a_1 = b_1$, $a_0 = b_0$

So $(a = b) = (a_1 \odot b_1)(a_0 \odot b_0)$

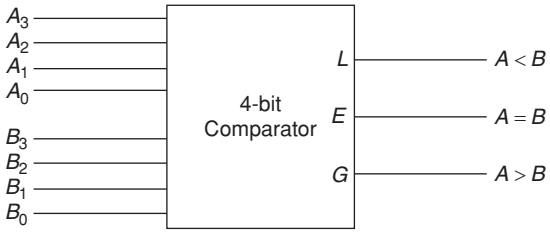


Figure 13 4-bit comparator will compare 2 input numbers each of 4-bits, as A_3 A_2 A_1 A_0 and B_3 B_2 B_1 B_0 ($A = B$) output will be 1 when each bit of input *A* is equal to corresponding bit in input *B*.

So we can write $(A = B) = (A_3 \odot B_3)\ (A_2 \odot B_2)\ (A_1 \odot B_1)\ (A_0 \odot B_0)$.

To determine whether A is greater or less than B, we inspect the relative magnitudes of pairs of significant bits, starting from MSB. If the two bits of a pair are equal, we compare the next lower significant pair of bits. The comparison continues until a pair of unequal bits is reached.

for $A < B$, $A = 0$, $B = 1$

for $A > B$, $A = 1$, $B = 0$

$$A < B = A_3'B_3 + (A_3 \odot B_3)\ A_2'B_2 + (A_3 \odot B_3)(A_2 \odot B_2) \times A_1'B_1 + (A_3 \odot B_3)(A_2 \odot B_2)\ (A_1 \odot B_1)\ A_0'B_0$$

$$A > B = A_3B_3' + (A_3 \odot B_3)\ A_2B_2' + (A_3 \odot B_3)(A_2 \odot B_2) \times A_1B_1' + (A_3 \odot B_3)(A_2 \odot B_2)\ (A_1 \odot B_1)\ A_0B_0'$$

4-bit comparator will have total 8 inputs and $2^8 = 256$ input combinations in truth table.

For 16 combinations $(A = B) = 1$, and for 120 combinations $A < B = 1$.

For remaining 120 combination $A > B = 1$.

Parity Bit Generator and Parity Bit Checker

When digital information is transmitted, it may not be received correctly by the receiver. To detect 1-bit error at receiver, we can use parity checker.

For detection of error, an extra bit known as parity bit is attached to each code word to make the number of 1's in the code even (in case of even parity) or odd (in case of odd parity).

For n-bit data, we use n-bit parity generator at the transmitter end. With 1 parity bit and n-bit data, total $n + 1$ bit will be transmitted. At the receiving end, $n + 1$ parity checker circuit will be used to check correctness of the data.

For even parity transmission, parity bit will be made 1 or 0 based on the data, so that total $n + 1$ bits will have even number of 1's. For example, if we want to transmit data 1011 by even parity transmission, then we will use parity bit as 1, so data will have even number of 1's, i.e., data transmitted will be 11011. At the receiving end, this data will be received and checked for even number of ones.

To transmit data $B_3B_2B_1B_0$ using even parity, we will transmit sequence $PB_3B_2B_1B_0$, where $P = B_3 \oplus B_2 \oplus B_1 \oplus B_0$ (equation for parity generator).

At the receiving end, we will check data received $PB_3B_2B_1B_0$ for error, $E = P \oplus B_3 \oplus B_2 \oplus B_1 \oplus B_0$ (equation for parity checker). If $E = 0$ (no error), or if $E = 1$ (1 bit error).

We use EX-OR gates for even parity generator/checker as EX-OR of bits gives output 1 if there are odd number of 1's else EX-OR output is 0.

Odd parity generator/checker is complement of even parity generator/checker. Odd parity circuits check for presence of odd number of 1's in data.

Code Converters

There are many situations where it is desired to convert from one code to another within a system. For example, the information from output of an analog to digital converter is often in gray code, before it can be processed in arithmetic unit, conversion to binary is required.

Let us consider simple example of 3-bit binary to gray code converter. This will have input lines supplied by binary codes and output lines must generate corresponding bit combination in gray code. The combination circuit code converter performs this transformation by means of logic gates.

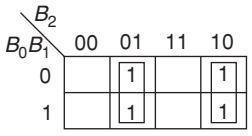
The output logic expression derived for code converter can be simplified by using the usual techniques, including don't-care, if any present. For example, *BCD* code uses only codes from 0000 to 1001, remaining combinations are treated as don't care combinations, similarly. EXS-3 uses only combinations from 0011 to 1100 remaining combinations are treated as don't care.

The relationship between the two codes is shown in the following truth table:

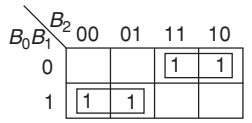
Decimal	B_2	B_1	B_0	G_2	G_1	G_0
0	0	0	0	0	0	0
1	0	0	1	0	0	1
2	0	1	0	0	1	1
3	0	1	1	0	1	0
4	1	0	0	1	1	0
5	1	0	1	1	1	1
6	1	1	0	1	0	1
7	1	1	1	1	0	0

For conversion, we require to find out minimized functions of

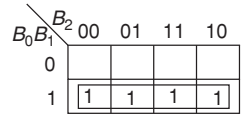
$$G_2(B_2, B_1, B_0) = \sum m(4, 5, 6, 7)$$
$$G_1(B_2, B_1, B_0) = \sum m(2, 3, 4, 5)$$
$$G_0(B_2, B_1, B_0) = \sum m(1, 2, 5, 6)$$



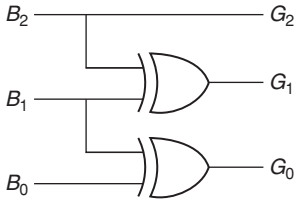
$$G_0(B_2, B_1, B_0) = B'_1B_0 + B_1B'_0 = B_1 \oplus B_0$$



$$G_1(B_2, B_1, B_0) = B'_1B_2 + B_1B'_2 = B_2 \oplus B_1$$



$$G_2(B_2, B_1, B_0) = B_2$$



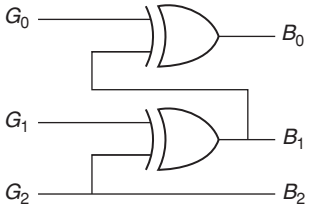
In similar fashion, we can derive n-bit binary to gray code conversion as

$$G_n = B_n$$
$$G_{n-1} = B_{n-1} \oplus B_n$$
$$G_{i-1} = B_{i-1} \oplus B_i$$

Thus conversion can be implemented by $n - 1$ X-OR gates for n-bits.

For reverse conversion of gray to binary, by following similar standard principle of conversion, we will get

$$B_0 = G_0 \oplus G_1 \oplus G_2, B_1 = G_1 \oplus G_2, B_2 = G_2$$



In general for n-bit gray to binary code conversion

$$B_i = G_n \oplus G_{n-1} \oplus G_{n-2} \ldots \oplus G_{i-1} \oplus G_i$$

$B_n = G_n$ (MSB is same in gray and binary). It also requires $n-1$ X-OR gates for n-bits.

Example 2: Design 84-2-1 to XS-3 code converter

Solution: Both 84-2-1 and XS-3 are *BCD* codes, each needs 4-bits to represent. The following table gives the relation between these codes. 84-2-1 is a weighted code, i.e., each position will have weight as specified. XS-3 is non weighted code; the binary code is three more than the digit in decimal.

Decimal	84-2-1 $B_3B_2B_1B_0$	XS-3 $X_3X_2X_1X_0$
0	0000	0011
1	0111	0100
2	0110	0101
3	0101	0110
4	0100	0111
5	1011	1000
6	1010	1001
7	1001	1010
8	1000	1011
9	1111	1100

We will consider minterm don't care combinations as 1, 2, 3, 12, 13, 14. For these combinations 84-2-1 code will not exist, and the remaining minimum terms can be found from truth table.

$$X_0(B_3, B_2, B_1, B_0) = \sum m(0, 4, 6, 8, 10) + \sum \Phi(1, 2, 3, 12, 13, 14) = \overline{B_0}$$

$$X_1(B_3, B_2, B_1, B_0) = \sum m(0, 4, 5, 8, 9, 15) + \sum \Phi(1, 2, 3, 12, 13, 14) = \overline{B_1}$$

$$X_2(B_3, B_2, B_1, B_0) = \sum m(4, 5, 6, 7, 15) + \sum \Phi(1, 2, 3, 12, 13, 14) = B_2$$

$$X_3(B_3, B_2, B_1, B_0) = \sum m(8, 9, 10, 11, 15) + \sum \Phi(1, 2, 3, 12, 13, 14) = B_3$$

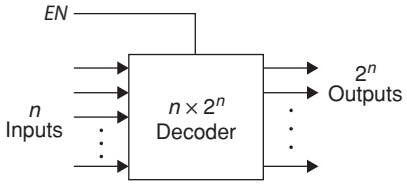
Decoder

A binary code of n-bits is capable of representing up to 2^n elements of distinct elements of coded information.

The three inputs are decoded into eight outputs, each representing one of the minterms of the three input variables.

A decoder is a combinational circuit that converts binary information from n input lines to a maximum 2^n unique output lines.

A binary decoder will have n inputs and 2^n outputs.



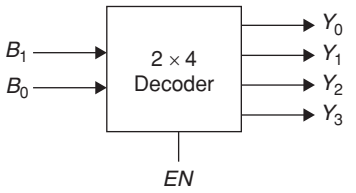


Figure 14 2 × 4 decoder

Truth Table

EN	B_1	B_0	Y_3	Y_2	Y_1	Y_0
0	X	X	0	0	0	0
1	0	0	0	0	0	1
1	0	1	0	0	1	0
1	1	0	0	1	0	0
1	1	1	1	0	0	0

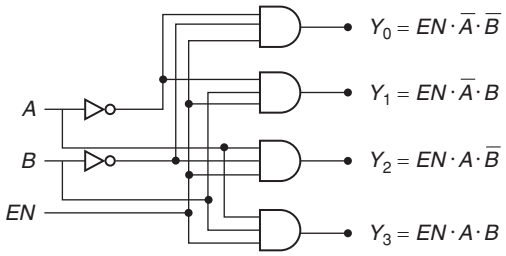
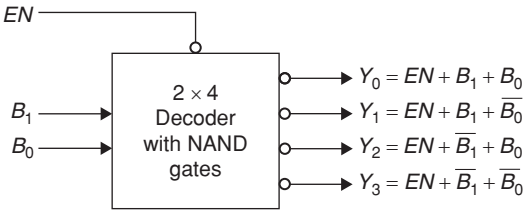


Figure 15 2 × 4 decoder

Decoder outputs are implemented by AND gates, but realization of AND gates at circuit level is done by the NAND gates (universal gates). So, the decoders available in IC form are implemented with NAND gates, i.e., the outputs are in complemented form and outputs are maximum terms of the inputs rather than minimum terms of inputs as in AND gate decoders.

Furthermore, decoders include one or more enable inputs to control the circuit operation. Enable can be either active low/high input.

Active low 2 × 4 decoder:



Truth Table

EN	B_1	B_0	Y_3	Y_2	Y_1	Y_0
1	X	X	1	1	1	1
0	0	0	1	1	1	0
0	0	1	1	1	0	1
0	1	0	1	0	1	1
0	1	1	0	1	1	1

The block diagram shown here is 2 × 4. Decoder with active low output and active low enable input.

The logic diagram is similar to the previous 2 × 4 decoder, except, all AND gates are replaced by NAND gates and *EN* will have inverter, *EN* is connected to all NAND inputs, as $\overline{EN}$ is active low input for this circuit.

The decoder is enabled when *EN* is equal to 0.

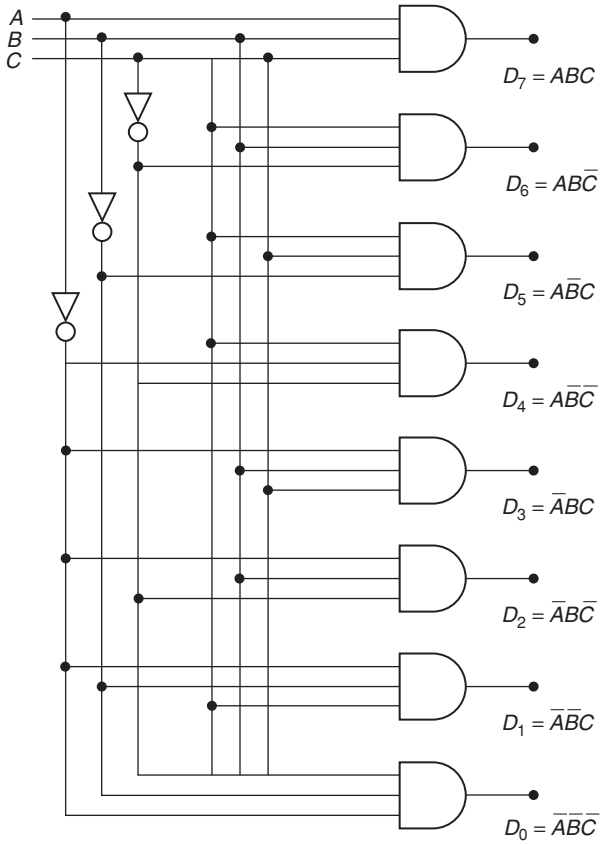
As shown in the truth table, only one output can be equal to 0 at any given time, all other outputs are equal to 1. The output whose value is equal to 0 represents the minimum term selected by inputs, enable.

Consider a 3 to 8 line decoder

Truth Table

Inputs			Outputs							
A	*B*	*C*	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

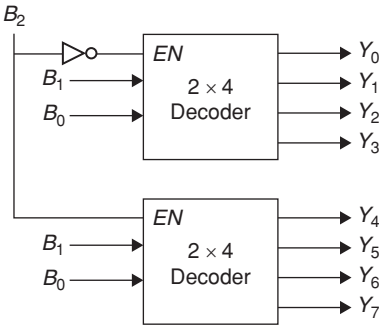
A 3 to 8 decoder has 3 input lines and 8 output lines, based on the combination of inputs applied for the 3 inputs, one of the 8 output lines will be made logic 1 as shown in the truth table. So, each output will have only one minimum term.



Designing High Order Decoders from Lower Order Decoders

Decoder with enable input can be connected together to form larger decoder circuit.

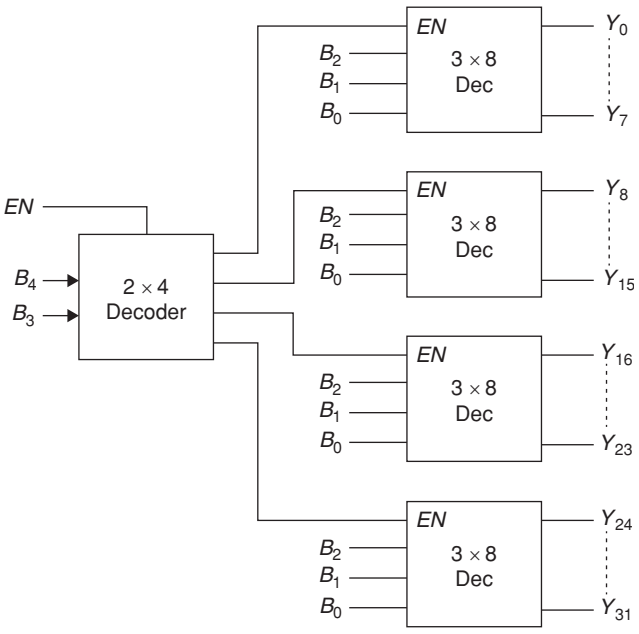
The following configuration shows 3 × 8 decoder with 2 × 4 decoders.



When $B_2 = 0$, top decoder is enabled and other is disabled, for 000 to 011 inputs, outputs are Y_0 to Y_3, respectively, and other outputs are 0.

For $B_2 = 1$, the enable conditions are reversed.

The bottom decoder outputs generates minterms 100 to 111, while the outputs of top decoder are all 0's. 5 × 32 decoder with 3 × 8 decoders, 2 × 4 decoders.



A 5 × 32 decoder will have 5 inputs $B_4\,B_3\,B_2\,B_1\,B_0$. A 3 × 8 decoder will have 8 outputs, so 5 × 32 requires four 3 × 8 decoders, and we need one of the 2 × 4 decoders to select one 3 × 8 decoders and the connections are as shown in the circuit above.

Combinational Logic Implementation

An $n \times 2^n$ decoder provides 2^n minimum terms of n input variables. Since any Boolean function can be expressed in sum of minimum terms form, a decoder that generates the

minimum terms of the function, together with an external OR gate that forms their logical sum, provides a hardware implementation of the function.

Similarly, any function with n inputs and m outputs can be implemented with $n \times 2^n$ decoders and m OR gates.

Example 3: Implement full adder circuit by using 2 × 4 decoder.

Sum = Σ(1, 2, 4, 7), Carry = Σ(3, 5, 6, 7)

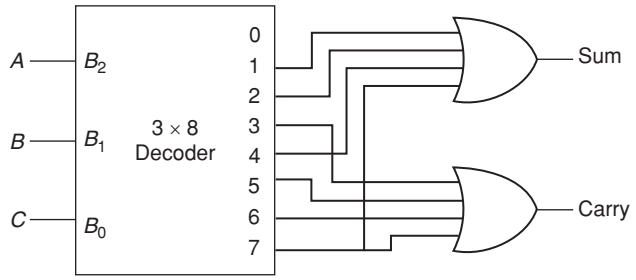


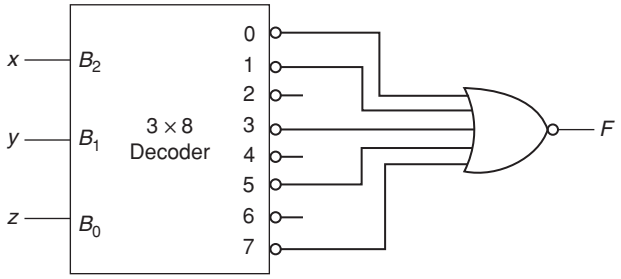
Figure 16 Implementation of full adder circuit with decoder

The 3 × 8 decoder generates the 8 minimum terms for A, B, and C. The OR gate for output sum forms the logical sum of minimum terms 1, 2, 4 and 7. The OR gate for output carry forms the logical sum of minimum terms 3, 5, 6 and 7.

Example 4: The minimized SOP form of output $F(x, y, z)$ is

(A) $x'y + z'$ (B) $x'y' + z'$

(C) $x'y' + z'$ (D) $x' + y'z$



Solution: (C)

The outputs of decoder are in active low state. So, we can express outputs as $\overline{Y_7}, \overline{Y_6} \ldots \overline{Y_0}$

Outputs 0, 1, 3, 5, 7 are connected to NAND gate to form function $F(x, y, z)$

So, $$F = \overline{\overline{Y_0} \cdot \overline{Y_1} \cdot \overline{Y_3} \cdot \overline{Y_5} \cdot \overline{Y_7}}$$
$$= Y_0 + Y_1 + Y_3 + Y_5 + Y_7$$
$$= \Sigma(0, 1, 3, 5, 7)$$

By using K-maps

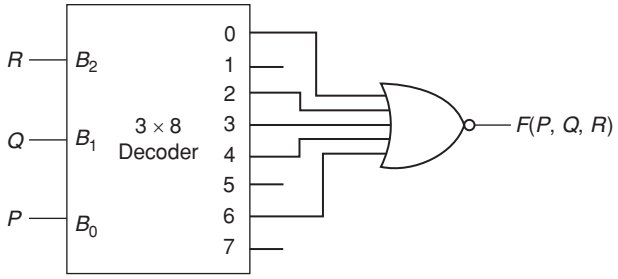
x \ yz	00	01	11	10
0	1	1	1	
1		1	1	

$$F = z + x'y'$$

Example 5: The minimal POS form of output function $f(P, Q, R)$ is

(A) $P\overline{Q} + PR$ (B) $P + \overline{Q}R$

(C) $P(\overline{Q} + R)$ (D) $Q(\overline{P} + R)$



Solution: (C)

The outputs of decoder are in normal form. 0, 2, 3, 4, 6 outputs are connected to NOR gate to form $F(P, Q, R)$

So $$F = \overline{Y_0 + Y_2 + Y_3 + Y_4 + Y_6}$$
$$= \overline{Y_0} \cdot \overline{Y_2} \cdot \overline{Y_3} \cdot \overline{Y_4} \cdot \overline{Y_6}$$

$Y_0, Y_1 \ldots Y_7$ indicate minimum terms, whereas $\overline{Y_0}, \overline{Y_1} \ldots \overline{Y_7}$ are maximum terms.

So $F = \pi(0, 2, 3, 4, 6)$

Here, from the decoder circuit MSB is R, LSB is P.

By using K-map

R \ QP	00	01	11	10
0	0		0	0
1	0			0

$$F(P, Q, R) = P(R + \overline{Q})$$

ENCODERS

It is a digital circuit that performs the inverse operation of a decoder.

An encoder has 2^n (or fewer) input lines and n output lines.

It is also known as an octal to binary converter.

Consider an 8–3 line encoder:

Truth Table

Inputs								Outputs		
D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	A	B	C
1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0
0	0	0	1	0	0	1	0	0	1	1
0	0	0	0	1	0	0	1	1	0	0
0	0	0	0	0	1	0	0	1	0	1
0	0	0	0	0	0	1	0	1	1	0
0	0	0	0	0	0	0	1	1	1	1

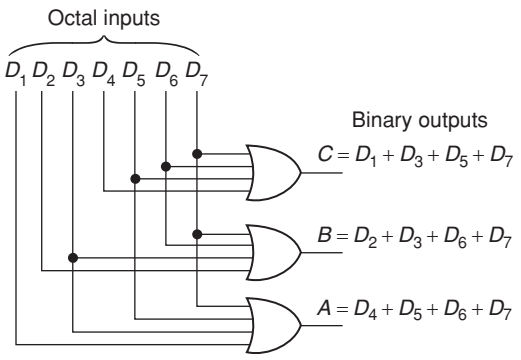


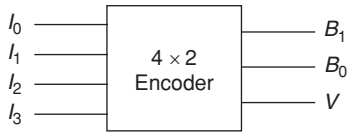
Figure 17 Logic diagram

Priority Encoder

A priority encoder is an encoder circuit that includes the priority function.

When two or more inputs are present, the input with higher priority will be considered.

Consider the 4 × 2 priority encoder.



I_3	I_2	I_1	I_0	B_1	B_0	V
1	X	X	X	1	1	1
0	1	X	X	1	0	1
0	0	1	X	0	1	1
0	0	0	1	0	0	1
0	0	0	0	X	X	0

I_3–I_0 are inputs and B_1 B_0 are binary output bits, valid (V) output is set to 1 when at least one input is present at input (I_3–I_0).

When there is no input present, (I_3–I_0 = 0000) then $V = 0$, for this combination the output B_1B_0 will not be considered.

The higher the subscript number, the higher the priority of the input. Input I_3 has the highest priority, I_2 has the next priority level. Input I_0 has lowest priority level. The Boolean expressions for output B_1 B_0 are

$$B_1 = I_3 + \overline{I}_3 I_2$$
$$= I_3 + I_2$$
$$B_0 = I_3 + \overline{I}_3 \overline{I}_2 I_1$$
$$= I_3 + \overline{I}_2 I_1$$
$$V = I_3 + I_2 + I_1 + I_0$$

Multiplexer

A multiplexer (MUX) is a device that allows digital information from several sources to be converted on to a single line for transmission over that line to a common destination.

The MUX has several data input lines and a single output line. It also has data select inputs that permits digital data on any one of the inputs to be switched to the output line.

Depending upon the binary code applied at the selection inputs, one (out of 2^n) input will be gated to single output. It is one of the most widely used standard logic circuits in digital design. The applications of multiplexer include data selection, data routing, operation sequencing, parallel to serial conversion, and logic function generation.

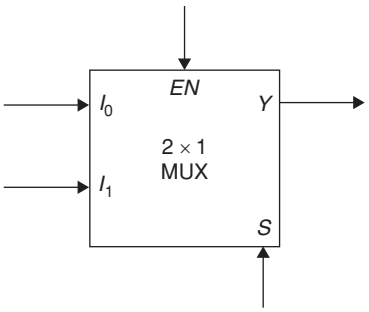
2^n inputs will be controlled by n selection lines and multiplexer will have 1 output, we denote it as $2^n \times 1$ multiplexer (data selector).

In other words, a multiplexer selects 1 out of n input data sources and transmits the selected data to a single output channel, this is called as multiplexing.

Basic 2 × 1 Multiplexer

The figure shows 2 × 1 multiplexer block diagram; it will have 2 inputs I_0 and I_1, one selection line S, and one output Y. The function table is as shown here.

EN	S	Y
0	x	0
1	0	I_0
1	1	I_1

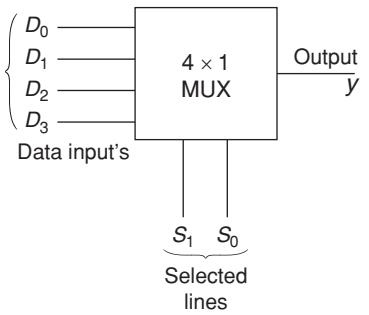


The output equation of 2 × 1 multiplexer is $Y = EN(I_0\overline{S} + I_1S)$.

When enable is 1, the multiplexer will work in normal mode, else the multiplexer will be disabled.

Sometimes enable input will be active low enable $\overline{EN}$, then $Y = \overline{EN}(I_0\overline{S} + I_1S)$.

The 4 × 1 Multiplexer



If a binary zero $S_1 = 0$ and $S_0 = 0$ as applied to the data select line the data input D_0 appear on the data output line and so on.

S_1	S_0	y
0	0	D_0
0	1	D_1
1	0	D_2
1	1	D_3

$$y = \overline{S_1}\,\overline{S_0}D_0 + \overline{S_1}S_0D_1 + S_1\overline{S_0}D_2 + S_1S_0D_3$$

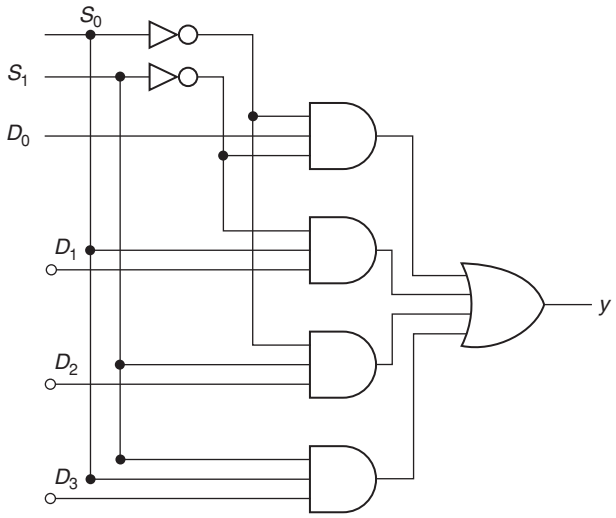


Figure 18 Logic diagram

For 8 × 1 multiplexer with 8 inputs from I_0 to I_7 based on selection inputs $S_2S_1S_0$, the equation for output

$$Y = I_0\overline{S_2}\,\overline{S_1}\,\overline{S_0} + I_1\overline{S_2}\,\overline{S_1}S_0 + I_2\overline{S_2}S_1\overline{S_0} + I_3\overline{S_2}S_1S_0$$
$$+I_4S_2\overline{S_1}\,\overline{S_0} + I_5S_2\overline{S_1}S_0 + I_6S_2S_1\overline{S_0} + I_7\overline{S_2}S_1S_0$$

From multiplexer equation we can observe, each input is associated with its minterm (in terms of selection inputs).

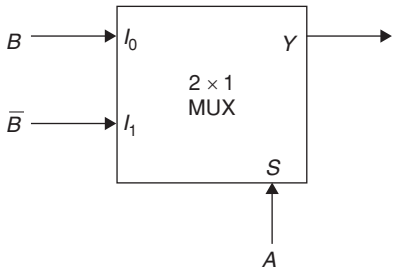


Figure 19 Basic gates by using MUX

$Y = \overline{A}B + A\overline{B} = X$-OR gate, we can interchange inputs A and B also,

By interchanging inputs I_0 and I_1, $Y = \overline{A}\,\overline{B} + AB$, X-NOR gate.

Similarly, we can build all basic gates by using 2 × 1 multiplexer.

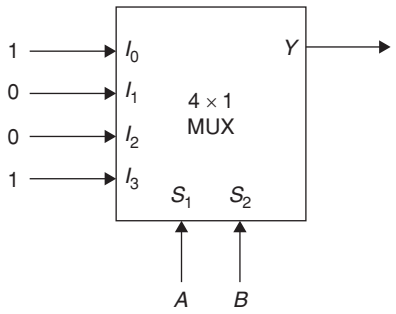
Example 6: If $I_0 = 1$, $I_1 = 0$, $S = A$, then Y is?

Solution: $= (I_0\overline{S} + I_1S) = \overline{A}$. It implements NOT gate.

Example 7: What should be the connections to implement NAND gate by using 2 × 1 MUX?

Solution: $Y = \overline{AB} = \overline{A} + \overline{B} = \overline{A} + A\overline{B} = 1\cdot\overline{A} + \overline{B}\cdot A$

By considering $I_0 = 1$, $I_1 = \overline{B}$, $S = A$, we can implement NAND gate, or by interchanging A and B also we can get the same answer.



For the above 4 × 1 multiplexer $Y = \overline{AB} + AB = X$-NOR gate. Similarly to implement 2-input gates by using 4 × 1 multiplexer, the inputs I_0, I_1, I_2, I_3 should be same as the terms in the truth table of that gate.

Logic Function Implementation by Using Multiplexer

Let us consider a full subtractor circuit (Barrow) to be implemented by using multiplexer.

Full subtractor Barrow (B) is a function of 3-inputs X, Y, Z. The truth table is

X	Y	Z	B	4 × 1 MUX	2 × 1 MUX
0	0	0	0	$B = Z$	$B = Y + Z$
0	0	1	1		
0	1	0	1	$B = 1$	
0	1	1	1		
1	0	0	0	$B = 0$	$B = YZ$
1	0	1	0		
1	1	0	0	$B = Z$	
1	1	1	1		

To implement Barrow by using 8 × 1 multiplexer, connect the three variables X, Y, Z directly to selection lines of the multiplexer, and connect the corresponding values of B to inputs, i.e., for $I_0 = 0$, $I_1 = 1$, $I_2 = 1$, etc. as per above truth table.

To implement Barrow by using 4 × 1 multiplexer, connect any two variables to selection lines (in this case X, Y) and write output (B) in terms of other variable, for $XY = 00$, output B is same as Z, so connect $I_0 = Z$, similarly 1, 0, Z for remaining inputs.

To implement the function by using 2 × 1 multiplexer, connect 1 variable as selection line (in this case consider X) and write output (B) in terms of other variables, for $X = 0$,

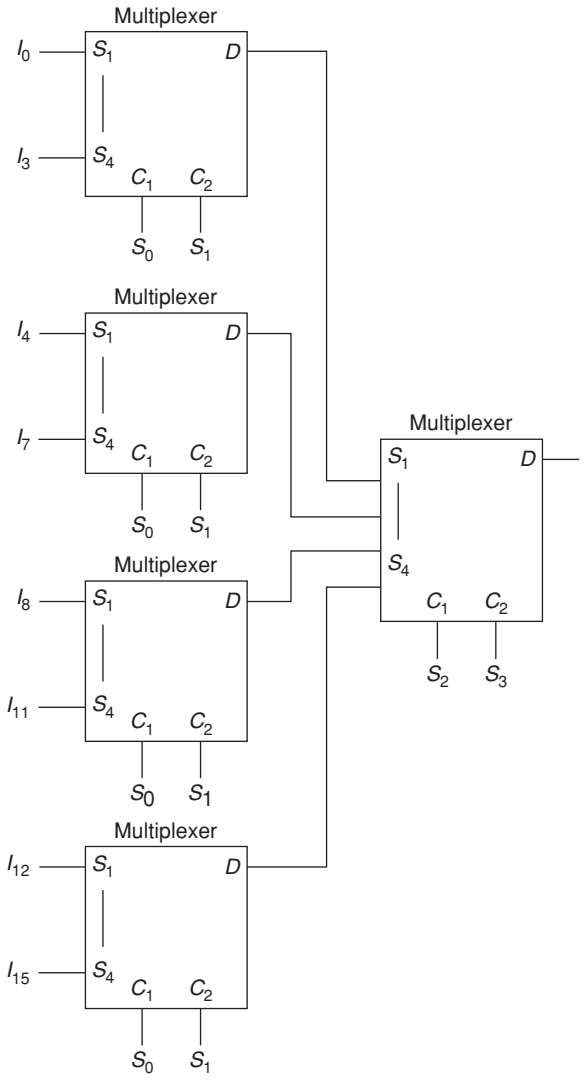
output B is varies as $B = Y + Z$, so connect $I_0 = Y + Z$. For $X = 1$, output B varies as $B = YZ$, connect $I_1 = YZ$.

N variable function can be implemented by using $2^{N-1} \times 1$ multiplexer without any extra hardware.

Implementation of Higher Order Multiplexer by Using Lower Order Multiplexers

By using lower order multiplexers, we can implement higher order multiplexers, for example by using 4×1 multiplexer, we can implement 8×1 MUX or 16×1 MUX or other higher order multiplexers.

Let us consider implementation of 16×1 MUX by using 4×1 MUX. 16×1 MUX will have inputs I_0 to I_{15} and selection lines S_0 to S_3, whereas 4×1 MUX will have only 4 input lines, and 2 selection lines, so we require four 4×1 MUX to consider all inputs I_0 to I_{15}, and again to select one of the four outputs of these four multiplexers one more 4×1 multiplexer is needed (for which we will connect higher order selection lines S_2 and S_3). So, total of 5, 4×1 multiplexers are required to implement 16×1 MUX.

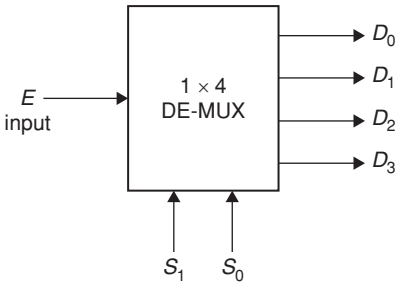


In a similar fashion, to design 4×1 MUX, we require three, 2×1 multiplexers, and to design 8×1 multiplexer, we require seven, 2×1 multiplexers.

De-multiplexer

The de-multiplexer [DeMUX] basically serves opposite of the multiplexing function. It takes data from one line and distributes them to a given number of output lines.

The other name for de-multiplexer is data distributor, as it receives information on a single line and distributes it to a possible 2^n output lines, where n is the number of selection lines, and value of n selects the line.



S_1	S_0	D_3	D_2	D_1	D_0
0	0	0	0	0	E
0	1	0	0	E	0
1	0	0	E	0	0
1	1	E	0	0	0

When $S_1S_0 = 10$; D_2 will be same as input E, and other outputs will be maintained at zero (0).

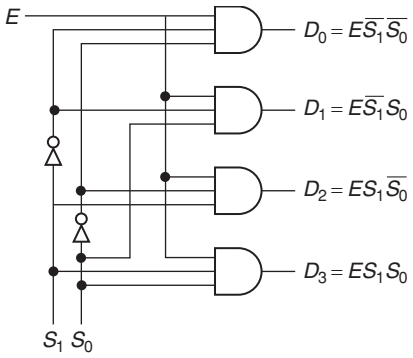


Figure 20 Logic diagram

Memory and Programmable Logic

A memory unit is a device to which binary information is transferred for storage and from which information is retrieved when needed for processing.

There are two types of memories that are used in digital systems. (1) Random Access Memory (RAM) and (2) Read Only Memory (ROM).

RAM stores new information for later use, RAM can perform both write and read operation, whereas ROM can perform only the read operation.

ROM is one example of a programmable logic device (PLD). Other such units are programmable logic array (PLA), programmable array logic (PAL) and field programmable logic array (FPGA).

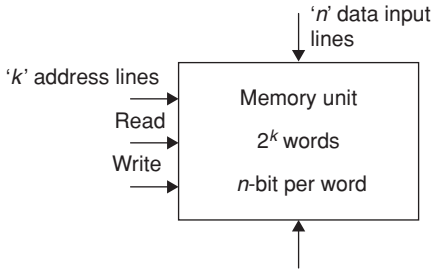
A PLD is an integrated circuit with internal logic gates connected through electronic paths that behave similar to fuses.

Random Access Memory

If the time taken to read or write data from any memory location is the same, then we call it as *random access memory*, whereas in *serial access memory* like magnetic tapes, the time taken to access different locations is different.

Any memory element will have address selection inputs to locate each word in memory, and the control input *Read/Write* specify the operation as well as data input or output lines for data writing or reading operation.

Each word in memory is assigned with an address, starting from 0 to 2^{k-1}, where k is the number of address lines.



The selection of a specific word inside memory is done by applying the k-bit address to the address lines.

For example, 4k × 16 memory has 12-bits in address and 16-bit in each word. Similarly, 32k × 16 memory has 15-bits as address lines and 16-bit in each word.

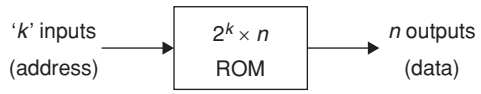
The memory enable (chip select) is used to enable the particular memory chip in a multichip implementation of a large memory.

Memory Enable	Read/Write	Memory Operation
0	X	No operation
1	0	Write to selected location
1	1	Read from selected location

Read-only Memory

A read-only memory (ROM) is a type of memory which stores data permanently or semi-permanently, i.e., data can be erasable. As the name indicates, data stored in ROM can only be read. The data that a user wanted to store on ROM will be given to manufacturer to fabricate the masked ROM which stores permanently. Or the user can programme the ROM in lab according to their specifications in the case of PROM. EPROM/EEPROM are the type of ROMs where user can erase data, and rewrite new data to ROM.

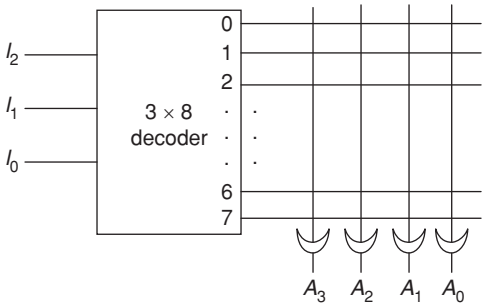
ROM also have address inputs similar to RAM, to access one of the memory location, and to read the data from that memory location, data output lines are available.



The number of words in a ROM is determined from the fact that k address input lines are needed to specify 2^k words.

ROM does not have data inputs because it does not have write operation.

Consider for example 8 × 4 ROM, the unit consists of eight words of 4-bit each. There are three input lines that from the binary numbers from 0 to 7 for address.



The above figure shows the internal logic construction of ROM, the three inputs are decoded into eight district outputs by means of 3 × 8 decoder, each output of the decoder represents a memory address. The eight outputs of decoder are connected to each of the four OR gates.

Each OR gate must be considered as having eight inputs. Each output of the decoder is connected to one of the inputs of each OR gate. Since each OR gate has eight input connections and there are four OR gates, the ROM contains 8 × 4 = 32 internal connections.

In general, a $2^{m \times n}$ ROM will have an internal $m \times 2^m$ decoder and n OR gates. Each OR gate has 2^m inputs, which are connected to each of the outputs of the decoder.

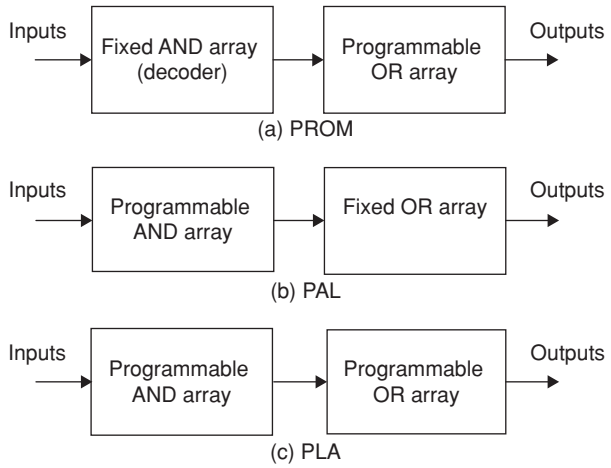
These intersections are programmable. A programmable connection between two lines is logically equivalent to a switch that can be altered to be either closed or open.

The internal binary storage of ROM is specified by a truth table that shows the word content in each address.

Combinational circuits can be implemented by ROM. For this, each output terminal of PROM is considered as separately as, the output of a Boolean function expressed as a sum of min terms.

The PROM is a combinational programmable logic device (PLD)—an integrated circuit with programmable gates divided into an AND array and an OR array to provide an AND–OR sum of product implementation.

There are three major types of combinational PLDS, differing in the placement of the programmable connections in the AND–OR array.



The PROM has fixed AND array constructed as a decoder, and a programmable OR array. The programmable OR gates implement the Boolean function in sum of min terms form.

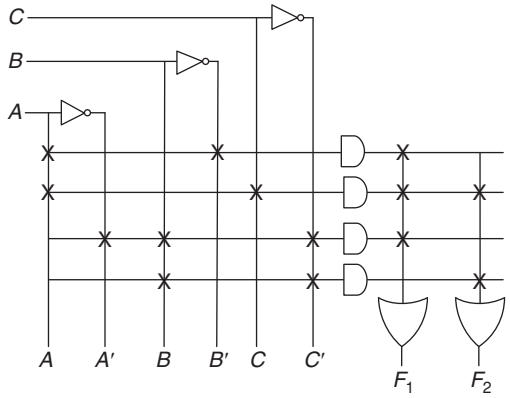
Most flexible PLD is PLA, in which both the AND and OR arrays can be programmed.

The PLA is similar in concept to the PROM, except that the PLA does not provide full decoding of the variables and doesn't generate all the min terms.

PLA for the functions,

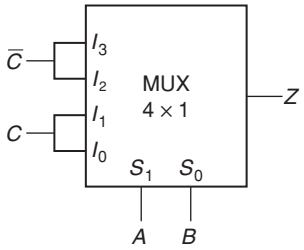
$F_1 = AB' + AC + A'\,B\,C'$

$F_2 = AC + BC'$ is shown below.



Solved Examples

Example 1: The multiplexer shown in the figure is a 4 : 1 multiplexer, where the output 'z' is



Solution:

A_1	B_0	Z
0	0	C
0	1	C
1	0	$\overline{C}$
1	1	$\overline{C}$

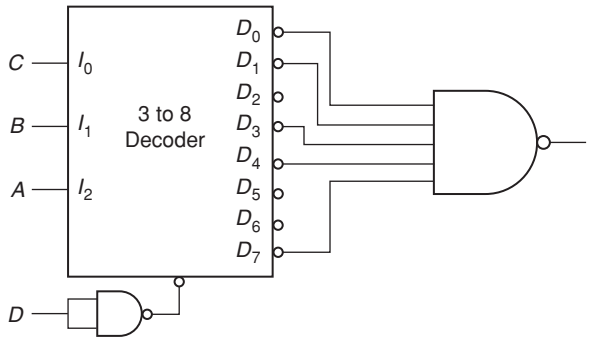
$$\therefore \quad Z = \overline{A}\overline{B}C + \overline{A}BC + AB\overline{C} + A\overline{B}\overline{C}$$

$$= \overline{B}(\overline{A}C + A\overline{C}) + B(\overline{A}C + A\overline{C})$$

$$= (\overline{A}C + A\overline{C})(B + \overline{B})(x + \overline{x} = 1)$$

$$\therefore \quad = \overline{A}C + A\overline{C} = A \oplus C$$

Example 2: The logic circuit shown in figure implements



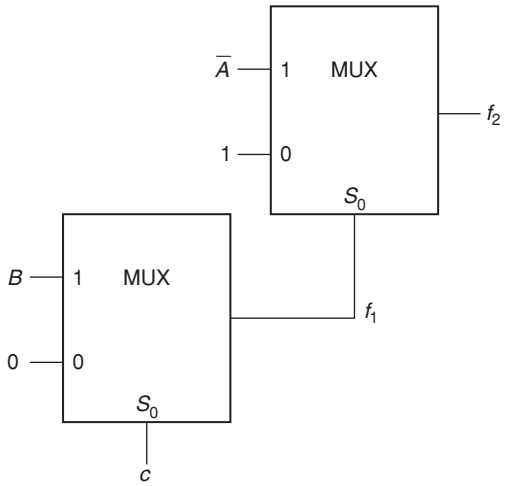
Solution: $z = D(\overline{A}\overline{B}\overline{C} + \overline{A}B\overline{C} + \overline{A}BC + A\overline{B}\overline{C} + ABC)$

$$= D(\overline{A}\overline{B}(C + \overline{C}) + BC(\overline{A} + A) + A\overline{B}\overline{C})$$

$$\times D(\overline{B}\overline{A} + B\overline{C} + BC)$$

$$= D(B \Theta C + \overline{A}\overline{B})$$

Example 3: The network shown in figure implements



Solution: $f_1 = \overline{C}_0 + CB = CB,\ f_1 = CB$

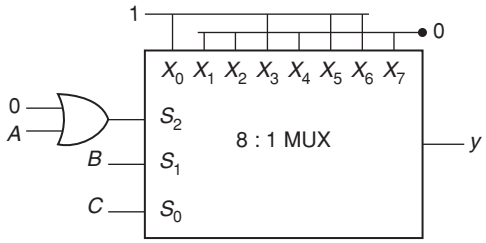
$$F_2 = \overline{f_1} + f_1\overline{A} = \overline{A} \cdot CB + \overline{CB}$$

$$= \overline{A} + \overline{CB}$$

$$= \overline{A} + \overline{C} + \overline{B} = \overline{ABC}$$

$\therefore$ NAND gate

Example 4: In the TTL circuit in figure, S_2 to S_0 are select lines and x_7 to x_0 are input lines. S_0 and X_0 are LSBs. The output y is

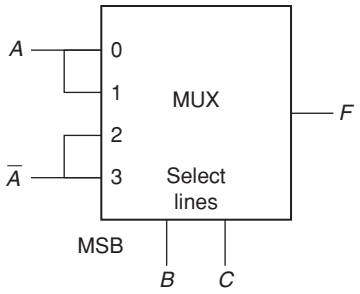


Solution: $S_2 = A, S_1 = B, S_0 = C$

$S_2(A)$	$S_1(B)$	$S_0(C)$	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

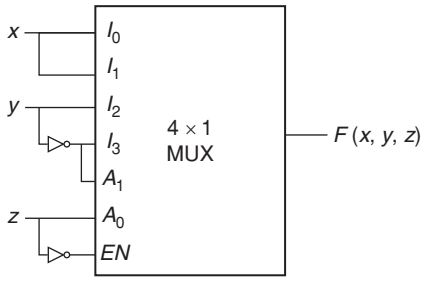
$$Y = \overline{A}\overline{B}\overline{C} + AB\overline{C} + A\overline{B}C + \overline{A}BC$$
$$= \overline{C}(\overline{A}B + AB) + C(A\overline{B} + \overline{A}B)$$
$$Y = \overline{C}(\overline{A} \oplus B) + C(A \oplus B) = A \oplus B \odot C$$

Example 5: The logic realized by the adjoining circuit is



Solution: $F = \overline{B}\overline{C}A + \overline{B}CA + B\overline{C}\overline{A} + BC\overline{A}$
$= \overline{C}(\overline{B}A + B\overline{A}) + C(\overline{B}A + B\overline{A})$
$= A\overline{B} + \overline{A}B(C + \overline{C}) = A \oplus B$

Example 6: Consider the following multiplexer, where I_0, I_1, I_2, I_3 are four date input lines selected by two address line combinations A_1A_0 = 00, 01, 10, 11, respectively and f is the output of the multiplexer. EN is the enable input, the function $f(x, y, z)$ implemented by the below circuit is



Solution: $A_1 = \overline{y} \cdot A_0 = z, EN = \overline{z}$

A_1	A_0	S	I
0	0	$(y\overline{z})$	x
0	1	$(y\ z)$	x
1	0	$(\overline{y}\overline{z})$	y
1	1	$(\overline{y}z)$	$\overline{y}$

$$f(x, y, z) = \text{SI} = (xy + 0 + \overline{y}z) \cdot EN$$
$$= xy \cdot \overline{z}$$

EXERCISES

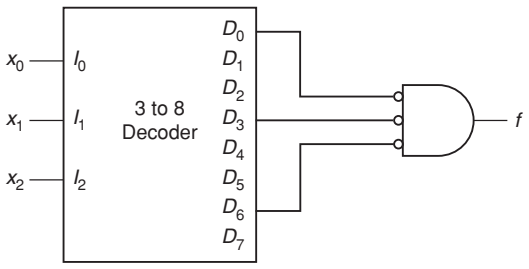
Practice Problems I

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The binary number 110011 is to be converted to gray code. The number of gates and type required are
(A) 6, AND (B) 6, X-NOR
(C) 6, X-OR (D) 5, X-OR

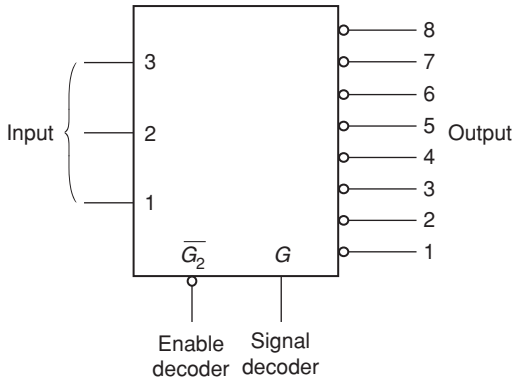
2. The number of 4- to 16-line decoder required to make an 8 to 256-line decoder is
(A) 16 (B) 17
(C) 32 (D) 64

3. $f(x_2, x_1, x_0) = ?$



(A) $\pi(1, 2, 4, 5, 7)$ (B) $\Sigma(1, 2, 4, 5, 7)$
(C) $\Sigma(0, 3, 6)$ (D) $\pi(0, 2, 3, 6)$

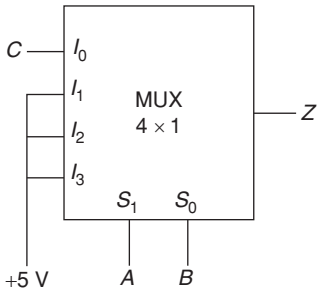
4. A 3 to 8 decoder is shown below:



All the output lines of the chip will be high except pin 8, when all the inputs 1, 2, and 3

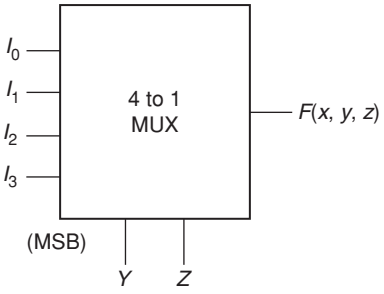
(A) are high; and G, G_2 are low
(B) are high; and G is low G_2 is high
(C) are high; and G, G_2 are high
(D) are high; and G is high G_2 is low

5. The MUX shown in figure is 4 × 1 multiplexer the output z is



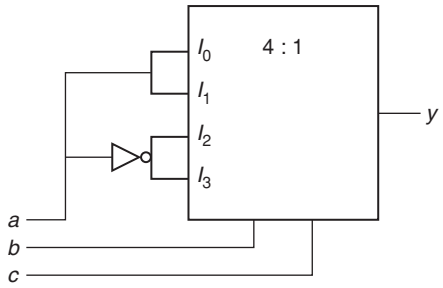
(A) $A\,B\,C$
(B) $A \oplus B \oplus C$
(C) $A\,\Theta\,B\,\Theta\,C$
(D) $A + B + C$

6. If a 4 to 1 MUX (shown below) realizes a three variable function $f(x, y, z) = xy + x\bar{z}$, then which of the following is correct?



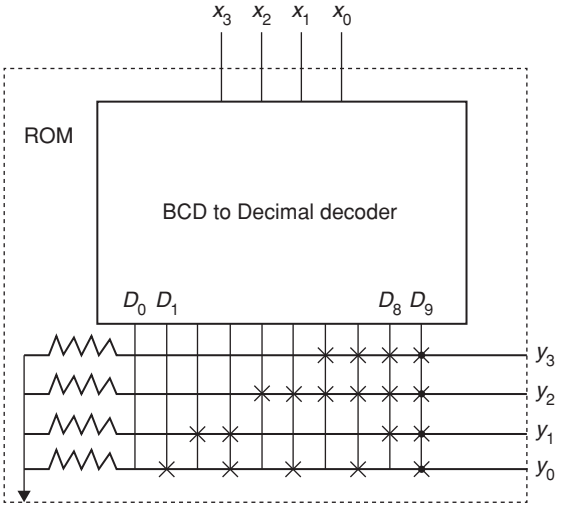
(A) $I_0 = X, I_1 = 0, I_2 = X, I_3 = X$
(B) $I_0 = 0, I_1 = 1, I_2 = Y_1, I_3 = X$
(C) $I_0 = X, I_1 = 1, I_2 = 0, I_3 = X$
(D) $I_0 = X, I_1 = 0, I_2 = X, I_3 = Z$

7. The circuit shown in the figure is same as



(A) Two input NAND gate with a and c inputs
(B) Two input NOR gate with a and c inputs
(C) Two input X-OR gates with a and b inputs
(D) Two input X-NOR gate with b and c as inputs.

8. If the input x_3, x_2, x_1, x_0 to the ROM in the figure are 8421 BCD numbers, then the outputs y_3, y_2, y_1, y_0 are

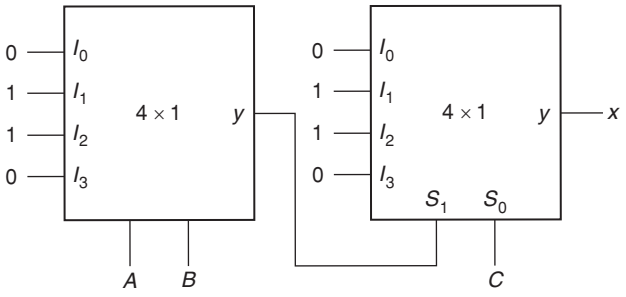


(A) Gray code numbers
(B) 2421 BCD
(C) Excess – 3 code numbers
(D) 84 – 2 – 1

9. A 4-bit parallel full adder without input carry requires

(A) 8 HA, 4 OR gates
(B) 8 HA, 3 OR gates
(C) 7 HA, 4 OR gates
(D) 7 HA, 3 OR gates

10. In the circuit find X.



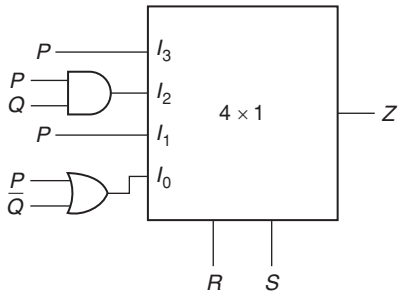
(A) $A\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}\bar{B}C + ABC$

(B) $\bar{A}BC + A\bar{B}C + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

(C) $AB + BC + AC$

(D) $\bar{A}\bar{B} + \bar{B}\bar{C} + \bar{A}\bar{C}$

11. Find the function implemented.



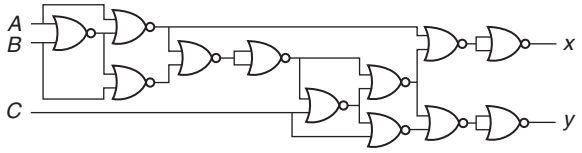
(A) $PQ + PS + \bar{Q}\bar{R}\bar{S}$

(B) $P\bar{Q} + PQ\bar{R} + \bar{P}\bar{Q}\bar{S}$

(C) $P\bar{Q}\bar{R} + \bar{P}QR + PQRS + \bar{Q}\bar{R}\bar{S}$

(D) $PQ\bar{R} + PQR\bar{S} + P\bar{Q}\bar{R}S + \bar{Q}\bar{R}\bar{S}$

12. Which function is represented by the given circuit?

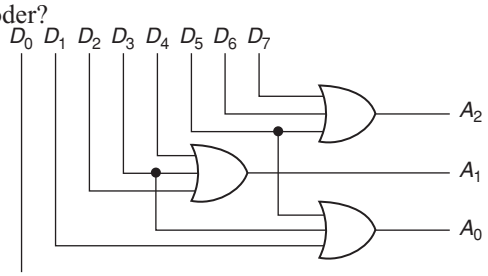


(A) Full adder (B) Full subtractor

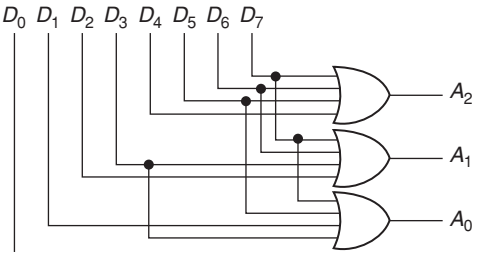
(C) Comparator (D) Parity generator

13 Which of the following represents octal to binary encoder?

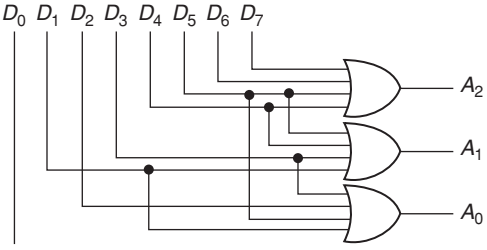
(A)



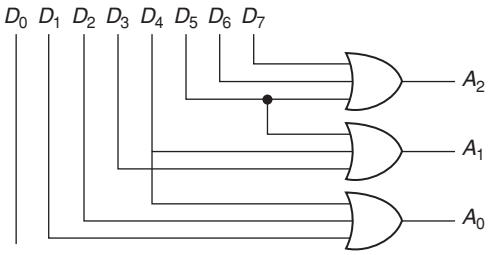
(B)



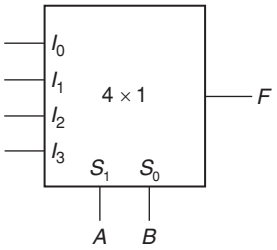
(C)



(D)



14. For a MUX to function as a full adder what should be the input provided to the I_0, I_1, I_2, I_3 if A and B are the select lines?



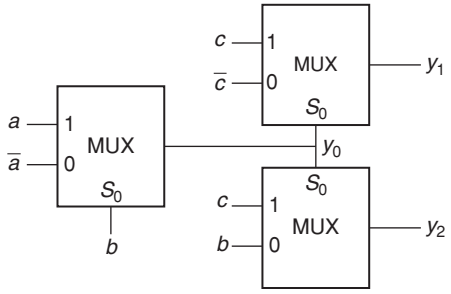
(A) $I_0 = I_1 = C_{in}; I_2 = I_3 = \overline{C_{in}}$

(B) $I_0 = I_1 = \overline{C_{in}}; I_2 = I_3 = C_{in}$

(C) $I_0 = I_3 = C_{in}; I_1 = I_2 = \overline{C_{in}}$

(D) $I_0 = I_3 = \overline{C_{in}}; I_1 = I_2 = C_{in}$

15. The given circuit act as



(A) full adder (B) half adder

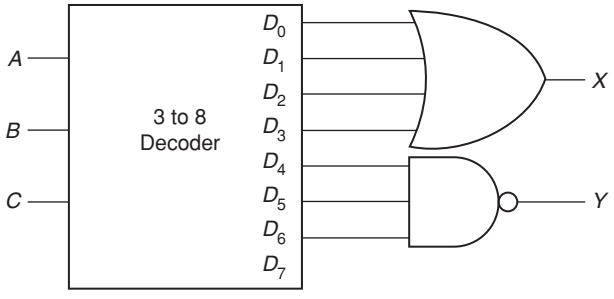
(C) full subtractor (D) half subtractor

16. For a 4 × 16 decoder circuit, the outputs of decoder (y_0, y_1, y_4 . y_5 . y_{10} . y_{11} . y_{14} . y_{15}) are connected to 8 input NOR gate. The expression of NOR gate output is?

(A) $A \oplus D$ (B) $A \odot D$

(C) $A \odot C$ (D) $A \oplus C$

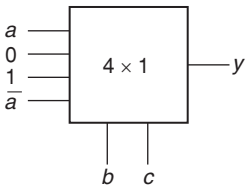
17. The function implemented by decoder is.

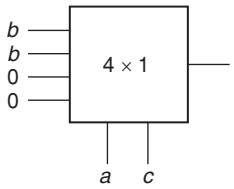


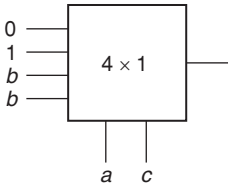
(A) $X = A'BC' + B'C', y = A + B$
(B) $X = A'C' + B'C', y = 1$
(C) $X = \overline{A}, y = 0$
(D) $X = \overline{A}, y = 1$

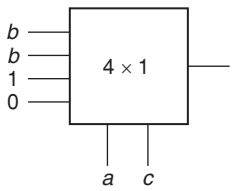
18. A relay is to operate with conditions that it should be on when the input combinations are 0000, 0010, 0101, and 0111. The states 1000, 1001, 1010 doesn't occur. For rest of the status relay should be off. The minimized Boolean expression notifying the relationship is
(A) $BC + ACD$ (B) $\overline{B}\overline{D} + \overline{A}BD$
(C) $BD + AC$ (D) $AB + CD$

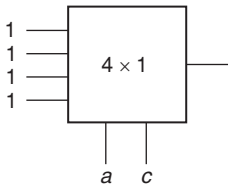
19. If a function has been implemented using MUX as shown, implement the same function with a and c as the select lines



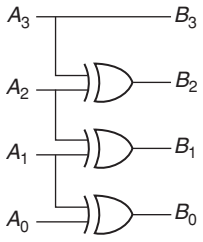
(A)


(B)


(C)


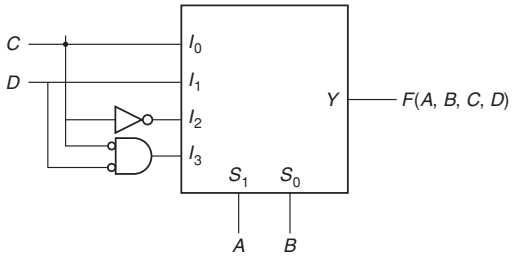
(D)


20. The circuit is used to convert one code to another. Identify it.



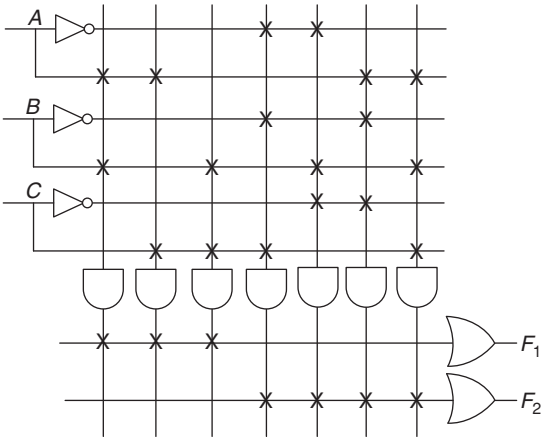
(A) Binary to grey
(B) Grey to binary
(C) Grey to XS – 3
(D) Grey to 8421

21. The Boolean function realised by logic circuit is



(A) $F = \Sigma m(0, 1, 3, 5, 9, 10, 14)$
(B) $F = \Sigma m(2, 3, 5, 7, 8, 12, 13)$
(C) $F = \Sigma m(1, 2, 4, 5, 11, 14, 15)$
(D) $F = \Sigma m(2, 3, 5, 7, 8, 9, 12)$

22. A circuit receives a 4-bit excess three code. The function to detect the decimal numbers 0, 1, 4, 6, 7, 8 is (assume inputs as A, B, C, D)?
(A) $ABC + \overline{A}C + BCD + AD$
(B) $CD + AD + AC + ACD$
(C) $CD + AD + \overline{A}\overline{C} + AC\overline{D}$
(D) $CD + AD + AC + \overline{A}\overline{C}\overline{D}$



Common Data for Questions 23 and 24:

23. Identify the function implemented.
(A) $f_1(A, B, C) = AB + BC + AC$
(B) $f_1(A, B, C) = \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\overline{C}$
(C) $f_1(A, B, C) = A + B + C$
(D) None of these

24. From the above, PLD implemented is
(A) PLA
(B) PROM
(C) PAL
(D) CPLD

25. The circuit implemented using the PLA in the above figure is
(A) full adder
(B) full subtractor
(C) half adder
(D) half subtractor

Practice Problems 2

Directions for questions 1 to 24: Select the correct alternative from the given choices.

1. For a binary half subtractor having two input A and B, the correct set of logical expression for the outputs $D = (A \text{ minus } B)$ and X (borrow) are

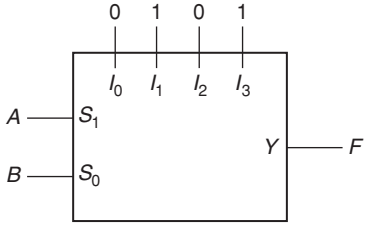
(A) $D = AB + \overline{AB},\ X = \overline{A}B$

(B) $D = \overline{AB} + A\overline{B} +,\ A\overline{B},\ X = \overline{A}B$

(C) $D = \overline{A}B + A\overline{B},\ X = \overline{A}B$

(D) $D = AB + \overline{A}\overline{B},\ X = \overline{A}B$

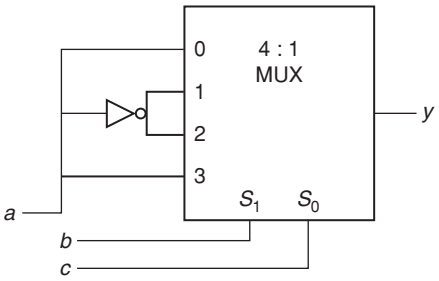
2. The function 'F' implemented by the multiplexer chip shown in the figure is



(A) A (B) B

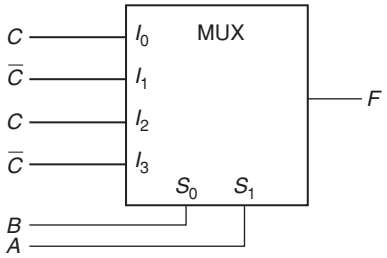
(C) $\overline{A}B$ (D) $A\overline{B} + \overline{A}B$

3. The following multiplexer circuit is equal to



(A) implementation of sum equation of full adder
(B) implementation of carry equation of full adder
(C) implementation of Borrow equation of full substractor
(D) All of the above

4. The output 'F' of the multiplexer circuit shown in the figure will be



(A) $AB + B\overline{C} + \overline{C}A + \overline{B}\overline{C}$ (B) $A \oplus B \oplus C$

(C) $A \oplus B$ (D) $B \oplus C$

5. Full subtractor can be implemented by using

(A) 3 to 8 line decoder only
(B) 3 to 8 line decoder and one OR gate
(C) 3 to 8 line decoder and two OR gates
(D) None

6. What are the difference and Borrow equations for the above circuit?

(A) $D = x\ \Theta\ y\ \Theta\ z,\ B = x'y + yz + zx'$
(B) $D = X \oplus y \oplus z,\ B = xy + yz + zx$
(C) $D = x \oplus y \oplus z,\ B = x'y + yz + zx'$
(D) A, C both

7. Combinational circuits are one in which output depends _____, whereas sequential circuit's output depends _____

(A) only on present input, only on past input
(B) only on present input, only on past and future input
(C) only on present input, only on present input and past output
(D) on present input, on past and present output

8. The sum output of the half adder is given by (assume A and B as inputs)

(A) $S = AB(\overline{A+B})$ (B) $S = (A+B)\overline{AB}$

(C) $S = (A+B)(AB)$ (D) $S = (\overline{A}+\overline{B})(\overline{A}B)$

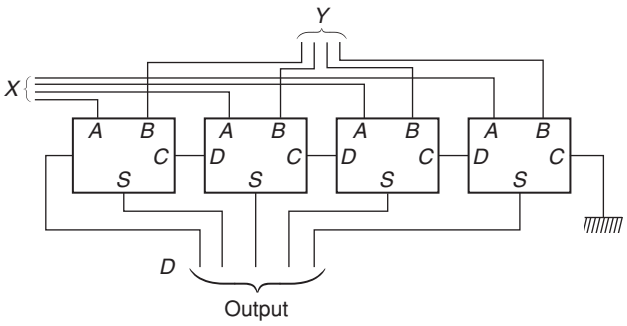
9. MUX implements which of the following logic?

(A) NAND-XOR (B) AND-OR
(C) OR-AND (D) XOR-NOT

10. A DeMUX can be used as a

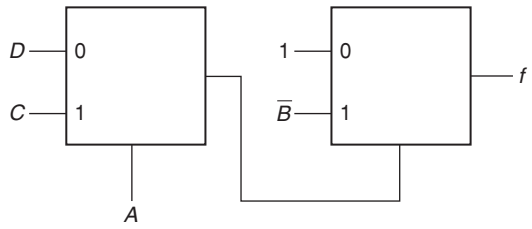
(A) comparator (B) encoder
(C) decoder (D) adder

11. If we have inputs as A, B and C and output as S and D. We are given that $S = A \oplus B \oplus C$. $D = BC + \overline{A}B + \overline{A}C$. Which of the circuit is represented by it?



(A) 4-bit adder giving $X + Y$
(B) 4-bit subtractor giving $X - Y$
(C) 4-bit subtractor giving $Y - X$
(D) 4-bit adder giving $X + Y + S$

12. The Boolean function f implemented in the figure using two input multiplexers is

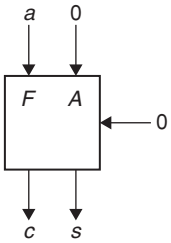


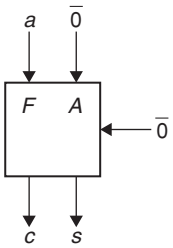
(A) $A\overline{C} + \overline{A}\overline{D} + \overline{D}C + \overline{A}\overline{B}D + A\overline{B}C$
(B) $\overline{A} + A\overline{C} + \overline{A}\overline{D} + \overline{D}\overline{C}$
(C) $\overline{B} + A\overline{C} + \overline{A}\overline{D} + \overline{D}\overline{C}$
(D) $AC + AD + A + B$

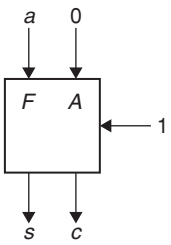
13. The Carry generate and Carry Propagate function of the look ahead carry adder is
(A) $CG = A + B$, $CP = A \oplus B$
(B) $CG = A \oplus B$, $CP = A + B$
(C) $CG = AB$, $CP = A \oplus B$
(D) $CG = AB$, $CP = A + B$

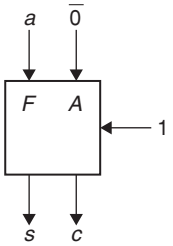
14. If we have a comparator and if E represents the condition for equality i.e., $(A_n \oplus B_n)$, if A_n and B_n are to be compared, then the expression $A_3\overline{B}_3 + E_3A_2\overline{B}_2 + E_3E_2A_1\overline{B}_1 + E_3E_2E_1A \cdot \overline{B}$. represents which of the condition for a 4-bit number?
(A) $A > B$ (B) $B > A$
(C) $A = B$ (D) None of these

15. When full adder is used to function as a 1-bit incrementor, which of the circuit configurations must be used?

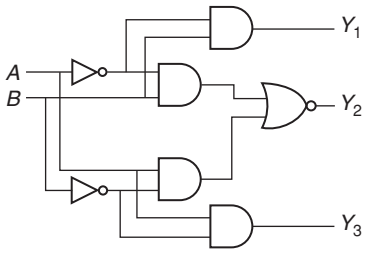
(A)


(B)


(C)


(D)


16. Identify the circuit.



(A) half adder
(B) full adder
(C) 1-bit magnitude comparator
(D) parity generator

17. In order to implement n variable function (without any extra hardware), the minimum order of MUX is
(A) $2^n \times 1$ (B) $2^n \times 1$
(C) $(2^n - 1) \times 1$ (D) $(2^{n-1}) \times 1$

18. A full adder circuit can be changed to full subtractor by adding a
(A) NOR gate (B) NAND gate
(C) Inverter (D) AND gate

19. The half adder when implemented in terms of NAND logic is expressed as
(A) $A \oplus B$ (B) $\overline{A \cdot \overline{AB} \cdot B \cdot \overline{AB}}$
(C) $\overline{\overline{A \cdot \overline{AB}} \cdot \overline{B \cdot \overline{AB}}}$ (D) $\overline{\overline{A \cdot \overline{ABB}} \cdot \overline{AB}}$

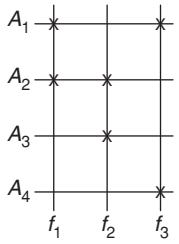
20. For a DeMUX to act as a decoder, what is the required condition?
(A) Input should be left unconnected and select lines behave as a input to decoder
(B) Input should be always 0 and select line behave as inputs to decoder
(C) Both are same
(D) Input should become enable and select lines behave as inputs to decoder

21. For a full subtractor, which of the combination will give the difference?
(A) $\overline{\overline{(A \oplus B)\overline{(A \oplus B)}b_i} \cdot \overline{b_i\overline{(A \oplus B)}b_i}}$
(B) $\overline{\overline{B \cdot \overline{AB}} \cdot \overline{b_i(A \oplus B)}}$
(C) $\overline{\overline{A} + \overline{B}} + \overline{\overline{b_i} + \overline{A \oplus B}}$
(D) None of these

22. A PROM contains
(A) a fixed AND array and programmable OR array
(B) a programmable AND array and OR array
(C) reprogrammable AND and OR array
(D) programmable AND array and fixed OR array

23. Which of the following is true of dynamic memories?
a. The power dissipation is slightly lower than that in static ROM
b. Refreshing operation of data is required to store data permanently
c. The clock will be needed
d. They will contain energy storage elements
(A) a, b, c, d (B) a, b, d
(C) a, b, c (D) c, b, d

24. For the given functions, we have to implement them using a OR gate array. When the input to the gate array must be product of one or two variables the terms A_1, A_2, A_3 should be



$f_1 = \bar{a}b\bar{c} + \bar{b}c + abc$

$f_2 = ac + a\bar{b}c + \bar{a}bc$

$f_3 = bc + \bar{a}b\bar{c} + \bar{a}b\bar{c}$

(A) $ab, ac, \bar{a}\bar{b}, bc$

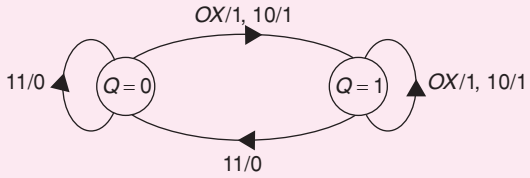
(B) $ac, \bar{a}b, \bar{a}\bar{b}, b\bar{c}$

(C) $\bar{a}\bar{b}, ac, \bar{a}b, bc$

(D) None of these

Previous Years' Questions

1. State diagram of a logic gate which exhibits a delay in the output is shown in the figure, where X is the don't care condition, and Q is the output representing the state. **[2014]**

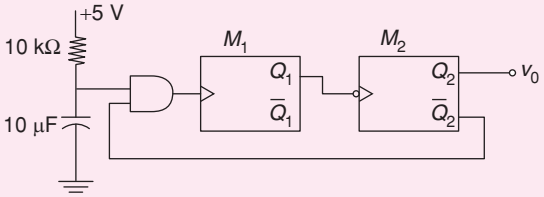


The logic gate represented by the state diagram is

(A) X-OR (B) OR

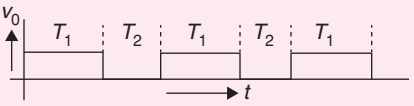
(C) AND (D) NAND

2. Two monoshot multivibrators, one positive edge triggered (M_1) and another negative edge triggered (M_2), are connected as shown in figure. **[2014]**

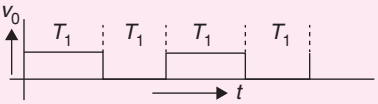


The monoshots M_1 and M_2 when triggered produce pulses of width T_1 and T_2, respectively, where $T_1 > T_2$. The steady state output voltage vo of the circuit is

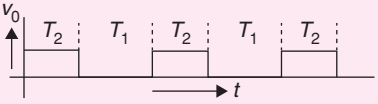
(A)



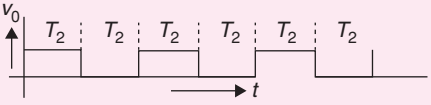
(B)



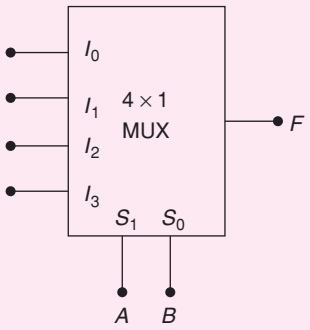
(C)



(D)



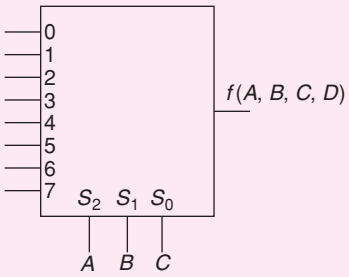
3. In the 4×1 multiplexer, the output F is given by $F = A + B$. Find the required input '$I_3\ I_2\ I_1\ I_0$'. **[2015]**



(A) 1010 (B) 0110

(C) 1000 (D) 1110

4. A Boolean function $f(A, B, C, D) = \prod(1, 5, 12, 15)$ is to be implemented using an 8×1 multiplexer (A is MSB). The inputs ABC are connected to the select inputs $S_2\ S_1\ S_0$ of the multiplexer respectively.



Which one of the following options gives the correct inputs to pins 0, 1,2,3,4,5,6,7 in order? **[2015]**

(A) $D, 0, D, 0, 0, 0, \bar{D}, D$

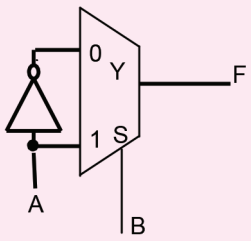
(B) $\bar{D}, 1, \bar{D}, 1, 1, 1, D, \bar{D}$

(C) $D, 1, D, 1, 1, 1, \bar{D}, D$

(D) $\bar{D}, 0, \bar{D}, 0, 0, 0, D, \bar{D}$

5. Consider the following circuit which uses a 2-to-1 multiplexer as shown in the figure below. The Boolean expression for output F in terms of A and B is

[2016]



(A) $A \oplus B$ (B) $\overline{A+B}$

(C) $A+B$ (D) $\overline{A \oplus B}$

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** B	**3.** B	**4.** D	**5.** D	**6.** A	**7.** C	**8.** B	**9.** D	**10.** A
11. A	**12.** B	**13.** B	**14.** C	**15.** C	**16.** D	**17.** D	**18.** B	**19.** C	**20.** A
21. D	**22.** D	**23.** A	**24.** A	**25.** A					

Practice Problems 2

1. C	**2.** B	**3.** A	**4.** D	**5.** C	**6.** C	**7.** C	**8.** B	**9.** B	**10.** C
11. B	**12.** C	**13.** C	**14.** A	**15.** C	**16.** C	**17.** C	**18.** C	**19.** C	**20.** D
21. A	**22.** A	**23.** A	**24.** C						

Previous Years' Questions

1. D **2.** C **3.** B **4.** B **5.** D

Chapter 10

Sequential Circuits

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Sequential circuits
- Basic storage elements
- Flip-flops
- Latches
- JK latch by using SR latch
- Flip-flops
- Triggering of flip-flop
- Applications of flip-flops
- Counters
- Identification of Up/down counters
- Asynchronous decode counter

Sequential Circuits

In sequential circuits, the output depends on the input as well as on the previous history of output, i.e., they contain memory elements.

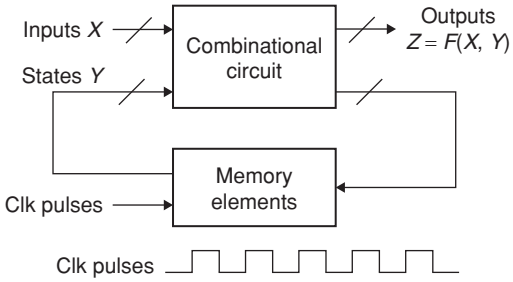


Table 1 *Comparison between combinational and sequential circuits*

Combinational Circuits	Sequential Circuits
1. Output at any time depends on the combine set of input applied to it simultaneously at that instant of time	Output depends on the present input as well as on the previous history of output
2. Contains no memory element	Contains at least one memory element
3. Easy to design due to absence of memory	Difficult to design
4. Totally described by the set of output values of it.	Its performance is totally described by the set of subsequent value as well as set of output values.
5. Faster in speed because all inputs are primary inputs and applied simultaneously	Slower in speed because secondary inputs are also needed which are applied after delay
6. It need more hardware for realization	Less hardware required
7. Expensive in Lost	Cheaper in Lost

Sequential circuits are of two types:

1. Clocked or synchronized
2. Unclocked or asynchronized

In synchronized sequential circuits, the logic circuits' action is allowed to occur in synchronization with the input clock pulse from a system clock.

In asynchronized sequential circuits, the logic sequential action is allowed to occur at any time.

Basic Storage Elements: Latches and flip-flops

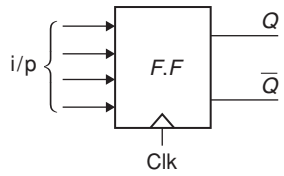
A storage element in digital circuit can maintain a binary state indefinitely until directed by an input signal to switch states. Storage elements that operate with signal levels (i.e., level triggering of signal inputs) are referred to as latches. Those controlled by a clock transition (i.e., edge triggering) are flip-flops.

Latches and filp-flops are related because latches are basic circuits from which all flip-flops are constructed. Latches are useful for storing binary information and for the design of asynchronous sequential circuits. However, latches are not practical for use in synchronous sequential circuits; hence, we use flip-flops.

Flip-flops

Flip-Flops are also known as bistable multivibrators. This is a basic memory element to store one-bit of information 0 or 1 and is used in storage circuits, counters, shift register and many other computer applications. It has two stable states: 1 and 0. The high state is called set state and zero as reset.

It has two outputs—one being the complement of the other usually designated by Q and $\overline{Q}$.

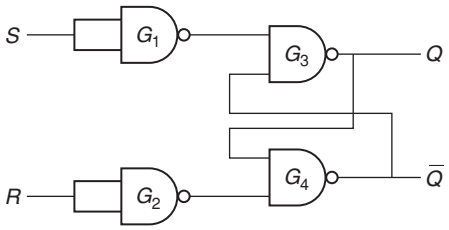


There are different types of flip-flops, such as *SR* flip-flop, *D* flip-flop, *T* flip-flop, *JK* flip-flops, etc.

LATCHES

S-R Latch

The simplest latch is called *SR* latch. *SR* means set, reset latch. They have two outputs, Q and $\bar{Q}$ and two inputs S and R, which represent set or reset signal.



The figure shows two cross-coupled gates G_3 and G_4 and inverters G_1 and G_2. Here, output of G_3 is connected to the input of G_4 and output of G_4 is applied to the input of G_3. $S = 1$, $R = 0$ output of $G_1 = 0$ and $G_2 = 1$. Since one of the input of G_3 is zero and its output will certainly one, consequently both input of G_4 will be one and the output $\bar{Q} = 0$.

For $S = 1$, $R = 0$, $Q = 1$, $\bar{Q} = 0$. $S = 0$, $R = 1$ the output will be $Q = 0$ and $\bar{Q} = 1$. The first of the input condition $S = 1$ and $R = 0$ makes $Q = 1$ which referred as the set state and the second condition $S = 0$ and $R = 1$ makes $Q = 0$ which is referred as reset state.

For $S = 0$ and $R = 0$, output of both G_1 and G_2 will be one and hence there will be no change in Q and $\bar{Q}$.

For $S = R = 1$, both the outputs Q and $\bar{Q}$ will try to become one which produces invalid results and should not be used for the above latch.

Input		Output		
S	*R*	*Q*	$\bar{Q}$	State
1	0	1	0	Set
0	1	0	1	Reset
0	0	0	0	No change
1	1	?	?	Invalid

SR Latch by Using NAND/NOR Gates

The *SR* latch is a circuit with two cross-coupled NOR gates or two cross-coupled NAND gates. Two inputs labelled S for set and R for reset. Latch will have two outputs: Q and Q_1;

Q – output state in normal form, and Q' – output state in complemented form.

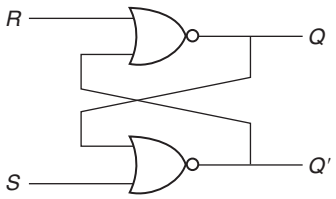


Figure 1 Logic diagram for *SR* latch

S	*R*	Q_{n+1}	Q'_{n+1}	
0	0	Q_n	Q'_n	(no change)
0	1	0	1	(Reset)
1	0	1	0	(set)
1	1	0	0	(invalid)

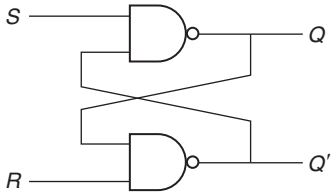


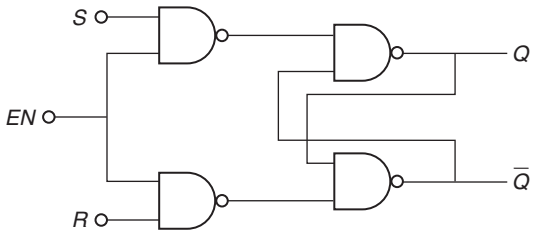
Figure 2 $\bar{S}\bar{R}$ Latch

S	*R*	Q_{n+1}	$\bar{Q}_{n+1}$	
0	0	1	1	(Invalid)
0	1	1	0	(Set)
1	0	0	1	(Reset)
1	1	Q_n	Q'_n	(No change)

$\bar{S}\bar{R}$ latch is active, low *SR* latch.

SR Latch with Control Input

The working of gated *SR* latch is exactly the same as *SR* latch when the *EN* pulse is present. When the *EN* pulse is not present (*EN* pulse = 0), the gates G_1 and G_2 are inhibited and will not respond to the input.

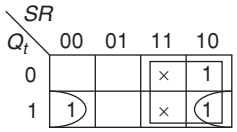


Characteristic table of *SR* latch shows the operation of latch in tabular form. Q_t stands as the binary state of the latch before the application of latch pulse and referred to as the present state. The S and R columns give the possible values of the inputs, and Q_{t+1} is the state of the latch after the application of a single pulse, referred to as next stage. *EN* input is not included in the characteristic table.

Characteristic table for *SR* latch is given below:

Q_t	S	R	Q_{t+1}
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	X
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	X

Characteristic equation of the latch is derived from the K-map:



$$\therefore \quad Q_{t+1} = S + \bar{R}Q_t$$

This equation specifies the value of the state as a function of the present state and the inputs.

Preset and Clear Inputs

For the latch/flip-flop, when the power is switched ON, the state of the circuit is uncertain. It can be either $Q = 0$ (reset) or $Q = 1$ (set) state.

In many applications, it is desired to set or reset the circuit, so that initial state of the circuit will be known. This is accomplished by using the asynchronous, inputs referred to as preset (Pr) and clear (Clr), inputs.

These inputs can be applied any time, and are not synchronized with *EN* input/Clr input.

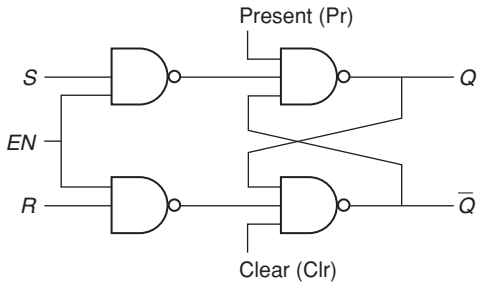


Figure 3 *SR* latch with *Pr* and Clr inputs.

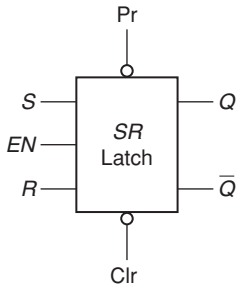
If Pr = Clr = 1, the circuits operates as of *S*-*R* latch explained previously.

If Pr = 0, Clr = 1, the output Q will become 1, which in turn changes $\bar{Q} = 0$.

If Pr = 1, Clr = 0 the, output $\bar{Q}$ will become 1, which in turn changes $Q = 0$.

If Pr = Clr = 0, both Q and $\bar{Q}$ will become 1, which is invalid case, so Pr = Clr = 0 condition must not be used.

Pr	Clr	Q_{n+1}	
1	1	Q	No change
0	1	1	Set
1	0	0	Reset
0	0	X	Invalid



D Latch (Transparent Latch)

One way to eliminate the invalid condition of *SR* latch (when $S = R = 1$) is to ensure that inputs *S* and *R* are never equal to 1 at the same time.

By connecting a NOT gate between *S* and *R* inputs. i.e., complement of *S* will be given to *R*, we can form *D* latch as shown in block diagram.

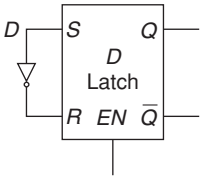


Figure 4 Block diagram for *D* latch

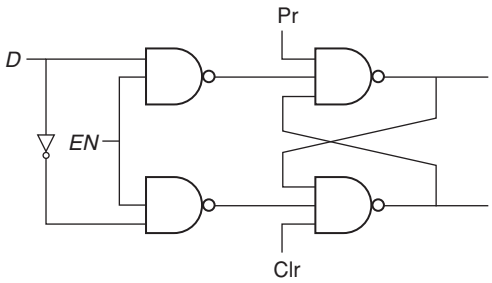


Figure 5 Logic diagram for *D* latch

EN	D	Q_{n+1}
0	X	Q_n No change (Disabled)
1	0	0 Reset state
1	1	1 Set state

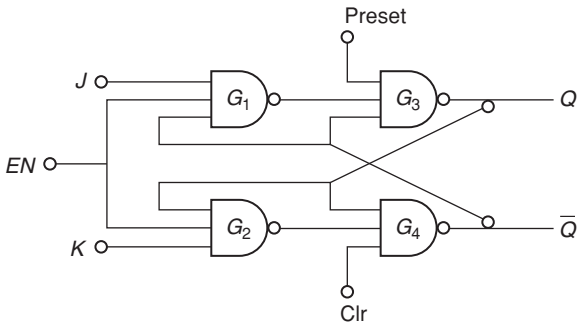
When $EN = 0$, the circuit will be disabled and input *D* will not have only effect on output, and output will be same as previous state.

When $EN = 1$, $D = 0$, i.e., $S = 0$, $R = 1$ which makes output $Q = 0$, and $\bar{Q} = 1$, (reset state).

When $EN = 1$, $D = 1$, i.e., $S = 1$, $R = 0$, which makes output $Q = 1$, and $\bar{Q} = 0$ (set state).

JK Latch

The function of *JK* latch is identical to that of *SR* latch except that it has no invalid state as that of *SR* latch, where $S = R = 1$. In this case, the state of the output is changed as complement of previous state.



EN	J	K	Q_{t+}	
1	0	0	Q_t	No change
1	0	1	0	Reset
1	1	0	1	Set
1	1	1	$\overline{Q}_t$	Toggle
0	X	x	Q_t	No change

JK latch by using SR latch

The uncertainty of *SR* flip-flop (when $S = 1$, $R = 1$) can be eliminated by converting it into *JK* latch.

The data inputs *J* and *K*, which are ANDed with $\overline{Q}$ and Q respectively, to obtain *S* and *R* inputs.

$$S = J \cdot \overline{Q}, \quad R = K \cdot Q$$

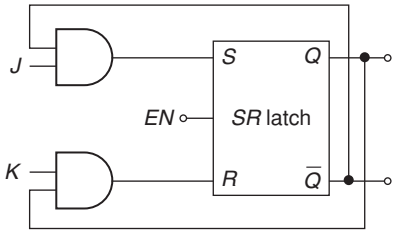
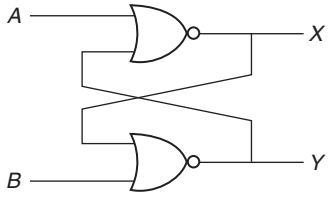


Figure 6 *JK* latch by using *SR* latch

J	K	S	R	Q_{n+1}	$\overline{Q}_{n+1}$	
0	0	0	0	Q_n	$\overline{Q}_n$	No change
0	1	0	Q_n	0	1	Reset
1	0	$\overline{Q}_n$	0	1	0	Set
1	1	$\overline{Q}_n$	Q_n	$\overline{Q}_n$	Q_n	Toggle

Example 1: The following binary values were applied to *A* and *B* inputs of NOR gate latch shown in the figure, in the sequence indicated below. $A = 1, B = 0$; $A = 1, B = 1$; $A = 0, B = 0$. The corresponding stable *X*, *Y* outputs will be



(A) 10, 01, 10 or 01 (B) 11, 00, 10
(C) 01, 00, 10 or 01 (D) 10, 11, 10 or 01

Solution: (C)

Given circuit is *RS* latch with NOR gates.

By comparing with *RS* latch $A = R$, $B = S$, and $X = Q$, $Y = \overline{Q}$. So from truth table of *RS* latch

S/B	R/A	Q/X	$\overline{Q}$/Y	
0	1	0	1	Reset (invalid)
1	1	0	0	
0	0	1	0	(Same as previous state)
		0	1	

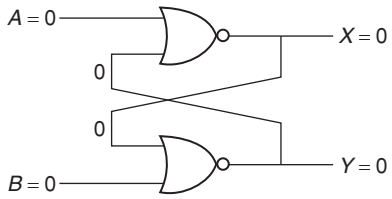
After invalid case $S = 1$, $R = 1$, i.e., $A = B = 1$,

The output $Q = 0$, $\overline{Q} = 0$, i.e., $X = Y = 0$

By applying $A = 0$, $B = 0$

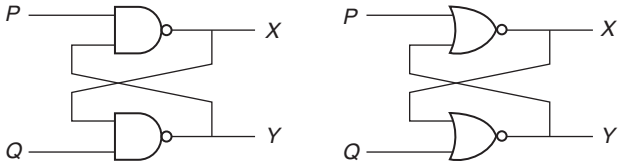
The output *X* becomes $\overline{(0+0)} = 1$ and which in turn changes:

$$Y = \overline{(B + X)} = \overline{(0+1)} = 0$$



or the output *Y* becomes $\overline{(0+0)} = 1$ and which in turn changes $X = \overline{(Y + A)} = \overline{(0+1)} = 0$. So, output (*X*, *Y*) cannot be predicted after the invalid condition. So, $X = 0$, $Y = 1$ or $X = 1$, $Y = 0$

Example 2: Refer to the NAND and NOR latches shown in the figure. The inputs (*P*, *Q*) for both the latches are first made (1, 0) and then after a few seconds, made (0, 0). The corresponding stable outputs (*X*, *Y*) are



(A) NAND: first (0, 1) then (0, 1); NOR: first (1, 0) then (1, 0)
(B) NAND: first (0, 1) then (1, 1); NOR: First (0, 1) then (0, 1)
(C) NAND: first (1, 0) then (0, 0); NOR: first (1, 0) then (1, 0)
(D) NAND: first (1, 0), then (1, 0); NOR: first (1, 0) then (1, 1)

Solution: (B)

From the truth table of *SR* latch and $\overline{S}\,\overline{R}$ latch *SR* latch with NOR gates

For $(P, Q) = (1, 0) = (R, S)$ output $(X, Y) = (Q, \overline{Q}) = (0, 1)$

Then (P, Q) are made (0, 0), i.e., $(R, S) = (0, 0)$, which results in no change at output. So, $(X, Y) = (Q, \bar{Q}) = (0, 1)$

SR latch with NAND gates

For $(P, Q) = (1, 0) = (S, R)$ output $(X, Y) = (Q, \bar{Q}) = (0, 1)$. Then (P, Q) are made (0, 0), i.e., $(S, R) = (0, 0)$ which is invalid conditions for $\bar{S}\,\bar{R}$ latch. So, $(X, Y) = (Q, \bar{Q}) = (1, 1)$.

Race Around Condition

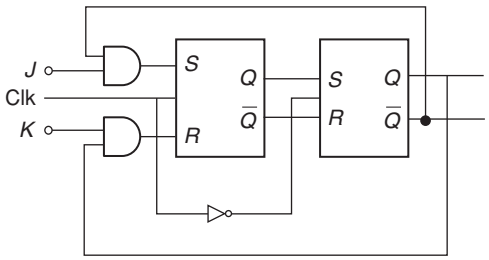
The difficulties of both the inputs ($S = R = 1$) being not allowed in an *SR* latch is eliminated in *JK* latch by using the feedback connection from the output to the input of the gate G_1 and G_2. In a normal *JK* latch, if $J = K = 1$ and $Q = 0$ and enable signal is applied without *RC* differentiator, after a time interval Δt, (the propagation delay through two NAND gate in series) the output will change to $Q = 1$. Now we have $J = K = 1$ and $Q = 1$ and after another time interval of Δt the output will change back to $Q = 0$. Hence for the duration of (t_p) of the enable signal, the output will oscillates back and forth between 0 and 1. At the end of the enable signal the values of Q are uncertain. This situation is referred to as race around condition.

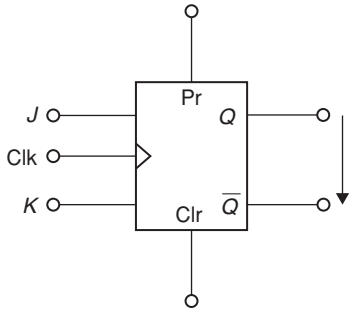
The race around condition can be avoided if enable time period $t_p < \Delta t$, but it may be difficult to satisfy this condition, as very small propagation delays in ICs. To solve this problem the enable signals are converted to narrows spike using *RC* differentiator circuit having a short time constant. Its output will be high during the high transmission time of the enable. Another method to avoid this problem is master-slave *JK* flip-flop.

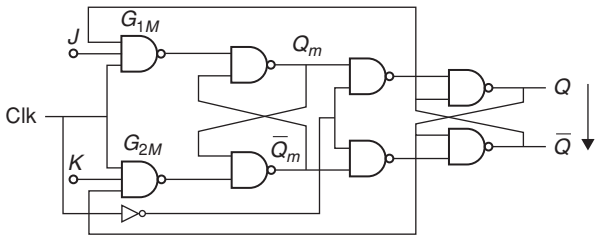
Flip-flops

Master-slave JK Flip-flop

This is a cascade of 2 *SR* latches with feedback from the output of the second *SR* latch to the inputs of the first as shown in the figure below.



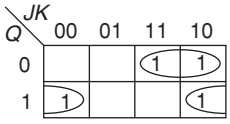




Positive clock pulse is applied to the first latch and the clock pulse will be inverted before its arrival at the second latch. When clk = 1, the first latch is enabled and the outputs Q_m and $\bar{Q}_m$ responds to the inputs J and K, according to the truth table of *JK* latch. At this time the second latch is inhibited because its clock is low ($\overline{\text{clk}} = 0$). When the clock goes low (clk = 0), the first latch is inhibited and the second is enabled. Hence, the outputs Q and $\bar{Q}$ follow the outputs Q_m and $\bar{Q}_m$, respectively. Since, the second latch simply follows the first one, it is referred to as slave and the first one as the master. Hence this configuration is known as master-slave JK flip-flop. In this circuit, the input to the gate G_{1m} and G_{2m} do not change, during the clock pulse levels. Therefore The race around condition does not exist.

State/characteristic Table

Clk	J	K	Q_t	Q_{t+1}	
↓	0	0	0	0	}
↓	0	0	1	1	}
↓	1	0	0	1	}
↓	1	0	1	1	} 1
↓	0	1	0	0	}
↓	0	1	1	0	} 0
↓	1	1	0	1	}
↓	1	1	1	0	} $\bar{Q}_t$



$$Q_{t+1} = J\bar{Q}_t + Q_t\bar{K}$$

Flip-flop Switching Time

In designing circuits with flip-flop the following parameters are important:

1. Set-up time
2. Hold time
3. Propagation delay

Set-up time

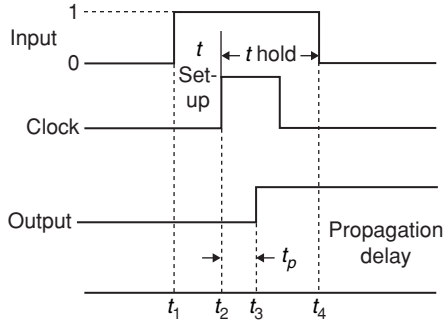
The minimum amount of time required for the data input to be present before the clock arrived.

Hold time

The minimum amount of time that the data input to be present after the clock trigger arrived.

Propagation delay

The amount of time it takes for the output to change states after an input trigger.



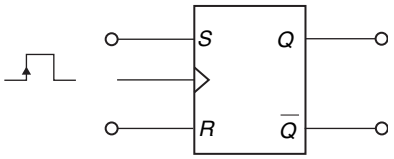
For example, t set-up = 50 m sec and t hold = 5 m sec, the data bit has to be the input at least 50 m sec before the clock bit arrives and hold at least 5 m sec after the clock edge.

Triggering of Flip-flop

The flip-flop can be triggered to set or reset either at one of the edges of the clock pulse. There are three types of triggering as described below:

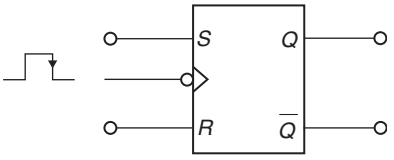
Positive edge triggering flip-flop

These set or reset at the positive (rising or leading) edge of the clock pulse depending upon the state of i/p signal and o/p remain steady for 1 clock period. Positive edge triggering is indicated by an arrow head at the clock terminal of the flip-flop.



Negative edge triggered flip-flop

There are flip-flops those in which state transmissions take place only at the negative edge (falling or trailing) of the clock signal. Negative edge triggering is indicated by arrow head with bubble at the clock terminal.

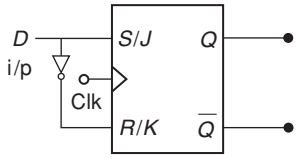


Level triggering

Level triggering means the specified action occurs based on the steady state value of the input. That is, when a certain level is reached (0 or 1) the output will change states level triggering will be used in latches.

D Flip-flop

It receives the designation from its ability to hold data into its internal storage. An *SR*/*JK* flip-flop has two inputs. It requires two inputs *S*/*J* and *R*/*K* to store 1-bit. This is a serious disadvantage in many application to overcome the difficulty *D* flip-flop has been developed which has only one input line. A *D* flip-flop can be realized using a *SR*/*JK* as show in the figure below.



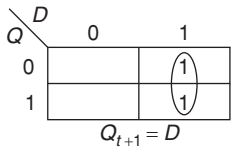
Truth Table

Clk	D	Q_{t+1}
↕	X	Q_t
↑	0	0
↑	1	1

There is no raising problem with *D* flip-flop. High or 1 state will set the flip-flop and a low or 0 state will reset the flip-flop. The presence of inverter at the input ensure that *S*/*J* and *R*/*K* inputs will always be in the opposite state.

Characteristic Table of *D* Flip-flop

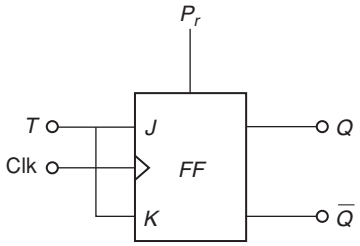
Q_t	D	Q_{t+1}
0	0	0
0	1	1
1	0	0
1	1	1



From the characteristic table of *D* flip-flop, the next state of the flip-flop is independent of the present state, since $Q_{t+1} = D$ whether $Q_t = 0$ or 1.

T Flip-flop

In a *JK* flip-flop, $J = K = 1$ and the resulting flip-flop is referred to as a *T* flip-flop.



Truth Table

Clk	T	Q_{n+1}
↑	0	Q_n
↑	1	$\overline{Q}_n$
↕	x	Q_n

If $T = 1$, it acts as a toggle switch for every Clk pulse with high input, the Q changes to its opposite state.

Characteristic Table

Q_t	T	Q_{t+1}
0	0	0
0	1	1
1	0	1
1	1	0
1	0	1

Q \ T	0	1
0		1
1	1	

$$Q_{t+1} = T\overline{Q}_t + Q_t\overline{T}$$

Excitation Table of Flip-flops

The truth table of flip-flop is also referred to as the characteristic table, which specifies the operational characteristic of flip-flop.

Sometimes we come across situations, in which present state and the next state of the circuit are known, and we have to find the input conditions that must prevail to cause the desired transition of the state.

Consider initially JK flip-flop output $Q_n = 1, \overline{Q}_n = 0$, after clock pulse it changed to $Q_{n+1} = 0$, $\overline{Q}_{n+1} = 1$,

The input conditions, which made this transition, can be

Toggle – for $J = 1,\ K = 1,\ Q_{n+1} = \overline{Q}_n$

or Reset – for $J = 0,\ K = 1,\ Q_{n+1} = 0,\ \overline{Q}_{n+1} = 1$

From the above conditions we can conclude that for transition $Q_n = 1$ to $Q_{n+1} = 0$ occurs when $J = 0$ (or) 1 (don't care) and $K = 1$.

Similarly, input conditions can be found out for all possible situations.

Present State	Next State	SR Flip-flop		JK Flip-flop		T Flip-flop	D Flip-flop
Q_n	Q_{n+1}	S	R	J	K	T	D
0	0	0	×	0	×	0	0
0	1	1	0	1	×	1	1
1	0	0	1	×	1	1	0
1	1	×	0	×	0	0	1

These excitation tables are useful in the design of synchronous circuits.

State Diagrams of Flip-flops

State diagram is a directed graph with nodes connected with directed arcs. State of the circuit is represented by the node, the directed arcs represent the state transitions, from present state (node) to next state (node) at the occurrence of clock pulse.

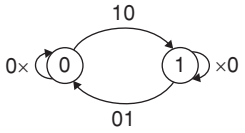


Figure 7 State diagram of SR flip-flop

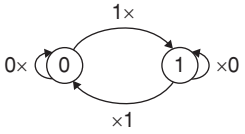


Figure 8 State diagram of JK flip-flop

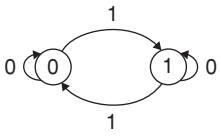


Figure 9 State diagram of T flip-flop

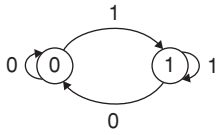


Figure 10 State diagram of D flip-flop

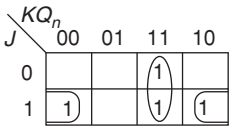
Conversion of One Flip-flop to Other Flip-flop

Conversion of T flip-flop to JK flip-flop

1. Write the characteristic table of required flip-flop (here JK).
2. Write the excitation table of available or given Flip-flop (here T).
3. Solve for inputs of given flip-flop in terms of required flip-flop inputs and output.

Table 2 *JK flip-flop characteristic and T flip-flop excitation table*

JK Flip-flop Characteristic Table		T Flip-flop Excitation Table		
J	K	Q_n	Q_{n+1}	T
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	1
1	0	1	1	0
1	1	0	1	1
1	1	1	0	1



$$T = J\overline{Q}_n + KQ_n$$

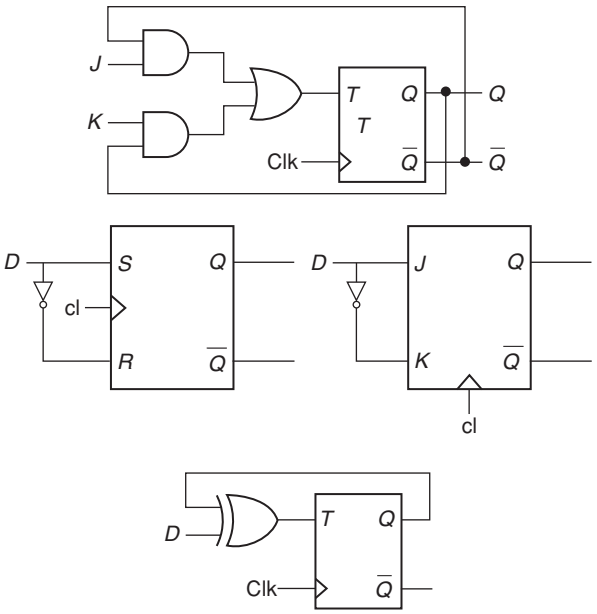


Figure 11 *D* Flip-flop by using other flip-flops

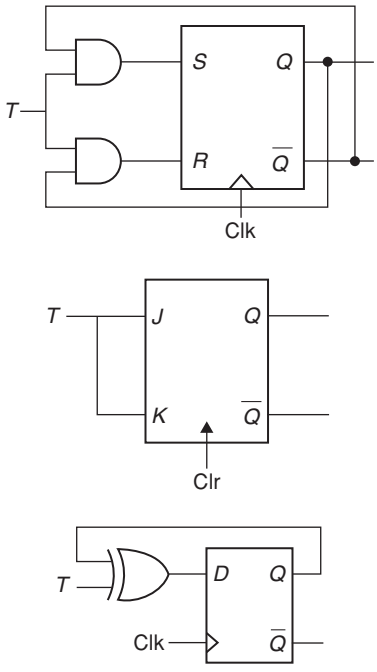
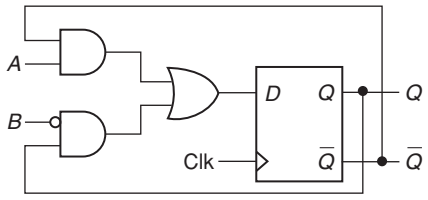


Figure 12 *T* flip-flop by using other flip-flops

Example 3: A sequential circuit using *D* flip-flop and logic gates is shown in the figure, where *A* and *B* are inputs and *Q* is output.



The circuit is

(A) *SR* flip-flop with inputs $A = S, B = R$

(B) *SR* flip-flop with inputs $A = \overline{R}, B = S$

(C) *JK* flip-flop with inputs $A = J, B = K$

(D) *JK* flip-flop with inputs $A = K, B = \overline{J}$

Solution: (C)

The characteristic equation of *D* flip-flop is $Q_{n+1} = D$

Here input $D = A\overline{Q}_n + \overline{B}Q_n$.

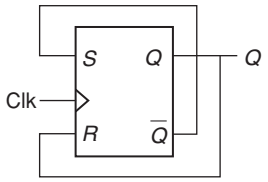
So, output $Q_{n+1} = A\overline{Q}_n + \overline{B}Q_n$.

By comparing this equation with characteristic equation of *JK*

$$Q_{n+1} = J\overline{Q}_n + \overline{K}Q_n$$

If $A = J, B = K$, then this circuit works like *JK* flip-flop.

Example 4: The input Clk frequency for the flip-flop given is 10 kHz, then the frequency of *Q* will be



(A) 10 kHz (B) 5 kHz

(C) 20 kHz (D) 2.5 kHz

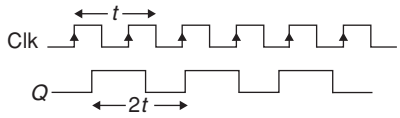
Solution: (B)

Form circuit we can say $S = \overline{Q}_n, R = Q_n$.

If initially $(Q_n, \overline{Q}_n) = (0, 1)$, then inputs $(S, R) = (1, 0)$, by applying clk pulse $(Q_{n+1}\overline{Q}_{n+1})$ becomes $(1, 0)\ldots$

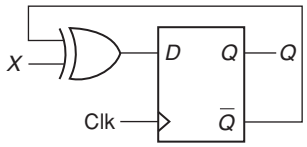
Clk	Q_n	$\overline{Q}_n$	S	R	Q_{n+1}	$\overline{Q}_{n+1}$
1	0	1	1	0	1	0
2	1	0	0	1	0	1
3	0	1	1	0	1	0
4	1	0	0	1	0	1

The output Q_{n+1} toggles for every clock pulse.



So frequency of $Q = \frac{1}{2t} = \frac{f}{2} = \frac{10}{2} = 5$ kHz

Examples 5: For the D flip-flop shown, if initially Q_n is set then what is the output state Q_{n+1} for $X = 0$, and for $X = 1$?



(A) 0, 0 (B) 0, 1
(C) 1, 0 (D) 1, 1

Solution: (B)
The characteristic equation of D is $Q_{n+1} = D$
Here $D = X \oplus \overline{Q_n}$
So $Q_{n+1} = X \oplus \overline{Q_n}$
We have $Q_n = 1$ (Q_n is set) for $X = 0$

$$Q_{n+1} = 0 \oplus 0 = 0$$

We have $Q_n = 1$ (Q_n is set), for $X = 1$

$$Q_{n+1} = 1 \oplus 0 = 1$$

Applications of Flip-flops

1. Data storage: A group of flip-flops (connected in series/parallel is called a register) to store a data of N-bits, N-flip-flops are required. Data can be stored in parallel or serial order. Similarly, serial to parallel conversion and parallel to serial conversion can be done by using registers.
2. Counting: A number of flip-flops can be connected in a particular fashion to count the pulses applied (Clk) electronically. One flip-flop can count two clk pulses, two flip-flops can count up to $2^2 = 4$ pulses, similarly n flip-flops can count up to 2^n pulses. Flip-flops may be used to count up/down.
3. Frequency division: Flip-flops may be used to divide input signal frequency by any number. A single flip-flop may be used to divide the input frequency by 2. Similarly n flip-flops may be used to divide the input frequency by 2^n. Output of a MOD-n counter (i.e., which counts n states) will divide input frequency by n.

Counters

Digital counters consist of a number of flip-flops. Their function is to count the number of clock pulses arriving at its clock input.

Counter Classification

Counters are classified according to their operational characteristic. Some of these characteristics include:

1. Counter triggering techniques
2. Frequency division characteristic
3. Counter modulus
4. Asynchronous or synchronous

In a synchronous counter all flip-flops are clocked simultaneously. In asynchronous counter the flip-flops are not clocked simultaneously. Each flip-flop is triggered by the previous flip-flop.

Asynchronous Counters (Ripple Counters)

Asynchronous counters do not have a common clock that controls all the flip-flop stages. The control clock is input to the first stage. The clock for each stage subsequent is obtained from the flip-flop of the prior stages. Let us analyze the 3-bit counter and its corresponding wave form diagram shown below.

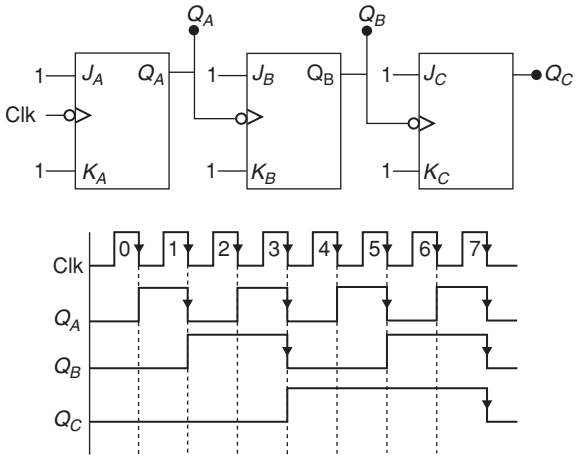


Figure 13 Timing diagrams

- The counter has three flip-flops and three output bits, therefore it is a three stage counter.
- The input clock does not trigger the three flip-flops; therefore, it is an asynchronous counter.
- The J and K inputs are tied together as kept high. So they are considered to be toggle flip-flops.
- The flip-flops are negative edge triggered.
- The wave form analysis reveals that Q_A is the LSB and that its frequency is $\frac{1}{2}$ the input clock frequency. Further more, Q_c is the MSB and its frequency is $\frac{1}{8}$ the input clock frequency.
- The count sequence is 000, 001, 010, 011, 100, 101, 110, 111 where the LSB is Q_A. Thus, it is MOD-8 binary up counter.
- Asynchronous counters are also known as ripple counters because the effect of the input clock ripples through the counter until it reaches the final stage.

Asynchronous counter design

Step 1: Write the counting sequence.
Step 2: Tabulate the values of reset signals. R for various state of counter.
Step 3: Obtain the minimal expression for R and $\overline{R}$ using K-map or any other method.
Step 4: Provide a feedback such that R or $\overline{R}$ resets all the flip-flops after the desired count.

Identification of up/down counters

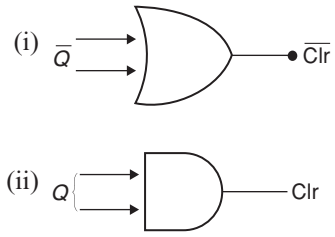
Clock Triggering	Q	Type
+ve edge	$\overline{Q}$	Up
+ve edge	Q	Down
–ve edge	$\overline{Q}$	Down
–ve edge	Q	Up

Clock is negative triggering pulse and Q is connected to next level clock, it is acting like a up counter.

Identification of gate to Clear the flip-flops

Input to the Gate	Output of the Gate	Type of Gate
$\overline{Q}$	$\overline{Clr}$	OR
$\overline{Q}$	Clr	NOR
Q	$\overline{Clr}$	NAND
Q	Clr	AND

Example:

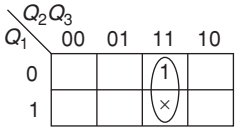


Example 6: Design and Implement a MOD-6 asynchronous counter using T flip-flops.

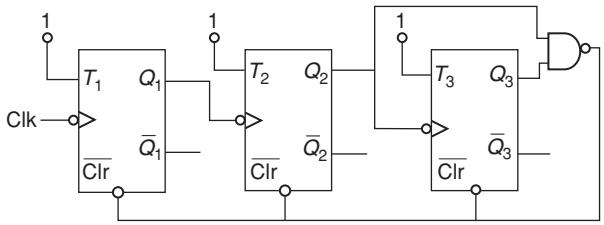
Solution: Counting sequence is 00, 001, 010, 011, 100, 101

After Pulses	States Q_3, Q_2, Q_1	Reset R
0	000	0
1	001	0
2	010	0
3	011	0
4	100	0
5	101	0
6	110	1
	↓↓↓	
7	000	0
	111	X

From the Truth table $R = Q_3 Q_2$
For active Low $\overline{R}$ is used.
$\therefore$ $R = 0$ for 000 to 101
$R = 1$ for 110
$R = X$ for 111
$\therefore$ K-map is



$\therefore$ $R = Q_2 Q_3$;
Logic diagram is



Asynchronous Decode Counter

A ripple counter is an asynchronous sequential circuit; clock is applying only for LSB side. Decade ripple counter it counts from 0 to 9 for up counter.

MOD-10 counter it counts starting from 0000 to 1001. If the NAND gate output is logic '0' at that instant the counter reset to initial state.

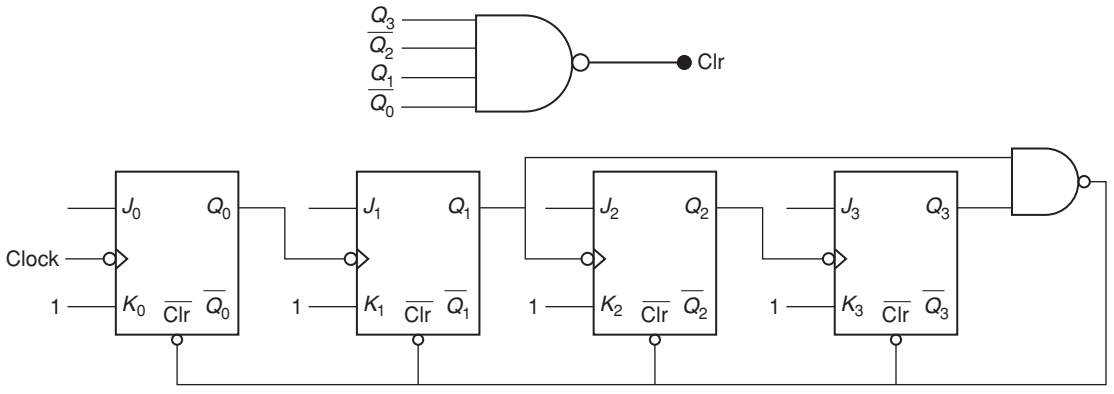


Figure 14 MOD-10 or decade counter

To design a MOD-N counter, minimum number of flip-flops required is

$N \le 2^n$

where, $N \rightarrow$ MOD

$n \rightarrow$ No. of flip-flops

Example:

MOD-5 counter

$5 \le 2^n$

$\therefore\ n = 3$

Operating, clock frequency

(i) Synchronous counter:

$$f_{clk} \le \frac{1}{t_{pd}}$$

(ii) Asynchronous counter:

$$f_{clk} \le \frac{1}{nt_{pd}}$$

Output frequency of the MOD-N counter is

$$\Rightarrow \quad f_o = \frac{f_{clk}}{N}.$$

Synchronous Counter

When counter is clocked such that each flip-flop in the counter is triggered at the same time, the counter is called as synchronous counter.

- Synchronous counters have the advantage of high speed and less severe decoding problems.
- Disadvantage is having more circuiting than that of asynchronous counter.

Synchronous Series Carry Counters

For normal ring counters to count N sequence total N flip-flops are required.

Unused states in ring counter = $2^N - N$.

Unused states in Johnson ring counter = $2^N - 2N$.

Asynchronous counters are slower than the synchronous counters. By using synchronous series carry adders, we can design MOD-N counter with n Flip-flops-only.

For non–binary counters $N \le 2^n$.

3-bit series carry up counter

It counts from initial state 000 to 111.

$\therefore$ MOD = 2^n = 8 states

$\therefore$ MOD-8

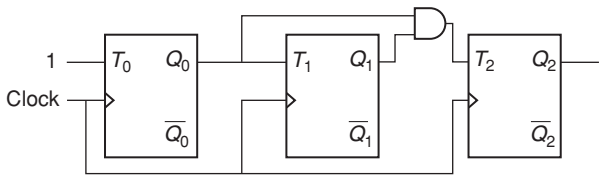


Figure 15 3-bit series carry counter

$$f_{clk} \le \frac{1}{t_{pd} + (n-2)t_{pdAND}}$$

Where,

$t_{pd} \rightarrow$ Propagation delay of each flip-flop.

$t_{pdAND} \rightarrow$ Propagation delay of AND gate.

$n \rightarrow$ No. of flip-flops.

In this, Q_o toggles for every clock pulse.

Q_1 toggles when Q_o is 1.

Q_2 toggles when o/p of AND gate is logic '1'.

Note: To design a synchronous series carry down counter, Connect $\overline{Q_o}$ to the next flip-flop input.

Design of Synchronous Counter

Step 1: Determine the required number of flip-flop.

Step 2: Draw the state diagram showing all possible states.

Step 3: Select the type of flip-flop to be used and write the excitation table.

Step 4: Obtain the minimal expressions for the excitations of the FFs using the *k*-maps.

Step 5: Draw a logic diagram based on the minimal expression.

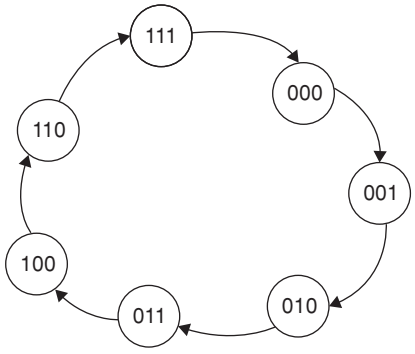
Let us employ these techniques to design a MOD-8 counter to count in the following.

Example 7: Sequence: 0, 1, 2, 3, 4, 5, 6, and 7, Design a synchronous counter by using *JK* flip-flops.

Solution:

Step 1: Determine the required number of flip-flops. The sequence shows a 3-bit up counter that requires 3 flip-flops.

Step 2: Draw the state diagram.

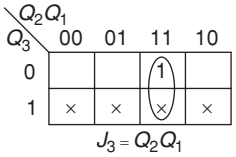


Step 3: Select the type of flip-flop to be used and write the excitation table.

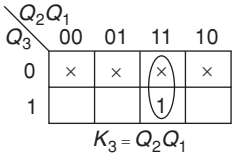
JK flip-flop is selected and excitation table of a 3-bit up counter is as follows:

PS	NS	Required Excitation
$Q_3\ Q_2\ Q_1$	$Q_3\ Q_2\ Q_1$	$J_3\ K_3\ J_2\ K_2\ J_1\ K_1$
0 0 0	0 0 1	0 x 0 x 1 x
0 0 1	0 1 0	0 x 1 x x 1
0 1 0	0 1 1	0 x x 0 1 x
0 1 1	1 0 0	1 x x 1 x 1
1 0 0	1 0 1	x 0 0 x 1 x
1 0 1	1 1 0	x 0 1 x x 1
1 1 0	1 1 1	x 0 x 0 1 x
1 1 1	0 0 0	x 1 x 1 x 1

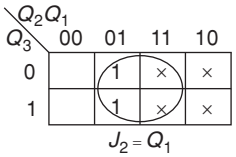
Step 4: Obtain the minimal expression using K-map.



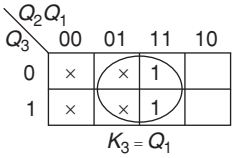
$J_3 = Q_2Q_1$



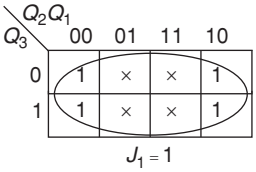
$K_3 = Q_2Q_1$



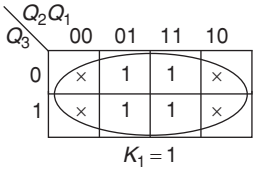
$J_2 = Q_1$



$K_3 = Q_1$



$J_1 = 1$



$K_1 = 1$

Step 5: Draw the logic diagram based on the minimal expression.

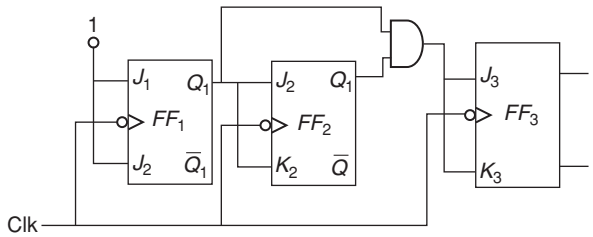


Table 3 *Comparison between asynchronous counter and synchronous counter*

Asynchronous Counter	Synchronous Counter
1. In this type of counter, flip-flops are connected in such a way that output of first flip-flop drives the clock for the next flip-flop.	In this type, there is no connection between output of first flip-flop and clock input of the next flip-flop.
2. All the flip-flops are not clocked simultaneously.	All the flip-flops are clocked simultaneously.
3. Logic circuit is very simple even for more number of states.	Design involves complex logic circuits as number of state increases.
4. Main draw back of these counters is their low speed as the clock is propagated through number of flip-flops before it reaches last flip-flop.	As clock is simultaneously given to all flip-flops, hence there is no problem of propagation delay. Hence they are preferred when number of flip-flops increases in the given design.

The main drawback of ripple counters is their high delays. If propagation delay of each flip-flop is assumed as x, then to get output of the first flip-flop it takes x, i.e., after x seconds the second flip-flop will get its clock pulse from previous stage, and output of second flip-flop will be out after another x seconds. Similarly, the final output of last flip-flop will be after nx seconds, where n is the number of flip-flops. So the propagation delay of ripple counter is nx, which is directly proportionate to the number of flip-flops.

The maximum frequency of operation of ripple counter is inverse of delay, $f_{max} = \frac{1}{nx}$.

Maximum operating frequency is the highest frequency at which a sequential circuit can be reliably triggered. If the clock frequency is above this maximum frequency, the flip-flops in the circuit cannot able to respond quickly and the operation will be unreliable.

In case of synchronous counters (synchronous circuits) as clock is applied simultaneously to all the flip-flops, the output of all the flip-flops change by x seconds (delay of one flip-flop) and this delay is independent of number of flip-flops used in circuit.

The maximum frequency of operation of synchronous counter is inverse of delay $f_{max} = \frac{1}{x}$.

Example 8: The maximum operation frequency of a MOD-64 ripple counter is 33.33 kHz, the same flip-flops are used to design a MOD-32 synchronous counter, and then the maximum operating frequency of the new counter is?

(A) 400 kHz (B) 200 kHz
(C) 40 kHz (D) 500 kHz

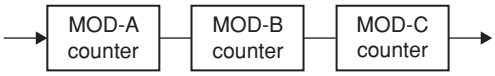
Solution: For ripple counter $f_{max} = \frac{1}{nx}$, given is a MOD-64 ripple counter, i.e., 2^6 states, so $n = 6$ flip-flops are required.

$$x = \frac{1}{33.33\ \text{K} \times 6} = 5\ \mu\text{S}$$

For synchronous counter

$$f_{max} = \frac{1}{x} = \frac{1}{5\ \mu\text{S}} = 0.2\ \text{MHz} = 200\ \text{kHz}.$$

When multiple counters are connected in cascade, then the total number of states of the new counter is $A \times B \times C$, i.e., it will work as MOD-A × B × C counter.



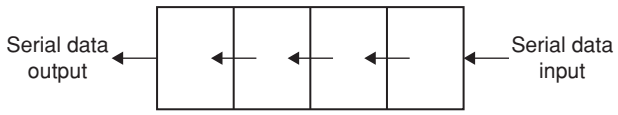
For example, decade counter counts from 0 to 9, 10 states – If two such decade counters are connected in cascade, then the total counting states will be 10 × 10 = 100, it will work as MOD-100 counter, which counts from 00 to 99.

Registers

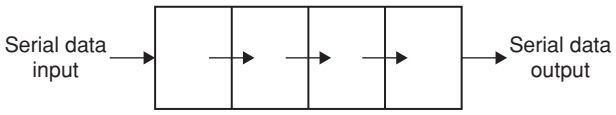
A number of flip-flops connected together such that data may be shifted into and shifted out of them is called a shift register. There are four basic type of shift register:

1. Serial in–Serial out
2. Serial in–Parallel out
3. Parallel in–Serial out
4. Parallel in–Serial out

Serial in–serial out

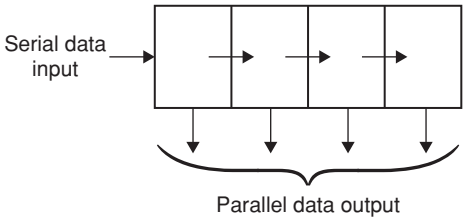


Serial –in, serial-out, shift-left, shift register

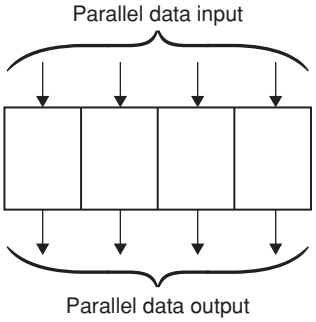


Serial in, serial out, shift right, shift register

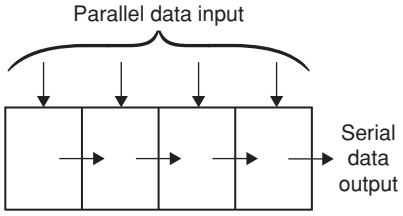
Serial in–parallel out



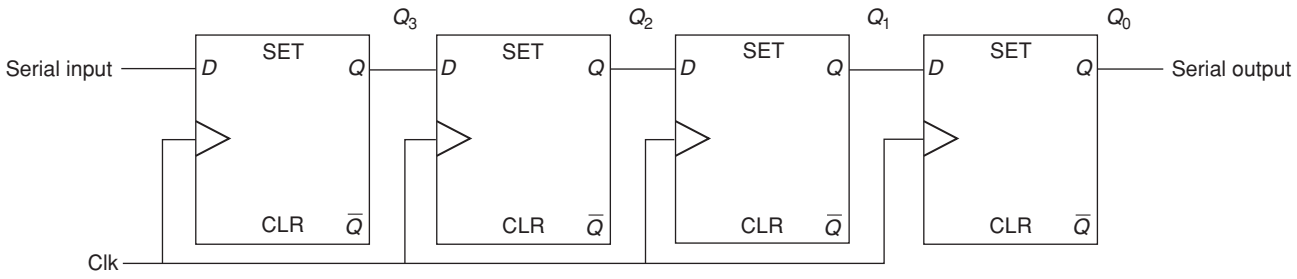
Parallel in–parallel out



Parallel in–serial out



Serial input and serial output register: This type of shift register accepts data serially, i.e., one bit at a time and also outputs data serially. The logic diagram of 4-bit serial input, serial output, shift-right, shift register is shown in the following figure. With four D flip-flops the register can store up to four bits of data.



If initially all flip-flops are reset, then by applying serial input 1101 the flip-flop states will change as shown in below table.

Clk	SI	Q_3	Q_2	Q_1	Q_0
0	1	0	0	0	0
1	0	1	0	0	0
2	1	0	1	0	0
3	1	1	0	1	0
4		1	1	0	1

The first data bit 1 will appear at serial output after 4 clock pulses.

Application of Shift Registers

1. Delay line: Serial input and serial output shift register can be used to introduce delay in digital signals.

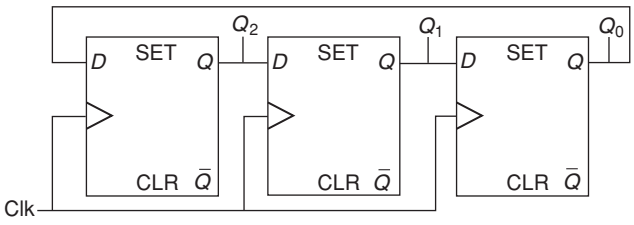
$$\text{Delay} = \text{no.of flip-flops} \times \frac{1}{\text{clk frequency}}$$

= No. of flip-flops × time period of clock pulse

2. Serial to parallel, parallel to serial converter: SIPO, PISO registers used for data conversion.
3. Sequence generator: A circuit which generates a prescribed sequence of bits, with clock pulses is called as sequence generator The minimum number of flip-flops 'n' required to generate a sequence of length 'S' bits is given by $S \le 2^n - 1$.

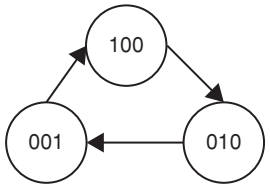
Shift register counters

One of the applications of the shift register is that they can be arranged to form as ring counters. Ring counters are constructed by modifying the serial-in, serial-out, shift registers. There are two types of ring counters: basic ring counter and twisted ring counter (Johnson counter). The basic ring counter is obtained from SISO shift register by connecting serial output to serial input.



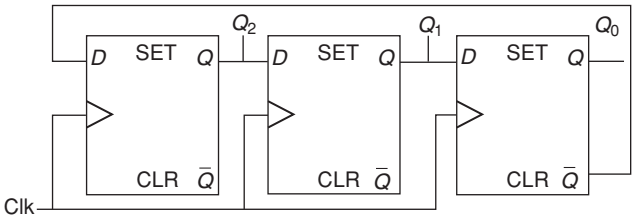
In most instances, only a single 1 or single 0 is in the register and is made to circulate around the register as long as the clock pulses are applied. Consider initially first flip-flop is set, and others are reset. After three clock pulses, again we will get initial state of 100. So this is a MOD-3 counter.

Clk	Q_2	Q_1	Q_0
0	1	0	0
1	0	1	0
2	0	0	1
3	1	0	0
4	0	1	0



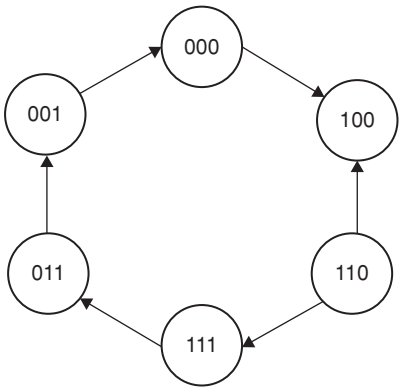
A ring counter with N flip-flops can count up to N states, i.e., MOD-N counter, whereas, N-bit asynchronous counter can count up to 2^N states. So ring counter is uneconomical compared to a ripple counter, but has the advantage of requiring no decoder. Since it is entirely synchronous operation and requires no gates for flip-flop inputs, it has further advantage of being very fast.

Twisted ring counter (Johnson counter): This counter is obtained from a SISO shift register by connecting the complement of serial output to serial input as shown in below figure:



Let initially all the FFs be reset, after each clock pulse the complement of last bit will appear as at MSB, and other bits shift right side by one bit. After six clock pulses, the register will come to initial state 000. Similarly, the 3-bit Johnson counter will oscillate between the states 101, 010.

Clk	Q_2	Q_1	Q_0
0	0	0	0
1	1	0	0
2	1	1	0
3	1	1	1
4	0	1	1
5	0	0	1
6	0	0	0



An n-bit Johnson counter can have $2n$ unique states and can count up to $2n$ pulses, so it is a MOD-$2n$ counter. It is more economical than basic ring counter but less economical than ripple counter.

Solved Examples

Example 1: Assume that 4-bit counter is holding the count 0101. What will be the count after 27 clock pulses?

Solution: Total clock pulses: 27 = 16 + 11
0101 + 1011 = 0000

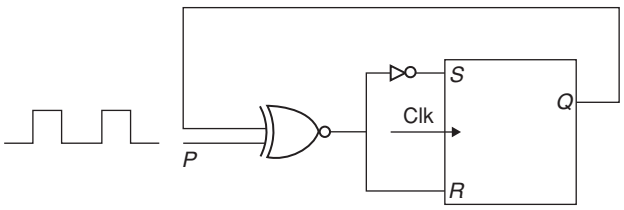
Example 2: A MOD-2 counter followed by MOD-5 counter is

Solution: A decade counter, counts 10 states (5 × 2).

Example 3: A 4-bit binary ripple counter uses flip-flops with propagation delay time of 25 m sec each. The maximum possible time required for change of state will be

Solution: The maximum time = 4 × 25 ms = 100 ms

Example 4: Consider the circuit, the next state Q^+ is



Solution

P	Q	S	R	Q^+
0	0	0	1	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	0

So, $Q^+ = P \oplus Q$

Example 5: A certain *JK FF* has t_{pd} = 12 n sec. What is the largest MOD counter, that can be constructed from these *FF* and still operate up to 10 MHz?

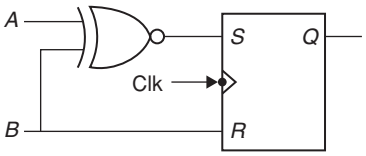
Solution: $N \le \dfrac{1}{f_{\max} \cdot t_{pd}}$

$F_{\max}$ = 10 MHz $N \le 8$
t_{pd} = 12 ns.

$$N \le \frac{1}{10 \times 10^6 \times 12 \times 10^{-9}}$$

MOD counter is = $2^N = 2^8 = 256$

Example 6: An *AB* flip-flop is constructed from an *SR* flip-flop as shown below. The expression for next state Q^+ is

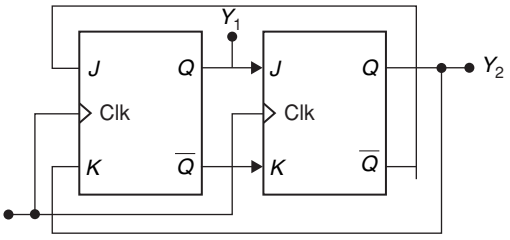


Solution:

A	B	Q	S	R	Q^+
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	0	0	0
1	0	1	0	0	1
1	1	0	1	1	×
1	1	1	1	1	×

$\therefore \quad Q^+ = \overline{A}\overline{B} + AQ = \overline{A}\overline{B} + \overline{B}Q$

Example 7: In the circuit shown below, the output y_1 and y_2 for the given initial condition $y_1 = y_2 = 1$ and after four input pulses will be



Solution:
After 1st pulse $y_1 = 0, y_2 = 1$
After 2nd pulse $y_1 = 0, y_2 = 0$
After 3rd pulse $y_1 = 1, y_2 = 0$
After 4th pulse $y_1 = 1, y_2 = 1$

Example 8: A ripple counter is to operate at a frequency of 10 MHz. If the propagation delay time of each flip-flop in the counter is 10 ns and the storbing time is 50 ns, how many maximum stages can the counter have?

Solution: $nt_{pd} + t_s \le \frac{1}{f}$

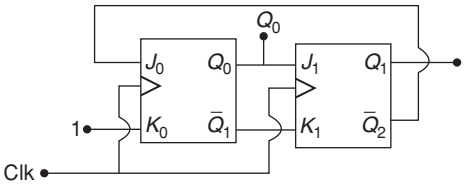
where, n = number of stages
T_{pd} = propagation delay time
T_s = strobing time
F = frequency of operation = $10 \times 10^{-9} n + 50 \times 10^{-9}$

$$\le \frac{1}{10 \times 10^6}$$

(or) $10n + 50 \le 100$
(or) $10n \le 50$

For max stages $n \frac{50}{10} = 5$

Example 9: In the circuit assuming initially $Q_0 = Q_1 = 0$. Then the states of Q_0 and Q_1 immediately after the 33rd pulse are

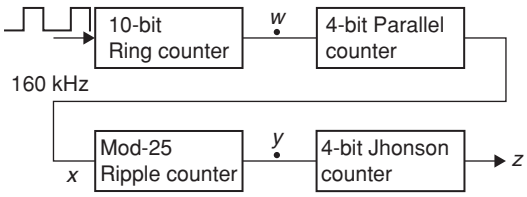


Solution:

J_0	K_0	J_1	K_1	Q_0	Q_1	Count
1	1	0	1	0	0	Initial
1	1	1	0	1	0	1st pulse
0	1	0	1	0	1	2nd
1	1	0	1	0	0	3rd
1	1	1	0	1	0	4th
0	1	0	1	0	1	5th pulse

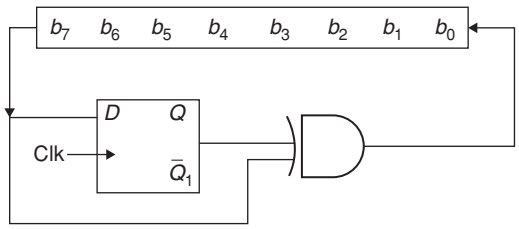
After 4th pulse, output is same as after 1st one, so, sequence gets repeated. So output after 33rd pulse would be same as after 3rd pulse. i.e., (00).

Example 10: The frequency of the pulse at z in the network shown in figure is



Solution: 10-bit ring counter is a MOD-10. So, it divides the 160 kHz input by 10. Therefore, w = 16 kHz. The 4-bit parallel counter is a MOD-16. Thus, the frequency at x = 1 kHz. The MOD-25 ripple counter produces a frequency at y = 40 Hz (1 kHz/25 = 40 Hz). The 4-bit Johnson counter is a MOD-8. The frequency at Z = 5Hz.

Example 11: The 8-bit shift left shift register, and D flip-flop shown in the figure is synchronized with the same clock. The D flip-flop is initially cleared. The circuit acts as

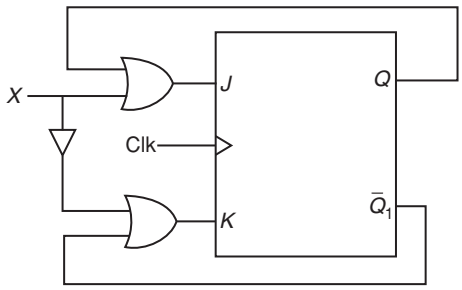


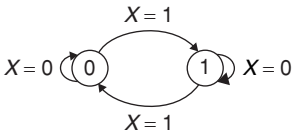
Solution: The output of X-OR gate is $Z = b_{i+1} \oplus b_i$ and this output shift the register to left. Initially, $Z = 0$.
After 2nd clock $Z = b_7 \oplus 0 = b_7$
After 2nd clock $Z = b_7 \oplus b_6$
3rd clock $Z = b_6 \oplus b_5$
4th clock $Z = b_5 \oplus b_4$
It is a binary to gray code converter.

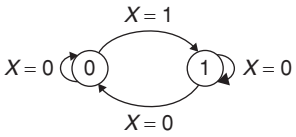
Example 12: A 4-bit MOD-16 ripple counter uses JK flip-flops. If the propagation delay of each flip-flop is 50 ns seconds, the maximum clock frequency that can be used is equal to

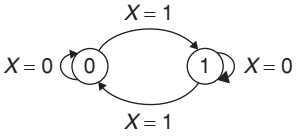
Solution: Max clock frequency $= \frac{1}{4 \times 50 \times 10^{-9}} = 5$ MHz

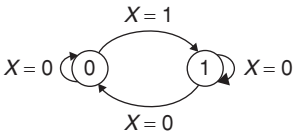
Example 13: What is the state diagram for the sequential circuit shown?



(A)


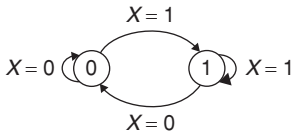
(B)


(C)


(D)


Solution: (D)
State diagram of a sequential circuit will have states (output) of all the flip-flops.

Present state	Next state	Q_{n+1}
Q_n	For $x = 0$	For $x = 1$
0	0	1
1	0	1



EXERCISES

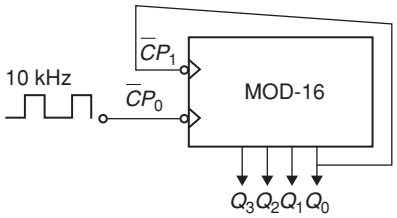
Practice Problems I

Directions for questions 1 to 22: Select the correct alternative from the given choices.

1. How many flip-flops are needed for MOD-16 ring counter and MOD-16 Johnson counter?
(A) 16, 16 (B) 16, 8
(C) 4, 3 (D) 4, 4

2. A 2-bit synchronous counter uses flip-flops with propagation delay time of 25 n sec, each. The maximum possible time required for change of state will be
(A) 25 n sec (B) 50 n sec
(C) 75 n sec D) 100 n sec

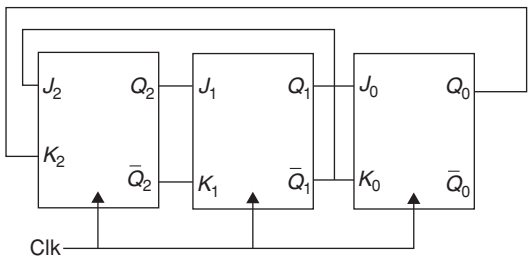
3. For given MOD-16 counter with a 10 kHz clock input, determine the frequency at Q_3



(A) 625 Hz (B) 10 kHz
(C) 2.5 kHz (D) 0 Hz

4. A 4-bit ripple counter and a 4-bit synchronous counter are made using flip-flops having a propagation delay of 10 n sec each. If the worst case delay in the ripple counter and the synchronous counter be R and S, respectively, then
(A) $R = 10$ ns, $S = 40$ ns (B) $R = 40$ ns, $S = 10$ ns
(C) $R = 10$ ns, $S = 30$ ns (D) $R = 30$ ns, $S = 10$ ns

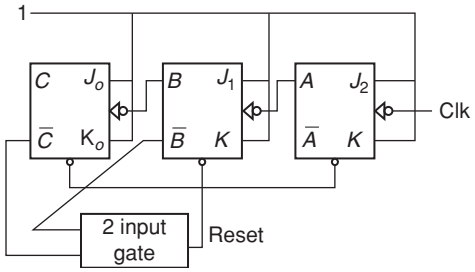
5. The counter shown in the figure has initially $Q_2Q_1Q_0 = 000$. The status of $Q_2Q_1Q_0$ after the first pulse is



(A) 001 (B) 010
(C) 100 (D) 101

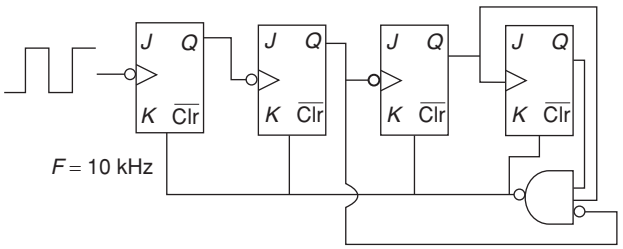
6. 12 MHz clock frequency is applied to a cascaded counter of MOD-3 counter, MOD-4 counter and MOD-5 counter. The lowest output frequency is?
(A) 200 kHz (B) 1 MHz
(C) 3 MHz (D) 4 MHz

7. In the modulo-6 ripple counter shown in the figure below, the output of the 2-input gate is used to clear the *JK* flip-flops. The 2-input gate is



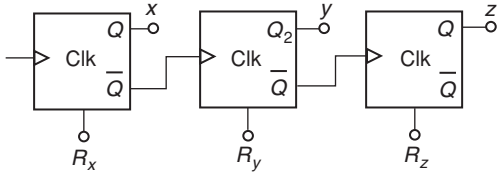
(A) a NAND gate (B) a NOR gate
(C) an OR gate (D) an AND gate

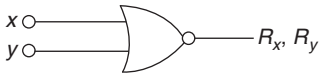
8. In figure, *J* and *K* inputs of all the 4 flip-flops are made high, the frequency of the signal at output *y* is

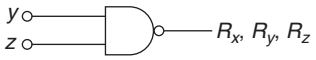


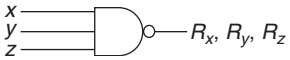
(A) 0.833 kHz (B) 1.0 kHz
(C) 0.91 kHz (D) 0.77 kHz

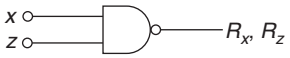
9. In a number system, a counter has to recycle to 0 at the sixth count. Which of the connections indicated below will realize this resetting? (a logic '0' at the *R* inputs resets the counters)



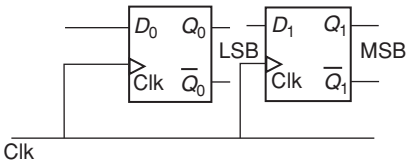
(A) 

(B) 

(C) 

(D) 

10. Two D flip-flops, as shown below, are to be connected as a synchronous counter that goes through the following Q_1Q_0 sequence $00 \rightarrow 01 \rightarrow 11 \rightarrow 10 \rightarrow 00$. The inputs D_0 and D_1 respectively should be connected as



(A) $\overline{Q}_1$ and Q_0 (B) $\overline{Q}_0$ and Q_1

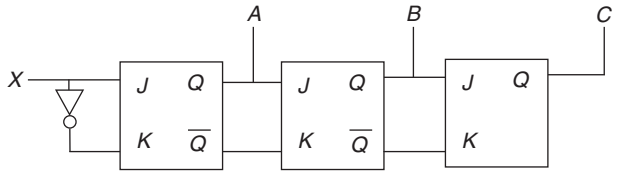
(C) $\overline{Q}_1\overline{Q}_0$ and $\overline{Q}_1Q_0$ (D) $\bar{Q}_1\bar{Q}_0$ and $\bar{Q}_1Q_0$

11. N flip-flops can be used to divide the input clock frequency by

(A) N (B) $2N$

(C) 2^N (D) 2^{N-1}

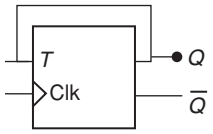
12. For a shift register as shown, $x = 1011$, with initially FF cleared, ABC will have value of after three clock pulses



(A) 101 (B) 011

(C) 001 (D) 111

13. If a FF is connected as shown what will be the output? (initially $Q = 0$)



(A) 11111... (B) 0000...

(C) 1010... (D) 0101...

14. The excitation table for a FF whose output conditions are, if $AB = 00$, no change of state occurs,
$AB = 01$, FF becomes 1 with next clock pulse,
$AB = 10$, FF becomes 0 with next clock pulse,
$AB = 11$, FF changes its state

(A)

Q_n	Q_{n+1}	A	B
0	0	0	x
0	1	1	x
1	0	x	1
1	1	x	0

(B)

Q_n	Q_{n+1}	A	B
0	0	1	x
0	1	0	x
1	0	x	0
1	1	x	1

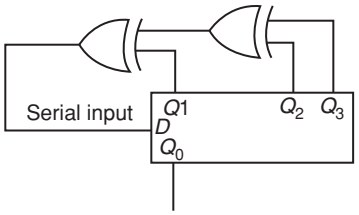
(C)

Q_n	Q_{n+1}	A	B
0	0	x	0
0	1	x	1
1	0	x	x
1	1	0	x

(D)

Q_n	Q_{n+1}	A	B
0	0	x	0
0	1	1	x
1	0	x	1
1	1	0	x

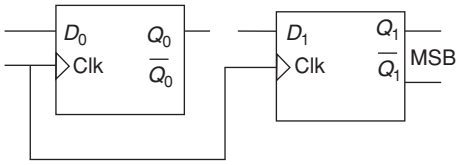
15. A shift register that shift the bits 1 position to the right at each clock pulse is initialized to 1100 (Q_0, Q_1, Q_2, Q_3). The outputs are combined using an XOR gate circuit and fed to the D input. After which clock pulse, will the initial pattern reappear at the output?



(A) 6th (B) 5th

(C) 4th (D) 7th

16. If we need to design a synchronous counter that goes through the states $00 \rightarrow 01 \rightarrow 11 \rightarrow 10 \rightarrow 00$ using D FF, what should be the input to the FF?



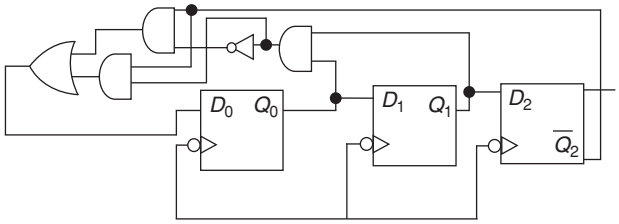
(A) $D_0 = Q_0$, $D_1 = \overline{Q}_1$

(B) $D_0 = \overline{Q}_1$, $D_1 = Q_0$

(C) $D_0 = \overline{Q}_1 \cdot Q_0$, $D_1 = \overline{Q}_1\overline{Q}_0$

(D) $D_0 = \overline{Q}_0$, $D_1 = Q_1$

17. Find the counter state sequence (Assume Q_0 as MSB).



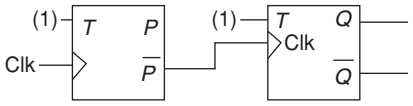
(A) 4, 6, 7, 3, 1, 0, 4 (B) 4, 6, 5, 3, 1, 0, 4

(C) 4, 5, 6, 7, 0, 4, 5 (D) 4, 6, 7, 1, 0, 4

18. If the propagation delay of each *FF* is 50 ns, and for the AND gate to be 20 ns. What will be the f_{max} for MOD-32 ripple and synchronous counters?

(A) 14.3 MHz, 4 MHz
(B) 14.3 MHz, 5 MHz
(C) 5 MHz, 14.3 MHz
(D) 3.7 MHz, 14.3 MHz

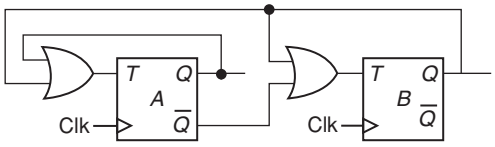
19. For a given counter, identify its behavior.



The output is taken from *PQ*.

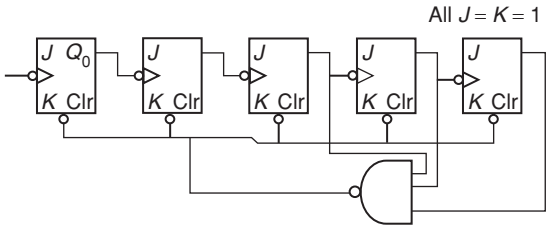
(A) MOD-4 up counter
(B) MOD-2 down counter
(C) MOD-4 down counter
(D) MOD-2 up counter

20. A circuit using *T FF* is given. Identify the circuit.



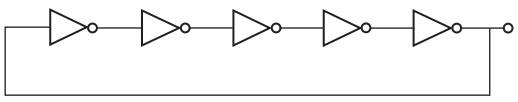
(A) MOD-2 counter
(B) MOD-4 counter
(C) MOD-3 counter
(D) MOD-2 generate 00, 10, 00

21. The MOD number of asynchronous counter shown as follow:



(A) 24 (B) 48
(C) 29 (D) 28

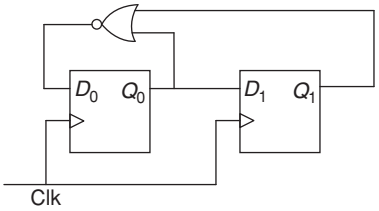
22. For the oscillator, find the fundamental frequency if propagation delay of each inverter is 1000 psec.



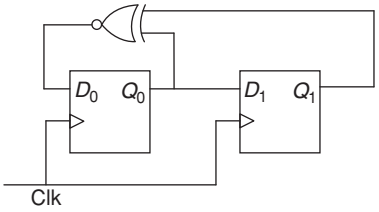
(A) 100 MHz (B) 10 MHz
(C) 1 GHz (D) 10 GHz

23. Which of the following represent MOD-3 synchronous up counter?

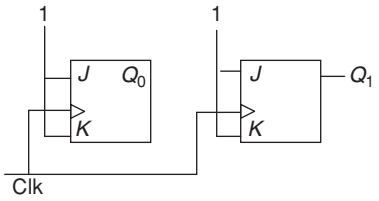
(P)



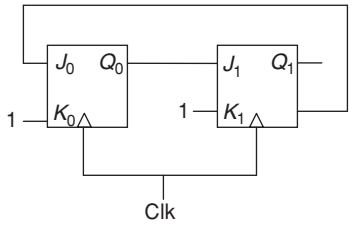
(Q)



(R)

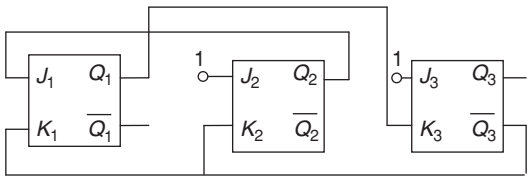


(S)



(A) *P*, *Q*, *R* and *S* (B) *P* and *S*
(C) *P*, *Q* and *S* (D) *P* and *Q*

24. Identify the output sequence (assume all outputs to be 1 initially), of synchronous counter?



(A) 7, 3, 4, 6 and 7 (B) 2, 4, 6, 7 and 2
(C) 7, 4, 7 and 3 (D) 7, 6, 7 and 3

25. If by using a modulo 1024 ripple counter we need to count a pulse train having a frequency of 1 MHz, what is the maximum permissible propagation delay of each flip-flop?

(A) 100 ns (B) 50 ns
(C) 20 ns (D) 10 ns

Practice Problems 2

Directions for questions 1 to 30: Select the correct alternative from the given choices.

1. Match List 1 (operation) with List 2 (associated device) and select the correct answer using the codes given below

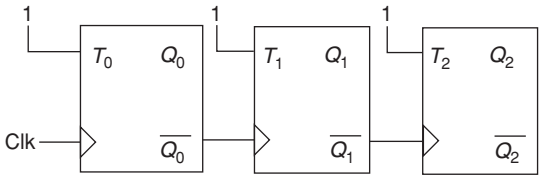
List 1	List 2
(a) Frequency Ddivision	(1) ROM
(b) Decoding	(2) Multiplexer
(c) Data selection	(3) Demultiplexer
(d) Code conversion	(4) Counter

(A) a – 3, b – 4, c – 2, d – 1
(B) a – 3, b – 4, c – 1, d – 2
(C) a – 4, b – 3, c – 1, d – 2
(D) a – 4, b – 3, c – 2, d – 1

2. A MOD-5 synchronous counter is designed by using *JK* flip-flop, the number of counts skipped by it will be
(A) 2 (B) 3
(C) 5 (D) 0

3. A counter starts off in the '0000' state, then clock pulses are applied. Some time later, the clock pulses are removed and the counter flip-flops read '0011'. How many clock pulses have occurred?
(A) 3 (B) 35
(C) 51 (D) Any of these

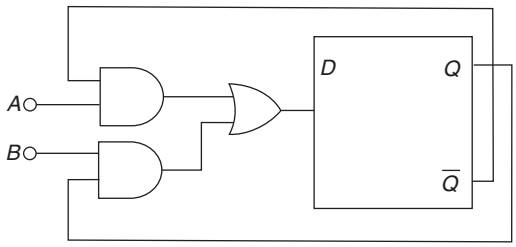
4. Figure below shown as ripple counter using positive edge triggered flip-flops. If the present state of the counters is $Q_2\,Q_1\,Q_0 = 011$, then its next state $(Q_2Q_1Q_0)$ will be



(A) 010 (B) 100
(C) 111 (D) 101

5. A synchronous sequential circuit is designed to detect a bit sequence 0101 (overlapping sequence include). Every time, this sequence is detected, the circuit produces output of '1'. What is the minimum number of states the circuit must have?
(A) 4 (B) 5
(C) 6 (D) 7

6. What is represented by digital circuit given below?



(A) An *SR* flip-flop with $A = S$ and $B = R$
(B) A *JK* flip-flop with $A = K$ and $B = J$
(C) A *JK* flip-flop with $A = J$ and $B = \overline{K}$
(D) An *SR* flip-flop with $A = R$ and $B = S$

7. In a ripple counter, the state whose output has a frequency equal to $\frac{1}{8}$th that of clock signal applied to the first stage, also has an output periodically equal to $\frac{1}{8}$th that of the output signal obtained form the last stage. The counter is
(A) MOD-8 (B) MOD-6
(C) MOD-64 (D) MOD-16

8. A flip-flop is popularly known as
(A) Astable multivibrator
(B) Bistable multivibrator
(C) Monostable multivibrator
(D) None of these

9. Which of the following represents the truth table for *JK* flip-flop?

(A)

J	*K*	Output
0	0	Q_0
0	1	0
1	0	1
1	1	$\overline{Q}_0$

(B)

J	*K*	Output
0	0	$\overline{Q}_0$
0	1	0
1	0	1
1	1	Q_0

(C)

J	*K*	Output
0	0	Q_0
0	1	0
1	0	1
1	1	Invalid

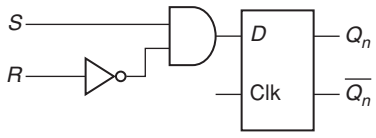
(D)

J	*K*	Output
0	0	1
0	1	0
1	0	1
1	1	0

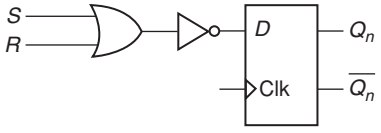
10. One disadvantage of master-slave *FF* is
(A) setup time becomes longer.
(B) it requires input to be held constant before clock transition.
(C) unpredictable output even if input held constant.
(D) hold time becomes longer.

11. Which of the following converts *D FF* to *SR FF*?

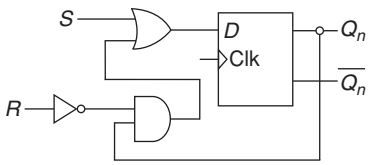
(A)



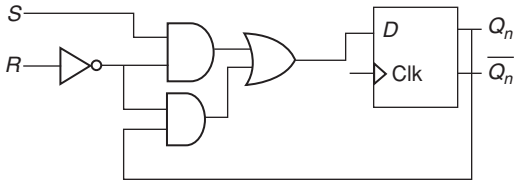
(B)



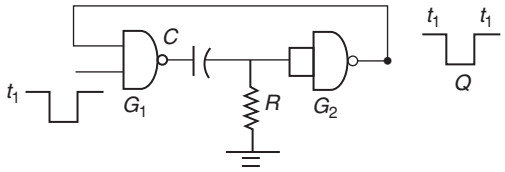
(C)



(D)



12. Which of the circuit is being represented by the figure?



(A) NAND gate
(B) Monostable multivibrator
(C) Astable multivibrator
(D) Schmitt trigger

13. Hold time is
(A) time for which output is held constant.
(B) time for which clock is to be held constant on applying input.
(C) time for which input should be maintained constant after the triggering edge of clock pulse.
(D) time for which input should be maintained constant prior to the arrival of triggering edge of clock pulse.

14. Shift registers are made up of
(A) MOS inverters (B) *FF*
(C) Latches (D) None of these

15. Data from a satellite is received in serial form. If the data is coming at 8 MHz rate, how long will it take to serially load a word in 40-bit shift register?
(A) 1.6 μs (B) 5 μs
(C) 6.4 μs (D) 12.8 μs

16. A *JK FF* can be converted into *T FF* by connecting
(A) $\overline{Q}$ to 0
(B) 0 to $\overline{Q}$
(C) 0 to Q
(D) by connecting both J and K inputs to T

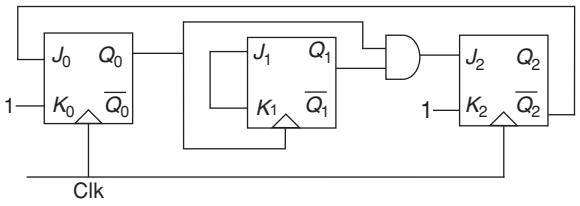
17. The flip-flop that is not affected by race around condition
(A) *T FF* (B) *JK FF*
(C) *SR FF* (D) None of these

18. The characteristic equation of *JK FF* is
(A) $J'Q(t) + KQ'(t)$ (B) $J'Q(t) + KQ(t)$
(C) $JQ'(t) + K'Q(t)$ (D) None of these

19. For a D.FF input, the $s\,\overline{Q}$ is connected. What would be the output sequence?
(A) 0000 (B) 1111
(C) 010101 (D) 101010

20. In order to implement a MOD-6 synchronous counter we have 3 *FF* and a combination of 2 input gate(*s*). Identify the combination circuit.
(A) One AND gate
(B) One OR gate
(C) One AND and one OR gate
(D) Two AND gates

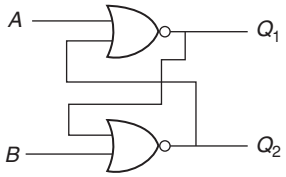
21. Given a MOD-5 counter. The valid states for the counter are (0, 1, 2, 3, 4). The propagation delay of each *FF* is T_F and that of AND gate is t_A. The maximum rate at which counter will operate satisfactorily.



(A) $\dfrac{1}{t_F + t_A}$ (B) $\dfrac{1}{3t_F}$

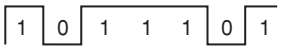
(C) $\dfrac{1}{2t_F + t_A}$ (D) $\dfrac{1}{3t_F - t_A}$

22. For a NOR latch as shown up A and B are made first (0, 1) and after a few seconds it is made (1, 1). The corresponding output (Q_1, Q_2) are



(A) first (1, 0) then (0, 0)
(B) first (1, 0) then (1, 0)
(C) first (1, 0) then (1, 1)
(D) first (1, 0) then (0, 1)

23. In order to design a pulse generator to generate the wave form using a shift register, what is the number of *FF* required?

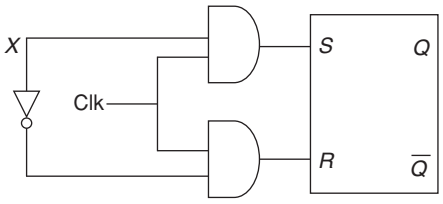


(A) 3 (B) 4
(C) 5 (D) 6

24. For what minimum value of propagation delay in each *FF* will a 10-bit ripple counter skip a count when it is clocked at 5 MHz?
(A) 10 ns (B) 20 ns
(C) 25 ns (D) 15 ns

25. A divide by 50 counter can be realized by using
(A) 5 no. of MOD-10 counter
(B) 10 no. of MOD-5 counter
(C) One MOD-5 counter followed by one MOD-10 counter
(D) 10 no. of MOD-10 counter

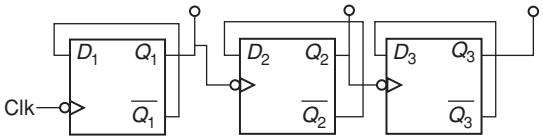
26. The following latch is



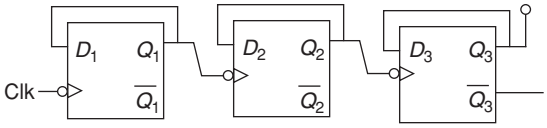
(A) *D* latch
(B) *T* latch
(C) *JK* latch
(D) *RS* latch

27. Which of the following represent a 3-bit ripple counter using *DFF*?

(A)

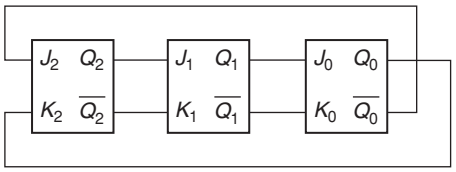


(B)



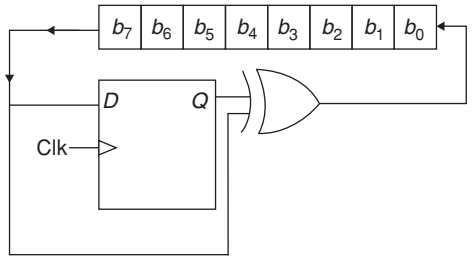
(C) Both (A) and (B)
(D) None of these

28. For the Johnson counter with initial Q_2, Q_1, Q_0 as 101, the frequency of the output is (Q_2, Q_1, Q_0)



(A) $\frac{f_c}{8}$
(B) $\frac{f_c}{6}$
(C) $\frac{f_c}{2}$
(D) $\frac{f_c}{4}$

29. For the given circuit the contents of register ($b_7 - b_0$) are 10010101, what will be the register contents after 8 clock pulses?



(A) 10010101
(B) 01101010
(C) 11011111
(D) 01101011

30. A latch is to be build with *A* and *B* as input. From the table find the expression for the next state Q^+.

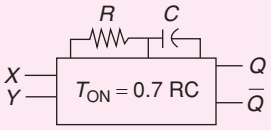
A	B	Q	Q^+
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(A) A
(B) B
(C) $A + \overline{B}$
(D) $A\overline{B} + AB$

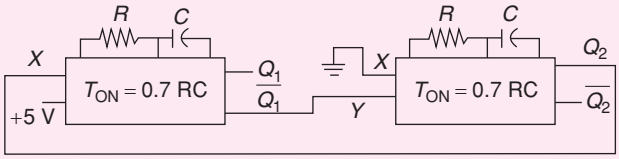
Previous Years' Questions

1. The truth table of a monoshot shown in the figure is given in the table below

X	Y	Q	$\overline{Q}$
0	↑	⊓	⊔
↓	1	⊓	⊔



Two monoshots, one positive edge triggered and other negative edge triggered, are connected as shown in the figure. The pulse widths of the two monoshot outputs, Q_1 and Q_2 are and, T_{ON_1} respectively.



The frequency and the duty cycle of the signal at Q_1 will respectively be **[2008]**

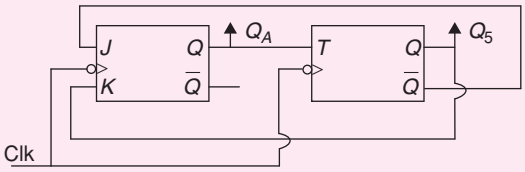
(A) $f = \dfrac{1}{T_{ON_1} + T_{ON_2}}, D = \dfrac{1}{5}$

(B) $f = \dfrac{1}{T_{ON_1} + T_{ON_2}}, D = \dfrac{T_{ON_2}}{T_{ON_1} + T_{ON_2}}$

(C) $f = \dfrac{1}{T_{ON_1}}, D = \dfrac{T_{ON_1}}{T_{ON_1} + T_{ON_2}}$

(D) $f = \dfrac{1}{T_{ON_2}}, D = \dfrac{T_{ON_1}}{T_{ON_1} + T_{ON_2}}$

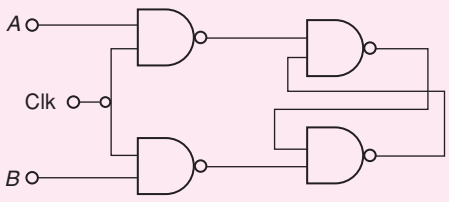
2. A two-bit counter circuit is shown below. **[2011]**



If the state Q_AQ_B of the counter at the clock time t_n is '10', then the state Q_AQ_B of the counter $t_n + 3$ (after three clock cycles) will be

(A) 00 (B) 01
(C) 10 (D) 11

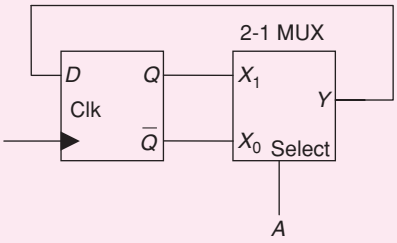
3. Consider the given circuit. **[2012]**

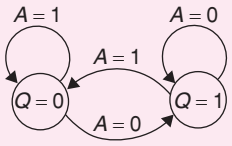


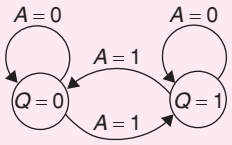
In this circuit, the race around.

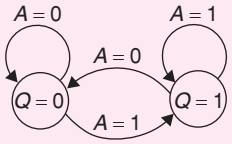
(A) does not occur
(B) occurs when Clk = 0
(C) occurs when Clk = 1 and $A = B = 1$
(D) occurs when Clk = 1 and $A = B = 0$

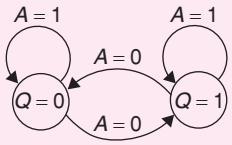
4. The state transition diagram for the logic circuit shown is **[2012]**



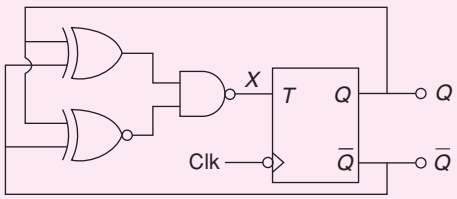
(A)


(B)


(C)


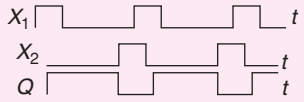
(D)


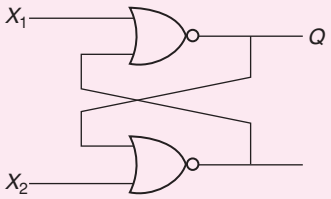
5. The clock frequency applied to the digital circuit shown in the figure below is 1 kHz. If the initial state of the output Q of the flip-flop is '0', then the frequency of the output waveform Q in kHz is **[2013]**

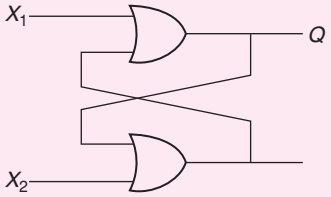


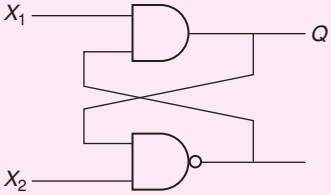
(A) 0.25 (B) 0.5
(C) 1 (D) 2

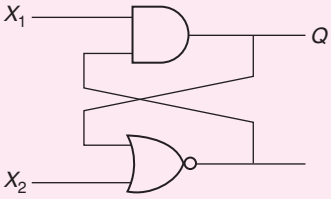
6. Select the circuit, which will produce the given output Q for the input signals X_1 and X_2 as given in the figure. **[2005]**



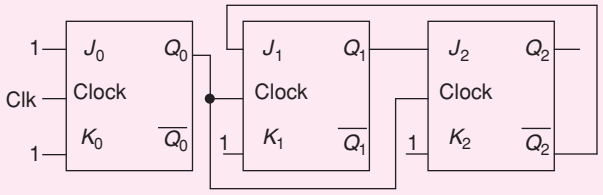
(A)


(B)


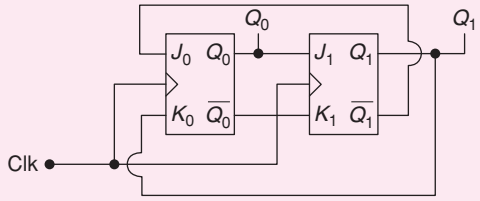
(C)


(D)


7. The figure shows a digital circuit constructed using negative edge triggered J-K flip flops. Assume a starting state of $Q_2Q_1Q_0 = 000$. This state $Q_2Q_1Q_0 = 000$ will repeat after _____ number of cycles of the clock CLK. **[2015]**

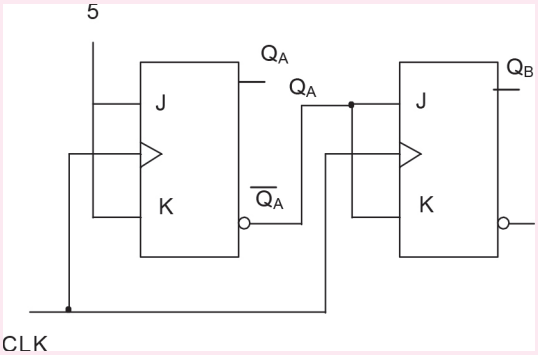


8. In the following sequential circuit, the initial state (before the first clock pulse) of the circuit is $Q_1\ Q_0 = 00$. The state (Q_1Q_0), immediately after the 333rd clock pulse is **[2015]**



(A) 00 (B) 01
(C) 10 (D) 11

9. The current state $Q_A\ Q_B$ of a two JK flip-flop system is 00. Assume that the clock rise-time is much smaller than the delay of the JK flip-flop. The next state of the system is_____ **[2016]**



(A) 00 (B) 01
(C) 11 (D) 10

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** A	**4.** B	**5.** C	**6.** A	**7.** C	**8.** A	**9.** B	**10.** A
11. C	**12.** B	**13.** B	**14.** C	**15.** D	**16.** B	**17.** A	**18.** D	**19.** A	**20.** C
21. D	**22.** A	**23.** B	**24.** A	**25.** A					

Practice Problems 2

1. D	**2.** B	**3.** D	**4.** B	**5.** A	**6.** C	**7.** C	**8.** B	**9.** A	**10.** B
11. C	**12.** B	**13.** C	**14.** B	**15.** B	**16.** D	**17.** C	**18.** C	**19.** C	**20.** D
21. C	**22.** A	**23.** D	**24.** B	**25.** C	**26.** A	**27.** A	**28.** C	**29.** C	**30.** A

Previous Years' Questions

1. B	**2.** C	**3.** A	**4.** D	**5.** B	**6.** A	**7.** 6	**8.** B	**9.** C

Chapter 11

Sample and Hold, A/D and D/A Circuits

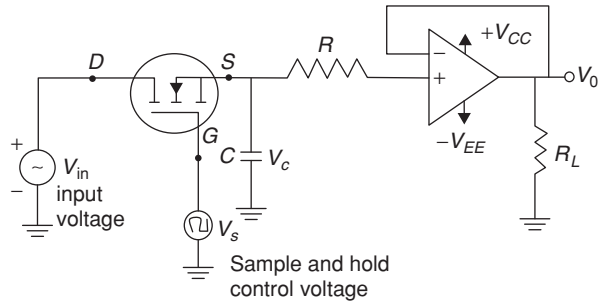
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Sample and hold circuit
- Digital to analog converter
- Weighted resistor DAC
- R – 2R ladder DAC
- Analog to digital converters
- Parallel comparator (Flash) A/D converter
- The counter type A/D converter
- Servo tracking A/D converter
- Successive approximation converter
- Charge balancing ADC
- Dual scope A/D converter

Sample and Hold Circuit

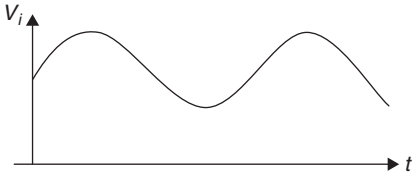
The sample and hold circuit as its name implies, samples an input signal and holds on to its last sampled value until the input is sample again.



The analog signal V_{in} to be sampled is applied to the drain, and sample and hold control voltage (V_s) is applied to the gate of E-MOSFET.

During positive portion of V_s, the MOSFET conducts and acts as a closed switch. This allows the input voltage to charge Capacitor C, i.e., input voltage appears across C and in turn at the output.

When V_s is zero, the E-MOSFET is off (non-conductive) and acts as an open switch, the only discharge path for C is through OP amp. However the input resistance of OP Amp is very high, So voltage across C is retained.



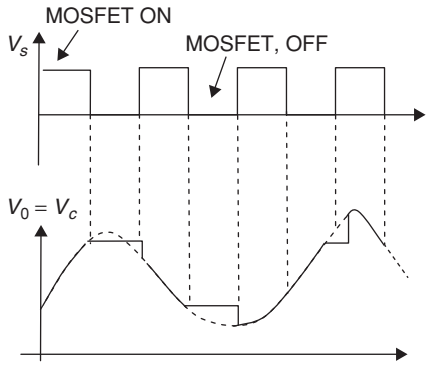
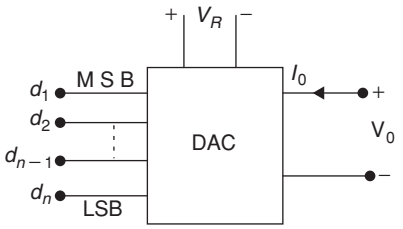


Figure 1 Input and output waveforms

To obtain close approximation of the input wave form, the frequency of the sample and hold control voltage must be significantly higher than that of the input. The sample and hold circuit is commonly used in digital interfacing and communications.

Digital to Analog Converter

The input is an n bit Binary word D and is combined with a reference voltage V_R to give an analog output signal.



For a voltage output (DAC) the D/A converter is mathematically described as

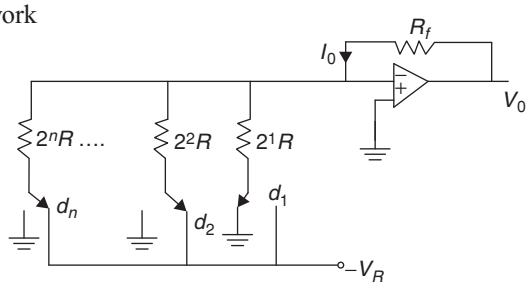
$$V_0 = K.V_{FS}\,(d_1\,2^{-1} + d_2\,2^{-2} + \cdots + d_n\,2^{-n})$$

V_0 – output voltage, V_{FS} – Full scale output voltage and K – Scaling factor,

Binary word $D = d_1\,d_2 \ldots d_n$

Weighted Resistor DAC

It uses a summing amplifier with a binary weighted resistor network

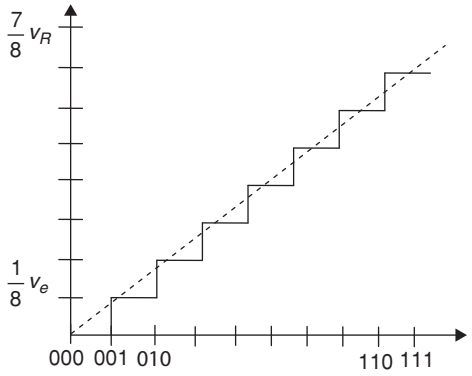


$$V_0 = I_0\,R_f$$

$$= V_R\,\frac{R_f}{R}(d_1 2^{-1} + d_2 2^{-2} + \cdots + d_n 2^{-n})$$

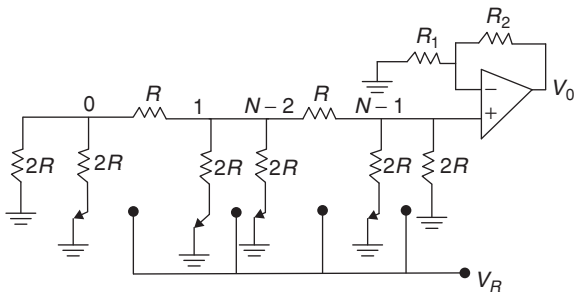
The OP Amp can be connected either in inverting or non inverting mode,

The OP Amp is simply working as a current to voltage converter.



This DAC uses wide range of resistances $2R$–2^nR, so it is not suitable for integrated circuits

R–2R Ladder DAC



This circuit utilizes twice the number of resistors as in binary weighted DAC, for the same number of bits (N) but of values R and $2R$ only.

The ladder used in this circuit is a current splitting device, at any of the ladder nodes the resistance is $2R$ looking, to the left or the right or towards the switch.

Example 1: The basic step of a 8-bit DAC is 12 mV. If 00000000 represents 0 V. What is the output for input 10110101?

Solution: Output is = 12 mV $(1 \times 2^7 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^2 + 1 \times 2^0) = 2.172$ V

Example 2: The values of LSB, MSB, and full scale output for an 8-bit DAC for 0 to 10 V range are?

Solution: $\text{LSB} = \frac{1}{2^8} = \frac{1\text{th}}{256}$ of full scale voltage

$$\text{LSB} = \frac{10}{256} = 39.\text{ mV}$$

$$\text{MSB} = \left(\frac{1}{2}\right) \text{full scale voltage} = 5\text{ V}$$

Full scale output = Full scale reading = (full scale voltage – 1 LSB)

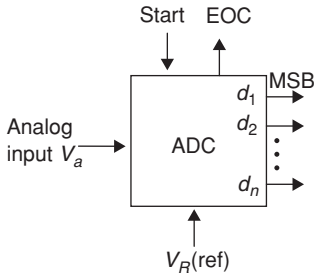
= 10 V – 0.039 V = 9.961 V

1. Resolution $= \frac{\text{FSV}}{2^n} = \frac{\text{FSR}}{2^n - 1}$
2. $\text{MSB} = \frac{\text{FSV}}{2}$

Analog to Digital Converters

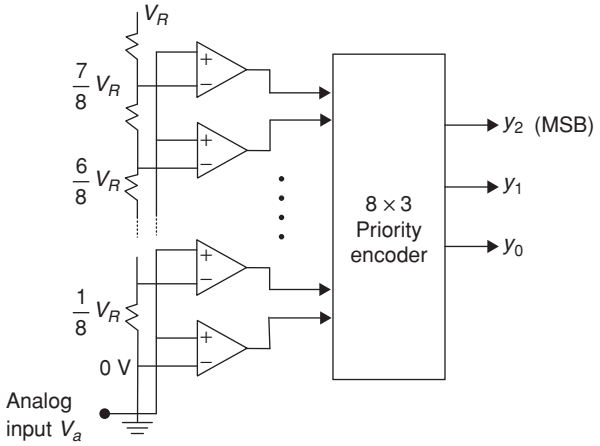
The block schematic of ADC is shown in the figure, it provides the function just opposite to that of a DAC. It accepts an analog input voltage V_a and produces an output binary word $d_1\,d_2 \ldots d_n$. of functional value D

So that



$$D = d_1\,2^{-1} + d_2\,2^{-2} + \cdots + d_n\,2^{-n}$$

Parallel Comparator (Flash) A/ D converter



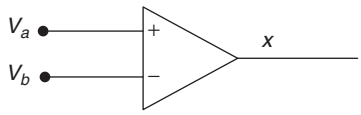


Figure 2 Basic circuit of a flash type A/D converter.

Table 1 *Comparator and its truth table*

Voltage Input	Logic Output
$V_a > V_b$	$X = 1$
$V_a < V_b$	$X = 0$
$V_a = V_b$	Previous value

This is the simplest possible A/D, converter, at the same time, the fastest and most expensive technique. At each node of the resistor divider, a comparison voltage is available, the purpose of this circuit is to compare the analog, input voltage V_a with each of the node voltages. In general the number of comparators required is $2^n - 1$ for n-bit ADC. Hence number of comparators doubles for each added bit.

The Counter Type A/D Converter

The clear pulse resets the counter to zero count, the counter then records in binary form the number of pulses form the clock line.

The principle is to adjust the DAC's input code until the DAC's output comes within $\pm\frac{1}{2}$LSB to the analog input V_a which is to converted to binary digital from the counter frequency must be low enough to give sufficient time for DAC to settle and for the comparator to respond, low speed is the major draw back in this method.

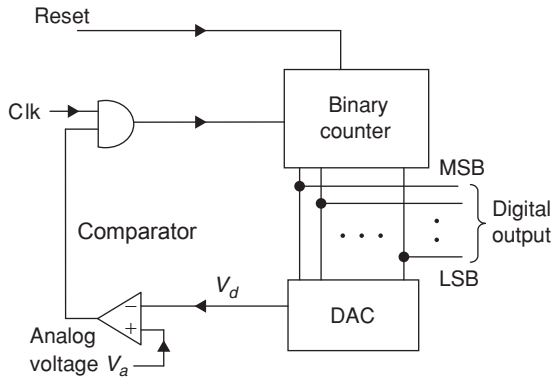


Figure 3 Counter type A/D converter

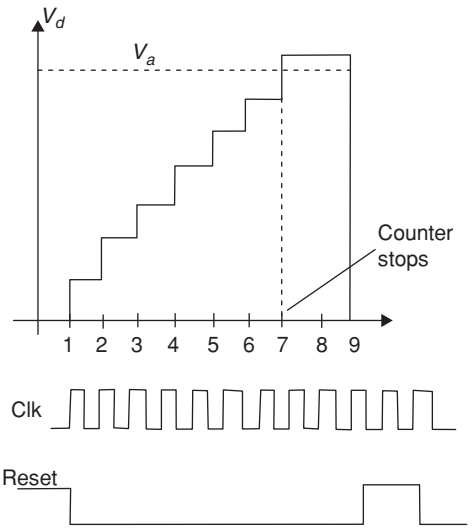


Figure 4 D/A output staircase waveform

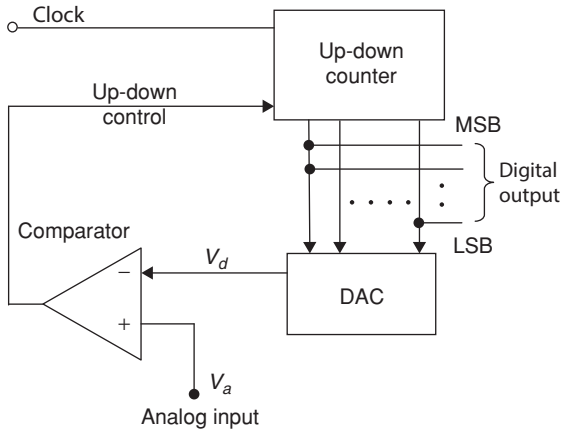
The conversion time can be as long as $(2^n - 1)$ clock periods depending upon the magnitude of the input V_a. For example, 12-bit system with 1MHz clock frequency, the counter will take $(2^{12} - 1)$ μs = 4.095 ms to convert a full scale input.

If the input signal is a sampled signal, the minimum interval between samples must be nT seconds for T – clock period, and maximum value of input voltage is represented by n-pulses.

Servo Tracking A/D Converter

An improved version of counting ADC is the tracking or a servo converter shown in the figure.

The circuit consists of an up/down counter with the comparator controlling the direction of the count



The analog output of the DAC is V_d and is compared with the analog input V_a.

If input $V_a > V_d$, the output of the comparator goes high and the counter is caused to count up.

The DAC output increase with each incoming clock pulse and when it becomes more than V_a, the counter reverses the direction and counts down.

The process goes on being repeated and digital output changes back and forth, by ±1 LSB around the correct value. The disadvantage is, however, the time needed to stabilize as a new conversion value is directly proportional to the rate at which the analog signal changes.

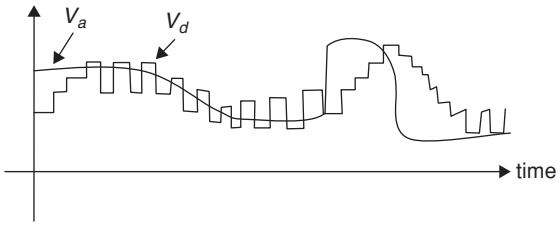
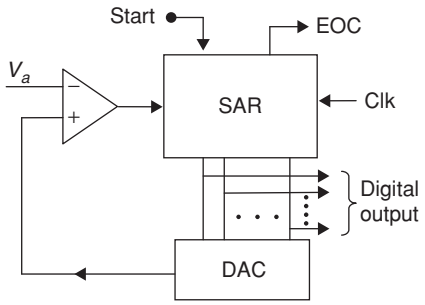


Figure 5 Wave forms associated with tracking ADC.

Successive Approximation Converter

The successive approximation technique uses a very efficient code search strategy to complete *n*-bit conversion in just n clock periods.

This circuit uses a successive approximation register (SAR) to find the required value of each bit by trial and error



Functional diagram of successive approximation ADC

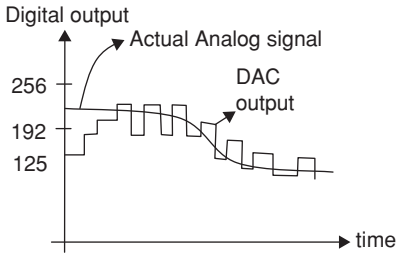


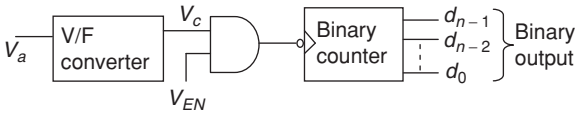
Figure 6 The D/A output voltage and analog input voltage

It can be seen that the D/A output voltage becomes successively closer to the actual analog input voltage. In general the successive approximation ADC technique is more versatile and superior to all other circuits

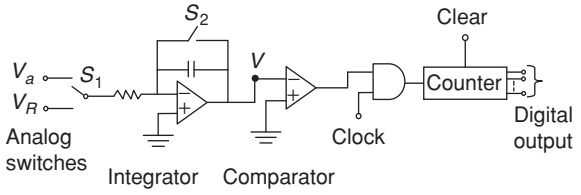
For all the ADCs which use DAC as one of their components, the major drawback is, the resolution of ADC will depend upon resolution of DAC.

Charge Balancing ADC

The principle of charge balancing ADC is to first convert the input signal to a frequency using a voltage to frequency (V/F) converter. This frequency is then measured by a counter and converted to an output code proportional to the analog input.

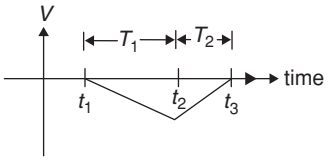


Dual Scope A/D Converter



$$V_a = V_R \cdot \frac{T_2}{T_1} = \frac{n_2 |V_R|}{n_1}$$

$$= n_2 \cdot \frac{|V_R|}{2^N}$$



This is widely used system with $V_a > 0$, $V_R < 0$, Initially counter cleared and at $t = t_1$, S_1 connects V_a and sampled for $n_1 = 2^N$ clock pulses. At the time t_2(at the end of the integration of v_a), all flip flops in the counter read 0.

Now the reference Voltage V_R is automatically connected to the input reference voltage V_R is automatically connected to the input of the integrator, at $t = t_2$, we have assumed that $|V_R| > V_a$, so that the integration time T_2 is less than T_1, as indicated. As long as V is negative the output of the comparator is positive and the AND gate allows clock pulses to be counted. When V falls to zero, at $t = t_3$, the AND gate is inhibited and no further clock pulses enter the counter.

$$V_a = n_2 \cdot \frac{|V_R|}{2^N},$$

V_a is proportional to counter reading n_2.

This technique can be very accurate; six digit digital voltmeters employ such signal processing, the dual scope system is inherently noise–immune because of input signal integration.

Conversion time = 2^{N+1}. Clk cycles

Resolution: the resolution of a converter is the smallest change in voltage which may be produced at the output or input of converter.

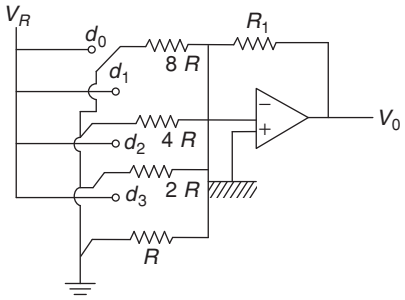
$$\text{Resolution} = \frac{V_{FS}}{2^N - 1} = 1 \text{ LSB} \quad \text{LSB increment}$$

EXERCISES

Practice Problems I

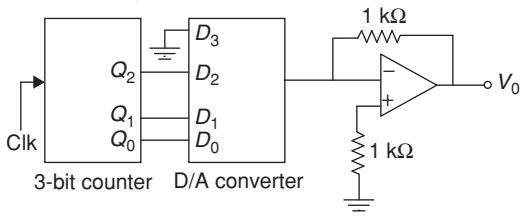
Directions for questions 1 to 10: Select the correct alternative from the given choices.

1. A 4-bit weighted DAC has $V_R = 2$ V and $\frac{R_1}{R} = 2$. For an input of 1000 the output will be

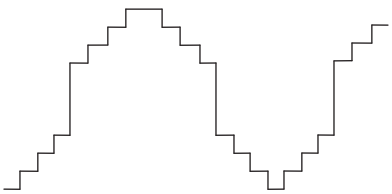


(A) −2 V (B) −4 V
(C) −8 V (D) 8 V

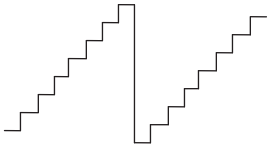
2. A 4-bit D/A converter is connected to free running 3-bit up-counter as shown. What will be the wave form obtained at v_0?



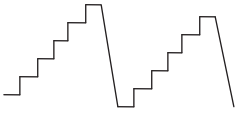
(A)



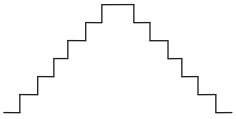
(B)



(C)



(D)



3. A 10-bit A/D converter is used to digitize an analog signal in the range 0 to 10 V. The maximum peak to peak ripple voltage that can be allowed in the dc supply voltage is____

(A) nearly 100 mV (B) nearly 10 mV
(C) nearly 25 mV (D) nearly 5 mV

4. For a dual ADC type $3\frac{1}{2}$ DVM, reference voltage is 200 mV and first integration time is 400 mS. For same input voltage de-integration is 470.2 mS. The DVM will indicate

(A) 200 mV (B) 235.1 mV
(C) 199.9 mV (D) 1.818 mV

5. In the case of a dual slope integrating type A/D converter if the output of 10-bit counter is clocked at 1 MHz, maximum frequency of analog signal that can be converted using A/D converter is

(A) 3 kHz (B) 2 kHz
(C) 20 kHz (D) 1 kHz

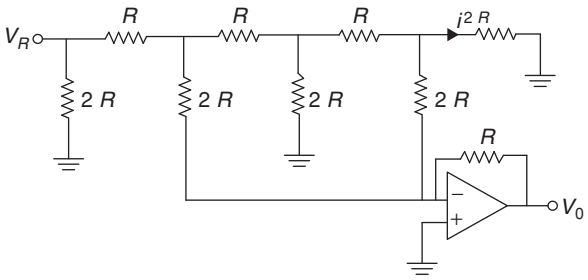
6. An 8-bit digital ramp ADC with a 50-mV resolution uses a clock frequency of 2.5 MHz and a comparator with $V_T = 1$ mV. The digital output for $V_A = 6$ V is

(A) 01111000 (B) 10010110
(C) 01011110 (D) 01011111

7. A sample and hold circuit having a holding capacitor of 0.1 nF is used at the input of a ADC. The conversion time is 1 μs and during this time the capacitor should not lose more than 0.1% of charge put across it during sampling time. The maximum value of the input signal to the *S/H* circuit is 5 V. The leakage current of *S/H* circuit is less than

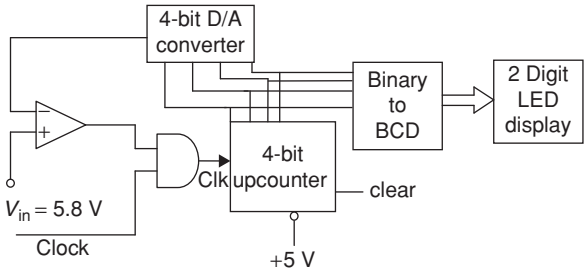
(A) 2.5 mA (B) 0.25 mA
(C) 5 μA (D) 0.5 μA

8. In the D/A converter, $V_R = 10$ V, R = 10 kΩ. Voltage V_0 is



(A) − 0.781 V (B) − 1.562 V
(C) − 3.125 V (D) − 6.250 V

Common Data for Questions 9 and 10: In the circuit comparator output is logic '1', if $V_1 > V_2$, and is logic '0' otherwise. The D/A conversion is done as per the relation. $V_{DAC} = \sum_{n=0}^{3} 2^{n-1} b_n$ V where b_3 (MSB), b_2, b_1 and b_0 are counter outputs. The counter starts from clear state.



9. The stable state of LED displays is
(A) 06 (B) 07
(C) 10 (D) 13

10. The magnitude of error between V_{DAC} and V_{in} at steady state in V.
(A) 0.2 V (B) 0.7 V
(C) 0.5 V (D) 0.1 V

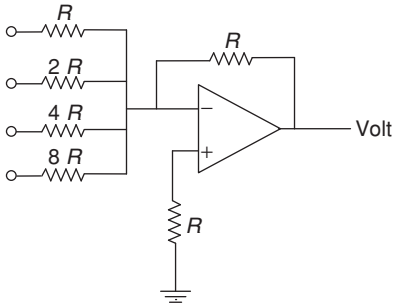
Practice Problems 2

Directions for questions 1 to 10: Select the correct alternative from the given choices.

1. The minimum number of comparators required to build n-bit flash ADC is
(A) 2^{n-1} (B) $2n$
(C) 2^n (D) $2^n - 1$

2. If the resolution of D/A converter is approximately 5% of its full scale range then it is an
(A) 8-bit converter (B) 10-bit converter
(C) 12-bit converter (D) 16-bit converter

3. The circuit shown is a 4-bit DAC. The inputs 0 correspond to 0 V and 1 to 5 V. The OP-AMP is ideal but all the resistances and 5 V input has a tolerance of ±10%. The specification of tolerance for the ADC is



(A) ± 35% (B) ± 20%
(C) ± 10% (D) ± 5%

4. A digital to analog converter with a full scale output voltage of 3.5 V has a resolution close to 20 mV. Its bit size is
(A) 4 (B) 8
(C) 16 (D) 32

5. An 8-bit DAC has a full scale output voltage of 15 V. The output voltage when the input is 10111010 is ____.
(A) 10.89 m V (B) 12 m V
(C) 12 V (D) 10.89 V

6. A 12-bit ADC is employed to convert an analog voltage 0 to 12 V, the resolution of the ADC is
(A) 2.44 mV (B) 83.3 mV
(C) 2.92 mV (D)1.0 mV

7. In A 4-bit weighted–resistor D/A converter the resistor value corresponding to LSB is 4 kΩ, then the resistor value corresponding to the MSB will be?
(A) 32 kΩ (B) 16 kΩ
(C) 1/4 kΩ (D) 1/2 kΩ

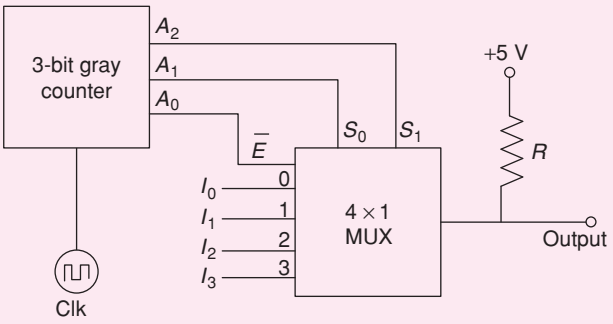
8. 6-bit DAC is having output voltage of 4.2 V for input 101010. The full scale reading of the DAC is?
(A) 6.4 V (B) 6.3 V
(C) 7.2 V (D) 8 V

9. Resolution of 4-bit ADC is 0.2 V, for the input of 2.55 V the output stable state will be?
(A) 1101 (B) 1011
(C) 1100 (D) 1110

10. The correct order for most accurate, fastest, simplest design of ADCs respectively,
(A) Dual scope ADC, Flash type ADC, SAR type ADC
(B) SAR type ADC, Dual scope ADC, Flash type ADC
(C) Flash type ADC, SAR type ADC, Counter type ADC
(D) Dual scope ADC, Flash type ADC, Counter type ADC

Previous Years' Questions

1. A 3-bit gray counter is used to control the output of the multiplexer as shown in the figure. The initial state of the counter is 000_2. The output is pulled high. The output of the circuit follows the sequence **[2014]**



(A) I_0, 1, 1, I_1, I_3, 1, 1, I_2 (B) I_0, 1, I_1, 1, I_2, 1, I_3, 1
(C) 1, I_0, 1, I_1, I_2, 1, I_3, 1 (D) I_0, I_1, I_2, I_3, I_0, I_1, I_2, I_3

2. An 8-bit, unipolar Successive Approximation Register type ADC is used to convert 3.5 V to digital equivalent output. The reference voltage is + 5 V. The output of the ADC, at the end of 3rd clock pulse after the start of conversion, is **[2015]**

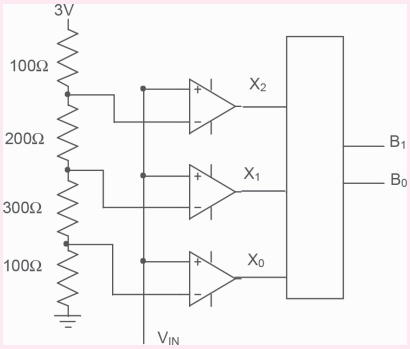
(A) 1010 0000 (B) 1000 0000
(C) 0000 0001 (D) 0000 0011

3. A temperature in the range of −40° C to 55° C is to be measured with a resolution of 0.1° C. the minimum number of ADC bits required to get a matching dynamic range of the temperature sensor is **[2016]**

(A) 8 (B) 10
(C) 12 (D) 14

4. A 2-bit flash Analog to Digital Converter (ADC) is given below. The input is 0 ″ V_{IN} ″ 3 Volts.

The expression for the LSB of the output B_0 as a Boolean function of X_2, X_1, and X_0 is **[2016]**



(A) $X_0\left[\overline{X_1 \oplus X_1}\right]$ (B) $\overline{X_0}\left[\overline{X_2 \oplus X_1}\right]$
(C) $X_0\left[\overline{X_1 \oplus X_1}\right]$ (D) $\overline{X_0}\left[X_2 \oplus X_1\right]$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** B	**3.** B	**4.** B	**5.** D	**6.** A	**7.** D	**8.** C	**9.** D	**10.** B

Practice Problems 2

1. D	**2.** A	**3.** A	**4.** B	**5.** D	**6.** C	**7.** D	**8.** B	**9.** A	**10.** D

Previous Years' Questions

1. A	**2.** A	**3.** B	**4.** A

Chapter 12

Microprocessor 8085

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Organization of microprocessor based system
- Microprocessor 8085
- Instructions classifications
- Instruction word size
- Operations in micro-processor
- 8085 signals
- Peripheral – mapped I/O
- Memory mapped I/O
- Additional instructions
- Stack
- Subroutine
- Restart (RST) instructions

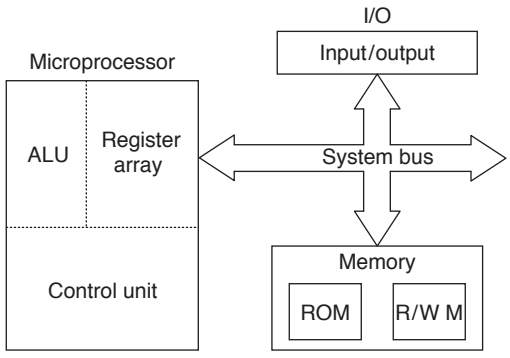
Introduction

The microprocessor is a programmable integrated device that has computing and decision-making capability similar to that of the central processing unit (CPU) of computer.

The microprocessor communicates and operates in the binary numbers 0 and 1 called bits.

Each microprocessor has a fixed set of instruction in the form of binary patterns called machine language to make it easier to understand the binary instructions that are given abbreviated names, called mnemonics, which form the assembly language for a given microprocessor.

Organization of Microprocessor-Based System



Microprocessor-based systems include three components: microprocessor, input/output (I/O) and memory

These components are organized around a common communication path called bus.

Microprocessor

The microprocessor is a clock driven semiconductor device consisting of electric logic circuits manufactured by using either LSI, or VLSI technique.

Arithmetic Logic Unit

The ALU unit performs arithmetic operations as addition, subtraction, and logic operations like AND, OR, exclusive OR.

Register Array

Microprocessor consists of various registers identified by *B*, *C*, *D*, *E*, *H* and *L*. These registers are primarily used to store data temporarily during the execution of a program.

Control Unit

The control unit provides the necessary timing and control signals to all the operations in the microcomputer.

Microprocessor 8085

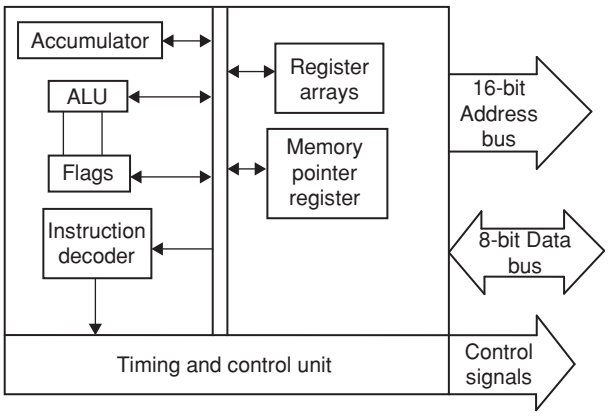


Figure 1 8085 Hardware model

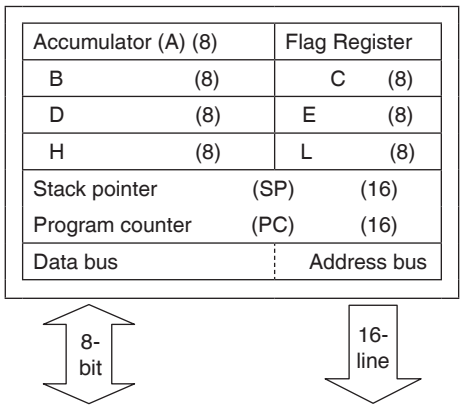


Figure 2 8085 Programming model

Table 1 *Flag register*

D_7	D_6	D_5	D_4	D_3	D_2	D_1	D_0
S	*Z*		*AC*		*P*		*CY*

Registers

The 8085 has six general purpose registers to store 8-bit data. These are identified as *B*, *C*, *D*, *E*, *H* and *L*, They can be combined as register pairs *BC*, *DE*, and *HL* to perform some 16-bit operations.

Accumulator

It is an 8-bit register, part of ALU. The result of an operation is stored in the accumulator. It is also identified as register *A*.

Flags

The ALU includes five flip-flops these flags are set or reset after the execution of an arithmetic or logic operation.

Z—Zero flag is set to 1 when result is zero, otherwise it is reset

CY—Carry flag—if an arithmetic operation results in carry or borrow, the carry flag is set, otherwise reset

S—Sign flag is set if bit D_7 (MSB) of the result = 1, otherwise reset

P—Parity flag is set, if result has an even number of 1's, for odd number of 1's the flag is reset

AC—Auxiliary carry flag is set, when (in arithmetic operation) a carry is generated by digit D_3 and passed to D_4 (carry transition from lower nibble to higher nibble). This flag is used internally for BCD operations.

Program Counter (PC) and Stack Pointer (SP)

These are two 16-bit registers used to hold memory addresses.

The function of program counter is to point to memory address from which the next byte is to be fetched and executed

When a byte (machine code) is being fetched, the program counter is incremented by one to point to the next memory location

Stack pointer points to a memory location in R/W memory, called stack. The beginning of the stack is defined by loading a 16-bit adder in the stack pointer.

Instructions Classifications

An instruction is a binary pattern designed inside a microprocessor to perform a specific function. The entire group of instructions, called instructions set, determines what functions the microprocessor can perform.

Data Transfer (Copy) Operations

These instructions perform the following six operations

Load an 8-number in a register

Load 16-bit number in a register pair

Copy from register to register

Copy between register and memory

Copy between I/O and Accumulator

Copy between registers and stack memory

Mnemonics	Operations
MVI reg., 8-bit	Load 8-bit data in a register
MOV Rd, Rs	Copy data from source (Rs) to destination (Rd) register
LXI Rp, 16-bit	Load 16-bit Num in a register pair(Rp)
OUT 8-bit (port addr)	Send data byte from accumulator
IN 8-bit (port Addr)	Accept data byte from input device into accumulator
LDA 16-bit	Load accumulator with data from 16-bit memory address
STA 16-bit	Store contents of A in 16-bit memory address
LDAX Rp	Load 'A' with data from memory address specified by (Rp) register pair
STAX Rp	Store data in A into the memory location specified by (Rp)
MOV R, M	Copy the data byte into register (R) from the memory specified by the address by HL pair
MOV M, R	Copy data byte into memory specified by HL pair from the register (R)

Arithmetic Instructions

The frequently used arithmetic operations are add, subtract, increment, decrement, the result will be stored in accumulator.

Mnemonics	Operation
ADD R	Add contents of reg. to contents of A
ADI 8-bit	Add 8-bit data to contents of A
ADD M	Add the contents of memory (in HL reg.) to A
SUB R	Subtract the contents of reg. from contents of A
SUI 8-bit	Subtract 8-bit data from contents of A
SUB M	Subtract the contents of memory HL from A
INR R/M	Increment reg./contents in memory (specified in HL) by 1
DCR R/M	Decrement reg./contents in memory (specified in HL) by 1
INX Rp	Increment the contents of register pair by 1
DCX Rp	Decrement the contents of register pair by 1

Logic and Bit Manipulation Instructions

These instructions include the following operations AND, OR, EX-OR, compare rotate bits, result will be stored in accumulator.

Mnemonic	Operations
ANA R/M	Logically AND the contents of reg./memory with contents of A
ANI 8-bit	Logically AND 8-bit data with A
ORA R/M	Logically OR the contents of reg./memory with contents of A
ORI 8-bit	Logically OR 8-bit data with A
XRA R/M	Ex-OR, the contents of a register/memory with A
XRI 8-bit	Ex-OR the contents of 8-bit data with A
CMP R	Compare the contents of R with A
CPI 8-bit	Compare 8-bit data with A
CMA	Complement accumulator

Branch Instructions

The following instructions change the program sequence

Mnemonics	Operation
JMP 16-bit address	Unconditional change the program sequence to specified address
JZ 16-bit address	Change the program sequence to specific address if zero flag is set
JNZ 16-bit address	If zero flag is reset change program sequence
JC 16-bit address	If carry flag is set change the program sequence
JNC 16-bit address	Change program sequence if carry flag is reset
CALL 16-bit address	Change the program sequence to the location of a subroutine specified
RET	Return to the called program after subroutine sequence

Machine Control Instructions

These instructions affect the operation of the processer

HLT—Stop processing and wait

NOP—Do not perform any operation

In data transfer, the contents of the source are not destroyed. Only the contents of the destination are changed. The data copy instructions do not affect the flags.

Arithmetic and logical operations are performed with the contents of accumulator and results are stored in the accumulator

Instruction word size

1-byte instructions These include OP code and operand in the same byte, each instructions, requires 1-byte memory space

Example: MOV C, A; ADD B; CMA
Mnemonic followed by letters representing register or memory.

2-byte instructions The first byte specifies the operation code and the second byte specifies operand. A mnemonic followed by 8-bit

Example: MVI A, 32H; MVI B, F2H;
These instructions would require two memory locations

3-byte instructions These instructions would require three memory locations, each to store the binary codes. The first byte specifies the op code, and the following two bytes specify the 16-bit address, the second byte is the low order address and third byte is the higher order address

These instructions have mnemonic followed by 16-bit
Example:

LDA 2050 H	3A first byte (op code) 50 second byte 20 third byte
JMP 2085 H	C3 first byte (op code) 85 second byte 20 third byte

An instructions has two parts, OP code (operation to be performed) and operand (data to be operated on)

The operand can be data (8 or 16-bit), address or register or it can be implicit.

The method of specifying an operand (directly or indirectly etc) is called the addressing mode.

Operations in Micro-processor

The microprocessor (MPU) primarily performs four operations

1. memory read
2. memory write
3. I/O read
4. I/O write

For each operations, it generates the appropriate control signals. To communicate with a peripheral (and memory) the MPU identifies the location by its address, transfers data, and provides timing signals.

The 8085 microprocessor has 16 address lines, 8-data lines, and control bus to provide timing of various operations. The 8085 can respond to four externally initiated operations: reset, interrupt, ready and hold.

Memory is a group of registers arranged in a sequence to store bits: The 8085 MPU requires an 8-bit wide memory word and uses 16-bit address to select a memory location. The memory addressing assigned to a memory chip in a system is called the memory mapping.

The assignment of memory address is done through the chip select logic.

Memory can be classified into two groups.

1. Read/write memory is volatile and can be used to read and write information this is also called user memory
2. The ROM is a non volatile memory and the information written into this memory is permanent

Input/output devices or peripherals can be interfaced with 8085 in two days. Peripheral I/O and memory mapped I/O.

In peripheral I/O, the MPU uses an 8-bit address to identify an I/O, and IN and OUT instructions for data transfer. In memory mapped I/O, the MPU uses a 16-bit address to identify an I/O, and memory related instructions for data transfer.

To execute an instruction, the MPU places the 16-bit address on the address bus, sends the control signal to enable the memory chip and fetches the instruction. The instruction is then decoded and executed.

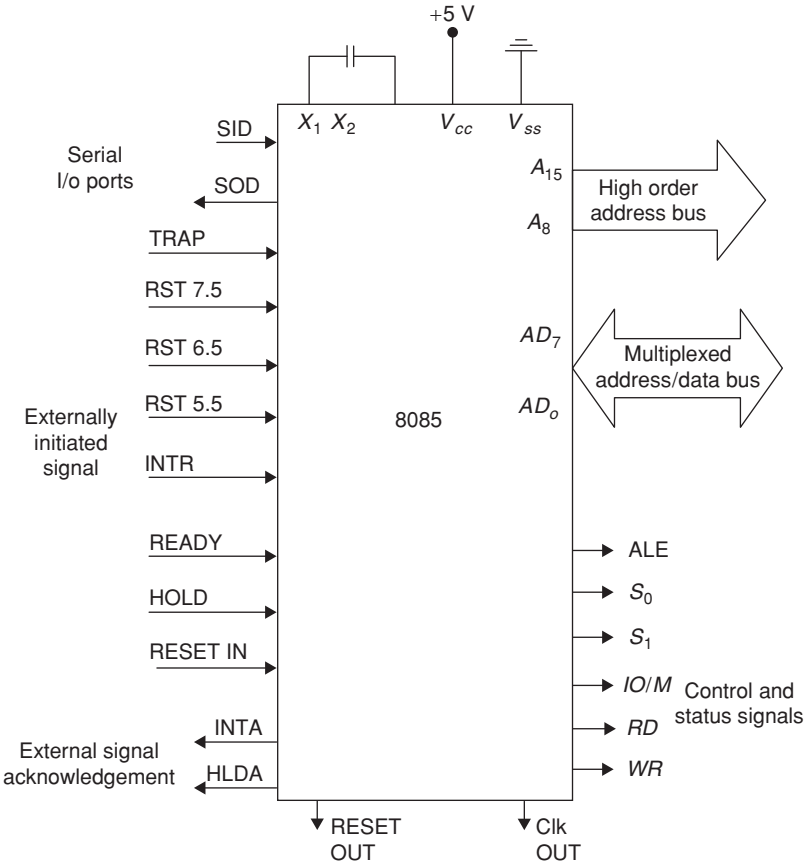


Figure 3 The 8085 microprocessor signals

8085 Signals

8085 is an 8-bit general purpose microprocessor.

Addressing capacity: 64 k bytes

It is an integrated circuit with 40 pins and uses a +5 V power supply, operates at 3 MHz single phase clock. (8085 A-2 version can operate at the maximum frequency of 5 MHz)

The signals of 8085 microprocessor are classified into 6 groups.

1. Address bus: The 8085 has 16 lines ($A_{15} - A_8$) and ($AD_7 - AD_0$) as address bus. Higher order address lines are unidirectional, lower order address lines ($AD_7 - AD_0$) are used for data and lower order address bus.
2. Data bus: The signal lines ($AD_7 - AD_o$) are bi-directional. In executing an instruction, during earlier part of the cycle, these lines are used as the lower address bus. During later part of the cycle, these lines are used as data bus.
3. Control and status signals:
   ALE: Address latch enable: It is active high signal generated every time the 8085 begin an operation.

(Machine Cycle): It indicates the bits $AD_7 - AD_0$ are address bits.

$\overline{RD}$ -Read: This active low signal indicates the selected IO/memory device to be read, and data are available on data bus.

$\overline{WR}$ -write: This active low signal indicates the data available on data bus are to be written into a selected memory or I/O.

$IO/\overline{M}$ This signal used to indicate IO(1) or memory(0) operation.

S_1 and S_0: These status signals can identify various operations as per the table below.

Machine Cycle	$IO/\overline{M}$	S_1	S_0	Control Signals
op code fetch	0	1	1	$\overline{RD} = 0$
Memory read	0	1	0	$\overline{RD} = 0$
Memory write	0	0	1	$\overline{WR} = 0$
I/O read	1	1	0	$\overline{RD} = 0$
I/O Write	1	0	1	$\overline{WR} = 0$
Interrupt acknowledge	1	1	1	$\overline{\text{INTA}} = 0$
Halt	Z	0	0	$\overline{RD}, \overline{WR} = z, \overline{\text{INTA}} = 1$
Hold	Z	X	X	$\overline{RD}, \overline{WR} = Z$
Reset	Z	X	X	$\overline{\text{INTA}} = 1$

4. Power supply and clk frequency
   V_{CC}: +5 V power supply, V_{SS}: Ground X_1, X_2: A crystal (RL or RC) is connected between these two pins; the frequency is internally divided by two.
   Clk (out): This is system clock for other devices.
5. Externally initiated signals:
   INTR (input): general purpose interrupt request.
   $\overline{\text{INTA}}$ (output): Interrupt Acknowledgement
   RST 7.5: } Restart interrupts, vectored interrupts,
   RST 6.5: } That transfer the program control to
   RST 5.5: } Specific memory location (8x)
   TRAP (input): non maskable interrupt with highest priority.
   HOLD (input): This signal indicates that a peripheral such as DMA is requesting the use of address/data bus.
   HLDA (output): Hold request acknowledgement.
   READY (input): When this signal goes low, microprocessor waits for an integral number of clock cycles until it goes high.
   $\overline{\text{RESETN}}$: When the signals goes low, PC is set to zero, the buses are tri-stated and microprocessor is reset.
   RESET OUT: This signal indicates, MPU is being rest and used to rest other devices.
6. Serial IO ports:
   SID—serial input data,
   SOD—serial output data.

The 8085 microprocessor has a multiplexed bus $AD_7 - AD_0$ used as lower order address bus and the data bus.

The bus $AD_7 - AD_0$ can be demultiplexed by using a latch and the ALE signal

The 8085 has a status signal $IO/\overline{M}$ and two control signals $\overline{RD}, \overline{WR}$, by ANDing these signals, four control signals can be generated $\overline{\text{MEMR}}$, $\overline{\text{MEMW}}$, $\overline{\text{IOR}}$, $\overline{\text{IOW}}$,

To execute an instruction the MPU

- Places the memory address of instruction on address bus
- Indicates the operation status on the status lines
- Sends $\overline{\text{MEMR}}$ control signal to enable the memory, fetches the instruction byte, and places it in the instruction decoder.
- Executes the instruction.
- Each instruction of 8085 microprocessor can be divided into a few basic operations called machine cycles, and each machine cycle can be divided into T-states.

The frequently used machine cycles are Opcode fetch, memory read and write, and I/O read and write.

When 8085 performs any of the operations, it asserts the appropriate control signal and status signal.

The opcode fetch and the memory read are operationally similar, the 8085 reads from memory in both machine cycles. However, the 8085 reads opcode during the opcode fetch cycle, and it reads 8-bit data during the memory read cycle. In the memory write cycle, the processer writes data into memory.

The 8085 performs there basic steps in any of these machine cycles, it places the address on address bus. Sends appropriate control signals, and transfers data via data bus.

8085 Machine Cycles and Bus Timings

1. Instruction cycle: It is the time required to complete the execution of an instruction, one instruction cycle consists of 1 to 6 machine cycles.
2. Machine cycle: It is the time required to complete one operation like accessing memory or IO or acknowledging external request. This cycle may consist of 3 to 6 T-states.
3. T-states: It is the part of the operation performed in one clock period. Each T-state is exactly equal to one clock period.

In each instruction cycle, the first operation is always opcode fetch, this cycle can be of four to six T-states duration.

Memory read is similar to opcode fetch but requires 3 T-states, and these two differentiated by the status signals.

Memory write cycle consists of 3 – T-states.

I/O read or I/O write consists of 3 – T-states.

Interrupt acknowledgement machine cycle consists of 6 T-states.

Consider instruction STA 2016 H, → store accumulator contents in memory location 2016 H.

This instruction includes → opcode fetch and memory read machine cycles to read address 2016 H, and then memory write machine cycle. Total of 4 machine cycles, and it is a 3-byte instruction.

Total T-states are = opcode fetch + memory read + memory read + memory write = 4 + 3 + 3 + 3 = 13 T-states.

Consider another instruct OUT 20 H, → transfer the data from accumulator to output device at 20 H.

This instruction consists of op code fetch, +memory read (reading 20 H) + I/O write.

Total T-states are = 4 + 3 + 3 = 10 T-states

Consider CALL 2006 H instruction, This instruction is a 3 byte instruction, which takes the program control to the address specified 2006 H and it stores the address of next instruction on the top of stack, so that RET instruction can take execution to previous program.

So call instruction consists of 5 machine cycles. Opcode fetch + memory read + memory read + memory write + memory write (storing program counter) = 4 + 3 + 3 + 3 + 3 = 18 T-states

To interface a memory chip with 8085, the necessary low order address lines of the 8085 address bus are connected to the address lines of the memory chip.

The high order address lines are decoded to generate $\overline{CS}$ signals to enable the chip.

In the absolute decoding technique, all the address lines that are not used for the memory chip to identify a memory register must be decoded, thus the chip select can be asserted by only one address. In the particular decoding technique, same address lines are left don't care. This technique reduces hardware, but generates multiple address resulting in fold back memory space.

Peripheral-mapped I/O

A latch is commonly used to interface output devices. A tri-state buffer is commonly used to interface input devices. To interface an output or an input device, the low order address bus $A_7 - A_0$ (or high order bus) needs to be coded to generate the device address plus, which must be combined with the control signal $\overline{IOR}$ or $\overline{IOW}$ to select the device.

Memory Mapped I/O

Memory related instructions are used to transfer data, to interface I/O device, the entire bus must be decoded to generate the device address pulse, which must be combined with the control signal $\overline{MEMR}$ or $\overline{MEMW}$ to generate the I/O select pulse. This pulse is used to enable the I/O device and transfer the data.

Memory interfacing

The primary function of memory interfacing is that the microprocessor should able to read and write into a given register of a memory element. To perform read/write operation with memory, the microprocessor should.

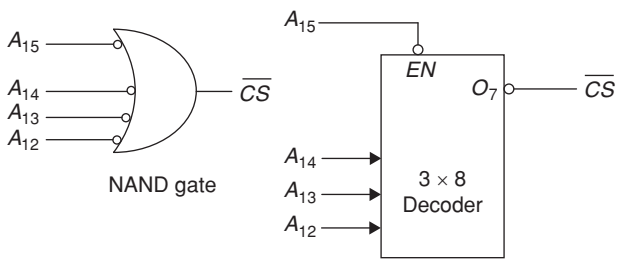
1. be able to select the chip (through $\overline{CS}$)
2. identify the register (using address bus)
3. enable read/write operation (using $\overline{RD}$ or $\overline{WR}$ signals)

Address decoding and memory addresses The process of address decoding should result in identifying a register for a given address. We should able to generate a unique pulse for a given address.

Let us consider 4k × 8 R/W memory has to interface with 8085 microprocessor.

Microprocessor uses 16-bit address, but $4k \times 8 = 2^{12} \times 8$ memory uses 12-bit address ($A_{11} - A_0$).

To interface with microprocessor, lower order address $A_{11} - A_0$ are connected to memory chip and remaining four address lines ($A_{15} - A_{12}$) of 8085 microprocessor must be decoded, this can be done by a gate/decoder/demultiplexer.

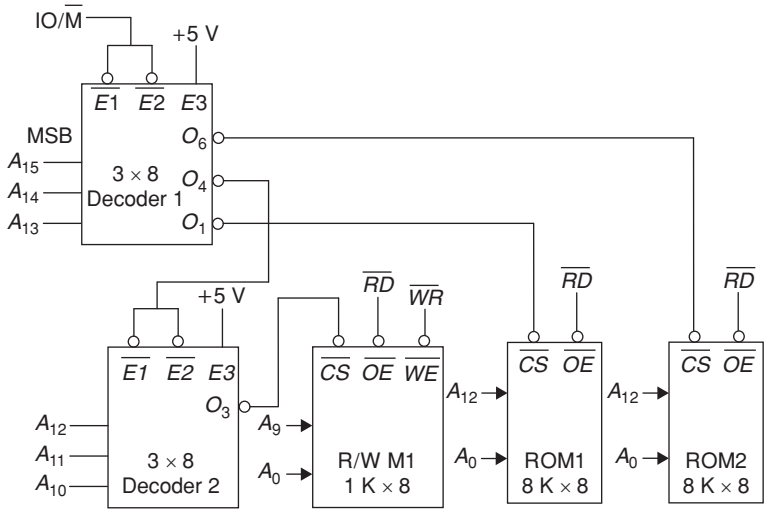


We can obtain the address range of this memory chip 4k × 8 by analyzing the possible logic levels on the 16 address lines.

As per the NAND gate, (or) decoder shown to select $\overline{CS}$, the address inputs $A_{15} - A_{12}$ should be 1111, and the address lines $A_{11} - A_0$ can assume any combinations from all 0 to all 1s. So memory address of this chip ranges from F000 H to FFFF H.

A_{15}	A_{14}	A_{13}	A_{12}	A_{11}	A_{10}	A_9	A_8	A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0	
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	= F000H
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	= FFFFH
Chip select				Register select												

Example: Identify the address range of the memory chips ROM1, ROM2 and R/WM1? For the memory interfacing circuit with 8085 microprocessor shown in the next page.



Solution: ROM1, ROM2 are read only memories $\overline{OE}$ —output enable is connected to $\overline{RD}$ signal of microprocessor. The chip select of ROM1, ROM2 are given from 3 × 8 decoder 1 outputs O_2 and O_6.

The enable inputs $\overline{E1}, \overline{E2}$ are connected $IO/\overline{M}$ so only memory operations will enable the decoder 1.

The range of address for ROM1 (ROM1 is selected with O_1 of decoder, so $A_{15}\,A_{14}\,A_{13} = 001$)

A_{15}	A_{14}	A_{13}	A_{12}	A_{11}	A_{10}	A_9	A_8	A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0	
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	= 2000 H
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	= 3FFFH

$A_{15} - A_{13}$ – used for chip select

$A_{12} - A_0$ – address inputs for ROM for all zero to all one

8k × 8 ROM = $2^{13} \times 8 \rightarrow$ 13 address lines $(A_{12} - A_0)$

ROM2 is selected with O_6 of decoder 1

So $A_{15}\,A_{14}\,A_{13} = 110$

A_{15}	A_{14}	A_{13}	A_{12}	A_{11}	A_{10}	A_9	A_8	A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0	
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
											= C000H					
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	= DFFFH

Read or write memory 1 of size 1 k × 8 = $2^{10} \times 8$ has 10 address lines $A_9 - A_0$, its chip select is connected from O_3 of decoder 2 with inputs $A_{12}\,A_{11}\,A_{10}$ (= 011)

Decoder 2 is enabled by decoder 1, O_4 output, so $A_{15}\,A_{14}\,A_{13} = 100$.

Output enable $(\overline{OE})$, write enable $(\overline{WE})$ are connected to microprocessor $\overline{RD}, \overline{WR}$ signals. So the range of address for the R/w memory is

A_{15}	A_{14}	A_{13}	A_{12}	A_{11}	A_{10}	A_9	A_8	A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0	
1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	
																= 800H
1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	= 8FFFH
Decoder 1			Decoder 2			Address llines of 1 k × 8										

Range of address for ROM1 = 2000H – 3FFFH

Range of address for ROM2 = C000H – DFFFH

Range of address for R/w M1 = 8C00H – 8FFFH

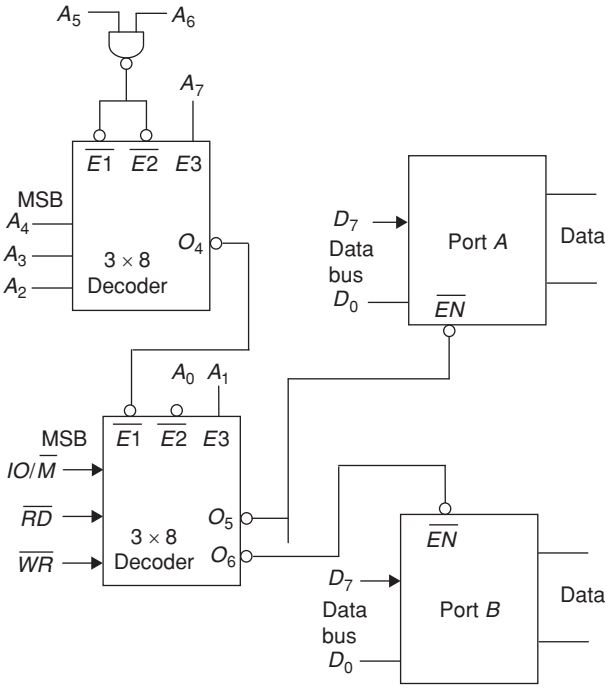
Peripheral I/O instructions The I/O devices such as key boards, and displays can be interfaced with 8085 microprocessor with 8-bit addresses in peripheral mapped I/O. In memory mapped I/O, 16-bit address will be used and memory related instructions for peripheral mapped I/O 8-bit address will be used.

- IN 8-bit port adder → This is 2-byte instruction, which transfers the data from input port to accumulator.
- OUT 8-bit port adder → This is a 2-byte instruction which transfers the data from accumulator to output port.

When an I/O instruction is executed, the 8085 microprocessor places the device address (port number) on the demultiplexed lower order as well as higher order address bus.

Address can be decoded to generate the pulse corresponding to the device address on the bus, and it will be added with appropriate signal IOR or IOW, and when both signals asserted, the I/O port is selected.

Example: In the following I/O interfacing circuit identify ports A and B as input or output ports and addresses of the ports? And find the instruction to be used?



Solutions: Two ports port A, and port B are connected in the above figure with two decoders.

First decoder O_4 is used to enable $(\overline{E1})$ of second decoder, so $A_4 A_3 A_2 = 100$.

First decoder will be enabled if $\overline{E1}\overline{E2}E3 = 001$, so $A_5 A_6$ = 11 (NAND output = 0), $A_7 = 1$

Second decoder will be enabled $\overline{E2}\,E3 = 01$

So $A_1 = 1, A_0 = 0$

The address of the port is

A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0	
1	1	1	1	0	0	1	0	= F2H

So both port A, and port B will have same address F2H,

But if we select IO Read operation

$IO/\overline{M} = 1, \overline{RD} = 0, \overline{WR} = 1$, inputs for second decoder. So O_5 will be selected, port A is used for IO read operation IN F2 H is the instruction for port A. (input)

For IO write operation

$IO/\overline{M} = 1, \overline{RD} = 1, \overline{WR} = 0$, 110 input for decoder will select O_6 output, so port B will be enabled for write operation.

OUT F2 H is the instruction for output port B.

Table 2 *Comparison between Memory Mapped I/O and Peripheral I/O*

	Memory Mapped I/O		Peripheral I/O
1	device address is 16-bit.	1	device address is 8-bit.
2	$\overline{\text{MEMR}}/\overline{\text{MEMR}}$ control signals for input/output.	2	$\overline{\text{IOR}}/\overline{\text{IOW}}$ control singles for input/output.
3	The memory map (64 k) is shared between IOs and system memory.	3	I/O map is independent of the memory map, 256 input devices and 256 output devices can be connected.
4	More hardware needed to decode 16-bit memory address but arithmetic/logical operations can be performed.	4	Less hardware needed to decode 8-bit address, Arithmetic and logical operations not available.

Additional Instructions

Table 3 *Additional Jump Instructions*

JP 16-bit	Jump on Plus (if $D_7 = 0$ and $S = 0$)
JM 16-bit	jump on minus (if $D_7 = 1$ and $S = 1$)
JPE 16-bit	jump on even party ($p = 1$)
JPO 16-bit	jump on odd party ($p = 0$)

Table 4 *Additional Data Transfer and 16-bit Arithmetic Instructions*

LXI, Rp, 16-bit	load reg. pair immediate
LDAX B/D	load accumulator indirect
LDA 16-bit	load accumulator direct
STAX B/D	store accumulator indirect
STA 16-bit	store accumulator direct
INX Rp,	increment register pair
DCX RP,	decrement register pair.

Table 5 *Logic Operation: Rotate*

RLC	Rotate accumulator left
RAL	Rotate accumulator left through carry
RRC	Rotate accumulator right
RAR	Rotate accumulator right through carry

Logic operations: Compare

CMP R/M: Compare register/memory with accumulator
CPI 8-bit: Compare immediate with accumulator

If $A <$ R/M $CY = 1, Z = 0$
If $A >$ R/M $CY = 0, Z = 0$
If $A =$ R/M $CY = 0, Z = 1$

Remaining flags effected according to the result of subtraction.

Table 6 16-*bit data transfer and data exchange instructions*

LHLD 16-bit	Load HL registers direct
SHLD 16-bit	Store HL registers direct
XCHG 16-bit	Exchange the contents of HL with DE
XTHL	Exchange the top of the stack with HL
SPHL	Copy HL reg. pair into stack pointer
PCHL	Copy HL reg. pair into the program counter
ADC R/M	Add reg./memory contents with carry to accumulator
SBB R/M	Subtract reg./memory contents with barrow from accumulator
DAA	Decimal adjust accumulator, it adjusts the 8-bit number in accumulator to form 2BCD numbers. It works with addition when BCD numbers are used
ACI 8-bit	Add immediate to accumulator with carry
CMC	Complement carry
DAD RP	Add register pair to HL registers
DI	Disable interrupts
EI	Enable interrupts
POP RP	POP off stack to reg. pair
PUSH Rp	Push reg. pair on to stack
SBI 8-bit	Subtract immediate with barrow
STC	Set carry
RIM	Read interrupt mask
SIM	Set interrupt mark
RST	Restart

Stack

Memory locations in R/W memory can be employed as temporary storage for information by initializing 16-bit address in the stack pointer (SP) register, these memory locations are called stack.

The stack space grows upward in the numerically decreasing order of memory address.

The contents of the stack pointer can be interpreted as the address of the memory location that is already used for storage. The retrieval of bytes begins at the address in the stack pointer; however the storage begins at the next memory location (in the decreasing order)

The storage and retrieval of data bytes on the stack should follow the LIFO (last in first out sequence) information in stack location is not destroyed until new information is stored in those location

PUSH Rp: Store register pair on stack: In copies the contents of the specified register pair on the stack. The stack pointer register is decremented and the contents of high order register (e.g., register B, D/H) are copied in the location shown by stack pointer register. The stack pointer register again decremented and the contents of the low order register (ex-register C/E/L) are copied in that location.

Operand PSW represents program status word, meaning the contents of the accumulator and the flags.

POP Rp: Retrieve register pair from stock: It copies the contents of the top two memory locations of stack into specified register pair.

First the contents of the memory location indicated by the stack pointer register are copied into the low order register (ex: register C/E/L) and then the stack pointer register incremented by 1. The contents of the next memory location are copied into the high order register (e.g., register H/B/D) and the stack pointer register is again incremented by 1.

Subroutine

It's a group of instructions written separately from the main program to perform a function that occurs repeatedly in the main program.

8085 has two instructions to implement subroutines. CALL 16-bit: call subroutine unconditionally. It transfer the program sequence to a subroutine address specified in instruction. First it saves the contents of the program counter (the address of next instruction) on the stack. Decrements the stack pointer by two, jumps unconditionally to the memory location specified by the second and third bytes in instruction. This instruction is accompanied by a return instruction in the subroutine.

RET: return from subroutine unconditionally. It inserts the two bytes from the top of the stack into the program counter and increments stack pointer by two. Unconditionally returns from a subroutine.

Conditional Call and Return Instructions

The conditional call and return instructions are based on four data conditions (flags) carry, zero, sign, parity.

If the call instruction in the main program is conditional the return instruction in the subroutine can be conditional or unconditional.

CC 16-bit: call subroutine if carry flag is set ($CY = 1$)

CNC 16-bit: call subroutine if carry flag is reset ($CY = 0$)

CZ 16-bit: call subroutine if zero flag is set ($z = 1$)

CNZ 16-bit: call subroutine if zero flag is reset ($z = 0$)

CM 16-bit: call subroutine if sign flag is set ($S = 1$ negative number)

CP 16-bit: call subroutine if sign flag is reset ($S = 0$ positive number)

CPE 16-bit: call subroutine if parity flag is set ($p = 1$ even parity)

CPO 16-bit: call subroutine if parity flag is reset ($p = 0$, odd parity)

Conditional RETURN

RC – return if $CY = 1$
RNC – return if $CY = 0$
RZ – return if $Z = 1$,
RNZ – return if $Z = 0$,
RM – return if S = 1
RP – return if $S = 0$
RPE – return if $P = 1$
RPO – return if $P = 0$

The programming technique of a subroutine calling another subroutine is called nesting; this process is limited only by the number of available stack locations. When a subroutine calls another subroutine, all return addresses are stored on the stack. One call can have multiple return instructions.

Restart (RST) Instructions

RST instructions are 1-byte call instructions that transfer the program execution to a specific location, on page 00 H.

RST 0 – call 0000 H
RST 1 – call 0008 H
RST 2 – call 0010 H
RST 3 – call 0018 H
RST 4 – call 0020 H
RST 5 – call 0028 H
RST 6 – call 0030 H
RST 7 – call 0038 H

The interrupt is an asynchronous process of communication with the microprocessor, initiated by an external peripheral.

The 8085 has a maskable interrupt that can be enabled or disabled using the instructions EI and DI.

The 8085 has four additional interrupt inputs, one non-maskable and three maskable, these three interrupts are implemented without any external hardware and are known as RST 7.5, 6.5, and 5.5.

The instruction SIM is necessary to implement the interrupt 7.5, 6.5, 5.5.

The instruction RIM can be used to check whether any interrupt requests are pending.

The direct memory access (DMA) is a process of high-speed data transfer under the control of external devices such as a DMA controller.

Table 7 *Summary of Interrupts in 8085 in the Order of Their Priority*

Interrupts	Type	Trigger
TRAP	Non-maskable – vectored (0024 H)	Level and edge sensitive
RST 7.5	Maskable vectored (003C H)	Edge-sensitive
RST 6.5	Maskable vectored (00034 H)	Level sensitive
RST 5.5	Maskable vectored (0002 CH)	Level sensitive
INTR	Maskable non-vectored (RST code from external hardware (0000-0038 H)	Level sensitive

Solved Examples

Example 1: Consider the following set of instructions:

LXI B, 7E45 H
STC
MOV A, B
ORA A
RAL
MOV B, A
This set of instructions

(A) Doubles the number in register B.
(B) Divides the number in register by 2.
(C) OR contents of B and A.
(D) Adds A and B register contents.

Solution: (A)

LXI B, 7E45 H—move 7E45 H to BC register pair, i.e., $B = 7EH$, $C = 45$ H.
STC—set carry, $CY = 1$
MOV A, B—Move contents of B to A.
ORA A—OR content of A with A itself, this makes $CY = 0$, and other flags will be effected as per result in A, and contents of A will not be altered as A OR $A = A$ only.
RAL—rotate accumulator left arithmetically, i.e., include carry bit in rotation, this instructions moves the contents of accumulator to shift left by 1-bit, and carry bit = (0) will be added at LSB side. This rotation is equal to multiplication by 2. Left shift by n-bit = multiplication by 2^n.
MOV B, A—move contents of A to B, i.e., store result in B again. Now the contents of B are doubled.

Common Data for Questions 2–4: Consider the following program of 8085 microprocessor and a subroutine at memory location 1010 H.

2020: LXI H, 2020 H
2023: SPHL
2024: MOV A, C
2025: XRA C
2026: RAL
2027: CZ 1010 H
202A: NOP

1010: PUSH B
1011: PUSH D
1012: POP B
1013: POP D
1014: XTHL
1015: RET

Example 2: What is the operation of subroutine at 1010 H?

(A) Interchanges contents of B and D registers
(B) Swaps the contents of SP and HL register pair

(C) Interchanges the contents of *B* and *D* register pairs.
(D) Copies the top of the stack to HL register pair.

Example 3: What are the contents of SP when the microprocessor is executing instruction at 1014 H?
(A) 201E H (B) 2020 H
(C) 1015 H (D) 202A H

Example 4: For how many number of times the instruction at 202AH will execute?
(A) Infinite times (B) 2
(C) 1 (D) Never executes

Solutions for questions 2 to 4:

2. (C) **3.** (A) **4**. (D)

2020: LXI H, 2020 H: Load HL register path with 2020 H, HL = 2020 H

2023: SPHL—Move HL register pair to SP stack pointer, SP = 2020 H

2024: MOV A, C—Move contents of register C to accumulator

2025: XRA C: XOR accumulator with C, now A = 00 H CY = 0, Z = 1.

2026: RAL—Rotate accumulator left through carry now CY = 0, A = 00 H

2027: CZ 1010 H-jump to 1010 if zero flag = 1, as Z = 1 now control execution go to 1010 H

But to come back to main program, CALL instruction stores the present PC = 202 AH on top of the stack, so now contents of top of stack are 202A, SP = 201E H and PC will be charged to 1010 H

1010: PUSH B—move contents of *BC* register pair on top of stack

1011: PUSH D—move contents of *DE* register pair on top of stack

1012: POP B—retrieve the top of stack contents to *BC* register pair i.e., *BC* = *DE*

1013: POP D—retrieve the top of stack contents to *DE* register pair i.e., *DE* = *BC*

These 4 instructions will interchange *B*, *D* register pairs, as stack works with last in first out principle, after these 4 instructions top of stack will have 202 AH and SP = 201E H

1014: XTHL—Exchange top of stack with HL register pair contents,

Now HL = 202AH, top of stack = 2020 H

1015:RET Return instruction returns to main program by taking the stored address of next instruction of CALL instruction on top of the stack i.e., by retrieving top of the stack to *PC*.

Now PC—2020 H. So the execution starts from 2020 H abs *CZ* will be always 1 and CALL(*CZ*) will be executed infinite number of times.

Example 5: Consider the following program for an 8085 microprocessor

```
    MVI A, 8-bit data
    ANA A
    JP SUB1
    XRA A
SUB1: OUT 8-bit port address
      HLT
```

At output port
(A) Only numbers with even number of 1's will appear
(B) Only numbers with odd number of 1's will appear
(C) No negative numbers will appear
(D) Only negative numbers will appear

Solution: (C)
MVI A, 8-bit data—Move 8-bit data to accumulator
ANA A—AND accumulator with accumulator itself.
JP SUB1—if sign flag is 0, i.e., for positive number, jump to SUB1, or else go to next instruction.
XRA A—XOR A with A i.e., contents of A − 00 H
SUB1: OUT 8-bit port address—output the accumulator contents to the 8-bit port address.

HLT—stop execution.

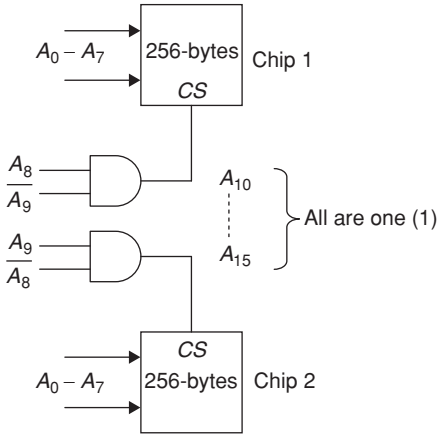
The program outputs only positive numbers. If the jump on positive condition fails then contents of accumulator will be reset, and 00 H will be the output at output port.

EXERCISES

Practice Problems I

Directions for questions 1 to 16: Select the correct alternative from the given choices.

1. Given two memories what is the memory address range represented by chip 1 or chip 2. (CS – chip select, $A_0 - A_{15}$ are address lines)



(A) 0100 – 02FF (B) F500 – F6FF
(C) F900 – FAFF (D) FD00 – FEFF

2. If we have an 8085 program as shown such that the program starts at location 0100 H, the content of accumulator when PC reaches 010C H is _____

LXI SP, 00 EF
LXI H, 0700
MOV A, M
XRA M
MOV M, A
MVI A, 30H
SUB M

(A) 30 H (B) 02 H
(C) 00 H (D) FF H

3. From the given program, what will be the content of SP on completion of RET execution?

LXI SP, EFFFH
CALL 2000 H
2000 H LXI H, 3CF4 H
PUSH PSW
SPHL
POP PSW
RET

(A) EFFD (B) EFFF
(C) 3CF6 (D) 3CF8

4. Given a 8085 program. Indicate the status of *CY* and *Z* flags after the execution of line 7 in the program.

Line 1: MVI A, B5 H
2: MVI B, 0EH
3: XRI 69 H
4: ADD B
5: ANI FBH
6: CPI 9F H
7: STA 3010 H
8: HLT

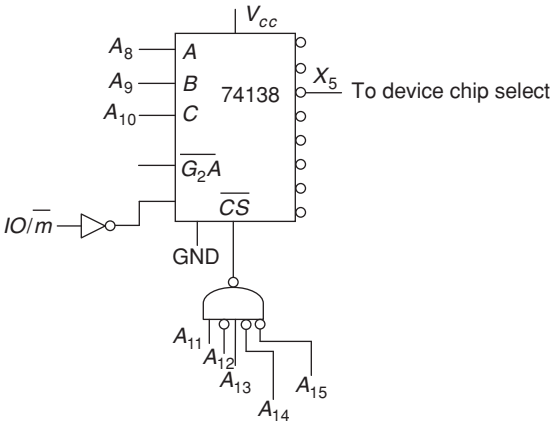
(A) $CY = 0, z = 0$ (B) $CY = 0, z = 1$
(C) $CY = 1, z = 0$ (D) $CY = 1, z = 1$

5. Predict the content of PC and HL after the execution of part of program.

2710 LXI H, 30A2H
2713 DAD H
2714 PCHL

(A) PC = 2715 H HL = 30 A2 H
(B) PC = 30 A2 H HL = 2715 H
(C) PC = 6144 H HL = 6144 H
(D) PC = 6144 H, HL = 2715 H

6. Determine the address range for the device connected to X_5.



(A) 2D00 – 2DFF (B) 2E00 – 2EFF
(C) FD00 – FDFF (D) 2F00 – 2FFF

7. For the program, find the content of accumulator after its complete execution

MVI A, 40 H
MOV B, A
STC
CMC
RAR
XRA B

(A) 60 H (B) 67 H
(C) 20 H (D) 40 H

8. Serial input data of 8085 can be loaded into bit-7 of accumulator by
(A) executing a RIM instruction
(B) using TRAP
(C) executing a RST 1
(D) None of these

9. Consider the program

```
    LXI D, 0009 H
    XRA A
LOOP: DCX D
    JNZ LOOP
```

The loop will be executed
(A) 9 times (B) 8 times
(C) 1 time (D) None of these

10. If we have an I/O port having address of 01 H what will be the content of flag register after the execution of instructions? (Assume data is 75 H)

```
IN  01 H
ANI 81 H
```

(A) 1 0 * 1 * 1 * 0
(B) 0 1 * 1 * 0 * 0
(C) 0 1 * 0 * 0 * 0
(D) 0 1 * 0 * 1 * 0

11. The content of SP and HL after the execution of instruction.

```
1000 LXI SP, 27FE
1003 CALL 1006
1006 POP H
```

(A) SP = 27FE, HL = 1003
(B) SP = 27FD, HL = 1003
(C) SP = 27FE, HL = 1006
(D) SP = 27FC, HL = 1006

12. In order to design a memory system of size 16 k bytes using chips with 11 address lines and 4 data lines each, how many chips are required
(A) 8 (B) 2
(C) 16 (D) 4

13. If the A contain 39 BCD, B contain 12 BCD then what will be the output of instructions?

```
ADD B
DAA
```

(A) 4 BH (B) 51 H
(C) B1H (D) 18 H

14. What are events followed by the instruction POP PSW in 8085 microprocessor?
(A) flag register ← [SP]
SP ← SP + 1
A ← [SP]
SP ← SP + 1
(B) SP ← SP + 1
flag register ← [SP]
SP ← SP + 1
A ← [SP]
(C) SP ← SP – 1
flag register ← [SP]
SP ← SP – 1
A ← [SP]
(D) flag register ← [SP]
SP ← SP – 1
SP ← SP – 1
[SP] ← A

Common Data for Questions 15 and 16: An 8085 assembly language program is given below.

```
Line 1:  MVI A, B5 H
     2:  MVI B, 0EH
     3:  XRA  A
     4:  ADD B
     5:  ANI 8CH
     6:  CPI 8DH
     7:  STA 3010 H
     8:  HLT
```

15. The contents of the accumulator just after execution of the ADD instruction in line 4 will be
(A) 00 H (B) C3 H
(C) 0CH (D) 0EH

16. After execution of line 7 of the program, the status of the *CY* and *Z* flags are
(A) $CY = 0, Z = 0$ (B) $CY = 0, Z = 1$
(C) $CY = 1, Z = 0$ (D) $CY = 1, Z = 1$

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

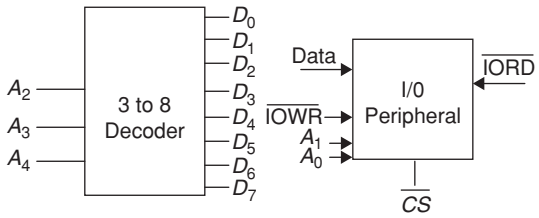
1. In an 8085 microprocessor, if we execute the instruction CMP B with content of B less than that of accumulator. As a result
(A) both carry flag and zero flag are set
(B) carry flag is set and zero flag is reset
(C) both carry flag and zero flag are reset
(D) carry flag is reset and zero flag is set

2. Match the number of machine cycles required to execute the instructions.

(i)	LHLD 2050 H	(1)	5
(ii)	LDA 2050 H	(2)	4
(iii)	LXI H, 2050 H	(3)	2
(iv)	XCHG	(4)	3
		(5)	1

(A) i – 1, ii – 2, iii – 4, iv – 5
(B) i – 1, ii – 2, iii – 4, iv – 3
(C) i – 1, ii – 5, iii – 1, iv – 3
(D) i – 2, ii – 1, iii – 1, iv – 5

3. An input/output peripheral is to be interfaced to an 8085 microprocessor. To select the input/output device in address range *E*1 to E3H, its $\overline{CS}$ should be connected to which of the decoder output pins?



(A) D_0 (B) D_7
(C) D_1 (D) D_3

4. An instruction that can be used so as to select only the higher 4-bits of accumulator in 8085 programming language
(A) ANI FOH (B) ANI OFH
(C) XRI OFH (D) XRI FOH

5. It is desired to multiply 0AH by 0BH and store the result in accumulator. The numbers are available in *C* and *D*. The steps required are
(A) MVI A, 00 H
LOOP: ADD C
DCR D
JNZ LOOP
HLT
(B) MVI A, 00 H
LOOP: JNZ LOOP
ADD C
DCR D
HLT
(C) MVI A, 00
LOOP: ADD C
JNZ LOOP
DCR D
HLT
(D) None of these

6. The content of accumulator after the execution of instruction
MVI A A7 H
ORA A
RLC
(A) CFH (B) 4FH
(C) 4EH (D) CEH

7. In the given program the number of times the first and second JNZ instruction causes the control to be transferred to LOOP are
MVI H, 02 H
MVI L, 05 H
LOOP: DCR L
FIRST: JNZ LOOP
DCR H
SECOND: JNZ LOOP
HLT
(A) 5 and 2 (B) 4 and 1
(C) 259 and 1 (D) 260 and 2

8. If the accumulator content is 27 H and if carry flag is already set, the instruction ACI 16 H will result in
(A) 3F H, $CY = 1$ (B) 39 H, $CY = 0$
(C) 3E H, $CY = 0$ (D) 3E H, $CY = 1$

9. Match the vectored interrupts with the address

A.	TRAP	(1) 002CH
B.	RST 7.5	(2) 003CH
C.	RST 6.5	(3) 0024 H
D.	RST 5.5	(4) 0034 H

(A) A – 2, B – 3, C – A, D – 1
(B) A – 3, B – 1, C – 2, D – 4
(C) A – 4, B – 2, C – 1, D – 3
(D) A – 3, B – 2, C – 4, D – 1

10. For the given program of 8085 µP what is the time interval between two MEMW signals.
START: MOV A, B
OUT 55
DCR B
STA FFF8 H
JMP START
(A) 5.8 µs (B) 6.2 µs
(C) 7.8 µs (D) 5.2 µs

11. Consider the following microprocessor programming code
L X I H, 5050 H
MOV A, M

```
MOV B , 05 H
ADD A, B
CMA
MOV M,A
HLT
```

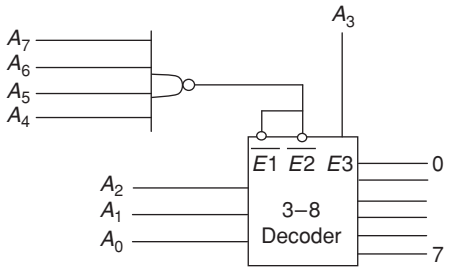
Memory location 5050 H is loaded with $(0A)_H$. The status of zero flag and carry flag after execution of CMA is

(A) $Z = 1, C = 1$ (B) $Z = 0, C = 1$
(C) $Z = 1, C = 0$ (D) $Z = 0, C = 0$

12. The last address location of a 1 KB memory chip is given as F080. The starting address will be

(A) EC 80 H (B) EC 81 H
(C) F000 H (D) EC00 H

13. A 3 – 8 decoder is used to decode the addresses and interface it to the microprocessor. Address lines are connected to the decoder as shown in the figure. The range of address possible to be selected by the given schematic is



(A) 88 H – 8 FH (B) F 0 H – F8 H
(C) F8 H – FFH (D) 00 H – FFH

14. Consider the following assembly language program.

```
MVI A, B3 H
MVI B, 10 H
XRI  69 H
ADD B
STA 3000 H
HLT
```

The content of accumulator after executing ADD instruction will be

(A) 69 H (B) DAH
(C) EAH (D) CIH

15. The status of control buses of an 8085 program is given below.

$IO/\overline{M}$ – HIGH

$\overline{RD}$ – LOW

$\overline{WR}$ – HIGH

The data in memory location 2500 H is 08 H. The output port is 25 H and input port is 20 H. The assembly language instruction being executed is

(A) IN 08 H (B) OUT 25 H
(C) OUT 08 H (D) IN 20 H

Previous Years' Questions

1. If the following program is executed in a microprocessor, the number of instruction cycles it will take from START TO HALT is **[2004]**

```
START MV1A, 14H  :  Move 14 H to register A
SHIFT RLC        :  Rotate left without carry
JNZ SHIFT        :  Jump on non-zero to SHIFT
HALT
```

(A) 4 (B) 8
(C) 13 (D) 16

2. The 8085 assembly language instruction that stores the contents of H and L registers into the memory locations 2050_H and 2051_H, respectively is: **[2005]**

(A) SPHL2050_H (B) SPHL 2051_H
(C) SHLD 2050_H (D) STAX 2050_H

3. A software delay subroutine is written as given below: **[2006]**

```
DELAY:  MVI  H,   255 D
        MVI  L,   255 D
LOOP:   DCR  L
        JNZ  LOOP
        DCR  H
        JNZ  LOOP
```

How many times DCR L instruction will be executed?

(A) 255 (B) 510
(C) 65025 (D) 65279

4. In an 8085, A microprocessor based system, it is desired to increment the contents of memory location whose address is available in (D.E) register pair and store the result in same location. The sequence of instructions is **[2006]**

(A) XCHG
INR M
(B) XCHG
INX H
(C) INX D
XCHG
(D) INR M
XCHG

5. Which one of the following statements regarding the INT (interrupt) and the BRQ (but request) pins in a CPU is true? **[2007]**

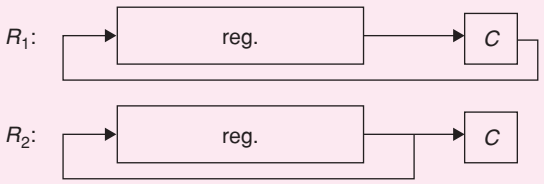
(A) The BRQ pin is sampled after every instruction cycle, but the INT is sampled after every machine cycle

(B) Both INT and BRQ are sampled after every machine cycle

(C) The INT pin is sampled after every instruction cycle, but the BRQ is sampled after every machine cycle

(D) Both INT and BRQ are sampled after every instruction cycle

Common Data for Questions 6 and 7: The associated figure shows the two types of rotate right instructions R_1, R_2 available in a microprocessor where reg. is a 8-bit register and C is the carry bit. The rotate left instruction L_1 and L_2 are similar except that C now links the most significant bit of reg. instead of the least significant one.



6. Suppose reg. contains the 2's complement number 11010110. If this number is divided by 2 the answer should be **[2007]**

(A) 01101011 (B) 10010101
(C) 11101001 (D) 11101011

7. Such a division can be correctly performed by the following set of operations **[2007]**

(A) L_2, R_2, R_1 (B) L_2, R_1, R_2
(C) R_2, L_1, R_1 (D) R_1, L_2, R_2

8. An input device is interfaced with Intel 8085A microprocessor as memory mapped I/O. The address of the device is 2500 H. In order to input data from the device to accumulator, the sequence of instruction will be **[2008]**

(A) LXI H, 2500 H
MOV A, M

(B) LXI H, 2500 H
MOV M, A

(C) LHLD 2500 H
MOV A,M

(D) LHLD 2500 H
MOV M,A

9. The contents (in Hexadecimal) of some of the memory locations in an 8085A based system are given below **[2008]**

Address	Contents
..	..
26FE	00
26FF	01
2700	02
2701	03
2702	04
..	..

The contents of stack pointer (SP), program counter (PC) and (H, L) are 2700 H, 2100 H and 0000 H respectively. When the following sequence of instructions are executed,

2100 H: DAD SP
2101 H: PCHL

the contents of (SP) and (PC) at the end of execution will be

(A) (PC) = 2102 H, (SP) = 2700 H
(B) (PC) = 2700 H, (SP) = 2700 H
(C) (PC) = 2800 H, (SP) = 26 FEH
(D) (PC) = 2A02 H, (SP) = 2702 H

10. The increasing order of speed of data access for the following devices is **[2009]**

(i) Cache memory
(ii) CDROM
(iii) Dynamic RAM
(iv) Processor registers
(v) Magnetic tape

(A) (v), (ii), (iii), (iv), (i)
(B) (v), (ii), (iii), (i), (iv)
(C) (ii), (i), (iii), (iv), (v)
(D) (v), (ii), (i), (iii), (iv)

11. In an 8085 microprocessor, the contents of the accumulator, after the following instructions are executed will become **[2009]**

XRA A
MVIB F0 H
SUB B

(A) 01 H (B) 0F H
(C) F0 H (D) 10 H

12. When a "CALL Addr" instruction is executed, the CPU carries out the following sequential operations internally where, **[2010]**

(R) means content of register R
((R)) means content of memory location pointed to by R
PC means Program Counter
SP means Stack Pointer

(A) (SP) incremented
(PC) ← Addr
((SP)) ← (PC)

(B) (PC) ← Addr
((SP)) ← (PC)
(SP) incremented

(C) (PC) ← Addr
(SP) incremented
((SP)) ← (PC)

(D) ((SP)) ← (PC)
(SP) incremented
(PC) ← Addr

13. A portion of the main program to call a subroutine SUB in an 8085 environment is given below. **[2011]**

```
        :
        :
    LXI D, DISP
LP: CALL SUB
        :
        :
```

It is desired that control be returned to LP + DISP + 3 when the RET instruction is executed in the subroutine. The set of instructions that precede the RET instruction in the subroutine are.

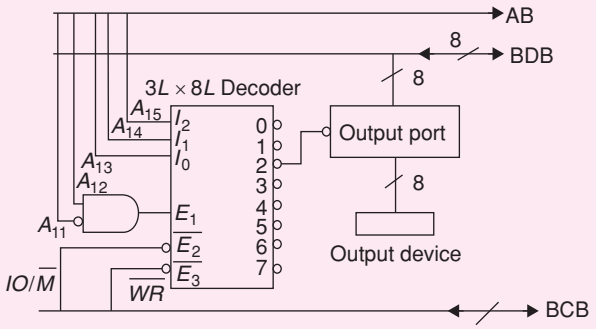
(A)
```
POP  D
DAD  H
PUSH D
```

(B)
```
POP  H
DAD  D
INX  H
INX  H
INX  H
PUSH H
```

(C)
```
POP  H
DAD  D
PUSH H
```

(D)
```
XTHL
INX  D
INX  D
INX  D
XTHL
```

14. An output device is interfaced with 8-bit microprocessor 8085A. The interfacing circuit is shown in figure **[2014]**



The interfacing circuit makes use of 3 line to 8 line decoder having 3 enable lines $E_1, \overline{E}_2, \overline{E}_3$. The address of the device is

(A) 50_H (B) 5000_H
(C) $A0_H$ (D) $A000_H$

15. In an 8085 microprocessor, the following program is executed **[2014]**

Address Location	Instruction
2000 H	XRA A
2001 H	MVI B, 04 H
2003 H	MVI A, 03 H
2005 H	RAR
2006 H	DCR B
2007 H	JNZ 2005
200A H	HLT

At the end of program, register A contains

(A) 60 H (B) 30 H
(C) 06 H (D) 03 H

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** A	**3.** D	**4.** A	**5.** C	**6.** A	**7.** A	**8.** A	**9.** C	**10.** B
11. C	**12.** C	**13.** B	**14.** A	**15.** D	**16.** C				

Practice Problems 2

1. C	**2.** A	**3.** A	**4.** A	**5.** A	**6.** B	**7.** C	**8.** C	**9.** D	**10.** C
11. D	**12.** B	**13.** C	**14.** C	**15.** D					

Previous Years' Questions

1. A	**2.** C	**3.** D	**4.** A	**5.** A	**6.** D	**7.** A	**8.** A	**9.** B	**10.** B
11. D	**12.** D	**13.** B	**14.** B	**15.** A					

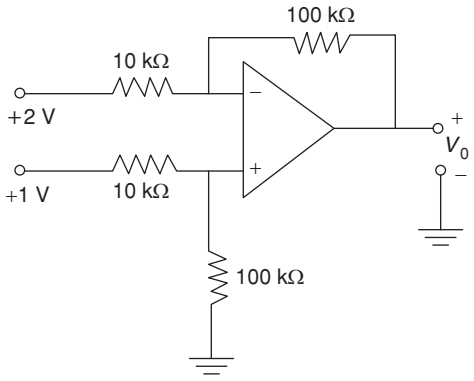
TEST
ANALOG AND DIGITAL ELECTRONICS

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. Consider the following OP-Amp circuit:

What is the output voltage V_0 in the OP-Amp circuit?



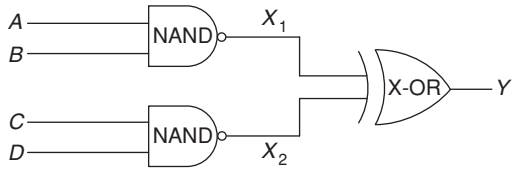
(A) +10 V
(B) –10 V
(C) +11 V
(D) –11 V

2. A 555 a stable oscillator with $R_1 = 40$ kΩ, $R_2 = 120$ kΩ and $C = 4{\cdot}0$ μF will oscillate at
(A) 1.288 Hz
(B) 443.5 Hz
(C) 443.5 kHz
(D) 1.288 kHz

3. The simplified SOP form for the Boolean expression $Y(A, B, C, D) = \Sigma\,(0, 1, 2, 3, 4, 5, 7)$ is

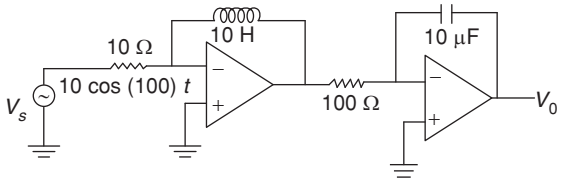
(A) $\overline{A}\overline{B} + \overline{A}\overline{C} + \overline{A}D$
(B) $\overline{A}\overline{B} + \overline{A}\overline{C} + \overline{A}B + CD$
(C) $\overline{A}\overline{C} + \overline{A}D + \overline{A}\overline{B} + C\overline{D}$
(D) $\overline{A}\overline{C} + \overline{A}D$

4. Which one of the following statements regarding the figure below is correct? A, B, C, D are input bits and Y is the output bit



(A) Output is zero when $A = B = C = D$
(B) $\overline{\overline{A}\overline{B} + \overline{C}\overline{D}} = Y$
(C) Y is always zero
(D) All are correct

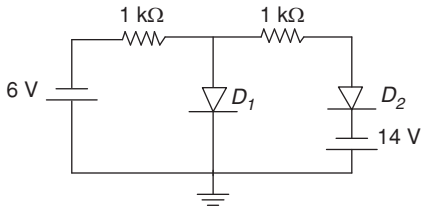
5.



In the figure, assume the OP-Amps to be ideal. The output V_0 of the circuit (in m V) is

(A) $10\cos(100\,t)$
(B) $10\int_0^t \cos(100\tau)d\tau$
(C) $10^{-4}\int_0^t \cos(100\tau)d\tau$
(D) $10^{-4}\dfrac{d}{dt}\cos(100t)$

6. The current through diode D_1 is [assume the diodes are ideal]

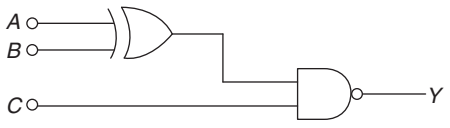


(A) 4 mA
(B) 6 mA
(C) 0 mA
(D) –2 mA

7. An amplifier circuit has an overall current gain of 100 and an input resistance of 20 kΩ with a load resistance of 2 kΩ. The overall voltage gain of the amplifier is _____.
(A) 2 dB
(B) 10 dB
(C) 20 dB
(D) 40 dB

8. The resolution of a 4-bit counting ADC is 0.5. For an analog input of 5.8 V, the digital output of the ADC will be _____.
(A) 1011
(B) 1100
(C) 1101
(D) 1110

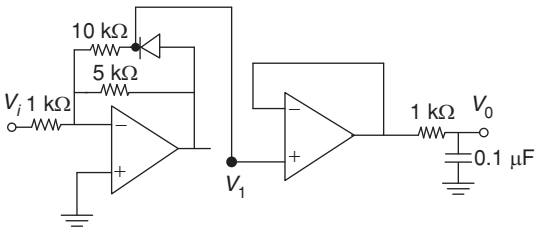
9. The Boolean expression for the output of the logic circuit shown in the following figure is



(A) $Y = \overline{A}\overline{B} + AB + \overline{C}$
(B) $Y = \overline{AB} + AB + \overline{C}$
(C) $Y = \overline{A}B + A\overline{B} + C$
(D) $Y = \overline{A}B + \overline{AB} + \overline{C}$

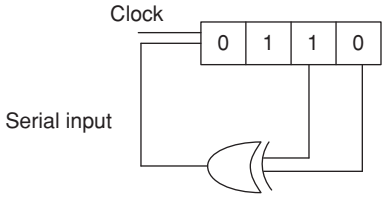
10. The number of memory chips of size 1 k × 4-bits required to build a memory bank of size 16 k × 8-bits is
(A) 64
(B) 32
(C) 16
(D) 8

11. The input voltage V_i in the circuit is a 2 kHz sine wave of 1 V amplitude. Assuming ideal operational amplifiers with ±15 V DC supply the average value of output voltage V_0 is (assume ideal diode)



(A) 3.33 V (B) 1.06 V
(C) 2.35 V (D) 4.71 V

12.



Initial contents of the 4-bit serial-in-parallel out right shift, shift register as shown in figure are 0110. After three clock pulses, the contents of the shift register will be

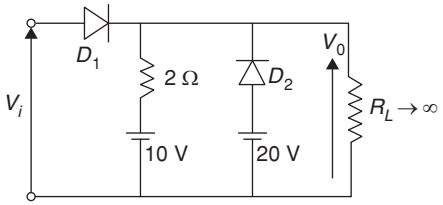
(A) 0000 (B) 0101
(C) 1010 (D) 1110

13. In an 8085 microprocessor, the contents of the accumulator, after following instructions are executed, will become

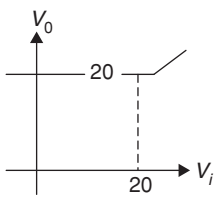
XRA	A
MVI	B, F0 H
SUB	B

(A) 01 H (B) 0F H
(C) F0 H (D) 10 H

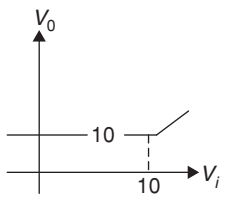
14. Assuming diodes D_1 and D_2 of the circuit shown in figure to be ideal ones, the transfer characteristics of the circuit will be



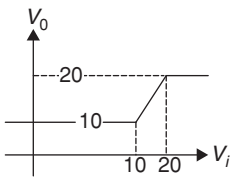
(A)



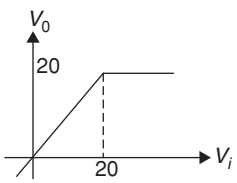
(B)



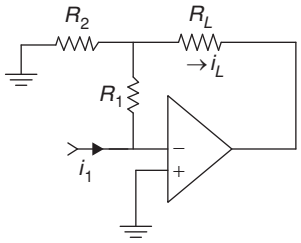
(C)



(D)



15. The below circuit shows a current controlled current source. The expression for $\frac{i_L}{i_1}$ is



(A) $\left(1+\frac{R_1}{R_2}\right)$

(B) $\left(1+\frac{R_L}{R_2}\right)$

(C) $\left(1+\frac{R_1+R_2}{R_L}\right)$

(D) $\frac{R_1}{R_2}$

16. The following program is written for an 8085 microprocessor to add two bytes, located at memory addresses 1 FFE and 1 FFF

```
LXI  H, 1FFE
MOV B, M
INR L
MOV A, M
ADD B
INR L
MOV M, A
XOR A
```

On completion of the execution of the program, the result of addition is found.

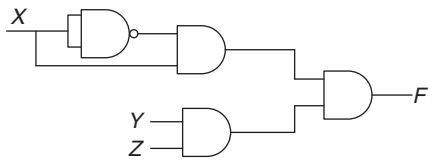
(A) in the register *A*
(B) at the memory address 1000
(C) at the memory address 1F00
(D) at the memory address 2000

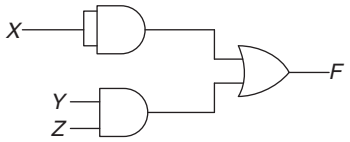
17. The following Karnaugh map represents a function *F*.

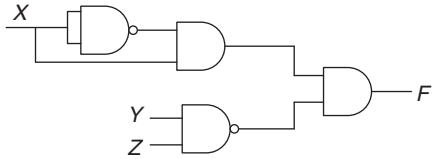
F

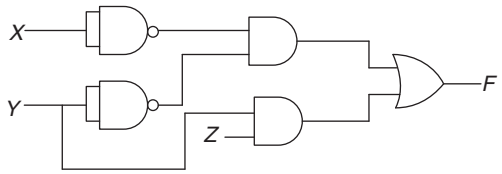
YZ / X	00	01	11	10
0	1	1	1	0
1	0	0	1	0

Which of the following circuits is a realization of the above function F?

(A)


(B)


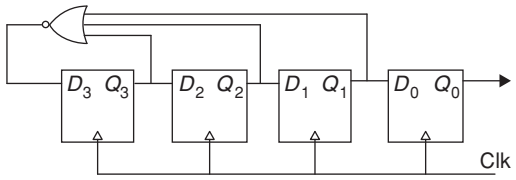
(C)


(D)


18. If $X = 1$ in the logic equation
$[X + Z\{\overline{Y} + (\overline{Z} + X\overline{Y})\}]\{\overline{X} + \overline{Z}(X + Y)\} = 1$, then,

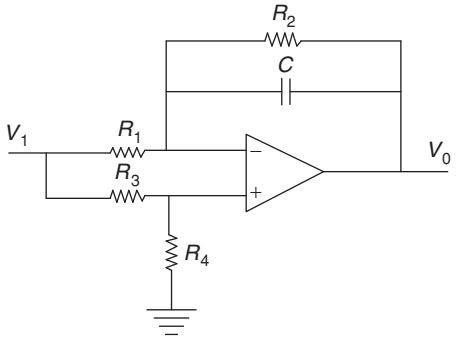
(A) $Y = Z$ (B) $Y = \overline{Z}$
(C) $Z = 1$ (D) $Z = 0$

19. For the ring counter in the figure, the initial state of the counter is 1110 (i.e., Q_3, Q_2, Q_1, $Q_0 = 1110$). The MOD number of the counter is



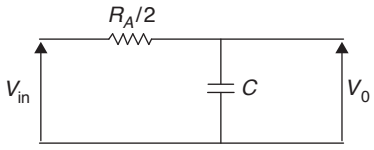
(A) 2 (B) 4
(C) 7 (D) 8

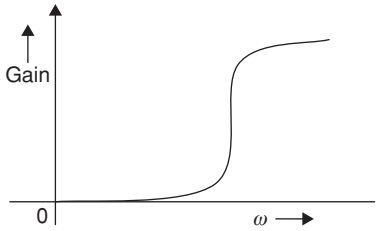
Common Data for Questions 20 and 21: A general filter circuit is shown in the following figure:

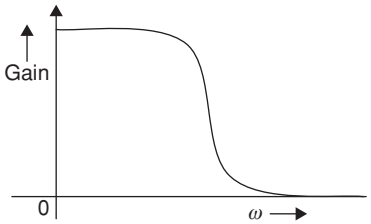


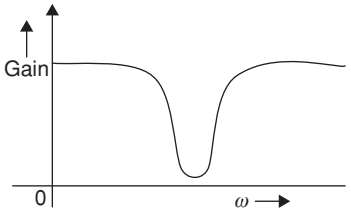
20. If $R_1 = R_2 = R_A$ and $R_3 = R_4 = R_B$, the circuit acts as
(A) all pass filter (B) band pass filter
(C) high pass filter (D) low pass filter

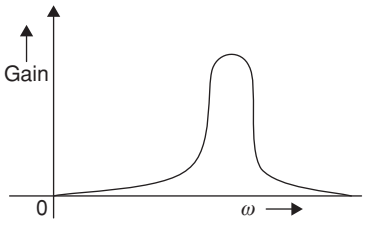
21. If the output of the above filter is given to the circuit shown below, the gain *Vs* frequency characteristic of the output V_0 will be

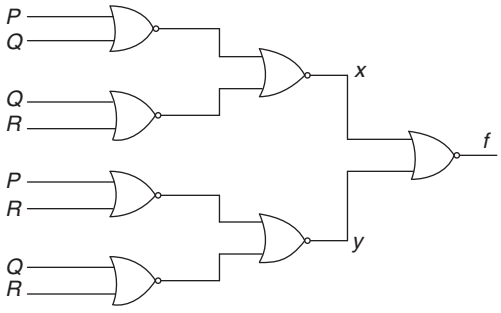


(A)


(B)


(C)


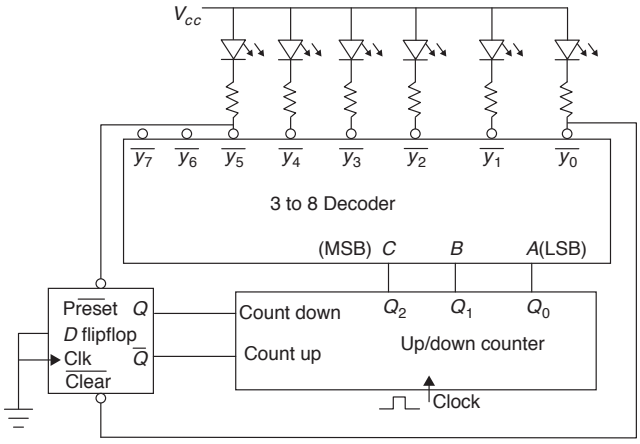
(D)


22.


What is the Boolean expression for the output f of the combinational logic circuit of NOR gates given above?

(A) $\overline{Q + R}$ (B) $\overline{P + Q}$
(C) $\overline{P + R}$ (D) $\overline{P + Q + R}$

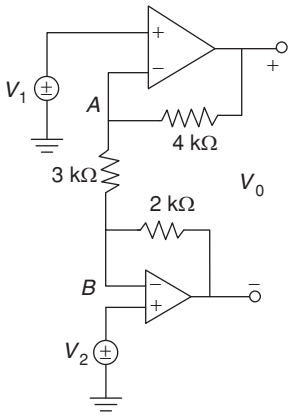
23. The circuit below shows an up/down counter working with a decoder and a flip flop. Preset and clear of the flip flop are asynchronous active low inputs.



Assume that initial values of counter output $(Q_2Q_1Q_0)$ as zeros. The counter output in decimal for next 12 clock cycles are

(A) 0, 1, 2, 3, 4, 4, 3, 2, 1, 1, 2, 3, 4
(B) 0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, 0
(C) 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, 0, 0
(D) 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0, 1, 2

24. The gain of the amplifier $A_V = \dfrac{V_0}{(V_1 - V_2)}$ is?



(A) –9 (B) +9
(C) –6 (D) +3

25. A BJT is specified to have a maximum power dissipation P_{D0} of 5 W at an ambient temperature of 25°C and a maximum junction temperature of 150°C. Then how much power can be safely dissipated at an ambient temperature of 50°C.

(A) 6 W (B) 5 W
(C) 4 W (D) 3 W

Answer Keys

1. B	**2.** A	**3.** A	**4.** A	**5.** A	**6.** C	**7.** C	**8.** B	**9.** A	**10.** B
11. B	**12.** C	**13.** D	**14.** C	**15.** A	**16.** C	**17.** D	**18.** D	**19.** B	**20.** C
21. D	**22.** A	**23.** D	**24.** D	**25.** C					

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Electric Circuits and Fields

UNIT III

Chapter 1:	Network Elements and Basic Laws	3.361
Chapter 2:	Network Theorems	3.387
Chapter 3:	Transient Analysis (AC and DC)	3.409
Chapter 4:	Two Port Networks	3.440
Chapter 5:	Resonance	3.471
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Chapter 1

Network Elements and Basic Laws

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Basic concepts
- Basic quantities
- Classification of network elements
- Independent sources
- Network terminology
- DC network
- AC network
- Kirchhoff's laws
- Circuit elements are connected in parallel
- Nodal analysis
- Mesh analysis

Basic Concepts

The most basic quantity used in the analysis of electrical circuits is the electric charge.

Basic Quantities

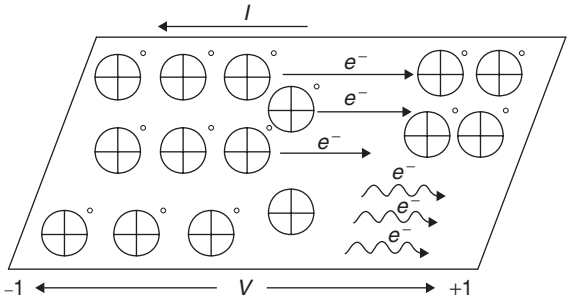
Electron

Electron is a mobile charge carrier. The electron (e^-) is measured in coulombs (C). $1\ e^- = 1.6 \times 10^{-19}$ C

- Multiples of electrons constitute charge (q)
- The movement of charge (q) over time causes current.

Current

There are free electrons available in all semi-conductive and conductive materials. These free e^-'s move at random in all directions with in the structure in the absence of external pressure or voltage. If a certain amount of voltage is applied across the material, all the free e^-'s move in one direction depending on the polarity of the applied voltage.



The voltage

According to the structure of an atom, there are two types of charges; positive and negative charge. A force of attraction exists between these charges. A certain amount of energy is required to overcome the force and move the charges through a specific distance. All opposite charges possess a certain amount of potential energy because of the separation between them. The difference in potential energy of the charges is called the potential difference.

The potential difference in electrical terminology is known as voltage and is denoted by V. It is expressed in terms of energy (W) per unit charges (Q)

$$\therefore \quad V = \frac{W}{Q} \text{ or } \upsilon = \frac{dW}{dQ}$$

The voltage is defined as the work (or) energy required to move a unit charge through an element.

The time rate of change of charge produces an electrical current

$$i(t) = \frac{dq(t)}{dt}$$

The electric current is measured in Ampere (A).

$$1\text{ A} = \frac{1\text{ C}}{1\text{ sec}}$$

Power and energy

Energy is the capacity for doing work, i.e., energy is nothing but stored work.

Energy can be expressed as,

$$W(t) = \int_{t_1}^{t_2} p(t) \cdot dt = \int_{at_1}^{t_2} v(t) \cdot i(t) \cdot dt$$

Power is the rate of change of energy and is denoted by P. If W amount of energy is used over a t amount of time, then

$$\text{Power } (P) = \frac{\text{Energy}}{\text{Time}} = \frac{W}{t} = \frac{dW}{dt'}$$

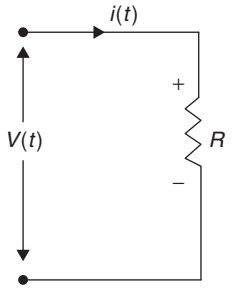
$$P = \frac{dW}{dt} = \frac{dW}{dq} \times \frac{dq}{dt}$$

$$\therefore P = V \cdot I \text{ W.}$$

$\Rightarrow$ One watt is the amount of power generated when one Joule of energy is consumed in one second.

Positive sign convention

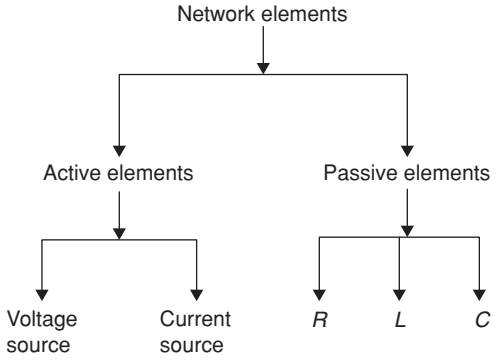
Current flows from the positive to the negative terminals.



$\Rightarrow$ Power is absorbed by elements if the sign of power is positive. That is, current enters from the +ve terminal and leaving from –ve terminal of the element. Power is supplied or delivered by element or source if the sign of power is negative i.e., current enters from the –ve terminal of the element.

Classification of Network Elements

The network elements can be classified as follows:

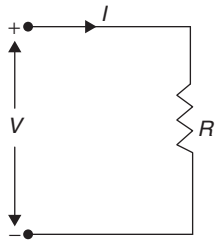


Where, $R \rightarrow$ resistance
$L \rightarrow$ inductance and
$C \rightarrow$ capacitance

Circuit elements: The basic elements of circuits are resistance, inductance, and capacitance.

Resistance (R)

Electrical resistance is the property of material. It opposes the flow of electrons through the material. Thus, resistance restricts the flow of current through the material.



The unit of resistance (R) is 'ohm' (Ω).

According to Ohm's law,

$$J = \sigma E$$

$$J = \frac{I}{A} \text{ and electric field } E = \frac{V}{\ell}$$

$$\frac{I}{A} = \sigma \times \frac{V}{\ell} \quad \Rightarrow \quad V = \frac{\ell}{\sigma} * \frac{I}{A}$$

$$V = \frac{\rho \ell}{A} \cdot I$$

$$\therefore V = R \cdot I \Rightarrow$$

Ohm's law in circuit theory

$$\therefore \quad R = \frac{\rho \ell}{A} \Omega$$

Where,

$\rho \rightarrow$ Resistivity (Ω–m)
$\ell \rightarrow$ Length of conductor (m)
A$\rightarrow$ cross sectional area (m^2)

When current flows through any conductor, heat is generated due to collision of free electrons with atoms.

The power absorbed by the resistor is given by:

$$P = V \cdot I = I^2 R = \frac{V^2}{R} \text{ W}$$

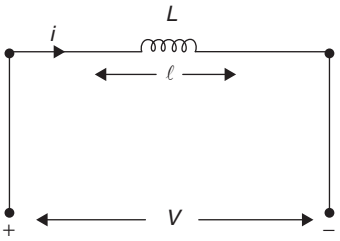
$$P = \frac{dW}{dt}$$

$$dW = P \cdot dt$$

$$W = \int_0^t P \cdot dt = \int_0^t V \cdot I dt = \frac{V^2 t}{R} \text{ J.}$$

Inductance (L)

A wire of finite length, when twisted into a coil, it becomes an inductor.



When a time varying current is flowing through the coil, magnetic flux will be produced.

The total flux $\psi(t) = N\phi$

$\therefore \quad \psi(t) = N \cdot \phi(t) wb$

$$V = \frac{d\psi(t)}{dt}$$

$$\psi(t) = L \cdot i$$

$$V = \frac{d}{dt}\{L \cdot i\}$$

$$V_L = L \cdot \frac{di_L(t)}{dt} \text{ V}$$

$$\Rightarrow \quad i_L = \frac{1}{L}\int_{-\infty}^{t} V \cdot dt \text{ A}$$

$\Rightarrow$ power $P = V \cdot i.$

$$W = \int P \cdot dt$$

$$W = \int L \cdot i \cdot di$$

$$\Rightarrow \quad \boxed{W = \frac{1}{2} Li^2 \text{ J.}}$$

$\therefore$ The energy stored in the inductor at any instant will depend on the current flow through the inductor at that instant.

$$L = \frac{\mu_0 N^2 A}{\ell}$$

Where,

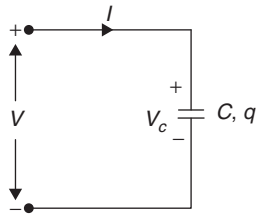
$\ell \rightarrow$ Length of the inductor

$N \rightarrow$ Number of turns

$A \rightarrow$ Cross-sectional area of coil

Capacitance (C)

It is the capability of an element to store electric charge within it



$$i = \frac{dq}{dt} \text{ A.}$$

$q \propto V \quad \Rightarrow \quad q = CV$

$$C = \frac{q}{V} \text{ F}$$

$$\Rightarrow \quad i_c = \frac{d}{dt} CV$$

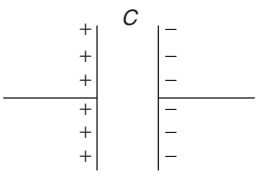
$$\boxed{\Rightarrow \quad i_c = C\frac{dV}{dt} \text{ A}}$$

$$V_c = \frac{1}{C}\int i_c dt \text{ V}$$

$$\Rightarrow P = V i_c$$

$$\boxed{W_c = \int P dt = \frac{1}{2} CV^2 \text{ J}}$$

For parallel plate capacitance



$$\boxed{C = \frac{\varepsilon A}{d} \text{ F}}$$

Where,

$\varepsilon \rightarrow$ permittivity of material

$A \rightarrow$ cross-sectional area of plates

$d \rightarrow$ distance between plates.

Table 1 *Summary of relationships for the parameters*

Parameter	Basic Relationship	Voltage-current Relationships	Energy
R $G = \frac{1}{R}$	$V = iR$	$V_R = i_R \cdot R, i_R = G \cdot V_R$	$W_R = \int_{-\infty}^{t} p \cdot dt; W_R = \int_{-\infty}^{T} V_R \cdot i_R dt$
L	$\Psi = L \cdot i$	$V_L = L \cdot \frac{di}{dt}; i_L = \frac{1}{L}\int_{-\infty}^{t} V_L dt$	$W_L = \frac{1}{2} Li^2 = \frac{1}{2} \psi \cdot i$
C	$q = CV$	$V_C = \frac{1}{C}\int_{-\infty}^{t} i_c \cdot dt; i_C = C \cdot \frac{dV_C}{dt}$	$W_C = \frac{1}{2} CV^2 = \frac{1}{2} q \cdot V$

Note: *R*, *L*, and C elements are a linear, passive, bilateral and time-invariant, at a constant temperature.

Solved Example

Example 1: In the interval $0 > t > 4\pi$ ms, a 10 μF capacitance has a voltage $V = 25 \sin 200t$ V. The charge, power, and energy are:

Solution:

Charge $q = C \cdot V$

$= 10 \times 25 \sin 200t$ (μC)

$= 250 \text{ Sin } 200t$ (μC)

Power, $P = V \cdot i$

But, $i_c = C \cdot \dfrac{dV_c}{dt}$

$= 10 \times 25 \times 200 \cos 200t$ μA

$i_c = 50 \cos 200t$ mA

$\therefore \quad P = V \cdot i = 25 \sin 200t \cdot 50 \cos 200t$ mW

$= 25 \times 25 \sin 400t$ mW

$= 0.625 \sin 400t$ W

Energy, $W_c = \int_{t1}^{t2} P \cdot dt$

$$W_c = \int_0^{4\pi \times 10^{-3}} 0.625 \sin 400t \, dt$$

$$= -\frac{0.625}{400}[\cos 400t]_0^{4\pi \times 10^{-3}}$$

$= 1.5$ nJ.

Example 2: A capacitor of 100 μF stores 10 mJ of energy. Obtain the amount of charge stored in it. How much time does it take to build up this charge if the charging current is 0.2A?

(A) $Q = 1.414$ mC, $t = 1.4$ ms

(B) $Q = 1.414$ mC, $t = 7.07$ ms

(C) $Q = 2$ μC, $t = 0.1$ ms

(D) $Q = 2$ m*C*, $t = 10$ ms

Solution: (B)

The energy stored in a capacitor is given by,

$$W_c = \frac{CV^2}{2} = \frac{1}{2}QV = \frac{Q^2}{2C}$$

$$\therefore \quad Q = \sqrt{2CW} = \sqrt{2 \times 10 \times 10^{-3} \times 100 \times 10^{-6}}$$

$= 1.414\ mC$. We know, $i = \dfrac{dq}{dt}$

$\therefore \ Q = I \cdot t$

$$t = \frac{Q}{I} = \frac{1.414 \times 10^{-3}}{0.2} = 7.07 \text{ ms.}$$

Example 3: The strength of current in 2 H inductor changes at a rate of 3 A/s. The voltage across it and the magnitude of energy stored in an inductor after 4 seconds are

(A) $V_L = 6$ V, $W_L = 144$ J

(B) $V_L = 1.5$ V, $W_L = 12$ J

(C) $V_L = 6$ V, $W_L = 77$ J

(D) $V_L = 1.5$ V, $W_L = 144$ J

Solution: (A)

From the given data

$$L = 2H, \frac{di_L}{dt} = 3 \text{ A/sec}$$

$$V_L = L \cdot \frac{di_L}{dt} = 2 \times 3 = 6 \text{ V.}$$

$$W = \frac{1}{2} L \cdot i^2$$

$$\frac{di}{dt} = 3 \text{ A/sec}$$

$$di = 3\, dt$$

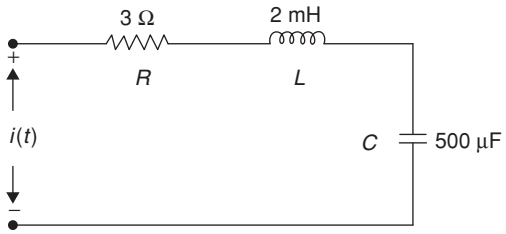
$I = 3t$ A/sec But $t = 4$ sec

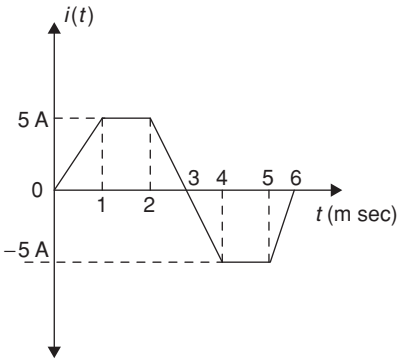
$I = 3 \times 4 = 12$ A

$$\text{W} = \frac{1}{2} \times 2 \times (12)^2$$

$= 144$ J.

Example 4: A current source $i(t)$ is applied to a series RLC circuit shown in below figures





The maximum voltage across the resistor is

(A) 10 V (B) 15 V

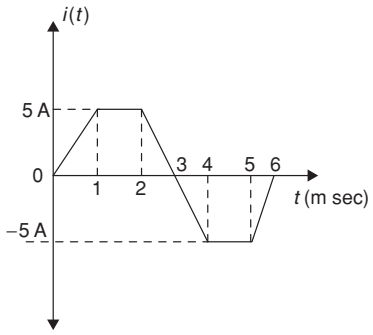
(C) 5 V (D) 20 V

Solution: (B)

The drop across resistance $V_R(t) = i(t)\, R$

$$V_{R(\max)} = i_{\max} \cdot R \text{ V}$$

$V(t) = 5 \times 3 = 15$ V



Example 5: The total voltage across indicator is

(A) 0 V (B) –10 V

(C) 10 V (D) None of these

Solution: (C)

$$V_L = L \cdot \frac{di(t)}{dt}$$

(i) For $0 \le t \le 1$ ms

$$i(t) = \frac{5}{1\times10^{-3}} t \text{ A} = 5\times10^{3} t \text{ A}$$

$$V_L = 2 \times 10^{-3} \times 5 \times 10^{3} = 10 \text{ V}$$

(ii) For 1 ms $\le t \le$ 2 ms:

$i(t) = 5$ A constant

$\therefore V_L = 0$ V

(iii) For $2 \le t \le 4$ ms:

$A(2 \times 10^{-3}, 5)$ $B(4 \times 10^{-3}, -5)$

$$I(t) = \frac{-5-5}{2\times10^{-3}}(t - 2\times10^{-3}) + 5$$

$$i(t) = 5 - 5 \times 10^{3} (t - 2 \times 10^{-3}) \text{ A}$$

$$= 15 - 5 \times 10^{3}\, t \text{ A}$$

$$V_L = L$$

$$\frac{di(t)}{dt} = 2\times10^{-3}[0 - 5\times10^{3}] = -10 \text{ V}$$

(iv) For 4 ms $\le t \le$ 5 ms:

$i(t) = 5$ and $\Rightarrow$ constants

so $V_L = 0$

(v) For 5 ms $\le t \le$ 6 ms:

$A\ (5 \times 10^{-3}, -5)$ $B\ (6 \times 10^{-3}, 0)$

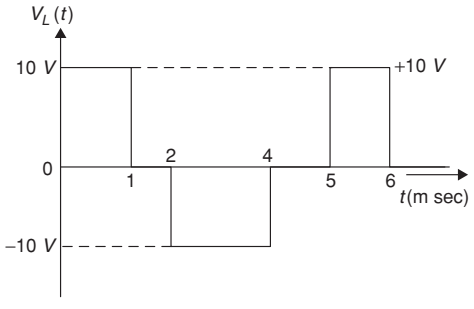
$i_L(t) = 5 \times 10^{3}\, t$ A

$$V_L(t) = L \frac{di_{L(t)}}{dt}$$

$$= 2 \times 10^{-3} \times 5 \times 10^{3}$$

$$= 10 \text{ V}$$

$\therefore V_L(t)$ shows below



$$\Rightarrow \quad \sum V_L = 10 + 0 + (-10) + 0 + 10$$

$$V_L = 10 \text{ V}$$

Example 6: A voltage $V(t) = 2 \sin \omega t$ V is applied across a capacitor having time varying capacitance given by $C(t) = (2 + 0.5 \sin t)$ F. Find $i(t) = ?$

Solution: $Q = C \cdot V \quad \Rightarrow \quad q(t) = C(t) \times V(t)$

$$i(t) = \frac{dq(t)}{dt}$$

$$i(t) = \frac{d}{dt}\{(2 + 0.5\sin t) \times 2 \sin \omega t\}$$

$$= \frac{d}{dt}\{4 \sin \omega t + \sin t \times \sin \omega t\}$$

$$= 4 \cos \omega t \times \omega + \sin t \times \cos \omega t \times \omega + \sin \omega t \times \cos t$$

$$I(t) = \sin \omega t \times \cos t + \omega\,[4 + \sin t] \cos \omega t$$

Example 7: If $\omega = 2$ rad/sec, the value of $i(t)$ at $i(t)$ at $t = \frac{1}{2}$ sec in the above problem is,

(A) 2 A (B) 4 A

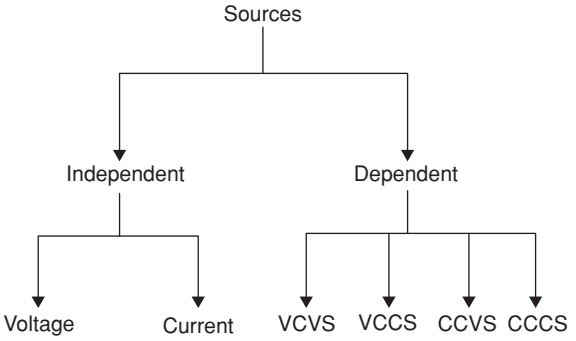
(C) 6 A (D) 8 A

Solution: $i(t) = [4 + \sin t]\ \omega \cos \omega t + \sin \omega t \cdot \cos t$

$= [4 + \sin 0.5\}\ 2 \times \cos 1 + \sin 1 \times \cos 0.5$

$= 8.016 + 0.174 = 8.033 \approx 8$ A.

Energy Sources



Independent Sources

Ideal voltage source

Terminal voltage of an ideal voltage source is independent of the current supplied by it. Internal resistance of an ideal voltage source is zero.

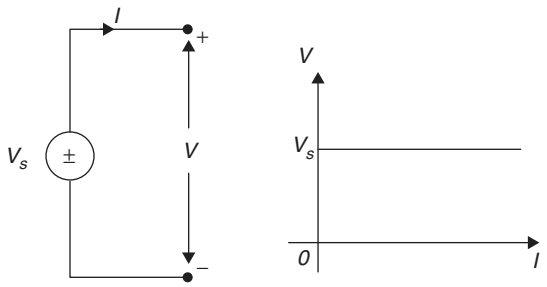


Figure 1 Ideal voltage source and *V–I* characteristics

Practical voltage source

Practical voltage source has some finite internal resistance. Due to the presence of an internal resistance, the terminal voltage of a practical voltage source reduces with the increase in current supplied.

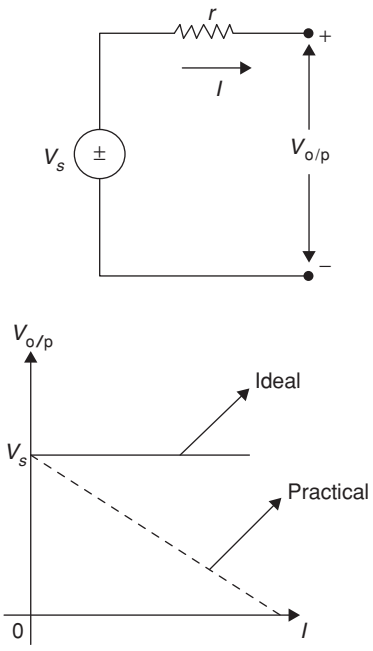


Figure 2 Practical voltage Source and *V–I* characteristics

By KVL

$$V_s - I \cdot r = V_{o/p}$$
$$V_{o/p} = V_s - I \cdot r$$

When current through any element is zero, then the potential difference is also zero.

Ideal current source

Current delivered by any ideal current source is independent of voltage across its terminals. Internal resistance of an ideal current source is infinite.

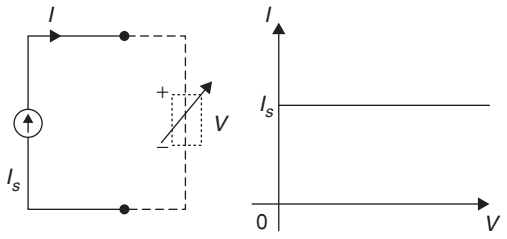
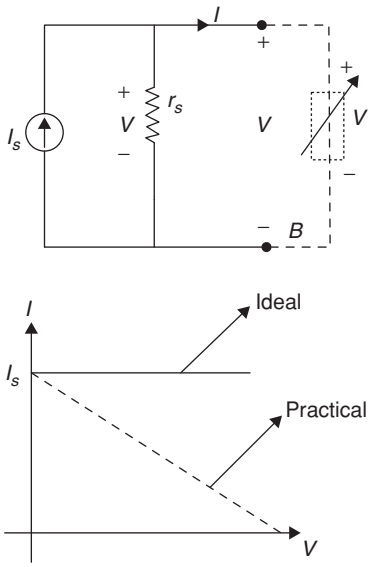


Figure 3 Ideal current source and *I* – *V* characteristics

That is, $I = I_s$ for all V.

Practical current source

Practical current source has some finite internal resistance. Due to the presence of an internal resistance, current delivered by the practical current source reduces with increase in its terminal voltage.



By KCL

$$I_s = \frac{V}{r_s} + I$$

$$\boxed{I = I_s - \frac{V}{r_s}}$$

→ current always choose minimum resistance path.

Dependent Sources

Voltage controlled voltage source (VCVS)

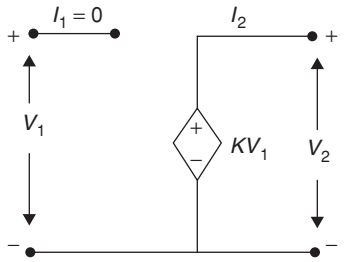
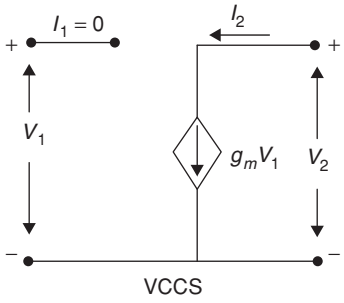


Figure 4 VCVS

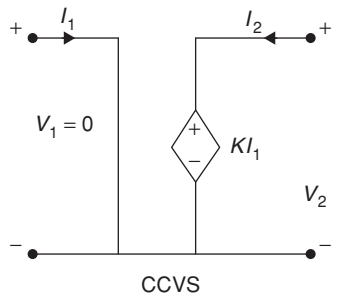
Voltage controlled current source (VCCS)



That is, $I_2 = g_m V_1$

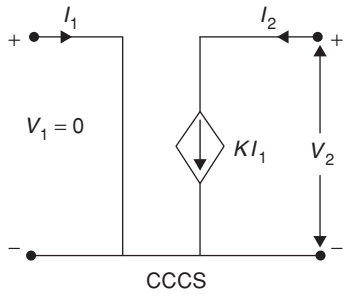
Because, o/p current depends on the i/p voltage

Current controlled voltage source



i.e., $V_2 = KI_1$

Current controlled current source



$$\therefore \quad I_2 = KI_1$$

Because o/p current depends on the i/p current so it is called current controlled current source

Where, $K \Rightarrow$ constant.

Basic Definitions

Network Terminology

In this section some of the basic terms which are commonly associated with a network are defined.

Network element

Network elements can be either active elements (or) passive elements.

Active elements $\Rightarrow$ which supply power or energy to the network (outside world).

Example: Voltage source and current source.

Passive elements $\Rightarrow$ which either store the energy or dissipate energy in the form of heat

Example: R, L and C

Branch

A part of the network which connects the various points of the network with one another is called a branch

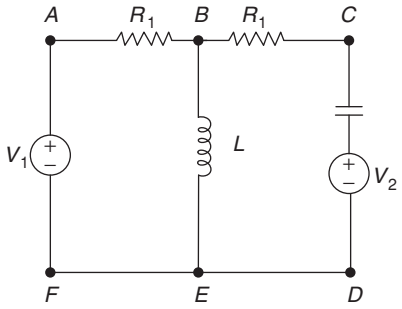
Node

A point at which two or more elements are connected together is called node. The junction points are also the nodes of network.

Mesh or Loop

Mesh is a set of branches forming a closed path in a network

Example:



Branches $\Rightarrow A - B, B - E, \ldots$ etc.

Nodes $\Rightarrow A, B, C, \ldots$ etc.

Mesh $\Rightarrow ABEFA, ABCDEFA, BCDEB$

Types of Elements

1. Linear and Non-linear
2. Active and passive
3. Bilateral and unilateral
4. Distributed and lumped
5. Time invariant and time variant

Linear and non-linear

A two terminal element is said to be linear for all time 't', if its characteristics is a straight line through the origin, otherwise it is non-linear.

Example: A linear element must satisfy superposition and homogeneity principles.

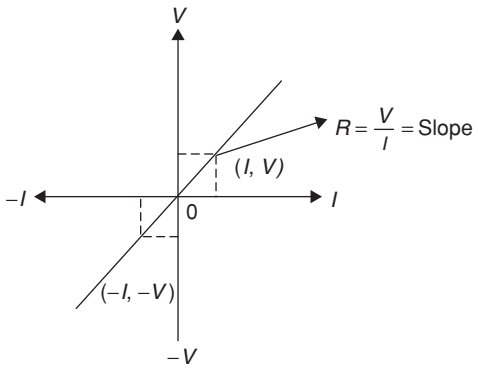


Figure 5 A bilateral, linear characteristics

Active network

A circuit which contains at least one source of energy is called Active. An energy source may be a voltage or current.

An element is said to be active if it deliveres a net amount of energy to the outside world is non-zero.

Example: Transistors, op-amps, batteries, … etc.

Passive network

A circuit which contains no energy source is called passive circuit. These networks consists of passive elements only.

Example: R, L, C and thermistors ... etc.

Bilateral and unilateral networks

A circuit whose characteristics, behaviors is same irrespective of the direction of current through various elements it is called bilateral network. Otherwise it is said to be unilateral.

Example: Diode (Unilateral)
Resistors (Bilateral)

Lumped and distributed networks

A network in which all the network elements are physically separable is known as lumped network.

Example: simple RLC circuits.
Otherwise it is called distributed network

Example: Transmission lines.

DC network A network consist of DC sources which are fixed polarity sources with time invariant is called a DC network.

AC network A network consist of AC sources which are alternating sources and periodically varying with time, is called an AC network.

Kirchhoff's Laws

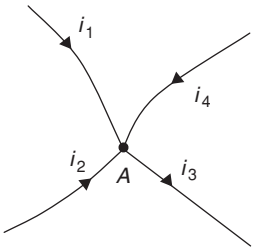
In 1847, a German physicist, Kirchhoff, formulated two fundamental Laws as given below:

1. Kirchhoff's Current law (KCL)
2. Kirchhoff's Voltage law (KVL)

Kirchhoff's current law (KCL)

In any network, the algebraic sum of currents meeting at a point or node is always zero i.e., the total current leaving a junction is equal to the total current entering that junction.

Example:



Assume currents entering a junction is positive and currents leaving away from the junction is negative. (vice-versa)

Apply KCL at node A

By KCL $\Rightarrow \Sigma$ Leaving currents = 0

$$i_1 + i_2 + i_4 - i_3 = 0 \Rightarrow i_3 = i_1 + i_2 + i_4.$$

we know, $i = \dfrac{dq}{dt}$

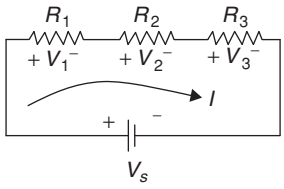
$$\frac{dq_3}{dt} = \frac{dq_1}{dt} + \frac{dq_2}{dt} + \frac{dq_4}{dt}$$

$$\boxed{q_3 = q_1 + q_2 + q_4}$$

$\therefore$ sum of entering charges is equal to sum of leaving charges, KCL is based on the principle of law of conservation of charge.

Kirchoff's voltage law (KVL)

The algebraic sum of all branch voltages around any closed path is always zero at all instants of time.



Apply KVL to the above circuit

$$-V_1 - V_2 - V_3 + V_S = 0$$

or

$$V_S = V_1 + V_2 + V_3$$

$$V_S = IR_1 + IR_2 + IR_3$$

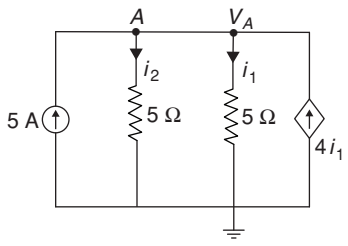
$$V_S = I(R_1 + R_2 + R_3)$$

Properties

1. KCL and KVL can be applied to any lumped electric circuits.
2. KCL expresses conservation of charge at each and every node. And KVL expresses conservation of flux or energy in every loop of electric circuit.

Note: Ohm's law is not applicable for active elements. It is applicable only for linear, passive elements.

Example 8:



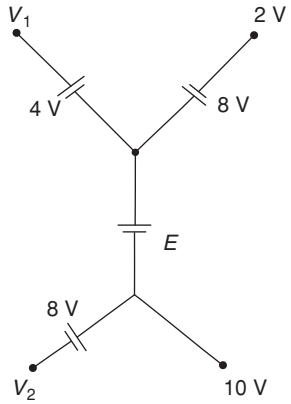
The values of i_1 and i_2 are respectively,

(A) $i_1 = -10$ A, $i_2 = 5$ A
(B) $i_1 = i_2 = -2.5$ A
(C) $i_1 = 2.5$ A and $i_2 = -2.5$ A
(D) None of the above

Solution: (B)
Applying KCL at node 'A'

$$5 + 4i_1 = i_1 + i_2$$
$$3i_1 - i_2 + 5 = 0$$
$$3\left[\frac{V_A}{5}\right] - \frac{V_A}{5} + 5 = 0$$
$$\frac{2V_A}{5} + 5 = 0$$
$$V_A = -12.5 \text{ V}$$
$$i_1 = -\frac{12.5}{5} = -2.5 \text{ A}$$
$$i_2 = i_1 = -2.5 \text{ A}$$

Example 9: Consider the following circuit



The node voltages V_1, V_2 and E are respectively,
(A) $V_1 = -14$ V, $V_2 = 18$ V, $E = -2$ V
(B) $V_1 = +14$ V, $V_2 = -2$ V, $E = -2$ V
(C) $V_1 = +14$ V, $V_2 = 2$ V, $E = 0$ V
(D) $V_1 = -14$ V, $V_2 = -2$ V, $E = 0$ V

Solution: (C)
Apply KVL in loop1

$$V_2 + 8 - 10 = 0$$
$$V_2 = 2 \text{ V}$$

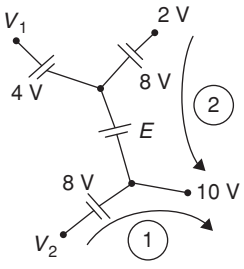
Apply KVL in loop 2

$$2 + 8 - E - 10 = 0$$
$$E = 0 \text{ V}$$

Apply KVL in Loop 3

$$V_1 - 4 - 0 - 10 = 0$$
$$V_1 = 14 \text{ V}$$

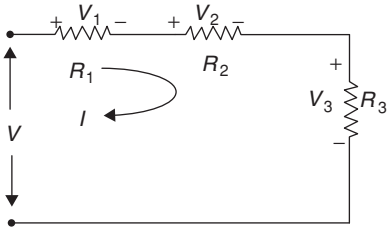
$\therefore$ $V_1 = 14$ V, $V_2 = 2$ V and $E = 0$ V



Circuit Elements in Series

When sending end of an element is connected to receiving end of another element and no other element is connected at that node then those two elements are said to be connected in series. Current flowing through series connected elements is equal.

Series Connected Resistors



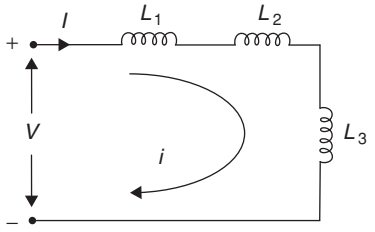
In the above network R_1, R_2, R_3 are connected in series. By apply KVL for the loop,

$$V = V_1 + V_2 + V_3$$
$$= IR_1 + IR_2 + IR_3$$
$$= [R_1 + R_2 + R_3] \cdot I$$
But $V = I.\ R_{eq}$.
$$\therefore I \cdot R_{eq} = I\,[R_1 + R_2 + R_3]$$

$$\boxed{R_{eq} = R_1 + R_2 + R_3 + \cdots R_n}$$

Series Connected Inductors

L_1, L_2, L_3 are the three inductances connected in series as shown in the fiure.



Apply KVL to the circuit,

$$V = L_1 \frac{di}{dt} + L_2 \cdot \frac{di}{dt} + L_3 \frac{di}{dt}$$
$$= (L_1 + L_2 + L_3)\frac{di}{dt}$$

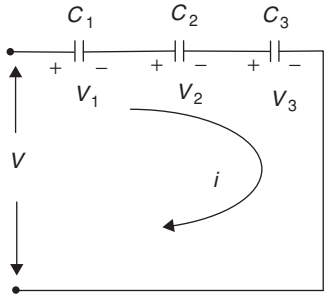
But we know,

$$V_L = L_{eq} \cdot \frac{di}{dt}$$

$$\boxed{\therefore \quad L_{eq} = L_1 + L_2 + L_3}$$

Series Connected Capacitors

If three circuit elements are capacitances connected in series, assuming zero initial charges. Apply KVL for the circuit



$$V = V_1 + V_2 + V_3$$

$$= \frac{1}{C_1}\int idt + \frac{1}{C_2}\cdot\int i\cdot dt + \frac{1}{C_3}\int idt$$

$$= \left[\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right]\int idt$$

$$V = \frac{1}{C_{eq}}\cdot\int idt$$

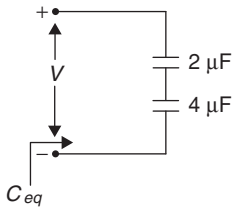
$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}}$$

Example 10: Two capacitors C_1 = 2 μF, C_2 = 4 μF are connected in series. The equivalent capacitance is

(A) 6 μF (B) 8 μF

(C) 2 μF (D) $\frac{4}{3}$ μF

Solution: (D)



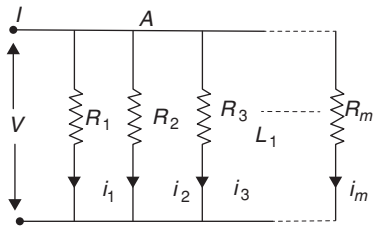
$$\Rightarrow \quad \frac{1}{C_{eq}} = \frac{1}{2} + \frac{1}{4} \quad \Rightarrow \quad C_{eq} = \frac{2\times 4}{2+4} = \frac{8}{6}\,\mu F$$

C_{eq} = 4/3 μF.

Circuit Elements are Connected in Parallel

Resistors in Parallel

Resistors R_1, R_2, R_3…R_n are connected in parallel as shown in the figure update.



Apply KCL at node A,

$$I = i_1 + i_2 + i_3 + \cdots + i_m$$

$$\frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2} + \cdots + \frac{V}{R_m}$$

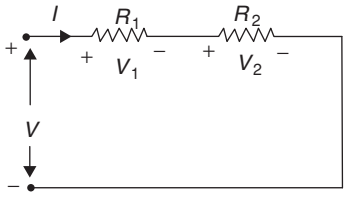
$$\boxed{\frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2} + \cdots + \frac{V}{R_m}}$$

Let $m = 2$

$$\boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

Voltage division

A set of series connected resistors as shown in figure is referred as a voltage divider.



Applied voltage V is divided into V_1 and V_2 across R_1 and R_2 respectively.

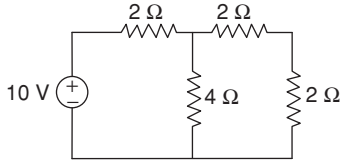
$$I = \frac{V}{R_1 + R_2}$$

$$V_1 = I\cdot R_1\ ;\ V_2 = I\cdot R_2$$

$$\boxed{V_1 = \frac{V}{R_1 + R_2}\cdot R_1\ \text{V}}$$

$$V_2 = \frac{V}{R_1 + R_2}R_2\,\text{V}$$

Example 11:

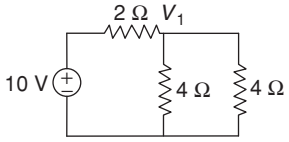


The voltage across 4Ω resistance is

(A) 5 V (B) 2.5 V

(C) 7.5 V (D) None of the above

Solution: (A)

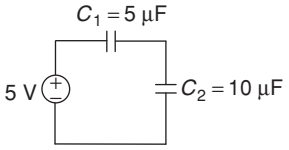


$$\frac{V_1 - 10}{2} + \frac{V_1}{4} + \frac{V_1}{4} = 0$$

$$2(V_1 - 10) + 2V_1 = 4V_1 = 20$$

$$V_1 = 5 \text{ V}.$$

Example 12:



The value of V_{C2} and V_{C1} are

(A) $V_1 = \frac{5}{3}$ V, $V_2 = \frac{10}{3}$ V.

(B) $V_1 = \frac{10}{3}$ V, $V_2 = \frac{5}{3}$ V.

(C) $V_1 = 2.5$ V, $V_2 = 2.5$ V

(D) $V_1 = \frac{3}{5}$ V, $V_2 = \frac{10}{3}$ V.

Solution: (B)

$$V_{c2} = \frac{V}{C_1 + C_2} \times C_1$$

$$= \frac{5}{15} \times 5$$

$$= \frac{5}{3} \text{ V}$$

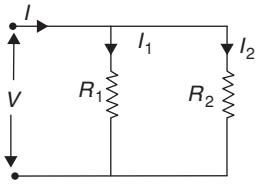
$$V_{c1} = \frac{V}{C_1 + C_2} \times C_2$$

$$= \frac{5}{15} \times 10$$

$$= \frac{10}{3} \text{ V}$$

Current division

A parallel arrangement of resistors shown in figure results in a current divider.

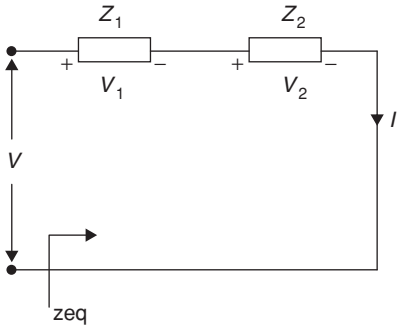


Total current I is divided into I_1 and I_2 through R_1 and R_2 respectively. I_1 and I_2 are expressed as shown below.

$$\boxed{I_1 = \frac{R_2}{R_1 + R_2} \cdot I}$$

$$I_2 = \frac{R_1}{R_1 + R_2} \cdot I$$

Note 1: When two impedances are connected in series, voltage division across those impedances depends on elements in it.



(i) If $Z = R$ or $Z = L$

$$Z_{eq} = Z_1 + Z_2$$

$$V_1 = \frac{V}{Z_{eq}} \times Z_1 \text{ V}$$

$$V_2 = \frac{V}{Z_{eq}} \times Z_2 \text{ V}$$

(ii) If $Z_1 = C_1$ and $Z_2 = C_2$

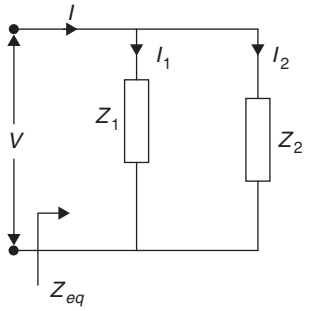
then

$$Z_{eq} = \frac{C_1 \times C_2}{C_1 + C_2}$$

$$\boxed{V_2 = \frac{V}{C_1 + C_2} \times C_1}$$

$$\boxed{V_1 = \frac{V}{C_1 + C_2} \times C_2}$$

Note 2: When two impedances are connected in parallel, current division through those impedances depends on the elements present in it.



$I = I_1 + I_2$.

(i) If $Z_1 = R_1\ \Omega$ and $Z_2 = R_2\ \Omega$ or

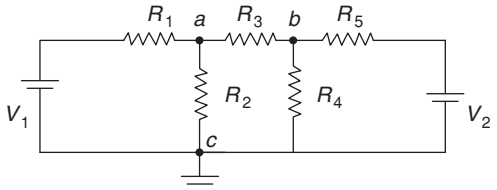
$$Z_{eq} = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2} Z_1 = L_1 \text{ and } Z_2 = L_2$$

(ii) If $Z_1 = C_1$ and $Z_2 = C_2$

Then,

$Z_{eq} = C_1 + C_2$

Nodal Analysis



KCL at node *a*,

$$\frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_b}{R_3} = 0$$

$$V_a\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right] - \frac{V_b}{R_3} = \frac{V_1}{R_1} \quad (1)$$

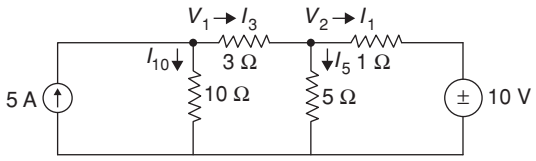
KCL at node *b*,

$$\frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} + \frac{V_b - V_2}{R_5} = 0$$

$$\left(\frac{-1}{R_3}\right)V_a + \left[\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right]V_b = \frac{V_2}{R_5} \quad (2)$$

Solving (1) and (2), the currents can be estimated.

Example 13: Write the node voltage equations and determine the currents in each branch of the network shown in figure



Solution:

KCL at node 1

$$5 = \frac{V_1}{10} + \frac{V_1 - V_2}{3}$$

$$5 = V_1\left[\frac{1}{10} + \frac{1}{3}\right] - V_2\left[\frac{1}{3}\right] \quad (3)$$

KCL at node 2

$$\frac{V_2 - V_1}{3} + \frac{V_2}{5} + \frac{V_2 - 10}{1} = 0$$

$$-V_1\left[\frac{1}{3}\right] + V_2\left[\frac{1}{3} + \frac{1}{5} + 1\right] = 10 \quad (4)$$

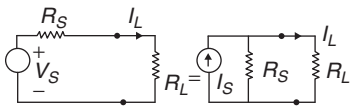
Solving (3) and (4)

$V_1 = 19.85$ V, $V_2 = 10.9$ V

$$I_{10} = \frac{V_1}{10} = 1.985 \text{ A}, I_3 = \frac{V_1 - V_2}{3} = 2.98 \text{ A}$$

$$I_5 = \frac{V_2}{5} = \frac{10.9}{5} = 2.18 \text{ A}, I_1 = \frac{V_2 - 10}{1} = 0.9 \text{ A}$$

Source Transformation



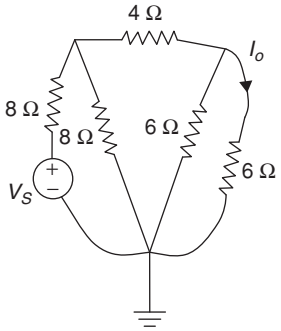
V_S is the voltage and R_S is the series resistance.

$$I_L = \frac{V_S}{R_S + R_L}$$

When transformed to a current source

$$I_S = \frac{V_S}{R_S} \text{ and } I_L = \frac{V_S}{R_S + R_L}$$

Example 14: For the network shown in fig. find V_s when $I_o = 0.5$ A

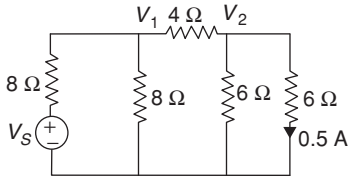


(A) $V_s = 22$ V (B) $V_s = 20$ V

(C) $V_s = -22$ V (D) $V_s = +70$ V

Solution: (A)

Redraw the above network



Given $I_o = 0.5$ A

So $$I_o = \frac{V_2}{6} \Rightarrow V_2 = 6 \times 0.5 = 3 \text{ V}$$

Apply KCL at node V_1

$$\frac{V_1 - V_S}{8} + \frac{V_1}{8} + \frac{V_1 - 3}{4} = 0$$

$$V_1 - V_S + V_1 + 2(V_1 - 3) = 0$$

$$4V_1 - V_S = 6 \quad (5)$$

Apply KCL at node (6)

$$\frac{V_2 - V_1}{4} + \frac{V_2}{3} = 0$$

$$3(V_2 - V_1) + 4V_2 = 0$$

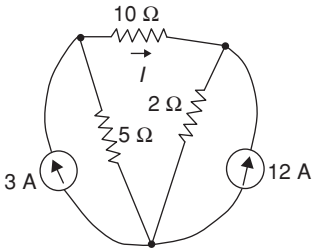
$$7V_2 = 3V_1$$

$$V_1 = \frac{7 \times 3}{3} = 7 \text{ V}$$

$$V_S = 4\ V_1 - 6$$

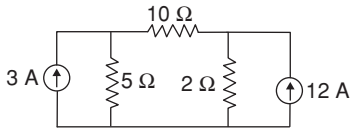
$$= 4 \times 7 - 6 = 22 \text{ V}$$

Examples 15: In the network shown in figure find the current in the 10 Ω resistor

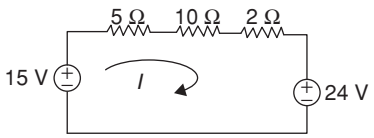


(A) $I = 0.53$ A (B) $I = 9$ A
(C) $I = -1$ A (D) $I = -0.53$ A

Solution: (D)
Redraw the above circuit



Apply source transformation



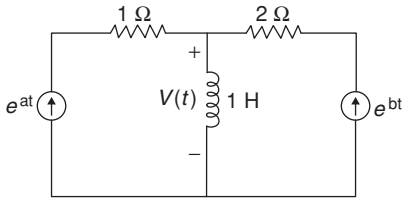
$$I = \frac{15 - 24}{17} = -\frac{9}{17}$$

$$= -0.53 \text{ A}$$

Notes:

1. **Simplenode:** It is an Inter connection of only two branches
2. **Principle node:** It is an inter connection of at least three branches.
3. If a branch between two essential non-reference node contain a voltage source this is called 'super-node'

Example 16: In the circuit given below, the voltage $V(t)$ is,



(A) $e^{at} - e^{bt}$ (B) $e^{at} + e^{bt}$
(C) $a.e^{at} - b.e^{bt}$ (D) $a.e^{at} + b \cdot e^{bt}$

Solution: (D)

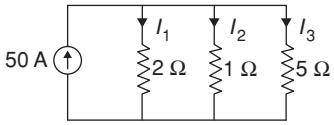
$$V_L = L \cdot \frac{di_L}{dt}$$

$$I_L = e^{at} + e^{bt}$$

$$V_L(t) = 1 \times \frac{d}{dt}[e^{at} + e^{bt}]$$

$$= a.e^{at} + b.e^{bt}.$$

Example 17: Determine the current in all resistors in the circuit shown in figure



Solution:
By KCL

$$I = I_1 + I_2 + I_3$$

$$I = \frac{V}{2} + \frac{V}{1} + \frac{V}{5}$$

$$50 = V\left[\frac{1}{2} + \frac{1}{1} + \frac{1}{5}\right]$$

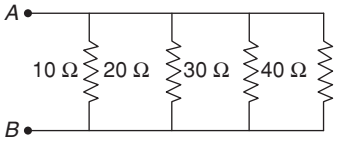
$$V = \frac{50}{1.7} = 29.41 \text{ V}$$

$$I_1 = \frac{29.41}{2} = 14.705 \text{ A}$$

$$I_2 = \frac{29.41}{1} = 29.41 \text{ A}$$

$$I_3 = \frac{29.41}{5} = 5.882 \text{ A}$$

Example 18:



Determine the parallel resistance between points A and B of the circuit shown in figure.

(A) 8 Ω (B) 6 Ω
(C) 4.8 Ω (D) 3 Ω

Solution: (C)

$$\frac{1}{R_T}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}$$

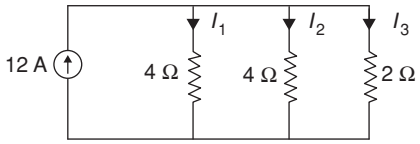
$$=\frac{1}{10}+\frac{1}{20}+\frac{1}{30}+\frac{1}{40}$$

$$=0.1+0.05+0.033+0.025$$

$$\frac{1}{R_T}=0.208$$

$$R_T=4.8\ \Omega$$

Example 19: Determine the current through each resistor in the circuit shown in figure



Solution:

$$I_1=12\times\frac{4\,||\,2}{4\,||\,2+4}$$

$$=12\times\frac{\frac{4\times2}{4+2}}{\frac{4\times2}{4+2}+4}$$

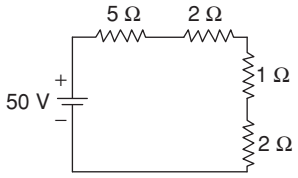
$$=12\times\frac{1.3333}{1.3333+4}=3\text{ A}$$

$$I_2=3\text{ A}$$

$$I_3=12-3-3$$

$$I_3=6\text{ A}$$

Example 20: Determine the total amount of power dissipated in the circuit shown in figure



(A) 100 W (B) 250 W (C) 150 W (D) 200 W

Solution: (B)

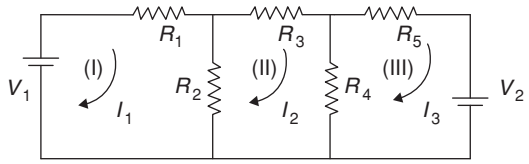
Total resistance, $R = 5 + 2 + 1 + 2 = 10\ \Omega$

$$\text{Power}=\frac{V^2}{R}$$

$$=\frac{(50)^2}{10}$$

$$=250\text{ W}$$

Mesh Analysis



Consider the network shown in figure. V_1 and V_2 are the voltage sources. The loop currents are I_1, I_2 and I_3 in their direction as shown in figure

KVL in loop I

$$-I_1R_1-I_2R_2+V_1=0$$

$$V_1=I_1(R_1+R_2)-I_2R_2 \qquad (7)$$

KVL in loop II ,

$$-I_2R_3-I_2R_4+I_3R_4-I_2R_2+I_1R_2=0$$

$$-I_1R_2+I_2(R_2+R_3+R_4)-I_3R_4=0 \qquad (8)$$

KVL in loop III,

$$-I_3R_5-V_2-I_3R_4+I_2R_4=0$$

$$-I_2R_4+I_3(R_4+R_5)=-V_2 \qquad (9)$$

from (7), (8) and (9)

$$\begin{bmatrix}R_1+R_2 & -R_2 & 0\\ -R_2 & R_2+R_3+R_4 & -R_4\\ 0 & -R_4 & R_4+R_5\end{bmatrix}$$

$$\begin{bmatrix}I_1\\I_2\\I_3\end{bmatrix}=\begin{bmatrix}V_1\\0\\V_2\end{bmatrix}$$

$$I_1=\frac{\begin{vmatrix}V_1 & -R_2 & 0\\ 0 & R_2+R_3+R_4 & -R_4\\ -V_2 & -R_4 & R_4+R_5\end{vmatrix}}{|R|}$$

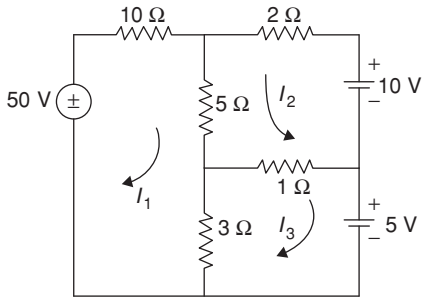
Where,

$$[R]=\begin{bmatrix}R_1+R_2 & -R_2 & 0\\ -R_2 & R_2+R_3+R_4 & -R_4\\ 0 & -R_4 & R_4+R_5\end{bmatrix}$$

$$I_2=\frac{\begin{vmatrix}R_1+R_2 & V_1 & 0\\ -R_2 & 0 & -R_4\\ 0 & -V_2 & R_4+R_5\end{vmatrix}}{|R|}$$

$$I_3=\frac{\begin{vmatrix}R_1+R_2 & -R_2 & V_1\\ -R_2 & R_2+R_3+R_4 & 0\\ 0 & -R_4 & -V_2\end{vmatrix}}{|R|}$$

Example 21: Determine the mesh currents I_1, I_2 and I_3 is the circuit shown in figure



Solution:

KVL in loop I

$$50 = (10 + 5 + 3)\, I_1 + 5I_2 - 3I_3 \qquad (10)$$

KVL in loop II

$$10 = 2I_2 + 5(I_1 + I_2) + 1(I_2 + I_3)$$

$$10 = 5\, I_1 + 8\, I_2 + I_3 \qquad (11)$$

KVL in loop III

$$1(i_2 + i_3) + 5 + 3\,(i_3 - i_1) = 0$$

$$-3i_1 + i_2 + 4i_3 = -5 \qquad (12)$$

By Cramer's rule

$$I_1 = \frac{\begin{vmatrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}}$$

$I_1 = 3.3$ A

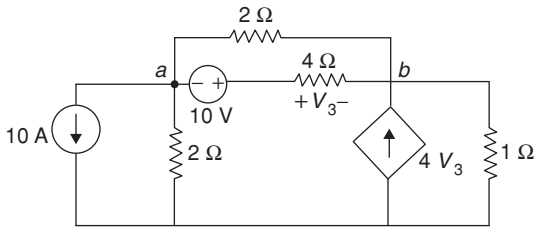
$$I_2 = \frac{\begin{vmatrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}}$$

$I_2 = -0.997$ A

$$I_3 = \frac{\begin{vmatrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}}$$

$I_3 = 1.47$ A

Example 22:



Write nodal equations for the circuit shown in figure and find the power supplied by the 10 V source.

Solution:

KCL at node 'a':

$$10 + \frac{V_a}{2} + \frac{V_a - V_b}{2} + \frac{V_a + 10 - V_b}{4} = 0$$

$$1.25\, V_a - 0.75\, V_b = -12.5 \qquad (13)$$

KCL at node 'b':

$$\frac{V_b - 10 - V_a}{4} + \frac{V_b - V_a}{2} - 4V_3 + \frac{V_b}{1} = 0$$

$$-4.75\, V_a + 5.75\, V_b = 42.b \qquad (14)$$

$$V_3 = V_a + 10 - V_b \qquad (15)$$

Solving (13) and (14) and (15)

$V_a = -11.03$ V, $V_b = -1.724$ V

The current deliverred by 10 V source is

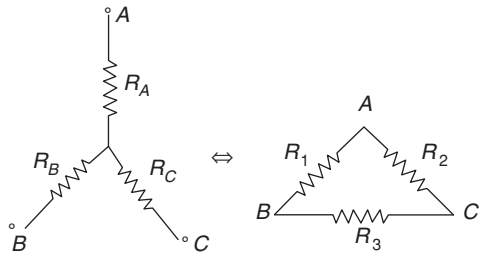
$$I_{10} = \frac{V_a - V_b + 10}{4}$$

The power supplied by the 10 V source is

$$P_{10} = (10)I_{10} = 10\left(\frac{V_a - V_b + 10}{4}\right)$$

$$= 1.735 \text{ W}$$

Star–Delta transformation



If delta is given, corresponding star elements are:

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3}$$

$$R_B = \frac{R_1 R_3}{R_1 + R_2 + R_3}$$

$$R_C = \frac{R_2 R_3}{R_1 + R_2 + R_3}$$

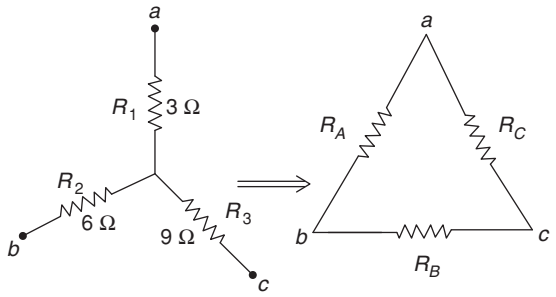
If star elements are given, corresponding delta network elements are:

$$R_1 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_C}$$

$$R_2 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_B}$$

$$R_3 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_A}$$

Example 23: A star connected network which is equivalent to the delta network is shown in the below figure. The R_A, R_B and R_C (in ohms) are respectively,



(A) 99 Ω, 33 Ω, 16.5 Ω
(B) 11 Ω, 16.5 Ω, 33 Ω
(C) 11 Ω, 33 Ω, 16.5 Ω
(D) 1 Ω, 3 Ω, 1.5 Ω

Solution: (C)

$$\Sigma R_A = R_1R_2 + R_2R_3 + R_3R_1$$
$$= 6 \times 3 + 3 \times 9 + 9 \times 6$$
$$= 99$$

$$R_A = \frac{\Sigma R}{R_3} = \frac{99}{9} = 11\,\Omega$$

$$R_B = \frac{\Sigma R}{R_1} = \frac{99}{3} = 33\,\Omega$$

$$R_C = \frac{\Sigma R}{R_2} = \frac{99}{6} = 16.5\,\Omega.$$

Note: Twelve Z Ω impedances are used as edges to form a cube. The equivalent impedance seen between the two diagonally opposite corners of the cube is

(i) If $Z = R\,\Omega$, then $z_{eq} = \frac{5}{6}R\,\Omega$

(ii) If $Z = L$, then $z_{eq} = \frac{5}{6}L$ H

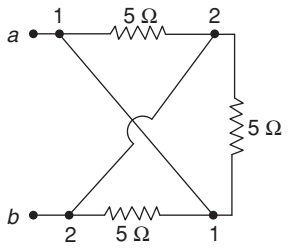
(iii) If $Z = C$ then $z_{eq} = \frac{6}{5}C$ F

Example 24: Twelve 3H inductors are used as edges to form a cube, determine the equivalent inductance seen b/w the two diagonally opposite corners of the cube.

Solution:
We know that,

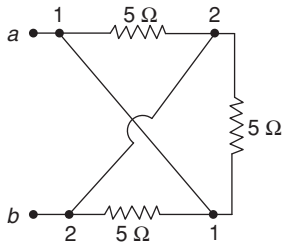
$$Z_{eq} = \frac{5}{6}\text{L} = \frac{5}{6} \times 3 = 2.5\text{ H}$$

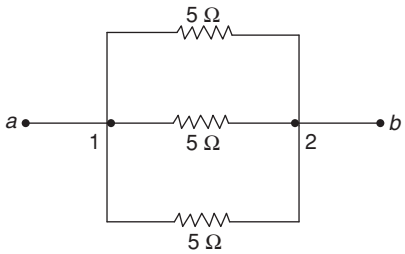
Example 25: Consider the circuit shown in figure below and determine R_{ab},



(A) 2.5 Ω
(B) $\frac{5}{3}\Omega$
(C) 7.5 Ω
(D) None of the above

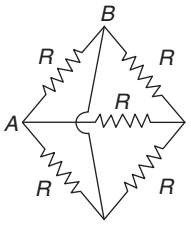
Solution: (B)
Redraw the above circuit





$$R_{ab} = (5 \parallel 5 \parallel 5)\,\Omega$$
$$= \frac{2.5 \times 5}{7.5} = \frac{5}{3}\,\Omega.$$

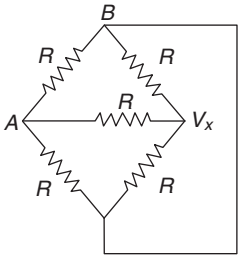
Example 26: Consider the circuit shown in figure determine R_{AB}



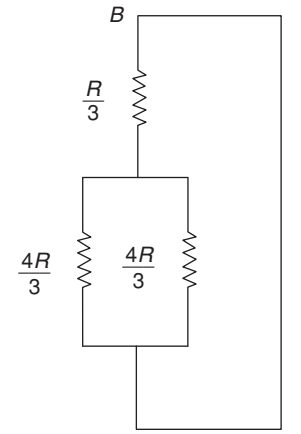
(A) $R_{AB} = 0\ \Omega$ (B) $R_{AB} = R\ \Omega$

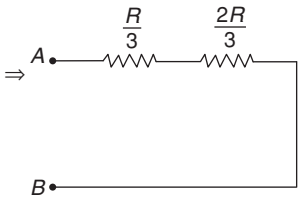
(C) $R_{AB} = 2R\ \Omega$ (D) $R_{AB} = \frac{R}{2}\ \Omega$

Solution: (B)
Redraw the given circuit



Convert $\Delta - y$

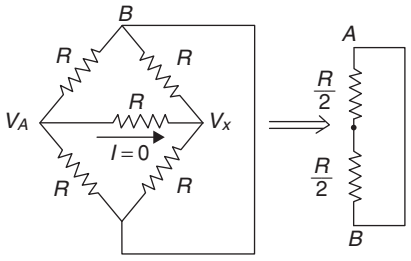




$R_{AB} = R\Omega$

for all values of 'R'

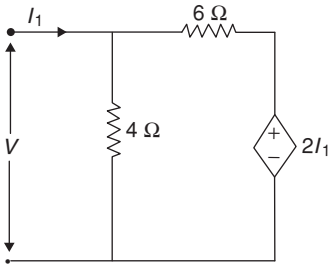
2nd Method:
From the given circuit
Since $R_1 \times R_4 = R_2 R_3$
Bridge is in Balanced condition
$R \times R = R \times R$



$$\Rightarrow \quad R_{AB} = \frac{R}{2} + \frac{R}{2} = R\ \Omega$$

Here $V_x = V_A$
So $I = 0$.

Example 27: The circuit shown in figure will act as a load resistor of



(A) $\frac{1}{5}\ \Omega$ (B) $\frac{16}{5}\ \Omega$

(C) $\frac{5}{16}\ \Omega$ (D) None of the above

Solution: (B)
From given circuit

$$I_1 = \frac{V}{4} + \frac{V - 2I_1}{6}$$

$$12I_1 = 3V + 2V - 2V - 4I_1$$

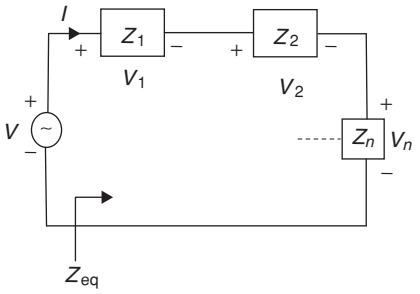
$$16_1 = 5V$$

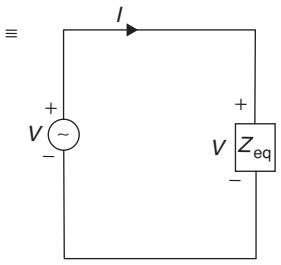
$$\frac{V}{I_1} = R = \frac{16}{5}\ \Omega.$$

Equivalent Circuits wrt Passive R, L, Cs

Two elements are said to be in series only when currents through the elements are same and two elements are said to be in parallel only when voltages across the elements are same

Series circuits





Let $n = 2$

$\Rightarrow Z = Z_R = R\ \Omega$

$\Rightarrow Z = Z_L = j\omega L\ \Omega$

$$\Rightarrow \quad Z = Z_C = \frac{1}{j\omega c}\ \Omega$$

$$z_{eq} = Z_1 + Z_2$$

If Z equal to

1. R: $R_{eq} = R_1 + R_2$
1. L: $L_{eq} = L_1 + L_2$
3. C: $\dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$

Voltage division

$$V = Z_{eq}\, I$$

$$I = \frac{V}{Z_{eq}}$$

$\Rightarrow V_1 = I\, Z_1$ (by ohm's law)

$$\Rightarrow \quad \boxed{V_1 = \frac{V}{Z_1 + Z_2} Z_1}$$

$\Rightarrow V_2 = Z_2\, I$

$$\Rightarrow \quad \boxed{V_2 = \frac{V}{Z_1 + Z_2} Z_2}$$

1. $Z = R$:

$$V_1 = \frac{V \cdot R_1}{R_1 + R_2}:$$

$$V_2 = \frac{V \cdot R_2}{R_1 + R_2}$$

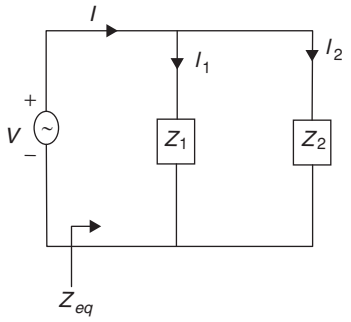
2. $Z = j\omega L$: $V_1 = \dfrac{V \cdot L_1}{L_1 + L_2}; V_2 = \dfrac{V \cdot L_2}{L_1 + L_2}$

3. $Z = \dfrac{1}{j\omega c}$

$$V_1 = \frac{V \cdot C_2}{C_1 + C_2}:$$

$$V_2 = \frac{V \cdot C_1}{C_1 + C_2}$$

Parallel circuits



$\equiv$

$$\frac{1}{Z_{eq}} = \frac{1}{z_1} + \frac{1}{z_2}$$

$$Z_{eq} = \frac{z_1 z_2}{z_1 + z_2}$$

If $Z = R$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

If $R_1 = R_2 = R\ \Omega$

$$R_{eq} = \frac{R}{2}\ \Omega$$

If $Z = j\omega L: \dfrac{1}{L_{eq}} = \dfrac{1}{L_1} + \dfrac{1}{L_2}$

$$L_{eq} = \frac{L_1 \cdot L_2}{L_1 + L_2}$$

If $Z = Z_c$: $C_{eq} = C_1 + C_2$

Current division

$$V = z_{eq} \cdot I$$

$$\Rightarrow \quad V = \frac{Z_1 Z_2}{Z_1 + Z_2} \times I$$

$$\boxed{I_1 = \frac{V}{Z_1}}$$

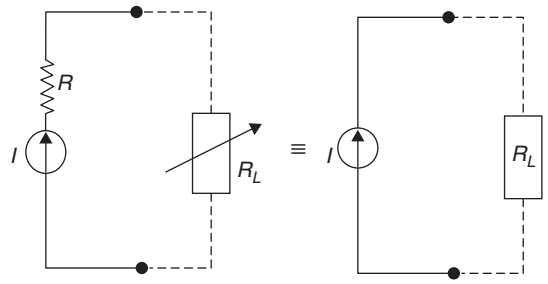
Examples:

$$Z_1 = \frac{1}{j\omega c1}; Z_2 = \frac{1}{j\omega c2}$$

$$I_1 = \frac{I \cdot C_1}{C_1 + C_2}; I_2 = \frac{I \cdot C_2}{C_1 + C_2}$$

Notes:

1. Resistance connected in series with an ideal current source (internal resistance of an ideal current source) does not have any effect on current supplied by it. So it can be neglected.

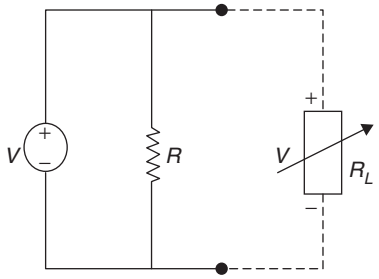


$R \neq \infty$

That is, the load current is independent of R value

$i^2R \neq 0$

2. Resistance connected in parallel with an ideal voltage source does not have any effect on voltage offered by it. So it can be neglected.



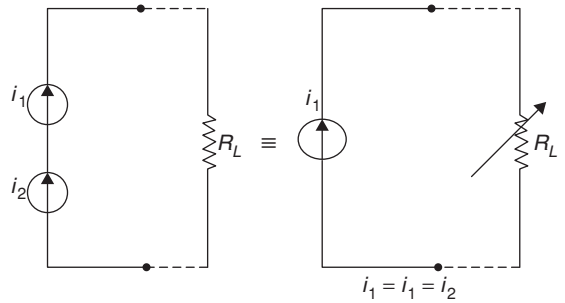
Here $R \neq 0$

So, a resistor in parallel with an ideal voltage source can be neglected in the analysis.

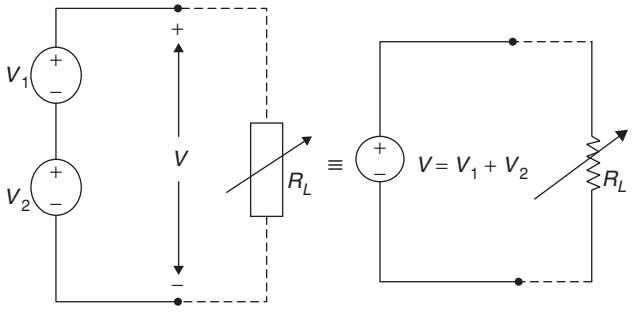
That is, the load voltage is independent of R value

$$\frac{V^2}{R} \neq 0$$

3. Equivalent of series connected current sources is a single current source and are of same magnitude.



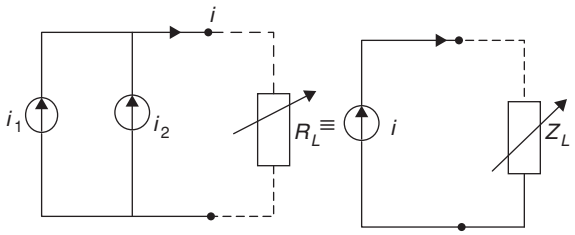
4. Equivalent of two series connected voltage sources is sum of those two depends on their polarity.



By KVL

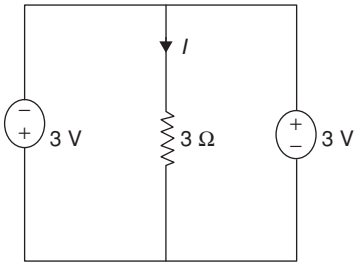
$$\Rightarrow V_2 + V_1 - V = 0$$
$$V = V_1 + V_2$$

5. Equivalent of two parallel connected current sources is equal to sum of those two with their relevant polarity.



$\therefore i = i_1 + i_2$

Example 28:



The current I is

(A) 1 A

(B) –1 A

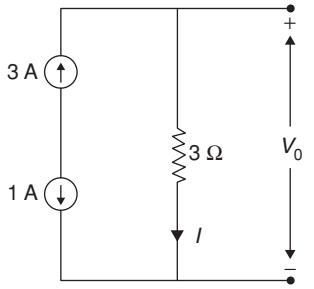
(C) 2 A

(D) Indeterminate

Solution: (D)

Two voltage sources are in parallel and different this is violates KVL and KCL, so indeterminate.

Examples 29:



Determine voltage V_o

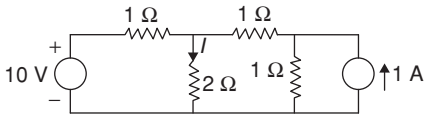
$$V_o = I \times 3$$
$$= (3 - 1)\,3$$
$$= 6 \text{ V}$$

EXERCISES

Practice Problems I

Directions for questions 1 to 21: Select the correct alternative from the given choices.

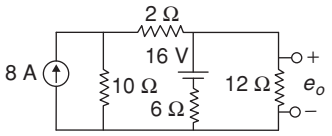
1. The current in the 2 Ω resistor '*I*' is



(A) $\frac{1}{8}$ A (B) $\frac{3}{8}$ A

(C) $\frac{21}{8}$ A (D) $\frac{23}{8}$ A

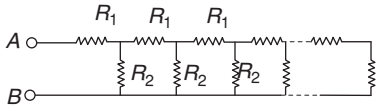
2. The voltage e_o in the figure is



(A) 48 V (B) 24 V
(C) 36 V (D) 28 V

3. The driving point impedance of the infinite ladder network shown in the figure is _____.

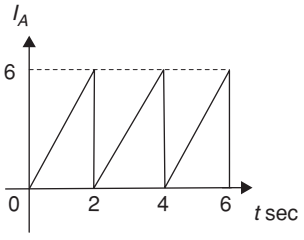
Given $R_1 = 3\ \Omega$, $R_2 = 2\ \Omega$



(A) $\frac{3}{2}\ \Omega$ (B) $\frac{2}{3}\ \Omega$

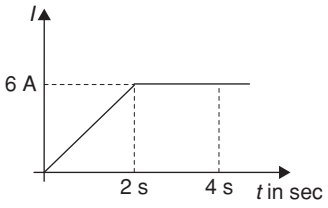
(C) $\left[3+\frac{2}{3}\right]\Omega$ (D) $\sqrt{21}$

4. The current wave formed in a pure resistor of 5 Ω is shown in the fig. The power dissipated in the resistor is



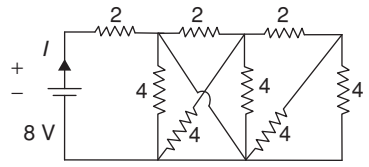
(A) 20 W (B) 45 W
(C) 60 W (D) 90 W

5. Figure shows the waveform of the current passing through an inductor of resistance 1 Ω and inductance 2H. The energy absorbed by the inductor in the first four seconds is _____.



(A) 132 J (B) 98 J
(C) 144 J (D) 168 J

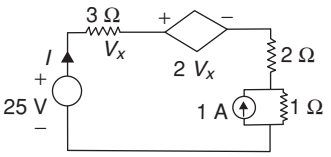
6. In the circuit of the given figure the source current '*I*' is



(A) 2 A (B) 4 A

(C) 1 A (D) $\frac{8}{3}$ A

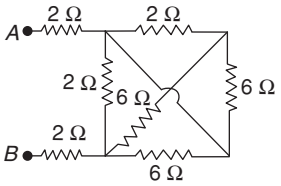
7. In the circuit shown in The figure the current *I* is



(A) $\frac{25}{3}$ A (B) $\frac{25}{6}$ A

(C) 2 A (D) 3 A

8. In the circuit shown in figure below

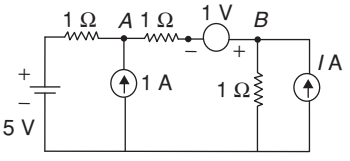


R_{AB} =?

(A) $\frac{21}{4}\ \Omega$ (B) $\frac{5}{6}\ \Omega$

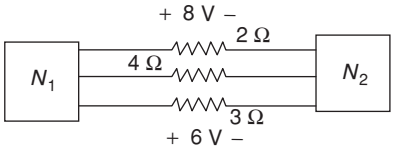
(C) 10 Ω (D) 8 Ω

9. What should be the value of current '*I*' to have zero current flowing through *AB*.



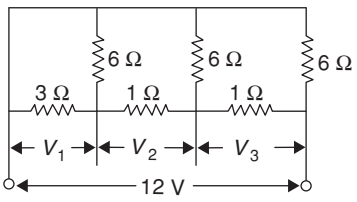
(A) 2 A (B) 4 A
(C) –4 A (D) –2 A

10. The two electrical sub networks N_1 and N_2 are connected through three resistors as shown in the figure. The voltage across the 2 Ω resistor is 8 V and the 3 Ω resistor is 6 V. The voltage across the 4 Ω resistor is



(A) 24 V (B) –24 V
(C) 8 V (D) –8 V

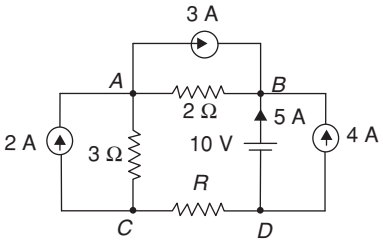
11. In the circuit shown in figure the voltages V_1, V_2 and V_3 are



(A) $\frac{8}{3}, \frac{16}{3}, 4$ (B) $\frac{16}{3}, \frac{8}{3}, \frac{8}{3}$

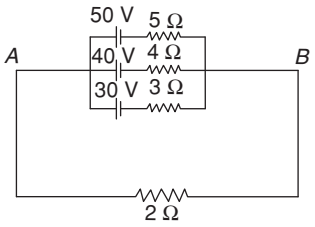
(C) $\frac{16}{3}, \frac{8}{3}, 4$ (D) $4, \frac{16}{3}, \frac{8}{3}$

12. In the network shown in the figure the current in resistor R is



(A) 2 A (B) 3 A
(C) 4 A (D) 9 A

13. In the circuit shown the voltage AB is

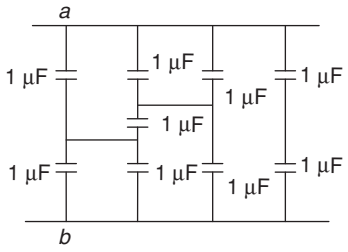


(A) 35 V (B) 28.2 V
(C) 38.3 V (D) 42.6 V

14. A network contains linear resistors which are connected in series across an ideal voltage source. If all the resistances are halved and the voltage is doubled then the voltage across each resistor becomes

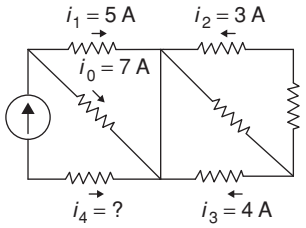
(A) Doubled (B) halved
(C) not changed (D) none

15. Obtain the equivalent capacitance of the network given



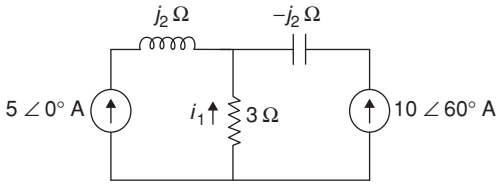
(A) 1 μF (B) 0.8 μF
(C) 1.9 μF (D) 2.6 μF

16. The current i_4 in the circuit of the figure equal to



(A) 12 A (B) −12 A
(C) 4 A (D) None of these

17.

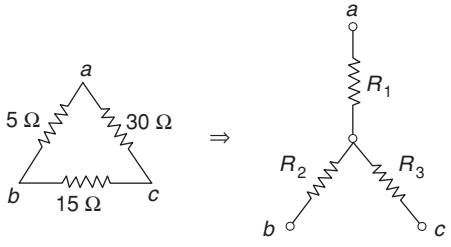


For the circuit shown in the figure, the instantaneous current $i_1(t)$ is

(A) $\frac{10\sqrt{3}}{2}\angle 90°$A (B) $\frac{10\sqrt{3}}{2}\angle -90°$A

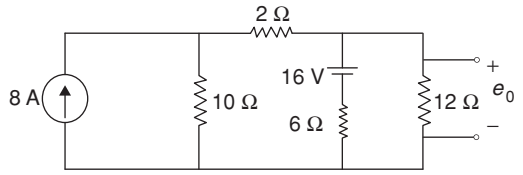
(C) $5\angle 60°$ A (D) $5\angle -60°$ A

18. A Delta - connected network with its Wye - equivalent is shown in the given figure. The resistance R_1, R_2 and R_3 (in Ohm) are respectively



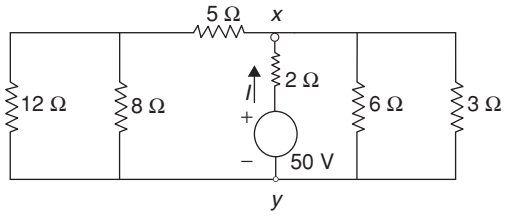
(A) 1.5, 3 and 9 (B) 3, 9 and 1.5
(C) 9, 3 and 1.5 (D) 3, 1.5 and 9

19. The voltage e_0 in the figure



(A) 48 V (B) 24 V (C) 36 V (D) 28 V

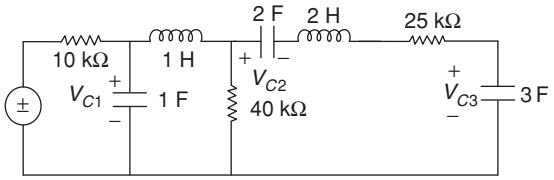
20.



The current I supplied by the source 50 V is

(A) 25 A (B) 13.7 A
(C) 9.8 A (D) 3.66 A

21. The voltages V_{C1}, V_{C2}, and V_{C3} across the capacitors in the circuit in the given figure, understeady state are respectively



(A) 80 V, 32 V, 48 V (B) 80 V, 48 V, 32 V
(C) 20 V, 8 V, 12 V (D) 20 V, 12 V, 8 V

Practice Problems 2

Directions for questions 1 to 19: Select the correct alternative from the given choices.

1. A network contains linear resistors which are connected in series across an ideal voltage source. If all the resistances are halved and the voltage is doubled then the voltage across each resistor becomes

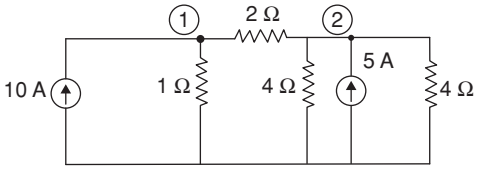
(A) doubled (B) halved
(C) not changed (D) none

2. Twelve similar conductors of 1 Ω resistance form a cubical frame work. Then the resistance between two adjacent corners, two opposite corners of one face and two opposite corners of the cube are

(A) $\frac{3}{4}, \frac{5}{6}, \frac{7}{12}$ (B) $\frac{7}{4}, \frac{5}{6}, \frac{3}{4}$

(C) $\frac{7}{12}, \frac{3}{4}, \frac{5}{6}$ (D) $\frac{12}{7}, \frac{4}{3}, \frac{6}{5}$

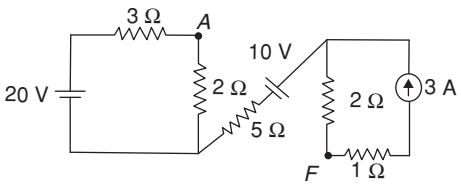
3. In the network shown in the figure



The voltage at node 2 is

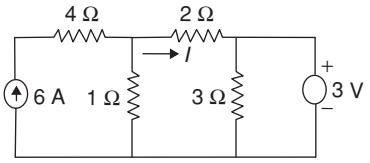
(A) 2 V (B) 10 V (C) 6 V (D) 4 V

4. In the network shown in the figure the voltage AF is



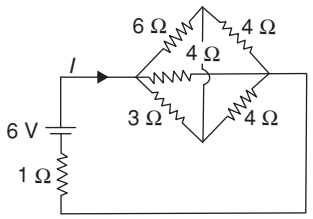
(A) 4 V (B) –4 V
(C) 6 V (D) 2 V

5. For the circuit shown in the figure the current 'I' is given by



(A) 2 A (B) 3 A
(C) 1 A (D) zero

6. The current 'I' supplied by the source in the figure is



(A) 2 A (B) 3 A
(C) $\frac{3}{2}$ A (D) $\frac{2}{3}$ A

7. A resistance of 10 Ω is connected in series with two resistances of 20 Ω arranged in parallel what resistance should be shunted across this parallel combination so that the total current taken shall be 2 A with 30 V applied

(A) 5 Ω (B) 10 Ω
(C) 20 Ω (D) 25 Ω

8. The resistance of a strip of conductor is R Ω. If the strip is elongated such that its length is doubled the resistance of the strip is given by

(A) $4R$ (B) $2R$
(C) $\frac{R}{2}$ (D) R

9. The current $i(t)$ through a 10 Ω resistor in series with an inductance, is given by $i(t) = 3 + 4\sin(100t + 45°) + 4\sin(300t + 60°)$ Amperes

The RMS (root mean square) value of the current and the power dissipated in the circuit are

(A) $\sqrt{41}$ A, 410 W, respectively

(B) $\sqrt{35}$ A, 350 W, respectively

(C) 5 A, 250 W, respectively

(D) 11 A, 1210 W, respectively

10. The nodal method of circuit analysis is based on

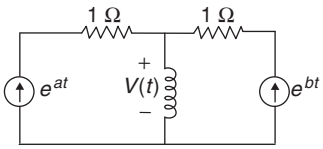
(A) KVL and Ohm's law

(B) KCL and Ohm's law

(C) KCL and KVL

(D) KCL, KVL and Ohm's law

11. In the given circuit, the voltage $v(t)$ is



(A) $e^{at} - e^{bt}$ (B) $e^{at} + e^{bt}$

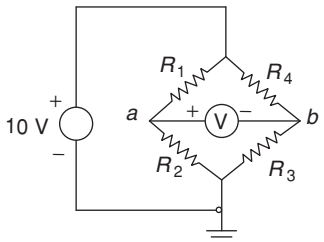
(C) $ae^{at} - be^{bt}$ (D) $ae^{at} + be^{bt}$

12. The rms value of the voltage defined by $v(t) = 5 + 5\sin\left(314t + \frac{\pi}{6}\right)$ is

(A) 5 V (B) 2.5 V

(C) 6.12 V (D) 10 V

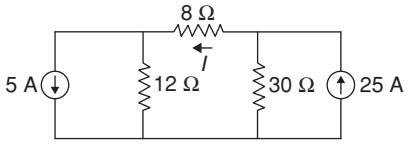
13. If $R_1 = R_2 = R_4 = R$ and $R_3 = 1.1\,R$ in the bridge circuit shown in the figure, then the reading in the ideal voltmeter connected between a and b is



(A) 0.238 V (B) 0.138 V

(C) −0.238 V (D) 1 V

14.

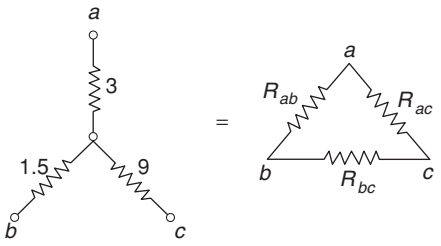


The current I in the above circuit is

(A) 20 A (B) −20 A

(C) 16.2 A (D) −16.12 A

15.

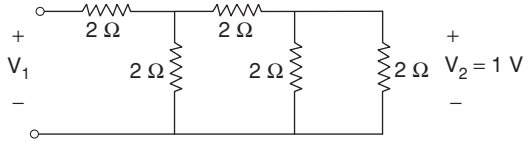


The value of R_{ab}, R_{ac} and R_{bc} are

(A) 30 Ω, 15 Ω, 5 Ω (B) 5 Ω, 30 Ω, 15 Ω

(C) 5 Ω, 30 Ω, 15 Ω (D) 15 Ω, 5 Ω, 30 Ω

16.



If $V_2 = 1$ V in the network shown, the value of V_1 will be

(A) 2.5 V (B) 4 V

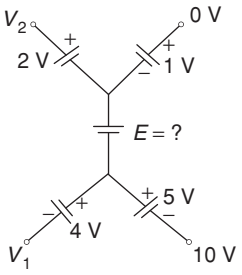
(C) 5 V (D) 8 V

17. If each branch of a Delta circuit has impedance $\sqrt{3}Z$, then each branch of the equivalent Wye circuit has impedance

(A) $\frac{Z}{\sqrt{3}}$ (B) $3Z$

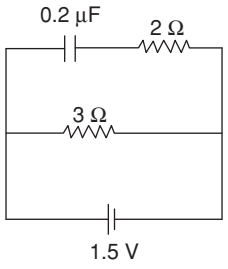
(C) $3\sqrt{3}Z$ (D) $\frac{Z}{3}$

18. In the given circuit, the value of the voltage source E is



(A) −16 V (B) 4 V (C) −6 V (D) 16 V

19. What is the current through resistor 2 Ω in the circuit given?

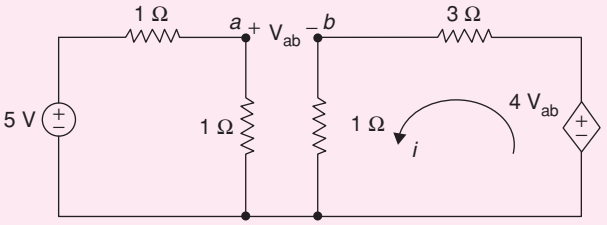


(A) 0.6 V (B) 1.8

(C) 0.9 V (D) 0 V

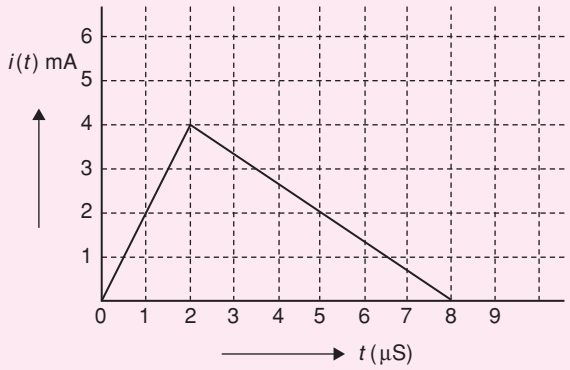
Previous Years' Questions

1. In the circuit shown in the figure, the value of the current i will be given by **[2008]**



(A) 0.31 A (B) 1.25 A
(C) 1.75 A (D) 2.5 A

Common Data for Questions 2 and 3: The current $i(t)$ sketched in the figure flows through an initially uncharged 0.3 nF capacitor.



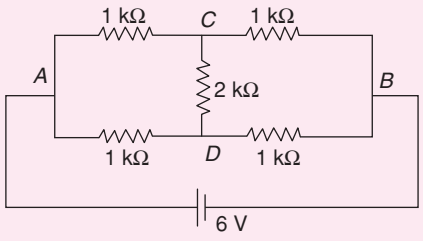
2. The charge stored in the capacitor at $t = 5$ μs, will be **[2008]**

(A) 8 nC (B) 10 nC
(C) 13 nC (D) 16 nC

3. The capacitor charged upto 5 μs, as per the current profile given in the figure, is connected across an inductor of 0.6 mH. Then the value of voltage across the capacitor after 1μs will approximately be **[2008]**

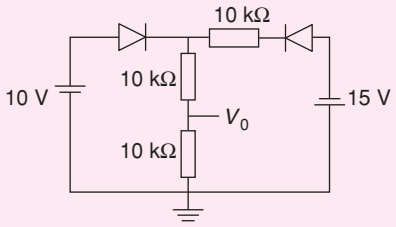
(A) 18.8 V (B) 23.5 V
(C) −23.5 V (D) −30.6 V

4. The current through the 2 kΩ resistance in the circuit shown is **[2009]**



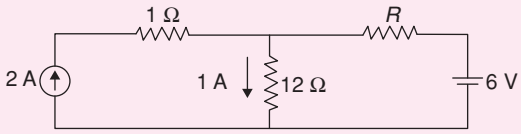
(A) 0 mA (B) 1 mA
(C) 2 mA (D) 6 mA

5. Assuming that the diodes in the given circuit are ideal, the voltage V_0 is **[2010]**



(A) 4 V (B) 5 V
(C) 7.5 V (D) 12.12 V

6. If the 12 Ω resistor draws a current of 1 A as shown in the figure, the value of resistance R is **[2010]**

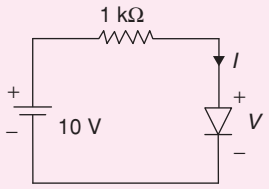


(A) 4 Ω (B) 6 Ω
(C) 8 Ω (D) 18 Ω

7. The $I-V$ characteristics of the diode in the circuit given below are **[2012]**

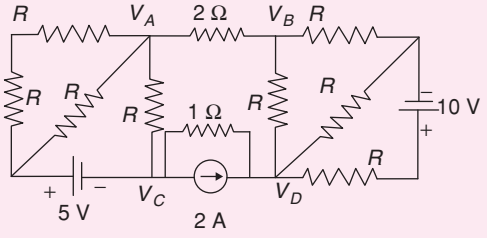
$$I = \begin{cases} \dfrac{V-0.7}{500}\ \text{A}, V \geq 0.7\ \text{V} \\ 0\ \text{A}, V < 0.7\ \text{V} \end{cases}$$

The current in the circuit is



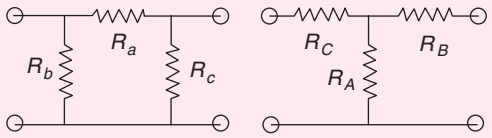
(A) 10 mA (B) 9.3 mA
(C) 6.67 mA (D) 6.2 mA

8. If $V_A - V_B = 6$ V, then $V_C - V_D$ is **[2012]**



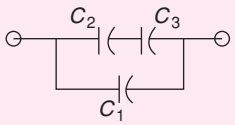
(A) −5 V (B) 2 V
(C) 3 V (D) 6 V

9. Consider a delta connection of resistors and its equivalent star connection as shown below. If all elements of the delta connection are scaled by a factor k, $k > 0$, the elements of the corresponding star equivalent will be scaled by a factor of **[2013]**



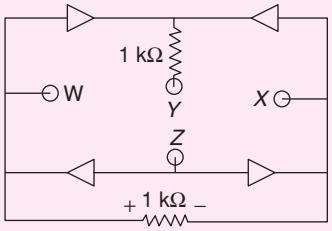
(A) k^2 (B) k
(C) $1/k$ (D) $\sqrt{k}$

10. Three capacitors C_1, C_2 and C_3, whose values are 10 µF, 5 µF, and 2 µF respectively, have breakdown voltages of 10 V, 5 V, and 2 V respectively. For the interconnection shown, the maximum safe voltage in Volts that can be applied across the combination and the corresponding total charge in µC stored in the effective capacitance across the terminals are respectively, **[2013]**



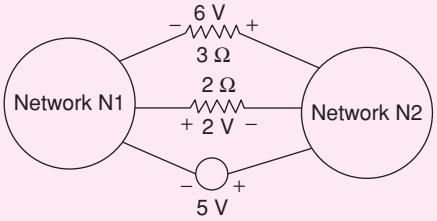
(A) 2.8 and 36 (B) 7 and 119
(C) 2.8 and 32 (D) 7 and 80

11. A voltage 1000 sin ωt V is applied across YZ. Assuming ideal diodes, the voltage measured across WX in volts is **[2013]**



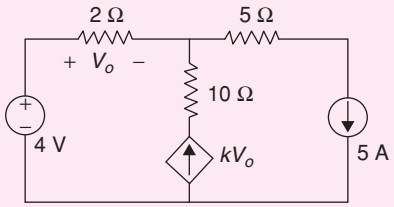
(A) sin ωt
(B) (sin ωt + |sin ωt|)/2
(C) (sin ωt – |sin ωt|)/2
(D) 0 for all t

12. The voltages developed across the 3 Ω and 2 Ω resistors shown in the figure are 6 V and 2 V respectively, with the polarity as marked. What is the power (in Watt) delivered by the 5 V voltage source? **[2015]**

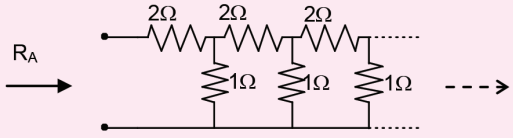


(A) 5 (B) 7
(C) 10 (D) 14

13. In the given circuit, the parameter k is positive, and the power dissipated in the 2 Ω resistor is 12.5 W. The value of k is _____ **[2015]**

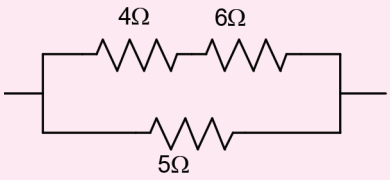


14. R_A and R_B are the input resistances of circuits as shown below. The circuits extend infinitely in the direction shown. Which one of the following statements is TRUE? **[2016]**



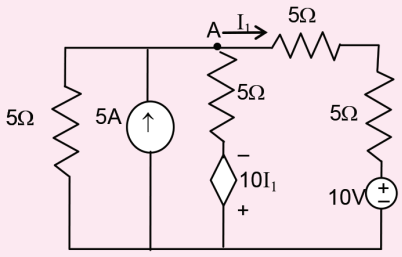
(A) $R_A = R_B$ (B) $R_A = R_B = 0$
(C) $R_A < R_B$ (D) $R_B = R_A/(1+R_A)$

15. In the portion of a circuit shown, if the heat generated in 5 Ω resistance is 10 calories per second, then heat generated by the 4 Ω resistance, in calories per second, is _____. **[2016]**

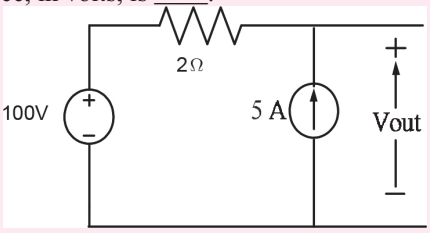


16. In the given circuit, the current supplied by the battery, in ampere, is _____. **[2016]**

17. In the circuit shown below, the node voltage VA is_____V. **[2016]**



18. In the circuit shown below, the voltage and current sources are ideal. The voltage (V_{out}) across the current source, in volts, is _____. **[2016]**

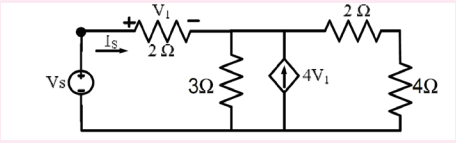


(A) 0 (B) 5
(C) 10 (D) 20

19. The graph associated with an electrical network has 7 branches and 5 nodes. The number of independent KCL equations and the number of independent KVL equations, respectively, are **[2016]**

(A) 2 and 5 (B) 5 and 2
(C) 3 and 4 (D) 4 and 3

20. The driving point input impedance seen from the source V_s of the circuit shown below, in Ω is_____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. C	**2.** D	**3.** D	**4.** C	**5.** C	**6.** B	**7.** C	**8.** A	**9.** B	**10.** B
11. C	**12.** D	**13.** C	**14.** A	**15.** C	**16.** B	**17.** A	**18.** D	**19.** D	**20.** B
21. B									

Practice Problems 2

1. A	**2.** C	**3.** B	**4.** A	**5.** B	**6.** A	**7.** B	**8.** A	**9.** C	**10.** B
11. D	**12.** C	**13.** C	**14.** C	**15.** C	**16.** D	**17.** A	**18.** A	**19.** D	

Previous Years' Questions

1. B	**2.** C	**3.** D	**4.** A	**5.** B	**6.** B	**7.** D	**8.** A	**9.** B	**10.** C
11. D	**12.** A	**13.** 0.5	**14.** D	**15.** 2	**16.** 0.5	**17.** 11.428	**18.** D	**19.** D	**20.** 20

Chapter 2

Network Theorems

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Superposition theorem
- Thevenin's theorem
- Norton's theorem
- Maximum power transfer theorem
- Reciprocity theorem
- Millman's theorem
- Tellegen's theorem
- Substitution theorem
- Compensation theorem

Superposition Theorem

Statement: Whenever a linear bilateral circuit is excited by more than one independent sources. The total response is the algebraic sum of individual responses due to all independent sources.

Steps to Apply Superposition Theorem

Step 1: Select a single source acting alone short the other voltage sources and open the current sources (deactivate).

Step 2: Find the current through or the voltage across the required element due to the sources under consideration.

Step 3: Repeat the above steps for all the independent sources.

Step4: Add all the individual effects produced by individual sources, to obtain the total current through or voltage across the element.

Notes:

1. Dependent sources are never deactivated.
2. When an independent voltage source is deactivated, it is set to zero.

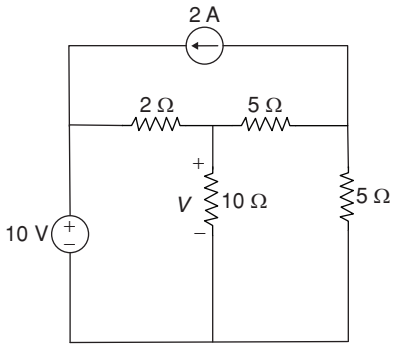
   $\Rightarrow$ replaced by short circuit
3. When an Independent current source is deactivated, it is set to zero.

   $\Rightarrow$ replaced by open circuit.

   $\therefore\ I = 0, \quad \Rightarrow \quad$ open circuit.

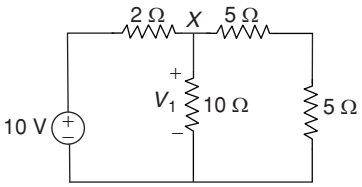
Solved Examples

Examples 1: Use superposition to find $V = ?$



Solution:

(i) Consider the independent voltage source acting alone:



Apply KCL (Kirchoff's Current Law) at node X:

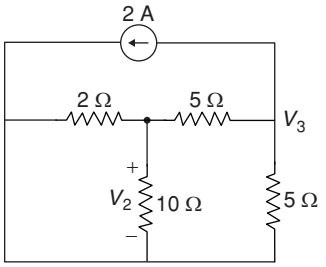
$$\frac{V_1 - 10}{2} + \frac{V_1}{10} + \frac{V_1}{10} = 0$$

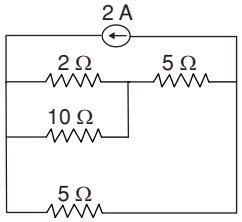
$$5(V_1 - 10) + 2V_1 = 0$$

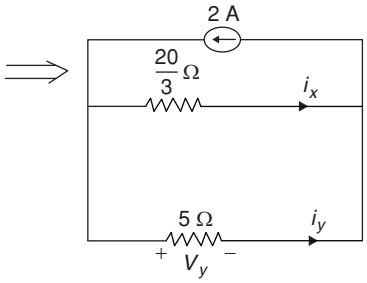
$$7V_1 = 50$$

$$V_1 = \frac{50}{7}\ \text{V}$$

(ii) Consider the independent current source acting alone:







$$i_x = \frac{2\times 5}{\frac{20}{3}+5} = \frac{10\times 3}{35}$$

$$= \frac{30}{35} = \frac{6}{7}\text{ A}$$

$$i_y = 2 - \frac{6}{7}$$

$$= \frac{8}{7}\text{ A}$$

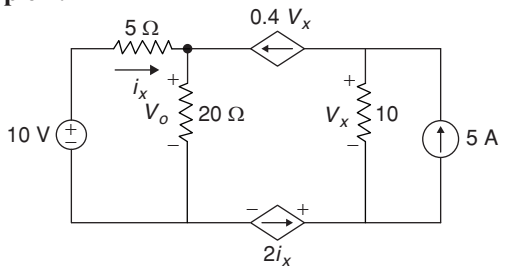
$$V_y = \frac{5\times 8}{7} = \frac{40}{7}\text{ V}$$

$$V_2 = V_{10\,\Omega} = \frac{40}{7} - 5\times\frac{6}{7}$$

$$= \frac{10}{7}\text{ V}$$

$$V = V_1 + V_2 = \frac{50}{7} + \frac{10}{7} = \frac{60}{7}\text{ V}$$

Example 2:



The value of V_o is

(A) −8 V (B) 16 V

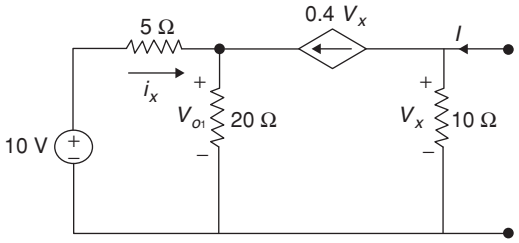
(C) −16 V (D) 24 V

Solution:

(i) Activate independent voltage source only

∴ Independent current source deactivated

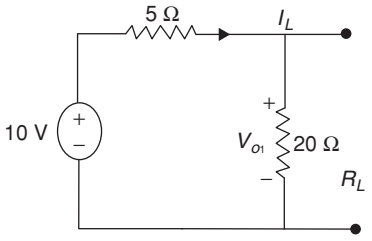
i.e., open circuit



$V_x = -10 \times 0.4\,V_x$

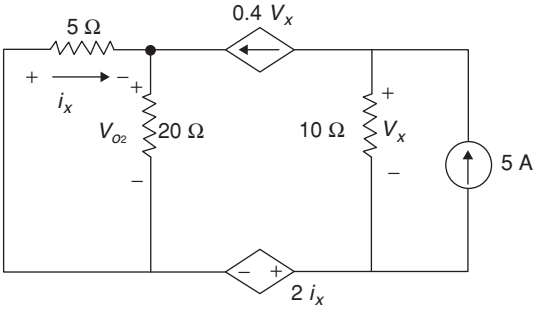
$V_x = -4V_x \quad \Rightarrow V_x = 0$

$\Rightarrow V_x = 0$, so dependent current $0.4V_x$ equal to zero, so it acts like an open circuit.



$$V_{01} = 10\times\frac{20}{25} = 8\text{ V}$$

(ii) Activate dependent current source only.



KCL at node A,

$$\frac{V_{o2}}{20} + \frac{V_{o2}}{5}0.4\,V_x = 0$$

$$5\,V_{o2} = 8\,V_x$$

At node B

$$5 = \frac{V_x}{10} + 0.4\,V_x$$

$$50 = V_x + 4\,V_x$$

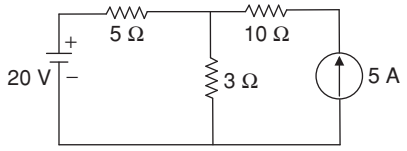
$$V_x = 10 \text{ V}$$

$$V_{o2} = \frac{80}{5} = 16 \text{ V}$$

$$\therefore\ V_o = V_{o1} + V_{o2}$$

$$= 8 + 16 = 24 \text{ V}$$

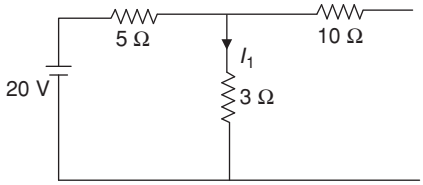
Example 3:



Find the current through the 3 Ω resistor using superposition theorem.

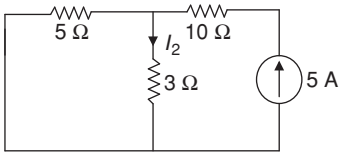
Solution:

Considering 20 V source alone, i.e., 5 A current source is open circuited



$$I_1 = \frac{20}{5+3} = \frac{20}{8} = 2.5 \text{ A}$$

Considering 5 A source alone, i.e., 20 V voltage source is short circuited.

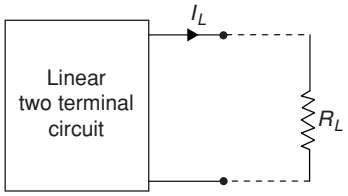


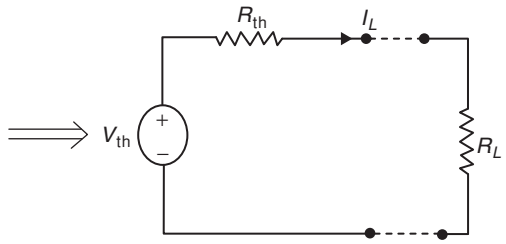
$$I_2 = 5 \times \frac{5}{5+3}$$

$$I_2 = 3.125 \text{ A}$$

$$I = I_1 + I_2 = 2.5 + 3.125 = 5.625 \text{ A}$$

THEVENIN'S THEOREM





Statement: Any two terminal bilateral linear circuit can be replaced by an equivalent circuit consisting of a Thevenin's voltage source and Thevenin's series resistor. Thevenin's voltage source is the open circuit voltage across the terminals and Thevenin's resistance is the equivalent resistance across the terminals.

Note:

1. Circuit consisting only independent sources; V_{th}, R_{th} are calculated conventionally.
2. A circuit consisting of both dependent and independent sources; V_{th}, I_{sc}

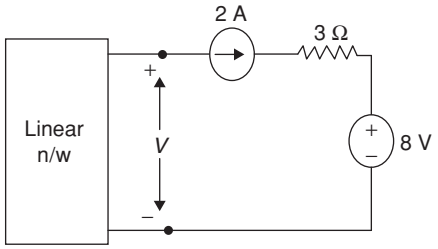
$$\Rightarrow\ R_{th} = \frac{V_{th}}{I_{sc}}\ \Omega$$

3. A circuit consisting only dependent sources.

$$R_{th} = \frac{V_{DC}}{I_{DC}}\ \Omega$$

$$V_{th} = 0 \text{ V and } I_{th} = 0 \text{ A}$$

Examples 4: Consider the following circuit



The linear network contains only DC sources and resistances. The value of V is

(A) 14 V (B) 2 V

(C) 8 V (D) indeterminate

Solution: (D)

From the given circuit

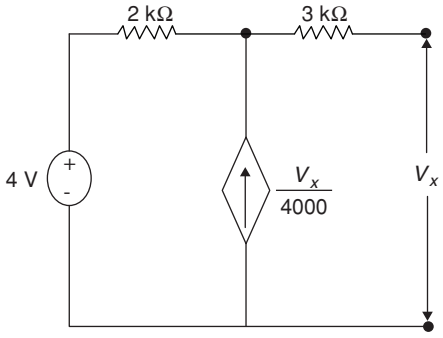
$$V = V_{2A} + 3 \times 2 + 8$$

$$V = V_{2A} + 14$$

But voltage across current source is unknown.

So, V = 14 + any value $\Rightarrow$ indeterminate.

Example 5: Obtain the Thevenin's equivalent of the network below:



(A) $V_{th} = 0$ V, $R_{th} = 5$ kΩ
(B) $V_{th} = 2.4$ V, $R_{th} = 1.2$ kΩ
(C) $V_{th} = 8$ V, $R_{th} = 10$ kΩ
(D) None of the above

Solution: (C)
Under open circuit condition,

$$I_{3\Omega} = 0$$

∴ The node equation is given by,

$$\frac{V_x - 4}{2\text{ k}\Omega} - \frac{V_x}{4000} = 0$$

$$\frac{V_x - 4}{2\text{ k}\Omega} - \frac{V_x}{4000}$$

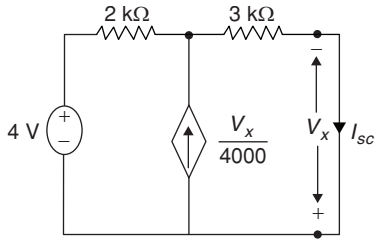
$$2(V_X - 4) = V_x$$

$$\Rightarrow V_X = V_{th} = 8\text{ V}$$

But we know,

$$R_{th} = \frac{V_{th}}{I_{sc}} \text{ case}$$

Compute the short circuit current



But, $V_x = 0$ from the above circuit

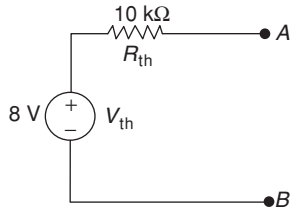
∴ dependent current source $\frac{V_x}{4000} = 0$

$\Rightarrow$ open circuit

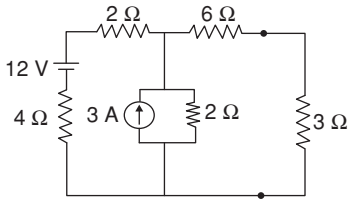
i.e., $I_{sc} = \frac{4}{(2+3)}$ mA = 0.8 mA

$$\therefore \quad R_{th} = \frac{V_{th}}{I_{sc}} = \frac{8}{0.8}\text{ k}\Omega = 10\text{ k}\Omega$$

∴ Thevenin's equivalent is

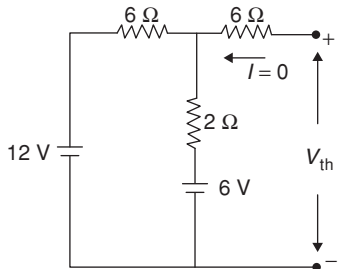


Example 6: Find the current flowing through the 3 Ω resistor.



(A) 1 A (B) 3 A
(C) 5 A (D) 0.714 A

Solution: (D)
Applying source transformation to current source it becomes.



Apply nodal Analysis at node A

$$\frac{V_{th} - 12}{6} + \frac{V_{th} - 6}{2} = 0$$

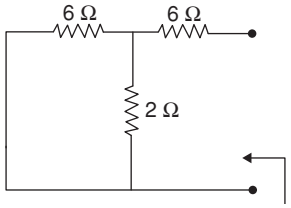
$$V_{th} - 12 + 3(V_{th} - 6) = 0$$

$$4V_{th} = 12 + 18$$

$$V_{th} = \frac{30}{4} = 7.5\text{ V}$$

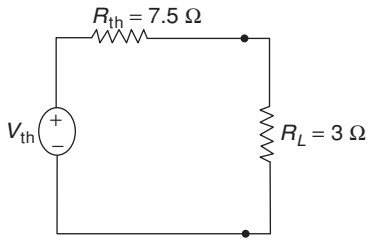
R_{th}:

(i) All independent voltage sources are short circuited current sources are open circuit.
∴ it becomes



$$R_{th} = 6 + 6||2$$
$$= 6 + 1.5$$
$$= 7.5\ \Omega$$

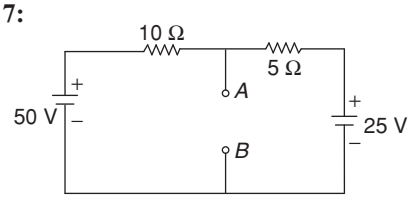
The Thevenin's equivalent network is shown below:



$$I_L = \frac{V_{th}}{R_{th} + R_L} = \frac{7.5}{7.5+3}$$

$$I_L = 0.71428 \text{ A.}$$

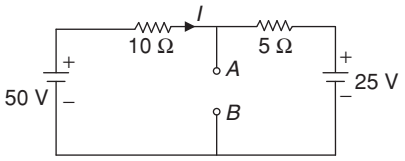
Example 7:



Determine the Thevenin's equivalent circuit across AB for the above network shown in figure.

Solution:

To find V_{th}:



$$I = \frac{50-25}{10+5}$$

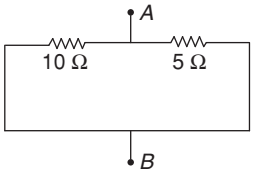
$$I = \frac{25}{15} = 1.67 \text{ A}$$

$$V_{AB} = 1.67 \times 5 + 25$$

$$V_{th} = 33.35 \text{ V}$$

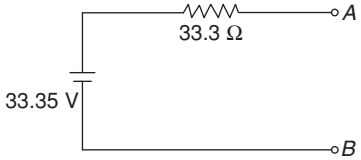
To find R_{th}:

Deactivate the voltage sources

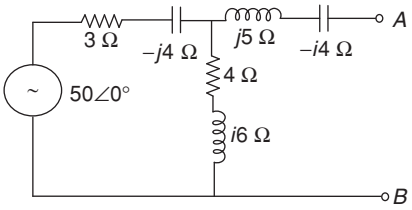


$$R_{th} = 10 \parallel 5 = 3.33\ \Omega$$

The Thevenin equivalent circuit is shown in the figure:



Example 8: For the circuit shown in figure, determine the Thevenin's equivalent between the output terminals.



Solution:

$$V_{th} = 50\angle 0° \times \frac{(4+j6)}{(4+j6)+(3-j4)}$$

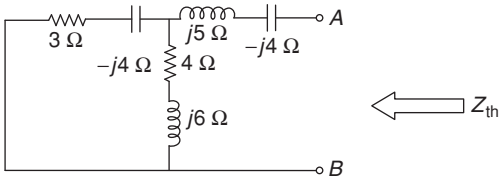
$$= 50\angle 0° \times \frac{(4+j6)}{7+j2}$$

$$= 50\angle 0° \times \frac{7.21\angle 56.3°}{7.28\angle 15.95°}$$

$$V_{th} = 49.5\angle 40.35° \text{V}$$

To find Z_{th}:

Short circuit the source $50\angle 0°$



$$Z_{th} = (j5 - j4) + \frac{(3-j4)(4+j6)}{(3-j4)+(4+j6)}$$

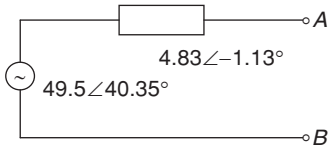
$$= j1 + \frac{5\angle 53.13° \times 7.21\angle 56.3°}{7.28\angle 15.95°}$$

$$= j1 + 4.95\angle -12.78°$$

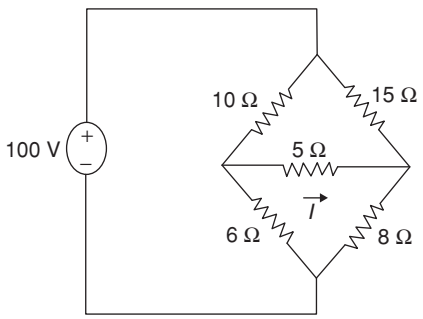
$$= j1 + 4.83 - j1.095$$

$$= 4.83 - j0.095$$

$$Z_{th} = 4.83\angle -1.13°$$

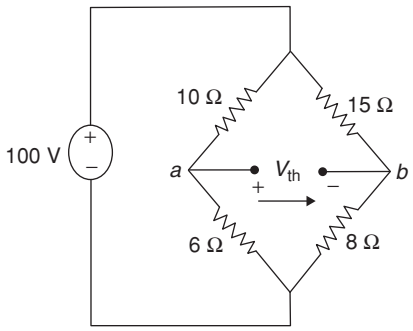


Example 9: Find the current through the 5Ω using Thevenin's Theorem,



(A) $I = 0.194$ A
(B) $1 = 0.3$ A
(C) $I = O$ A
(D) None of the above

Solution: (A)
Balanced condition
$R_1 R_4 = R_2 R_3$
$10 \times 8 \neq 6 \times 15$
$\therefore$ The bridge is unbalanced so current flowing through the 5 Ω resistor is not zero.



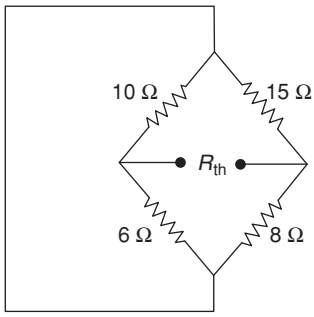
$$V_{th} = V_{ab} = V_a - V_b$$

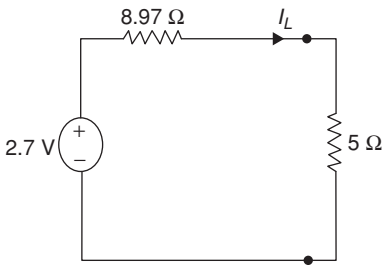
$$V_a = \frac{100 \times 6}{6+10} = 37.5 \text{ V}$$

$$V_b = 100 \times \frac{8}{15+8} = 34.78 \text{ V}$$

$$V_{th} = 37.5 - 34.78 = 2.717 \text{ V}$$

R_{th}:

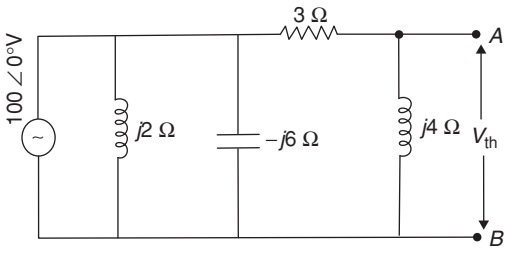




$$R_{th} = (10 \parallel 6) + (15 \parallel 8)$$
$$= 3.75 + 5.21 \approx 8.97 \ \Omega$$

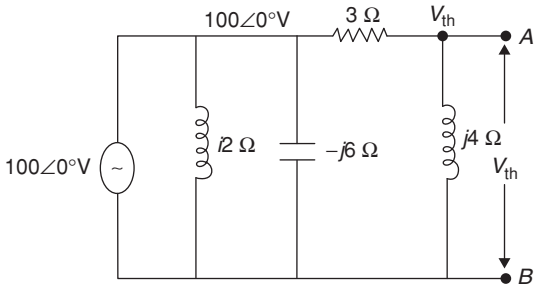
$$I_L = \frac{2.717}{13.97} = 0.194 \text{ A}.$$

Example 10: The Thevenin's equivalent voltage V_{th} across the terminal A and B of the network shown in the figure is given by



(A) $j16(3 - j4)$
(B) $j16(3 + 4)$
(C) $16(3 + j4)$
(D) $16(3 - j4)$

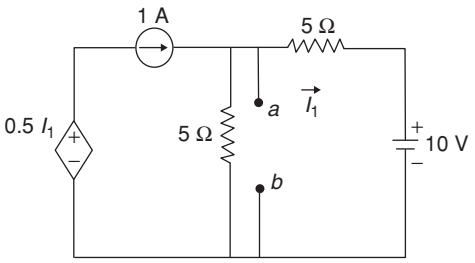
Solution: (A)



$$V_{th} = V_{j4} = 100 \times \frac{j4}{j4+3}$$

$$= \frac{100 \times 4(3 - j4)}{25} = j16(3 - j4).$$

Example 11: For circuit shown in the figure, Thevenin's voltage and Thevenin's equivalent resistance at terminals $a - b$ is



(A) 5 V and 2 Ω
(B) 7.5 V and 2.5 Ω
(C) 4 V and 2 Ω
(D) 3 V and 2.5 Ω

Solution: (B)

$$V_{th} = V_{ab}$$

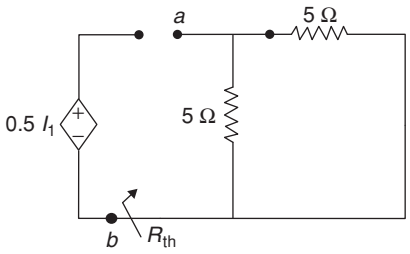
For V_{th}:

$$\frac{V_{th} - 10}{5} + \frac{V_{th}}{5} - 1 = 0$$

$$2V_{th} - 10 - 5 = 0$$

$$V_{th} = 7.5 \text{ V}.$$

For R_{th}: Deactivate the independent sources

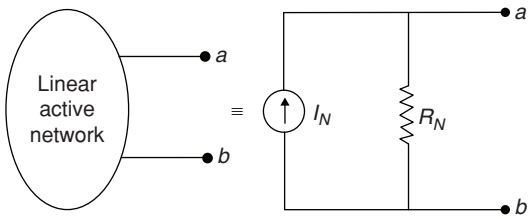


$$R_{th} = R_{ab} = 5 \,||\, 5 = 2.5\ \Omega$$

$$\therefore\ V_{th} = 7.5\ \text{V and } R_{th} = 2.5\ \Omega$$

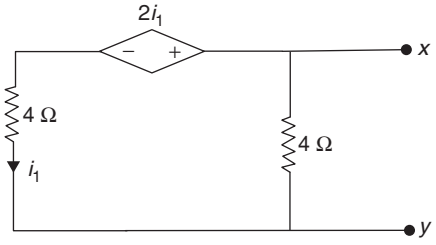
Norton's Theorem

A one port linear, active, resistive network, which contains one or more voltage or current sources can be replaced by a single current source in parallel with a resistance.



$$I_n = \frac{V_{th}}{R_{th}} \text{ and } R_N = R_{th}$$

Example 12:



The Norton's equivalent of the network is

(A) $I_{SC} = 0,\ R_N = 1.5\ \Omega$

(B) $V_{th} = 0\ \text{V},\ R_{th} = \frac{4}{3}\ \Omega$

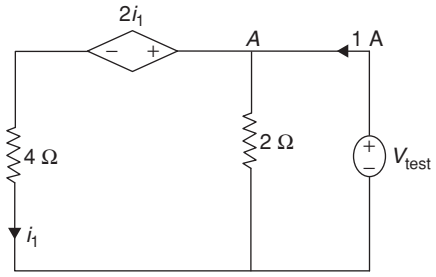
(C) $I_{SC} = 0,\ R_N = \frac{4}{3}\ \Omega$

(D) None of the above

Solution: (A)

The network does not have any active independent sources,

$I_{sc} = 0$



Applying KCL at node A,

$$1\ \text{A} = \frac{V_{test}}{2} + i_1$$

But $$i_1 = \frac{V_{test} - 2i_1}{4}$$

$$4i_1 + 2i_1 = V_{test}$$

$$i_1 = \frac{V_{test}}{6}\ \text{A}$$

$$1\ \text{A} = \frac{V_{test}}{2} + \frac{V_{test}}{6}$$

$$6 = 3V_{test} + V_{test}$$

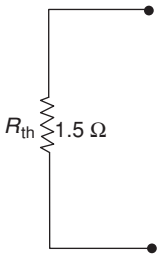
$$V_{test} = 1.5\ \text{V}$$

$$R_{th} = \frac{V_{test}}{1\ \text{A}} = 1.5\ \Omega$$

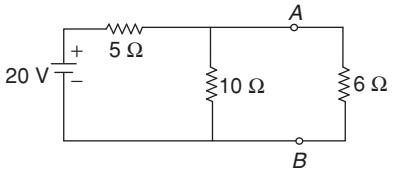
Example 13: For the above circuit find the Thevenin's equivalent circuit.

Solution:

We know, $R_{th} = R_N = 1.5\ \Omega$, so the network does not have any active independent sources. Hence, $V_{th} = 0$ V

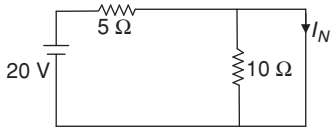


Example 14:



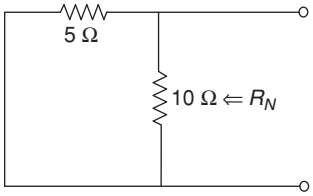
Using Norton's theorem, find the current through the 6 Ω load.

Solution:
To find I_N, short circuit load terminals,



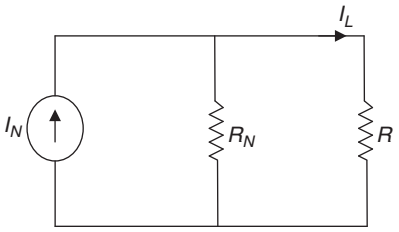
$$I_N = \frac{20}{5} = 4 \text{ A}$$

To find R_N, remove load, short circuit the voltage source 20 V



$$R_N = 5 \| 10 = \frac{5 \times 10}{5+10} = 3.33\ \Omega$$

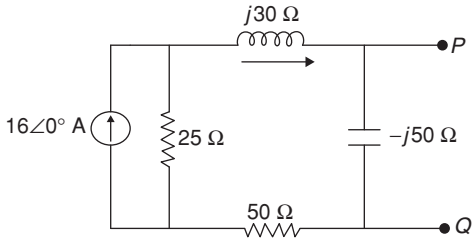
The Norton equivalent circuit is



$$I_L = I_N \frac{R_N}{R_N + R_L} = 4 \times \frac{3.33}{3.33+6}$$

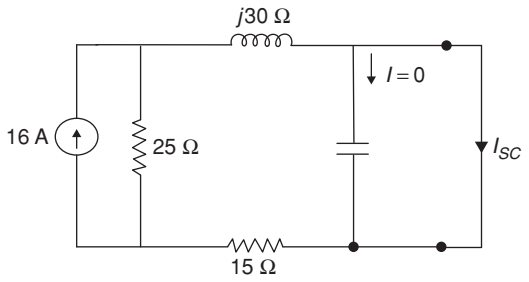
$$I_L = 1.43 \text{ A}$$

Example 15: In the circuit shown below, the Norton's equivalent current, with respect to the terminals P and Q is



(A) $(6.4 + j4.8)$ A
(B) $(16 + j0)$ A
(C) $(6.4 - j4.8)$ A
(D) $(6.56 - j7.87)$ A

Solution: (C)
For Norton equivalent current short circuiting the terminals PQ



$$I_{sc} = 16\angle 0° \times \frac{25}{25+15+j30}$$

$$= \frac{25}{40+j30} \times 16\angle 0°$$

$$= \frac{25 \times 16}{\sqrt{(40)^2+(30)^2}} \cdot \frac{1}{\angle 36.86°}$$

$$= 8\angle -36.86°$$

$$= 8\,[\cos 36.86° - j\sin 36.86°]$$

$$I_{sc} = (6.4 - j4.8) \text{ A}.$$

Maximum Power Transfer Theorem

This theorem is used to find the value of load resistance for which there would be maximum amount of power transfer from source to load.

Statement: A resistance load connected to a DC network, receives maximum power when the load resistance is equal to the internal resistance (Thevenin's equivalent resistance) of the source network as seen from the load terminals.

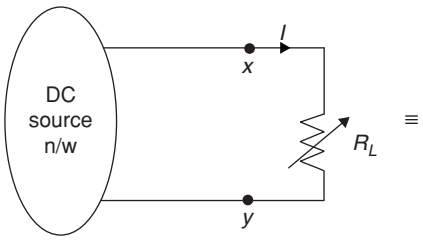


Figure 1 Load connected to the DC source network

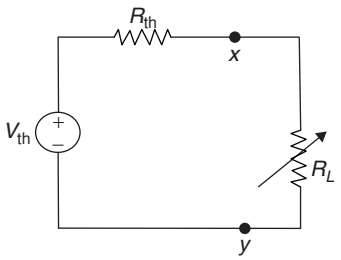
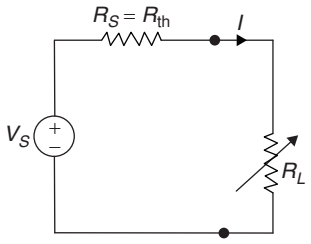


Figure 2 Equivalent network

⇒ This theorem is applicable only for linear networks
⇒ This theorem is applicable only when load is variable

Case 1: Load is variable resistance R_L



$$I = \frac{V_S}{R_S + R_L}$$

$$P_L = V \cdot I = I^2 \cdot R_L \text{ W}$$

Max power deliver to R_L is

$$P_L = \frac{V_s^2}{(R_S + R_L)^2} \cdot R_L \text{ W}$$

To determine the value of R_L for maximum power transferred to the load.

$$\therefore \quad \frac{dP_L}{dR_L} \cdot = 0$$

$$\frac{dP_L}{dR_L} \cdot = \frac{v_s^{\,2}[(R_S + R_L)^2 \cdot 1 - 2R_L(R_S + R_L)]}{(R_S + R_L)^2} = 0$$

$$R_s^2 + R_L^2 + 2R_sR_L - 2R_L \cdot R_s - 2R_L^2 = 0$$

$$\boxed{R_s = R_L}$$

for max power transfer condition.

$$P_{\max} = V_s^2 \cdot \frac{R_s}{(R_s + R_s)^2}$$

$$\boxed{P_{\max} = \frac{V_s^2}{4R_s} \text{ W}}$$

$$P_{\text{total}} = I^2 R_S + I^2 R_L$$

$$= 2I^2 R_s \quad \Rightarrow \quad \frac{V_s^2}{2R_s} \text{ W}$$

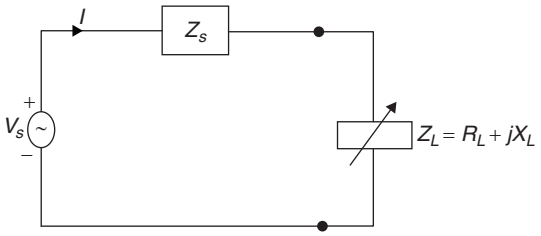
$$\text{Efficiency of MPT } \eta = \frac{\text{usefull power (loadpower)}}{\text{Total power}}$$

$$\eta\% = \frac{\frac{V_s^2}{4R_s}}{\frac{V_s^2}{2R_s}} \times 100$$

$$\eta_{\max} = 50\%$$

$\Rightarrow$ That is, the efficiency of a circuit at Maximum power transfer condition is 50% only.

Case 2: Load is variable Impedance Z_L and source impedance Z_s.



$$I = \frac{V_S}{Z_L + Z_S} = \frac{V_S}{(R_S + jX_S) + (R_L + jX_L)}$$

$$P_L = |I^2|\, R_L(W)$$

$$|I| = \frac{V_S}{\sqrt{(R_S + R_L)^2 + (X_S + X_L)^2}} \text{ A}$$

$$\boxed{P_L = \frac{V_S^2 \cdot R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}}$$

1. Only R_L is variable:

   For maximum power transfer $\frac{dP_L}{dR_L} = 0$

   $\therefore$ Condition for MPT

   $$R_L = \sqrt{R_S^2 + (X_S + X_L)^2} \ \Omega$$

   $$\boxed{\therefore \quad P_{\max} = \frac{P_L}{R_L} = \sqrt{R_S^2 + (X_S + X_L)^2}}$$

2. Only 'X_L' is variable:

   For MPT $\frac{dP_L}{dX_L} = 0$

   Condition for maximum power transfer

   $\Rightarrow \quad X_L = -X_S$

   $\Rightarrow \qquad \boxed{X_L + X_S = 0}$

3. Both R_L and X_L are varied simultaneously:

   In this case consider the above two conditions.

   $\therefore Z_L = R_L + jX_L$

   $= R_s - jX_s = Z_s^*$

   $$\therefore \quad \boxed{Z_L = Z_S^* \ \Omega}$$

   $P_{\max} = P_L$ at $R_L = R_s$ and at $X_L = -X_S$

   $$P_{\max} = \frac{P_L}{Z_L} = Z_s^*$$

   $$P_{\max} = \frac{V_s^2}{4R_L} = \frac{V_s^2}{4R_s} \text{ W}$$

Note:

In $Z_L = R_L + jX_L$

If $X_L = 0$

Then $R_L = \sqrt{R_S^2 + (X_S + X_L)^2}\ \Omega$

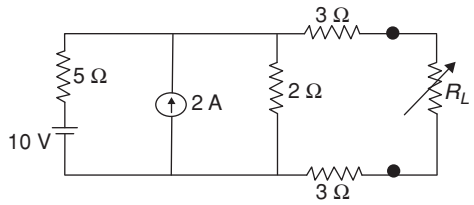
Subjected to $X_L = 0$ above

$$R_L = \sqrt{R_s^2 + X_s^2}\ \Omega$$

$$\boxed{\therefore \quad R_L = |Z_S|\ \Omega}$$

$$P_{max} = P_L \text{ at } R_L = |Z_s|\ \Omega$$

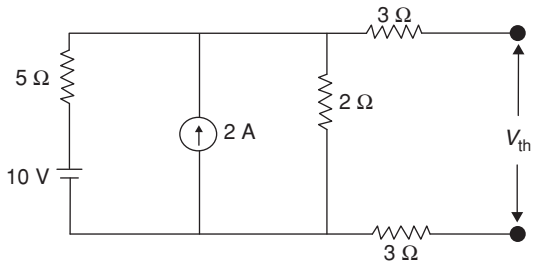
Example 16: Find the maximum power that can be transferred to R_L



Solution:

Find the Thevenin's equivalent circuit

(i) V_{th}:



Applying nodal Analysis,

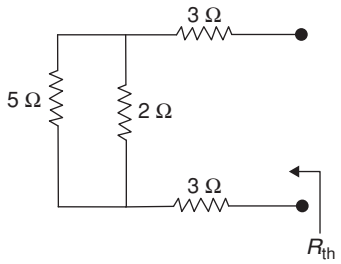
$$\frac{V_{th} - 10}{5} + \frac{V_{th}}{2} - 2 = 0$$

$$2(V_{th} - 10) + 5\,V_{th} = 20$$

$$7V_{th} = 40$$

$$V_{th} = \frac{40}{7}\ \text{V}$$

(ii) R_{th}:



$$R_{th} = (5||2) + 3 + 3$$

$$= 6 + \frac{10}{7}$$

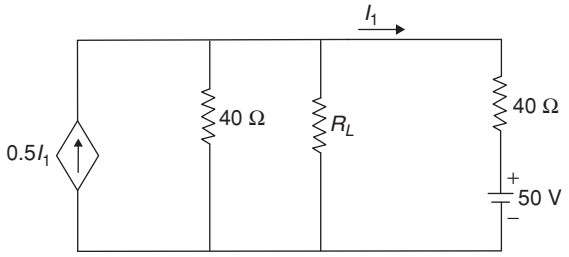
$$= 7.423\ \Omega$$

$$P_{max} = \frac{V_{th}^2}{4R_{th}} = \frac{36.65}{29.69}$$

$$= 1.099\ \text{W}$$

$$\approx 1\ \text{W}$$

Example 17: In the network of the figure, the maximum power is delivered to R_L if its value is



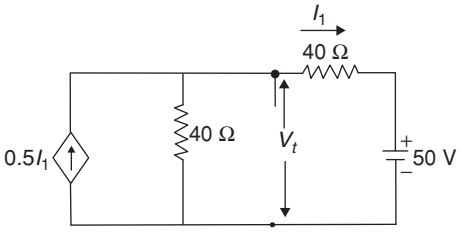
(A) $R_L = 16.66\ \Omega$ (B) $R_L = 20\ \Omega$

(C) $R_L = 26.66\ \Omega$ (D) None of the above

Solution:

$$R_{th} = R_L = \frac{V_{th}}{I_{sc}}$$

V_{th}:



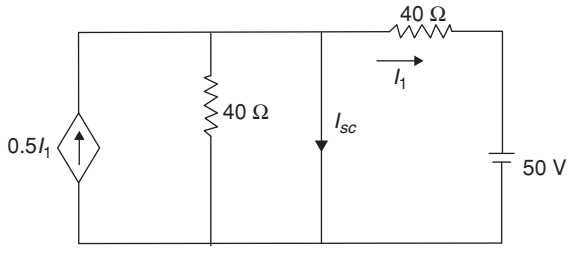
$$\frac{V_{th} - 50}{40} + \frac{V_{th}}{40} = 0.5\left[\frac{V_{th} - 50}{40}\right]$$

$$\frac{1}{2}\left[\frac{V_{th} - 50}{40}\right] + \frac{V_{th}}{40} = 0$$

$$V_{th} - 50 + 2\,V_{th} = 0$$

$$3V_{th} = 50 \quad \Rightarrow \quad V_{th} = \frac{50}{3}\ \text{V}$$

I_{sc} :



$$I_{SC} = -0.5I_1$$

$$= -0.5\left[\frac{0 - 50}{40}\right]$$

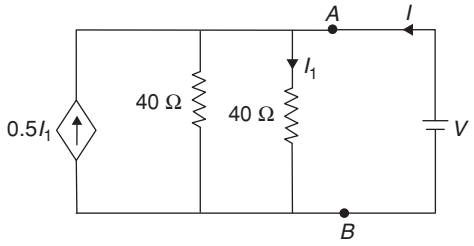
$$= 0.625\ \text{A}$$

$$\therefore \quad R_{th} = \frac{V_{th}}{I_{sc}} = \frac{50}{3 \times 0.625}\ \Omega$$

$$= 26.66\ \Omega$$

2nd method: For maximum power delivered to R_L open circuit R_L.

R_{th} Across AB.



All independent sources deactivated

KCL at Node A

$$0.5I_1 + I = \frac{V}{40} + \frac{V}{40}$$

But $$I_1 = \frac{V}{40}$$

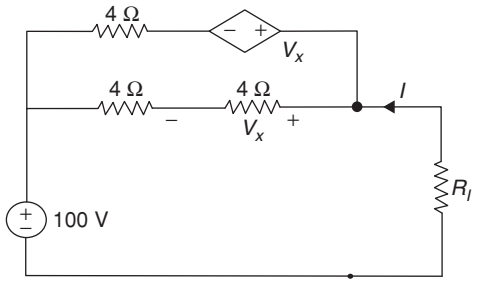
$$0.5\frac{V}{40} + I = \frac{2V}{40}$$

$$I = \frac{1.5\ \text{V}}{40}$$

$$\frac{V}{I} = \frac{40}{1.5} = 26.66\ \Omega$$

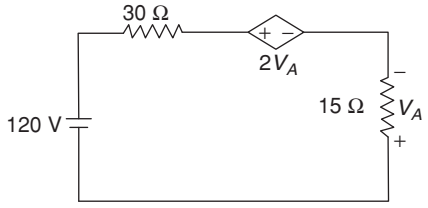
Solution:

In the circuit shown, what value of R_L maximizes the power delivered to R_L?



Note: The value of R_L maximizes the power delivered to the load equal to Thevenin's resistance find the Thevenin's resistance by using case 3.

Example 18: In the circuit shown in the below figure, the power absorbed by each element is



Solution:

Applying KVL for the loop

$$120 = 30I + 2V_A = +15I$$

$$V_A = -15I$$

$$120 = 30I - 2V_A + 15I = 0$$

$$I = 8\ \text{A}$$

$$P_{120\text{V}} = V \cdot I = -8 \times 120 = -960\ \text{W}$$

Delivered by 960 W

$$P_{30\Omega} = 30 \times 8^2 = 1920\ \text{W} \Rightarrow \text{absorbed}$$

$$P_{2VA} = +8 \times (-2 \times 15 \times 8) = -1920\ \text{W}$$

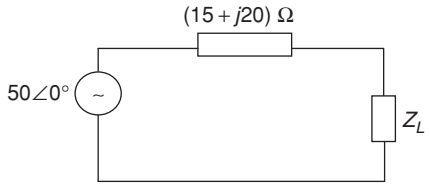
$$P_{15\Omega} = 82 \times 15 = 960\ \text{W}$$

$$\text{Total power} = -960 + 1920 + 960 - 1920$$

$$= 0\ \text{W}.$$

$\therefore$ Total power delivered = total power absorbed.

Example 19:



For the circuit shown in figure, find the value of load impedance for which the source delivers maximum power. Calculate the value of the maximum power

(A) 41.33 W (B) 50 W

(C) 75 W (D) −30 W

Solution: (A)

For maximum power transfer

$$Z_L = Z_S^*$$

$$Z_L = (15 - j20)\ \Omega$$

When $Z_L = (15 - j20)\ \Omega$, the current passing through the circuit is

$$I = \frac{V_S}{Z_S + Z_L} = \frac{50\angle 0°}{15 + j20 + 15 - j20}$$

$$I = \frac{50\angle 0°}{30\angle 0°} = 1.66\angle 0°$$

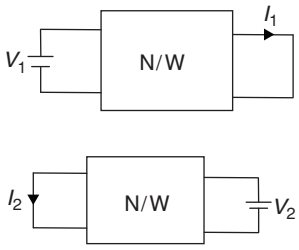
The maximum power transferred to load is

$$P_{max} = I^2 R_L$$

$$= (1.66)^2 \times 15 = 41.33\ \text{W}$$

Reciprocity Theorem

Statement: In a linear, passive, bilateral and time invariant network, the ratio of output to input (source) is constant even though the source is interchanged from input terminals to output terminals.



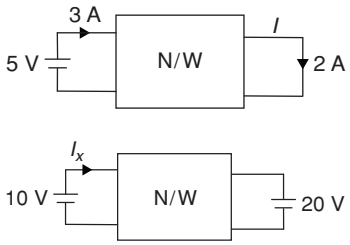
i.e., $\frac{I}{V} = \text{constant}$

$\Rightarrow \quad \frac{I_1}{V_1} = \frac{I_2}{V_2}$

Note: The presence of the dependent sources makes the network active and hence the reciprocity theorem is not applicable.

$\therefore$ Reciprocity theorem is not applicable to active networks.

Example 20:



The network contains only resistances. Use the data given in Figure 1 and find the current I_x in Figure 2.

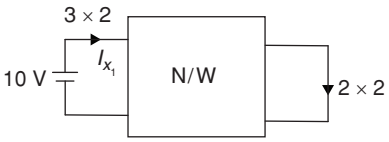
(A) 2 A (B) 6 A
(C) –2 A (D) 1 A

Solution: (C)

From the Reciprocity theorem

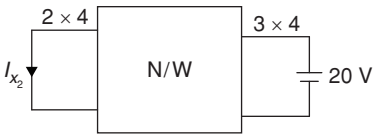
$$\frac{I_1}{V_1} = \frac{I_2}{V_2} = \text{constant}$$

(i) Consider that the 10 V is activated



$I_{x_1} = 6$ A.

(ii) Consider 20 V source is activated



$\therefore I_{x_2} = 8$ A

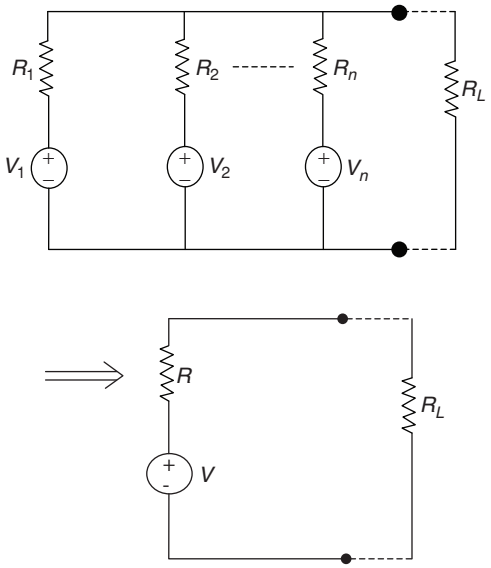
$\therefore$ By using Superposition theorem

$I_x = I_{x_1} - I_{x_2}$

$= 6 - 8 = -2$ A

Millman's Theorem

Statement: When a number of voltage sources ($V_1, V_2 \ldots V_n$) are in parallel having Internal resistances ($R_1, R_2, \ldots R_n$) respectively. The arrangement can be replaced by a single equivalent voltage source V in series with an equivalent series resistance R as given below.



As per Millman's Theorem

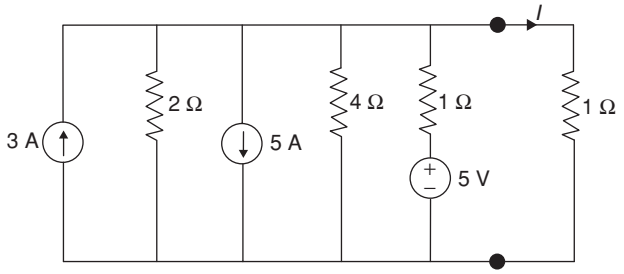
$$V = \frac{\pm V_1 G_1 \pm V_2 G_2 \pm \cdots \pm V_n G_n}{G_1 + G_2 + \cdots + G_n}$$

$$R = \frac{1}{G} = \frac{1}{G_1 + G_2 + \cdots + G_n}$$

i.e., $$\boxed{V = \frac{\sum_{k=1}^{n} V_K G_k}{\sum_{k=1}^{n} G_k}}$$

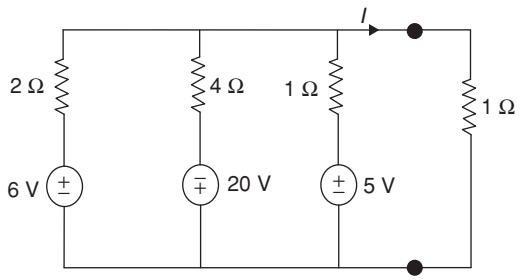
$\Rightarrow$ This theorem is applicable to only linear networks.

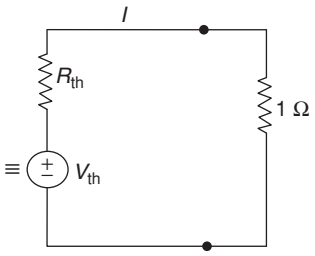
Examples 21: Find the current through the 1 Ω resistor using Millman's Theorem.



(A) $I = 3$ A (B) $I = 1.09$ A
(C) $I = 7$ A (D) $I = -1$ A

Solution: (B)
Converting current source to an equivalent voltage source





$$I = \frac{V_{th}}{R_{th}+1}\text{ A}$$

$$\Rightarrow \quad V_{th} = \frac{\Sigma V_k G_k}{G_k}$$

$$V_{th} = \frac{\frac{6}{2}-\frac{20}{4}+\frac{5}{1}}{\frac{1}{2}+\frac{1}{4}+\frac{1}{1}}$$

$$= \frac{3-5+5}{0.5+0.25+1} = \frac{3}{1.75} = 1.714\text{ V}$$

$$R = \frac{1}{\frac{1}{2}+\frac{1}{4}+\frac{1}{1}} = \frac{1}{1.75} = 0.5714\ \Omega$$

$$\Rightarrow \quad I = \frac{1.714}{1+0.57} = 1.09\text{ A.}$$

Tellegen's Theorem

Statement: In an arbitary network, the algebraic sum of powers at any given instant is zero. That is, the power delivered by some elements is equal to power absorbed by other elements present in the network.

$$\therefore \quad \sum_{j=1}^{n} V_j \times i_j = 0$$

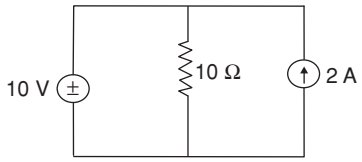
Where n = Total number of branches

Notes:

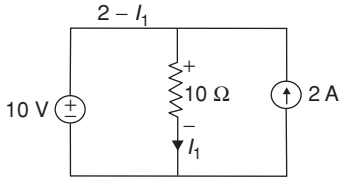
1. When current enters at the positive terminals of an element, then that element will absorbs the power, otherwise it will delivers the power.
2. Sources can deliver power or absorb power, whereas the passive elements will always absorb power since the current will enter at the positive terminal in the respective R, L, C's.

Properties: This Theorem is independent of the nature of the elements.

Example 22: For the circuit shown in figure, verify the Tellengen's Theorem.



Solution:



$$I_1 = \frac{10}{10} = 1$$

$\therefore \quad P_{10\Omega} = V \cdot I = I^2 \cdot R$

$= 10$ W (absorbed)

$V_{2A} = 10$ V

$\Rightarrow \quad P_{2A} = -2 \times 10 = -20$ W (delivered)

$I_{10V} = 2 - 1 = 1$ A

$\Rightarrow \quad P_{10V} = 10 \times 1 = 10$ W (absorbed)

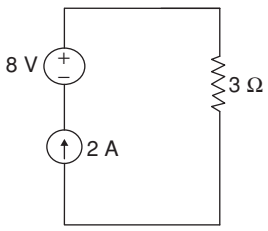
$\therefore \Sigma V \cdot I = 0$

Total absorbed power = Total delivered power

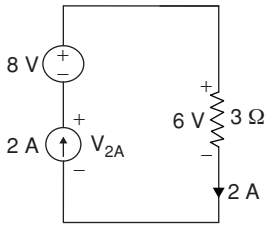
$$P_{2A} + P_{10V} + P_{10\Omega} = 0$$

$\therefore$ Tellegen's Theorem is verified.

Example 23:



Solution:
Verify Tellengen's Theorem?



$V_{2A} = 6 - 8 = -2$ V

$P_{8V} = V \cdot I = -8 \times 2 = -16$ W (delivers)

$P_{2A} = 2 \times 2 = 4$ W (absorbed)

$P_{3\Omega} = \text{V} \cdot I = 12$ W (absorbed)

$\therefore \sum \text{power} = 0$

$-16 + 4 + 12 = 0$

$\therefore$ Tellegen's Theorem is verified.

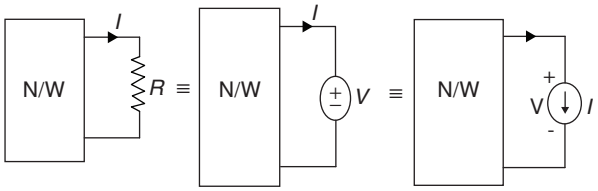
Substitution Theorem

The voltage across the current through any branch of a DC bilateral network can be replaced by any combination of elements that will make the same voltage across and current though the chosen branch.

In a linear network any passive element can be equivalently substituted by an ideal voltage source or an ideal current source provided all the other branch current and voltages are kept constant.

Any branch in a linear network can be substituted by a different branch without disturbing the voltage and currents in the entire network provided the new branch has the same set of terminal voltage and currents as the original network.

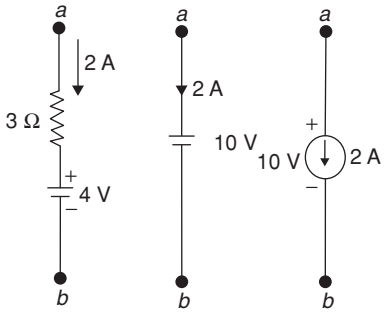
$\Rightarrow$ This theorem is applicable for any LTI and bilateral networks.



Where $V = R \cdot I$

$P = V \cdot I = I^2 R(W)$

Example:



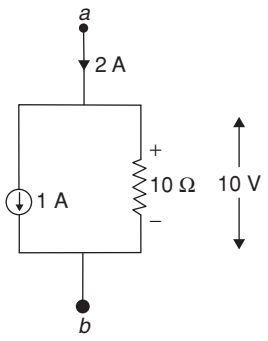


Figure 3 Equivalent circuits of branch $a - b$

Compensation Theorem

Statement: In an LTI network when the resistance 'R' of an uncoupled branch, carrying a current (I), is changed by ΔR, the current in all the branches would change and can be obtained by assuming that an ideal voltage source of (V_s) has been connected in series with ($R + \Delta R$) when all other sources in the network are replaced by their internal resistances $V_s = I \cdot \Delta R$.

This theorem is useful in determining the current and voltage changes in circuit element when the value of its impedance is changed.

Example: Bridge and potentiometer circuit.

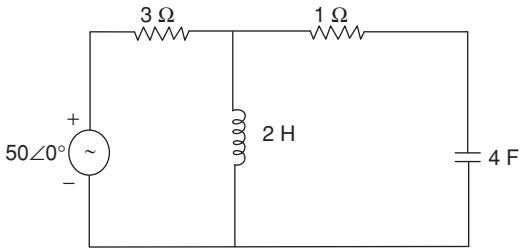
Duality of circuits Two linear circuits are said to be duals of one another if they are described by the same characteristic equations with dual quantities interchanged.

Network and its dual are same only with respect to the performance, but the elements and connects point of view are not equal.

Table 1 *Dual pairs*

R	$\leftrightarrow$	G
L	$\leftrightarrow$	C
Z	$\leftrightarrow$	Y
V	$\leftrightarrow$	I
Voltage Source	$\leftrightarrow$	Current Source
KCL	$\leftrightarrow$	KVL
Star	$\leftrightarrow$	Delta
Node	$\leftrightarrow$	Mesh
Series	$\leftrightarrow$	Parallel
Open circuit	$\leftrightarrow$	Short circuit
$L \times \frac{di(t)}{dt}$	$\leftrightarrow$	$C \cdot \frac{dv(t)}{dt}$
$\frac{1}{c}\int i(t)dt$	$\leftrightarrow$	$\frac{1}{L}\int v(t) \cdot dt$
Thevenin's	$\leftrightarrow$	Norton's
$R \cdot i(t)$	$\leftrightarrow$	$G \cdot V(t)$

Example 24:

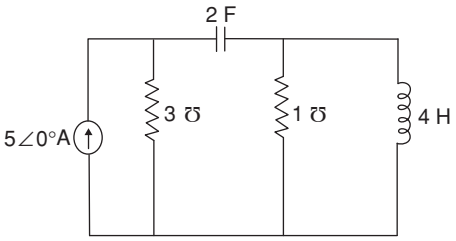


The dual of the network is

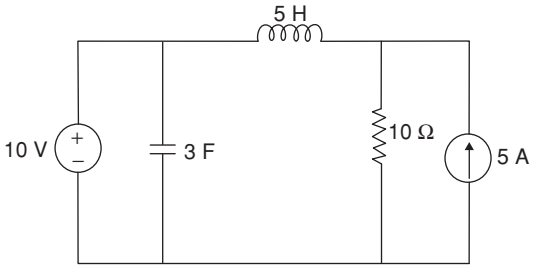
Solution:

Dual of the given network,

$V \leftrightarrow I$, $R \leftrightarrow G$; $L \leftrightarrow C$ and series $\leftrightarrow$ parallel



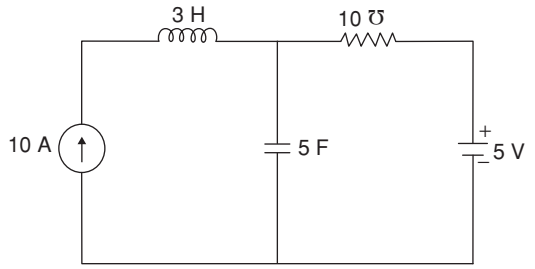
Example 25: Obtain the dual of the network shown in figure below.



Solution:

Dual of the above network

Series $\leftrightarrow$ parallel, $V \leftrightarrow I$, $L \leftrightarrow C$, $R \leftrightarrow G$.

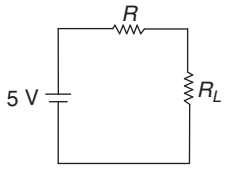


EXERCISES

Practice Problems I

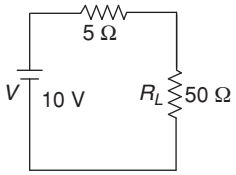
Directions for questions 1 to 28: Select the correct alternative from the given choices.

1. The maximum power transferred to the load in the circuit is given as 0.5 W. Get the values of R and R_L.



(A) 15 Ω, 10 Ω (B) 12.5 Ω, 12.5 Ω
(C) 10 Ω, 15 Ω (D) 10 kΩ, 10 kΩ

2. Find the efficiency of the circuit given for $R_L = 50\ \Omega$

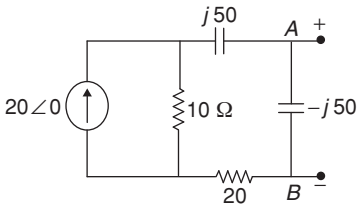


(A) 99% (B) 91%
(C) 80% (D) 87%

3. Current in the circuit is given by the equation $i(t) = 10\cos(20\pi t + 50)$ and the impedance of the load is given as $Z_L = 5 + j3$. Find the average power delivered to the load.

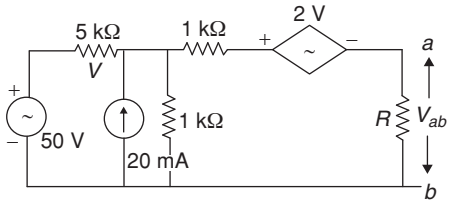
(A) 353.5 W (B) 291.5 W
(C) 250 W (D) 176.7 W

4. In the circuit shown below the Norton equivalent current in amps across $A - B$ is



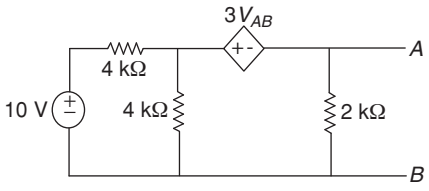
(A) $19.45 + j3.24$ (B) $6.48 - j1.08$
(C) $12.97 - j2.16$ (D) $20 + j0$

5. Find the Thevenin equivalent voltage external to the load R_L.



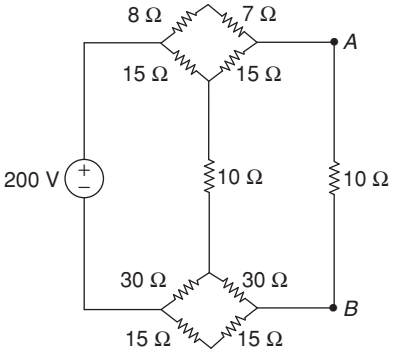
(A) 25 V (B) 50 V
(C) 49 V (D) 45 V

6. Find the Thevenin's resistance associated with the circuit.



(A) 1 kΩ (B) 0.45 kΩ
(C) 2 kΩ (D) 0.22 kΩ

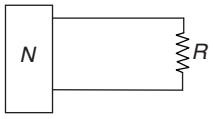
Common Data for Questions 7 and 8: Select the correct alternative from the given choices.



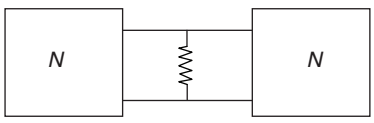
7. Find Thevenin's equivalent voltage of the circuit.
(A) 100 V (B) 120 V
(C) 125 V (D) 150 V

8. The resistance across $A - B$ is 10 Ω. Find the current through the 10 Ω resistor.
(A) 5.1 A (B) 6.45 A
(C) 3.35 A (D) 13.9 A

9. When a resistor R is fed from an electrical network 'N' consumes a power of 'P' W as shown in Figure (a). If an identical network is added as shown in Figure (b) the power consumed by R will be _____.



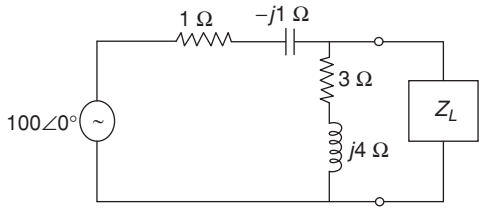
(a)



(b)

(A) 2 P (B) P
(C) $\frac{P}{2}$ (D) between P and 4 P

Common Data for Questions 10 and 11: Select the correct alternative from the given choices.

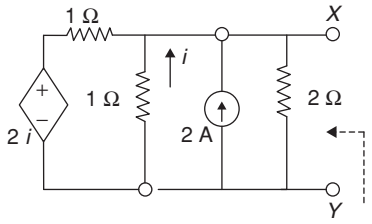


10. Find the value of Z_L at which maximum power is transferred to Z_L.
(A) (1.24 − j0.676) Ω (B) (1.24 + j0.676) Ω
(C) 1.31 Ω (D) 1.24 Ω

11. The maximum power transferred is
(A) 201.6 W (B) 617 W
(C) 2016 W (D) 6170 W

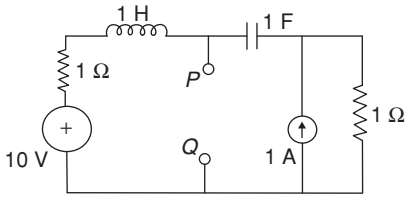
Common Data for Questions 12 to 15: Select the correct alternative from the given choices.

12. For the Circuit shown in the figure, the Thevenin voltage and resistance looking into $x - y$ are _____.



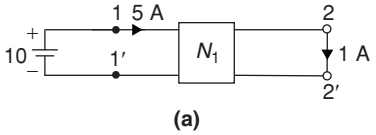
(A) 4/3V, 2 Ω (B) 4V, 2/3 Ω
(C) 4/3V, 2/3 Ω (D) 4V, 2 Ω

13. The Thevenin equivalent impedance Z_{th} between the nodes P and Q in the following circuit is _____.

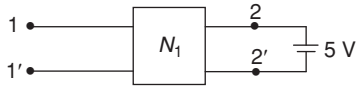


(A) 1 (B) $1+S+\frac{1}{S}$
(C) $2+S+\frac{1}{S}$ (D) $\frac{S^2+S+1}{S^2+2S+1}$

14. The short circuit test of a 2 port π network is shown in Figure (a). The voltage across the terminals 11′ in the network shown in Figure (b) will be



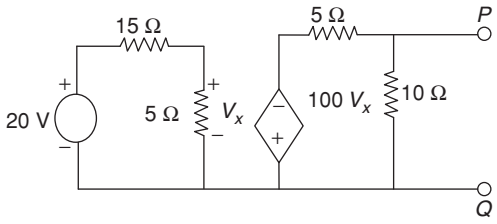
(a)



(b)

(A) 2 V (B) 5 V
(C) 10 V (D) 1 V

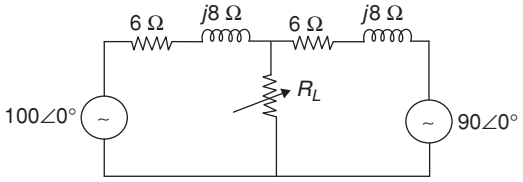
15. The Norton's equivalent circuit at terminals PQ has a current source and a Norton's resistance of _____.



(A) 10 A, 3.33 Ω (B) 100 A, $\frac{85}{12}$ Ω
(C) 100 A, 3.33 Ω (D) 500 A, 3.33 Ω

Common Data for Questions 16 and 17: Select the correct alternative from the given choices.

16.

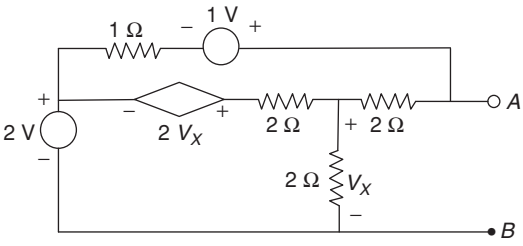


In the circuit shown in figure, under the maximum power transfer condition, the value of R_L is _____.
(A) 5 Ω (B) 20 Ω
(C) $\frac{25}{3}$ Ω (D) 6 Ω

17. The power absorbed by R_L at maximum power transfer condition is _____.
(A) 1000 W (B) 500 W
(C) 625 W (D) 2000 W

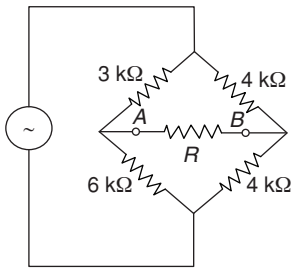
Common Data for Questions 18 to 22: Select the correct alternative from the given choices.

18. The Thevenin voltage at the terminals *AB* of the network shown in the figure is



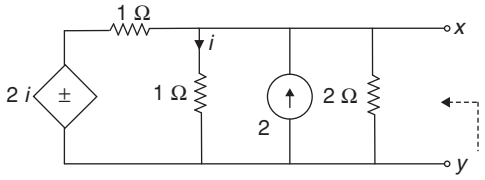
(A) 4 V (B) 2 V
(C) $\frac{3}{2}$ V (D) $\frac{1}{2}$ V

19. The value of the resistance *R*, connected across the terminals *A* and *B*, which will absorb the maximum power is



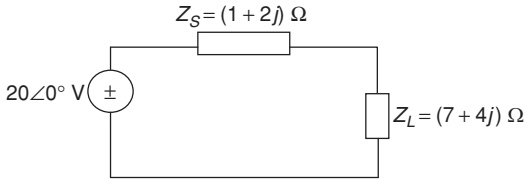
(A) 4 kΩ (B) 4.11 kΩ
(C) 8 kΩ (D) 9 kΩ

20. For the circuit shown in figure, the Thevenin voltage and resistance looking into $X - Y$ are



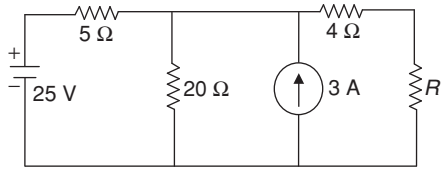
(A) $\frac{4}{3}$ V, 2 Ω (B) 4 V, $\frac{2}{3}$ Ω
(C) $\frac{4}{3}$ V, $\frac{2}{3}$ Ω (D) 4 V, 2 Ω

21. An AC source of RMS voltage 20 V with internal impedance $Z_S = (1 + 2j)$ Ω feeds a load of impedance $Z_L = (7 + 4j)$ Ω shown in the figure. The reactive power consumed by the load is



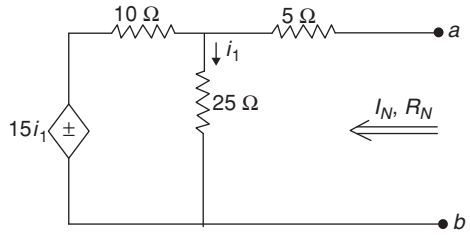
(A) 8 VAR (B) 16 VAR
(C) 28 VAR (D) 35 VAR

22. The value of *R* (in Ohms) required for maximum power transfer in the network shown in the figure is



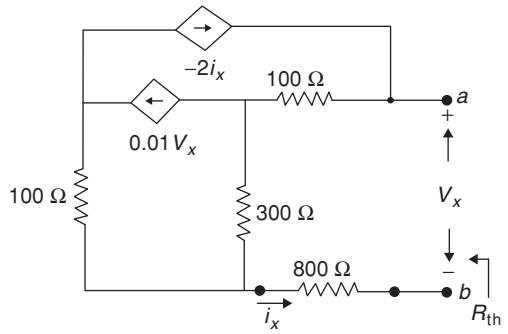
(A) 2 (B) 4
(C) 8 (D) 16

23. For the following circuit the value of i_N and R_N are



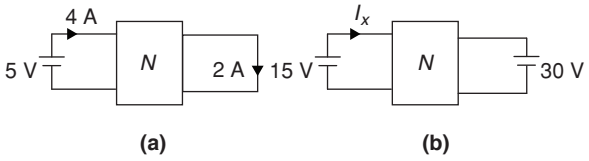
(A) 2 A, 12 Ω (B) 0 A, 20 Ω
(C) 0.5 A, 20 Ω (D) 0 A, 12 Ω

24. For the circuit shown in figure below the values of R_{th} and V_{th} are



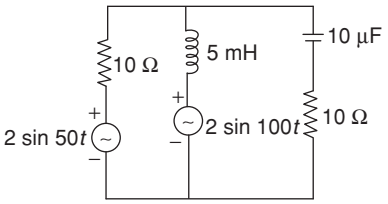
(A) 0 V, 100 Ω (B) 0 V, 1200 Ω
(C) 5 V, 100 Ω (D) None of the above

25. Consider the following circuits



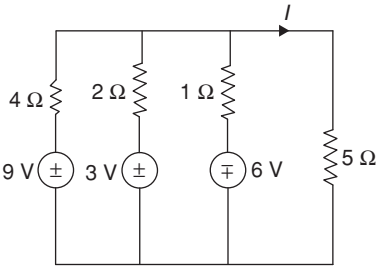
The network '*N*' contains only resistances. Use the data given in Figure (a) and find the current i in Figure (b)
(A) 0 A (B) 12 A
(C) −6 A (D) 6 A

26. In the circuit shown in figure, which one of the following theorem can be more conveniently used to evaluate the responses in the 10 Ω resistors.



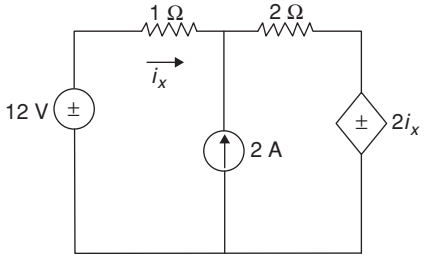
(A) Thevenin's Theorem
(B) M P T T (Maximum Power Transfer Theorem)
(C) Milliman's Theorem
(D) Superposition Theorem

27. Consider the network shown in the figure below:



The current *I* is
(A) 0.23 A (B) −0.23 A
(C) 2.25 A (D) −0.5 A

28. Consider the circuit shown in the below figure:

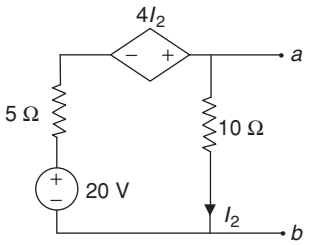


The current i_x would be
(A) $i_x = 1.6$ A (B) $i_x = 1.3$ A
(C) $i_x = -1.5$ A (D) $i_x = 0.8$ A

Practice Problems 2

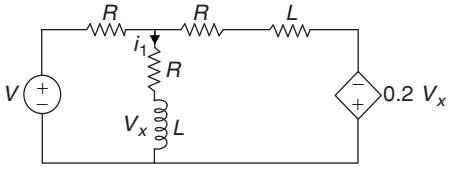
Directions for questions 1 to 16: Select the correct alternative from the given choices.

1. Find the Thevenin's equivalent of the Circuit given below:



(A) $V_{th} = 20$ V, $R_{th} = 3.3$ Ω
(B) $V_{th} = 16$ V, $R_{th} = 5$ Ω
(C) $V_{th} = 20$ V, $R_{th} = 5$ Ω
(D) $V_{th} = 4$ V, $R_{th} = 10$ Ω

2. Find the state equation for the circuit given.



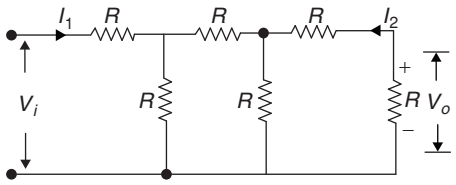
(A) $L\frac{di_2}{dt} = 0.7V_x - 1.5i_2R + .5$ V

(B) $L\frac{di_2}{dt} = -0.7V_x + 1.5i_2R - .5$ V

(C) $L\frac{di_2}{dt} = 0.7V_x - 1.5i_2R + .5$ V

(D) $L\frac{di_2}{dt} = 0.7V_x - 1.5i_2R - .5$ V

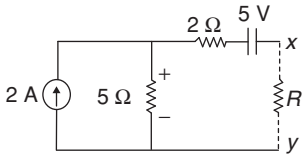
3. Find the transfer function of the network shown.



(A) $\frac{V_0}{V_i} = \frac{1}{5}$ (B) $\frac{V_0}{V_i} = \frac{1}{13}$

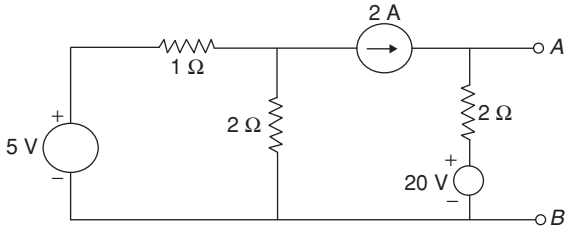
(C) $\frac{V_0}{V_i} = 0$ (D) $\frac{V_0}{V_i} = \frac{-1}{13}$

4. A network is shown below with an unknown load R. Find the value of R so that maximum power is delivered to the load.



(A) 5 Ω (B) 7 Ω
(C) 1.43 Ω (D) 2 Ω

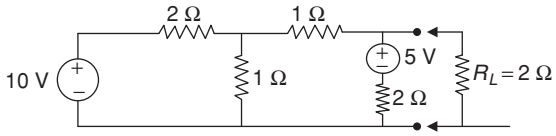
5. The Thevenin's resistance across the terminals AB of the figure is _____.



(A) $\frac{2}{3}\Omega$ (B) 2 Ω

(C) $\frac{7}{3}\Omega$ (D) 5 Ω

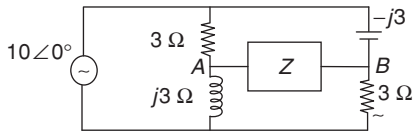
6. In the circuit shown in the figure, the power consumed by R_L is



(A) $\frac{25}{16}$ W (B) $\frac{25}{3}$ W

(C) $\frac{25}{8}$ W (D) $\frac{5}{2}$ W

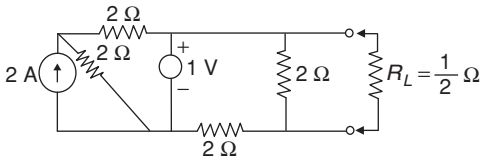
7. In the circuit shown in the given figure the Thevenin impedance between terminals A and B is _____.



(A) 6 Ω (B) 3 Ω

(C) $\frac{3}{2} - \frac{j3}{2}$ (D) $6 + j6$

8. In the circuit shown in the figure, the current through resistance R_L is _____.



(A) $\frac{2}{3}$ A (B) $\frac{3}{2}$ A

(C) $\frac{4}{3}$ A (D) $\frac{1}{3}$ A

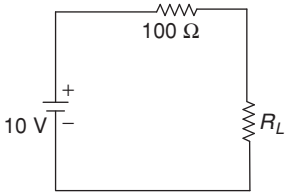
9. A source of angular frequency 1 rad/sec has source impedance consisting of 1 Ω resistance in series with 1 H inductance. The load that will obtain the maximum power transfer is

(A) 1 Ω resistance
(B) 1 Ω resistance in parallel with 1 H inductance
(C) 1 Ω resistance in series with 1 F capacitor
(D) 1 Ω resistance in parallel with 1 F capacitor

10. Superposition theorem is NOT applicable to networks containing

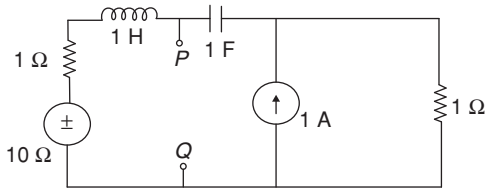
(A) non linear elements
(B) dependent voltage sources
(C) dependent current sources
(D) transformers

11. The maximum power that can be transferred to the load resistor R_L from the voltage source in the figure is



(A) 1 W (B) 10 W
(C) 0.25 W (D) 0.5 W

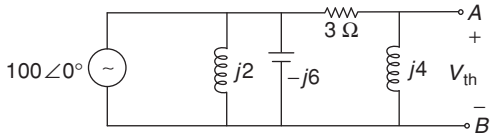
12. The Thevenin equivalent impedance Z_{th} between the nodes P and Q in the following circuit is



(A) 1 (B) $1+s+\frac{1}{s}$

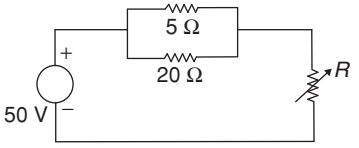
(C) $2+s+\frac{1}{s}$ (D) $\frac{s^2+s+1}{s^2+2s+1}$

13. The Thevenin equivalent voltage V_{th} appearing between the terminals A and B of the network shown in the figure is given by



(A) $j16\,(3-j4)$ (B) $j16\,(3+j4)$
(C) $16\,(3+j4)$ (D) $16\,(3-j4)$

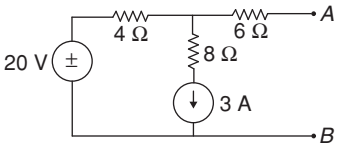
14.



In the circuit shown, the adjustable resistor R is set such that the power in the 5 Ω resistor is 20 W. The value of R is

(A) 6 Ω (B) 25 Ω
(C) 4 Ω (D) 16 Ω

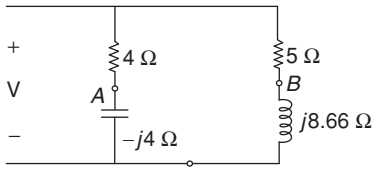
15.



The Norton equivalent of the above circuit is

(A) $I_N = 8$ A, $R_N = 10$ Ω
(B) $I_N = 0.8$ A $R_N = 10$ Ω
(C) $I_N = 3$ A $R_N = 8$ Ω
(D) $I_N = 8$ A $R_N = 3$ Ω

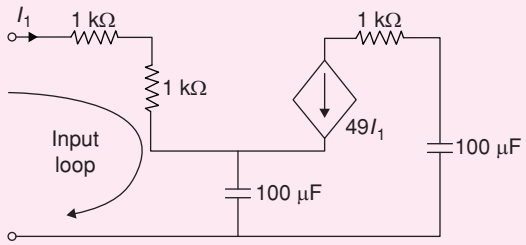
16. In the circuit shown in the below figure, $V_{AB} = 48.3\angle 30°$. The applied voltage V is



(A) $40\angle 90°$ (B) $100\angle 130°$
(C) $50\angle 135°$ (D) $100\angle 135°$

Previous Years' Questions

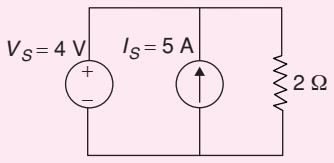
1. The equivalent capacitance of the input loop of the circuit shown is **[2009]**



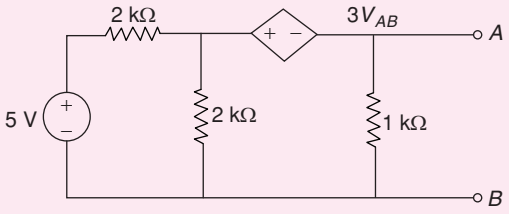
(A) 2 μF (B) 100 μF
(C) 200 μF (D) 4 μF

2. For the circuit shown, find out the current flowing through the 2 Ω resistance. Also identify the changes to be made to double the current through the 2 Ω resistance **[2009]**

(A) (5 A; Put $V_s = 20$ V) (B) (2 A; Put $V_s = 8$ V)
(C) (5 A; Put $I_s = 10$ A) (D) (7 A; Put $I_s = 12$ A)



Common Data for Questions 3 and 4:



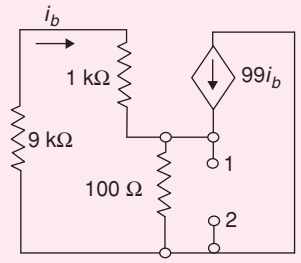
3. For the circuit given above, the Thevenin's resistance across the terminals A and B is **[2009]**

(A) 0.5 kΩ (B) 0.2 kΩ
(C) 1 kΩ (D) 0.11 kΩ

4. For the circuit given above, the Thevenin's voltage across the terminals A and B is **[2009]**

(A) 1.25 V (B) 0.25 V
(C) 1 V (D) 0.5 V

5. The impedance looking into nodes 1 and 2 in the given circuit is **[2012]**

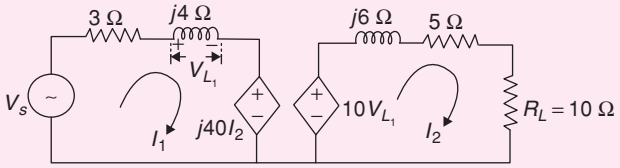


(A) 50 Ω (B) 100 Ω
(C) 5 kΩ (D) 10.1 kΩ

6. A source $v_s(t) = V \cos 100\ \pi t$ has an internal impedance of $(4 + j3)$ Ω. If a purely resistive load connected to this source has to extract the maximum power out of the source, its value is Ω should be **[2013]**

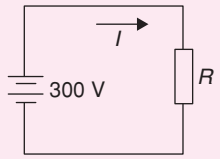
(A) 3 (B) 4
(C) 5 (D) 7

7. In the circuit shown below, if the source voltage $V_s = 100\angle 53.13^\circ$ V then the Thevenin's equivalent voltage in Volts as seen by the load resistance R_L is **[2013]**

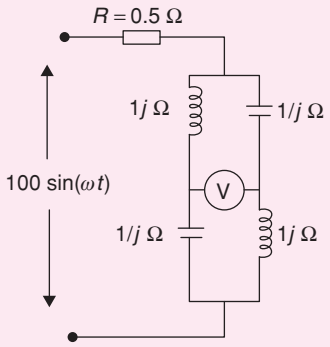


(A) $100\angle 90^\circ$ (B) $800\angle 0^\circ$
(C) $800\angle 90^\circ$ (D) $100\angle 60^\circ$

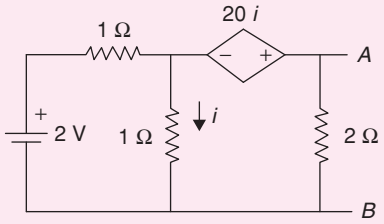
8. In the figure, the value of resistor R is $\left(25+\frac{I}{2}\right)$ ohms, where I is the current in amperes. The current I is _____. **[2014]**



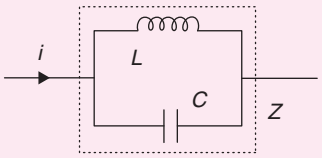
9. The reading of the voltmeter (rms) in Volts, for the circuit shown in the figure is_____ **[2014]**



10. For the given circuit, the Thevenin equivalent is to be determined. The Thevenin voltage, V_{Th} (in Volt), seen from terminal AB is _____. **[2015]**

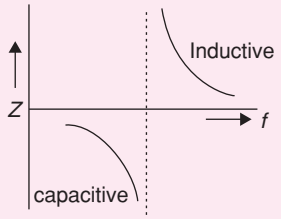


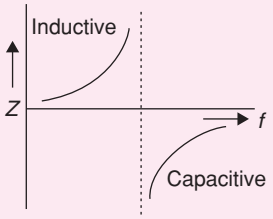
11. An inductor is connected in parallel with a capacitor as shown in the figure.



As the frequency of current i is increased, the impedance (Z) of the network varies as **[2015]**

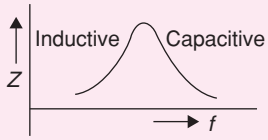
(A)



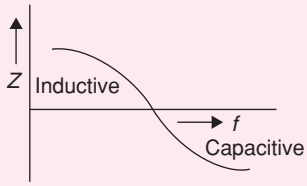


(B)

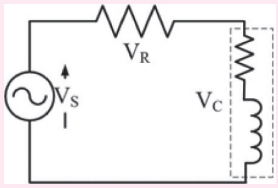
(C)



(D)



12. A resistance and a coil are connected in series and supplied form a single phase, 100V, 50Hz ac source as shown in the figure below. The rms values of possible voltages across the resistance (V_R) and coil (V_C) respectively, in volts, are **[2016]**



(A) 65, 35 (B) 50, 50
(C) 60, 90 (D) 60, 80

13. The voltage (V) and current (A) across a load are as follows.

$v(t) = 100\text{Sin}(\omega t)$,

$I(t) = 10\text{Sin}(\omega t - 60°) + 2\text{Sin}(3\omega t) + 5\text{Sin}(5\omega t)$

The average power consumed by the load, in W is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** B	**3.** C	**4.** B	**5.** A	**6.** D	**7.** C	**8.** B	**9.** D	**10.** B
11. C	**12.** D	**13.** A	**14.** D	**15.** C	**16.** A	**17.** B	**18.** A	**19.** A	**20.** D
21. A	**22.** C	**23.** B	**24.** A	**25.** A	**26.** C	**27.** B	**28.** A		

Practice Problems 2

1. A	**2.** A	**3.** B	**4.** B	**5.** B	**6.** C	**7.** B	**8.** D	**9.** C	**10.** A
11. C	**12.** A	**13.** A	**14.** D	**15.** B	**16.** C				

Previous Years' Questions

1. A	**2.** B	**3.** B	**4.** D	**5.** A	**6.** C	**7.** C	**8.** 10	**9.** 141.42
10. 3.36 V		**11.** B	**12.** D	**13.** 250				

Chapter 3

Transient Analysis (AC and DC)

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Classification of transients
- Singularity functions
- Step response of an *RC* circuit
- Transient response
- Steady-state response
- Higher-order circuits
- AC transients
- Sinusoidal steady-state analysis
- Phasor
- Inductor
- Sinusoidal steady state analysis of *RLC* circuits
- AC transients
- Sinusoidal steady-state analysis
- Sinusoidal steady state analysis of *RLC* circuits

Introduction

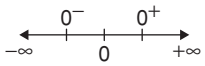
Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transitional period during which the branch currents and element voltages change from their initial values to new ones. This period is called the transient period. After the transient period has passed, the circuit is said to be in steady state.

Transient in the system is because of the presence of energy storing elements (i.e., L and C).

Since the energy stored in a memory element cannot change instantaneously, i.e., within zero time.

The network consists of only resistances, no transients in the system at the time of switching. Since the resistor can accommodate any amount of voltage and currents.

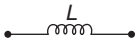
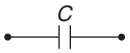
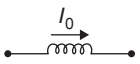
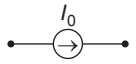
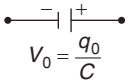
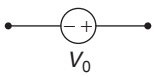
The equivalent form of the elements in terms of the initial condition of the elements.



1. $Z_L = sL\ \Omega$
2. $Z_C = \dfrac{1}{sC}\ \Omega$

At $t = 0^+ \Rightarrow f = \infty \Rightarrow Z_L = \infty \Rightarrow L \Rightarrow 0.C$

$Z_C = \dfrac{1}{2\pi fc} = 0\ \Omega \Rightarrow C \Rightarrow$ Short circuit

S. No	Element (Initial Condition)	Equivalent Circuit at $t = 0^+$
1.		
2.		
3.		
4.		
5.	 $V_0 = \dfrac{q_0}{C}$	 V_0

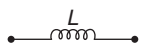
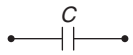
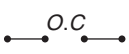
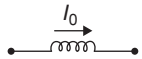
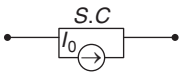
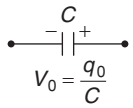
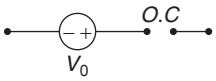
A long time after the switching action ($t \to \infty$) is the steady state. In *S.S*, the inductor behaviours is a short circuit and capacitor behaviours is an open circuit.

$t \to \infty \Rightarrow f = 0.$

$Z_L = sL\ \Omega \Rightarrow Z_L = 0\ \Omega \Rightarrow S.C$

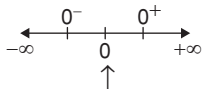
$Z_C = \dfrac{1}{sC}\ \Omega \Rightarrow Z_C = \infty \Rightarrow$ Open circuit.

* The equivalent form of the elements in terms of the final condition of the element.

S.No	Element (and Initial Condition)	Equivalent Circuit at $t = \infty$. (Steady State Values)
1.		
2.		
3.		
4.		
5.		

Notes:

1. Inductor current at $t = 0^-$ and $t = 0^+$ instants



Current flowing through an inductor is given as

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)\cdot dt$$

$$= \frac{1}{L}\int_{-\infty}^{0^-} V_L(t)dt + \frac{1}{L}\int_{0^-}^{0^+} V_L(t)dt$$

$$i_L(0^+) = i_L(0^-) + 0$$

$$\therefore \quad i_L(0^+) = i_L(0^-)$$

Therefore, inductor current cannot change instantaneously.

2. Capacitor voltage at $t = 0^-$ and at $t = 0^+$ instants

$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C \cdot dt$$

$$= V_C(0^-) + \frac{1}{C}\int_{0_-}^{t} i_C(t)\cdot dt$$

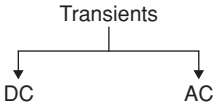
At $t = 0^+$

$$V_C(0^+) = V_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_C(t)\cdot dt$$

$$V_C(0^+) = V_C(0^-)\ \text{V}$$

Classification of Transients

The transient effects are more for DC as compared to AC and the transient free condition is possible to only for AC excitations.



DC Transients

Source free circuits

1. *RL* Circuit → initial current through the inductor ($L \to I_0$)
2. *RC* Circuit → initial voltage (V_0) across the capacitor ($C \to V_0$)
3. *RLC* Circuit → initial current through inductor or voltage across capacitor.

Source free RC circuits: A source free *RC* circuit occurs when its DC source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor as shown in figure 1.

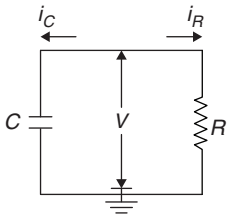


Figure 1 A source-free *RC* circuit.

The capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

$$V(0) = V_0$$

Applying KCL at the node of circuit in figure 1

$$i_C + i_R = 0$$

$$C\cdot\frac{dV}{dt} + \frac{V}{R} = 0 \quad \Rightarrow \quad \frac{dV}{dt} + \frac{1}{RC}\cdot V = 0$$

$$\frac{dV}{V} = -\frac{1}{RC}\cdot dt$$

Integrating both sides, we get

$$\ln\ V = \frac{-t}{RC} + \ln\ A$$

where A is the integration constant. Thus,

$$\ln\frac{V}{A} = \frac{-t}{RC} \quad \Rightarrow \quad \frac{V}{A} = e^{\frac{-t}{RC}}$$

$$\therefore \quad V(t) = A\cdot e^{-t/RC}$$

But from initial conditions at $t = 0$

$$V(0) = A\cdot 1\ A = V(0) = V_0$$

$$\boxed{\therefore \quad V(t) = V_0\cdot e^{\frac{-t}{RC}}\text{V}}$$

i.e., it is called the natural response of the circuit. The natural response of a circuit refers to the behaviour of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in figure 2.

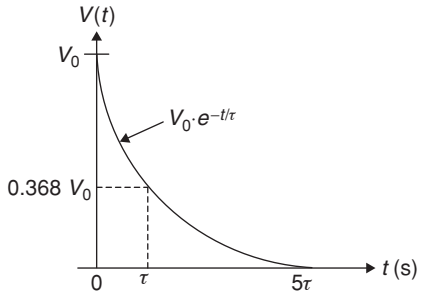


Figure 2 The voltage response of the *RC* circuit.

Time constant: The time constant of a circuit is the time required for the response to decay to a factor of 1/*e* or 36.8% of its initial value.

At $t = \tau$

$$V_0 \cdot e^{-\tau/RC} = V_0 \cdot e^{-1} = 0.368\ V_0$$

$\therefore \quad \tau = RC$

$$\therefore \quad V(t) = V_0 \cdot e^{\frac{-t}{RC}} = V_0 \cdot e^{\frac{-t}{t}}\ \text{V}$$

Note: In finding the time constant $\tau = RC$, R is obtained the thevenin's equivalent resistance at the terminals of the capacitor

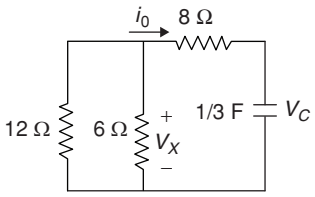
$$\therefore \quad \tau = R_{eq} \cdot C = R_{th} \cdot C$$

$$i_R(t) = \frac{V(t)}{R} = \frac{V_0}{R} \cdot e^{\frac{-t}{\tau}}\ \text{A}$$

Power dissipated in the resistor (R) is

$$P(t) = V \cdot i_R = \frac{V_0^2}{R} \cdot e^{\frac{-2t}{\tau}}$$

Examples 1: The circuit shown in the below figure. Let $V_C(0) = 15$ V



The value of $V_C(t)$, at $t = 2$ s

(A) 12 V (B) 13.62 V

(C) 2.04 V (D) 9.09 V

Solution: (D)

We know for source free *RC* circuits

$V_c(t) = V_0 \cdot e^{-t/RC}$

$\tau = R_{eq} \cdot C \quad \Rightarrow \quad R_{eq} = 8 + (6 \,||\, 12) \quad \Rightarrow \quad 12\ \Omega$

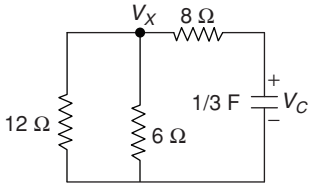
$\tau = 12 \times 1/3 = 4$ s

$V_c(t) = 15 \cdot e^{-t/4}$ V

at $t = 2$ s

$V_c(2) = 15 \cdot e^{-2/4} \quad \Rightarrow \quad 15 \cdot e^{-0.5} = 9.09$ V

Examples 2: The value of V_x for $t \geq 0$



(A) $V_x = 5 \cdot e^{-4t}$ V (B) $V_x = 12 \cdot e^{-0.25t}$ V

(C) $V_x = 5 \cdot e^{-0.25t}$ V (D) $V_x = 15 \cdot e^{-0.71t}$ V

Solution: (C)

Applying KCL at node V_x.

$$\frac{V_x}{12} + \frac{V_x}{6} + \frac{V_x - V_c}{8} = 0$$

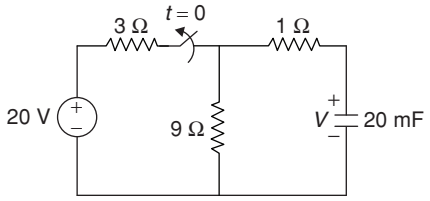
$2V_x + 4V_x + 3(V_x - V_c) = 0$

$9V_x = 3V_c$

$V_x = 1/3\ V_c \quad \Rightarrow \quad V_x = 1/3 \times 15 \cdot e^{-t/4}$ V.

$V_x = 5 \cdot e^{-0.25t}$ V.

Examples 3: The switch in the circuit in the below figure has been closed for a long time, and it is opened at $t = 0$



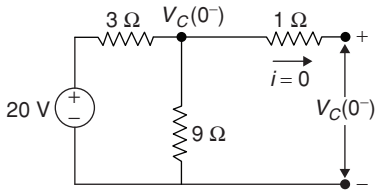
(i) The $V_C(t)$ for $t \geq 0$ is

(A) $V_C(t) = 15 \cdot e^{-t}$ V (B) $V_C(t) = 12 \cdot e^{-0.2t}$ V

(C) $V_C(t) = 15 \cdot e^{-5t}$ V (D) $V_C(t) = 12 \cdot e^{-5t}$ V

Solution: (C)

For $t < 0$: The switch is closed, the capacitor is open circuit to DC.



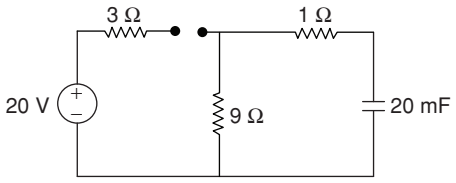
Circuit is in *S* and *S*.

$$\therefore \quad V_c(0^-) = 20 \times \frac{9}{9+3} = 15\ \text{V for } t < 0.$$

Since the voltage across capacitor does not change instantaneously,

$V_c(0^-) = V_c(0^+) = 15$ V

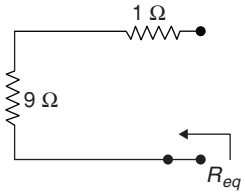
For $t \geq 0$. The switch is opened. The circuit shown in the following figure.



$\therefore \quad V_C(t) = V_0 \cdot e^{-t/\tau}$

$\tau = R_{eq} \cdot c.$

$R_{eq} \Rightarrow$



$R_{eq} = (9 + 1) = 10\ \Omega$

$\tau = 10 \times 20 \times 10^{-3}\ \tau = 0.2$ s.

Thus, the voltage across the capacitor for $t \geq 0$ is

$V(t) = V_0 \cdot e^{-t/\tau}$

$= 15 \cdot e^{-t/0.2}$ V

$\therefore\ V(t) = 15 \cdot e^{-5t}$ V

(ii) The initial energy stored in the capacitor is

(A) 4.5 J (B) 0 J

(C) 2.25 J (D) 4 J

Solution: (C)

$$W_c(0) = \frac{1}{2}CV_c^2(0) = \frac{1}{2}CV_o^2$$

$$= 1/2 \times 20 \times 10^{-3} \times (15)^2 = 2.25 \text{ J}$$

The source free RL circuits: Consider the series connection of a RL circuit as shown in figure.3.

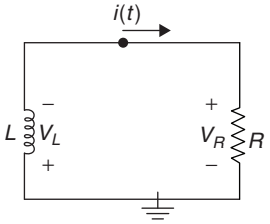


Figure 3 A source free *RL* circuit.

At $t = 0$, we assume that the inductor has on initial current I_0.

$\therefore\ i(0) = I_0$

Applying KVL around the loop in figure 3.

$$V_L + V_R = 0$$

$$L \cdot \frac{di}{dt} + iR = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$\text{Let } \frac{d}{dt} = D$$

$$\left(D + \frac{R}{L}\right)i = 0$$

$D = -R/L.$

So $i(t) = A \cdot e^{-Dt} \quad \Rightarrow \quad i(t) = A \cdot e^{-Rt/L}$

$\therefore\ i_L(t) = A \cdot e^{-t/(L/R)}$

at $t = 0 \quad \Rightarrow \quad i_L(0) = I_0$

$i_L(0) = I_0 = A \cdot e^{-0} \quad \Rightarrow \quad A = I_0$

$$\boxed{i_L(t) = I_0 \cdot e^{-\frac{Rt}{L}} \text{ A}}$$

$$\therefore \quad \tau = \frac{L}{R} \text{ s}$$

$$\tau = \frac{L_{eq}}{R_{eq}} \text{ s}$$

$\therefore\ i(t) = I_0.e^{-t/\tau}$ A

This shows that the natural response of the *RL* circuit is an exponential decay of the initial current.

The current response shown in figure 4.

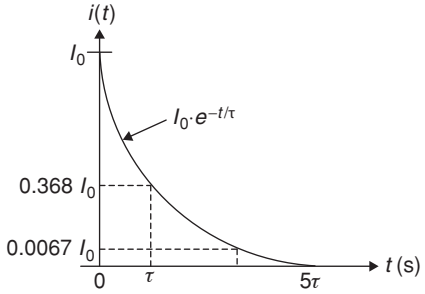
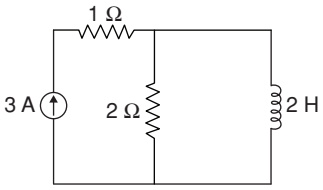


Figure 4 The current response of the *RL* circuit

Note: When $\tau = RC = \frac{L}{R}$

$$R^2 = \frac{L}{C} \quad \Rightarrow \quad R = \sqrt{\frac{L}{C}}\ \Omega$$

Examples 4: The time constant of the given circuit τ is



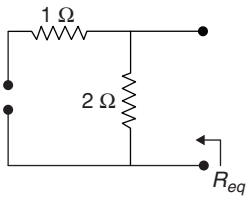
(A) 1 s (B) 1.5 s

(C) $\frac{2}{3}$ s (D) 4 s

Solution: (A)

$$\tau = \frac{L_{eq}}{R_{eq}}.$$

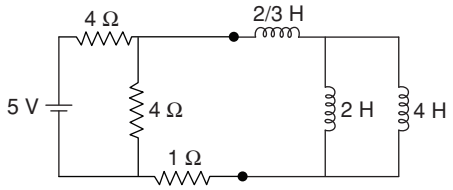
$R_{eq} \Rightarrow R_{th}$



$$\Rightarrow \quad R_{eq} = 2\ \Omega.$$

$$\tau = \frac{2}{2} = 1\ \text{s}$$

Examples 5:



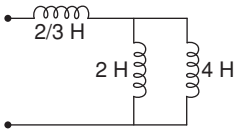
The time constant τ is

(A) 1 s (B) 1.5 s

(C) $\frac{2}{3}$ s (D) 4 s

Solution: (C)

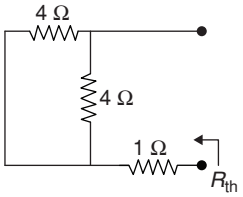
$$\tau = \frac{L_{eq}}{R_{eq}}\ \text{s}$$



$$L_{eq} = \frac{2}{3} + (2 \,||\, 4)$$

$$= \frac{2}{3} + \frac{4}{3} = 2\ \text{H}.$$

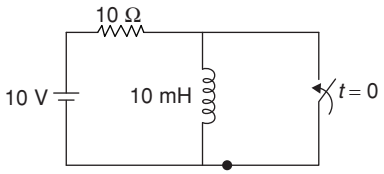
$$R_{eq} \quad \Rightarrow \quad R_{th}$$



$$R_{th} = 1 + (4||4) = 3\ \Omega$$

$$\tau = \frac{2}{3}\ \text{s}$$

Examples 6: The circuit shown in the figure is in a steady state. When the switch is closed at $t = 0$. The current through the inductor at $t = 0^+$ is



(A) 0 A (B) 0.5 A

(C) 1 A (D) 2 A

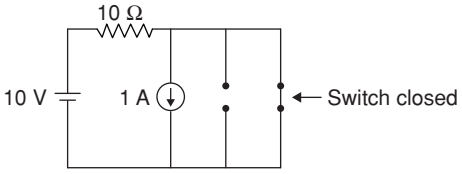
Solution: For $t < 0$:

at $t = 0^-$; the switch is opened.

In steady state. L behaves as a short circuit.

$$i_L(0^-) = \frac{10}{10} = 1\ \text{A}.$$

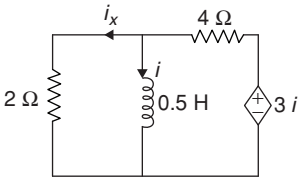
at $t = 0^+$; the circuit shown in the following figure.



Therefore, inductor does not allow sudden change in current

$\therefore\ i_L(0^+) = i_L(0^-) = 1\ \text{A}.$

Examples 7: Assume that $i(0) = 10$ A. Calculate the $i(t)$ and $i_x(t)$ in the circuit (for $t \geq 0$).



(A) $i(t) = 10 \cdot e^{-2t/3}$ A; $i_x(t) = -5/3 \cdot e^{-2t/3}$ A

(B) $i(t) = 10 \cdot e^{-3t/8}$ A; $i_x(t) = 5/3\ e^{-3t/8}$ A

(C) $i(t) = 10 \cdot e^{-4t}$ A; $i_x(t) = -20 \cdot e^{-3t/8}$ A

(D) None of the above

Solution: (A)

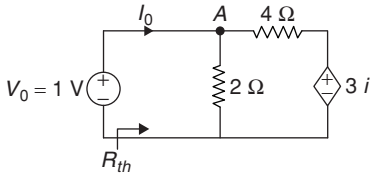
$$i(t) = I_0 \cdot e^{\frac{-t}{\tau}}$$

Given $I_0 = 10$ A

$$\tau = \frac{L_{eq}}{R_{eq}}$$

$R_{eq} = R_{th}$:

The equivalent resistance is the same as the thevenin's resistance at the inductor terminals.



Here $I_0 = -i$

Applying KCL at node A.

$$\frac{V_0}{2} - I_0 + \frac{V_0 - 3i}{4} = 0$$

$$2V_0 - 4I_0 + V_0 + 3I_0 = 0$$
$$3V_0 = I_0$$
$$\frac{V_0}{I_0} = R_{th} = \frac{1}{3}\Omega$$
$$\tau = \frac{L_{eq}}{R_{th}} = \frac{0.5}{\left(\frac{1}{3}\right)} = 1.5 \text{ s}$$

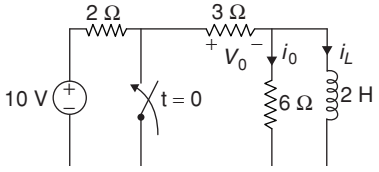
Therefore, thus, the current through the inductor is $i(t)$

$$i(t) = 10 \cdot e^{-t/1.5} \text{ A}$$
$$i(t) = 10 \cdot e^{-2t/3} \text{ A for } t \geq 0.$$
$$V_L(t) = L \cdot \frac{di(t)}{dt}$$
$$= 0.5 \times 10 \frac{d}{dt} e^{\frac{-2t}{3}}$$
$$= 5 \times \left(\frac{-2}{3}\right) \cdot e^{\frac{-2t}{3}} V$$
$$= \frac{-10}{3} \cdot e^{\frac{-2t}{3}} V.$$

Voltage across inductor equal to voltage across 2 Ω resistors.

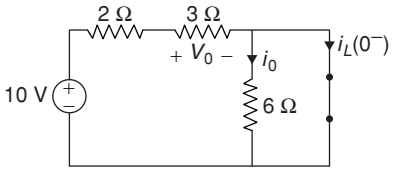
$$\therefore \quad i_x(t) = \frac{V_L(t)}{2} = \frac{-5}{3} \cdot e^{\frac{-2t}{3}} \text{ A. for } t > 0.$$

Examples 8: In the circuit shown in figure assuming that the switch was open for along time.



Find i_0, V_0, and i_L for all times $(-\infty \leq t \leq \infty)$.

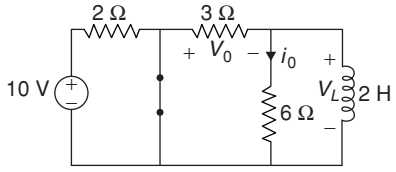
Solution: $t < 0$:
for $t < 0$ the switch is open.
At $t = 0^-$
the inductor acts like a short circuit to DC



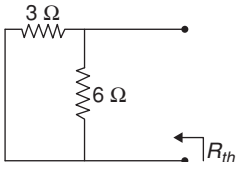
The current through 6 Ω $\Rightarrow$ $i_0 = 0$ A.

$$V_{6\Omega} = 0 \text{ V} \quad \Rightarrow \quad i_0 = 0 \text{ A}$$
$$i_{L(0^-)} = \frac{10}{5} = 2 \text{ A.}$$
$$V_0 = 3 \times 2 = 6 \text{ V.}$$

(i) for $t > 0$: The switch is closed
$i_L(t) = I_0 \cdot e^{-t/\tau}$ A.



find R_{th}:



$$R_{th} = 6 \| 3 \quad \Rightarrow \quad \frac{6 \times 3}{9}$$
$$R_{th} = 2 \ \Omega.$$
$$\tau = \frac{L}{R_{th}} = \frac{2}{2} = 1 \text{ s}$$

thus $i_L(t) = i(0^+) \cdot e^{\frac{-t}{1}}$
$= 2.e^{-t}$ A. for $t > 0$.

(ii) $i_0(t) = \frac{V_L(t)}{6}$

$$\Rightarrow \quad V_L(t) = L \cdot \frac{di(t)}{dt} = 2 \times \frac{d}{dt}(2 \times e^{-t})$$
$$= -4.e^{-t} \text{V.}$$
$$i_0(t) = \frac{-2}{3} \cdot e^{-t} \text{A. for } t > 0.$$

(iii) $V_0 = -V_L$
$\Rightarrow V_0 = 4.e^{-t}$ V for $t > 0$
thus for all time

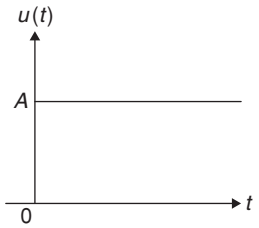
$$i_0(t) = \begin{cases} 0A : t < 0 \\ \frac{-2}{3} \cdot e^{-t} A, t \geq 0 \end{cases}$$
$$V_0(t) = \begin{cases} 6 \text{ V}, t < 0 \\ 4 \cdot e^{-t} V, t > 0. \end{cases}$$
$$i(t) = \begin{cases} 2 \ A; t < 0 \\ 2 \cdot e^{-t} A, t \geq 0 \end{cases}$$

Singularity Functions

A basic understanding of singularity functions will help us make sense of the response of first order circuits to a sudden application of an independent DC voltage or current source.

The most widely used singularity functions in circuit analysis are the step ramp and impulse functions.

Step function



The mathematical terms:

$$u(t) = \begin{cases} 0 & t < 0 \\ A; & t > 0 \end{cases}$$

The unit step function is undefined at $t = 0$

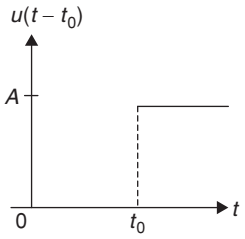


Figure 5 The step function delayed by t_0 units.

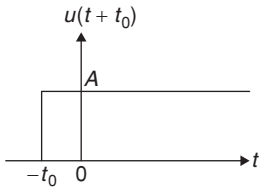
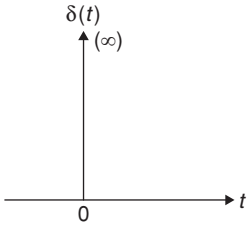


Figure 6 The step function advanced by t_0

Impulse function $\delta(t)$

The derivative of the step function $u(t)$ is the impulse function $\delta(t)$.

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} \text{undefined; } (\infty) : \text{for } t = 0 \\ 0; \text{ else where} \end{cases}$$



1. $\int_{0-}^{0+} \delta(t)dt = 1$
2. $\int_{t1}^{t2} f(t) \cdot \delta(t - t_0)dt = f(t_0); t_1 < t_0 < t_2 = 0$; else where

Ramp function

Integrating the unit step function $u(t)$ results in the unit ramp function $r(t)$.

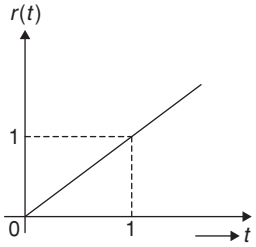


Figure 7 The unit ramp function

$$r(t) = \int_{-\infty}^{t} u(t) \cdot dt = t \cdot u(t)$$

$$r(t) = \begin{cases} 0 & t < 0 \\ \text{At} & t \geq 0 \end{cases}$$

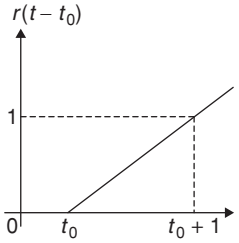


Figure 8 Unit ramp function delayed by t_0.

$$r(t - t_0) = \begin{cases} 0 & t < t_0 \\ t - t_0 & t \geq t_0 \end{cases}$$

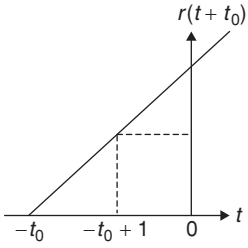


Figure 9 The unit ramp function advanced by to

$$r(t + t_0) = \begin{cases} 0 & t < -t_0 \\ t + t_0 & t \geq -t_0 \end{cases}$$

Step Response of an RC Circuit

When the DC source of an *RC* circuit is applied, the voltage or current source can be modelled as a step function, and the response is known as a step response.

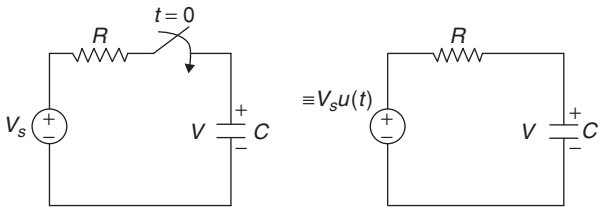


Figure 10 An *RC* circuit with voltage source.

We assume an initial voltage V_0 on the capacitor, since the voltage of the capacitor does not change instantaneously.

$$V(0^-) = V(0^+) = V_0$$

Applying KCL for $t \geq 0$.

$$C \cdot \frac{dV}{dt} + \frac{V - V_S}{R} = 0.$$

$$\frac{dV}{dt} + \frac{V}{RC} = \frac{V_S}{RC} \qquad (1)$$

$$\frac{dV}{dt} = -\frac{(V - V_S)}{RC}$$

$$\frac{dV}{V - V_s} = -\frac{1}{RC} \cdot dt$$

Integrating both sides and introducing the initial conditions.

$$\ln t[V - V_S]_{V_0}^{V(t)} = \left[-\frac{t}{RC}\right]_0^t$$

$$\ln t[V(t) - V_s] - \ln t[V_0 - V_s] = \frac{-t}{RC} + 0$$

$$\ln t\left[\frac{v(t) - v_s}{v_0 - v_s}\right] = \frac{-t}{RC} \qquad (2)$$

$$\frac{V(t) - V_s}{V_0 - V_s} = e^{\frac{-t}{RC}}$$

We know $\tau = RC$

$$V(t) - V_s = (V_0 - V_s) \cdot e^{-t/\tau}$$

$$V(t) = V_S + (V_0 - V_S) \cdot e^{-t/\tau}$$

for $t > 0$.

Thus

$$V(t) = \begin{cases} V_0; t \leq 0 \\ V_S + (V_0 - V_s) \cdot e^{\frac{-t}{\tau}}; t > 0. \end{cases}$$

This is known as the complete response of the *RC* circuit.

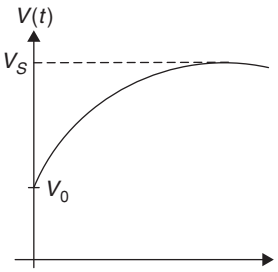


Figure 11 Response of an *RC* circuit With initially charged capacitor.

$$i(t) = C \cdot \frac{dV(t)}{dt}$$

$$i(t) = \frac{V_s}{R} \cdot e^{\frac{-t}{\tau}}; t > 0.$$

with initial values zero.

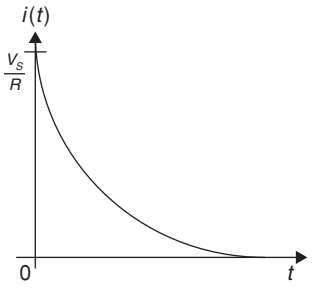


Figure 12 Current response.

Complete response = natural response + forced response.

$\therefore V = V_n + V_f$

where $V_n = V_0 \cdot e^{\frac{-t}{\tau}}$

$$V_f = V_S(1 - e^{\frac{-t}{\tau}})$$

$V_n \Rightarrow$ transient response
$V_f \Rightarrow$ steady state response

Transient response: The transient response is the circuit's temporary response that will dies out with time.

Steady-state response: The steady-state response is the behaviour of the circuit a long time after an external excitation is applied.

Therefore, complete response may be written as

$$\Rightarrow \quad V(t) = V(\infty) + [V(0) - V(\infty)].e^{-t/\tau}\,\text{V}$$

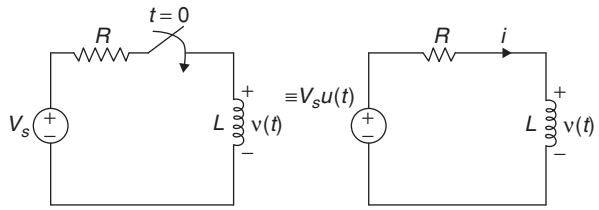
Where $V(0) = V_0 \Rightarrow$ The initial capacitor voltage

$V(\infty) \Rightarrow$ final capacitor voltage (or) *S.S.* value
$\tau \Rightarrow$ time constant

Note: If the switch changes position at time $t = t_0$ instead of $t = 0$, there is a time delay in response. So that above equation becomes

$$\boxed{V(t) = V(\infty) + [V(t_0) - V(\infty)].e^{-(t - t0)/\tau}\,\text{Volts.}}$$

Step Response of an *RL* Circuit



An RL circuit with a step input voltage

Let the response be the sum of the transient response and the steady state response.

$$i = i_{tr} + i_s \tag{3}$$

We know that i_{tr} is always a decaying exponential.

$$\text{i.e., } i_{tr} = A \cdot e^{\frac{-t}{\tau}}, \tau = \frac{L}{R}$$

The steady-state is the value of current along time after the switch is closed. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_S appears across R.

Thus, the steady state response is

$$i_{ss} = \frac{V_s}{R} A.$$

$$i = A \cdot e^{\frac{-t}{\tau}} + \frac{V_s}{R} \tag{4}$$

We know $i(0^-) = i(0^-) = I_0$

Thus at $t = 0$, Equation (4) becomes

$$i(0) = A + \frac{V_s}{R} = I_0$$

$$A = I_0 - \frac{V_s}{R}$$

Sub A in Equation (2), we get

$$i(t) = \left(I_0 - \frac{V_s}{R} \right) \cdot e^{\frac{-t}{\tau}} + \frac{V_s}{R}$$

This is the comp ute response of the *RL* circuit

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)] \cdot e^{\frac{-t}{\tau}} A.}$$

Note: If the switching takes place at time $t = t_0$ instead of $t = 0$,

$$i(t) = i(\infty) + [i(t_0) - i(\infty)] \cdot e^{\frac{-(t\ t_0)}{\tau}}$$

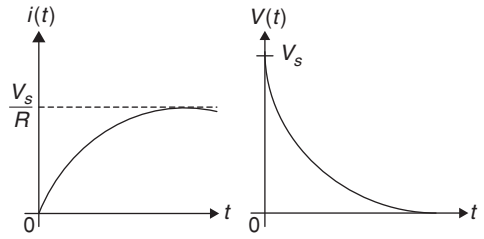


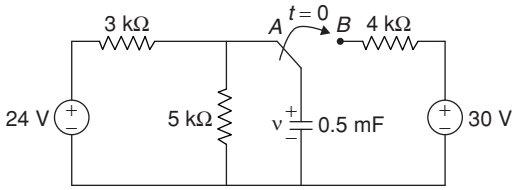
Figure 13 Step response of an RL circuit with no initial inductor current.

Note: With sources, elements behaviour at

$$t = 0^+ \text{ and } t \to \infty$$

1. At $t = 0^+ \Rightarrow$ L = open circuit
   C = short circuit
2. At $t \to \infty \Rightarrow f = 0$
   $\Rightarrow X_L = j\omega L = 0\ \Omega \Rightarrow$ short circuit
   $X_C = \frac{1}{j\omega C} = \infty\Omega \Rightarrow$ open circuit

Examples 9: The switch in figure has been in position A for a long time. At $t = 0$, the switch moves to B. determine $V(t)$ for $t > 0$ and calculate its value at $t = 2$ s.



(A) $V_C(2) = 24.5$ V
(B) $V_C(2) = 5.5$ V
(C) $V_C(2) = -14.5$ V
(D) $V_C(2) = 18.4$ V

Solution: (A)

For $t < 0$, the switch is at position A.

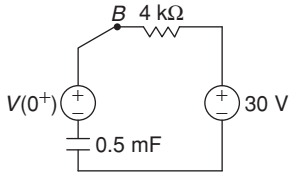
at $t = 0^-$: $\rightarrow$ capacitor acts like open circuit

$$v(0^-) = 5\ \text{k}\Omega \times \frac{24\ \text{V}}{8\ \text{k}\Omega} = 15\ \text{V}.$$

we know $v(0^+) = v(0^-) = 15$ V.

(capacitor voltages can't change instantaneously)

For $t > 0$, the switch is a position B



$\tau = RC = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2$ s.

at $t \to \infty$

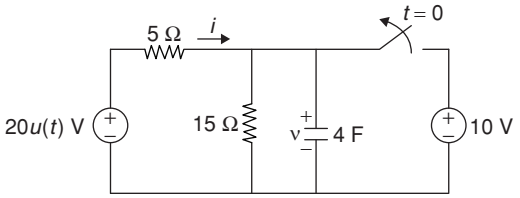
Capacitor acts like an open circuit to dc at steady state.
Thus, $V(\infty) = 30$ V

$$V_C(t) = V(\infty) + [V(0^+) - V_C(\infty)] \cdot e^{-t/t}\ V$$
$$= 30 + (15 - 30) \cdot e^{-t/2}\ \text{s.}$$
$$= 30 - 15 \cdot e^{-0.5t}\ \text{s.}$$

at $t = 2$ s

$$V_C(2) = 30 - 15 \cdot e^{-1}$$
$$= 30 - \frac{15}{e} = 24.5\ \text{V}$$

Examples 10: The switch has been closed for a long time and is opened at $t = 0$

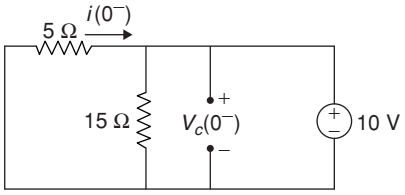


The value of $V_C(t)$ for $t > 0$ would be
(A) $V_C(t) = 15 - 5 \cdot e^{-t/15}$ V
(B) $V_C(t) = 5 - 15 \cdot e^{-t/15}$ V
(C) $V_C(t) = 15 + 5 \cdot e^{-0.25t}$ V
(D) $V_C(t) = 15 + 10 \cdot e^{-t/15}$ V

Solution: (A)
For $t < 0$:
at $t = 0^-$, switch is closed. And $C \Rightarrow$ open circuit

$$u(t) = \begin{cases} 0; t < 0 \\ 1; t > 0. \end{cases}$$

The shown in below at $t < 0$.

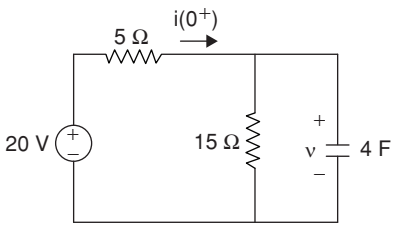


circuit is in steady state from the above circuit.

$$i(0^-) = \frac{10}{5} = 2\ \text{A.}$$

$$V_C(0^-) = 10\ \text{V.} = V_C(0^+)$$

for $t > 0$: Switch opened.
at $t = 0^+$



$$\begin{cases} \tau = R_{eq} \cdot c \\ R_{eq} = \dfrac{5 \times 15}{5 + 15} = \dfrac{5 \times 15}{20} = \dfrac{15}{4}\ \Omega \\ \tau = \dfrac{15}{4} \times 4 = 5\ \text{s} \end{cases}$$

$C \rightarrow$ Short circuit

$$i(0^+) = \frac{20}{5} = 4\ \text{A.}$$

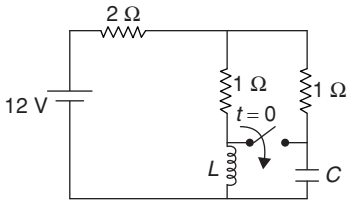
at $t \rightarrow \infty$; C open circuit.

$$V_C(\infty) = V_{15\ \Omega} = 20 \times \frac{15}{15 + 5} = 15\ \text{V}$$

We know total response $V_C(t)$ is

$$V_C(t) = V_C(\infty) + [V_C(0^+) - V_C(\infty)] \cdot e^{\frac{-t}{\tau}}$$
$$= 15 + [10 - 15] \cdot e^{-t/15}$$
$$V_C(t) = 15 - 5\ e^{-t/15}\ \text{V}$$

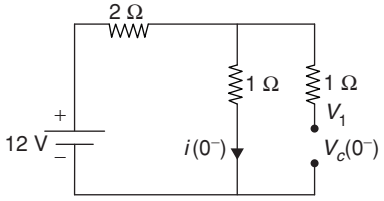
Examples 11: In the circuit shown below, the switch is closed at $t = 0$. What is the Initial value of the current through the capacitor?



(A) 0.8 A (B) 1.6 A
(C) 2.4 A (D) 3.2 A

Solution:
For $t < 0$
At $t = 0^-$; The switch is opened
$L \Rightarrow$ Short circuit
$C \Rightarrow$ Open circuit

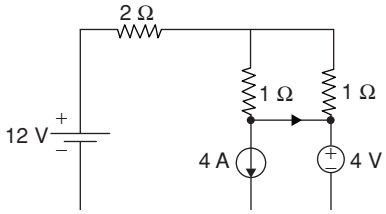
The equivalent circuit shown below:

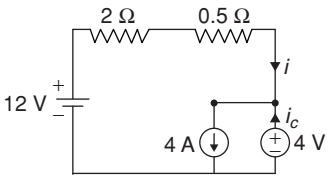


$$i(0^-) = \frac{12}{3} = 4\ \text{A.} = i_L(0^+)$$

$$V_C(0^-) = 1 \times 4 = 4\ \text{V} = V_C(0^+)$$

at $t = 0^+$, the equivalent circuit is shown below:

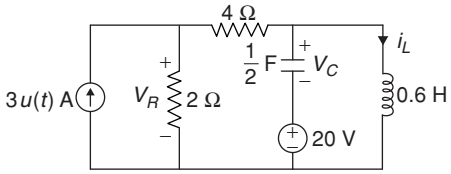




$$i = \frac{12-4}{2.5} = 3.2 \text{ A}$$

$$i + i_C = 4 \quad \Rightarrow \quad i_C = 0.8 \text{ A}.$$

Common Data for Questions 12 to 14:



Examples 12: The values of $i_L(0^+)$, $V_C(0^+)$, and $V_R(0^+)$ would be.

(A) $i_L(0^+) = 0$ A; $V_C(0^+) = 20$ V; and $V_R(0^+) = 4$ V

(B) $i_L(0^+) = 5$ A; $V_C(0^+) = -15$ V; and $V_R(0^+) = 4$ V

(C) $i_L(0^+) = 3$ A; $V_C(0^+) = -20$ V; and $V_R(0^+) = 4$ V

(D) $i_L(0^+) = 0$ A; $V_C(0^+) = -20$ V; and $V_R(0^+) = +4$ V

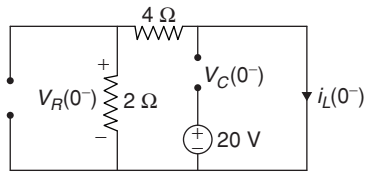
Solution: (D)

For $t < 0$, 3. $u(t) = 0$; at $t = 0^-$

Since the circuit is in steady state.

$\therefore$ L $\Rightarrow$ short circuit

C $\Rightarrow$ open circuit



from this figure:

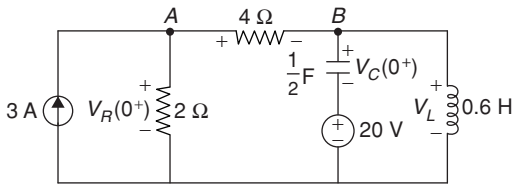
$i_L(0^-) = 0$; $V_R(0^-) = 0$; and $V_C(0^-) = -20$ V

for $t > 0$:

the circuit shown in below at $t = 0^+$

$\therefore$ $L \Rightarrow$ open circuit

$C \Rightarrow$ short circuit



The inductor current and capacitor voltages cannot change instantaneously

$i_L(0^+) = 0$; $i_L(0^-) = 0$ and $V_C(0^+) = V_C(0^-) = -20$ V

apply KCL at node A.

$$3 = \frac{V_R(0^+)}{2} + \frac{V_0(0^+)}{4}$$

Applying KVL to the middle Loop.

$$V_R(0^+) - V_0(0^+) - V_c(0^+) - 20 = 0$$

$$V_C(0^+) = -20 \text{ V}$$

$\therefore$ $V_R(0^+) = V_0(0^+)$.

$$3\ V_R(0^+) = 12$$

$\Rightarrow$ $V_R(0^+) = 4$ V

$\therefore$ $V_R(0^+) = 4$ V, $V_C(0^+) = -20$ V. $I_L(0^+) = 0$ A.

Examples 13: The steady state values of the I_L V_C, and V_R would be

(A) $V_R(\infty) = 4$ V, $V_C(\infty) = -20$ V, $I_L(\infty) = 0$ A

(B) $V_R(\infty) = 4$ V, $V_C(\infty) = -20$ V, $I_L(\infty) = 1$ A

(C) $V_R(\infty) = -4$ V, $V_C(\infty) = 20$ V, $I_L(\infty) = 4$ A

(D) None of the above

Solution: (B)

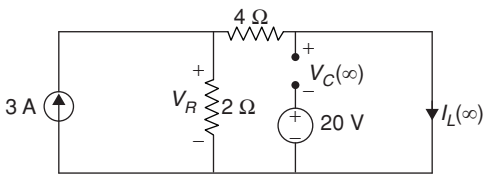
As $t \to \infty$. The circuit reaches steady-state.

In steady state

$\therefore$ $L \Rightarrow$ short circuit

$C \Rightarrow$ open circuit

The circuit becomes.



$$i_L(\infty) = 2 \times \frac{3}{6} = 1 \text{ A}.$$

$$V_C(\infty) = -20 \text{ V}.$$

$$V_R(\infty) = 2 \times 2 = 4 \text{ V}.$$

$\therefore$ $V_R(\infty) = 4$ V, $V_C(\infty) = -20$ V and $i_L(\infty) = 1$ A.

Examples 14: The values of $\frac{di_L(0^+)}{dt}$ and $\frac{dV_C(0^+)}{dt}$ would be

(A) 0 A/s, 2 V/s (B) 0 A/s, 0.5 V/s
(C) 2 A/s, 2 V/s (D) −2 A/s, −2 V/s

Solution: (A)
for $t > 0$:
at $t = 0^+$.

$$V_L(0^+) = L \cdot \frac{di_L(0^+)}{dt} \Rightarrow \frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L}$$

But $V_L(0^+) = 0$ V $\Rightarrow \frac{di_L(0^+)}{dt} = 0\,\frac{\text{A}}{\text{s}}$

$$i_C = C \cdot \frac{dV_c}{dt} \Rightarrow \frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

Apply KCL at node B,

$$\frac{V_0(0^+)}{4} = i_C(0^+) + i_L(0^+)$$

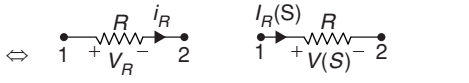
$$i_C(0^+) = \frac{4}{4} = 1\text{ A}$$

$$\frac{dV_C(0^+)}{dt} = \frac{1}{\frac{1}{2}} = 2\text{ V/s}$$

Equivalent Circuits for *R*, *L*, and *C* in *S* Domain
Laplace Transform

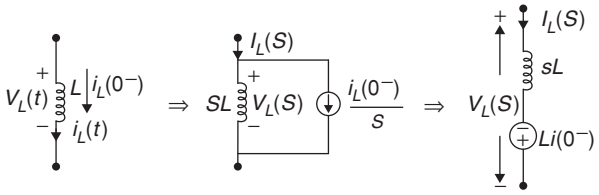
$$i(t) \leftrightarrow I(s)$$
$$V(t) \leftrightarrow V(s)$$
$$R \leftrightarrow R$$
$$L \leftrightarrow sL\ \Omega$$
$$C \leftrightarrow \frac{1}{sC}\ \Omega$$

(i) *R*: *R*esistor (*R*):

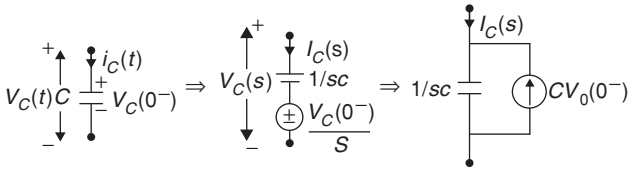


1. Time domain 2. Frequency domain.

(ii) Inductor (*L*):



(iii) Capacitor (*C*):

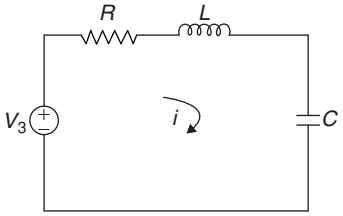


1. Time domain
2. s-domain
3. s-domain Voltage source current source.

Higher–order Circuits

When two or more energy storage elements are present, the network equations will result in second order differential equations.

Series RLC circuits



Apply KVL

$$R_i + L \cdot \frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i \cdot dt = V_s. \quad (5)$$

To eliminate the integral, differentiate with respect to *t*.

$$R \cdot \frac{di}{dt} + L \cdot \frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L} \cdot \frac{di}{dt} + \frac{i}{LC} = 0. \quad (6)$$

Apply Laplace transform to Equation (6)

$$\Rightarrow \quad s^2 + \frac{R}{L} \cdot s + \frac{1}{LC} = 0.$$

Characteristic equation of the differential equation.

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \quad (7)$$

Second order characteristic equation
From Equation (7)

$$\omega_n = \frac{1}{\sqrt{LC}} \text{ and } 2\xi\omega_n = \frac{R}{L}$$

$$\xi\omega_n = \frac{R}{2L}$$

$$\tau = \frac{1}{\xi\omega_n} = 2\frac{L}{R}$$

$$\boxed{\xi = \frac{R}{2}\cdot\sqrt{\frac{C}{L}}}$$

The roots of the characteristic equation are

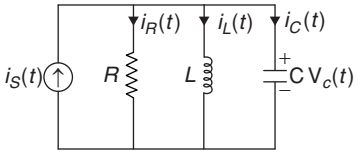
$$s = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $\alpha = \frac{R}{2L} = \frac{1}{\tau}$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$$

1. If $\alpha > \omega_n$ (or) $\xi > 1$, we have over damped oscillations.
2. $\xi = 1$ or $\alpha = \omega_n$, $\Rightarrow$ Critically damped
3. $0 < \xi < 1$ or $\alpha < \omega_n$, we have under damped oscillators.

Parallel RLC Circuit



$$i_S(t) = i_R(t) + i_L(t) + i_C(t)$$

Let us assume that the voltage across the capacitor is $V_C(t)$

$$\therefore \quad \frac{V_C(t)}{R} + \frac{1}{L}\int_0^t V_C(t)dt + C\cdot\frac{dV_C(t)}{dt} = i_s(t)$$

simplifying the above equation, it becomes

$$\frac{d^2V_C(t)}{dt} + \frac{1}{RC}\cdot\frac{dv_c(t)}{dt} + \frac{1}{LC}\cdot V_C(t) = 0$$

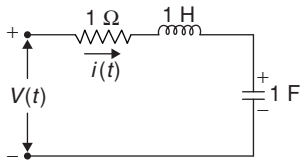
In S-domain

$$s^2V_C(s) + \frac{1}{RC}\cdot sV_c(0) + \frac{1}{LC}\cdot V_c(s) = 02$$

$$s^2 + \frac{1}{RC}\cdot s + \frac{1}{LC} = 0.$$

$$\boxed{\xi = \frac{1}{2R}\cdot\sqrt{\frac{L}{C}};}\ \omega_n = \frac{1}{\sqrt{LC}}.$$

Example 15: The circuit shown in figure has initial current $i_L(0^-)$ = 1 A. Through the inductor and an initial voltage $V_C(0^-) = -1$ V across the capacitor for input $V(t) = u(t)$, the Laplace transform of the current $i(t)$ for $t \geq 0$ is



(A) $\frac{s}{s^2+s+1}$ (B) $\frac{s+2}{s^2+s+1}$

(C) $\frac{s-1}{s^2+s+1}$ (D) $\frac{s-1}{s^2+s+1}$

Solution: (B)
Apply KVL, the loop equation is

$$V(t) = R\cdot i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt.$$

but $V(t) = u(t)$, $i_L(0^-) = 1$ A and $V_C(0^-) = -1$ V

Take LPF both sides.

$$L[sI(s) - I(0^-)] + RI(s) + \frac{1}{sC}\cdot I(S) + \frac{V_C(0^-)}{s} = \frac{1}{s}$$

Sub R = 1 Ω, and L = 1 H
C = 1 F

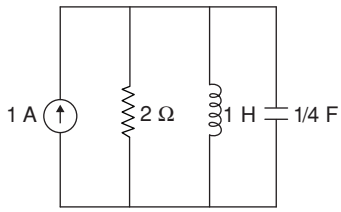
$$sI(s) - 1 + I(s) + \frac{1}{s}I(S) \quad \frac{1}{s} = \frac{1}{s}$$

$$I(s)\left[s + \frac{1}{s} + 1\right] = 1 + \frac{2}{s}$$

$$I(s)\,[s^2 + s + 1] = s + 2$$

$$I(s) = \frac{s+2}{s^2+s+1}$$

Example 16: The circuit is



(A) Critically damped (B) Undamped
(C) Under damped (D) Over damped

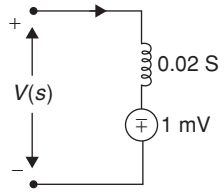
Solution: (D)
For RLC parallel circuit

$$\xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$= \frac{1}{2\times 2}\times\sqrt{\frac{4}{1}}\ ;\ \xi = \frac{1}{2}.$$

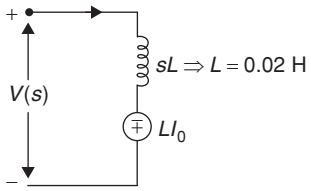
$\therefore\ \xi < 1 \Rightarrow$ under damped system.

Example 17: The value of Initial current is



(A) 0.5 A (B) 0.05 A
(C) 0.01 A (D) 0.2 μA.

Solution: (B)

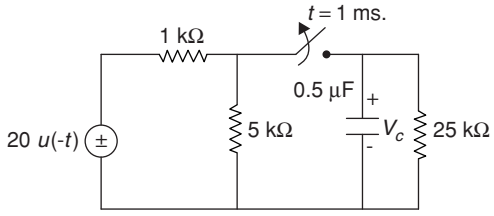


$$L.\,I_0 = 1 \times 10^{-3}$$

$$I_0 = \frac{1\times10^{-3}}{0.02} = 0.05 \text{ A}$$

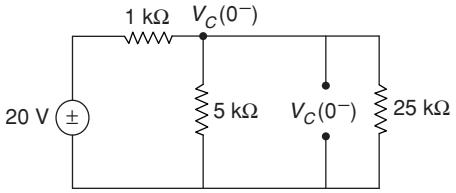
$$I_0 = 50 \text{ mA}$$

Example 18: In the following circuit, the 20 V source has been applied for a long time. The switch is opened at t = 1 m s.



At t = 3 ms the value of $V_C(t)$ is
(A) 1.314 V (B) 1.128 V
(C) 16.13 V (D) None of the above.

Solution: (B)
For $t < 0$:
at $t = 0^-$ switch was closed
Capacitor behaves like an open circuit.



$$\frac{V_C(0^-)-20}{1\text{ k}\Omega} + \frac{V_C(0^-)}{5\text{ k}\Omega} + \frac{V_C(0^-)}{25\text{ k}\Omega} = 0$$

$$25\,(V_C(0^-) - 20) + 5\,V_C(0^-) + V_C(0^-) = 0$$

$$31\,V_C(0^-) = 25 \times 20$$

$$\Rightarrow \quad V_C(0^-) = \frac{25\times20}{31} = 16.13 \text{ V}$$

$$R_{eq} = (1\text{ k}\Omega \parallel 5\text{ k}\Omega \parallel 25\text{ k}\Omega)$$

$$\frac{1}{R_{eq}} = \frac{1}{1\text{ k}\Omega} + \frac{1}{5\text{ k}\Omega} + \frac{1}{25\text{ k}\Omega}$$

$$= \frac{25+5+1}{25\text{ k}\Omega}$$

$$R_{eq} = \frac{25}{31}\text{ k}\Omega = 806\ \Omega$$

$$\tau = R_{eq} \times C = 806 \times 0.5 \times 10^{-6}$$

$$\tau = 0.4 \text{ ms.}$$

$$V_C(t) = V_0.e^{-t/\tau} = 16.13 \times e^{-2500t}\text{ V}$$

$$V_C(1\text{ ms}) = 16.13 \times e^{-2500\times1\times10^{-3}}\text{ V}$$

$$= 16.13.\ e^{-2.5} = 1.324 \text{ V for } 0 < t < 1 \text{ ms}$$

$$V_C(t) = 1.324 \times e^{\frac{-(t-1m)}{25\times0.5\times10^{-3}}}\text{ V}$$

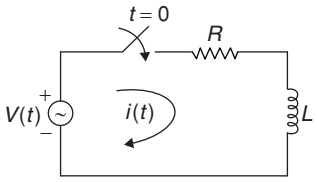
$$V_C(3\text{ m}) = 1.324 \times e^{-80\times2\times10^{-3}}$$

$$= 1.128 \text{ V}$$

AC Transients

Transient Response with Sinusoidal Excitation

Series R – L Circuit



Let $V(t) = V_m \cdot \cos(\omega t + \phi)$ V.
for $t > 0$: Switch is closed.
Apply KVL

$$R \cdot i + L \cdot \frac{di}{dt} = V_m \cdot \cos(\omega t + \phi)$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V_m}{L} \cdot \cos(\omega t + \phi) \qquad (8)$$

Let $$\frac{d}{dt} = D.$$

$$\left(D + \frac{R}{L}\right) i = \frac{V_m}{L} \cdot \cos(\omega t + \phi) \qquad (9)$$

The complete solution = complement function + particular solution.

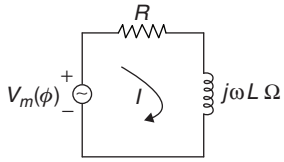
That is, C.S. = C.F. + P.I.

The complementary function of Equation (2) is

$$i_{tr}(t) = A \cdot e^{-\frac{R}{L}t} \text{ A}$$

Next, we are to obtain the particular solution of current $i(t)$: $\Rightarrow i_{ss}(t)$

Transform the above network into phasor domain.



Network is phasor domain

Apply KVL to above circuit

$$V_m \angle\phi - R.I - j\omega LI = 0.$$

$$I = \frac{V_m \angle\phi}{R + j\omega L}$$

$$\Rightarrow \quad I = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \angle\phi - \tan^{-1}\frac{\omega L}{R}$$

I in time domain $i_{ss}(t)$.

$$i_{ss}(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) \text{A}.$$

$$i(t) = i_{tr}(t) + i_{ss}(t)$$

= Transient + Steady state response

Summary:

S.No	Excitation	Circuit	Transient free condition (at $t = t_0$)
(i)	$V(t) = V_m \sin(\omega t + \varphi)$	Series *RL* (or)	$\omega t_0 + \phi = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\omega\tau.$
		Series *RC*	$\omega t_0 + \phi = \tan^{-1}\omega RC = \tan^{-1}\omega\tau.$
(ii)	$V(t) = V_m \cos(\omega t + \varphi)$	Series *RL* (or) Series *RC*	$\omega t_0 + \phi = \tan^{-1}\omega t + \frac{\pi}{2}$ where $t = \frac{L}{R}$ for *RL* $\tau = RC$ for *RC*.
(iii)	$i(t) = I_m \sin(\omega t + \varphi)$	Parallel *RL* (or) Parallel *RC*	$\omega t_0 + \phi = \tan^{-1}\omega\tau$ where $\tau = RC$ for *RC* circuits. $\tau = \frac{L}{R}$ for *RL* circuits.
(iv)	$i(t) = I_m \cos(\omega t + \varphi)$	Parallel *RL* (or) Parallel *RC*	$\tau = \frac{L}{R}$ for *RL* circuits, *RC* for *RC* circuits $\omega t_0 + \phi = \tan^{-1}\omega t + \frac{\pi}{2}$

$$= A \cdot e^{-\frac{R}{L}t} + \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) \text{A}$$

at $t = 0^- \Rightarrow i(0^-) = 0 = i(0^+)$.

$$A + \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) = 0$$

$$A = \frac{-V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) \text{A}.$$

$\Rightarrow$ As $A << 1$, the transient effects are less for AC at $t = 0$.

Transient free condition

That is, $A = 0$.

$$\Rightarrow \quad \cos\left(\phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\phi - \tan^{-1}\frac{\omega L}{R} = \frac{\pi}{2}$$

$$\phi = \frac{\pi}{2} + \tan^{-1}\frac{\omega L}{R}; \text{ at } t = 0.$$

Case 1: If the switch is closed at $t = t_0$. Then the condition for transient free response is

$$\omega t_0 + \phi = \frac{\pi}{2} + \tan^{-1}\frac{\omega L}{R}$$

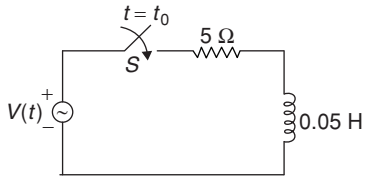
Note: If the excitation is $V(t) = V_m.\ \sin(\omega t + \phi)$.

Then the condition for transient free response.

$\Rightarrow \quad \omega t_0 + \phi = \tan^{-1}\omega\tau$ at $t = t_0$

$\therefore\ i(t) = i_{ss}(t)$

Example 19:



If $V(t) = 5\cos\left(100\pi t + \frac{\pi}{4}\right)$ V, the value of t_0 which results in a transient free response.

Solution: We know for *RL* series circuits transient free condition.
Given input co sinusoidal

$$\therefore \quad \omega t_0 + \phi = \tan^{-1}\frac{\omega L}{R} + \frac{\pi}{2}$$

from the given data.

$$\omega = 100\pi \text{ and } \phi = \frac{\pi}{4}$$

$$100\pi t + \frac{\pi}{4} = \tan^{-1}\left(\frac{100\pi \times 0.05}{5}\right) + \frac{\pi}{2}$$

$$100\pi t = 117.34$$

$$t = 0.3735 \text{ s}$$

Sinusoidal Steady-State Analysis

A sinusoidal forcing function produces both a transient response and steady state response.

The transient response dies out with time so that only steady state response remains. When the transient response has become negligibly small compared to steady-state response, we say that the circuit is operating at sinusoidal steady state.

Sinusoids

Let us consider a general expression for the sinusoidal.

$$V(t) = V_m \cdot \sin(\omega t + \varphi)$$

where

$V_m \Rightarrow$ amplitude
$\omega \Rightarrow$ angular frequency
$\omega t + \varphi \Rightarrow$ argument of the sinusoidal
$\varphi \Rightarrow$ phase

A sinusoid can be expressed in either sine or cosine form.

This is achieved by using the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Note:

$$\sin(\omega t \pm 180°) = -\sin \omega t$$

$$\cos(\omega t \pm 180°) = -\cos \omega t$$

$$\sin(\omega t \pm 90°) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90°) = \pm \sin \omega t.$$

Examples 20: Given the sinusoidal signal

$$V(t) = 10\sin\left(4\pi t + \frac{\pi}{6}\right).$$

Calculate its power and period.
(A) 100 W, 2 s
(B) 50 W, 0.5 s
(C) 10 W, 0.5 s
(D) 0 W, 2 s

Solution: (B)
for sinusoidal signals

$$\text{power} = \frac{V_m^2}{2R} (\because \quad V_m = \text{maximum voltage})$$

but *R* is not given
So Let $R = 1\ \Omega$

$$P = \frac{V_m^2}{2} = \frac{(10)^2}{2} = 50 \text{ W}$$

$$\omega = 2\pi f = 4\pi$$

$$T = \frac{2\pi}{4\pi} = \frac{1}{2} \quad \Rightarrow \quad 0.5 \text{ s}$$

Example 21: Calculate the phase angle between $V_1 = -10\cos(\omega t + 40°)$ and

$$V_2 = 8\sin(\omega t - 20°).$$

(A) 30° (B) 60° (C) −60° (D) 20°

Solution: In order to compare V_1 and V_2, we must express them in same form.

$$V_1 = -10\cos(\omega t + 40°)$$
$$= +10\sin(\varpi t + 40° - 90°).$$
$$= 10\sin(\omega t - 50°).$$
$$V_2 = 12\sin(\omega t - 20°)$$

V_2 Leads V_1 by 30°.

Phasor

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

$$Z = x + jy \Rightarrow \text{rectangular form}$$

$$Z = r \angle \varphi \Rightarrow \text{polar form}$$

$$Z = r.\, e^{j\varphi} \Rightarrow \text{exponential form}$$

Euler's identity

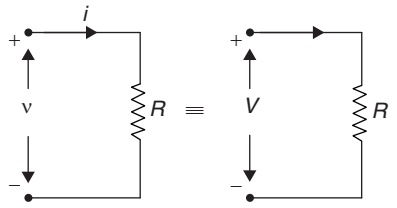
$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

Table 1 *Sinusoidal phasor transformation:*

S.No	Time domain	Phasor domain
(i)	(i) $V_m \cos(\omega t + \varphi)$	$V_m \angle\varphi$
(ii)	(ii) $V_m \sin(\omega t + \varphi)$	$V_m \angle\varphi - 90°$
(iii)	(iii) $I_m \cos(\omega t + \varphi)$	$I_m \angle\varphi$
(iv)	(iv) $I_m \sin(\omega t + \varphi)$	$I_m \angle\varphi - 90°$
(v)	(v) $\frac{dv}{dt} \Leftrightarrow$	$j\omega v$
(vi)	(vi) $\int v.dt \Leftrightarrow$	$\frac{V}{j\omega}$

Phasor relationships for circuit elements:

Resistor



(i) Time domain (ii) Frequency domain

If the current through a resistor R is

$$i = I_m \cos(\omega t + \varphi).$$

The voltage across it is given by

$$V = iR = R.I_m \cos(\omega t + \varphi)$$

The phasor form of this voltage is

$$V = R \times I_m \angle\varphi$$

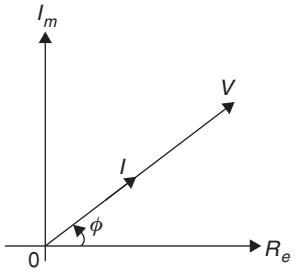
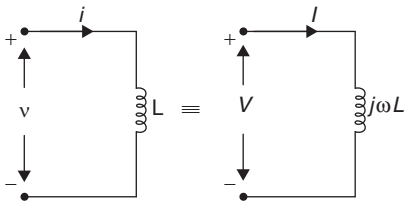


Figure 14 Phasor diagram for the resistor.

Inductor (L)



Time domain
$T - D$

$$V = L \cdot \frac{di}{dt}$$

Frequency domain
$F - D$

$$V = j\omega L \cdot I$$

The voltage across the inductor is

$$V = L \cdot \frac{di}{dt} = -\omega L \cdot I_m \sin(\omega t + \phi)$$

$$V = \omega L.I_m \cos(\omega t + \varphi + 90°)$$

which transform to the phasor

$$V = \omega L \cdot I_m.e^{j(\varphi + 90°)} = \omega L \cdot I_m.\angle\varphi + 90°$$

$$\text{but } I_m \cdot \angle\varphi = I \text{ and } e^{j90°} = j.$$

$$\therefore V = j\omega L.I.$$

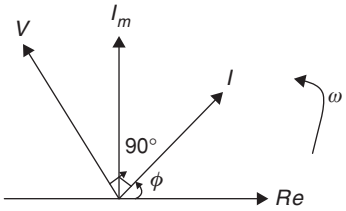
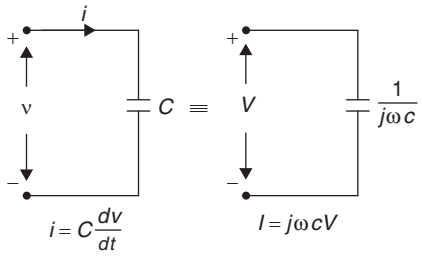


Figure 15 Phasor diagram for the inductor *I* lags V. (or) V leads *I*.

Capacitor (C)



For the capacitor C, assume the voltage across it is $V = V_m \cos(\omega t + \varphi)$.

$$i = C \cdot \frac{dV}{dt} \Leftrightarrow I = j\omega CV$$

$$T - D \qquad F - D$$

$$\therefore \quad V = \frac{\mathrm{I}}{j\omega c}$$

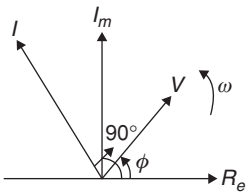


Figure 16 Phasor diagram for the capacitor. *I* leads *V*. or *V* lags *I*.

Note: Summary of $V - I$ relationships

Elements	$T - D$	$F - D$
R	$v = Ri$	$V = R.I$
L	$v = L \cdot di/dt$	$V = j\omega LI$
C	$I = C\ dv/dt$	$I = I/j\omega C$

Examples 22: The voltage $V = 5\cos(40t + 60°)$, is applied to a 0.5 H inductor. The steady state current through the inductor is

(A) $i(t) = 4\sin(40t + 60°)$ A
(B) $i(t) = 2\cos(40t + 30°)$ A
(C) $i(t) = 0.25\cos(40t - 30°)$ A
(D) $i(t) = 0.25\cos(\omega t + 60°)$ A

Solution: (C)
For the inductor, $V = j\omega L\, I$
form the given data.
$\omega = 40$ rad/s.
$L = 1/2$ *H*. and $V = 5\angle 60°$ V.

$$I = \frac{V}{j\omega L} = \frac{5\angle 60°}{j\times 40\times 0.5}$$

$$= \frac{1}{4}\angle 60° - 90° = 0.25\angle -30°.$$

$\therefore \quad i(t) = 0.25.\cos(40t - 30°)$ A.

Example 23: If voltage $V = 4\sin(50t + 30°)$ is applied to a 100 μF, capacitor. The steady state current through the capacitor is

(A) $i(t) = 20\sin(50t + 60°)$ mA
(B) $i(t) = 20\cos(50t + 30°)$ mA
(C) $i(t) = 20\cos(50t - 30°)$ mA
(D) $i(t) = 20\cos(50t - 60°)$ mA

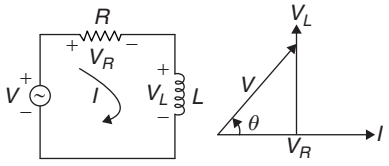
Solution: (B)
Given V = 4 sin (50 t + 30°)

$$i = C\cdot\frac{dV}{dt} \Leftrightarrow V = \frac{\mathrm{I}}{j\omega C};\ V = 4\angle \quad 60°$$

$I = j\omega V.C \quad \Rightarrow \quad j\times 50\times 100\times 10^{-6}\times 4\angle -60°.$
$I = 20\cos(50t - 60° + 90°)$ mA
$i(t) = 20\cos(50t + 30°)$ mA.

Sinusoidal Steady-State Analysis of RLC Circuits

RL Series Circuit



$$V = V_R + V_L$$
$$= I\cdot R + j\omega L\cdot I$$

$$V = \sqrt{V_R^2 + V_L^2}$$

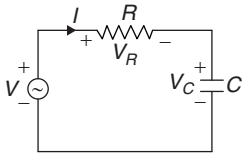
$$\theta = \tan^{-1}\left[\frac{V_L}{V_R}\right]$$

where $V_R = I\cdot R$

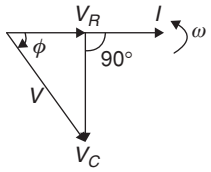
Power factor $= \cos\theta = \dfrac{V_R}{V}$

$$V_L = I\cdot X_L\angle 90°\text{Lag.}$$

Series RC Circuit



$$V = V_R + V_C$$
$$V_R = I\cdot R\ ;\ V_C = I\cdot Z_C = IX_C\angle -90°.$$

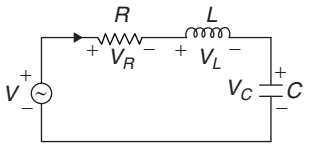


Phasor diagram

$$V = \sqrt{V_R^2 + V_C^2}$$

$$pf = \cos\phi = \frac{V_R}{V}\ \text{Lead}$$

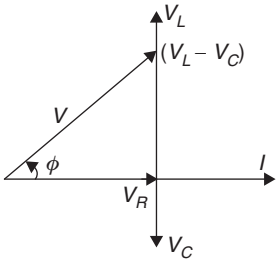
Series RLC Circuit



$$V = V_R + V_L + V_C$$
$$= I\cdot R + j\omega L\cdot I + \frac{1}{j\omega c}\cdot I$$

$$= I.R + I.X_L\angle 90° + I.X_C\angle -90°.$$

1. $V_L > V_C$:



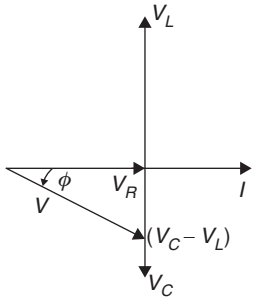
from the above phasor diagram

$$|V| = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$\phi = \tan^{-1}\left[\dfrac{V_L - V_C}{V_R}\right]$ $\Rightarrow$ Impedance or Admittance angle

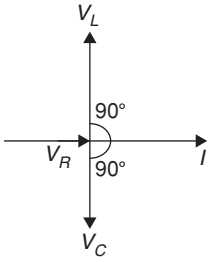
Power factor $= pf = \cos\phi = \dfrac{V_R}{V}$; Lag

2. If $V_L < V_C$:



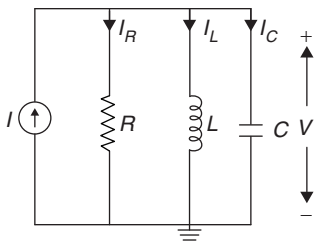
$$\cos\phi = \frac{V_R}{V}; \text{ Lead}$$

3. If $V_L = V_C$
In this case



$$\cos\phi = \frac{V_R}{V} = 1 \Rightarrow \text{unity power factor}$$

Parallel RLC Circuit

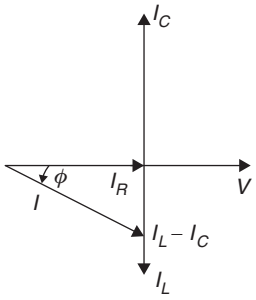


$$I = I_R + I_L + I_C$$

$$I = \frac{V}{R} + \frac{V}{j\omega L} + j\omega c \cdot V$$

$$\Rightarrow \quad I = \frac{V}{R} + \frac{V}{X_L}\angle -90° + \frac{V}{X_C}\angle 90°.$$

1. $I_L > I_C$

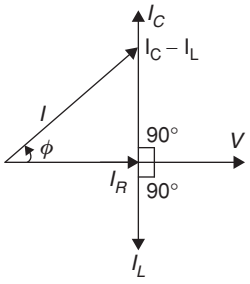


from the above phasor diagram

$$|I| = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$pf = \cos\phi = \frac{I_R}{I}; \text{ Lag}$$

2. $I_L < I_C$



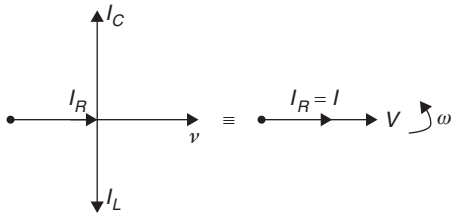
$$|I| = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$\phi = \tan^{-1}\left[\dfrac{I_L - I_C}{I_R}\right]$ $\Rightarrow$ Impedance (or) admittance angle

$$\cos\phi = pf = \frac{I_R}{I} \text{ Lead}$$

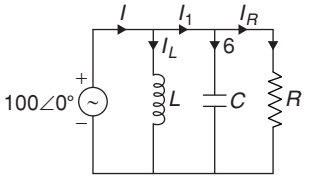
3. $I_L = I_C$

$|I| = I_R$.



$$\cos\phi = \frac{I_R}{I} = 1 \quad \Rightarrow \quad \text{unity power factor.}$$

Examples 24: Consider the following circuit:



If $|I_1|$ = 10 A and | I | = 12 A, then the values of I_L and I_R would be
(A) $I_L = -14.94$ A, $I_R = 8$ A
(B) $I_L = 2$ A, $I_R = 4$ A
(C) $I_L = 14.94$ A, $I_R = 8$ A
(D) None of the above

Solution: (A)

$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$I_1 = \sqrt{I_R^2 + I_C^2}$$

$$100 = I_R^2 + 36$$

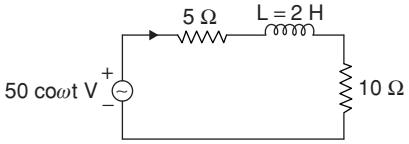
$$I_R^2 = 64 \quad \Rightarrow \quad I_R = 8 \text{ A}$$

$$12 = \sqrt{8^2 + (I_L - 6)^2}$$

$$(I_L - 6)^2 = 80$$

$$I_L = 6 + 8.944 = 14.94 \text{ A}$$

Example 25:



If the power dissipated in the 5 Ω resistor is 15 W, then the power factor of the circuit is
(A) $p.f = 0.68$
(B) $p.f = 0.854$
(C) $p.f = 0.52$
(D) None of the above.

Solution: (C)
For RL series circuit

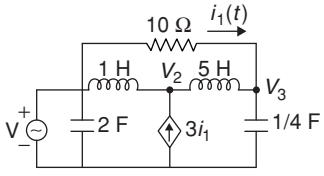
$$pf = \cos\phi = \frac{V_R}{V}$$

$P = V \cdot I = I^2 \cdot R$
$I^2 \times 5 = 15 \quad \Rightarrow \quad I^2 = 3$
$I = 1.732$ A
$\therefore \quad V_R = 15 \times 1.732 \approx 26$
$|V| = 50 \angle 0°$.

$$\cos\phi = \frac{26}{50} = 0.52 \quad \Rightarrow \text{ power factor}$$

Impedance angle $\varphi = 58.66°$

Example 26: In the circuit shown below it is known that $V_2(t) = 0.757 \cos(2t + 66.7°)$ V, $V_3(t) = 0.606 \cos(2t - 69.8°)$ V. then $i_1 t) = ?$



(A) $0.194 \cos(2t + 35.73°)$ A
(B) $0.318 \cos(2t + 177°)$ A
(C) $0.196 \cos(2t - 35.6°)$ A
(D) $0.318 (2t - 177°)$ A

Solution: (A)
Apply KCL at node V_3.

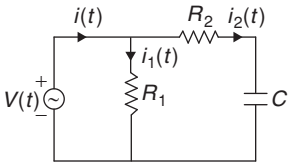
$$I_1 = \frac{V_3}{1}\left(\frac{j}{2}\right) + \frac{V_3 - V_2}{j \times 2 \times 5}$$

$I_1 = j\,0.1\,V_2 + j\,0.4 \times V_3$
$= 0.1 \times \angle 90° \times 0.757 \angle 66.7° + 0.4 \angle 90° \times 0.606 \angle -69.8°$
$= 0.0757 \angle 156.7° + 0.2424 \angle 20.2°$.
$I_1 = 0.1945 \angle 35.736°$A

Locus Diagrams

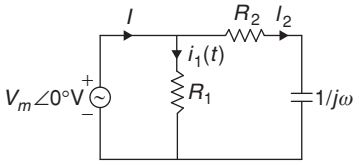
The locus diagram or circle diagram is the graphical representation of the electrical circuit. The frequency response of a circuit has been exhibited by drawing separately the angle and magnitude of a network function against variable parameter (e.g., ω, L, R, C)

Examples 27:



Consider $V(t) = V_m \cos \omega t$ V, if frequency "ω" of the source is varying from 0 to ∞. Draw the locus of the current phasor I_2 _____

Solution: Transform the given network into phasor domain



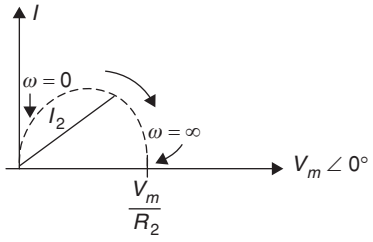
$$I = I_1 + I_2$$

$$I_1 = \frac{V_m \angle 0°}{R_1}; \; I_2 = \frac{V_m \angle 0°}{R_2 + \frac{1}{j\omega c}}$$

If $\omega = 0$, $I_2 = 0$ [$\because$ capacitor open circuit]

$$\omega = \infty, \; I_2 = \frac{V_m \angle 0°}{R_2}$$

Locus of I_2:



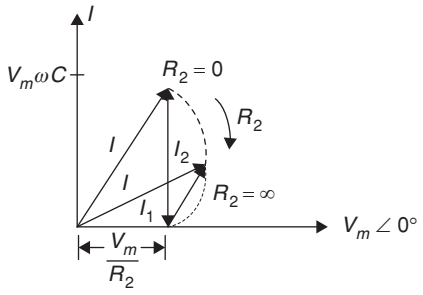
Case 2: In the above circuit instead of 'ω' If R_2 is varying from 0 to ∞.

If $R_2 = 0$; $I_2 = j\,\omega.C\,V_m = \omega \cdot CV_m < 90°$

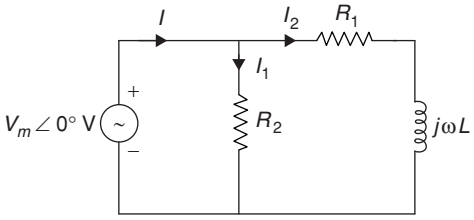
$R_2 = \infty$; $I_2 = 0$,

$$I_1 = \frac{V_m \angle 0°}{R_1}$$

Locus diagram of *I*:



Example 28:



If R_1 is varied from '0' to '∞', draw the locus diagram of *I*.

Solution: $I = I_1 + I_2$

$$I = \frac{V_m \angle 0°}{R_2} + \frac{V_m}{R_1 + j\omega L}$$

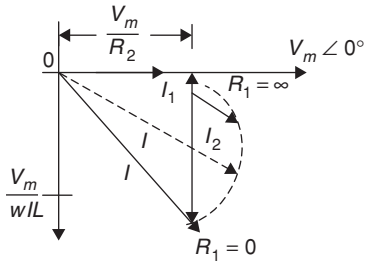
If $R_1 = 0$

$$I = \frac{V_m}{R_2} + \frac{V_m \angle -90°}{\omega L}$$

If $R_1 = \infty$

$$I = \frac{V_m \angle 0°}{R_2} + 0$$

$$\therefore \quad I = \frac{V_m \angle 0°}{R_2}$$



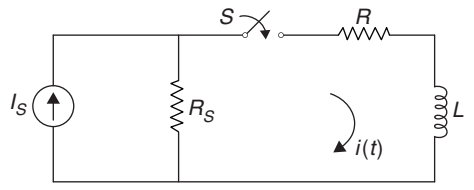
Locus diagram of *I*

EXERCISES

Practice Problems I

Directions for questions 1 to 32: Select the correct alternative from the given choices.

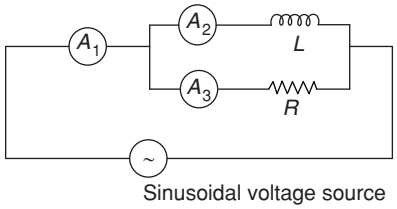
1. In the following circuit, the switch *S* is closed at $t = 0$. The rate of change of current $\frac{di}{dt}(0+)$ is given by



(A) 0 (B) $\frac{R_s L_s}{L}$

(C) $\frac{(R+R_s)I_s}{L}$ (D) ∞

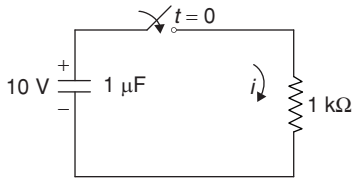
2. In the given figure, A_1, A_2, and A_3 are ideal ammeters. If A_2 and A_3 read 3 *A* and 4 *A*, respectively, then A_1 should read



(A) 1 A (B) 5 A

(C) 7 A (D) None of these

3.



The current in the circuit when the switch is closed at $t = 0$

(A) $10\,e^{-100t}$ (B) $0.01\,e^{-1000t}$

(C) $0.1\,e^{-1000t}$ (D) $10\,e^{-0.1t}$

4. An input voltage $V(t) = 10\sqrt{2}\cos(t + 10°) + 10\sqrt{5}\cos(2t + 10°)$V is applied to a series combination of resistance $R = 1\ \Omega$ and an inductance $L = 1$ H. The resulting steady state current $i(t)$ in ampere is

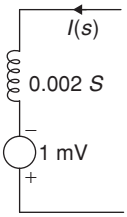
(A) $10\cos(t + 55°) + 10\cos(2t + 10° + \tan^{-1}2)$

(B) $10\cos(t + 55°) + 10\sqrt{\frac{3}{2}}\cos(2t + 55°)$

(C) $10\cos(t - 35°) + 10\cos(2t + 10° - \tan^{-1}2)$

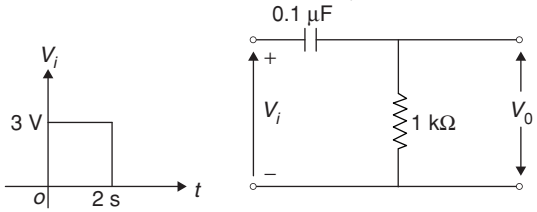
(D) $10\cos(t - 35°) + 10\sqrt{\frac{3}{2}}\cos(2t - 35°)$

5. A_2 mH inductor with some initial current can be represented as shown in the figure, where S is the Laplace transform variable. The value of initial current is



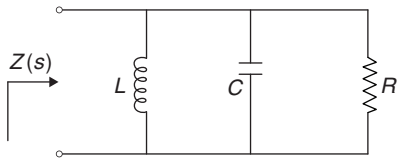
(A) 0.5 A (B) 2.0 A
(C) 1.0 A (D) 0 A

6. A square pulse of 3 V amplitude is applied to $C - R$ circuit shown in the figure. The capacitor is initially uncharged. The output voltage V_0 at time $t = 2$ s is



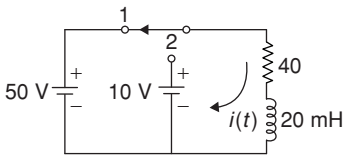
(A) 3 V (B) − 3 V
(C) 0 (D) − 4 V

7. The driving point impedance of the following network is given by $Z(s) = \frac{0.2s}{s^2 + 0.1s + 2}$. The component values are



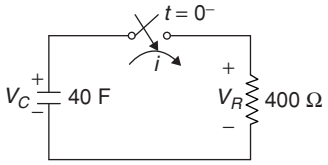
(A) $L = 5$ H, $R = 0.5\ \Omega$, $C = 0.1$ F
(B) $L = 0.1$ H, $R = 0.5\ \Omega$, $C = 5$ F
(C) $L = 5$ H, $R = 2\ \Omega$, $C = 0.1$ F
(D) $L = 0.1$ H, $R = 2\ \Omega$, $C = 5$ F

8. The switch has been in position 1 for a long time, it is moved to position 2 at $t = 0$. The expression for $i(t)$ for $t > 0$ is



(A) $0.25\ e^{-2000t}$ (B) $0.25 + e^{-2000t}$
(C) $0.5\ e^{-2000t}$ (D) $0.5 + e^{-2000t}$

9.



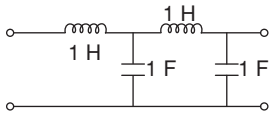
At $t = 0^-$, just before the switch is closed, $V_C = 100$ V. The current $i(t)$ for $t > 0$ is

(A) $100\ e^{-62.5t}$ (B) $50\ e^{-160t}$
(C) $0.25\ e^{-62.5t}$ (D) $50\ e^{-62.5t}$

10. A series RL circuit with $R = 5\ \Omega$ and $L = 2$ mH, has an applied voltage V = 150 sin5000 t, the resulting current i is

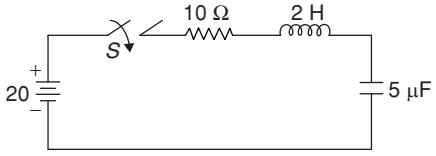
(A) 13.4 sin5000t
(B) 10 cos(5000t − 36.4°)
(C) 13.4 cos(5000t − 63.4°)
(D) 13.4 sin(5000t − 63.4°)

11. Driving point impedance of the network shown in the figure is



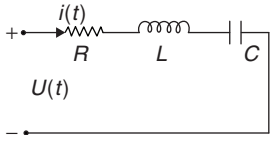
(A) $\frac{s^4 + 3s^2 + 1}{s^3 + 2s}$ (B) $\frac{s^2 + 1}{s(s^2 + 2)}$
(C) $\frac{s^4 + 3s^3 + 2s^2 + 1}{s^3 + 2s}$ (D) $\frac{s^2 + 1}{s + 1}$

12. Initially, the circuit shown in the given figure was relaxed. If the switch is closed at $t = 0$ the values of $i(0^+)$, $\frac{di(0^+)}{dt}$, $\frac{d^2i(0^+)}{dt^2}$ will, respectively, be _____.



(A) 0,10 and 100 (B) 0,10 and 50
(C) 0, 10 and −100 (D) 0,10 and −50

13. Transient response of the following circuit _____.

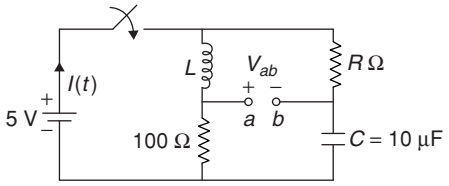


(A) Rises exponentially
(B) Decays exponentially
(C) Oscillatory and the oscillations die down with time
(D) Will have sustained oscillations

14. In the circuit shown below

$i_L(0^-) = 0$, $V_c(0^-) = 0$

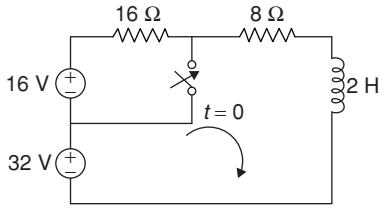
Switch S is closed at $t = 0$ $i(0^+) = 20$ mA, and V_{ab}=0 for $t \geq 0$



The value of R is _____.

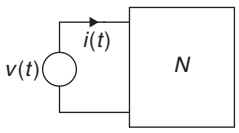
(A) $\frac{1}{4}$ kΩ (B) 250 Ω
(C) 350 Ω (D) 100 Ω

15. For the circuit shown in the following figure. If the switch is closed at $t = 0$; then $i(t)$ for $t \geq 0$ will be



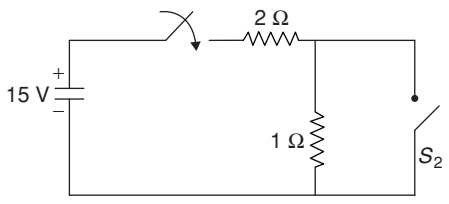
(A) $4 + 2e^{-4t}$ (B) $4 - 2e^{-4t}$
(C) $4 + 2e^{4t}$ (D) $4 - 2^{4t}$

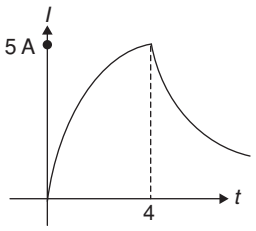
16. The network shown in the figure consists of only two elements. The response for unit step excitation is $i(t) = e^{-5t}$ then the elements are

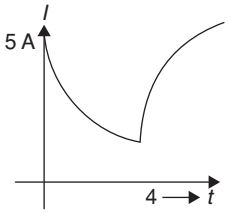


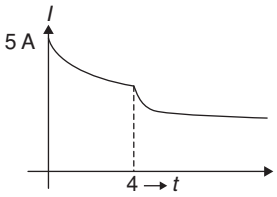
(A) $R = 1\ \Omega$, $L = 5$ H in series
(B) $R = 1\ \Omega$, $C = \frac{1}{5}$ F in series
(C) $R = 1\ \Omega$, $L = 5$ H in parallel
(D) $R = 1\ \Omega$, $C = \frac{1}{5}$ F in parallel

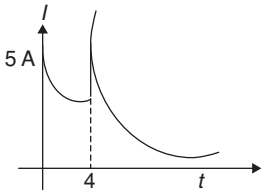
17. The capacitor in the circuit is initially charged to 15 V with S_1 and S_2 open. S_1 is closed at $t = 0$ while S_2 is closed at $t = 4$ s. The wave form of the capacitor current is



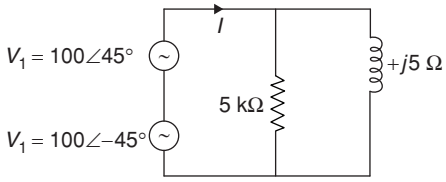
(A)


(B)


(C)


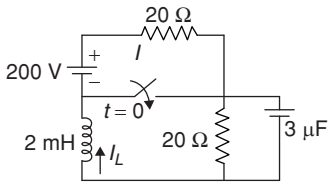
(D)


18. The phase angle of the current I with respect to the voltage V_2 in the circuit shown in the figure is



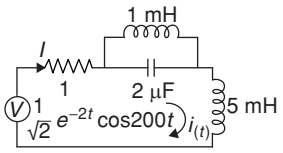
(A) 0° (B) −45°
(C) +45° (D) +90°

19. In the circuit of figure the switch S has been opened for along time. It is closed at $t = 0$. The values of $V_L(0^+)$ and $I_L(0^+)$ are _____.



(A) 200 V, −5 A
(B) 0 V, 5 A
(C) 100 V, 5 A
(D) 100 V, −5 A

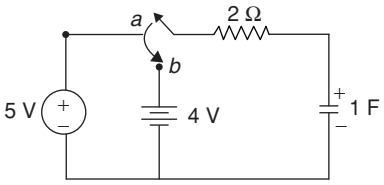
20. Obtain the value of current $i(t)$ in the given circuit at steady state.



(A) $\frac{1}{\sqrt{2}}\cos(200t+45)$ (B) $\frac{1}{\sqrt{2}}$

(C) 0 (D) ∞

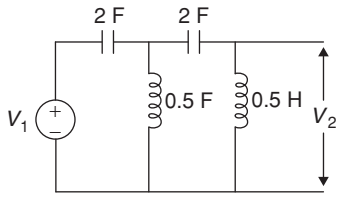
21. For the network shown, the switch is at position '*a*' initially. At steady state, the switch is thrown to position '*b*' Now $i(0^-)=2$ A, $V_c=2$ V are the initial conditions. Find the circuit current



(A) e^{-3t} (B) $-3e^{\frac{-3}{2}t}$

(C) $3e^{\frac{-3}{2}t}$ (D) $3e^{-2t}$

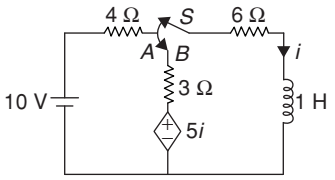
22. Obtain the driving point impedance of the network given in the diagram.



(A) $\frac{s^3+s^2+1}{(s+1)s}$ (B) $\frac{s^4+s^2+s}{1+2s^2}$

(C) $\frac{s^4+s^3+2s^2+1}{(s+1)s}$ (D) $\frac{s^4+s^2+1}{(1+2s^2)2s}$

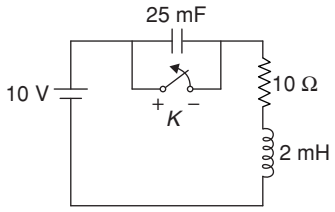
23. For the given circuit switch *S* is at position '*A*' when t < 0. At $t = 0$ the switch is thrown to position *B*. What will the value of current '*i*' in the circuit at the instant $t = 4$ s.



(A) 0.1 μA (B) .01 mA

(C) 10 μA (D) 200 mA

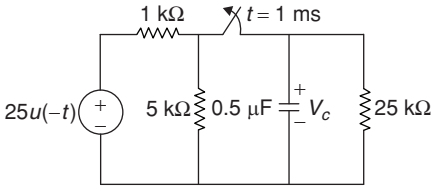
24. In the given network, the switch *K* is closed for a long time and circuit is in steady state. Now at $t = 0$ the switch is opened. Find $V_c(0^+)$ and $i(0^+)$ in the circuit.



(A) 1 A, 0 V (B) 0 A, 5 V

(C) 0, α (D) 0, 0

25. In the following circuit the 25 V source has been applied for a long time the switch is opened at $t = 1$ ms



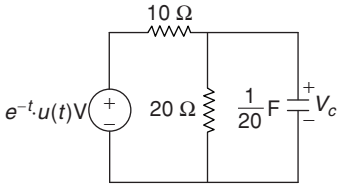
At $t = 5$ m s the value of V_c is

(A) 1.23 V (B) 20.16 V

(C) 1.69 V (D) −1.23 V

26. In the following circuit capacitor is initially uncharged.

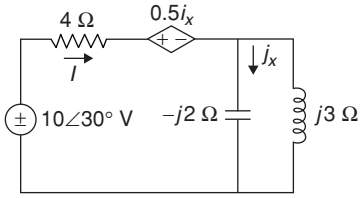
At $t = 0^+$ the value of $\frac{dV_c}{dt}$ and $\frac{d^2V_c}{dt^2}$



(A) 0 V/s, 8 V/s^2 (B) −2 V/s, 8 V/s^2

(C) 2 V/s, −8 V/s (D) None of these

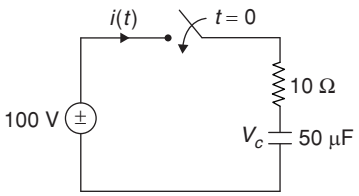
27. In the circuit shown below the current i_x is



(A) 3.94 ∠ 46.28° A (B) 4.62 ∠ 97.38° A

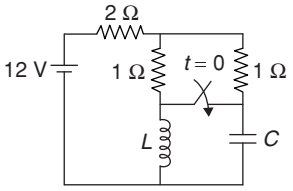
(C) 7.42 ∠ 92.49° A (D) 6.78 ∠ 49.27° A

28. In the circuit shown below. The initial charge on the capacitor is 2.5 mC, with the voltage polarity as indicated. The switch is closed at time $t = 0$. The current $i(t)$ at a time t after the switch is closed is



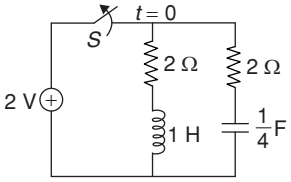
(A) $i(t) = 15 \exp(-2 \times 10^3 t)$ A
(B) $i(t) = 5 \exp(-2 \times 10^3 t)$ A
(C) $i(t) = 10 \exp(-2 \times 10^3 t)$ A
(D) $i(t) = -5 \exp(-2 \times 10^3 t)$ A

29. In the circuit shown below the switch is closed at $t = 0$. What is the initial value of the current through the capacitor?



(A) 0.8 A (B) 2.4 A
(C) 1.6 A (D) 3.2 A

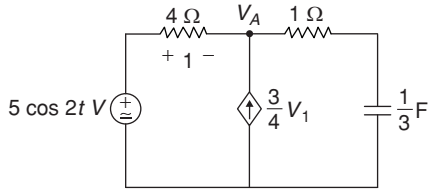
30.



The time constant of the circuit after the switch is opened would be

(A) 2 s (B) 0.5 s
(C) 1 s (D) None of these

31. The power factor seen by the voltage source is



(A) 0.8 (lagging) (B) 0.8 (leading)
(C) 36.9 (Lag) (D) − 36.9 (Leading)

32. An input voltage
$V(t) = 10\sqrt{2}\cos(t + 10°) + 10\sqrt{5} \cdot \cos(2t + 10°)$ V is applied to a series combination of resistance $R = 1\ \Omega$ and an inductance $L = 1$ H. the resulting steady state current $i(t)$ is

(A) $10\cos(t + 55°) + 10\cos(2t + 10° + \tan^{-1}2)$ A

(B) $10\cos(t + 55°) + 10\sqrt{\frac{3}{2}} \cdot \cos(2t + 55°)$ A

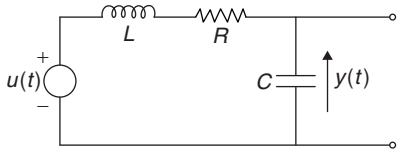
(C) $10\cos(t - 35°) + 10.\cos(2t + 10° - \tan^{-1}2)$ A

(D) $\frac{10\sqrt{3}}{2} \angle 90°$ A

Practice Problems 2

Directions for questions 1 to 24: Select the correct alternative from the given choices.

1. The condition on R, L and C such that the step response $y(t)$ in the figure has no oscillations, is



(A) $R \geq \frac{1}{2}\sqrt{\frac{L}{C}}$ (B) $R \geq \sqrt{\frac{L}{C}}$

(C) $R \geq 2\sqrt{\frac{L}{C}}$ (D) $R = \frac{1}{\sqrt{LC}}$

2. A series circuit consists of two elements has the following current and applied voltage $i = 4\cos(2000t + 11.32°)$ A
$\upsilon = 200\sin(2000t + 50°)$ V. The circuit elements are
(A) Resistance and capacitance
(B) Capacitance and inductance
(C) Inductance and resistance
(D) Both resistances

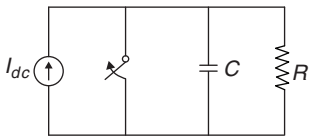
3. Transient current of an *RLC* circuit is oscillatory when

(A) $R = 2\sqrt{\frac{L}{C}}$ (B) $R = 0$

(C) $R > 2\sqrt{\frac{L}{C}}$ (D) $R < 2\sqrt{\frac{L}{C}}$

4. The transient response occurs
(A) Only in resistive circuits
(B) Only in inductive circuits
(C) Only in capacitive circuits
(D) Both in (B) and (C)

5.

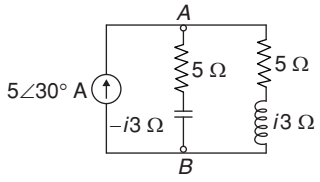


The initial voltage across the capacitor when the switch S is opened at $t = 0$

(A) zero (B) $C \cdot \frac{I_{dc}}{s}$

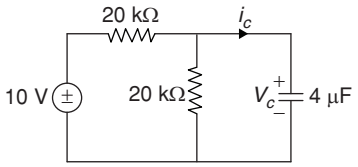
(C) $\frac{1}{Cs} I_{dc}$ (D) $Cs\, I_{dc}$

6. In the AC network shown in the figure, the phasor voltage V_{AB} (in V) is



(A) 0 (B) $5\angle 30°$
(C) $12.5\angle 30°$ (D) $17\angle 30°$

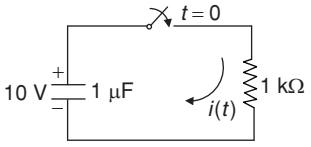
7. In the circuit shown, V_C is 0 V at $t = 0$ V at $t = 0$ s. For $t > 0$, the capacitor current $i_C(t)$, where 't' is in seconds, is given by



(A) 0.50 exp (– 25t) mA
(B) 0.25 exp (– 25t) mA
(C) 0.50 exp (–12.5t) mA
(D) 0.25 exp (– 6.25t) mA

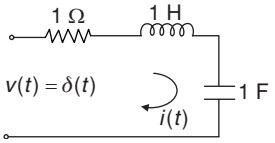
8. A series RL circuit, with $R = 10\ \Omega$ and $L = 1$ H, has a 100 V source applied at $t = 0$. The current for $t > 0$ is
(A) $10\ e^{-10t}$
(B) $10\ (1 - e^{-10t})$
(C) $100\ e^{-100t}$
(D) $100\ (1 - e^{-100t})$

9. The current in the circuit when the switch is closed at $t = 0$ is



(A) $10\ e^{-100t}$ (B) $0.01\ e^{-1000t}$
(C) $0.1\ e^{-1000t}$ (D) $10\ e^{0.1t}$

10.

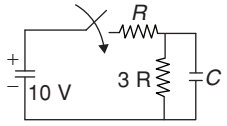


The circuit shown in the figure is initially relaxed. The Laplace transform of the current $i(t)$ is

(A) $\dfrac{s+1}{s^2+2s+1}$ (B) $\dfrac{s+1}{s^2+s+1}$

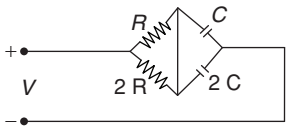
(C) $\dfrac{s}{s^2+s+1}$ (D) $\dfrac{s}{s^2+2s+1}$

11. The time constant of the network shown in the figure is



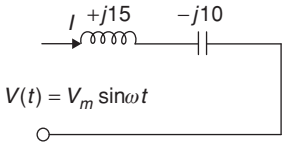
(A) 4 RC (B) $\dfrac{3}{4}RC$
(C) 3 RC (D) RC

12. The time constant of the network shown in the figure is _____.



(A) $\dfrac{2}{3}RC$ (B) $\dfrac{3}{2}RC$
(C) 2 RC (D) RC

13. The network shown in the figure draws a current of "I"

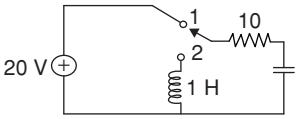


If the supply frequency is doubled then the current drawn by the circuit is _____.

(A) $\dfrac{I}{2}$ (B) 2 I

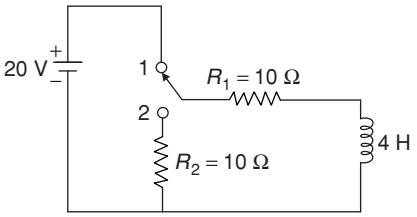
(C) $\dfrac{I}{5}$ (D) $\dfrac{2I}{5}$

14. In the circuit shown in the figure, the switch is thrown from position 1 to 2 at $t = 0$, after being at position 1 for a long time. The value of $\dfrac{d^2i(0^+)}{dt^2}$ is _____.



(A) 200 (B) – 200
(C) – 100 (D) 100

Common Data for Questions 15 and 16: The circuit shown in the figure is initially under a steady state condition.



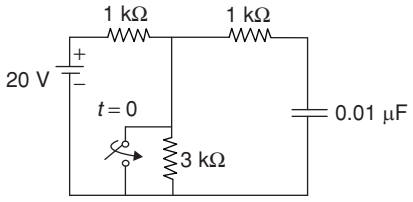
The switch is moved from position 1 to position 2 at $t = 0$.

15. The current through inductor immediately after switching is _____.

(A) 2 A (B) $\dfrac{1}{2}$ A
(C) 1 A (D) 5 A

16. The expression for current $i(t)$ is _____.
(A) e^{-5t} (B) $2e^{-5t}$
(C) $\dfrac{2}{5}e^{-5t}$ (D) $5e^{-2t}$

17. The switch in the circuit shown in the figure closes at $t = 0$. Find current i_c for all times.



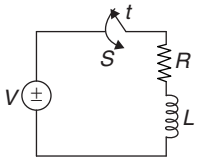
(A) $5\times10^{-3}\times e^{-105t}$ A

(B) $15\times10^{-2}\times e^{-105t}$ A

(C) $15\times10^{-3}\times e^{-105t}$ A

(D) $10\times10^{-3}\times e^{-105t}$ A

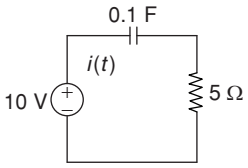
18. For the given circuit, the current passing through inductor 'L' at the instant $t = 0^+$ is



(A) $\frac{V}{R}$ (B) Infinity

(C) $\frac{V}{R+jwL}$ (D) 0

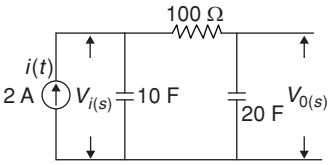
19. Determine the current i for $t \geq 0$, if $V_c(0) = 1$ V for the circuit shown



(A) $\frac{0.9e^{-5t}}{2}$ (B) $1.8\ e^{-2t}$

(C) $5e^{-2t}$ (D) 3.6

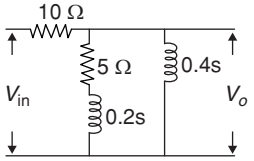
20. Find the transfer function of the given system $\left(\frac{V_0(s)}{V_1(s)}\right)$



(A) $\frac{1}{2000s+1}$ (B) $\frac{10s}{2000s+1}$

(C) $\frac{100}{1000s+1}$ (D) $\frac{1}{1000s+1}$

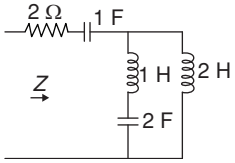
21. Obtain the transfer function of the following system:



(A) $\frac{0.4s}{0.2s+1s}$ (B) $\frac{s(2+0.4s)}{0.4s^2+8s+100}$

(C) $\frac{s(2+.08s)}{0.08s^2+8s+50}$ (D) $\frac{2+.04s}{s^2+8s+50}$

22. Find the driving point admittance of the network given below.



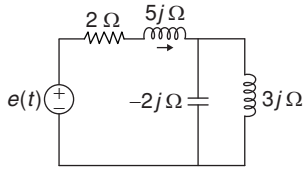
(A) $\frac{3s^3+2s}{2s^4+6s^3+7s^2+4s+2}$

(B) $\frac{3s^2+2}{2s^4+5s^2+4s+2}$

(C) $\frac{(3s^2+2)s}{2s^4+7s^3+6s^2+4s}$

(D) $\frac{5s^3+2s}{s^4+7s^3+6s^2+4s+2}$

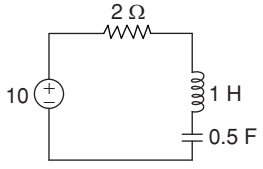
23. Find the voltage V_{ab} across the impedance of $(2 + 5j)\ \Omega$ in the network.
The supply voltage.
$e(t) = 10 \sin(2\pi t + 45)$



(A) $20\angle 44°$ (B) $24\angle -40°$

(C) $4.48\angle -108°$ (D) $2.25\angle 63.43°$

24. Find the current $i(t)$ through the circuit given



(A) $10e^{-t}\cos t$ (B) $\frac{10}{\sqrt{2}}e^{t}\cos\sqrt{2}t$

(C) $10e^{-t}\sin t$ (D) $\frac{10}{\sqrt{2}}e^{t}\sin\sqrt{2}t$

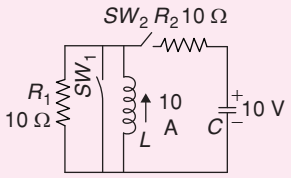
Previous Years' Questions

1. An ideal capacitor is charged to a voltage V_0 and connected at $t = 0$ across an ideal inductor L. (The circuit now consists of a capacitor and inductor alone). If we let $\omega_0 = \frac{1}{\sqrt{LC}}$, the voltage across the capacitor at time $t > 0$ is given by **[2006]**

(A) V_0 (B) $V_0 \cos(\omega_0 t)$

(C) $V_0 \sin(\omega_0 t)$ (D) $V_0 e^{-\omega 0 t} \cos(\omega_0 t)$

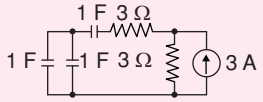
2. In the circuit shown in figure switch SW_1 is initially CLOSED and SW_2 is OPEN. The inductor L carries a current of 10 A and the capacitor is charged to 10 V with polarities as indicated. SW_2 is initially CLOSED at $t = 0-$and SW_1 is OPENED at $t = 0$. The current through C and the voltage across L at $t = 0+$ is **[2007]**



(A) 55 A, 4.5 V (B) 5.5 A, 45 V

(C) 45 A, 5.5 V (D) 4.5 A, 55 V

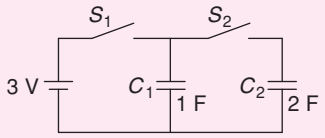
3. The time constant for the given circuit will be **[2008]**



(A) $\frac{1}{9}$ s (B) $\frac{1}{4}$ s

(C) 4 s (D) 9 s

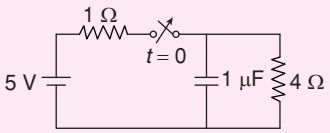
4. In the figure shown, all elements used are ideal. For time $t < 0$, S_1 remained closed and S_2 open. At $t = 0$, S_1 is opened and S_2 is closed. If the voltage V_{c2} across the capacitor C_2 at $t = 0$ is zero, the voltage across the capacitor combination at $t = 0^+$ will be **[2009]**



(A) 1 V (B) 2 V

(C) 1.5 V (D) 3 V

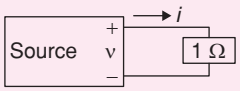
5. The switch in the circuit has been closed for a long time. It is opened at $t = 0$. At $t = 0^+$, the current through the 1 μF capacitor is **[2010]**



(A) 0 A (B) 1 A

(C) 1.25 A (D) 5 A

6. As shown in the figure, a 1 Ω resistance is connected across a source that has a load line $v + i = 100$. The current through the resistance is **[2010]**



(A) 25 A (B) 50 A

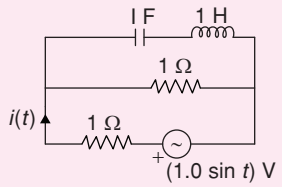
(C) 100 A (D) 200 A

7. The voltage applied to a circuit is $100\sqrt{2}\cos(100\pi)$ V and the circuit draws a current of $10\sqrt{2}\sin(100\pi + \pi/4)$ amperes. Taking the voltage as the reference phasor, the phasor representation of the current in amperes is **[2011]**

(A) $10\sqrt{2}\angle\frac{-\pi}{4}$ (B) $10\sqrt{2}\angle\frac{-\pi}{4}$

(C) $10\angle+\pi/4$ (D) $10\sqrt{2}\angle+\frac{\pi}{4}$

8. The rms value of the current $i(t)$ in the circuit shown below is **[2011]**



(A) $\frac{1}{2}$ A (B) $\frac{1}{\sqrt{2}}$

(C) 1 A (D) $\sqrt{2}$ A

9. The current I_C in the figure above is **[2011]**

(A) $j\,2$ A (B) $-j\frac{1}{\sqrt{2}}$ A

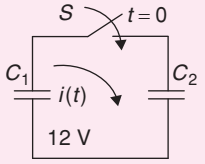
(C) $+j\frac{1}{\sqrt{2}}$ A (D) $+j2$ A

10. The power dissipated in the resistor R is **[2011]**

(A) 0.5 W (B) 1 W

(C) $\sqrt{2}$ W (D) 2 W

11. In the following figure, C_1 and C_2 are ideal capacitors. C_1 has been charged to 12 V before the ideal switch S is closed at $t = 0$. The current $i(t)$ for all t is **[2012]**



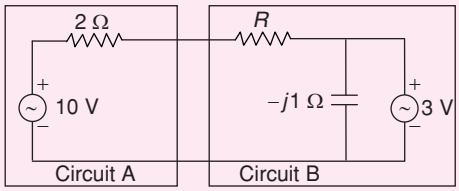
(A) Zero

(B) A step function

(C) An exponentially decaying function

(D) An impulse function

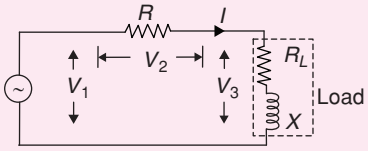
12. Assuming both the voltage sources are in phase, the value of R for which maximum power is transferred from circuit A to circuit B is **[2012]**



(A) 0.8 Ω (B) 1.4 Ω
(C) 2 Ω (D) 2.8 Ω

Common Data for Questions 13 and 14: In the circuit shown, the three voltmeter readings are V_1 = 220 V, V_2 = 122 V, V_3 = 136 V.

13. The power factor of the load is **[2012]**

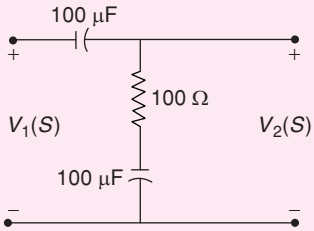


(A) 0.45 (B) 0.50
(C) 0.55 (D) 0.60

14. If R_L = 5 Ω, the approximate power consumption in the load is **[2012]**

(A) 700 W (B) 750 W
(C) 800 W (D) 850 W

15. The transfer function $\frac{V_2(s)}{V_1(s)}$ of the circuit shown below is **[2013]**



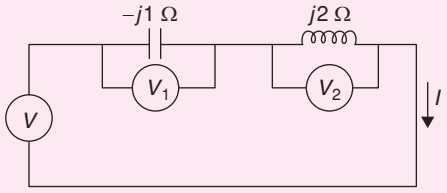
(A) $\frac{0.5s+1}{s+1}$ (B) $\frac{3s+6}{s+2}$

(C) $\frac{s+2}{s+1}$ (D) $\frac{s+1}{s+2}$

16. A single phase load is supplied by a single-phase voltage source. If the current flowing from the load to the source is $10\angle -150°$ A and if the voltage at the load terminals is $100\angle 60°$ V, then the **[2013]**

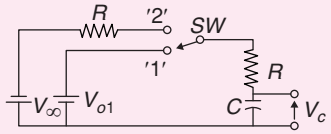
(A) Load absorbs real power and delivers reactive power.
(B) Load absorbs real power and absorbs reactive power.
(C) Load delivers real power and delivers reactive power.
(D) Load delivers real power and absorbs reactive power.

17. Three moving iron type voltmeters are connected as shown below. Voltmeter readings are V, V_1, and V_2, as indicated. The correct relation among the voltmeter readings is **[2013]**



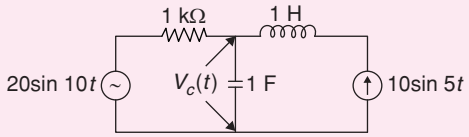
(A) $V = \frac{V_1}{\sqrt{2}} + \frac{V_2}{\sqrt{2}}$
(B) $V = V_1 + V_2$
(C) $V = V_1 V_2$
(D) $V = V_2 - V_1$

18. The switch SW shown in the circuit is kept at position '1' for a long duration. At t = 0+, the switch is moved to position '2'. Assuming $|V_{o2}| > |V_{o1}|$, the voltage $v_c(t)$ across the capacitor is **[2014]**



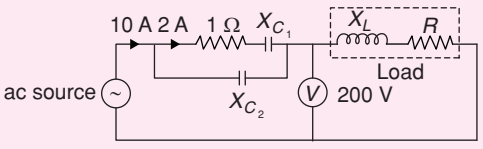
(A) $V_c(t) = -V_{o2}(1 - e^{-t/RC}) - V_{o1}$
(B) $V_c(t) = V_{o2}(1 - e^{-t/RC}) + V_{o1}$
(C) $V_c(t) = -(V_{o2} + V_{o1})(1 - e^{-t/RC}) - V_{o1}$
(D) $V_c(t) = (V_{o2} - V_{o1})(1 - e^{-t/RC}) + V_{o1}$

19. The voltage across the capacitor, as shown in the figure, is expressed as
$v_c(t) = A_1 \sin(\omega_1 t - \theta_1) + A_2 \sin(\omega_2 t - \theta_2)$ **[2014]**



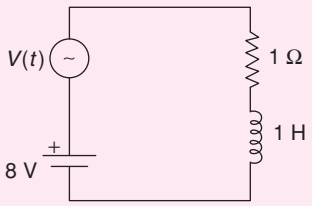
The values of A_1 and A_2, respectively, are
(A) 2.0 and 1.98
(B) 2.0 and 4.20
(C) 2.5 and 3.50
(D) 5.0 and 6.40

20. The total power dissipated in the circuit, shown in the figure, is 1 kW. **[2014]**

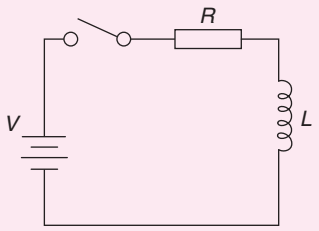


The voltmeter, across the load, reads 200 V. The value of X_L is_____.

21. The circuit shown in the figure has two sources connected in series. The instantaneous voltage of the AC source (in Volts) is given by $v(t) = 12 \sin t$. If the circuit is in steady state, then the rms value of the current (in Ampere) flowing in the circuit is _____. **[2015]**



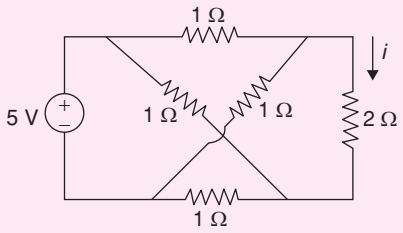
22. A series RL circuit is excited at $t = 0$ by closing a switch as shown in the figure. Assuming zero initial conditions, the value of $\frac{d^2t}{dt^2}$ at $t = 0^+$ is **[2015]**



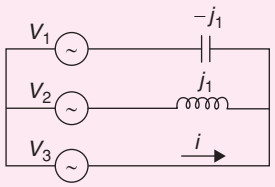
(A) $\frac{V}{L}$ (B) $\frac{-V}{R}$

(C) 0 (D) $\frac{-RV}{L^2}$

23. The current i(Ampere) in the 2 Ω resistor of the given network is _____. **[2015]**



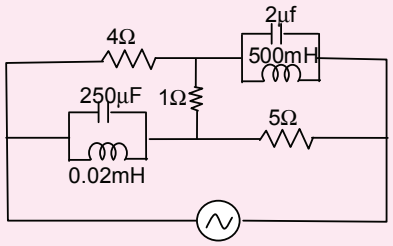
24. In the given network $V_1 = 100 \angle 0°$ V, $V_2 = 100 \angle -120°$ V, $V_3 = 100 \angle +120°$ V. The phasor current i (in Ampere) is **[2015]**



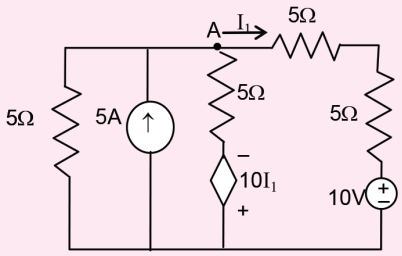
(A) $173.2 \angle -60°$ (B) $173.2 \angle 120°$

(C) $100.0 \angle -60°$ (D) $100.0 \angle 120°$

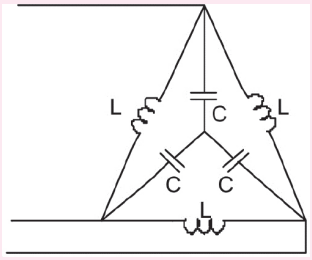
25. In the circuit shown below, the supply voltage is 10 sin(1000) volts. The peak value of the steady state current through the 1Ω resistor, in amperes, is _____. **[2016]**



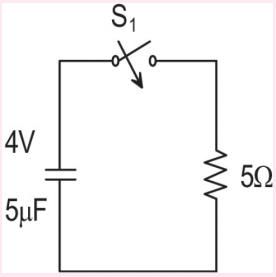
26. The circuit below is excited by a sinusoidal source. The value of R, in Ω, for which the admittance of the circuit becomes a pure conductance at all frequencies is_____. **[2016]**



27. In the balanced 3-phase, 50Hz, circuit shown below, the value of inductance (L) is 10mH. The value of the capacitance (C) for which all the line currents are zero, in millifarads, is _____. **[2016]**



28. In the circuit shown below, the initial capacitor voltage is 4V. switch S_1 is closed at $t = 0$. The charge (in μC) lost by the capacitor form $t = 25$μS to $t = 100$μS is _____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. B	**2.** B	**3.** B	**4.** C	**5.** A	**6.** C	**7.** D	**8.** B	**9.** C	**10.** D
11. A	**12.** D	**13.** C	**14.** B	**15.** B	**16.** B	**17.** D	**18.** A	**19.** C	**20.** C
21. B	**22.** D	**23.** A	**24.** A	**25.** A	**26.** C	**27.** B	**28.** A	**29.** A	**30.** B
31. B	**32.** C								

Practice Problems 2

1. C	**2.** D	**3.** D	**4.** D	**5.** A	**6.** D	**7.** A	**8.** B	**9.** B	**10.** C
11. B	**12.** C	**13.** C	**14.** B	**15.** A	**16.** B	**17.** C	**18.** D	**19.** B	**20.** A
21. C	**22.** A	**23.** B	**24.** C						

Previous Years' Questions

1. B	**2.** D	**3.** C	**4.** A	**5.** B	**6.** B	**7.** B	**8.** B	**9.** D	**10.** B
11. D	**12.** A	**13.** A	**14.** B	**15.** D	**16.** B	**17.** D	**18.** D	**19.** A	
20. 17.34	**21.** 9.99 A	**22.** D	**23.** 0 A	**24.** B	**25.** 1	**26.** 14.14	**27.** 3	**28.** 6.99	

Chapter 4

Two Port Networks

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Open circuit or impedance(z) parameters
- Equivalent circuit of z-parameters
- y-parameters or short circuit admittance parameters
- Hybrid parameters
- g-parameters or inverse hybrid parameters
- Transmission or abcd parameters
- inverse transmission parameters
- Inter connection of networks
- Terminated two-port network
- Network functions
- Network graphs
- Twigs and links
- Incidence matrix and formulation of KCL
- Tie-set matrix and branch currents

Introduction

A pair of terminals through which a current may enter or leave a network is known as a port.

The current entering one terminal leaves through the other terminals so that the net current in the port equals zer o.

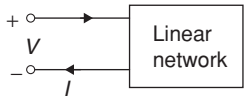


Figure 1 One-port network

A two-port networks is an electrical network with two separate ports for input and output.

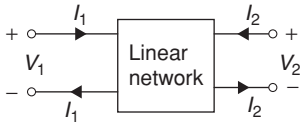


Figure 2 Two-port network

To characterize a two-port network required that we relate the terminal quantities V_1, V_2, I_1 and I_2.

Classification of Networks

1. **Linear circuits:**
   It is the circuit whose parameters remain constant with change in applied voltage or current ($V \propto I$ ohm's Law)
   Example: Resistance, inductance and capacitance
2. **Non–linear circuits:**
   It is a circuit whose parameters changed with voltage or current.
   Example: Diodes, transistor … etc
   Non linear circuits do not obey ohm's Law.
3. **Unilateral circuits Bilateral circuit:**
   When the direction of current is changed, the characteristic or properties of the circuit may change. This circuit is called unilateral circuits.
   Example: Diode, transistor, UJT … etc
   Otherwise, it is called bilateral circuit.
   Example: R, L, C circuits.

Active and Passive Elements

If a circuit element has the capability of enhancing the energy level of a signal passing through it, it is called an active element.
Example: Transistors, op-amp, vaccum tubes … etc. Otherwise it is called passive elements.
Example: Resistors, inductors, thermistors capacitors etc are passive elements.

Lumped and distributed network

Physically separable network elements are like R, L and C are known as lumped elements.

A transmission line on a cable in the other hand is an example of distributed parameter network. They are not physically separable. If the network is fabricated with its elements in lumped form, it is called a lumped network and if in distributed form it is called distributed network.

Recurrent and Non-recurrent networks

When a large circuit consists of similar networks connected one after another, the network is called as recurrent network or cascaded

network. It is also called as ladder network. Otherwise a single network is called non-recurrent network.

Symmetrical and asymmetrical network

If the network looks the same from both the ports then it is said to be symmetrical. Otherwise it is called asymmetric network.

Examples:

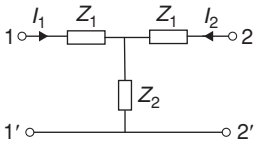


Figure 3 Symmetrical network

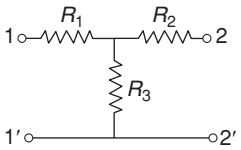


Figure 4 Symmetrical network

Reciprocal and Non-reciprocal networks

If the network obeys the reciprocity theorem then it is called reciprocal network. Otherwise it is called non-reciprocal network.

All the passive networks are always reciprocal and all the active networks are always non-reciprocal.

Network Configuration

T-section

When a network section looks like a 'T'. It is known as T-section.

Examples:

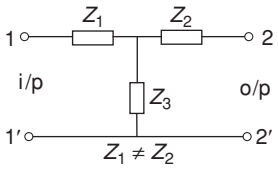


Figure 5 Un-symmetrical T-section

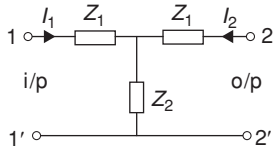


Figure 6 Symmetrical and unbalanced T-section

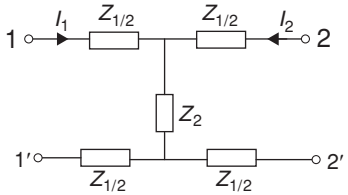


Figure 7 Balanced symmetrical T-section

π-section

Examples:

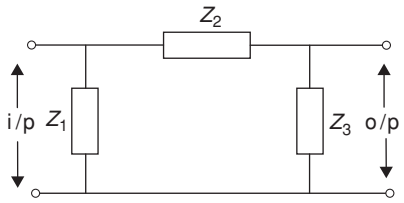


Figure 8 Unbalanced asymmetrical π-section ($Z_1 \neq Z_3$)

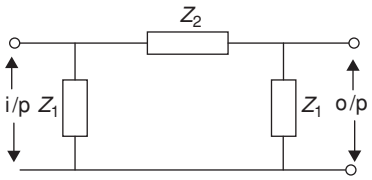


Figure 9 Unbalanced symmetrical π-section

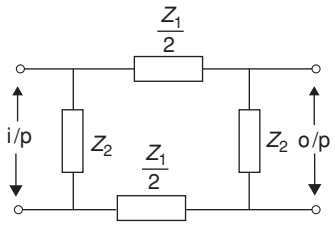
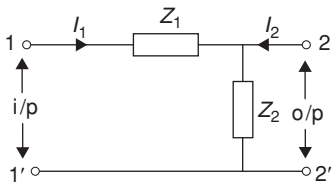


Figure 10 Balanced symmetrical π-section

L-section

When the network section looks like 'L' the configuration is termed as L-section.



Lattice Section

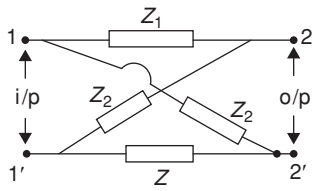
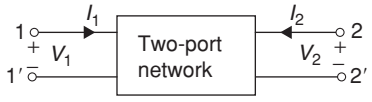


Figure 11 Symmetrical lattice section

Two-port netwoks



A two-port network has two pairs of accessible terminals; one pair represents the input and the other represents the output. Both the currents I_1 and I_2 enter the network and the polarities of the voltages are shown in the figure. There are four variable V_1, V_2, I_1 and I_2 of these four variables, two can

be taken as independent variables, the remaining two will be dependent variables.

Open Circuit or Impedance (Z) Parameters

Here the two voltages V_1 and V_2 are functions of I_1 and I_2

$V_1 = f(I_1, I_2)$

$V_2 = f(I_1, I_2)$

$[V] = [Z] \cdot [I]$

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \quad (1)$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \quad (2)$$

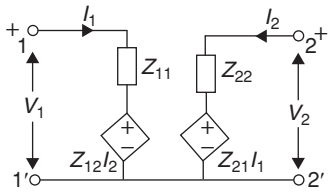
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I1=0}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \mathrm{I}_1 \\ \mathrm{I}_2 \end{bmatrix}$$

From (1) and (2), the network can be drawn as shown in figure

Equivalent Circuit of Z-Parameters



Condition of reciprocity and symmetry

Network must be reciprocal when ratio of response at port 2 to the excitation at port 1, is same as ratio of response at port 1 to port 2, then the network is called reciprocal.

$$\left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{V_1}{I_2} \text{ at } I_1 = 0$$

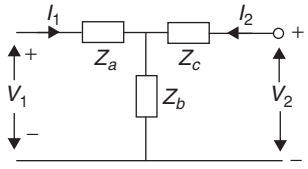
Condition for reciprocal:

$$Z_{12} = Z_{21}$$

Condition for symmetrical network:

$$\boxed{Z_{11} = Z_{22}}$$

Example 1:



Find the *Z*-parameters for the circuit shown in figure

Solution: $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

Let $I_2 = 0$

$V_1 = I_1 (Z_a + Z_b)$

$$Z_{11} = \frac{V_1}{I_1} = Z_a + Z_b$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$V_2 = I_1 Z_b$

$Z_{21} = Z_b$

Let $I_1 = 0$

$V_2 = I_2 (Z_c + Z_b)$

$Z_{22} = Z_c + Z_b$

$V_1 = I_2 Z_b$

$Z_{12} = Z_b$

$$\Rightarrow \quad [Z] = \begin{bmatrix} Z_a + Z_b Z_b \\ Z_b Z_b + Z_c \end{bmatrix}$$

Example 2: The following readings are obtained experimentally for an unknown two-port network:

	V_1	V_2	I_1	I_2
o/p open	80 V	60 V	10 A	0
i/p open	50 V	40 V	0	5 A

The *Z*-parameters are

(A) $\begin{bmatrix} 8 & 6 \\ 10 & 8 \end{bmatrix}$

(B) $\begin{bmatrix} 8 & 10 \\ 6 & 8 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 10 \\ 8 & 4 \end{bmatrix}$

(D) None of the above

Solution: (B)
We know $[V] = [Z]\,[I]$

If $I_2 = 0;\ Z_{11} = \dfrac{V_1}{I_1}$ and $Z_{21} = \dfrac{V_2}{I_1}$

So $Z_{11} = \dfrac{80}{10} = 8\ \Omega$

$$Z_{21} = \frac{60}{10} = 6\ \Omega$$

If $I_1 = 0$.

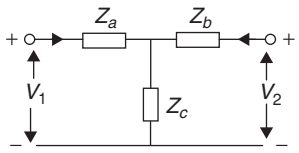
$$Z_{22} = \frac{V_2}{I_2} \text{ and } Z_{12} = \frac{V_1}{I_2}$$

$$Z_{22} = \frac{40}{5} = 8\ \Omega \text{ and } Z_{12} = \frac{50}{5} = 10\ \Omega$$

$$\therefore\quad [Z] = \begin{bmatrix} 8 & 10 \\ 6 & 8 \end{bmatrix}$$

$\therefore Z_{11} = Z_{22} = 8\ \Omega \Rightarrow$ Symmetrical network
$Z_{12} \neq Z_{21} \Rightarrow$ Non-reciprocal network

Example 3:



If $Z_a = 2\ \angle 0°$, $Z_b = 5\ \angle -90°$, and $Z_c = 3\ \angle 90°$. Then the above T-network, Z-parameters are

(A) Symmetrical and Reciprocal
(B) Symmetrical and Non-reciprocal
(C) Asymmetrical and Reciprocal
(D) Asymmetrical and Non-Reciprocal

Solution: (C)
Apply KVL, the loop equations are

$V_1 = 2I_1 + 3\ \angle 90°(I_1 + I_2)$
$V_1 = (2 + j3)\ I_1 + j3I_2$ (3)
$V_2 = 5\ \angle -90°I_2 + 3\ \angle 90°\ (I_1 + I_2)$
$V_2 = j3\ I_1 + (3j - 5j)\ I_2$
$V_2 = j3I_1 - 2jI_2$ (4)

From equation (3) and (4)

$$[Z] = \begin{bmatrix} 2 + j3 & j3 \\ j3 & -2j \end{bmatrix} = \begin{bmatrix} 3.6\angle 56° & 3\angle 90° \\ 3\angle 90° & 2\angle -90° \end{bmatrix}$$

$Z_{11} \neq Z_{22} \Rightarrow$ unsymmetrical
$Z_{12} = Z_{21} \Rightarrow$ Reciprocal network.

Y-Parameters or Short Circuit Admittance Parameters

In a two port network, the input currents I_1 and I_2 can be expressed in terms of input and output voltages V_1 and V_2 respectively as $[I] = [Y]\,[V]$
where $[Y]$ is the admittance matrix.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

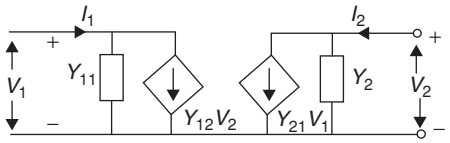
Here, $I_1 = f(V_1, V_2)$
$I_2 = f(V_1, V_2)$
$I_1 = Y_{11}V_1 + Y_{12}V_2$ (5)
$I_2 = Y_{21}V_1 + Y_{22}V_2$ (6)

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \quad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

From (5) and (6), the circuit can be drawn as shown in the figure.

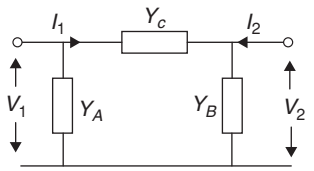
Equivalent Circuit of Y-Parameters



Condition for reciprocity and symmetrical

1. If $Y_{11} = Y_{22} \Rightarrow$ Symmetrical otherwise asymmetrical network
2. If $Y_{12} = Y_{21} \Rightarrow$ Reciprocal network. (Or) passive network otherwise Non-reciprocal or active network.

Example 4: Find the Y-parameters of the following π-circuit shown in below figure.



Solution: Using KCL at node a

$I_1 = V_1\ Y_A + (V_1 - V_2)\ Y_c$
$I_1 = (Y_A + Y_C)\ V_1 - (Y_C)\ V_2$ (7)

Apply KCL at node b

$I_2 = V_2.\ Y_B + (V_2 - V_1)\ Y_C$
$I_2 = -Y_c\ V_1 + (Y_B + Y_C)\ V_2$ (8)

From equation (7) and (8)

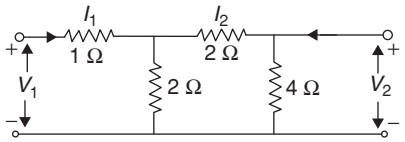
$$[Y]=\begin{bmatrix} Y_A+Y_C & -Y_C \\ -Y_C & Y_B+Y_C \end{bmatrix}$$

Where,

$$Y=\frac{1}{R}, Y_L=\frac{1}{SL}$$

And $Y_C = SC$

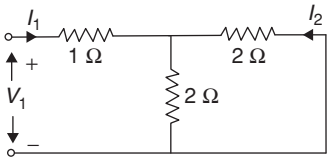
Example 5: Find the *Y*-parameters for the network shown in figure



Solution: $I_1 = Y_{11}V_1 + Y_{12}V_2$

$I_2 = Y_{21}V_1 + Y_{22}V_2$

With $V_2 = 0$



$V_1 = I_1 \times (1 + 2 \,\|\, 2)$

$V_1 = I_1 (1 + 1)$

$$\frac{I_1}{V_1}=\frac{1}{2}=0.5 \text{ mho}$$

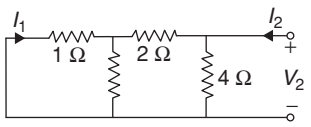
$$-I_2 = I_1 \times \frac{2}{2+2}$$

$$-I_2 = I_1 \times \frac{2}{4}$$

$$-I_2 = \frac{V_1}{2} \times \frac{1}{2}$$

$$Y_{21}=\frac{I_2}{V_1}=\frac{-1}{4} \text{ mho}$$

With $V_1 = 0$



$V_2 = I_2 [(1 \| 2 + 2) \| 4]$

$$V_2 = I_2\left[\left(\frac{1\times 2}{1+2}+2\right)\|4\right]$$

$$=\left[\left(\frac{2}{3}+2\right)\|4\right]$$

$$V_2 = I_2\left[\frac{\frac{8}{3}\times 4}{\frac{8}{3}+4}\right]=\frac{\frac{8}{3}}{\frac{8+12}{3}}=\frac{32}{20}=\frac{8}{5}$$

$$\frac{I_2}{V_2}=Y_{22}=\frac{5}{8} \text{ mho}$$

$$I_2' = I_2 \times \frac{4}{4+\frac{8}{3}}$$

$$I_2' = I_2 \times \frac{12}{20}$$

$$-I_1 = I_2' \times \frac{2}{2+1}$$

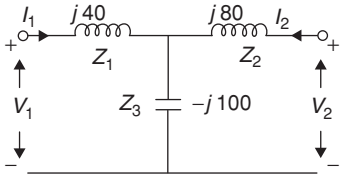
$$-I_1 = I_2 \times \frac{12}{20}\times\frac{2}{3}$$

$$-I_1 = I_2 \times \frac{2}{5}$$

$$-I_1 = \frac{5}{8}V_2 \times \frac{2}{5}$$

$$\frac{I_1}{V_2}=Y_{12}=\frac{-1}{4} \text{ mho}$$

Example 6. Find *Y*-parameters of network shown in figure.



Solution:

$Y = [Z]^{-1}$

$$[Z]=\begin{bmatrix} -j60 & -j100 \\ -j100 & -j20 \end{bmatrix} \Rightarrow \begin{bmatrix} j60 & j100 \\ j100 & j20 \end{bmatrix}$$

$[Y] = [Z^{-1}]$

$$=\frac{1}{|Z|}\{\text{adj}Z\}=\frac{1}{8800}\begin{bmatrix} -j20 & j100 \\ j100 & -j60 \end{bmatrix}$$

$$= \begin{bmatrix} -j2.27\times10^{-3}\ \Omega & j11.36\times10^{-3}\ \Omega \\ j11.36\times10^{-3}\ \Omega & -j6.8\times10^{-3}\ \Omega \end{bmatrix}$$

$$Y_{11} = -j2.27 \times 10^{-3}\ \Omega$$
$$Y_{12} = -j11.36 \times 10^{-3}\ \Omega$$
$$Y_{21} = -j11.36 \times 10^{-3}\ \Omega$$
$$Y_{22} = -j6.8 \times 10^{-3}\ \Omega$$

Hybrid Parameters

The Z and Y-parameters of a two-port network do not always exist.

A two port network can be represented using the h-parameters. The describing equations for h-parameters are

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

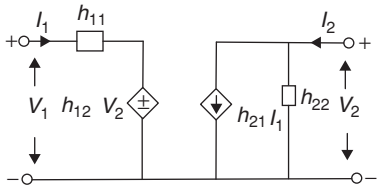
The value of the parameters are determined as

$$h_{11} = \frac{\frac{V_1}{I_1}}{V_2} = 0;\ h_{12} = \frac{\frac{V_1}{V_2}}{I_1} = 0$$

$$h_{21} = \frac{\frac{I_2}{I_1}}{V_2} = 0;\ h_{22} = \frac{\frac{I_2}{V_2}}{I_1} = 0$$

Where

$h_{11} \Rightarrow$ short circuit input impedance
$h_{12} \Rightarrow$ open circuit reverse voltage gain
$h_{21} \Rightarrow$ short circuit forward current gain
$h_{22} \Rightarrow$ open circuit o/p admittance



Condition of reciprocity

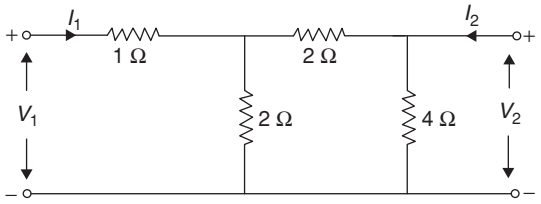
$$\boxed{h_{12} = -h_{21}}$$

Condition of symmetry

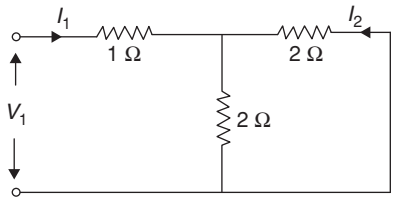
$$|h| = 1$$

$$\text{i.e., } h_{11} \cdot h_{22} - h_{12} \cdot h_{21} = 1$$

Example 7 Find the h-parameters of the network shown in figure



With $V_2 = 0$



$$\frac{V_1}{I_1} = h_{11} = 1 + 2 \,||\, 2$$
$$= 1 + 1 = 2\ \Omega$$
$$-I_2 = I_1 \times \frac{2}{2+2}$$
$$-I_2 = I_1 \times \frac{1}{2}$$
$$h_{21} = \frac{I_2}{I_1} = \frac{-1}{2}$$

With $I_1 = 0$

$$V_1 = V_2 \times \frac{2}{2+2}$$
$$\frac{V_1}{V_2} = h_{12} = \frac{1}{2}$$
$$V_2 = I_2 \times 4||(2+2)$$
$$V_2 = I_2 \times 2$$
$$\frac{I_2}{V_2} = h_{22} = \frac{1}{2}$$

G-Parameters or Inverse Hybrid Parameters

These are represented by

$I_1 = g_{11} V_1 + g_{12} I_2$
$V_2 = g_{21} V_1 + g_{22} I_2$

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

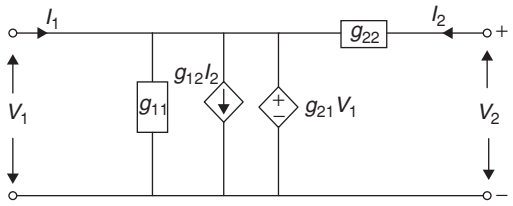


Figure 12 *G*-parameter equivalent circuit

The *G*-Parameters can be defined as

$$g_{11} = \frac{\frac{I_1}{V_1}}{I_2} = 0; \qquad g_{12} = \frac{\frac{I_1}{I_2}}{V_i} = 0$$

$$g_{21} = \frac{\frac{V_2}{V_1}}{I_2} = 0; \qquad g_{22} = \frac{\frac{V_2}{I_2}}{V_1} = 0$$

Condition for reciprocity

$$g_{12} = -g_{21}$$

Condition for symmetry

$$|g| = 1$$

$$\text{i.e., } g_{11} \cdot g_{22} - g_{12} \cdot g_{21} = 1$$

Transmission or ABCD Parameters

The transmission parameters express the required source variables V_1 and I_1 in terms of the existing destination variables V_2 and I_2. They are called ABCD or *T*-parameters and are defined by

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The transmission parameters are determined as

$$A = \frac{\frac{V_1}{V_2}}{I_2} = 0; \qquad \text{(No units)}$$

$$B = \frac{\frac{-V_1}{I_2}}{V_2} = 0 \qquad (\Omega)$$

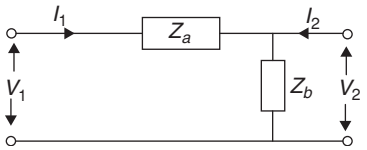
$$C = \frac{\frac{I_1}{V_2}}{I_2} = 0 \qquad \text{(mho)}$$

$$D = \frac{\frac{-I_1}{I_2}}{V_2} = 0 \qquad \text{(No units)}$$

Thus, the transmission parameters are called specifically
A = Open circuit voltage ratio
B = Negative short-circuit transfer impedance
C = Open circuit transfer admittance
D = Negative short circuit current ratio

Condition for reciprocity and symmetry

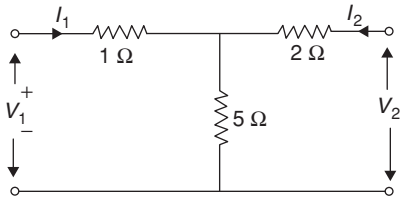
$AD - BC = 1 \Rightarrow$ Reciprocal
And Symmetrical $A = D$, if



Above circuit *T*-parameters definedas

$$[T] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_a}{Z_b} & Z_a \\ \frac{1}{Z_b} & 1 \end{bmatrix}$$

Example 8:



Find the transmission parameters for the circuit shown in figure

Solution:

$V_1 = AV_2 - BI_2$
$I_1 = CV_2 - DI_2$
With $I_2 = 0$

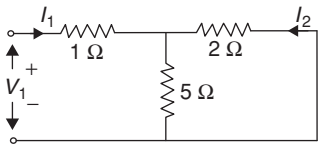
$$V_2 = V_1 \times \frac{5}{5+1}$$

$$A = \frac{V_1}{V_2} = \frac{6}{5}$$

$$V_2 = I_1 \times 5$$

$$C = \frac{I_1}{V_2} = \frac{1}{5}$$

With $V_2 = 0$



$$-I_2 = I_1 \times \frac{5}{5+2}$$

$$V_1 = I_1 \times \left[1 + \frac{5\times 2}{5+2}\right] = I_1\left[1+\frac{10}{7}\right]$$

$$V_1 = \frac{-7}{5} I_2 \times \frac{17}{10}$$

$$B = \frac{-V_1}{I_2} = \frac{-119}{50}$$

$$D = \frac{-I_1}{I_2}$$

$$D = \frac{7}{5}.$$

Inverse Transmission Parameters

$V_2 = A' V_1 - B' I_1$
$I_2 = C' V_1 - D' I_1$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

The inverse transmission parameters can be defined as

$$A' = \frac{\frac{V_2}{V_1}}{I_1} = 0$$

Forward voltage ratio with sending end open circuit is:

$$C' = \frac{I_2}{V_1}\Big|_{I_1=0}$$

Transfer admittance with sending end open circuit is:

$$B' = \frac{V_2}{-I_1}|V_1 = 0$$

Transfer impedance with sending end short circuit:

$$D = \frac{I_2}{-I_1}|V_1 = 0$$

$A' = D' \Rightarrow$ Symmetrical
$A'D' - B'C' = 1 \Rightarrow$ Reciprocal

Relationships between parameters

1. $[Y] = [Z]^{-1}$
2. $[g] = [h]^{-1}$
3. $[t] \neq [T]^{-1}$

Inter Connection of Networks

A large complex network may be divided into sub networks for the purpose of analysis and design. The sub networks are modelled as two port networks, inter connected to form the original network. The inter connection can be in series, in parallel, or in cascade. The inter connected network can be described by any of the six parameters sets. For example, when the networks are in series, their undivided Z-parameters add up to give the Z-parameter of the larger network.

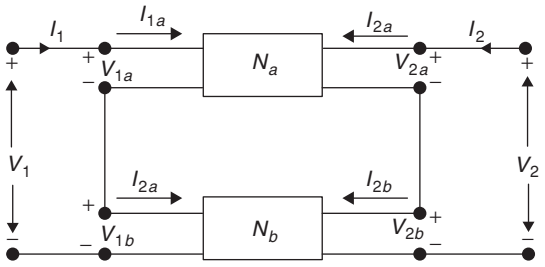


Figure 13 Series connection of two-port networks

$$[Z] = [Z_a] + [Z_b]$$

1. Two-port n/w are in parallel when their port voltages are equal and the port currents of the larger network are the sums of the individual port elements. The parallel connection of two two-port networks is shown in figure.

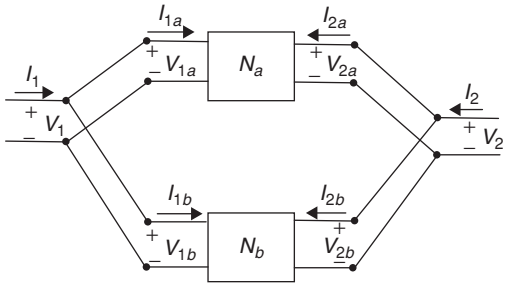


Figure 14 Parallel connection of two two-port networks

From the above figure,
$I_{1a} = Y_{11a} V_{1a} + Y_{12a} V_{2a}$
$I_{2a} = Y_{22a} V_{1a} + V_{22a} V_{2a}$
And
$I_{1b} = Y_{11b} V_{1b} + Y_{12b} V_{2b}$
$I_{2b} = Y_{21b} V_{1b} + V_{22b} . V_{2b}$
But from the figure
$V_1 = V_{1a} = V_{1b}$
$V_2 = V_{2a} = V_{2b}$
And
$I_1 = I_{1a} + I_{2b}$
$I_2 = I_{2a} + I_{2b}$

Thus, the *Y*-parameters of the overall network are

$$[Y] = [Y_a] + [Y_b]$$

2. Two networks are said to be cascaded when the O/P of one is the I/p of the other.

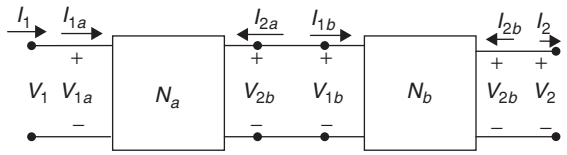
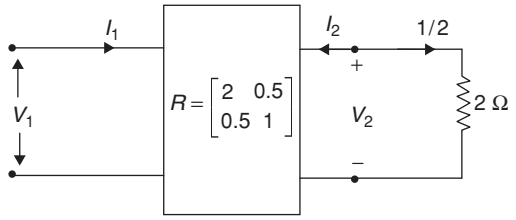


Figure 15 Cascade connection of two-port networks

$$[T] = [T_a]\,[T_b]$$

Example 7: Determine the value of V_1.



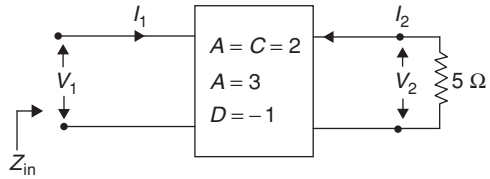
(A) 6 V
(B) 5.5 V
(C) 5.75 V
(D) None of the above

Solution: (C)
Form the given data
$I_2 = -1/2$ A
$V_2 = 1/2 \times 2 = 1$ V
$V_1 = 2I_1 + 0.5I_2$ (9)
$V_2 = 0.5I_1 + 1I_2$ (10)
Sub I_2 and V_2 in equation (10)

$$1 = 0.5I_1 \quad 0.5 \;\Rightarrow\; I_1 = \frac{1.5}{0.5} = 3 \text{ A}$$

$V_1 = 2 \times 3 + 0.5\,(-0.5)$
$= 6 - 0.25$
$= 5.75$ V

Example 8: (C)
Determine Z_{in}



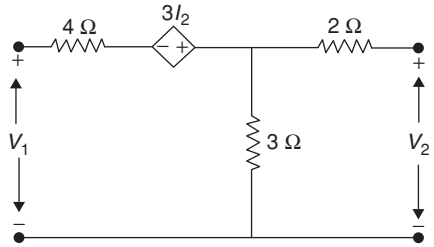
(A) −12/9
(B) 12/13
(C) 13/9
(D) 13/11

Solution: (C)

$$Z_{in} = \frac{AZ_L + B}{CZ_L + D}$$

$$= \frac{2\times 5 + 3}{2\times 5 - 1} = \frac{13}{9}.$$

Example 9:



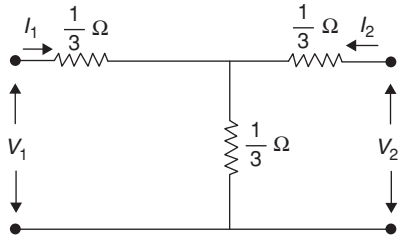
The *Z*-parameters of the two-port network are

(A) $\begin{bmatrix} 7 & 0 \\ -3 & 5 \end{bmatrix}$
(B) $\begin{bmatrix} 7 & 3 \\ 0 & 5 \end{bmatrix}$
(C) $\begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix}$
(D) $\begin{bmatrix} 5 & 3 \\ 7 & 0 \end{bmatrix}$

Solution: (C)
Applying KVL to the input loop
$V_1 = 4I_1 - 3I_2 + 3(I_1 + I_2)$
$V_1 = 7I_1 + 0.I_2$ (11)
Apply KVL to the O/P loop
$V_2 = 2I_2 + 3(I_1 + I_2)$
$V_2 = 3I_1 + 5I_2$ (12)
From equation (11) and (12)

$$[Z] = \begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix}$$

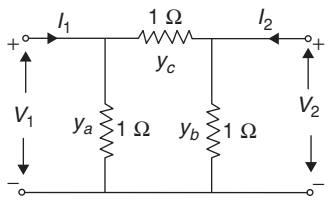
Example 10: A two-port network, shown in fig



The admittance parameters, Y_{11}, Y_{12}, Y_{21} and Y_{22} are

(A) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
(B) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$
(D) $\begin{bmatrix} 1 & -0.5 \\ \frac{-1}{3} & 2 \end{bmatrix}$

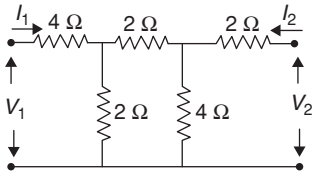
Solution: (A)
Convert star to delta connection



$$[Y] = \begin{bmatrix} y_a + y_c & -y_c \\ -y_c & -y_b + y_c \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Example 11:



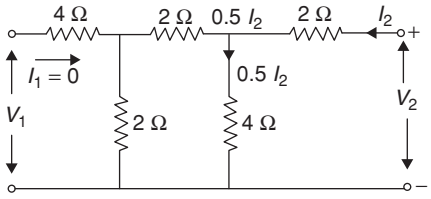
For the two-port network shown in figure the value of h_{12} is given by

(A) 0.125 (B) 0.167
(C) 0.625 (D) 0.25

Solution: (D)
H-parameters can be defined by

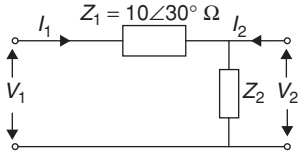
$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$
$$h_{12} = \frac{V_1}{V_2} \text{ at } I_1 = 0$$

So $I_1 = 0$
The circuit becomes



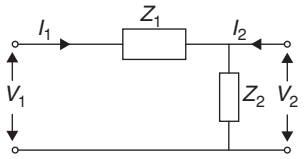
$$V_2 = 2I_2 + 4 \cdot \frac{I_2}{2}$$
$$V_2 = 4I_2$$
$$I_2 = \frac{V_2}{4}$$
$$V_1 = \frac{I_2}{2} \times 2 = I_2$$
$$V_1 = \frac{V_2}{4}$$
$$\Rightarrow \quad h_{12} = \frac{V_1}{V_2} = \frac{1}{4} = 0.25$$

Example 12: Two networks are connected in cascade as shown in the figure with the usual notations the equivalent A, B, C and D constants are obtained. Given that $C = 0.025\angle 45°$, the value of Z_2 is



(A) $10\angle 30°\ \Omega$ (B) $40\angle -45°\ \Omega$
(C) $1\ \Omega$ (D) $0\ \Omega$

Solution: (B)
We know



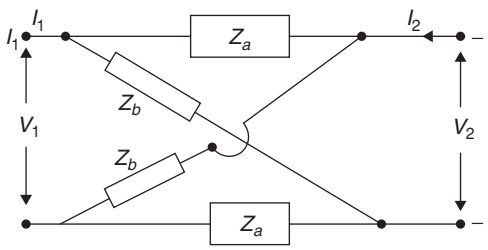
$$[T] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_1}{Z_2} & Z_1 \\ \frac{1}{Z_2} & 1 \end{bmatrix}$$

Given $C = 0.025\ \angle 45°$

So $C = \frac{1}{Z_2}$

$$Z_2 = \frac{1}{C} = \frac{1}{0.025} \angle \quad 45°$$
$$= 40\ \angle -45°.$$

Example 13: Find the Z-parameters of the two-port circuit of figure shown below.



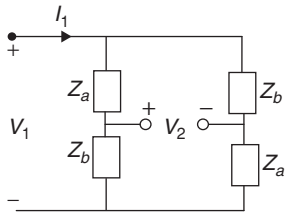
Solution:

$$Z_{11} = \frac{V_1}{I_1} \text{ at } I_2 = 0$$

Z_a and Z_b having same current

So $V_1 = (Z_a + Z_b) \parallel (Z_a + Z_b)$
$= 1/2\,(Z_a + Z_b)\ \Omega$
The circuit is symmetric so $Z_{11} = Z_{22} = 1/2[Z_a + Z_b]$
Similarly

$$Z_{21} = \frac{V_2}{I_1} \text{ at } = 0$$



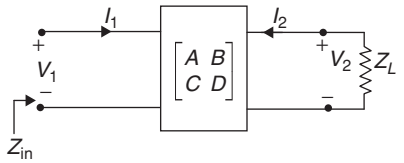
$$V_2 = Z_b\left(\frac{I_1}{2}\right) - \frac{Z_a}{2}\cdot I_1$$

$$\frac{V_2}{I_1} = \frac{1}{2}[Z_b - Z_a]$$

$\therefore$ For a symmetrical lattice network

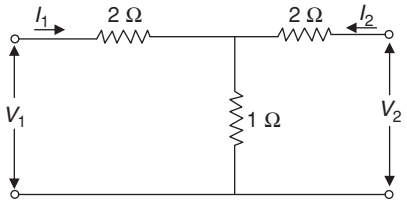
$$[Z] = \begin{bmatrix} \frac{Z_a + Z_b}{2} & \frac{Z_b - Z_a}{2} \\ \frac{Z_b - Z_a}{2} & \frac{Z_b + Z_a}{2} \end{bmatrix}$$

Note:



Then $Z_{in} = \dfrac{AZ_L + B}{CZ_L + D}$

Example 14:



The T-parameters of the network are

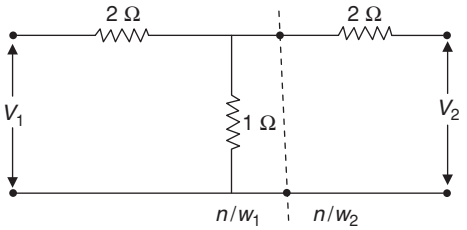
(A) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

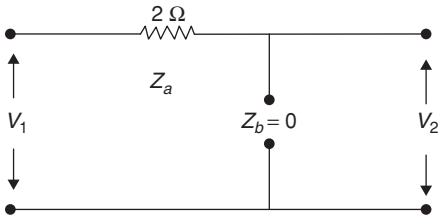
(D) $\begin{bmatrix} 7 & 2 \\ 7 & 3 \end{bmatrix}$

Solution: (B)



$$[T_1] = \begin{bmatrix} 1 + \frac{2}{1} & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$



$$[T_2] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

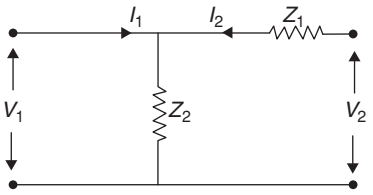
$$[T] = [T_1]\,[T_2]$$

$$= \begin{bmatrix} \rightarrow \\ 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \downarrow \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

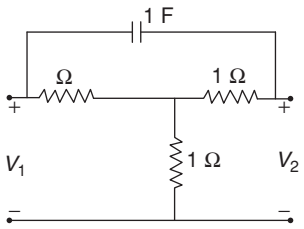
$\therefore A = D \Rightarrow$ Symmetrical
And $AD - BC = 1 \Rightarrow$ Reciprocal

Note:

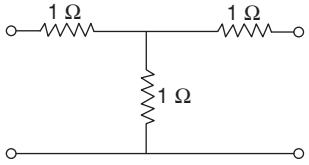


$$[T] = \begin{bmatrix} 1 & Z_1 \\ \frac{1}{Z_2} & 1 + \frac{Z_1}{Z_2} \end{bmatrix}$$

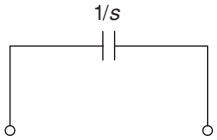
Example 15: Determine *Y*-parameters for the following networks



Solution:

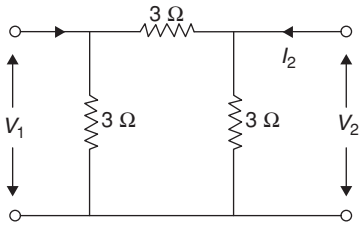


(a)



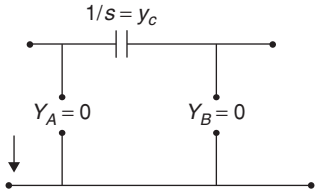
(b)

Figure (a) and figure (b) are in parallel, so
$[y] = [y_1] + [y_2]$
From the figure (a)
Y-Δ transformation



$$[Y_1] = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \Omega$$

From figure (b)



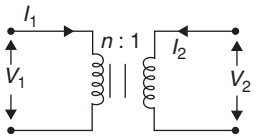
$$[Y_2] = \begin{bmatrix} \frac{1}{s} & \frac{-1}{s} \\ \frac{-1}{s} & \frac{1}{s} \end{bmatrix}$$

$$[Y] = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{s} & \frac{-1}{s} \\ \frac{-1}{s} & \frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}+\frac{1}{s} & -\left(\frac{1}{3}+\frac{1}{s}\right) \\ -\left(\frac{1}{3}+\frac{1}{s}\right) & \frac{2}{3}+\frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2s+3}{3s} & \frac{-(s+3)}{3s} \\ \frac{-(s+3)}{3s} & \frac{2s+3}{3s} \end{bmatrix}$$

Example 16: The *ABCD* parameters of an ideal *n*:1 transformer shown in figure are $\begin{bmatrix} n & 0 \\ 0 & x \end{bmatrix}$. The value of '*X*' will be



(A) n (B) $1/n$ (C) n^2 (D) $1/n^2$

Solution: (B)
For the given ideal transformer

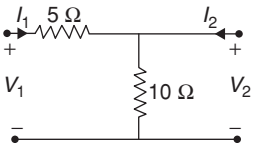
$$\frac{V_1}{V_2} = \frac{n}{1} = \frac{-I_2}{I_1}$$

$V_1 = nV_2 - 0.I_2$ (13)
$I_1 = 0.V_2 - 1/n \,.\, I_2$ (14)
From equation (13) and (14)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

$$\Rightarrow X = 1/n$$

Example 17: The *h*-parameters of the circuit shown in figure are



(A) $\begin{bmatrix} 5 & 1 \\ -1 & 0.1 \end{bmatrix}$ (B) $\begin{bmatrix} 0.2 & -1 \\ 1 & 10 \end{bmatrix}$

(C) $\begin{bmatrix} 5 & -1 \\ 0.1 & 1 \end{bmatrix}$ (D) None of the above

Solution:

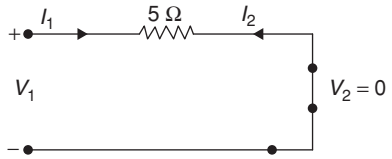
h-parameters are defined by

$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$

From $V_2 = 0 \quad \Rightarrow \quad h_{11} = \dfrac{V_1}{I_1}\,\Omega$

$$h_{21} = \frac{I_2}{I_1}$$

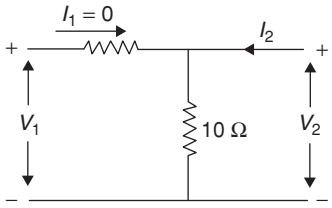


$$\frac{V_1}{I_1} = 5\,\Omega = h_{11}$$

$$I_1 = -I_2$$

$$\frac{I_2}{I_1} = h_{21} = -1$$

For $I_1 = 0$



$\therefore \quad V_1 = V_2$

$$h_{12} = \frac{V_1}{V_2} = 1$$

$$h_{22} = \frac{I_2}{V_2}$$

$$V_2 = 10 I_2$$

$$\frac{I_2}{V_2} = \frac{1}{10} = 0.1\,\Omega^{-1}$$

Relation between Z and Y parameters

$$Y_{11} = \frac{Z_{22}}{\Delta z};\ Y_{12} = \frac{-Z_{12}}{\Delta z}$$

$$Y_{21} = \frac{Z_{21}}{\Delta z};\ Y_{22} = \frac{+Z_{11}}{\Delta z}$$

Where $\Delta Z = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}$

Similarly

$$Z_{11} = \frac{Y_{22}}{\Delta y};\ Z_{12} = \frac{-Y_{12}}{\Delta y}$$

$$Z_{21} = \frac{-Y_{21}}{\Delta y};\ Z_{12} = \frac{Y_{11}}{\Delta y}$$

ABCD Parameters in Terms of Z-Parameters and Y-Parameters

$$A = \frac{Z_{11}}{Z_{21}} = \frac{-Y_{22}}{Y_{21}}$$

$$B = \frac{\Delta Z}{Z_{21}} = \frac{-1}{Y_{21}}$$

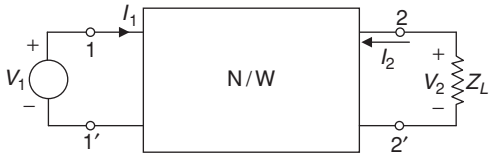
$$C = \frac{1}{Z_{21}} = \frac{-\Delta y}{Y_{21}}$$

$$D = \frac{Z_{22}}{Z_{21}} = \frac{-Y_{11}}{Y_{21}}$$

Terminated Two-port Network

Driving point impedance at the input port of a load terminated network.

Figure shows a two-port network connected to an ideal generator at the input port and to a load impedance at the output port.



$V_2 = -I_2 Z_L$

$V_1 = Z_{11} I_1 + Z_{12} I_2$

$V_2 = Z_{21} I_1 + Z_{22} I_2$

$-I_2 Z_L = Z_{21} I_1 + Z_{22} I_2$

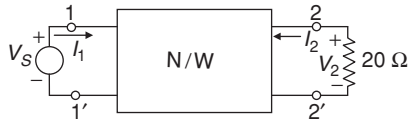
$$I_2 = \frac{-I_1 Z_{21}}{Z_L + Z_{22}}$$

$$V_1 = Z_{11} I_1 - \frac{Z_{12} Z_{21} I_1}{Z_L + Z_{22}}$$

$$V_1 = I_1 \left[Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}} \right]$$

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}}$$

Example 18:



The Z-parameter of a two port network shown in figure are $Z_{11} = Z_{22} = 10\ \Omega$, $Z_{12} = Z_{21} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, V_2, I_2 and input impedance

Solution:

$$V_1 = V_s = 20\ \text{V}$$

$$V_1 = I_1\left(Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}\right)$$

$$\therefore \quad 20 = I_1\left(10 - \frac{4\times 4}{20+10}\right)$$

$$I_1 = 2.11\ \text{A}$$

$$I_2 = -I_1\frac{Z_{21}}{Z_L + Z_{22}}$$

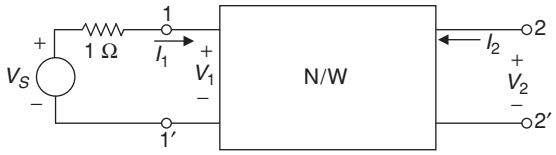
$$= -2.11\times\frac{4}{20+10} = -0.281\ \text{A}$$

$$V_2 = -I_2 \times 20 = 0.281 \times 20 = 5.626\ \text{V}$$

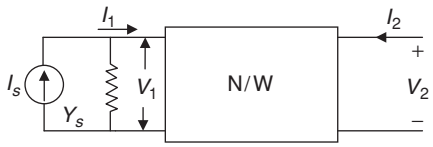
$$\text{Input impedance} = \frac{V_1}{I_1} = \frac{20}{2.11}$$

$$= 9.478\ \Omega$$

Example 19: The Y-parameter of the two-port network shown in figure are $Y_{11} = Y_{22} = 6$ mho; $Y_{12} = Y_{21} = 4$ mho. Determine the driving point admittance at port $2 - 2'$ if the source voltage is 100 V and has an impedance of 1 Ω.



Solution:



$$I_1 = I_s - V_1Y_s$$
$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$
$$I_s - V_1Y_s = Y_{11}V_1 + Y_{12}V_2$$
$$-V_1(Y_s + Y_{11}) = Y_{12}V_2 - I_s$$

$$-V_1 = \frac{Y_{12}V_2 - I_s}{Y_s + Y_{11}}$$

$$I_2 = -Y_{21}\left(\frac{Y_{12}V_2 - I_s}{Y_s + Y_{11}}\right) + Y_{22}V_2$$

$$I_2 = -\frac{-Y_{21}Y_{12}V_2}{Y_s + Y_{11}} + Y_{22}V_2$$

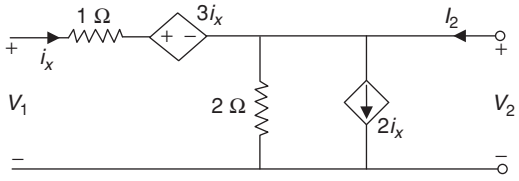
$$\frac{I_2}{V_2} = \frac{Y_{22}Y_5 + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_s + Y_{11}}$$

$$= \frac{6\times 1 + 6\times 6 - 4\times 4}{1+6}$$

$$= 3.714\ \text{mho}$$

Driving point impedance at port $2 - 2' = \frac{1}{3.714}\ \Omega$

Example 20:

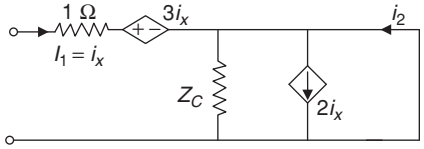


Find the H-parameters for the network shown in figure?

Solution:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For $V_2 = 0$



$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$V_1 = I_1 \times 1 + 3I_1$$

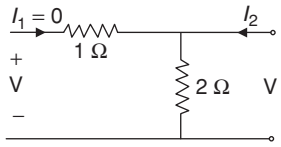
$$\frac{V_1}{I_1} = 4 = h_{11}$$

By KCL,

$$I_2 = 2I_1 - I_1$$
$$I_2 = I_1$$

$$\frac{I_2}{I_1} = 1 = h_{21}$$

For $I_1 = 0$



$V_2 = V_1$

$$\frac{V_1}{V_2} = h_{12} = 1$$

$V_2 = I_2 \times 2$

$$\frac{I_2}{V_2} = \frac{1}{2} = 0.5 = h_{22}$$

Network Functions

For a one-port network, the driving point impedance of the network is defined as

$$Z(s) = \frac{V(s)}{I(s)}$$

Similarly the driving point admittance

$$Y(s) = \frac{I(s)}{V(s)}$$

For a two-port network without internal sources, the driving point impedance at port $1 - 1'$ is

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

At port $2 - 2'$, it is

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

Similarly $Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$; $Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$

$Y_{11}(s)$ and $Y_{22}(s)$ are transfer admittances. Voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$

And $G_{12}(s) = \frac{V_1(s)}{V_2(s)}$

Current transfer ratio

$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$$

$$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)}$$

Transfer impedance

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$$

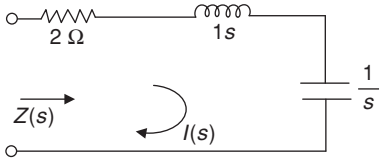
$$Z_{12}(s) = \frac{V_1(s)}{I_2(s)}$$

Transfer admittance

$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$$

$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)}$$

Example 21:



For the network shown in figure, obtain the driving point impedance.

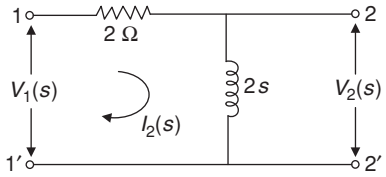
Solution:

$$Z(s) = \frac{v(s)}{I(s)}$$

$$= 2 + s + \frac{1}{s}$$

$$Z(s) = \frac{2s + s^2 + 1}{s} = \frac{s^2 + 2s + 1}{s}$$

Example 22: For the network shown in figure, obtain the transfer functions $G_{21}(s)$ and $Z_{21}(s)$ and the driving point impedance $Z_{11}(s)$.



Solution: $V_1(s) = I_1(s)\,[2 + 2s]$

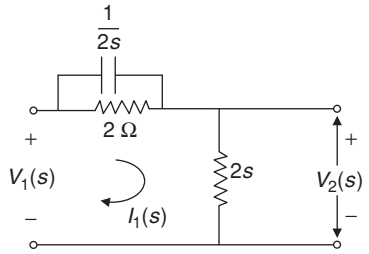
$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = 2(s+1)$$

$$V_2(s) = I_1(s) \times 2s$$

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{2s}{2s+2} = \frac{s}{s+1}$$

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = 2s$$

Example 23:



For the network shown in figure. Find $G_{21}(s)$. $Z_{21}(s)$ and $Z_{11}(s)$

Solution:

$$\frac{V_1(s)}{I_1(s)} = Z_{11}(s) = \frac{2 \times \frac{1}{2s}}{2 + \frac{1}{2s}} + 2$$

$$= \frac{2}{4s+1} + 2$$

$$Z_{11}(s) = \frac{8s+4}{4s+1}$$

$$V_2(s) = \frac{V_1(s) \cdot 2}{2 + \frac{2}{4s+1}}$$

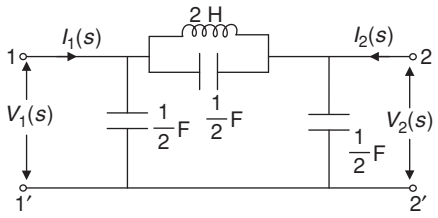
$$\frac{V_2(s)}{V_1(s)} = G_{21}(s) = \frac{2(4s+1)}{8s+4}$$

$$= \frac{8s+2}{8s+4}$$

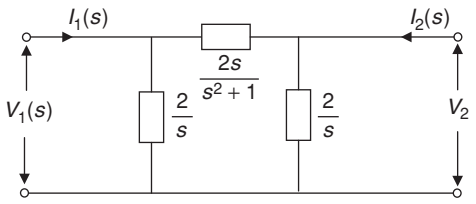
$$V_2(s) = I_1(s).2$$

$$\frac{V_2(s)}{I_1(s)} = Z_{21}(s) = 2$$

Example 24: For the network shown in figure, determine the transfer function $G_{21}(s)$ and $Z_{21}(s)$



Solution:
Transform the circuit into s-domain



$$V_2(s) = V_1(s) \cdot \frac{\frac{2}{s}}{\frac{2}{s} + \frac{2s}{s^2+1}}$$

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{2}{s}}{2\frac{(s^2+1)+2s^2}{(s^2+1)s}}$$

$$= \frac{2(s^2+1)}{4s^2+2}$$

$$G_{21}(s) = \frac{s^2+1}{2s^2+1}$$

$$V_2(s) = I_1(s) = \frac{\frac{2}{s}}{\frac{4}{s} + \frac{2s}{s^2+1}} \times \frac{2}{s}$$

$$\frac{V_2(s)}{I_1(s)} = Z_{21}(s)$$

$$= \frac{2(s^2+1)}{s(3s^2+2)}$$

Network Graphs

The solution of a linear network problem required the formation of a set of equations describing the response of the network first and then the manipulation of the co-efficient matrix so produced.

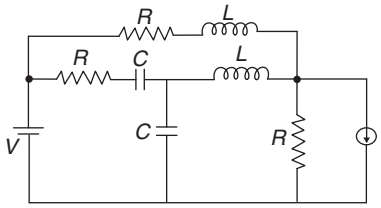
Networks topology deals with concepts involving inter connections in the networks, rather than the actual nature of the elements.

Graph

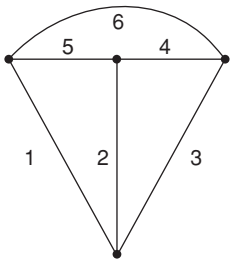
The connection of the network topology shown by replacing all physical elements by lines is called a graph.

While constructing a graph from the given network all passive elements and the ideal voltage sources are replaced by short circuit, all the ideal current sources are replaced by open circuit.

Example:
A network and it related graph is shown in figure a and figure b



(a)



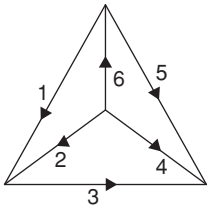
(b)

Directed graph

A graph in which each branch is assigned a direction is called a directed or oriented graph.

Complete graph (or) standard graph

For a standard graph, between any pair of nodes only one branch is connected for all combination.

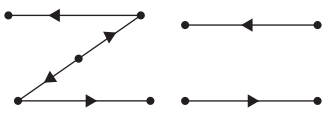


Example: The no. of edges in a complete graph with n-nodes in ${}^{n}C_{2} \Rightarrow \frac{n(n-1)}{2} = b.$

Connected graph

In a connected graph all the nodes are connected by at least one branch, otherwise it is said to be unconnected.

Example:



(i) (ii)

Connected graph

Sub graph

It is a graph with less no. of branches as compared with the original graph.

Example:



Sub graph

Planar and non-planar graphs

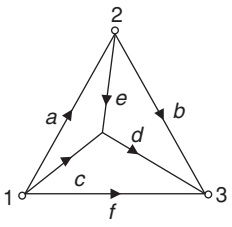
A graph is said to be planar if it can be drawn on a plane surface such that no two branches cross each other.

A non-planar graph can not be drawn on a plane surface without a crossover.

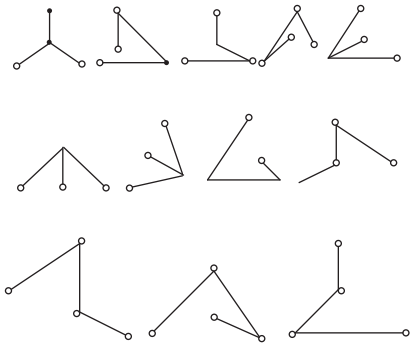
Tree and co-tree

A tree is a connected of a network which consists of all the nodes of the original graph but no closed paths. The number of nodes in the graphs is equal to the number of nodes in the tree.

Example 25: For the given graph shown in figure, draw the number of possible trees

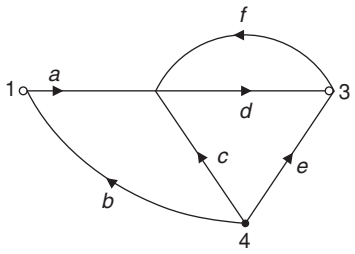


Solution:
There are four nodes. The possible trees are shown in the figure

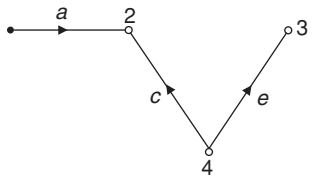


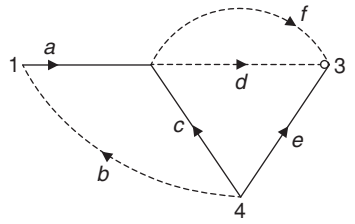
Twigs and Links

The branches of a tree are called its 'twigs'. For a given branch, the complementary set of branches of the tree is called the co-tree of the graph. The branches of co-tree are called links, i.e., those elements of the connected graph that are not included in the tree links and form a sub graph.



For the graph shown in the figure the tree branches are "ace" as shown in figure

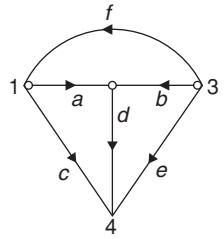




The set of branches (b, d, f) represented by dotted lines form a co-tree of the graphs. These branches are called links of this tree.

For a network with 'b' branches and 'n' nodes, the number of twigs for a selected tree is $(n-1)$ and the number of links 'l' with respect to this tree is $b-n+1$. The number twigs is called the rank of the tree

Incidence matrix (A)



For the oriented graph shown in the figure, the incidence matrix is

$$\downarrow \text{Nodes} \quad \text{Branches} \rightarrow$$

$$A = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} a & b & c & d & e & f \\ 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}_{n\times b}$$

In matrix A with 'n' rows and b columns an entry a_{ij} in the ith row and jth column has the following values

$a_{ij} = 1$, if the jth branch is incident to and oriented away from the ith node

$aij = 1$, if the jth branch is incident and oriented towards the ith node

$a_{ij} = 0$, if the jth branch is not incident to the ith node

Note: Consider incoming branches are "–ve" sign and outgoing branches are "+ve" sign.

Example 26: Draw the graph corresponding to the given incidence matrix

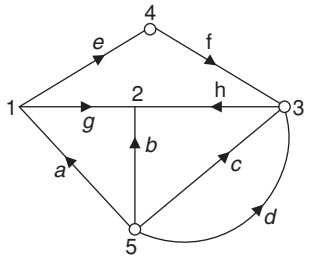
$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

There are five rows and eight columns which indicate that there are five nodes and eight branches

$$A = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} a & b & c & d & e & f & g & h \\ -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The graph can be drawn as shown in figure

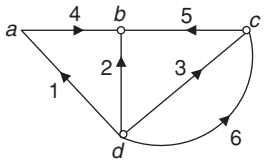


Incidence Matrix and Formulation of KCL

$A_I I = 0$

Where A_I is the Incidence matrix and I represents branch current vectors I_1, I_2…

Consider the graph shown in figure



It has four nodes a, b, c and d. Let node 'd' be taken as reference node. Let the branch currents be i_1, i_2,…, i_6

Applying KCL at nodes a, b, and c

$$-i_1 + i_4 = 0$$
$$-i_2 + i_5 - i_4 = 0$$
$$-i_3 - i_5 - i_6 = 0$$

In matrix form,
$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1- & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A_I I_b = 0$ … (KCL)

Relation Between Twigs and Links

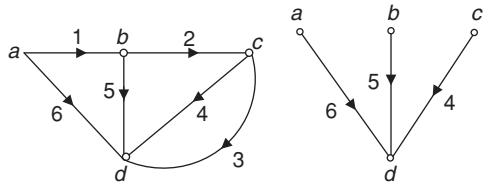
The number of twigs on a tree is always one less than the number of nodes.
i.e., twigs = $(n - 1)$
Let n = number of nodes
Also, if 'ℓ' represents the total number of links, while 'b' the total number of branches
$L = b - (n - 1)$
$L = b - n + 1$

Tie-set Matrix

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current.

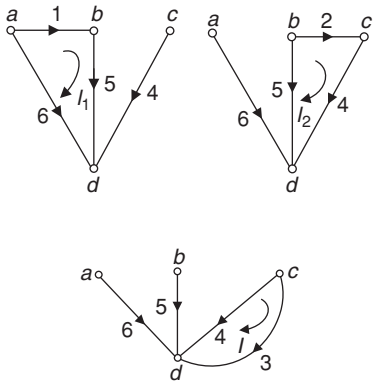
The fundamental loop formed by one link has a unique path in the tree joining the two nodes of the link. This loop is called a tie set.

Consider a connected graph shown in figure (a). It has four nodes and six branches. One of its trees is arbitrarily chosen and is shown in figure



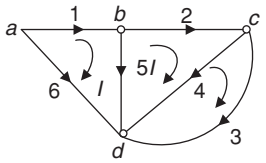
(a) (b)

No. of nodes $n = 4$
No. of branches $b = 6$
No. of tree branches or twigs = $n - 1$
No. of link branches $I = b - (n - 1) = 3$
Let i_1, i_2 …, i_6 be the branch currents with directions as shown in figure 'a'. When a link is added to the tree, a closed circuit is formed. The closed loops are as shown in the figures.



By convention a fundamental loop is given the same orientation as its defining link, i.e., the link current. I_1 coincides with the branch current i_1 direction in ab. Similarly I_2 coincides with the branch current direction in bc and I_3 coincides with the direction of cd.

Tie-set



Consider the above figure. Kirchoff's voltage law can be applied to the fundamental loops to get a set of linearly independent equation.

These are three fundamental loops I_1, I_2 and I_3 corresponding to the link branches 1, 2 and 3 respectively. If V_1, V_2, …V_6 are the branch voltages, the KVL equations for the three fundamental loops are

$$V_1 + V_5 - V_6 = 0$$
$$V_2 + V_4 - V_5 = 0$$
$$V_3 - V_4 = 0$$

The above equation can be written in matrix form.

$$\begin{matrix} \text{loop} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\text{branches} \rightarrow$$

i.e., $B\ V_b = 0$ … (KVL)
Where B is an $I \times b$ matrix called the tie-set matrix or fundamental loop matrix and V_b is a column vector of branch voltages.

Tie-Set Matrix and Branch Currents

$$[I_b] = [B^T]\,[I_L]$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$[I_b] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}; [I_L] = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are

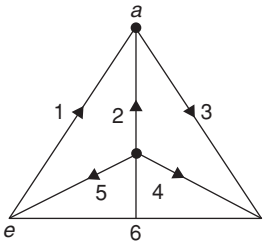
$i_1 = I_1 \quad i_2 = I_2 \quad i_3 = I_3$

$i_4 = I_2 - I_3 \quad i_5 = I_1 - I_2 \quad i_6 = -I_1$

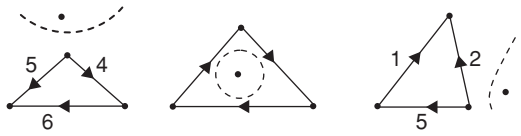
Cut-set

It is a set of branches of a connected graph (G), where in the removal of all the branches of the set causes remaining graph to have two unconnected sub-graphs, i.e., the cut set is a minimal set of branches of the graph, remove of which divides the graph in to two sub graphs.

Example:



Standard graph



C(1, 2, 3) C(2, 4, 5) C(3, 4, 6)

Note:

$$aij = \begin{cases} +1\text{: If branch } j \text{ leaves node } (i) \\ -1\text{: If branch } j \text{ enters node } (i) \\ 0\text{: If branch } j \text{ is not incident on } (i) \end{cases}$$

Properties of a Tree in a graph

1. It consists of all the nodes of the graph.
2. If the graph has N no. of nodes the tree will have $(N - 1)$ branches.
3. There will be no closed path in the tree.

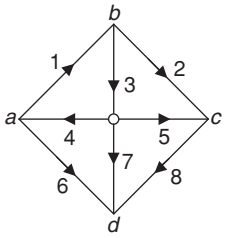
Note:

1. Rank of a graph = $(n - 1)$
   Where $n \Rightarrow$ number of nodes
2. The number of trees for a given standard graph = $(n)^{n-2}$
3. Total number of KCL equations equal to $n - 1$
4. Number of fundamental tie sets for a graph equal to number of Links
   i.e., $L = (b - n + 1)$
5. Rank of tie-set matrix = $(b - n + 1)$ = Links
6. Number of nodal equations in a given graph equal to $(n - 1) \Rightarrow$ f-cut sets.
7. Number of mesh equations = f-loops = $b - n + 1$

Fundamental cut-sets

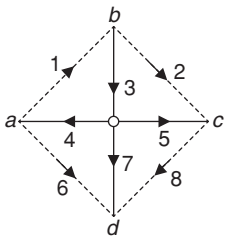
The fundamental cut-sets are defined for a given tree of the graph. From a connected graph, first a tree is selected, and then a twig is selected. Removing this twig from the tree separates the tree into two parts, All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will constitute a cut-set. This cut-set is called a fundamental cut-set of the graph.

Example 27: Obtain the fundamental cut-set matrix Q_f for the network shown in figure

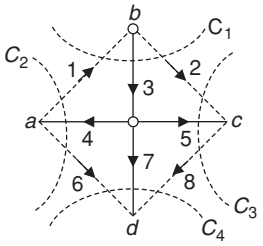


Solution:

A selected tree of the graph is shown is the figure



The twigs of the tree are {3, 4, 5, 7}. The remaining branches 1, 2, 6 and 8 are the links, corresponding to the selected tree.



The fundamental cut-set matrix is formed as fundamental

$$\begin{array}{c} \text{Cutset} \quad \text{branches}\rightarrow \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

The branch voltages in terms of twig voltages are

$V_1 = -V_3 - V_4 = -V_{t3} - V_{t4}$
$V_2 = -V_3 - V_5 = +V_{t3} - V_{t5}$
$V_3 = V_{t3}$
$V_4 = V_{t4}$
$V_5 = V_{t5}$
$V_6 = V_7 - V_4 = V_{t7} - V_{t4}$
$V_7 = V_{t7}$
$V_8 = V_7 - V_5 = V_{t7} - V_{t5}$

In matrix form

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_{t3} \\ V_{t4} \\ V_{t5} \\ V_{t7} \end{bmatrix}$$

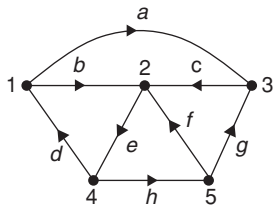
Example 28: A standard graph consists of 55 branches, the number of f-cut sets, Tie-sets, f-cut sets matrices and Tie -set matrices are

Solution:
Given $b = 55$

$$55 = \frac{n(n-1)}{2}$$

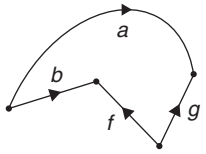
$n(n-1) = 11$
$n = 11$
$L = b - n + 1 = 55 - 11 + 1 = 45$
f-loops or Tie-sets = links = 45
f-cut set matrices = Tie-set matrices
f-loop matrices = $(n)^{n-2}$
$= (11)^9$

Example 29: Identify which of the following is NOT a tree of the graph, shown in figure

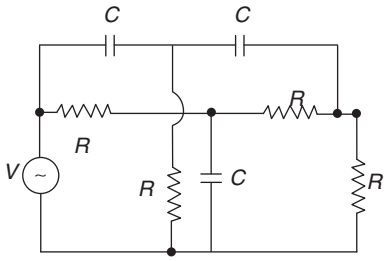


(A) $b\,c\,g\,h$ (B) $d\,e\,f\,g$
(C) $a\,b\,f\,g$ (D) $a\,e\,g\,h$

Solution: (C)
Tree is a connected graph, with out forming a closed path. From the given options (C) is not satisfied. Above statement

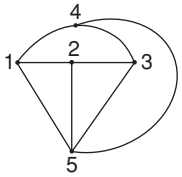


Example 30: Minimum no. of equation required to analyze the circuit shown in figure is



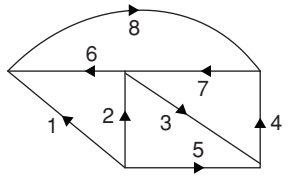
(A) 3 (B) 4
(C) 6 (D) 7

Solution: (B)



No. of nodes = 5
Nodal equations = $N - 1 = 4$
Number of mesh equations = $L = b - n + 1$
$= 8 - 5 + 1 = 4$
Minimum number of equations = min (nodal, Mesh equations)

Example 31: Match List I with List II for the co-tree branches, 1, 2, 3, and 8, of the graph shown in the figure and select the correct answer using the codes given below the lists.

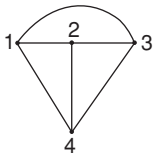


	List I		List II
p.	Twigs	1.	4, 5, 6, 7
q.	Links	2.	1, 2, 3, 8
r.	Fundamental cut set	3.	1, 2, 3, 4
s.	Fundamental loop	4.	6, 7, 8

(A) $p-1, q-2, r-3\ s-4$
(B) $p-3, q-2, r-1, s-4$
(C) $p-1, q-4, r-3, s-2$
(D) $p-3, q-4, r-1, s-2$

Solution: (A)
Total no. of branches = twigs + Links
From the given data
Links $\Rightarrow$ 1, 2, 3, 8
$\therefore$ Twigs = 4., 5, 6, 7 = $n - 1 = 4$
Given nodes = 5
Fundamental cut set having at a time only one tree branch
i.e., 1, 2, 3, 4 → twig
$\therefore$ f-loops $\Rightarrow$ having at a time only one link
$\Rightarrow$ 6, 7, 8 → Link

Example 32: What is the total number of trees for the graph shown below?



(A) 4
(B) 8
(C) 12
(D) 16

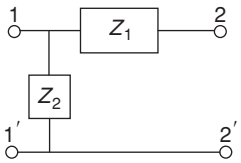
Solution: (D)
For a standard graph total number of trees = $(n)^{n-2}$
Where $n = 4$
$\therefore$ no. of trees = $(4)^{4-2} = 4^2 = 16$

EXERCISES

Practice Problems I

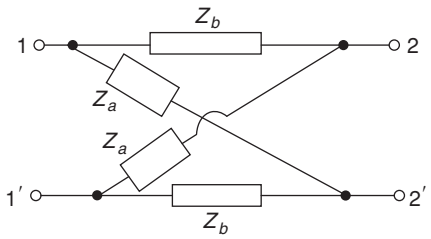
Directions for questions 1 to 31: Select the correct alternative from the given choices.

1. The z-parameter of the network shown in the figure is



(A) $\begin{bmatrix} z_1+z_2 & z_1 \\ z_2 & z_1+z_2 \end{bmatrix}$
(B) $\begin{bmatrix} z_1 & z_2 \\ z_1+z_2 & z_1-z_2 \end{bmatrix}$
(C) $\begin{bmatrix} z_2 & z_2 \\ z_2 & z_1+z_2 \end{bmatrix}$
(D) $\begin{bmatrix} z_1 & z_1 \\ z_1 & z_1+z_2 \end{bmatrix}$

2. For the lattice circuit shown in figure, $Z_a = j2\ \Omega$ and $Z_b = 2\ \Omega$. The values of the open circuit impedance parameters $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ are



(A) $\begin{bmatrix} 1-j & 1+j \\ 1-j & 1+j \end{bmatrix}$
(B) $\begin{bmatrix} 1-j & 1+j \\ -1+j & 1-j \end{bmatrix}$
(C) $\begin{bmatrix} 1+j & 1+j \\ 1-j & 1-j \end{bmatrix}$
(D) $\begin{bmatrix} 1+j & -1+j \\ -1+j & 1+j \end{bmatrix}$

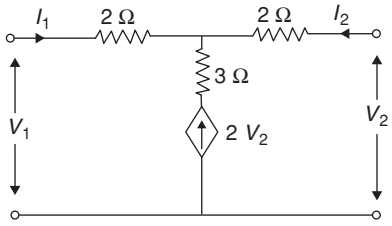
3. A two port network is represented by $ABCD$ parameters given by $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$

If port-2 is terminated by R_L, the input impedance seen at port-1 is given by _____.

(A) $\frac{A+BR_L}{C+DR_L}$ (B) $\frac{AR_L+C}{BR_L+D}$

(C) $\frac{DR_L+A}{BR_L+C}$ (D) $\frac{B+AR_L}{D+CR_L}$

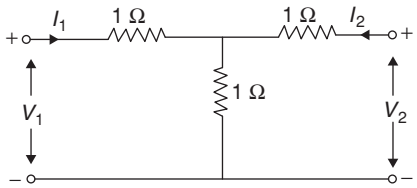
4. The admittance parameter of the network shown in the figure is _____.



(A) $\begin{bmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ (B) $\begin{bmatrix} \frac{-1}{4} & \frac{5}{4} \\ \frac{1}{4} & \frac{-3}{4} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{-1}{4} & \frac{-3}{4} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{4} & \frac{-5}{4} \\ \frac{-1}{4} & \frac{-3}{4} \end{bmatrix}$

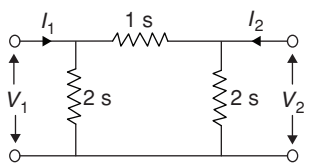
5. The transmission parameter $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ of the two port network shown in the figure is _____.



(A) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

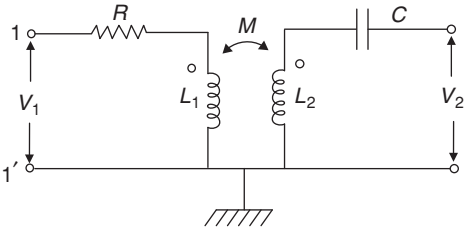
6. If two identical sections of the network shown in the figure are connected in parallel, the Y-parameter of the resulting network is given by _____.



(A) $\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$

7.

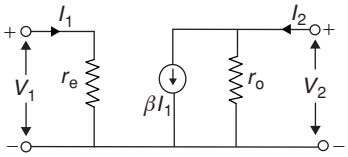


The element Z_{22} of the two port network shown in the above figure is

(A) $R + sL_1$ (B) $\frac{1}{Cs} + sL_2$

(C) $\frac{1}{Cs} + sL_1$ (D) sL_2

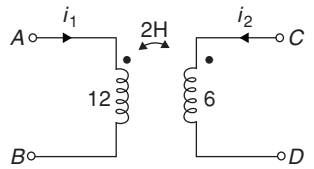
8. In the two-port network shown in the figure below, Z_{12} and Z_{21} respectively are



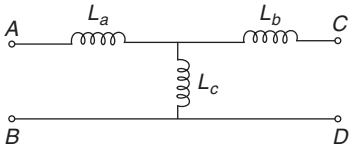
(A) r_e and βr_o (B) 0 and $-\beta r_o$

(C) 0 and βr_o (D) r_e and $-\beta r_o$

9. A linear transformer and its T equivalent circuit are shown in figure (a) and figure (b) respectively. The values of L_a, L_b and L_c respectively are _____.



(a)

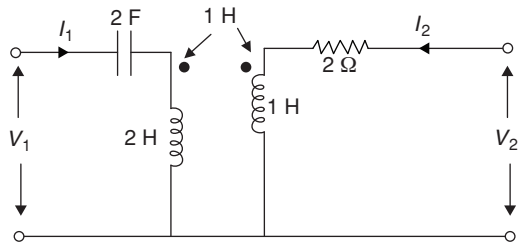


(a)

(A) 10 H, 4H, –2 H (B) 14 H, 8H, –2H

(C) 10 H, 4H, 2H (D) 14 H, 8H, +2H

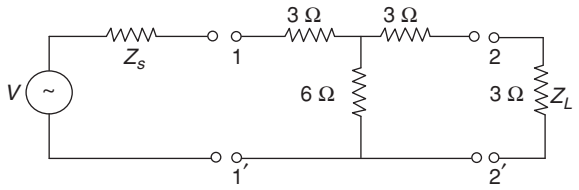
10. For the network shown below the 'Z' parameter will be



(A) $\begin{bmatrix} \frac{1}{2s}+2s & 2+s \\ 2+s & s \end{bmatrix}$ (B) $\begin{bmatrix} s & \frac{1}{2s}+2s \\ 2+s & s \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{2s}+2s & s \\ s & 2+s \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{2s}-s & s \\ s & 2-s \end{bmatrix}$

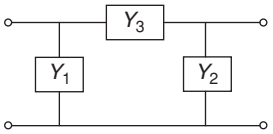
11. An impedance match is desired at the 1–1′ port of the two-port network shown in the given figure. The match will be obtained when Z_S equals



(A) 6 Ω (B) 3 Ω

(B) $\frac{3}{2}\Omega$ (D) $\frac{2}{3}\Omega$

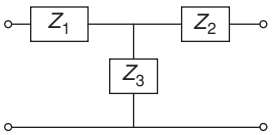
12. The admittance parameter of the 2-port network shown in the figure are Y_{11}=10 Ω, $Y_{12} = Y_{21} = 6$ Ω and $Y_{22} = 8$ Ω The values of y_1, y_2, and y_3 will be respectively



(A) 2, 4 and 6 (B) 4, 2 and –6

(C) 2, 4 and –6 (D) 16, 14 and –6

13.



To construct a High pass filter as in the above circuit

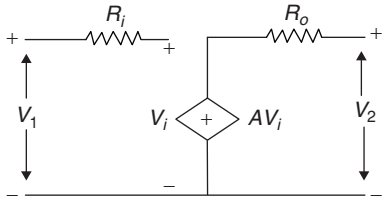
(A) z_1, z_2 are capacitors and z_3 inductor

(B) z_1 z_2 are resistors and z_3 capacitor

(C) z_1 z_2 are inductors z_3 capacitor

(D) z_1 z_2 are resistors z_3 inductor.

14. For the given equivalent circuit find input impedance, output impedance and output voltage



(A) 0, α, α (B) α, R_0, Av_i

(C) R_i, R_o, Av_i (D) R_i, R_0, A

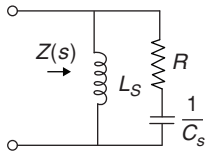
15. Find the driving point admittance of the network shown.

Given

$R = 1$ MΩ

$C = 10$ μF

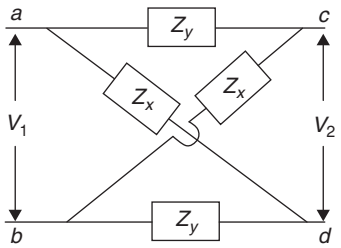
$L = 1000$ H



(A) $\frac{1\times10^3}{1+10s+10^4 s^2}$ (B) $\frac{(1+10s)10^5 s}{s^2+10^3 s+10^2}$

(C) $\frac{10s}{10^{-3}s^2+10s+1}$ (D) $\frac{10}{10^{-3}s^2+10s+1}$

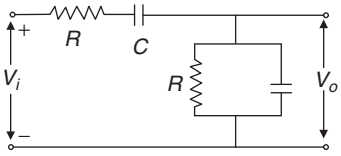
16. Circuit below shows a lattice circuit $z_x = 4j$ Ω and $z_y = 4$ Ω find the values of open circuit impedance $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$



(A) $\begin{bmatrix} 2+2j & 2-2j \\ -2+2j & -2-2j \end{bmatrix}$ (B) $\begin{bmatrix} 2+2j & -2+2j \\ -2+2j & 2+2j \end{bmatrix}$

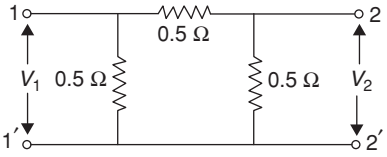
(C) $\begin{bmatrix} -2+2j & 2-2j \\ 2-2j & 2+2j \end{bmatrix}$ (D) $\begin{bmatrix} -2-2j & 2-2j \\ 2-2j & -2-2j \end{bmatrix}$

17. The RC circuit shown in the figure is



(A) A low- pass filter (B) A high- pass filter
(C) A band- pass filter (D) A band- reject filter

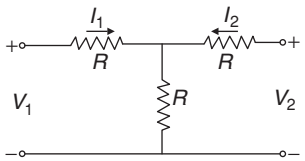
18. For the two-port network shown below, the short circuit admittance parameter matrix is



(A) $\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

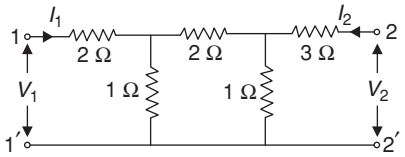
19. A two port network is shown in the figure. The parameter h_{21} for this network can be given by



(A) $\frac{-1}{2}$ (B) $+\frac{1}{2}$

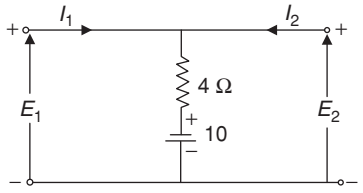
(C) $\frac{-3}{2}$ (D) $+\frac{3}{2}$

20. The impedance parameters Z_{11} and Z_{12} of the two-port network in the figure are



(A) $Z_{11} = 2.75\ \Omega$ and $Z_{12} = 0.25\ \Omega$
(B) $Z_{11} = 3\ \Omega$ and $Z_{12} = 0.5\ \Omega$
(C) $Z_{11} = 3\ \Omega$ and $Z_{12} = 0.25\ \Omega$
(D) $Z_{11} = 2.25\ \Omega$ and $Z_{12} = 0.5\ \Omega$

21. The Z-parameters Z_{11} and Z_{21} for the two-port network in the figure,



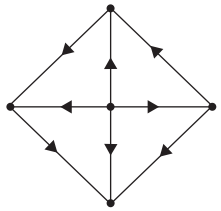
(A) $Z_{11} = \frac{-6}{11}\ \Omega;\ Z_{21} = \frac{16}{11}\ \Omega$

(B) $Z_{11} = \frac{6}{11}\ \Omega;\ Z_{21} = \frac{4}{11}\ \Omega$

(C) $Z_{11} = \frac{6}{11}\ \Omega;\ Z_{21} = \frac{-16}{11}\ \Omega$

(D) $Z_{11} = \frac{4}{11}\ \Omega;\ Z_{21} = \frac{4}{11}\ \Omega$

22. For the graph shown in the figure, the order of the tie set matrix is



(A) 4×4 (B) 4×8
(C) 8×4 (D) 8×8

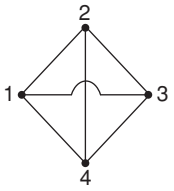
23. If V_b, Q, V_t represents branch voltage matrix, cut set matrix, and the twig voltage matrix then the relation between them is given by

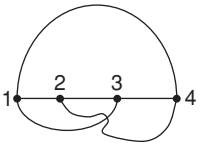
(A) $V_b = Q\,V_t^T$ (B) $V_b = Q^T V_t^T$
(C) $V_b = V_t\,Q$ (D) $V_b = Q^T V_t$

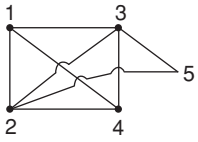
24. A planar graph has 5 nodes and 9 branches. The number of meshes in the dual graph is

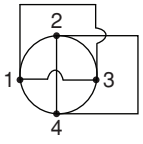
(A) 5 (B) 4
(C) 14 (D) none

25. From the graph given below, which of them is non planer?

(i) 

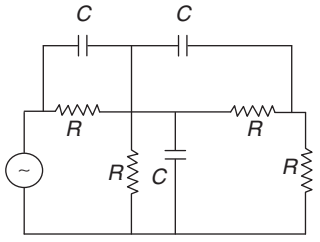
(ii) 

(iii) 

(iv) 

(A) i and ii (B) ii and iii
(C) iii only (D) iv only

26. The minimum number of equations required to analyze the circuit shown in the figure is

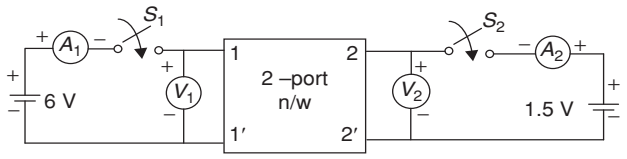


(A) 3 (B) 4
(C) 6 (D) 7

27. A two-part network shown below is excited by external dc sources. The voltages and the current are measured with voltmeters V_1, V_2 and ammeters A_1, A_2 as indicated. Under following switch conditions the readings obtained are:

(i) S_1-Open, S_2-closed
$A_1 = 0$A, $V_1 = 4.5$ V, $V_2 = 1.5$ V, $A_2 = 1$ V

(ii) S_1-closed, S_2-open
$A_1 = 4$ A, $V_1 = 6$ V, $V_2 = 6$ V, $A_2 = 0$A



The Z-parameter matrix for this network is

(A) $\begin{bmatrix} 1.5 & 1.5 \\ 4.5 & 1.5 \end{bmatrix}$ (B) $\begin{bmatrix} 1.5 & 4.5 \\ 1.5 & 4.5 \end{bmatrix}$

(C) $\begin{bmatrix} 1.5 & 4.5 \\ 1.5 & 1.5 \end{bmatrix}$ (D) $\begin{bmatrix} 4.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}$

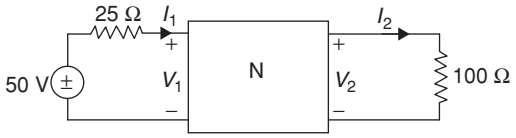
28. From the above questions data the H-parameter matrix for this network is

(A) $\begin{bmatrix} -3 & 3 \\ -1 & 0.67 \end{bmatrix}$ (B) $\begin{bmatrix} -3 & -1 \\ +3 & 0.67 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 3 \\ 1 & 0.67 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 1 \\ -3 & -0.67 \end{bmatrix}$

29. In the circuit shown below, the network N is described by the following Y matrix $[Y] = \begin{bmatrix} 0.1\,\Omega & -0.01\,\Omega \\ 0.01\,\Omega & 0.1\,\Omega \end{bmatrix}$

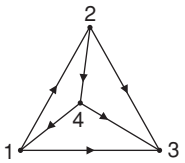
The voltage gain $\frac{V_2}{V_1}$ is

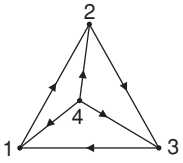


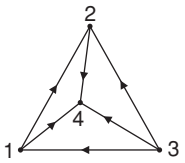
(A) 1/11 (B) −1/11
(C) −1/99 (D) 1/90

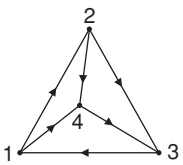
30. The incidence matrix of a graph is as given below

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$ the graph is

(A)


(B)


(C)


(D)


31. The incidence matrix of a graph is as given below

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The number of possible trees is
(A) 40 (B) 70
(C) 50 (D) 240

Practice Problems 2

Directions for questions 1 to 23: Select the correct alternative from the given choices.

1. Which parameters are used in the analysis of transistors?
(A) Z-parameters
(B) Y-parameters
(C) h-parameters
(D) Transmission parameters

2. If a transmission line is represented by a two port network whose parameters are A, B, C, D, then the sending and voltage end current are given by_____.

(A) $V_S = AV_r + BI_r$
$I_S = CV_r + DI_r$
(B) $V_S = AV_r + CI_r$
$I_S = BV_r + DI_s$
(C) $V_S = AV_r - BI_r$
$I_S = CV_r - DI_r$
(D) $V_S = AV_r - CI_r$
$I_S = BV_r - DI_r$

3. A two-port network is reciprocal if and only if
(A) $Z_{11} = Z_{22}$. (B) $Y_{12} = Y_{21}$.
(C) $BC - AD = -1$. (D) $h_{12} = h_{21}$.

4. A two-port network is symmetrical if

(A) $z_{11} = z_{22}$ (B) $z_{11}z_{22} - z_{12}z_{21} = 1$

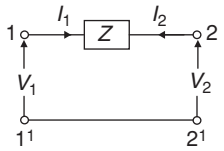
(C) $h_{11}h_{22} - h_{12}h_{21} = 1$ (D) Both A and C

5. A two-port network is reciprocal if

(a) $Z_{12} = Z_{21}$ (B) $A = D$

(C) $Y_{11} = Y_{22}$ (D) $BC - AD = 1$

6. For the two port network shown in the figure which of the following statements is true.



(A) It has Z-parameter.

(B) It has no Z-parameter.

(C) It has no Y-parameter.

(D) It has no transmission parameter.

7. Two two-port networks have Z-parameters

$$[Z]_x = \begin{bmatrix} Z_{11x} & Z_{12x} \\ Z_{21x} & Z_{22x} \end{bmatrix} \text{ and } [Z]_y = \begin{bmatrix} Z_{11y} & Z_{12y} \\ Z_{21y} & Z_{22y} \end{bmatrix}$$

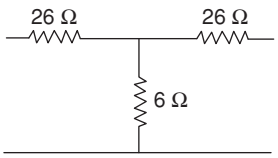
Then the open circuit transfer impedance of the cascaded network is

(A) $Z_{12x} + Z_{12y}$ (B) $Z_{21x} + Z_{21y}$

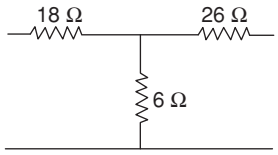
(C) $\dfrac{Z_{21x}Z_{21y}}{Z_{11x} + Z_{22y}}$ (D) $\dfrac{Z_{12x}Z_{12y}}{Z_{12x} + Z_{12y}}$

8. A two-port network is represented by $V_1 = 32I_1 + 6I_2$ and $V_2 = 6I_1 + 24I_2$. Which one of the following networks is represented by these equations.

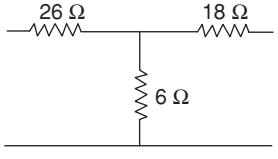
(A)



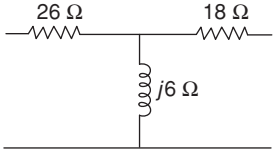
(B)



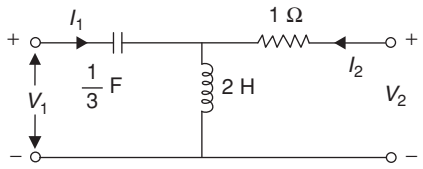
(C)



(D)



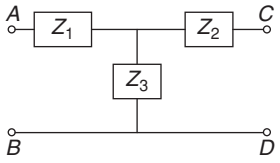
9. Z-matrix for the network shown in the given figure is



(A) $\begin{bmatrix} \dfrac{1}{3s} & 2s \\ 2s & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2s + \dfrac{3}{s} & -2s \\ -2s & 2s+1 \end{bmatrix}$

(C) $\begin{bmatrix} 2s + \dfrac{3}{s} & 2s \\ 2s & 2s+1 \end{bmatrix}$ (D) $\begin{bmatrix} 2s & 2s + \dfrac{3}{s} \\ 2s + \dfrac{3}{s} & 2s \end{bmatrix}$

10. Find values of Z_1, Z_2, Z_3 in the network shown



$Z_{11} = 16\ \Omega$, $Z_{12} = 10\ \Omega$, $Z_{22} = 15\ \Omega$, $Z_{21} = 10\ \Omega$

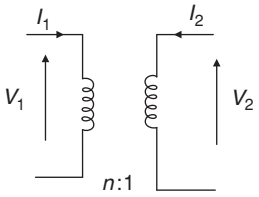
(A) $Z_1 = 10\ \Omega$, $Z_2 = 5\ \Omega$, $Z_3 = 6\ \Omega$

(B) $Z_1 = 26\ \Omega$, $Z_2 = 25\ \Omega$, $Z_3 = 10\ \Omega$

(C) $Z_1 = 6\ \Omega$, $Z_2 = 5\ \Omega$, $Z_3 = 10\ \Omega$

(D) $Z_1 = 25\ \Omega$, $Z_2 = 26\ \Omega$, $Z_3 = 10\ \Omega$

11. The $ABCD$ parameters of an ideal n : 1 transformer shown in the figure are $\begin{bmatrix} n & o \\ o & x \end{bmatrix}$. The value of x will be



(A) n (B) $\dfrac{1}{n}$

(C) n^2 (D) $\dfrac{1}{n^2}$

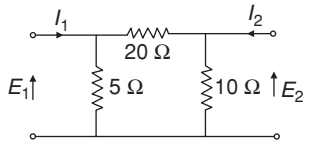
12. The short-circuit admittance matrix of a two port network is

$$\begin{bmatrix} 0 & \dfrac{-1}{2} \\ \dfrac{1}{2} & 0 \end{bmatrix}$$

The two-port network is

(A) Non-reciprocal and passive
(B) Non-reciprocal and active
(C) Reciprocal and passive
(D) Reciprocal and active

13. The admittance parameter Y_{12} in the two-port network shown in the figure is

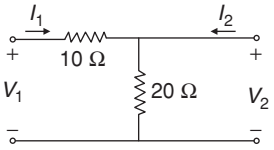


(A) −0.2 mho (B) 0.1 mho
(C) −0.05 mho (D) 0.05 mho

14. For a two-port network to be reciprocal
(A) $Z_{11} = Z_{22}$ (B) $Y_{21} = Y_{22}$
(C) $h_{21} = -h_{12}$ (D) $AD - BC = 0$

15. Which parameters are widely used in transmission line theory?
(A) Z-parameters (B) Y-parameters
(C) $ABCD$ parameters (D) H-parameters

16. The H-parameters of the circuit shown in the figure are:



(A) $\begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}$ (B) $\begin{bmatrix} 10 & -1 \\ 1 & 0.05 \end{bmatrix}$

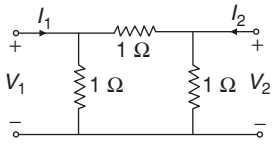
(C) $\begin{bmatrix} 30 & 20 \\ 20 & 20 \end{bmatrix}$ (D) $\begin{bmatrix} 10 & 1 \\ -1 & 0.05 \end{bmatrix}$

17. The impedance matrices of two, two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 5 & 25 \end{bmatrix}$. If the two networks are connected in series, the impedance matrix of the combination is

(A) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (B) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 7 \\ 25 & 3 \end{bmatrix}$

18.



The Y-parameters for the network is

(A) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

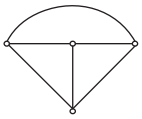
(C) $\begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}$ (D) $\begin{bmatrix} 0.5 & -1 \\ -1 & 0.5 \end{bmatrix}$

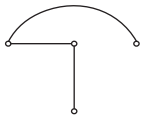
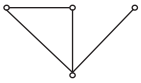
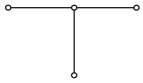
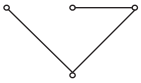
19. If the graph of an electrical network has 'N' nodes and 'B' branches. The number of links 'L' is given by
(A) $N - B + 1$ (B) $B - N + 1$
(C) $N + B$ (D) $B - N$

20. A connected network of $N > 2$ nodes has at the most one branch directly connecting any pair of nodes. The graph of the network
(A) Must have at least 'N' branches for one or more closed paths to exist.
(B) Can have an unlimited number of branches.
(C) Can only have at the most N branches.
(D) Can have a minimum number of branches not decided by N.

21. If B is tie-set matrix and 'I_L' Loop current matrix then branch current matrix I_b is given by
(A) $I_b = BI_L$ (B) $I_b = I_L B^T$
(C) $I_b = B^T I_L$ (D) $I_b = B^T I_L^T$

22. Consider the network graph shown in the figure. Which one of the following is NOT a 'tree' of this graph?

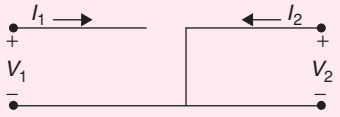


(A) 
(B) 
(C) 
(D) 

23. A network has 7 nodes and 5 independent loops. The number of branches in the network is
(A) 13 (B) 12
(C) 11 (D) 10

Previous Years' Questions

1. The parameter type and the matrix representation of the relevant two port parameters that describe the circuit shown are **[2006]**



(A) Z-parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

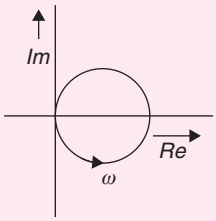
(B) H-parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) H-parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

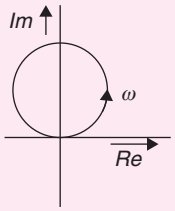
(D) Z-parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. The R-L-C series circuit shown is supplied from a variable frequency voltage source. The admittance-locus of the R-L-C network at terminals AB for increasing frequency ω is **[2007]**

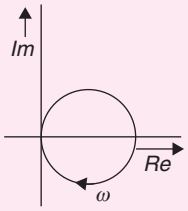
(A)



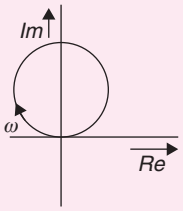
(B)



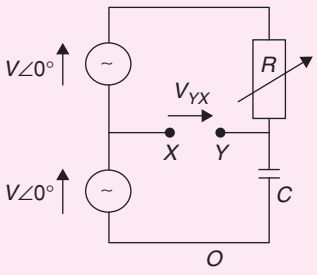
(C)



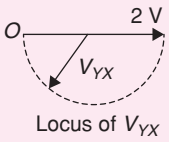
(D)



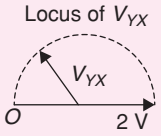
3. In the figure given below all phasors are with reference to the potential at point 'O'. The locus of voltage phasor V_{YX} as R is varied from zero to infinity is shown by **[2007]**



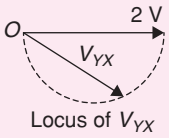
(A)



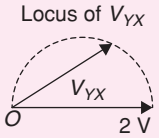
(B)



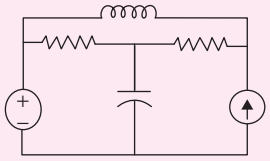
(C)



(D)

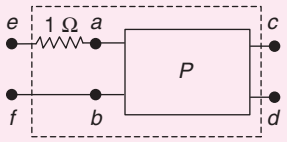


4. The number of chords in the graph of the given circuit will be **[2008]**



(A) 3 (B) 4
(C) 5 (D) 6

5. The two-port network P shown in the figure has ports 1 and 2, denoted by terminals (a, b) and (c, 0), respectively. It has an impedance matrix Z with parameters denoted by z_{ij}. A 1 Ω resistor is connected in series with the network at port 1 as shown in the figure. The impedance matrix of the modified two-port network (shown as a dashed box) is **[2010]**



(A) $\begin{pmatrix} z_{11}+1 & z_{12}+1 \\ z_{21} & z_{22}+1 \end{pmatrix}$ (B) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21} & z_{22}+1 \end{pmatrix}$

(C) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ (D) $\begin{pmatrix} z_{11}+1 & z_{12} \\ z_{21}+1 & z_{22} \end{pmatrix}$

Common Data for Questions 6 and 7: With 10 V dc connected at port A in the linear non reciprocal two-part network shown below, the following were observed:

(i) 1 Ω connected at port B draws a current of 3 A.
(ii) 2.5 Ω connected at port B draws a current of 2 A.



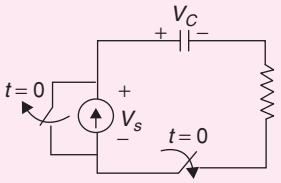
6. For the same network, with 6 V dc connected at port A, 1 Ω connected at port B draws 7/3 A. If 8 V DC is connected to port *A*, the open circuit voltage at port *B* is **[2012]**

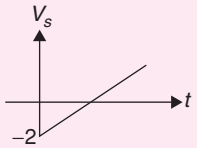
(A) 6 V (B) 7 V
(C) 8 V (D) 9 V

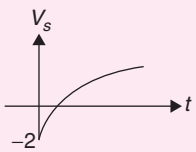
7. With 10 V dc connected at port A, the current drawn by 7 Ω connected at port *B* is **[2012]**

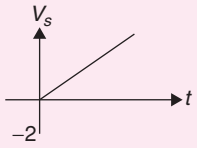
(A) 3/7 A (B) 5/7 A
(C) 1 A (D) 9/7 A

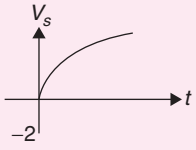
8. A combination of 1 μF capacitor with an initial voltage $v_c(0) = -2$ V in series with a 100 Ω resistor is connected to a 20 mA ideal dc current source by operating both switches at $t = 0$ s as shown. Which of the following graphs shown in the options approximates the voltage v_s across the current source over the next few seconds? **[2014]**



(A)


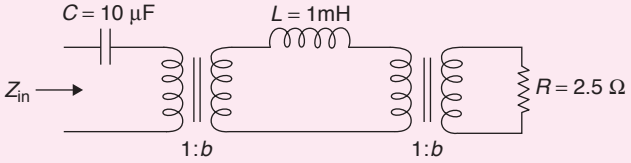
(B)


(C)


(D)


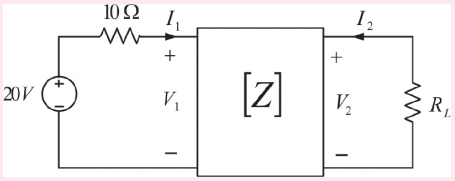
9. In a linear two-port network, when 10 V is applied to Port 1, a current of 4 A flows through Port 2 when it is short-circuited. When 5 V is applied to Port 1, a current of 1.25 A flows through a 1 Ω resistance connected across Port 2. When 3 V is applied to Port 1, the current (in Ampere) through a 2 Ω resistance connected across Port 2 is _____. **[2015]**

10. Find the transformer ratios a and b such that the impedance (Z_{in}) is resistive and equals 2.5 Ω when the network is excited with a sine wave voltage of angular frequency of 5000 rad/s. **[2015]**



(A) $a = 0.5, b = 2.0$
(B) $a = 2.0, b = 0.5$
(C) $a = 1.0, b = 1.0$
(D) a = 4.0, b = 0.5

11. The *Z*-parameters of the two port network shown in the figure are $Z_{11} = 40\Omega$, $Z_{12} = 60\Omega$, $Z_{21} = 80\Omega$, and $Z_{22} = 100\Omega$. The average power delivered to $R_L = 20\Omega$, in watts, is _____. **[2016]**



Answer Keys

Exercises

Practice Problems I

1. C	**2.** D	**3.** D	**4.** D	**5.** C	**6.** C	**7.** B	**8.** B	**9.** C	**10.** C
11. A	**12.** D	**13.** C	**14.** B	**15.** B	**16.** B	**17.** C	**18.** A	**19.** A	**20.** A
21. C	**22.** B	**23.** D	**24.** A	**25.** D	**26.** B	**27.** C	**28.** A	**29.** B	**30.** D
31. A									

Practice Problems I

1. C	**2.** A	**3.** C	**4.** D	**5.** A	**6.** B	**7.** C	**8.** C	**9.** C	**10.** C
11. B	**12.** B	**13.** C	**14.** C	**15.** C	**16.** D	**17.** B	**18.** B	**19.** B	**20.** D
21. C	**22.** B	**23.** C							

Previous Years' Questions

1. C	**2.** A	**3.** A	**4.** A	**5.** C	**6.** B	**7.** C	**8.** C	**9.** 0.5454
10. B	**11.** 35.55							

Chapter 5

Resonance

LEARNING OBJECTIVES

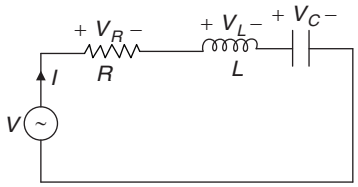
After reading this chapter, you will be able to understand:

- Series resonance
- Quality factor
- Frequency at which v_c is maximum
- Parallel resonance
- Filters
- Self-inductance
- Mutual inductance
- Coupling coefficient
- Magnetically coupled inductors in parallel
- Transformer coupling
- Laplace transform
- Network synthesis
- *L–C* emittance function
- Power relations in ac circuit

Introduction

Resonance in electrical circuits consisting of passive and active elements represents a particular state of the circuit when the current or voltage in the circuit is maximum or minimum with respect to the magnitude of excitation at a particular frequency. The circuit impedance being either minimum or maximum at the power factor unity.

Series Resonance



The circuit is said to be in resonance if the current is in phase with the applied voltage.

The impedance $Z = R + j\omega L - \dfrac{1}{j\omega C}$

$$Z = R + j\left(\omega L - \frac{1}{\omega c}\right)$$

At, resonance, the impedance is purely resistive

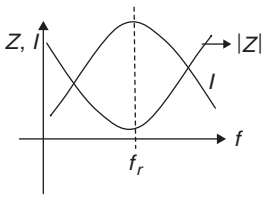
i.e., $\omega L - \dfrac{1}{\omega C} = 0$

$$\omega^2 = \frac{1}{LC}$$

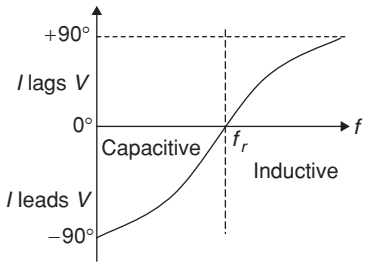
$$\omega = \frac{1}{\sqrt{LC}}$$

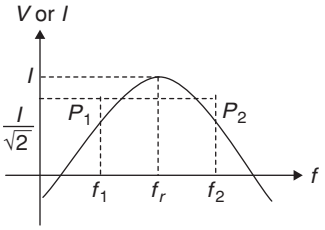
$f_r = \dfrac{1}{2\pi\sqrt{LC}}$ is the resonant frequency in Hz

At resonance the current, $I = \dfrac{V}{R}$, i.e., the current is maximum at resonance and the impedance is minimum.



- At zero frequency X_C and, Z are infinitely large, and X_L is zero because at zero frequency, the capacitor acts as an open circuit.
- As the frequency increases, X_C decreases and X_L increases. Since X_C is larger than X_L, at frequencies below the resonant frequency f_r, Z decreases along with X_C.
- At resonant frequency f_r, $X_C = X_L$ and $Z = R$.
- At frequencies above the resonant frequency f_r, X_L is larger than X_C, causing Z to increase.
- The phase angle as a function of frequency is shown in figure.





The bandwidth is the range of frequencies over which the current is equal to 70.7% of its value at the resonant frequency.

Bandwidth, $BW = f_2 - f_1$

Where f_2 and f_1 are cutoff frequencies or half power frequencies. At $f = f_1$ or f_2, the net reactance is R

i.e., $\frac{1}{\omega_1 C} - \omega_1 L = R$

$$\frac{1}{\omega_2 C} - \omega_2 L = R$$

$$\frac{1}{\omega_1 C} - \omega_1 L = \frac{1}{\omega_2 C} - \omega_2 L$$

$$L(\omega_1 + \omega_2) = \frac{1}{C}\left[\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right]$$

$$\omega_1 \omega_2 = \frac{1}{LC}$$

$$\omega_r^2 = \omega_1 \omega_2$$

$$\omega_r = \sqrt{\omega_1 \omega_2}$$

$$\omega_2 - \omega_1 = \frac{R}{L}$$

$$f_2 - f_1 = \frac{R}{2\pi L}$$

Bandwidth, $BW = \frac{R}{2\pi L}$

$$f_1 = f_r - \frac{R}{4\pi L}$$

$$f_2 = f_r + \frac{R}{4\pi L}$$

The quality factor (Q-factor) is the ratio of the reactance of the coil to its resistance at resonant frequency.

$$Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R}$$

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q}$$

$$Q = \frac{\omega L}{R} = \frac{1}{\omega CR}$$

$$\boxed{Q = \frac{f_r}{BW}}$$

Let V be the applied voltage across RLC circuit, at resonance, the current is I, the voltage across L is

$V_L = IX_L = \frac{V}{R}\omega_r L$

$V_L = QV$

Similarly $V_C = VQ$

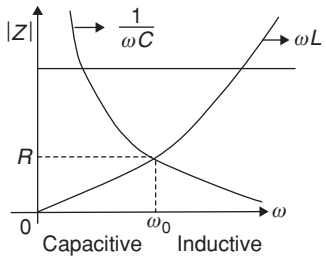
i.e., $Q = \frac{V_L}{V}$ or $\frac{V_C}{V}$

Quality Factor

A quality factor or figure of merit can be assigned to a component or to a complete circuit. It is defined as

$$Q = 2\pi\left(\frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}\right)$$

$$\boxed{Q = \frac{|X_L|}{R} = \frac{|X_C|}{R} = \frac{1}{R}\cdot\sqrt{\frac{L}{C}}}$$



- For $\omega < \omega_0 \Rightarrow$ series RLC circuit behaves like (RC ckt) capacitive circuit
  $\Rightarrow$ Leading power factor
  $\Rightarrow V_C = QV \angle -90°$
- for $\omega = \omega_0 \Rightarrow$ Resistive nature; $pf = 1$
- for $\omega > \omega_0 \Rightarrow$ inductive nature
  $\Rightarrow$ Lagging power factor
  $V_L = I \cdot Z_L$

$V_L = QV \angle 90°$

Where $Q = \dfrac{\omega_0 L}{R}$

- **At resonance**:

$|V_L| = |V_C|$ and these are 180° out of phase

Solved Examples

Example 1: In a series RLC circuit, the Q factor at resonance is 80. If all the component values are doubled, then the new Q factor is

(A) $Q' = 160$ (B) $Q' = 40$

(C) $Q' = 80°$ (D) None of the above

Solution: (B)

For a series RLC circuits

$$Q = \frac{1}{R}\cdot\sqrt{\frac{L}{C}}$$

Given $R' = 2R$

$L' = 2L$

$C' = 2C$

$$Q' = \frac{Q}{2} = 40.$$

Example 2: In a series RLC circuit, R = 8 Ω, X_L = 16, X_C = 16 and V_{in} = 80 V. The voltage across the capacitor is

(A) $V_C = 160 \angle 90°$ (B) $V_C = 40 \angle 90°$

(C) $V_C = 160 \angle -90°$ (D) $V_C = 80 \angle -90°$

Solution: (C)

$$V_C = QV \angle -90°$$

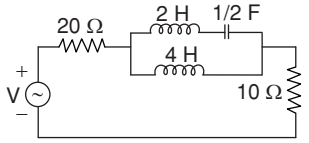
$$Q = \frac{|X_L|}{R} = \frac{16}{8} = 2$$

Given $V_{in} = V = 80$ V

$V_C = 2 \times 80 \angle -90°$

$= 160 \angle -90°$.

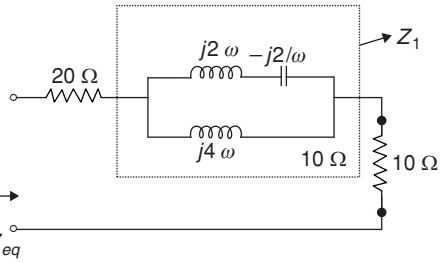
Example 3: For the circuit shown in the figure determine the resonant frequency (f_0).



(A) $f = 0$ Hz (B) $f = \dfrac{1}{2\pi}$ Hz

(C) $f = \dfrac{1}{\pi}$ Hz (D) None of the above

Solution: (B)



$$Z_{eq} = 20 + Z_1 + 10$$

$$Z_1 = (j4\omega) \,||\, [j(2\omega - 2/\omega)]$$

$$= \frac{j4\omega j\left(2\omega - \dfrac{2}{\omega}\right)}{j4\omega + j2\omega - \dfrac{j2}{\omega}}$$

$$= \frac{j4\omega\left[2\omega - \dfrac{2}{\omega}\right]}{6\omega - \dfrac{2}{\omega}} \Rightarrow \frac{j4\omega[2\omega^2 - 2]}{6\omega^2 - 2}$$

$$= \frac{j4\omega[\omega^2 - 1]}{3\omega^2 - 1}$$

$$Z_{eq} = 20 + \frac{j\omega(\omega^2 - 1)\times 4}{3\omega^2 - 1} + 10$$

At resonance imaginary part equal to zero.

At $\omega = \omega_r$

$$\frac{\omega(\omega^2 - 1)\times 4}{3\omega^2 - 1} = 0$$

$$\Rightarrow \omega_r^2 - 1 = 0$$

$$\omega_r = 1$$

$$2\pi f_r = 1$$

$$f_r = \frac{1}{2\pi} \text{ Hz.}$$

Example 4: A series resonant circuit has L = 10 mH and C = 100 μF. The required R for the BW 16 Hz is

(A) $R = 16$ Ω (B) $R = 0.16$ Ω

(C) $R = 160$ Ω (D) $R = 1.6$ Ω

Solution: (B)

For a series RLC circuit characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$BW = \frac{R}{L}$$

$$\frac{R}{10\times 10^{-3}} = 16 \Rightarrow R = 0.16\ \Omega$$

Selectivity The selectivity of a resonating circuit is defined by the ratio of f_0 (the resonance frequency) to the bandwidth of the circuit.

$$\text{Selectivity} = \frac{f_0}{f_2 - f_1} = \frac{f_0}{BW}$$

Frequency at which V_C is maximum V_C and V_L are not maximum at resonant frequency in case of series RLC resonance but at other frequency value. This can be obtained by

$$V_C = I \cdot X_C$$

$$= \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega c}\right)^2}} \frac{1}{\omega c}$$

Where,

$$I = \frac{V}{|Z|}$$

The frequency at which V_C is maximum can be obtained by differentiate V_C wrt to ω. i.e.,

$$\frac{dV_c^2}{d\omega} = 0$$

$$\frac{dV_c^2}{d\omega} = 0 \quad \Rightarrow \quad \frac{-V\left[2\omega_0 (RC)^2 + 2\left(\omega_0^2 LC - 1\right) \cdot 2\omega_0 LC\right]}{\left[(\omega_0 RC)^2 + \left(\omega_0^2 LC - 1\right)^2\right]^2}$$

But $V \neq 0$,

$2\omega_0^2 L^2 C + CR^2 - 2L = 0$

By simplification we get

$$\omega_0^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

$$\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}}$$

Where $f_0 \Rightarrow$ at which voltage across capacitor is maximum.

The frequency at which V_L is maximum $V_L = I \cdot X_L$

$$= \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega c}\right)^2}} \omega L$$

The frequency at which V_L is max is given by

$$\frac{dV_L^2}{d\omega} = 0$$

By the simplification we get

$2\omega^2 LC - \omega^2 (RC)^2 - 2 = 0$

$\omega^2 [2LC - (RC)^2] = 2$

$$\omega^2 = \frac{2}{2LC - (RC)^2}$$

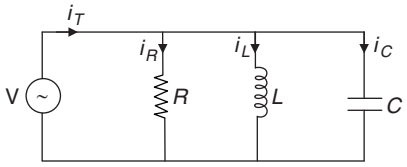
$$\boxed{f_{0L} = \frac{1}{2\pi} \times \frac{1}{\sqrt{LC - \frac{(RC)^2}{2}}}}$$

Where $f_{oL} \Rightarrow$ frequency at which the voltage across the inductor is max.

Parallel Resonance

A circuit consisting of a parallel connection of R, L and C is called a second order parallel resonant circuit, parallel resonance circuit is also called anti-resonance circuit. It acts as a band-reject filter.

Parallel Resonance



The admittance, $Y = G + \frac{1}{j\omega L} + j\omega C$

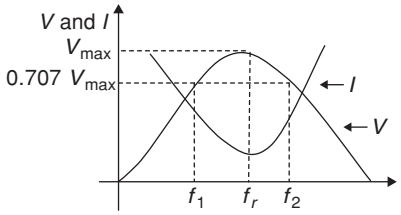
At resonance 'Y' is purely real, i.e.,

$$j\left(\omega C - \frac{1}{\omega L}\right) = 0$$

$$\Rightarrow \quad \omega C - \frac{1}{\omega L} = 0$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$f_r = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$



At resonance, the current is minimum and the impedance is maximum.

At lower half power frequency ω_1,

$$\omega_1 C - \frac{1}{\omega_1 L} = \frac{-1}{R}$$

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

At upper half power frequency ω_2,

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

Bandwidth, $BW = \omega_2 - \omega_1 = \frac{1}{RC}$

The quality factor $Q = \frac{\omega_r}{BW}$

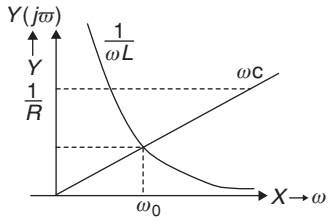
$$Q = \omega_r RC$$

$$Q = \frac{R}{\omega_r L}$$

At resonance $I_L = I_r Q \angle -90°$

$$I_C = I_r Q \angle 90°$$

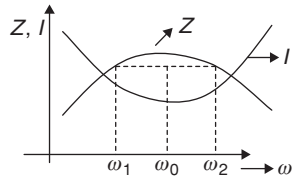
The resonant frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ Hz



1. Below resonance the circuit acts like an *RL* circuit ($\omega < \omega_o$), i.e., lagging power factor.
2. Above resonance the circuit acts like an *RC* circuit ($\omega > \omega_o$), i.e., leading power factor.
3. At $\omega = \omega_0$ the circuit acts like an resistive nature.
4. $BW = \frac{1}{RC}$.
5. Quality factor $Q = R\sqrt{\frac{C}{L}}$
6. $\xi = \frac{1}{2Q}$

Note: Impedance of parallel resonant circuit is maximum at resonance. Current of parallel resonance circuit is minimum at resonance. i.e., $Z_{in} = R$ at $\omega = \omega_o$ is maximum.

$$\frac{I}{\omega} = \omega_o \frac{V_s}{R} \Rightarrow \text{minimum}$$



Examples 6: A series resonant circuit has an inductive reactance of 1 kΩ, a capacitive reactance of 1 kΩ and a resistance of 0.1 Ω. If the resonant frequency is 10 MHz, then the bandwidth of the circuit will be

(A) 1 kHz (B) 10 kHz
(C) 1 MHz (D) 0.1 kHz

Solution: (A)

$$BW = \frac{f_0}{Q} = \frac{f_0}{\left(\frac{\omega_0 L}{R}\right)}$$

$$BW = \frac{0.1 \times 10 \times 10^6}{1000} = 1 \text{ kHz}$$

Example 7:

Assertion(A): A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it.

Reason(R): At resonance, the power factor of the circuit is zero.

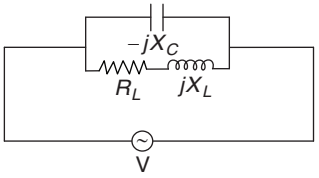
(A) Both A and R are true and 'R' is the correct explanation of A.
(B) Both 'A' and 'R' are true but 'R' is NOT the correct explanation of A.
(C) 'A' is true but 'R' is false.
(D) 'A' is false but 'R' is true.

Soltuion: (C)

At resonance the voltage across the circuit is in phase with the current through it and the power factor is '1'.

cos $\phi = 1 \Rightarrow$ unity power factor

Example 8:



Find the resonant frequency for the tank circuit shown in figure

Solution:

The admittance is given by

$$Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C} = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}$$

$$Y = \frac{R_L}{RL^2 + X_L^2} + j\left[\frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2}\right]$$

At resonance Y is purely real

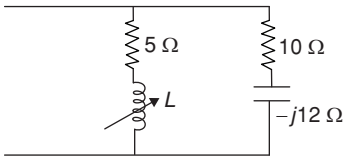
$$\therefore \quad \frac{1}{X_C} = \frac{X_L}{R_L^2 + X_L^2}$$

$$\omega C = \frac{\omega L}{R_L{}^2 + \omega^2 L^2}$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$

Example 9: Find the value of L at which the circuit resonates at a frequency of 1000 rad/sec in the circuit shown in figure.



Solution:

$$Y = \frac{1}{10 - j12} + \frac{1}{5 + jX_L}$$

$$Y = \frac{10 + j12}{10^2 + 12^2} + \frac{5 - jX_L}{25 + X_L^2}$$

$$Y = \frac{10}{10^2 + 12^2} + \frac{5}{25^2 + X_L^2} + j\left[\frac{12}{10^2 + 12^2} - \frac{X_L}{25 + X_L^2}\right]$$

At resonance Y is purely real, i.e.,

$$\frac{X_L}{25 + X_L^2} = \frac{12}{10^2 + 12^2}$$

$X_L^2 - 20.3X_L + 25 = 0$

$X_L = 18.98\ \Omega$ or $1.32\ \Omega$

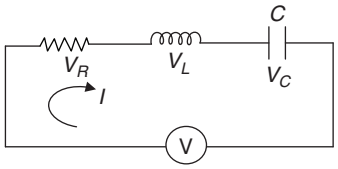
$X_L = \omega L = 18.98\ \Omega$ or $1.32\ \Omega$

$$L = \frac{18.98}{1000} \text{ or } \frac{1.32}{1000}$$

$L = 18.98$ mH or 1.32 mH

Example 10: A series RLC circuit is supplied by 220 V; 50 Hz; at resonance, the voltage across the capacitor = 550 V, $I = 1$ A; determine R, L and C.

Solution:



At resonance $X_L = X_C$

$$I = \frac{V}{R} = \frac{220}{R}$$

$$\Rightarrow R = 220\ \Omega$$

$$V_C = I_0 X_C$$

$$550 = 1 \cdot \frac{1}{\omega_0 C}$$

$$C = \frac{1}{550 \times 2\pi \times 50} = 5.75\ \mu\text{F}$$

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{C}\left[\frac{1}{2\pi fr}\right]^2$$

$L = 1.75$ H

Example 11: A series RLC circuit consists of a 50 Ω resistance, 0.2 H inductance and 10 μF capacitor with an applied voltage of 20 V. Determine the resonant frequency. Find the Q-factor of the circuit. Compute the lower and upper frequency limits and also find the band width of the circuit

Solution:

Resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}}$$

$= 112.5$ Hz

Quality factor $Q = \dfrac{\omega_r L}{R}$

$$= \frac{2\pi \times 112.5 \times 0.2}{50}$$

$= 2.83$

Lower frequency limit $f_1 = f_r - \dfrac{R}{4\pi L}$

$$= 112.5 - \frac{50}{4 \times \pi \times 0.2}$$

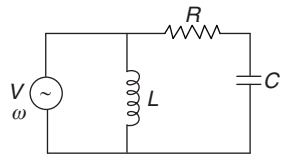
$= 92.6$ Hz

Upper frequency limit

$$f_2 = f_r + \frac{R}{4\pi L} = 112.5 + \frac{50}{4\pi \times 0.2}$$
$$= 112.5 + 19.89$$
$$= 132.39 \text{ Hz}$$

Bandwidth, $BW = f_2 - f_1 = 39.79$ Hz

Example 12: Consider the following circuit



For what value of ω, the circuit shown above exhibits unity power factor?

(A) $\frac{1}{\sqrt{LC}}$

(B) $\frac{1}{\sqrt{LC + R^2C^2}}$

(C) $\frac{1}{\sqrt{LC - R^2C^2}}$

(D) $\frac{1}{RC}$

Solution: (C)
For unity power factor, imaginary part of impedance should be zero.

$$Z_{eq} = \frac{j\omega L\left(R - \frac{j}{\omega c}\right)}{R + j\omega L - \frac{j}{\omega c}}$$

$$Z_{eq} = \frac{j\omega l\left(R - \frac{j}{\omega c}\right)}{\left(R + j\omega L - \frac{j}{\omega c}\right)} \times \frac{\left(R - j\omega L + \frac{j}{\omega c}\right)}{\left(R - j\omega l + \frac{j}{\omega c}\right)}$$

At resonance imaginary part equal to zero

$$R^2 - \frac{L}{C} + \frac{1}{\omega^2 C^2} = 0$$

By simplification we get

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC - R^2C^2}}}$$

Filters The general transfer functions for second order filters:

	Filters		Transfer Function
1	LPF	$\Rightarrow$	$\frac{P}{s^2 + as + b}$
2	HPF	$\Rightarrow$	$\frac{Ps^2}{s^2 + as + b}$
3	BPF	$\Rightarrow$	$\frac{Ps}{s^2 + as + b}$
4	BSF	$\Rightarrow$	$\frac{Ps^2 + q}{s^2 + as + b}$
5	APF	$\Rightarrow$	$\frac{s^2 + Ps + q}{s^2 + as + b}$

Coupled Circuits

Self Inductance

When a current changes in a circuit, the magnetic flux linking the same circuit changes (and vice-versa) and an emf is induced in the circuit.

This induced emf is proportional to the rate of change of current.

$$v = L \cdot \frac{di}{dt} \quad (1)$$

Where

$v \rightarrow$ induced voltage

$\frac{di}{dt} \rightarrow$ rate of change of current

$L \rightarrow$ self inductance

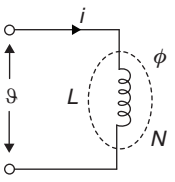
Inductance also expressed as

$$L = \frac{N\phi}{I}$$

$N \rightarrow$ no. of turns in the circuit.
$\phi \Rightarrow L$ flux linkage

$$v = L \cdot \frac{d\left(\frac{N\phi}{L}\right)}{dt}$$

$$= N \cdot \frac{d\phi}{dt} \quad (2)$$

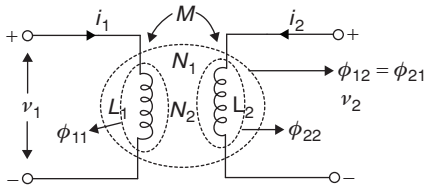


Equate the equation (1) and (2)

$$L \cdot \frac{di}{dt} = N \cdot \frac{d\phi}{di}$$

$$\boxed{L = N \cdot \frac{d\phi}{di}}$$

Mutual Inductance



The Induced voltage of coil-2 is given by

$$V_{L2} = N_2 \cdot \frac{d\phi_{12}}{dt} \tag{3}$$

ϕ_{12} is related to the current of coil-I and the induced voltage is proportional to the rate of change of i_1.

$$V_{L_2} = M \cdot \frac{di_1}{dt} \tag{4}$$

(3) = (4)
We get

$$\boxed{M = N_2 \cdot \frac{\phi_{12}}{i_1} = N_1 \cdot \frac{\phi_{21}}{i_2}}$$

Coupling Co-efficient

The amount of magnetic coupling is expressed by co-efficient of coupling (k)

$$K = \frac{\text{useful flux}}{\text{total flux}} = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

$$\boxed{M = K\sqrt{L_1 L_2}}$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

For ideal circuits $K = 1$ and practical circuits,range of K is $0 < K < 1$

Leakage factor

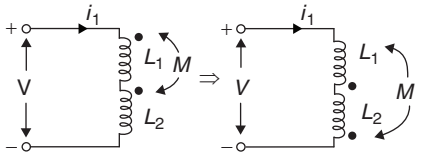
$$\text{Leakage factor} = \frac{\text{Total flux}}{\text{Useful flux}} = \frac{1}{K}$$

Dot conventions

1. When either both the currents are entering at dotted terminal or leaving the dotted terminal, mutual inductance will be added.
2. If one current is entering and other is leaving the dotted terminal or vice-versa then mutual inductance will be substracted.

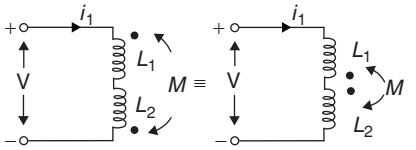
Series Connection of Coupled Coils

1.



$$\boxed{\text{Leq} = L = L_1 + L_2 + 2M} \Rightarrow \text{Aiding}$$

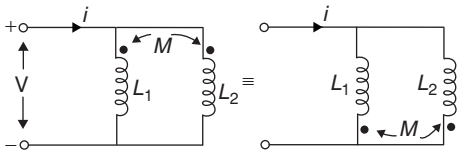
2.



$$\boxed{\text{Leq} = L = L_1 + L_2 - 2M} \Rightarrow \text{Opposition}$$

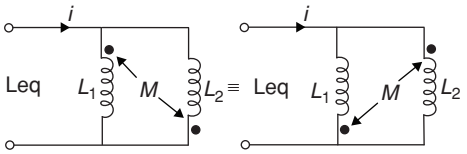
Magnetically Coupled Inductors in Parallel

1. Magnetically aiding



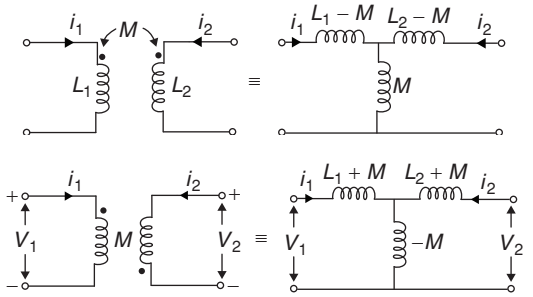
$$\boxed{\text{Leq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}}$$

2. Magnetically opposition



$$\boxed{\text{Leq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}} \Rightarrow$$

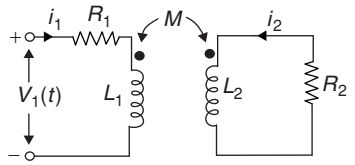
Equivalent circuit:



TRANSFORMER COUPLING

Two inductors with self inductances L_1 and L_2, mutual inductance M, and coefficient of coupling K, are shown in figure, with dot convention

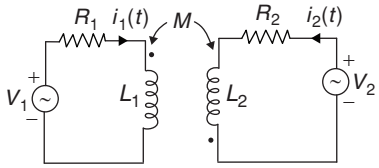
1.



$$V_1(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

$$0 = R_2 i_2(t) + L_2 \frac{di_2}{dt} + M \frac{di_1(t)}{dt}$$

2.

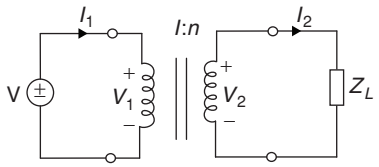


$$V_1 = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt}$$

$$V_2 = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} - M \frac{di_1(t)}{dt}$$

Ideal Transformer

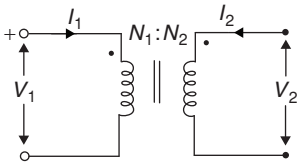
An ideal transformer is a unity coupled, lossless transformer in which the primary and secondary coils have infinite self inductances.



$$V_1 = N_1 \frac{d\phi}{dt} = v_2 = N_2 \frac{d\phi}{dt}$$

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = \frac{n}{1} = \frac{I_1}{I_2}$$

1. When $n = 1$, generally called the transformer an isolation transformer.
2. $n > 1 \Rightarrow$ step up transformer $(V_2 > V_1)$
3. $n < 1 \Rightarrow$ step-down transformer $(V_2 < V_1)$



$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = \frac{-I_1}{I_2}$$

The input impedance as seen by the source is

$$Z_{in} = \frac{V_1}{I_1} = \frac{1}{n^2} \cdot \frac{V_2}{I_2}$$

$$\boxed{Z_{in} = \frac{Z_L}{n^2}}$$

$$\boxed{Z_{in} = Z_L \left(\frac{N_1}{N_2}\right)^2}$$

Examples 13: Two couples coils have self inductances L_1 = 15 mH and L_2 = 18 mH. The coefficient of coupling (k) being 0.75 in the air, find voltage in the second coil and flux of first coil provided the second coil has 500 turns and the circuit current is given by $i_1 = 5.\ \sin 314t$ A.

Solution: $M = K\sqrt{L_1 K_2}$

$$= 0.75\sqrt{15 \times 18} \times 10^{-3}$$

$$= 12.32 \times 10^{-3} \text{ H}$$

The voltage induced in the second coil is

$$V_{L_2} = M \cdot \frac{di_1}{dt} = 12.32 \times 10^{-3} \cdot \frac{d}{dt}(5 \sin 314t)$$

$$= 12.32 \times 10^{-3} \times 5 \times 314 \times \cos 314t$$

$$V_{L2} = 19.34 \cos 314t \text{ V}$$

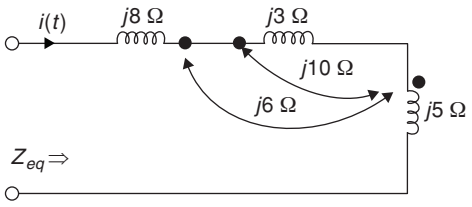
$$M = \frac{N_2 \phi_{12}}{i_1}$$

$$M = \frac{500 \times k \times \phi_1}{i_1}$$

$$\phi = \frac{12.32 \times 10^{-3} \times 5 \sin 314t \times 1}{0.75 \times 500}$$

$$\phi_1 = 16.42 \times 10^{-5} \sin 314t$$

Example 14: Impedance z as shown in figure is



(A) $j29\ \Omega$ (B) $j9\ \Omega$
(C) $j19\ \Omega$ (D) $j24\ \Omega$

Solution:

$$Z_{eq} = L_1 + L_2 + L_3 \pm 2M_{12} \pm 2M_{13} \pm 2M_{23}\ldots$$
$$\therefore Z_{eq} = j8 + j3 + j5 + 2 \times j10 - 2 \times j6$$
$$= j16 + j8 = j24\ \Omega$$

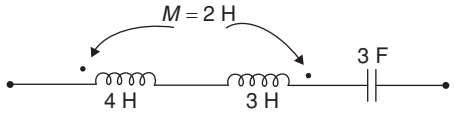
Example 15: Two coils have self-inductances of 0.09 H and 0.01 H and a mutual inductance of 0.015 H. The coefficient of coupling between the coil is

(A) 0.06 (B) 0.5
(C) 1.0 (D) 0.05

Solution:

$$K = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.015}{\sqrt{0.09 \times 0.01}}$$
$$= \frac{0.015}{3 \times 10^{-2}}$$
$$= 0.5$$

Example 16: The resonant frequency of the given series circuit is



(A) $f = \frac{1}{4\pi}$ Hz (B) $f = \frac{1}{3\pi}$ Hz

(C) $f = \frac{1}{6\pi}$ Hz (D) $f = \frac{1}{2\sqrt{3}\pi}$ Hz

Solution:

$$f = \frac{1}{2\pi\sqrt{L_{eq}c}}$$

$$\text{Leq} = L_1 + L_2 \pm 2M$$
$$\text{Leq} = 4 + 3 - 2 \times 2 = 3\ \text{H}.$$

$$f = \frac{1}{2\pi\sqrt{3 \times 3}}$$

$$f = \frac{1}{6\pi}\ \text{Hz}$$

Network Functions and Laplace Transform

Laplace Transform

Let $f(t)$ be a time function which is zero for $t \le 0$ and which is arbitrarily defined for $t > 0$.

Then the Laplace transform of $f(t)$, is defined by

$$L\{f(t)\} = F(s) = \int_{o^+}^{\infty} f(t)e^{-st} \cdot dt$$

In general

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt}$$

The inverse Laplace transform can also be expressed as an integral

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma_o - j\infty}^{\sigma_o + j\infty} F(s) \cdot e^{st} ds.$$

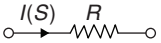
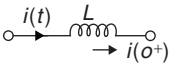
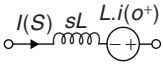
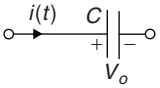
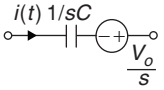
Table 1 *Laplace transform pairs*

$f(t)$	$F(s)$
1. 1	$\frac{1}{s}$
2. $Au(t)$	$\frac{A}{s}$
3. At	$\frac{A}{s^2}$
4. e^{-at}	$\frac{1}{s+a}$
5. $t \cdot e^{-at}$	$\frac{1}{(s+a)^2}$
6. $\sin at$	$\frac{a}{s^2+a^2}$
7. $\cos at$	$\frac{s}{s^2+a^2}$
8. $\frac{df(t)}{dt}$	$sF(s) - f(o^+)$
9. $\int_o^t f(t) \cdot dt$	$\frac{F(s)}{s}$
10. $f(t-a)$	$e^{-aS} \cdot F(s)$
11. $\int f_1(\tau) \cdot f_2(t-\tau) \cdot d\tau$	$F_1(s) \cdot F_2(s)$

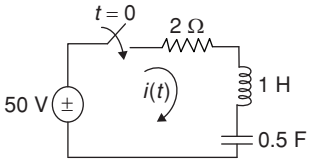
Initial-value and find-value theorems

1. $\underset{t \to o^+}{\text{Lt}}\ f(t) = f(o^+) = \underset{S \to \infty}{\text{Lim}}\{s \cdot F(s)\}$
2. $\underset{t \to \infty}{\text{Lt}}\ f(t) = f(\infty) = \underset{S \to o}{\text{Lim}}\{s \cdot F(s)\}$

Table 2 *Circuits in the S-domain*

	Time domain	*S*-domain	*S*-domain voltage
1.			R.I(s)
2.			sLI(s)–L.i(o+)
3.			$\frac{I(s)}{sC} + \frac{V_o}{s}$

Examples 17: In the RLC circuit shown in figure, there is no initial charge on the capacitor. If the switch is closed at $t = 0$, the resultant current is



(A) $i(t) = j25 \cdot e^{-t}.\cos t$ A (B) $i(t) = 50 \cdot e^{-t} \sin t$ A
(C) $i(t) = 25 \cdot e^{-t} \sin 2t$ A (D) None of the above

Solution: (B)
Given $i(o^+) = 0$ and $V_c(o^+) = 0$
For $t > 0$ apply KVL
The time domain equation of the given circuit is

$$R \cdot i(t) + L \cdot \frac{di(t)}{dt} + \frac{1}{c}\int_0^t i(t)dt = V. \qquad (5)$$

Convert Eqn (5) into LPF

$$R \cdot I(s) + sL \cdot I(s) + \frac{1}{sC} \cdot I(s) = \frac{V}{s}$$

$$\left(2 + S + \frac{2}{s}\right) \cdot I(s) = \frac{50}{s}$$

$$I(s) = \frac{50}{s^2 + 2s + 2}$$

Apply inverse Laplace transform to $I(s)$

$$i(t) = j25\{e^{-1-jt} - e^{-1+jt}\}$$

$$= 50 \cdot e^{-t} \cdot \sin t \text{ A}$$

Network Synthesis

Consider a network as shown in figure

Excitation	Network	Response
$E(s)$ or $e(t)$	$H(s)$	$R(s)$ or $r(t)$

Figure 1 General Network

The system function is defines as

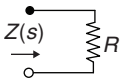
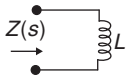
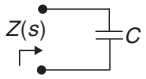
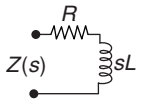
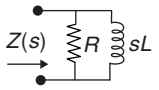
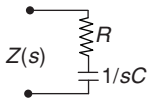
$$\Rightarrow \quad H(s) = \frac{R(s)}{E(s)}$$

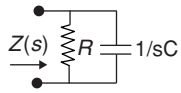
Example:

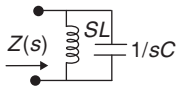
1. Driving point impedance $Z(s) = \dfrac{V(s)}{I(s)}$
2. Driving point admittance $V(S) = \dfrac{I(s)}{V(s)}$

Note: Immitance = Impedance (or) admittance

Driving point immitance of simple networks

1.  $\Rightarrow \quad Z(s) = R; Y(s) = \dfrac{1}{R}$
2.  $\Rightarrow \quad Z(s) = Ls; Y(s) = \dfrac{1}{sL}$
3.  $\Rightarrow \quad Z(s) = \dfrac{1}{sC}; Y(s) = sC$
4.  $\Rightarrow \quad Z(s) = R + Ls; Y(s) \dfrac{1}{R + sL}$
5.  $\Rightarrow \quad Z(S) = \dfrac{RLs}{R + Ls}; y(s) = \dfrac{\frac{1}{L} + \frac{1}{R}s}{s}$
6.  $\Rightarrow \quad Z(s) = R + 1/sC$

   $Y(s) = \dfrac{sC}{1 + RCs}$
7.  $\Rightarrow \quad Z(S) = \dfrac{1}{Cs + \frac{1}{R}}$

   $Y(s) = \dfrac{1}{R} + sC$
8.  $\Rightarrow \quad Z(s) = \dfrac{sL}{1 + s^2LC}$

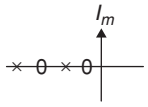
   $Y(s) = \dfrac{1 + s^2LC}{sL}$

R-L Driving point impedance The impedance function is of the form

$$Z(s) = K_o + K_v s + \sum_{i=1}^{n} \frac{K_i}{s + \sigma_i}$$

Properties:

1. Poles and zeros of an R-L impedance lie on σ axis and alternate

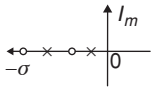


2. The residues are real and negative.
3. The singularity closest to the origin is a zero.

R-C driving point impedance The R-C impedance function has the general form $Z(s) = \dfrac{K_o}{s} + K_v + \sum_{i=1}^{n} \dfrac{K_i}{s + \sigma_i}$

Properties:

1. The poles and zeroes lie on the negative real zxis and alternate.



2. The residues of the poles, K_i, are real and +ve.
3. R-C impedance ⇔ R-L admittance.
4. The singularity nearest the origin being a pole.

L-C immitance function The general expansion is given by

$$Z(S)=\frac{K_o}{s}+\sum_{i=1}^{n}\frac{2K_i s}{s^2+w_i^2}+\cdots+K_v s$$

Properties:

1. $Z(s)$ or $Y(s)$ is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$ axis and alternate.
3.

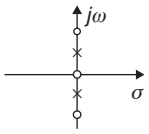
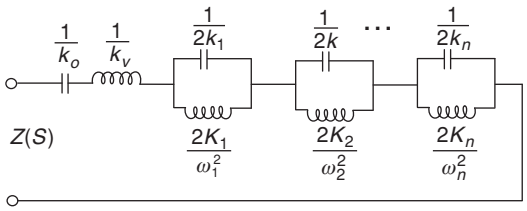


Figure 2 L-C pole-zero realization

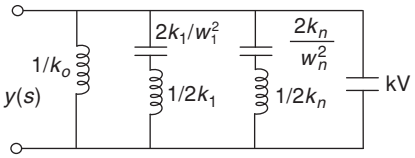
4. The highest power of the numerator and denominator differ by unity.
5. The lowest power of the numerator and denominator differ by unity.
6. There is either a pole or a zero at the origin and at infinity.

Network Function Have Four Canonical Forms

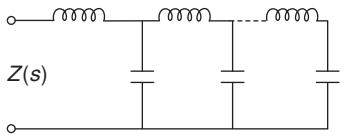
Foster Form I



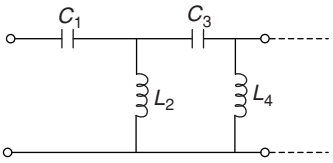
Foster form II



Cauer form I



Cauer from II

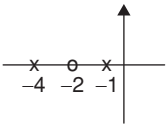


Examples 15: The network function

$F(s)=\dfrac{(s+2)}{(s+1)(s+4)}$ represents an

(A) R-L impedance
(B) R-C impedance
(C) R-C impedance and an R-L admittance
(D) R-C admittance and R-L impedance

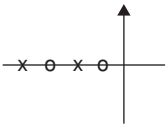
Solution: (C)
Pole zero realization:



⇒ pole nearby origin so R-C impedance
⇒ R-C impedance ⇔ R-L admittance and vice-versa.

Example 18: The first critical frequency nearest the origin of the complex frequency plane for an R-L driving point impedance function will be
(A) A zero in the left-half plane
(B) A zero in the right-half plane
(C) A pole in the LH-plane
(D) Either a pole or zero in the LH-plane

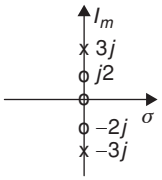
Solution: (A)
For R-L impedance function



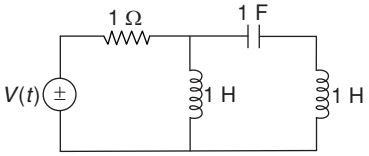
∴ Critical frequency is a zero nearby origin in the LH-plane.

Example 19: Driving point impedance $Z(S)=\dfrac{s(s^2+4)}{(s^2+9)}$ is not realizable because the
(A) Number of zeros is more than the no. of poles.
(B) Poles and zeros lie on the imaginary axis.
(C) Poles and zeros are not located on the real axis.
(D) Poles and zeros do not alternated on imaginary axis.

Solution: (D)
The pole-zero realization of $Z(S)$ is shown below zeros $\Rightarrow Z = 0$ and $Z = \pm j2 = S$
Poles $S^2 + 9 = 0 \Rightarrow S = \pm j3$



Example 20: The driving point impedance of the network shown below is



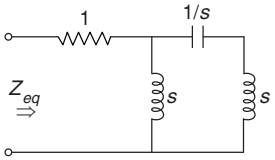
(A) $Z(S) = s + 1$

(B) $Z(S) = \dfrac{2s^2 + 1}{s^3 + 2s^2 + s + 1}$

(C) $Z(S) = \dfrac{1}{s+1}$

(D) $Z(S) = \dfrac{s^3 + 2s^2 + s + 1}{2s^2 + 1}$

Solution: (D)



$$Z_{eq} = 1 + s||(s + 1/s)$$

$$= 1 + \frac{s\left(s + \frac{1}{s}\right)}{s + s + \frac{1}{s}}$$

$$= \frac{s^3 + 2s^2 + s + 1}{2s^2 + 1}$$

Power Relations in AC Circuit

In a passive AC circuit, let the instantaneous voltage be $V(t) = V_m \sin \omega t$. The current is given by $i(t) = I_m \sin(\omega t - \phi)$, ϕ being the phase difference between voltage and current at any instant.

The instantaneous power p is given by $P = vi = V_m I_m \cdot \sin\omega t \cdot \sin(\omega t - \phi)$

$$= \frac{1}{2} \cdot V_m \cdot I_m \cdot \cos\phi \cdot -\frac{1}{2} V_m \cdot I_m \cos(2\omega t - \phi) \qquad (6)$$

In Eqn (6) contains a double frequency term and the magnitude of the average value of this term is zero.

∴ The avrage power in the passive circuit is given by

$$P_{avg} = \frac{1}{2} \cdot V_m \cdot I_m \cdot \cos\phi$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cdot \cos\phi$$

$$P = V_{rms} \cdot I_{rms} \cdot \cos\phi = V \cdot I \cdot \cos\phi (W)$$

Loads	Power Factor(cosϕ)
R	Unity
L	zero lag
C	zero lead
R – L	lagging
R – C	leading

Apparent power

S = voltage × current
$= V \cdot I = V_{rms} I_{rms}$
Relative power $Q = V \cdot I \sin\phi$ (VAR)

$$S = \sqrt{P^2 + Q^2} VA.$$

Average value

$$X_{avg} = \frac{1}{T} \int_o^T x(t) \cdot dt$$

Where T is the time period of periodic function $x(t)$

RMS or effective value

$$X_{rms} = \sqrt{\frac{1}{T} \int_o^T [x(t)]^2 \cdot dt}$$

1. RMS value of $A \sin\omega t$ and $A \cos\omega t$ is $\dfrac{A}{\sqrt{2}}$
2. If $x(t) = a_o + (a_1 \cos \omega t + a_2 \cos 2 \omega t + \cdots) + (b_1 \sin \omega t + b_2 \sin 2 \omega t + \cdots)$

$$X_{rms} = \sqrt{a_o^2 + \frac{1}{2}\left(a_1^2 + a_2^2 + \cdots\right) + \frac{1}{2}\left(b_1^2 + b_2^2 + \cdots\right)}$$

EXERCISES

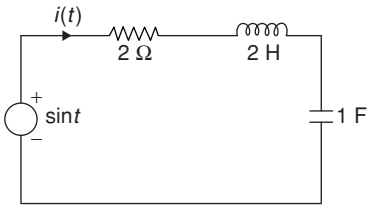
Practice Problems I

Directions for questions 1 to 30: Select correct alternative from the given choices.

1. A series RLC circuit has a resonance frequency of 1 kHz and a quality factor $Q = 100$. If each of R, L and C is doubled from its original value, the new Q of the circuit is

(A) 25 (B) 50
(C) 100 (D) 200

2. The differential equation for the current $i(t)$ in the circuit of the figure is



(A) $2\frac{d^2i}{dt^2}+2\frac{di}{dt}+i(t)=\sin t$

(B) $2\frac{d^2i}{dt^2}+2\frac{di}{dt}+2i(t)=\cos t$

(C) $2\frac{d^2i}{dt^2}+2\frac{di}{dt}+i(t)=\cos t$

(D) $\frac{d^2i}{dt^2}+2\frac{di}{dt}+2i(t)=\sin t$

3. In a series RLC circuit, $R = 2$ kΩ, $L = 1$ H and $C=\frac{1}{400}$ μF. The resonant frequency is

(A) 2×10^4 Hz (B) $\frac{1}{\pi}\times10^4$ Hz
(C) 10^4 Hz (D) $2\pi\times10^4$ Hz

4. The current $i(t)$ through a 10 Ω resistor in series with an inductance, is given by $i(t) = 3 + 4\sin(100t + 45°) + 4\sin(300t + 60°)$ Amperes

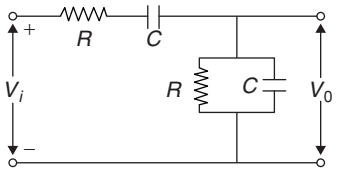
The RMS value of the current and the power dissipated in the circuit are

(A) $\sqrt{41}$ A, 410 W, respectively
(B) $\sqrt{35}$ A, 350 W, respectively
(C) 5 A, 250 W, respectively
(D) 11 A, 1210 W, respectively

5. Two 2 H inductance coils are connected in series and are also magnetically coupled to each other, the co-efficient of coupling being 0.1. The total inductance of the combination can be

(A) 0.4 H (B) 3.2 H
(C) 4.0 H (D) 3.6 H

6. The R-C circuit shown in the figure is

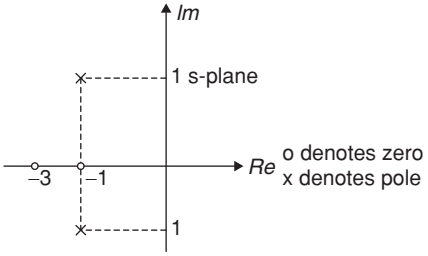


(A) A low-pass filter (B) A high-pass filter
(C) A band-pass filter (D) A band-reject filter

7. A certain series resonant circuit has a band width of 1000 Hz. If the existing coil is replaced by a coil with lower Q, what happens to the bandwidth

(A) It increases (B) It is zero
(C) It decreases (D) It remains the same

8. The driving point impedance $Z(s)$ of a network has the pole-zero locations as shown in the figure If $Z(0) = 3$, then $Z(s)$ is



(A) $\frac{3(s+3)}{s^2+2s+3}$ (B) $\frac{2(s+3)}{s^2+2s+2}$
(C) $\frac{3(s-3)}{s^2-2s-2}$ (D) $\frac{2(s-3)}{s^2-2s-3}$

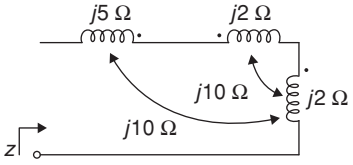
9. Consider the following statements S_1 and S_2

S_1: At the resonant frequency the impedance of series R-L-C circuit is zero.

S_2: In a parallel G-L-C circuit, increasing the conductance G results in increase in its Q-factor. Which one of the following is correct?

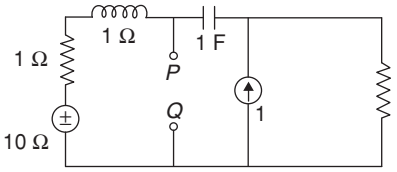
(A) S_1 is FALSE and S_2 is TRUE
(B) Both S_1 and S_2 are TRUE
(C) S_1 is TRUE and S_2 is FALSE
(D) Both S_1 and S_2 are FALSE

10. Impedance Z as shown in the given figure is



(A) $j29$ Ω (B) $j9$ Ω
(C) $j19$ Ω (D) $j39$ Ω

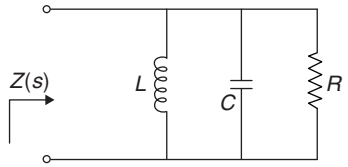
11. The Thevenin equivalent impedance Z_{th} between the nodes P and Q in the following circuit is



(A) 1 (B) $1+s+\frac{1}{s}$

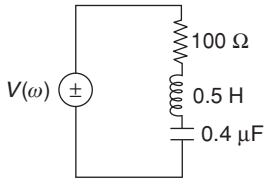
(C) $2+s+\frac{1}{s}$ (D) $\frac{s^2+s+1}{s^2+2s+1}$

12. The driving point impedance of the following network is given by $Z(s)=\frac{0.2s}{s^2+0.1s+2}$. The component values are



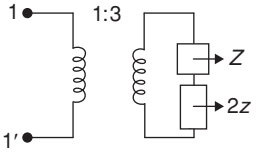
(A) $L = 5$ H, $R = 0.5\ \Omega$, $C = 0.1$ F
(B) $L = 0.1$ H, $R = 0.5\ \Omega$, $C = 5$ F
(C) $L = 5$ H, $R = 2\ \Omega$, $C = 0.1$ F
(D) $L = 0.1$ H, $R = 2\ \Omega$, $C = 5$ F

13. The resonant frequency for the series R-L-C circuit shown in the figure is



(A) 550 Hz (B) 670 Hz
(C) 1100 Hz (D) 355 Hz

14. If an ideal centre tapped 1:3 transformer is loaded as shown in the figure, the impedance measured across the terminals 11′ would be_____.



(A) $\frac{2}{3}Z$ (B) Z

(C) $\frac{Z}{9}$ (D) $\frac{Z}{3}$

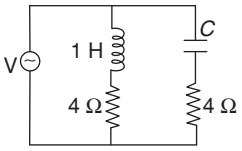
15. What is the ratio of the heating effects of two current waves of equal peak value, one being sinusoidal and the other rectangular in wave form?

(A) 1 : 2 (B) $1:\sqrt{2}$
(C) $\sqrt{2}:1$ (D) 2 : 1

16. The form factor of a half wave and a full wave rectified sine wave are _____.

(A) 1.11, 1.57 (B) 1.57, 1.11
(C) 1.414, 1.11 (D) 1.11, 1.414

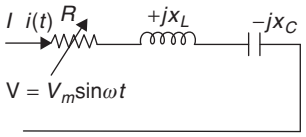
17. The value of capacitance C in the given AC circuit, to make it a pure resistive circuit and for the supply current to be independent of its frequency is _____.



(A) $\frac{1}{12}$ F (B) $\frac{1}{8}$ F

(C) $\frac{1}{4}$ F (D) $\frac{1}{16}$ F

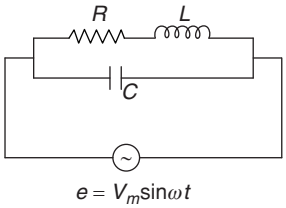
18.



In the circuit shown in the figure. if R is varied from 0 to ∞ the locus of the tip of the current phasor is

(A) Circle (B) Semi circle
(C) Exponential curve (D) Sine curve

19.

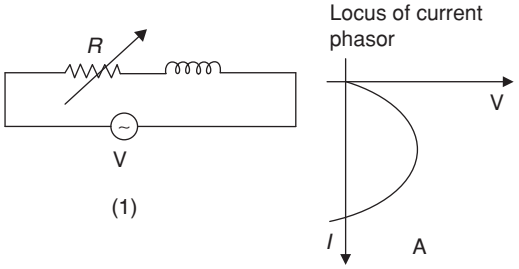


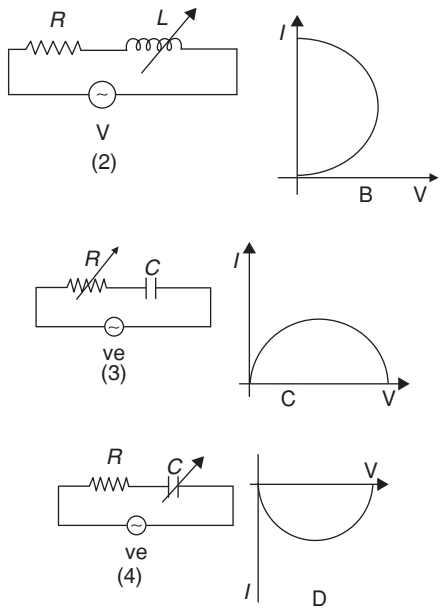
The dynamic impedance of the above circuit at resonance is

(A) $\frac{R}{LC}$ (B) $\frac{1}{R}\sqrt{\frac{L}{C}}$

(C) $\frac{L}{RC}$ (D) $\frac{C}{RL}$

20. Match the following:





(A) 1 – D, 2 – A, 3 – B, 4 – C
(B) 1 – A, 2 – D, 3 – B, 4 – C
(C) 1 – B, 2 – C, 3 – A, 4 – D
(D) 1 – C, 2 – B, 3 – D, 4 – A

Common Data for Questions 21 and 22: A low pass π section filter consists of an inductance of 25 mH in the series arm and two capacitors 0.2 uF in the shunt arms.

21. The cut-off frequency is _____.

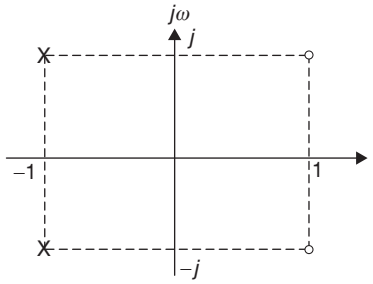
(A) $\frac{10^4}{\pi}$ (B) $\frac{1}{10^4\pi}$

(C) $\frac{5000}{\pi}$ (D) $\frac{1}{5000\pi}$

22. Design impedance is _____.

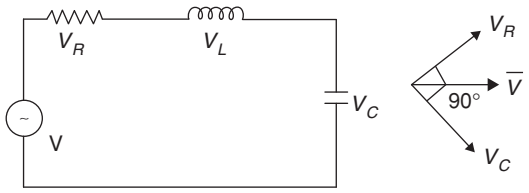
(A) 25 Ω (B) 250 Ω
(C) 15 Ω (D) 2.5 Ω

23. An impedance has the pole-zero patterns shown in figure. It must be composed of



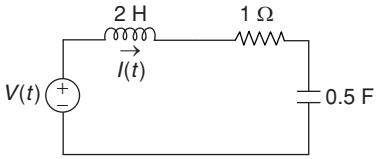
(A) RLC elements (B) RL elements only
(C) RC elements only (D) LC elements only

24. For the series R-L-C circuit of Figure I shown below, phasor diagram (involving some phasors) is shown in Figure II. The operating frequency of the circuit is



(A) Equal to resonant frequency
(B) Less than resonant frequency
(C) Twice the resonant frequency
(D) Greater than the resonant frequency

25. Which one of the following represents the state equation of the given R-L-C series circuit?



(A) $\begin{bmatrix}\dot{\phi}\\ q'c\end{bmatrix}=\begin{bmatrix}-2 & -0.5\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}\phi\\ qc\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}v(t)$

(B) $\begin{bmatrix}\dot{\phi}\\ \dot{q}_c\end{bmatrix}=\begin{bmatrix}-2 & 2\\ -05 & 0.5\end{bmatrix}\begin{bmatrix}\phi\\ qc\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v(t)$

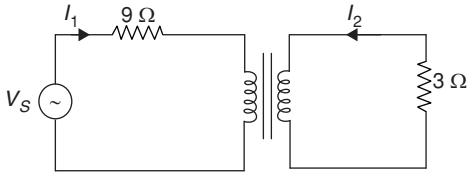
(C) $\begin{bmatrix}\dot{\phi}\\ \dot{q}_c\end{bmatrix}=\begin{bmatrix}-0.5 & -2\\ 0.5 & 0\end{bmatrix}\begin{bmatrix}\phi\\ qc\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}v(t)$

(D) $\begin{bmatrix}\dot{\phi}\\ \dot{q}_c\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0.5 & 2\end{bmatrix}\begin{bmatrix}\phi\\ qc\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}v(t)$

26. When two inductances are connected in series aiding the equivalent inductance of 14 H and in opposition is 6 H. Find out co-efficient of coupling '*K*'

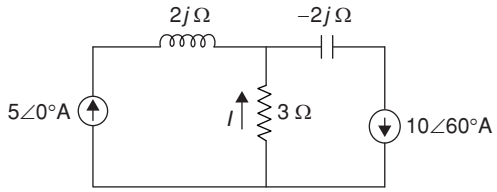
(A) 0.408 (B) 0.534
(C) 0.628 (D) 0.707

27. If the secondary winding of the ideal transformer shown in the circuit of figure has 50 turns, the number of turns in the primary winding for maximum power transfer to the 3 Ω resister will be



(A) 80 (B) 87
(C) 90 (D) 100

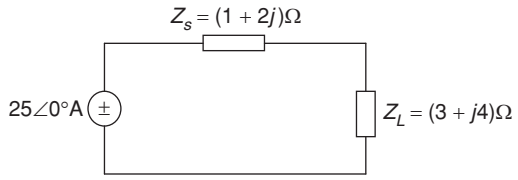
28. For the circuit in figure the instantaneous current $i(t)$ is



(A) $\frac{10\sqrt{3}}{2}\angle 90°\text{A}$ (B) $5\angle -60°\text{A}$

(C) $5\angle 60°\text{A}$ (D) $\frac{10\sqrt{3}}{2}\angle -90°\text{A}$

29. An AC sources of RMS voltage 25 V with internal impedance $Z_s = (1 + 2j)\Omega$ feeds a load of impedance $Z_L = (3 + j4)\Omega$, in the figure below. The reactive power consumed by the load is



(A) 60 VAR (B) 70 VAR

(C) 62.5 VAR (D) 78 VAR

30. The n/w function $\frac{(s+1)(s+4)}{s(s+2)(s+6)}$ is a

(A) RL impedance function

(B) RC impedance function

(C) LC impedance function

(D) All the above

Practice Problems 2

Directions for questions 1 to 28: Select correct alternative from the given choices.

1. For parallel RLC circuit, which one of the following statements is not correct?

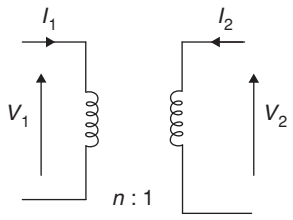
(A) The bandwidth of the circuit decreases if R is increased

(B) The bandwidth of the circuit remains same if 'L' is increased

(C) At resonance, input impedance is a real quantity

(D) At resonance, the magnitude of the input impedance attains its minimum value

2. The ABCD parameters of an ideal n : 1 transformer shown in the figure are $\begin{bmatrix} n & o \\ o & x \end{bmatrix}$. The value of x will be



(A) n (B) $\frac{1}{n}$

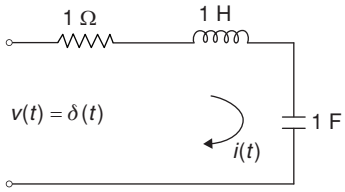
(C) n^2 (D) $\frac{1}{n^2}$

3. What is the total reactance of a series RLC circuit at resonance

(A) Equal to X_L (B) Equal to X_C

(C) Equal to R (D) Zero

4.

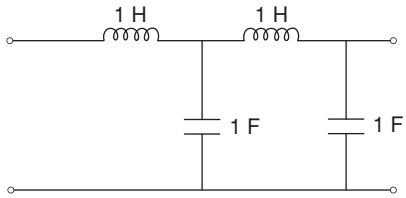


The circuit shown in the figure is initially relaxed. The Laplace transform of the current $i(t)$ is

(A) $\frac{s+1}{s^2+2s+1}$ (B) $\frac{s+1}{s^2+s+1}$

(C) $\frac{s}{s^2+s+1}$ (D) $\frac{s}{s^2+2s+1}$

5. Driving point impedance of the network shown in the figure is



(A) $\frac{s^4+3s^2+1}{s^3+2s}$ (B) $\frac{s^2+1}{s(s^2+2)}$

(C) $\frac{s^4+3s^3+2s^2+1}{s^3+2s}$ (D) $\frac{s^2+1}{s+1}$

6. The two windings of a transformer have an inductance of 3 H each. If the mutual inductance between them is also 3 H, then

(A) Transformation ratio is 3.

(B) It is an ideal transformer.

(C) It is a perfect transformer as the coefficient of coupling is 1.

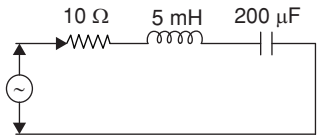
(D) None of the above

7. The complex power in a single phase AC circuit is given by _____.
(A) VI (B) V^*I
(C) VI^* (D) $V\ ^*I^*$

8. In a series resonance circuit which of the following statements are true.
(i) *PF* is unity.
(ii) Voltage magnification takes place.
(iii) Current magnification takes place.
(iv) Resonant frequency depends on resistance.
(A) (i) and (ii) (B) (ii) and (iv)
(C) (i), (ii), and (iv) (D) (i) and (iii)

9. The condition for parallel resonance is
(A) Net reactance is zero
(B) Net susceptance is zero
(C) Net reactive power is zero
(D) Reactive component of net current is zero

10. In the circuit of the given figure, the magnitudes of V_L and V_c are equal at what value of supply frequency _____.



(A) $\frac{1}{15\pi}$ (B) $\frac{2}{15}\pi$
(C) $\frac{10^3}{2\pi}$ (D) 2π

11. If the voltage and current in an AC circuit are given by $v = 200 \sin(\omega t + 30)$ and $i = 10 \sin(\omega t - 60)$ then the p.f of circuit is
(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$
(C) 0 (D) $\frac{1}{\sqrt{2}}$

12. If $v = 200 \sin(2\ \omega t + 30)$, $i = 10 \sin(\omega t + 30)$ then the phase difference between v and i is
(A) 0 (B) 30
(C) 60° (D) None

13. In a series R-L-C circuit for frequencies less than the resonant frequency the circuit is _____.
(A) Inductive (B) Capacitive
(C) Resistive (D) None

14. In a parallel resonance circuit for frequencies greater than the resonant frequency the circuit is _____.
(A) Inductive (B) Capacitive
(C) Resistance (D) None

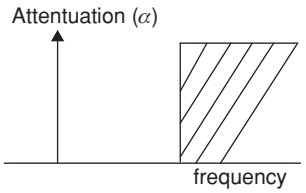
15. The Q of a circuit can be increased by _____.
(A) Increasing the BW
(B) Decreasing the BW
(C) Increasing the R
(D) None

16. In a R-L-C series circuit the bandwidth is increased by _____.
(A) Decreasing L (B) Decreasing C
(C) Increasing R (D) Decreasing R

17. In a R-L-C. series, resonant circuit at the half power points.
(A) The current is half of the current at resonance.
(B) The impedance is half of the impedance at resonance.
(C) The resistance is equal to the resultant reactance.
(D) None of the above

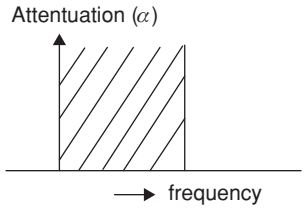
18. Which of the following statements are true
(i) The higher the value of Q, the more selective will be the circuit and lesser will be the Bandwidth.
(ii) Impedance of parallel resonant circuit is maximum at resonance.
(A) (i) and (ii) (B) (i) only
(C) (ii) only (D) Both are false

19. Match the following:

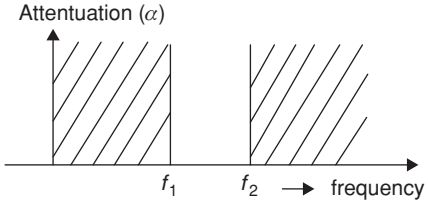


(1)

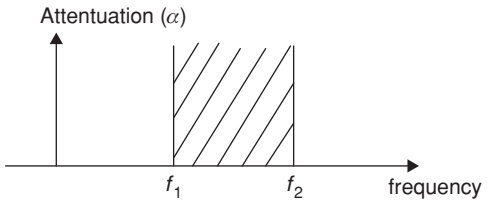
(A) Band pass (B) Low pass
(C) High pass (D) Band elimination



(2)



(3)



(4)

(A) 2 – B, 1 – C, 3 – D, 4 – A
(B) 1 – B, 2 – C, 3 – A, 4 – D
(C) 1 – A, 2 – D, 3 – B, 4 – C
(D) 1 – A, 2 – D, 3 – C, 4 – B

20. Given that

$f_1 \rightarrow$ lower cut-off frequency

$f_2 \rightarrow$ higher cut-off frequency

Then resonant frequency 'f_r', quality factor at resonance and selectivity are given by

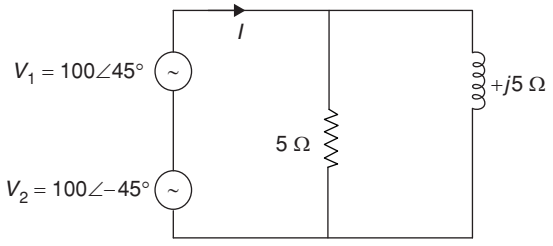
(A) $\sqrt{f_1 f_2}, \dfrac{f_2 - f_1}{f_r}, \dfrac{f_r}{f_2 - f_1}$

(B) $f_1 f_2, \dfrac{f_r}{f_2 - f_1} \dfrac{f_2 - f_1}{f_r},$

(C) $\sqrt{f_1 f_2}, \dfrac{f_r}{f_2 - f_1}, \dfrac{f_2 - f_1}{f_r}$

(D) $\dfrac{1}{\sqrt{f_1 f_2}}, \dfrac{f_r}{f_2 - f_1}, \dfrac{f_2 + f_1}{f_2 - f_1}$

21. The phase angle of the current I with respect to the voltage V_2 in the circuit shown in the figure is



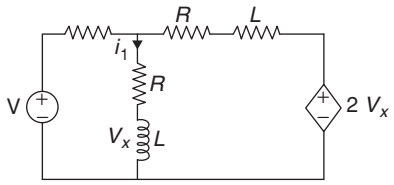
(A) 0° (B) –45°
(C) +45° (D) +90°

22. The conditions for defining driving point functions are.

(i) Response and excitation are applied to different terminals.
(ii) The network should not contain independent sources.
(iii) The network should be initially relaxed

(A) (i), (ii) and (iii) (B) (i) and (iii)
(C) (ii) and (iii) (D) (i) and (ii)

23. Find the state equation for the circuit given.



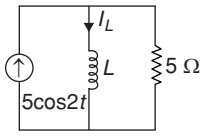
(A) $L\dfrac{di_2}{dt} = 0.7V_x - 1.5i_2R + .5$ V

(B) $L\dfrac{di_2}{dt} = -0.7V_x + 1.5i_2R - .5$ V

(C) $L\dfrac{di_2}{dt} = 0.7V_x - 1.5i_2R + .5$ V

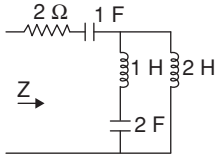
(D) $L\dfrac{di_2}{dt} = 0.7V_x - 1.5i_2R - .5$ V

24. What is the value of the inductance if current through it is given as $i(t) = 0.707 < -45°$. The input current is $5\cos 2t$.



(A) 2 H (B) 1 H
(C) 5 H (D) 3 H

25. Find the driving point admittance of the network given below.



(A) $\dfrac{3s^3 + 2s}{2s^4 + 6s^3 + 7s^2 + 4s + 2}$

(B) $\dfrac{3s^2 + 2}{2s^4 + 5s^2 + 4s + 2}$

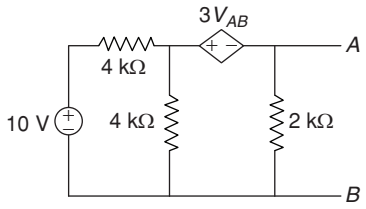
(C) $\dfrac{(3s^2 + 2)s}{2s^4 + 7s^3 + 6s^2 + 4s}$

(D) $\dfrac{5s^3 + 2s}{s^4 + 7s^3 + 6s^2 + 4s + 2}$

26. An unknown impedance Z is connected across a voltage source $V = \sqrt{50}\cos(\omega t + 50)$. The current flowing through the circuit is $\sqrt{18}\cos(10t + 75)$. What is the value of Z.

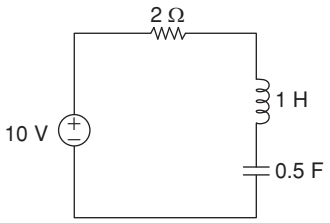
(A) 1.18 Ω Resistor in series with 84.7 mH inductor.
(B) 1.18 Ω Resistor in parallel with 84.7 mH inductor.
(C) 1.18 Ω Resistor in series with 84.7 mF Capacitor.
(D) 1.18 Ω resistor in parallel with 84.7 mF capacitor.

27. Find the Thevenins resistance associated with the circuit.



(A) 1 kΩ (B) 0.45 kΩ
(C) 2 kΩ (D) 0.22 kΩ

28. Find the current $i(t)$ through the circuit given



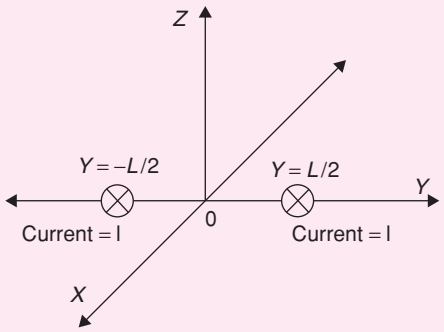
(A) $10e^{-t}\cos t$ (B) $\frac{10}{\sqrt{2}}e^{t}\cos\sqrt{2}t$

(C) $10e^{-t}\sin t$ (D) $\frac{10}{\sqrt{2}}e^{t}\sin\sqrt{2}t$

Previous Years' Questions

1. Two identical coupled inductors are connected in series. The measured inductances for the two possible series connections are 380 μH and 240 μH. Their mutual inductance in μH is _____. **[2014]**

2. A steady current I is flowing in the $-x$ direction through each of two infinitely long wires at $y = \pm\frac{L}{2}$ as shown in the figure. The permeability of the medium is μ_0. The $\vec{B}$-field at (0, L, 0) is **[2015]**



(A) $-\frac{4\mu_0 t}{3\pi L}\hat{Z}$ (B) $+\frac{4\mu_0 I}{3\pi L}\hat{Z}$

(C) 0 (D) $-\frac{3\mu_0 I}{4\pi L}\hat{Z}$

3. The self inductance of the primary winding of a single phase, 50 Hz, transformer is 800 mH, and that of the secondary winding is 600 mH. The mutual inductance between these two windings is 480 mH. The secondary winding of this transformer is short circuited and the primary winding is connected to a 50 Hz, single phase, sinusoidal voltage source. The current flowing in both the windings is less than their respective rated currents. The resistance of both windings can be neglected. In this condition, what is the effective inductance (in mH) seen by the source? **[2015]**

(A) 416 (B) 440
(C) 200 (D) 920

4. A symmetrical square wave of 50% duty cycle has amplitude of ±15 V and time period of 0.4π ms. This square wave is applied across a series RLC circuit with $R = 5\ \Omega$, $L = 10$ mH, and $C = 4$ μF. The amplitude of the 5000 rad/s component of the capacitor voltage (in Volt) is _____. **[2015]**

5. Two identical coils each having inductance L are placed together on the same core. If an overall inductance of αL is obtained by interconnecting these two coils, the minimum value of α is _____. **[2015]**

6. The flux linkage (λ) and current (i) relation for an electromagnetic system is λ = $\left(\sqrt{i}\right)$ /g. when i = 2A and g(air - gap length) = 10 cm, the magnitude of mechanical force on the moving part, in N, is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** C	**3.** B	**4.** C	**5.** D	**6.** C	**7.** A	**8.** B	**9.** D	**10.** B
11. A	**12.** D	**13.** D	**14.** D	**15.** A	**16.** B	**17.** D	**18.** B	**19.** A	**20.** B
21. A	**22.** B	**23.** D	**24.** B	**25.** C	**26.** C	**27.** B	**28.** A	**29.** C	**30.** B

Practice Problems 2

1. D	**2.** B	**3.** D	**4.** C	**5.** A	**6.** C	**7.** C	**8.** A	**9.** B	**10.** C
11. C	**12.** D	**13.** B	**14.** B	**15.** B	**16.** A	**17.** C	**18.** A	**19.** B	**20.** C
21. A	**22.** C	**23.** A	**24.** C	**25.** A	**26.** C	**27.** D	**28.** C		

Previous Years' Questions

1. 35	**2.** A	**3.** A	**4.** 190 to 192 V	**5.** 0	**6.** 188

Chapter 6

Filters and Field Theory

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Ideal filters
- Filter fundamentals
- Low pass filter
- High pass filer
- Circuit parameters
- Band pass filter
- Band elimination filter
- Poly phase circuits
- Inter connection of three phases
- Delta or mesh connection
- Power in delta connected system
- Star to delta and delta to star transformation
- Electrical fields
- Gradient, divergence and curl in all the co-ordinate system
- Electric field of continuous space distribution of charges
- Gauss's law
- Dielectrics inductance and capacitance
- Magneto statics
- Magnetic field on the axis of a circular coil
- Induced EMF
- Maxwells equations

Basic Filter Concepts

A network which freely passes a desired band of frequency while suppressing other band of frequency is called a filter.

It has the ability to discriminate between signals which differ in frequency.

Classification of Filters

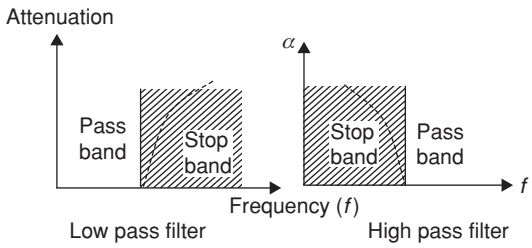
1. **Active filters:** These filters consist of active elements such as transistor, op-amp along with resistors, capaitors and inductors as active elements. In these filters voltage, current and power gain is possible. But the main disadvantage of this filter is that they require additional power supplies for the operation of active elements.
2. **Passive filters:** These filters use only passive elements like resistors, capacitors and inductors. In passive filters, none of the gain is possible. As inductors are very heavy and bulky, they are very costly. They don't require any additional power supplies.

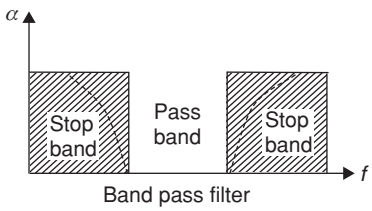
Basic Filter Section

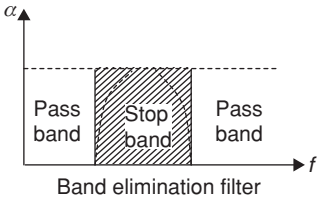
1. **Low pass filter:** It passes the frequency upto the cut off frequency and attenuates all frequencies above it.
2. **High pass filter:** It attenuates all frequencies upto cut off frequency and passes all frequencies above it.
3. **Band pass filter:** This type of filter passes all frequencies between two cut off frequencies and attenuates all other frequencies.
4. **Band elimination filter:** This type of filter attenuates all frequencies between two cut off frequencies and passes all other frequencies.

Ideal Filters

An ideal filter should have zero attenuation in the pass band and infinite attenuation in the top band. But practically in stop band attenuation gradually changes. Dark lines represent ideal characteristics and dotted lines show practical characteristics.







Filter Fundamentals

For T Network

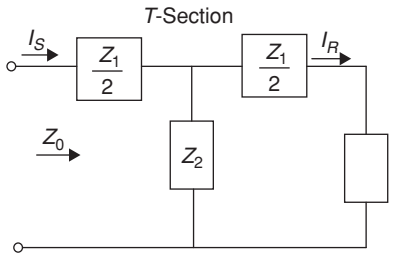
1. Characteristic impedence

$$Z_{\sigma T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

If we consider all elements of $Z_{OT} = j\sqrt{\frac{X_1^2}{4} + X_1 X_2}$ T network are reactive.

2. Propagation constant (γ)

$$\cos h\,\gamma = 1 + \frac{Z_1}{2Z_2}$$



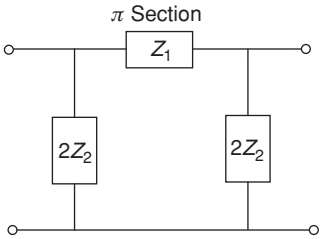
For π Network

1. Characteristic impedance

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}}$$

2. Propagation constant (γ)

$$\cosh\gamma = 1 + \frac{Z_1}{2Z_2}$$



Cut-off Frequency

$$Z_0 = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

If $Z_1 = jX_1$ and $Z_2 = jX_2$

$$Z_0 = \sqrt{\frac{-X_1^2}{4} - X_1 X_2} = \sqrt{-X_1\left(\frac{X_1}{4} + X_2\right)}$$

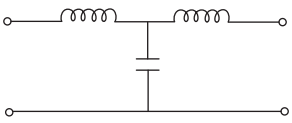
If X_1 and $\left(\frac{X_1}{4} + X_2\right)$ have same sign Z_0 is imaginary and gives STOP BAND.

If X_1 and $\left(\frac{X_1}{4} + X_2\right)$ have oppositie sign Z_0 is real (purely resistive) and gives a PASS BAND.

CUT-off Frequency: It is the frequency at which Z_0 changes from real to imaginary.

$$\text{At } f_c : X_1\left(\frac{X_1}{4} + X_2\right) = 0 \quad \Rightarrow \quad X_1 = -4X_2$$

Low-pass Filter



Circuit Parameters

1. Total series impedance $Z_1 = j\omega L$
2. Total shunt impedance $Z_2 = \frac{j}{\omega C}$

$$Z_1 Z_2 = (j\omega L)\left(\frac{-j}{\omega C}\right) = \frac{L}{C}$$

$$R_0^2 = \frac{L}{C} \quad \Rightarrow \quad R_0 = \sqrt{\frac{L}{C}}.$$

3. Cut-off frequency

$$Z_1 = j\omega L \quad X_1 = \omega L$$

$$Z_2 = \frac{-j}{\omega C} \quad X_2 = \frac{-1}{\omega C}$$

$$\therefore \quad \left(\frac{X_1}{4} + X_2\right) = j\left(\frac{\omega L}{4} - \frac{1}{\omega C}\right)$$

Point A marks cut off frequency at which $\omega = \omega_C$

Hence $\frac{\omega_c L}{4} - \frac{1}{\omega_c C} = 0$

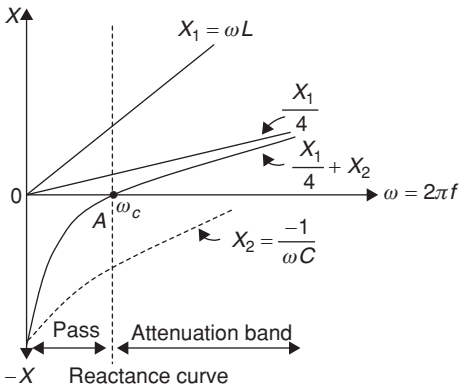
$$\omega_c^2 = \frac{4}{LC}$$

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

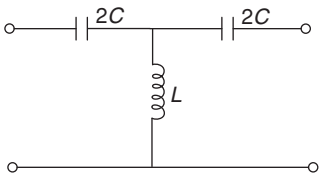
4. Characteristic impedance

$$Z_{OT} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{\frac{-\omega^2 L^2}{4} + \frac{L}{C}}$$

$$Z_{OT} = R_0\sqrt{1 - \frac{\omega^2 LC}{4}}$$



High-pass Filter



T-section prototype high-pass filter

Circuit Parameters

1. Total series arm impedance $Z_1 = \frac{-j}{\omega C}$
2. Total shunt arm impedance $Z_2 = j\omega L$

$$Z_1 Z_2 = \left(-\frac{j}{\omega C}\right)(j\omega L) = \frac{L}{C}$$

$$R_0^2 = \frac{L}{C} \quad \Rightarrow \quad R_0 = \sqrt{\frac{L}{C}}$$

3. Reactance curve and cut-off frequency

$$Z_1 = \frac{-j}{\omega C} X_1 = \frac{-1}{\omega C}$$

$$Z_2 = j\omega L, X_2 = \omega L$$

$$\frac{X_1}{4} + X_2 = \frac{-1}{4\omega C} + \omega L = \omega L - \frac{1}{4\omega C}$$

At point B the curve $\left(\frac{X_1}{4} + X_2\right)$ crosses the frequency axis.

$$\omega_c L - \frac{1}{4\omega_c C} = 0$$

$$\omega_c^2 = \frac{1}{4LC}$$

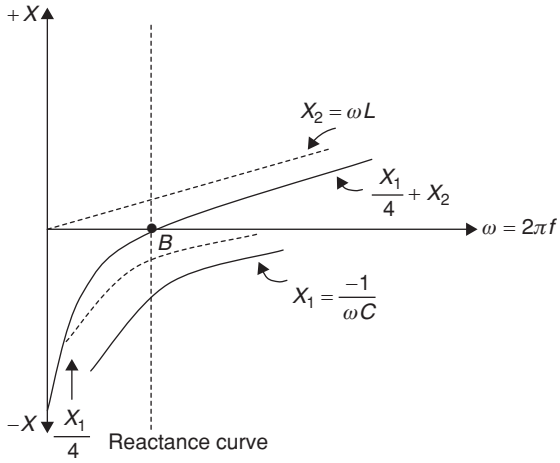
$$f_c = \frac{1}{2\sqrt{LC}}$$

4. Characteristic impedance

$$Z_{OT} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{\frac{-1}{4\omega^2 C^2} + \frac{L}{C}}$$

$$= \sqrt{\frac{L}{C}}\sqrt{1 - \frac{1}{4\omega^2 LC}}$$

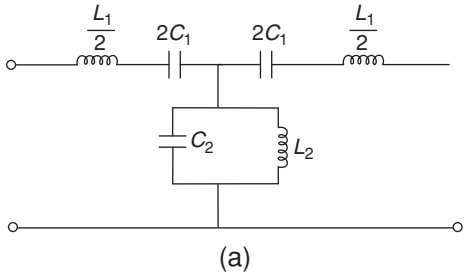
$$Z_{OT} = R_o \sqrt{1 - \frac{1}{4\omega^2 LC}}$$



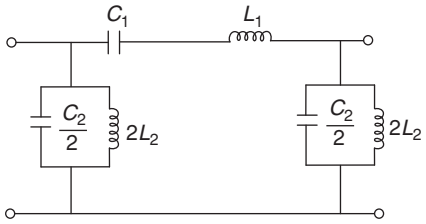
Reactance Curve

All frequencies below $B \rightarrow$ stop Band
Above $B \rightarrow$ Pass band
At point $B \rightarrow$ cut off frequency

Band Pass Filter



(a)



(b)

Circuit Parameters

1. Series arm impedance $Z_1 = j\omega L_1 - \frac{j}{\omega C_1}$

$$= j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)$$

2. Shunt arm impedence Z_2

$$= \frac{j\omega L_2 \cdot \frac{1}{j\omega C_2}}{j\omega L_2 + \frac{1}{j\omega C_2}} = \frac{j\omega L_2}{1-\omega^2 L_2 C_2}$$

$$Z_1 Z_2 = \frac{L_2}{C_1}$$

$$= \frac{L_1}{C_2} = K^2$$

3. Condition for resonance and cut-off frequency for series arm $\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1} \omega_0^2 L_1 C_1 = 1$ for shunt arm

$$\frac{1}{\omega_0 C_2} = \omega_0 L_2 \omega_0^2 L_2 C_2 = 1$$

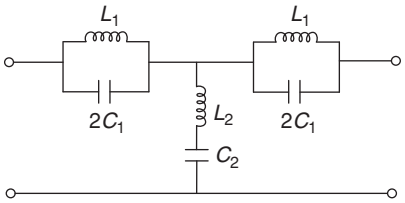
$$L_1 C_1 = L_2 C_2$$

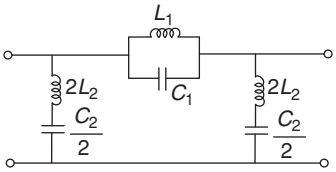
Cut-off frequency $f_0 = \sqrt{f_1 f_2}$.

4. Circuit components

$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2}$	$C_2 = \frac{L_1}{k^2} = \frac{1}{\pi(f_2 - f_1)k}$
$L_1 = \frac{K}{\pi(f_2 - f_1)}$	$L_2 = C_1 K^2 = \frac{(f_2 - f_1)k}{4\pi f_1 f_2}$

Band Elimination Filter





Condition for Equal Resonance Frequencies

For series arm $\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1} \Rightarrow \omega_0^2 = \frac{1}{L_1 C_1}$

For shunt arm $\omega_0 L_2 = \frac{1}{\omega_0 C_2} \Rightarrow \omega_0^2 = \frac{1}{L_2 C_2}$

$$\frac{1}{L_1 C_1} = \frac{1}{L_2 C_2} = K$$

1. Cut-off frequency $f_0 = \sqrt{f_1 f_2}$
2. Circuit components

$$C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi k(f_2 - f_1)}$$

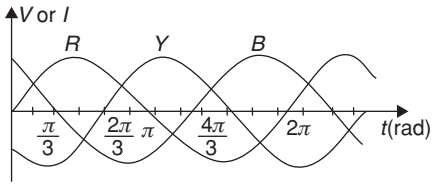
$$L_1 = k^2 C_2 = \frac{K}{\pi}\left(\frac{f_2 - f_1}{f_1 f_2}\right)$$

$$C_2 = \frac{1}{k\pi}\left|\frac{f_2 - f_1}{f_1 f_2}\right|$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)}$$

Poly Phase Circuits

A three phase system of voltages (currents) is a combination of three single phase system of voltages (currents) of which the three voltages differ in phase by 120 electrical degrees from each other in a particular sequence.



Advantages of Three Phase System

1. Power in 1ϕ system is pulsating. If *pf* is unity power pulsates at a frequency of 100 Hz. The total three phase power supplied to a balanced three phase circuit is constant at every instant of time.
2. Three phase transmission circuit requires less conductor material than a single phase circuit.
3. In a given frame size, a three phase motor or generator produces more output than the single phase counter part.

Phase Sequence

If the three phases are named *R*, *Y*, *B* then the order in which there phase attain their maximum values is the phase sequence of the system. In the system given, the phase sequence is *RYB*. The sequence depends on the rotation of the field in the generator.

Note that the sequence *RYB*, *BRY* and *YBR* are the same.

If the direction of rotation of field is changed, the phase sequence changes. The other phase sequence is *RBY*, *BYR* or *YRB*.

In phase sequence *RYB*

$V_R = V_m \sin \omega t$

$V_Y = V_m \sin(\omega t - 120°)$

$V_B = V_m \sin(\omega t - 240°)$

Note: The sum of three vectors equal is magnitude but displaced from each other by 120° is zero.

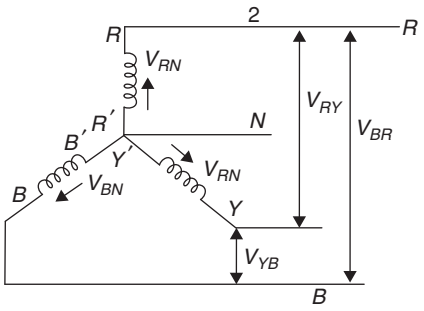
So in a balanced three phase system, sum of phase as well as line quantities is zero.

Inter Connection of Three Phases

The coils in a three phase system may be connected to form a wye (Y) or delta (Δ) system to achieve economy and to reduce the number of conductors.

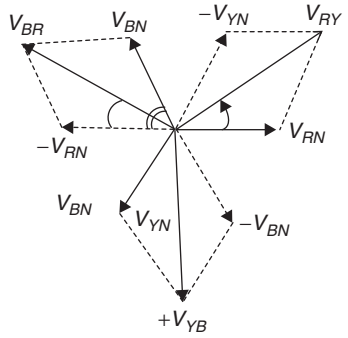
Wye or Star Connection

In this connection similar ends (start or finish) of the three phases are joined together. The common terminal is called is the neutral. The terminals R, Y, B are called the line terminals. The voltage between any line and the neutral point is called phase voltage (V_{RN}, V_{YN}, V_{BN}), while voltage between any two lines is called line voltage (V_{RY}, V_{YB}, V_{BR}).



The double subscript is purposefully used to represent currents and voltage is a 3ϕ system.

Voltage relation in Y connected circuit



$$V_{RY} = V_{RN} - V_{YN} = V_{RN} + (-V_{YN})$$
$$V_{YB} = V_{YN} - V_{BN} = V_{YN} + (-V_{BN})$$
$$V_{BR} = V_{BN} - V_{RN} = V_{BN} + (-V_{RN})$$

Since $|V_{RN}| = |-V_{YN}| = V_{PL}$

$$V_{RY} = 2V_{Ph}\cos\left(\frac{60}{2}\right) = \sqrt{3}\,V_{Ph}$$

$$\therefore \quad |V_{RY}| = |V_{YB}| = |V_{BR}| = \sqrt{3}\,V_{Ph}$$

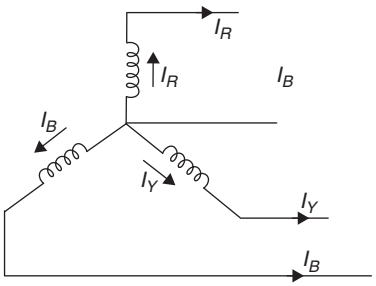
In a star connected circuit

1. Line voltage $= \sqrt{3} \times$ phase voltage
2. All line voltages are equal in magnitude but displaced in phase by 120°.
3. All line voltages are 30° ahead of their respective phase voltages.

Current Relation

All the phase currents are displaced by 120° from each other. For a balanced load, all the phase currents are equal in magnitude. Each live conductor is connected in series with its individual phase winding. So current in a line conductor is the same as that is phase to which the line conductor is connected.

$$I_L = I_{ph} = I_R = I_Y = I_B$$



If load is lagging, the angle between line (phase) current and line voltage is $(30 + \phi)°$ and it is $(30 - \phi)$ if the load is leading.

Power in a star connected network

Total power in the three phase = 3 × power each phase

$$V_L = \sqrt{3}V_{ph}I_L = I_{ph}$$

$$P = \sqrt{3}V_L I_L \cos\phi \text{ W}$$

Total reactive power in the circuit is

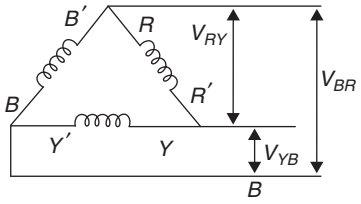
$$Q = \sqrt{3}V_L I_L \sin\phi \text{ VAR}$$

Apparent power (or) Active Power

$$S = \sqrt{3}V_L I_L$$

Delta or Mesh Connection

Here, the dissimilar ends are joined together as in figure. Here there is no common terminal only there line voltages V_{RY}, V_{YB}, and V_{BR} are available. These voltages are displaced 120° from each other.



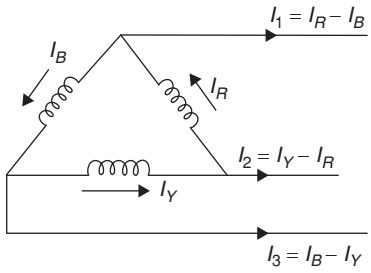
Voltage relation in Delta circuit

Voltage between two lines $V_L = V_{ph}$ the phase voltage.

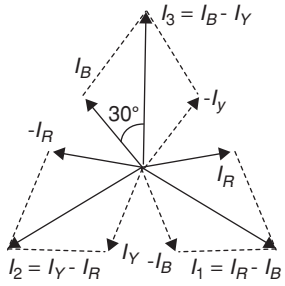
$$\therefore \quad V_{RY} = V_L = V_{Ph} = V_{YB} = V_{BR}$$

Since the system is balanced, all phase voltages are equal but displaced by 120° from one another.

Current Relation



$$I_1 = I_R - I_B = \sqrt{3} I_{ph}$$



The line currents are 30° behind the respective phase currents.

Power in delta connected system

The power per phase = $V_{Ph}\, I_{ph} \cos \phi$ is the phase angle between phase voltage and phase current.

Total power $P = 3 \times V_{ph}\, I_{ph} \cos \phi$

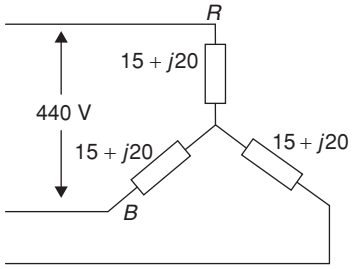
In terms of line quantities

$$P = \sqrt{3} V_L I_L \cos\phi$$

$$V_{ph} = V_L \text{ and } I_{ph} \frac{I_L}{\sqrt{3}}$$

Solved Examples

Example 1: A balanced star connected load having impedance (15 + 20 *j*) Ω per phase is connected to a three phases 440 V, 50 Hz supply. Find the line currents and power aborted by the load.



Solution: $V_{RN} = \dfrac{440\angle 0°}{\sqrt{3}} = 254\angle o°$

$$V_{YN} = 254 \angle -120°$$
$$V_{BN} = 254 \angle -240°$$

Impedance per phase

$$Z_{ph} = 15 + j\,20 = 25\angle 53.13°$$

The phase currents are

$$I_R = \frac{V_{RN}}{Z_{ph}} = \frac{254\angle o}{25\angle 53.13°}$$

$$= 10.16 \angle -53.13° \text{ A}$$
$$I_Y = 10.16 \angle -\text{A}3.13° \text{ A}$$
$$I_B = 10.16 \angle -293.13° \text{ A}$$

Since the load is star connected, these current also represent the line quantities.

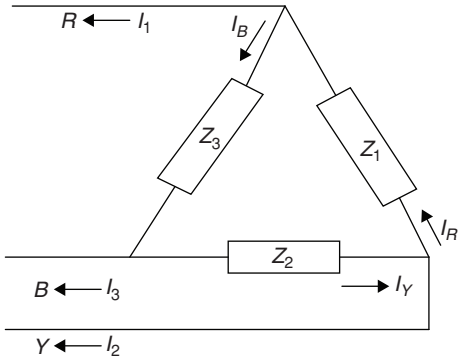
$$S = 3 \text{ VI} = 3 \times 254 \times 10.16 \angle 53.13°$$
$$= 7741.92\angle 53.13°$$
$$= 4645 + j\ 61\ 93$$
$$\therefore \quad P = Re\,(S) = 4645 \text{ W}$$

Example 2: Three impedances $Z_1 = 20\ \angle 30°\ \Omega$, $Z_2 = 40\ \angle 60°\ \Omega$ and $Z_3 = 10\ \angle 90°\ \Omega$ are delta connected to a 400 V. 3ϕ system as shown in figure. Determine the

(i) Phase currents

(ii) Line currents and

Total power consumed by the load.



Solution: Let the phase currents be I_R, I_Y, I_B and the line currents are I_1, I_2, and I_3.

Let

$$V_{RY} = 400\ \angle 0°$$
$$V_{YB} = 400\ \angle -120°$$
$$V_{BR} = 400\ \angle -240°$$

$$I_R = \frac{V_{RY}}{Z_1} \frac{400\angle 0}{40\angle 30°} = 20\angle -30°\text{A}$$

$$I_Y = \frac{V_{YB}}{Z_2} = \frac{400\angle -120}{40\angle 60°} = 10\angle -180°\text{A}$$

$$I_B = \frac{V_{BR}}{Z_3} = \frac{400\angle -240}{10\angle -90} = 40\angle -150°\text{A}$$

The line currents are

$$I_1 = I_R - I_B = 20\angle -30° - 40\angle -150°$$
$$= 52.91\angle 10.89° \text{ A}$$
$$I_2 = I_Y - I_R = 20.09\angle 159.89 \text{ A}$$
$$I_3 = I_B - I_Y = 31.73\angle -140.94 \text{ A}$$

The total power is calculated by adding the powers in the individual phases [$R = Z\cos\phi$]

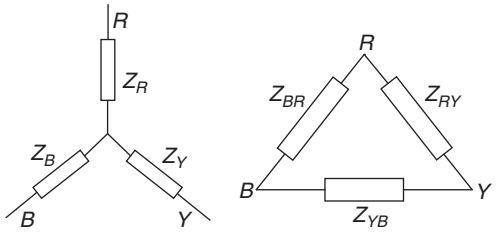
Power in R phase $= I_R^2 \times R_R = 20^2 \times 17.32 = 6928$ W

Power in Y phase $= I_Y^2 \times R_Y = 10^2 \times 20 = 2000$ W

Power in B phase $= I_B^2 \times R_B = 40^2 \times 0 = 0$ W

Total power in the load = 6928 + 2000 = 8928 W

Star to Delta and Delta to Star Transformation



Delta impedances in terms of star impedances are

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B} = Z_R + Z_Y + \frac{Z_R Z_Y}{Z_B}$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R} = Z_Y + Z_B + \frac{Z_y Z_B}{Z_R}$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y} = Z_B + Z_R + \frac{Z_B Z_R}{Z_Y}$$

Similarly we can replace the delta load of figure by equivalent star load

$$Z_R = \frac{Z_{RY} Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$Z_Y = \frac{Z_{RY} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

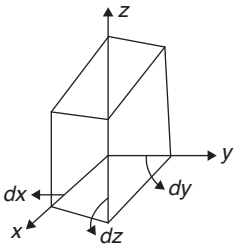
$$Z_B = \frac{Z_{BR} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}.$$

Electrical Fields

Co-ordinate System and Vector Calculus

Cartesian coordinate system (x, y, z)

Unit vectors are $\hat{a}_x$, $\hat{a}_j$, $\hat{a}_z$



Differential length

$$\vec{d\ell} = (dx)\hat{a}_x + (dy)\hat{a}_y + (dz)\hat{a}_z$$

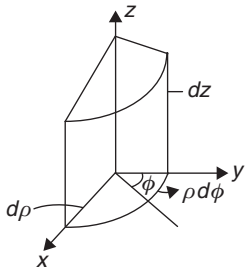
Differential surface

$$\vec{ds_1} = (dxdy)\hat{a}_z, \vec{ds_2} = (dydz)\hat{a}_x, \vec{ds_3} = (dxdz)\hat{a}_y$$

Differential volume: $dv = dx\,dy\,dz$

Cylindrical co-ordinate system (ρ, ϕ, z)

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = Z$$
$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$



Differential length

$$\vec{d\ell} = (d\rho)\hat{a}_\rho + (\rho\, d\phi)\hat{a}_\phi + (dz)\hat{a}_z$$

Differential surface

$$ds = (\rho\, d\phi\, dZ)\hat{a}_\rho + (d\rho\, dz)\hat{a}_\phi + (\rho\, d\rho\, d\phi)\hat{a}_z$$

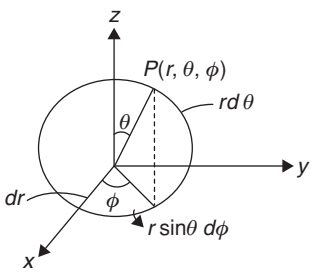
Differential volume

$$dv = \rho d\rho d\phi dz$$

Spherical coordinate system (r, θ, ϕ)

$$\vec{d\ell} = (dr)\hat{a}_r + (rd\theta)\hat{a}_\theta + (r\sin\theta\, d\phi)\hat{a}_\phi$$
$$\vec{ds} = (r^2\sin^2\theta\, d\theta\, d\phi)\hat{a}_r + (r\sin\theta\, drd\phi)\hat{a}_\theta + (rdrd\theta)\hat{a}_\phi$$

$$\vec{dv} = r^2\sin\theta\, d\theta\, dr d\phi$$



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$r = \sqrt{x^2+y^2+z^2}\,\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\frac{Z}{\sqrt{x^2+y^2+z^2}}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

The del operator (∇)

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\hat{k}$$

$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}\hat{k}$ where ϕ is a scalar quantity.

Gradient of a scalar field

Gradient is defined as the vector quantity whose magnitude is given by maximum rate of change of scalar quantity with respect to space variable and its direction is where maximum change occurs.

If ϕ is any scalar field, then its gradient is given by

Grad $\phi = \nabla\phi$

Divergence of a vector

Divergence of a vector quantity is a scalar quantity whose magnitude is equal to net outflow of flux from a closed surface when volume shrinks to zero.

If $\vec{f}$ is a vector function whose first partial derivatives exist, then

$$\text{div } f = \vec{\nabla}\cdot\vec{f} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

Curl of a vector

Curl of a vector is a vector quantity whose magnitude is given by maximum circulation per unit area when area tends to zero and its direction is normal to the surface.

If V is vector function who's first partial derivative exist, then

$$\text{Curl } V = \nabla\times V = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Gradient, Divergence and Curl in all the Co-ordinate System

Gradient

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_j + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_p + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\hat{a}_\phi$$

Divergence

$$\nabla\cdot\vec{V} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$$

$$\nabla\cdot\vec{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V\phi}{\partial\phi} + \frac{\partial V_z}{\partial_z}$$

$$\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r^2\sin\theta}\frac{\partial V\phi}{\partial\phi}$$

Curl

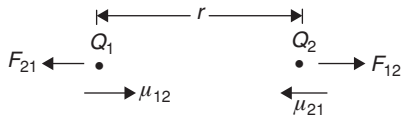
$$\nabla\times\vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\nabla\times\vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho v_\phi & V_z \end{vmatrix}$$

$$\nabla\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_\rho & r\hat{a}_\phi & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta V\phi \end{vmatrix}$$

Coulomb's Law

The force between two charged particles at rest is proportional to product of charges and inversely proportional to square of distance between them. The force acts along the line joining the mid point of the particles.



$F_{12} = \frac{1}{4\pi\varepsilon_o\varepsilon_r}\frac{Q_1Q_2}{r^2}\cdot U_{12}$ where U_{12} and U_{21} are unit vectors ε_r is relative permittivity of medium. ε_o is absolute permittivity = 8.854×10^{-12} F/n

Example 3: Calculate the distance of separation between two electrons in vacuum for which electric force between them is equal to gravitational force on one of them at earth surface.

Solution: Gravitational force on one electron = mg Newton

Electrostatic force in vaccum $= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$

$$\therefore \quad \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} = \text{mg} \quad \therefore \quad r = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\text{mg}}}$$

$$= 5.08 \text{ m}$$

Example 4: Find the force are Q_1 ($20\mu\ c$) due to Q_2 ($-300\ \mu c$) Q_1 is at (0, 1, 2) and Q_2 is at (2, 0, 0)

Solution: $F_2 = \frac{1}{4\pi\in_o}\frac{q_1 q2}{r^2}\hat{U}_2$

$$\hat{U}_{21} = \frac{-2\hat{i}+\hat{j}+2\hat{k}}{3} = \frac{-2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$F_{21} = \frac{1}{4\pi\in_o}\frac{20\times -300\times 10^{-12}}{3^2}\times\left(\frac{-2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$$

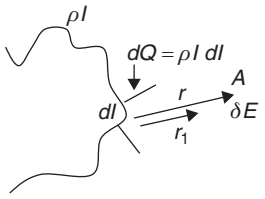
$$= -6\left(\frac{-2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right) = 4\hat{i}-2\hat{j}-4\hat{k}N.$$

Electric Field of Continuous Space Distribution of Charges

Electric field intensity $E = \left[\frac{Q}{4\pi\varepsilon_0 r^2}\right]\hat{r}_1$ Newtons/Coloumb

Line Distribution of Charges

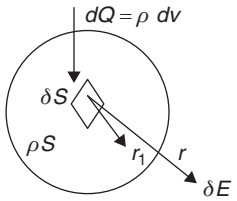
$$E = \frac{1}{4\pi\varepsilon_0}\left[\int_l\left\{\frac{\rho l dl}{r^2}\right\}\right]\hat{r}_1$$



$\rho_l \rightarrow$ charge per unit length

Surface Distribution of Charges

$$E = \frac{1}{4\pi\varepsilon_0}\left[\iint_s\left\{\frac{\rho s ds}{r^2}\right\}\right]\hat{r}_1$$

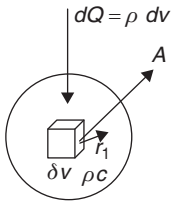


ρ_s = charge/unit area

Volume Distribution of Charges

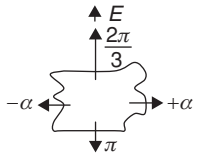
$$E = \frac{1}{4\pi\varepsilon_0}\left[\iiint_v\left\{\frac{\rho_c dv}{r^2}\right\}\right]\hat{r}_1$$

$$\rho_c = \frac{\text{charge}}{\text{unit volume}}.$$

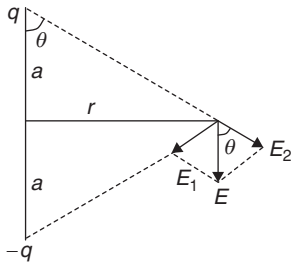


Infinite Plane Charge

$$\vec{E} = \frac{\rho s}{2E_0}\hat{a}_x$$



Electric Field Due to Dipole



$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

$$E_1 = E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{(a^2+r^2)}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+r^2}}$$

$$\therefore \quad \bar{E} = 2\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2+r^2}\cdot\frac{a}{\sqrt{a^2+r^2}}$$

$$\bar{E} = \frac{2aq}{4\pi\varepsilon_0(a^2+r^2)^{\frac{3}{2}}} = \frac{\bar{P}}{4\pi\varepsilon_0(a^2+r^2)^{\frac{3}{2}}}$$

For $r >> a$

$$E = \frac{2aq}{4\pi\varepsilon_0 r^3} = \frac{\bar{P}}{4\pi\varepsilon_0 r^3}$$

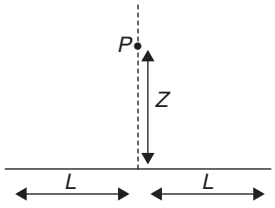
$$|E| \alpha \frac{1}{r^3}.$$

Electric Field at a point P on the axis of a circular loop of radius a, carrying charge density λ (c/m)

$$\vec{E} = \frac{qz}{4\pi\in_0}\cdot\frac{1}{(r^2+z^2)^{\frac{3}{2}}}\cdot\hat{a}_z \text{ at } z = 0\,|\,\vec{E}\,| = 0$$

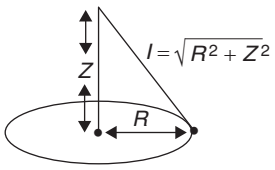
Electric field at point P from a line of charge of length $2L$ carrying a uniform charge λ

$$E = \frac{1}{2\pi\in_0}\frac{\lambda L}{z\sqrt{z^2+L^2}}$$



Electric Field at a distance z above the center of a flat disc of radius R carrying uniform surface charge σ.

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2 + Z^2}}\right)$$



Gauss's Law

The total electric flux enclosed by a surface surrounding charge is equal to the amount of charge enclosed.

$\int_s \bar{D} \cdot d\bar{s} = Q$ where Q is the total charge and D is the electric flux density.

$$\int_s \epsilon_0 \bar{E} \cdot d\bar{s} = Q$$

$$\therefore \int_s \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_v \rho dv$$

Where ρ is the charge/unit volume?

The total flux emanating from the surface is independent of

1. Shape of the closed surface
2. Position of charge within the surface

Electric Potential

Electric potential at any point in an electric field near a charged body is defined as the amount of work done in bringing a unit positive charge from α to that point against the electric field.

If a charge q is moved through a distance dl is a uniform electric field E.

Electric force on the charge $= q\bar{E}$

Work done $= \bar{F} \cdot \overline{dl} = -q\bar{E} \cdot d\bar{l}$

Work done in moving the test charge from ∞ to a point P

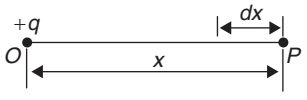
$$W = \int_\infty^p -q\bar{E} \cdot dl = -q\int_\infty^P E \cdot dl$$

Potential of point $P, V_p = \int_{-\alpha}^{\rho} E \cdot dl$

i.e., work done by unit positive charge.

Electric field E = grad$V$

Potential due to a point charge



Consider a point charge $+q$ situated at O and let a unit positive charge is placed at P, x meters from O.

Force of repulsion experienced by unit positive charge at P is $|F| = \frac{q}{4\pi\epsilon_0 x^2}$.

If this unit charge is moved by a short distance dx, work done in doing so against the force of repulsion is $dW = \frac{-q}{4\pi\epsilon_0 x^2} dx$ Joule.

If the point is at ∞, $W = \int_\infty^r \frac{-q}{4\pi\epsilon_0 x^2} dx$ Joules

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{x}\right]_\infty^r = \frac{q}{4\pi\epsilon_0 r}$$

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

Potential due to a number of charges

If charges $q_1, q_2, \ldots$ etc. are at distances $r_1, r_2, r_3, \ldots$ etc from the point at which potential is to be found, then the resultant potential at this point,

$$V = \frac{q_1}{4\pi\epsilon_0 r_1} + \frac{q_2}{4\pi\epsilon_0 r_2} + \frac{q_3}{4\pi\epsilon_0 r_3} + \cdots$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right)$$

$$V = \frac{1}{4\rho\varepsilon_0}\sum\frac{q}{p}$$

Potential due to a line charge distribution

When the charge is uniformly distributed along a line then potential at any point P due to this linear charge distribution of λ coulombs/meter is given by

$$V_p = \frac{1}{4\pi\epsilon_0}\int_l \frac{\lambda}{r} dl$$

Potential due to surface charge distribution

Potential at any point P due to surface charge distribution ρ C/sq m is

$$V_p = \frac{1}{4\pi\epsilon_0}\int_s \frac{\sigma}{r} ds$$

Potential due to volume charge distribution

Potential due to uniform volume charge distribution ρ coulombs per cubic meter is

$$V_p = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho}{r} dv$$

Dielectrics Inductance and Capacitance

Behavior of Conductors and Dielectrics in Electrostatic Fields

In conductors, there is abundance of conduction or free electrons which give rise to conduction current under the influence of applied electric field. However in dielectric substances, electrons are tightly bound to nuclei of atoms and so dielectrics do not conduct current through them. In dielectrics, charges are not able to move freely and they are bound by finite forces.

Polarisation

When a dielectric is placed in an electric field, the nucleus gets displaced in the direction of electric field and the electrons opposite to the field. In this way, a dipole is formed by a positive and negative charge with distance d separating them. This gives a dipole moment

$$P = qd$$

In each atom of the dielectric material, a tiny dipole moment is induced tending to align the atoms in the direction of electric field. The dielectric is said to be polarized and polarization P, is defined as the dipole moment per unit volume.

$$P \alpha \bar{E}$$

$\bar{E}$ = total field, ie field due to polarization and external field.

Flux density in dielectrics

$$D = \in_0 \bar{E} + P \frac{C}{m^2}$$

Inductance

It is defined as the property of the electric circuit by the virtue of which a varying current induces an emf in the neighbouring circuit or the property of a conductor or circuit is that establishes magnetic flux linkages.

Inductance of simple geometries

1. Coaxial conductor $L = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right)$

   Where l – length of conductor
   b – outer radius
   a – inner radius

2. Toroid $L = \frac{\mu_0 N^2 a}{2\pi} \ell_n \left(\frac{r_2}{r_1}\right)$

   N – number of turns
   a – axial thickness
   r_2 – outer radius
   r_1 – inner radius
   If average radius is given

   $L = \frac{\mu_0 N^2 s}{2\pi r}$, r – average radius

3. Parallel conductors of radius a,

   $L = \frac{\mu_0 l}{\pi} \cosh^{-1}\left(\frac{d}{2a}\right)\left(\frac{H}{m}\right)$

   $L = \frac{\mu_0 l}{\pi} \ln\left(\frac{d}{a}\right)$ (H/m) when d is larger

4. Long solenoid of small cross-sections

   $L = \frac{\mu_0 N^2 S}{l} Hl >> a$

   N – number of turns
   S – area (πr^2)
   l – length

Capacitance

Capacitance is the ability of the device to store energy in an electric field.

For a two conductor capacitor it is the ratio of charge on one of the conductors to the magnitude.

i.e., $C = \frac{Q}{V}\frac{C}{V}$ or Farad

For a parallel plate of surface A and separated by a distance d.

$C = \frac{\in A}{d}$ $\in$ is the permittivity of the medium $\in = \in_0 \in_r$.

For concentric spheres with inner radius a and outer radius b

$$C = 4\pi \in \left(\frac{ab}{b-a}\right)$$

For concentric cylinders of length L and inner and outer radii a and b.

$$C = \frac{2\pi \in dL}{\ln\left(\frac{b}{a}\right)}$$

Energy stored in the capacitor

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$$

Magnetostatics

Electric current flowing through a conductor produces a magnetic field which is concentric circles around it.

Magnetic Flux (ϕ)

It is the number of magnetic lines of force passing through a given area perpendicular to the area. If the lines of force

are not normal to the area, we have to take the component which is normal to the area. It is denoted by ϕ and unit is Webers (Wb)

Magnetic Flux Density (B)

It is the number of times of force passing per unit area i.e., magnetic flux per unit area is flux density

$B = \frac{\phi}{A}$. It is measured in Webers/m^2 or Tesla (T).

$$\phi = \int_s \bar{B} \cdot d\bar{s} Wb$$

Magnetic Field Intensity or Field Strength (H)

It is the force experienced by a unit North pole placed at that point in the magnetic field.

$$H = \frac{NI}{l}\frac{A}{m}$$

Also $\vec{B} = \mu \vec{H}$

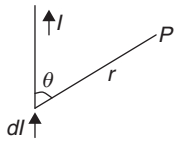
Where μ is the permeability of the medium $\mu = \mu_0 \mu_r$

Biot Swat's Law

The Biot savart law gives the magnetic flux density at a point P at a certain distance from a small current carrying element.

$$\Delta B = \left(\frac{\mu_0 I}{4\pi}\right)\left(\frac{\bar{dl} \times \bar{u}}{r^2}\right)$$

$$B = \frac{\mu_0 I}{4\pi}\int_c \left(\frac{\bar{dl} \times \bar{u}}{r^2}\right)$$

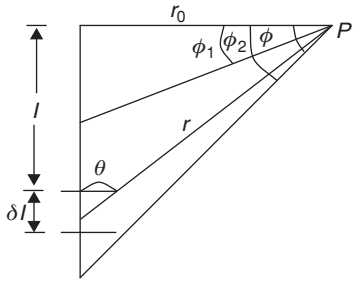


Where unit vector $\bar{u}$ is directed towards the fixed point at which the flux density is determined.

The magnitude of flux density is

$$|\Delta B| = \frac{\mu_0 I dl \sin\theta}{4\pi r^2}$$

Magnetic field of a short straight length or wire



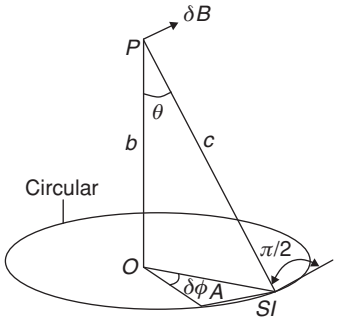
$$B_P = \frac{\mu_0 I}{4\pi r_0}(\sin\phi_2 - \sin\phi_1)$$

The direction of B_P would be normal to plane containing dl and r. The magnetic field due to an infinitely long wire carrying a current I, we have to substitute $\phi_1 = \frac{-\pi}{2}, \phi_2 = \frac{\pi}{2}$, so,

$$B_P = \frac{\mu_0 I}{2\pi r_0}$$

Magnetic field on the axis of a circular coil

The magnetic field ∂B at P due to circumferential element ∂l is



$$\partial B = \frac{\mu_0 I \partial l \sin\left(\frac{\pi}{2}\right)}{4\pi c^2} = \frac{\mu_0 I \partial l}{4\pi c^2}$$

The components of B normal to the axis OP cancel out and components along the axis add up

$$B_{ax} = \frac{\mu_0 I}{2a}\sin^3\theta$$

When P is on centre of the coil, i.e., at o

$$B = \frac{\mu_0 I}{2r}$$

Ampere's law

The magnetic field B at a point P at a distance r from an infinitely long straight conductor

$$B = \frac{\mu_0 I}{2\pi r} \text{ or } B \times (2\pi r) = \mu_0 I$$

i.e., B multiplied by the length of the contour is proportional to the current in the wire.

Vector H is such that $H = \frac{B}{\mu}\left(\frac{B}{\mu_0} \text{ in free space}\right)$

$$\Rightarrow H \times 2\pi r = I$$

A contour of any shape can be built up of infinite number of circumferential elements and for a contour of any arbitrary shape.

$$\oint_c H \cdot dl = I$$

I is the current enclosed by contour C.

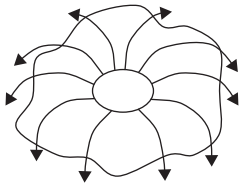
By using the principle of superposition, we now extend the law to conductors of finite cross section and any conducting medium having current distribution in it.

$$\oint_c H \cdot dl = \Sigma I$$

This is Ampere's law or Magnetic circuit law.

Example 5: What net flux crosses the surface S shown in figure which contain a charge distribution in the form of a plane disk of radius 4 m and $\rho_s = \frac{\sin^2\phi}{2r}\frac{C}{m^2}$.

Solution:



$$\psi = Q_{enc} = \int_s \rho_s d_s$$

$$= \int_0^{2\pi}\int_0^4 \frac{\sin^2\phi}{2r} r dr d\phi$$

$$= \int_0^{2\pi} \frac{\sin^2\phi}{2}[r]_0^4 d\phi = 2\int_0^{2\pi} \sin^2\phi d\phi$$

$$= 2\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi} = 4\pi C.$$

Example 6: A point Q is at origin of spherical coordinate system. Find the flux which crosses the portion of a spherical shell described by $\alpha \le \theta \le \beta$

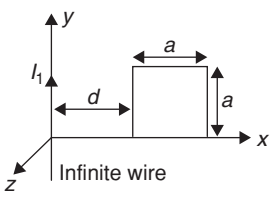
Solution: Area A of the shell $A = \int_0^{2\pi}\int_\alpha^\beta r^2 \sin\theta d\theta d\phi$

$$= 2\pi r^2 (\cos\alpha - \cos\beta)$$

$$\psi_{net} = \frac{A}{4\pi r^2}\cdot Q = \frac{Q}{2}(\cos\alpha - \cos\beta).$$

Mutual inductance between infinite current carrying conductor and square current carrying loop separated by a distance

$$M_{ab} = \frac{\mu a}{2\pi}\ln\left(\frac{d+a}{d}\right) H$$



Force and torques in magnetic fields

$$\vec{F} = q\vec{V}\times\vec{B}$$

$\vec{F} = q(\vec{E} + \vec{V}\times\vec{B})$ is called as the Lorentz force.

Induced EMF

Faraday's law (varying field and fixed conductor)

(i) $\phi\vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = \int_s \left(\frac{-\partial B}{\partial t}\right)\cdot d\vec{s}$$

Field is fixed but conductor is moving

$$V_{ab} = \int_0^a \overline{E_m}\cdot\overline{dl} = \int_0^a (\vec{V}\times\vec{B})\cdot d\vec{l}$$

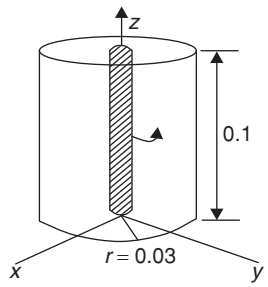
$$\Rightarrow V = \oint (\vec{V}\times\vec{B})\cdot d\vec{l}$$

Conductor in motion through time dependent fields

$$V = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s} + \oint (\vec{V}\times\vec{B})\cdot d\vec{l}$$

Example 7: Work and power required to move the conductor shown in figure, one full revolution in 0.02 sec. If the magnetic flux density $\vec{B} = 2.5\times10^{-3} a\hat{r}\ T$ and current 45 A.

Solution:



$$\vec{F} = I(\vec{1}\times\vec{B}) = -1.13\times10^{-2}\hat{a}_\phi N$$

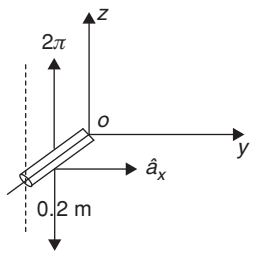
$$W = \int_{2\pi} \overrightarrow{Fa}\cdot d\vec{l} = \int_0^{2\pi} (-1.13\times10^{-2})\hat{a}_\phi \cdot r d\phi \hat{a}_\phi$$

$$= -2.13\times10^{-3} J$$

$$P = \frac{W}{t} = -0.107 \text{ W}.$$

Example 8: Find the induced voltage in the conductor of the figure where $\vec{B} = 0.04\hat{a}_y\ T$ and $V = 2.5\sin(10^3 t)\hat{a}_z$.

Solution:

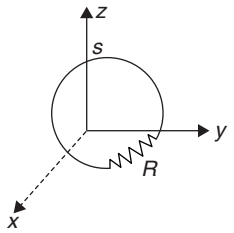


$E_m = \vec{V} \times \vec{B} = 0.1\sin(10^3 t)(-\hat{a}_x)$

So, induced voltage $V = \int \overrightarrow{E_m} \cdot d\vec{l}$

$= -0.02 \sin(10^3 t)$ V.

Example 9: The circular loop conductor shown in the figure lies in the $z = 0$ plane, has a radius of 0.1 m and resistance of 5 Ω. Given $\vec{B} = 0.02\sin(10^3 t)a_z$ determine the current in the resistance R.



Solution:

$$\phi = \vec{B} \cdot \vec{S}$$

$$\Rightarrow \quad \phi = 0.00628\sin(10^3 t)$$

$$V = -\frac{d\phi}{dt} = -6.28\cos(10^3 t)$$

$$i = \frac{V}{R} = -1.256\cos(10^3 t)\text{ A}$$

MAXWELLS EQUATIONS

1. Total displacement through the surface enclosing a volume is equal to the total charge within the volume. It is given in differential and integral form as

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \text{ and } \oint_s \bar{E} \cdot \overline{dS} = \frac{1}{\varepsilon_0}\int_v \rho dv$$

2. Net magnetic field emerging through any closed surface is zero.

$$\nabla \cdot \bar{B} = 0 \text{ and } \oint Bd\bar{s} = 0$$

3. Electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any surface bounded by the path.

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \text{ and } \oint_s \bar{E} \cdot \overline{dS} = \frac{1}{\varepsilon_0}\int_v \rho dv$$

4. M.M.F around a closed path is equal to the conduction current and the time derivative of electric displacement though any surface bounded by the path.

$$\nabla \cdot \bar{B} = 0 \text{ and } \oint Bd\bar{s} = 0$$

$$\oint \bar{B} \cdot \overline{dl} = \int_s \left(\mu_0 \bar{j} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right) \cdot d\bar{s}$$

EXERCISES

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. An air filled parallel plate capacitor has a stored energy W_0 and a charge of Q_0 when connected to a voltage source V_0. If air space is now filled completely by a dielectric having a dielectric constant of 2 and connected to a voltage source of 2 V_0, then stored energy will be

(A) W_0 (B) $2W_0$

(C) $4W_0$ (D) $8W_0$

2. An insulated metal sphere of 10cm radius is charged by rubbing with a charge of 2×10^{-8}C. Potential developed will be

(A) 1800 V (B) 900 V

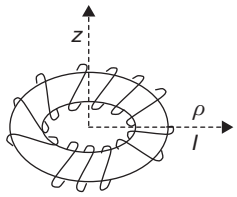
(C) 0 V (D) 200 V

3. The energy stored in the magnetic field of a solenoid 30 cm long and 3 cm diameter wound with 100 turns of wire carrying a current of 100 A is

(A) 0.015 J (B) 0.15 J

(C) 0.5 J (D) 1.15 J

4. For the N term toroid with I amperes, H within the toroid and outside toroid is



(A) $\frac{NI}{\pi l}\hat{a}_\phi$ and 0 (B) 0 and $\frac{NI}{\pi l}\hat{a}_\phi$

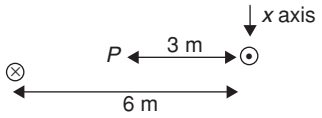
(C) $\frac{NI}{2\pi l}\hat{a}_\phi$ and 0 (D) 0 and $\frac{Nl}{2\pi l}\hat{a}_\phi$

5. If a conductor of cross-section A and carrying a current $I\ \hat{a}_y$ is oriented along the y-axis in the magnetic field $B = B_0\hat{a}_x + B_0\hat{a}_y$, the force density exerted on the conductor is

(A) $\frac{B_0 I}{a}\hat{a}_z \frac{N}{m^2}$ (B) $\frac{B_0 I}{a}\hat{a}_x \frac{N}{m^2}$

(C) $\frac{-B_0 I}{a}\hat{a}_z \frac{N}{m^2}$ (D) $\frac{B_0 I}{a}\hat{a}_y \frac{N}{m^2}$

6. Two infinitely long wires separated by distance 6 m carry currents I in opposite directions along z axis as shown in figure. Magnetic field intensity at point P is given by



(A) $\frac{I}{3\pi}\hat{a}_\phi \frac{A}{m}$ (B) $\frac{I}{6\pi}\hat{a}_\phi \frac{A}{m}$

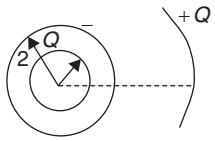
(C) $\frac{I}{12\pi}\hat{a}_\phi$ (D) 0

7. If in a 1 μF capacitor, an instantaneous displacement current of 1 A is to be established between its plates, then it is possible by

(A) 10^6 A (B) 10^6 A/s

(C) 10^6 V/s (D) 10^6 V

8. An infinite number of concentric rings carry a charge Q each alternately positive and negative. Their radii are 1, 2, 4, 8 meter etc in geometric progression as shown. Potential at the centre of the rings will be



(A) $\frac{Q}{2\pi \in_0}$ (B) $\frac{Q}{4\pi \in_0}$

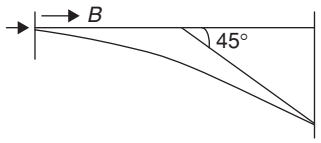
(C) 0 (D) $\frac{Q}{6\pi \in_0}$

9. In the infinite plane, y = 6 m, there exists a uniform surface charge density of $\left(\frac{1}{600\pi}\right)\frac{\mu C}{m^2}$. The associated electric field strength is

(A) $30\hat{i}\frac{V}{m}$ (B) $30\hat{j}\frac{V}{m}$

(C) $30\hat{k}\frac{V}{m}$ (D) $60\hat{j}\frac{V}{m}$

10. A slab of uniform magnetic field deflects a moving charged particle by 45° as shown in the figure. The kinetic energy of the charged particle at the entry and exit points in the magnetic fields will change in the ratio of



(A) $1:\sqrt{2}$

(B) $\sqrt{2}:1$

(C) 1 : 1

(D) 1 : 2

11. A radial field $\vec{H} = \frac{2.39\times105}{r}\cos\phi\hat{a}_r \frac{A}{m}$ exists in free space. Magnetic flux ϕ crossing the surface defined by $-\frac{\pi}{4} \le \phi \le \frac{\pi}{4}$, $0 \le z \le 1$ m will be

(A) 6.3 Wb

(B) 8.48 Wb

(C) 4.24 Wb

(D) 3.16 Wb

12. A plane electromagnetic wave in free space is specified by the electric field $\hat{a}_x[(20\cos(\omega t - \beta z) + 5\cos(\omega t + \beta z)]$ V/m

The associated magnetic field is

(A) $\frac{\hat{a}_x}{120\pi}[20\cos(\omega t - \beta z) + 5\cos(\omega t + \beta z)]$ A/m

(B) $\frac{\hat{a}_x}{120\pi}[20\cos(\omega t - \beta z) - \cos(\omega t + \beta z)]$ A/m

(C) $\frac{\hat{a}_y}{120\pi}[20\cos(\omega t - \beta z) + 5\cos(\omega t + \beta z)]$ A/m

(D) $\frac{\hat{a}_y}{120\pi}[20\cos(\omega t - \beta z) - \cos(\omega t + \beta z)]$ A/m

13. A rectangular loop in the x-z plane bounded by the lines $x = 0$, $x = a$, $z = 0$ and $z = b$, is in a time varying magnetic field given by $\vec{B} = B_0 \cos\omega t\, \hat{a}_y$

B_0 is a constant, ω is the angular frequency and $\hat{a}_y$ is the unit vector in the y direction. The emf induced in the loop is given by

(A) $ab\, B_0 \cos\omega t$

(B) $ab\, B_0\, \omega \sin\omega t$

(C) $B_0\, \omega \sin\omega t$

(D) Zero

14. A parallel plate air capacitor has plates of 1500 cm^2 separated by 5 mm. If a layer of dielectric 2 mm thick and relative permittivity 3 is now introduced between the plates then new separation in mm between the plates so that capacitive value is unchanged will be

(A) 7 (B) 6.33

(C) 4.33 (D) 8

15. Let $\overline{D} = 1\cdot\hat{a}_r \frac{c}{m^2}$. Find ρ_v, hence calculate amount of flux leaving the spherical surface $r = 1$.

(A) π C

(B) 2π C

(C) 6π C

(D) 4π C

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. The equation that is desired from Gauss law is

(A) $\nabla \cdot \vec{D} = \rho_v$ (B) $\vec{D} = \in_0 \vec{E}$

(C) $\nabla \times \vec{E} = 0$ (D) None of these

2. There is a charged metal sphere and a thin circular plate. Distribution of charge around the surface is

(A) Uniform in both
(B) Uniform in sphere and Non uniform in plate
(C) Non uniform in sphere and uniform in plate
(D) Non-uniform in both

3. A charged hollow sphere is suspended in a uniform electric field of strength E, then the field strength

(A) Inside the sphere is zero
(B) Gradually decreases to zero inside the sphere
(C) E and V inside the sphere are zero
(D) All of these

4. As a result of reflections from a plane conducting wall, electro magnetic waves acquire an apparent velocity greater than the velocity of light in space. This is called the

(A) Velocity of propagation
(B) Normal velocity
(C) Group velocity
(D) Phase velocity

5. Two infinite parallel metal plates are charged with equal surface charge density of the same polarity. The electric field in the gap between the plates is

(A) The same as that produced by one plate
(B) Double of the field produced by one plate
(C) Dependent on coordinates of the field point
(D) Zero

6. Copper behaves as a

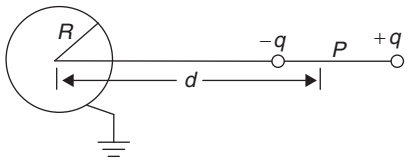
(A) Conductor always
(B) Conductor or dielectric depending on applied electric field strength
(C) Conductor or dielectric depending on the frequency
(D) Conductor or dielectric depending on the electric current density

7. Match the following

List I	List II
P. $\nabla \times \vec{H} = J + \frac{\partial s}{\partial t}$	X Gauss Law.
Q. $\nabla \cdot \vec{D} = \rho$	Y Biot Savart Law
R. $d\vec{H} = \frac{\mu I d\vec{l} \times \vec{a}_{12}}{4\pi d^2}$	Z Ampere's Law

	X	Y	Z		X	Y	Z
(A)	P	Q	R	(B)	R	P	Q
(C)	Q	P	R	(D)	R	Q	P

8. An electric dipole of moment P is placed in front of a grounded sphere as shown in the figure. The charge induced on the surface of the sphere is



(A) Zero (B) $\frac{PR}{d^2}$

(C) $\frac{PR}{d^3}$ (D) $\frac{PR^3}{d^4}$

9. Maxwell's divergence equation for the magnetic field is given by

(A) $\nabla \times B = 0$ (B) $\nabla \cdot B = 0$

(C) $\nabla \times \beta = \rho$ (D) $\nabla \cdot B = \rho$

10. When a lossy capacitor with a dielectric of permittivity $\in$ and conductivity σ operates at a frequency ω, the loss tangent for the capacitor is given by

(A) $\frac{\omega\sigma}{\in}$ (B) $\frac{\omega \in}{\sigma}$

(C) $\frac{\sigma}{\omega \in}$ (D) $\sigma\omega \in$

11. Which of the following is Laplace's equation?

(A) $\nabla^2 v = 0$ (B) $\nabla^2 v = \frac{-\rho_v}{\in}$

(C) $\nabla \cdot D = \rho_v$ (D) $\nabla \cdot B = 0$

12. The potential difference V_{LM}, the potential at M with reference to L, is

(A) $V_{LM} = -\int_L^M E \cdot dl$ (B) $V_{LM} = \int_L^M E \cdot dl$

(C) $V_{LM} = -\int_L^M E \cdot dl$ (D) $V_{LM} = \int_L^M E \cdot dl$

13. Work done in moving a charge from one point to another along an equipotential line or surface is

(A) Infinite (B) Unity
(C) Zero (D) None

14. Which of the following is Ohm's law

(A) $D = \in E$ (B) $B = \mu H$

(C) $J = \frac{E}{\sigma}$ (D) $J = \sigma E$

15. Identify, which of the following expressions are not Maxwell's equation for time – varying fields

(A) $\nabla \cdot D = \rho_v$ (B) $\nabla \cdot J + \frac{\partial \rho_v}{\partial t} = 0$

(C) $\nabla \times E = \frac{-\partial B}{\partial t}$ (D) $\nabla \times H = J + \frac{\partial D}{\partial t}$

Previous Years' Questions

1. A capacitor consists of two metal plates each 500 × 500 mm^2 and spaced 6 mm apart. The space between the metal plates is filled with a glass plate of 4 mm thickness and a layer of paper of 2 mm thickness. The relative permittivity of the glass and paper and 8 and 2 respectively. Neglecting the fringing effect, the capacitance will be (Given that $E_0 = 8.85 \times 10^{-12}$ F/m) **[2008]**

(A) 983.33 pF (B) 1475 pF
(C) 6637.5 pF (D) 9956.25 pF

2. A coil of 300 turns is wound on a non-magnetic core having a mean circumference of 300 mm and a cross-sectional area of 300 mm^2. The inductance of the coil corresponding to a magnetizing current of 3A will be **[2008]**

(Given that $\mu_0 = 4\pi \times 10^{-7}$ *H/m*)

(A) 37.68 μH (B) 113.04 μH
(C) 37.68 *mH* (D) 113.04 *mH*

3. $F(x, y) = (x^2 + xy)\hat{a}_x + (y^2 + xy)\hat{a}_y$. It's line integral over the straight line from $(x, y) = (0, 2)$ to $(x, y) = (2, 0)$ evaluates to **[2009]**

(A) –8 (B) 4
(C) 8 (D) 0

4. Divergence of the three-dimensional radial vector field $\bar{r}$ is **[2010]**

(A) 3 (B) $1/r$
(C) $\bar{i} + \bar{j} + \bar{k}$ (D) $3(\bar{i} + \bar{j} + \bar{k})$

5. A capacitor is made with a polymeric dielectric having an ε_r of 2.26 and dielectric breakdown strength of 50 kV/cm. The permittivity of free space is 8.85 pF/m. If the rectangular plates of capacitor have a width of 20 cm and a length of 40cm, then the maximum electric charge in capacitor is **[2011]**

(A) 2 μC (B) 4 μC
(C) 8 μC (D) 10 μC

6. The direction of vector $\bar{A}$ is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot A = 0$ is **[2012]**

(A) –2 (B) 2
(C) 1 (D) 0

7. The flux density at a point in space is given by $B = 4a_x + 2kya_y + 8a_z$, Wb/m^2. The value of constant k must be equal to **[2013]**

(A) –2 (B) –0.5
(C) +0.5 (D) +2

8. A dielectric slab with 500 mm × 500 mm cross-section is 0.4 *m* long. The slab is subjected to a uniform electric field of $E = 6a_x + 8a_y$ *kV*/mm. The relative permittivity of the dielectric material is equal to 2. The value of constant ε_0 is 8.85×10^{-12} *F*/m. The energy stored in the dielectric in Joules is **[2013]**

(A) 8.85×10^{-11} (B) 8.85×10^{-5}
(C) 88.5 (D) 885

9. C_0 is the capacitance of a parallel plate capacitor with air as dielectric (as in Figure (a)). If, half of the entire gap as shown in Figure (b) is filled with a dielectric of permittivity $\in_r$, the expression for the modified capacitance is **[2014]**



Figure (a) Figure (b)

(A) $\frac{C_0}{2}(1 + \in_r)$ (B) $(C_0 + \in_r)$

(C) $\frac{C_0}{2} \in_r$ (D) $C_o(1 + \in_r)$

10. The undesirable property of an electrical insulating material is **[2014]**

(A) High dielectric strength
(B) High relative permittivity
(C) High thermal conductivity
(D) High insulation resistively

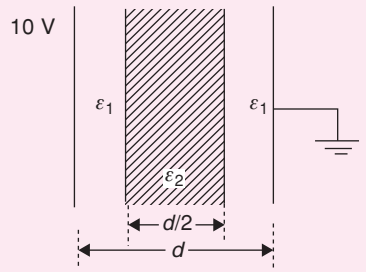
11. The line integral of function $F = yzi$, in the counter-clockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is **[2014]**

(A) $-\pi$ (B) $-\pi$
(C) π (D) 2π

12. The following four vector fields are given in Cartesian co-ordinate system. The vector filed which does not satisfy the property of magnetic flux density is **[2014]**

(A) $y^2a_x + z^2a_y + x^2a_z$
(B) $z^2a_x + x^2a_y + y^2a_z$
(C) $x^2a_x + y^2a_y + z^2a_z$
(D) $y^2z^2a_x + x^2z^2a_y + x^2y^2a_z$

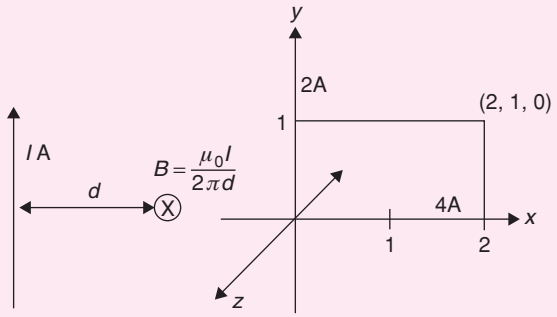
13. A parallel plate capacitor consisting two dielectric materials is shown in the figure. The middle dielectric slab is placed symmetrical with respect to the plates. **[2014]**



If the potential difference between one of the plates and the nearest surface of dielectric interface is 2 Volts, then the ratio $\varepsilon_1 : \varepsilon_2$ is **[2014]**

(A) 1 : 4 (B) 2 : 3
(C) 3 : 2 (D) 4 : 1

14. The magnitude of magnetic flux density $(\vec{B})$ at a point having normal distance d meters from an infinitely extended wire carrying current of I A is $\frac{\mu_0 I}{2\pi d}$ (in SI units). An infinitely extended wire is laid along the x-axis and is carrying current of 4 A in the $+ve$ x direction. Another infinitely extended wire is laid along the y-axis and is carrying 2 A current in the +ve y direction. μ_0 is permeability of free space. Assume $\hat{i}, \hat{j}, \hat{k}$ to be unit vectors along x, y and z axes respectively. **[2014]**



Assuming right handed coordinate system, magnetic field intensity, $\vec{H}$ at coordinate (2, 1, 0) will be **[2014]**

(A) $\frac{3}{2\pi}\hat{k}$ Wb/m (B) $\frac{4}{3\pi}\hat{i}$ A/m

(C) $\frac{3}{3\pi}\hat{k}$ A/m (D) 0 A/m

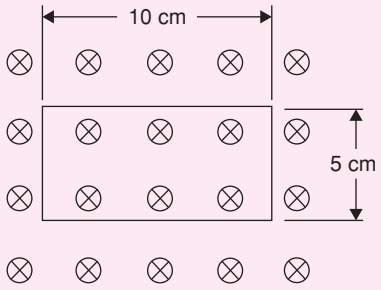
15. The horizontally placed conductors of a single phase line operating at a 50 Hz are having outside diameter of 1.6 cm, and spacing between centers of the conductors is 6 m. The permittivity of free space is 8.854×10^{-12} F/m. The capacitance to ground per kilometer of each line is **[2014]**

(A) 4.2×10^{-9} F (B) 8.4×10^{-9} F
(C) 4.2×10^{-12} F (D) 8.4×10^{-12} F

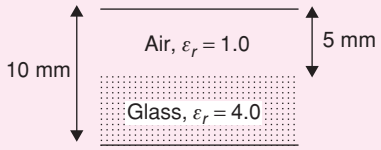
16. Consider a function $\vec{f} = \frac{1}{r^2}\hat{r}$, where r is the distance from the origin and $\hat{r}$ is the unit vector in the radial direction. The divergence of this function over a sphere of radius R, which includes the origin, is **[2015]**

(A) 0 (B) 2π
(C) 4π (D) $R\pi$

17. Consider a one-turn rectangular loop of wire placed in a uniform magnetic field as shown in the figure. The plane of the loop is perpendicular to the field lines. The resistance of the loop is 0.4 Ω, and its inductance is negligible. The magnetic flux density (in Tesla) is a function of time, and is given by $B(t) = 0.25 \sin \omega t$, where $\omega = 2\pi \times 50$ radian/second. The power absorbed (in Watt) by the loop from the magnetic field is _____. **[2015]**



18. A parallel plate capacitor is partially filled with glass of dielectric constant 4.0 as shown below. The dielectric strengths of air and glass are 30 kV/cm and 300 kV/cm, respectively. The maximum voltage (in kilovolts), which can be applied across the capacitor without any breakdown, is _____. **[2015]**



19. Match the following. **[2015]**

P. Stoke's Theorem 1. $\oiint D.ds = Q$

Q. Gauss's Theorem 2. $\oint f(z)dz = 0$

R. Divergence Theorem 3. $\iiint (\nabla.A)dv = \oiint A.ds$

S. Cauchy's Integral Theorem 4. $\iint (\nabla \times A).ds = \oint A.dl$

(A) P–2 Q–1 R–1 S–3
(B) P–4 Q–1 R–3 S–2
(C) P–4 Q–3 R–1 S–2
(D) P–3 Q–4 R–2 S–1

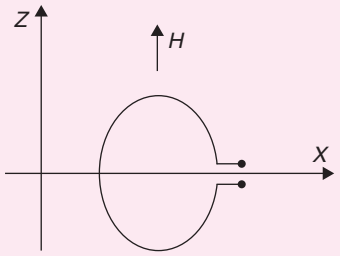
20. Two semi-infinite dielectric regions are separated by a plane boundary at $y = 0$. The dielectric constants of region 1($y < 0$) and region 2($y > 0$) are 2 and 5, respectively. Region 1 has uniform electric field $\vec{E} = 3\hat{a}_y + 2\hat{a}_z$, where $\hat{a}_x, \hat{a}_y$ and $\hat{a}_z$ are unit vectors along the x, y and z axes, respectively. The electric field in region 2 is **[2015]**

(A) $3\hat{a}_x + 1.6\hat{a}_y + 2\hat{a}_z$ (B) $1.2\hat{a}_x + 4\hat{a}_y + 2\hat{a}_z$

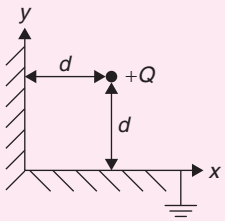
(C) $1.2\hat{a}_x + 4\hat{a}_y + 0.8\hat{a}_z$ (D) $3\hat{a}_x + 10\hat{a}_y + 0.8\hat{a}_z$

21. A circular turn of radius 1 m revolves at 60 rpm about its diameter aligned with the x-axis as shown in the figure. The value of μ_0 is $4\pi \times 10^{-7}$ in SI unit. If a uniform magnetic field intensity

$\vec{H} = 10^7\ \hat{z}$ A/m is applied, then the peak value of the induced voltage, V_{turn}(in Volts), is_____. **[2015]**



22. Two semi-infinite conducting sheets are placed at right angles to each other as shown in the figure. A point charge of +Q is placed at a distance of d from both sheets. The net force on the charge is $\frac{Q^2}{4\pi \in_0}\frac{K}{d^2}$. where K is given by **[2015]**



(A) 0

(B) $-\frac{1}{4}i-\frac{1}{4}j$

(C) $-\frac{1}{8}i-\frac{1}{8}j$

(D) $\frac{1-2\sqrt{2}}{8\sqrt{2}}i+\frac{1-2\sqrt{2}}{8\sqrt{2}}j$

23. In cylindrical coordinate system, the potential produced by a uniform ring charge is given by $\phi = f(r, z)$, where f is continuous function of r and z. Let $\vec{E}$ be the resulting electric field. Then the magnitude of $\nabla \times \vec{E}$ **[2016]**

(A) increase with r (B) is 0

(C) is 3 (D) decrease with z

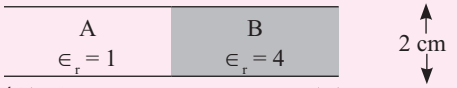
24. A soft-iron toroid is concentric with a long straight conductor carrying a direct current/. If the relative permeability μ_r of soft-iron is 100, the ratio of the magnetic flux densities at two adjacent points located just inside and just outside the toroid, is _____. **[2016]**

25. Two electric charges z and −2z are placed at (0,0) and (6,0) on the x-y plane. The equation of the zero equipotential curve in the x-y plane is _____. **[2016]**

(A) $x = -2$ (B) $y = 2$

(C) $x^2 + y^2 = 2$ (D) $(x+2)^2 + y^2 = 16$

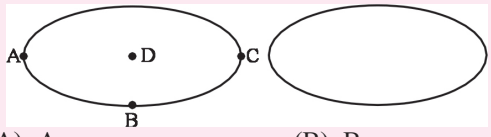
26. A parallel plate capacitor filled with two dielectrics is shown in the figure below. If the electric field in the region A is 4kV/cm, the electric field in the region B, in kV/cm, is ____. **[2016]**



(A) 1 (B) 2

(C) 4 (D) 16

27. Two electrodes, whose cross-sectional view is shown in the figure below, are at the same potential. The maximum electric field will be at the point **[2016]**



(A) A (B) B

(C) C (D) D

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** A	**3.** B	**4.** C	**5.** C	**6.** A	**7.** C	**8.** D	**9.** B	**10.** C
11. C	**12.** D	**13.** B	**14.** B	**15.** D					

Practice Problems 2

1. A	**2.** B	**3.** A	**4.** D	**5.** D	**6.** A	**7.** B	**8.** B	**9.** B	**10.** C
11. A	**12.** A	**13.** C	**14.** D	**15.** B					

Previous Years' Questions

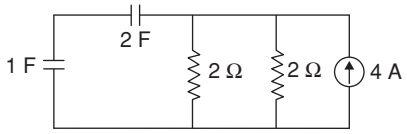
1. B	**2.** B	**3.** D	**4.** A	**5.** C	**6.** D	**7.** A	**8.** C	**9.** A	**10.** B
11. B	**12.** C	**13.** C	**14.** C	**15.** B	**16.** A	**17.** 0.193	**18.** 17 to 20		**19.** B
20. A	**21.** 246 to 250		**22.** D	**23.** B	**24.** 100	**25.** D	**26.** C	**27.** A	

Test
Electric Circuits and Fields

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The time constant for the given circuit will be



(A) $\frac{2}{3}s$ (B) $3s$

(C) $\frac{4}{3}s$ (D) $\frac{1}{4}s$

2. The V-I characteristic as seen from the terminal pair (A, B) of the network of Figure (a) is shown in the figure (b). If an inductance of 5 mH is connected across the terminal pair (A, B), the time constant of the system will be

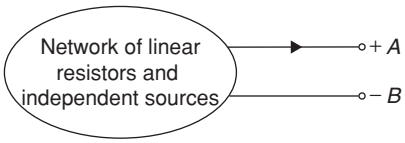


Figure: (a)

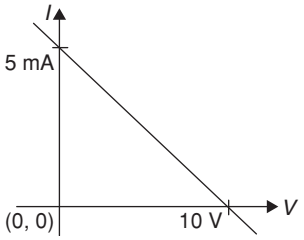


Figure: (b)

(A) 2.5 μsec (B) 10 μsec

(C) 5 μsec (D) 15 μsec

3. In a R-L-C series resonant circuit at the half power points,

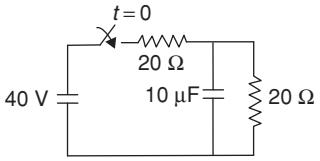
(A) the impedance is half the impedance at resonance

(B) the current is half of the current at resonance

(C) power factor is 0.5

(D) the resistance is equal to resultant reactance

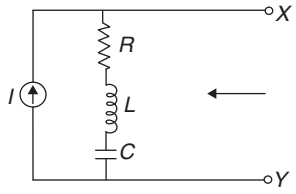
4. In the figure given below, the initial capacitor voltage is zero. The switch is closed at $t = 0$. The final steady state voltage across the capacitor is



(A) 5 V (B) 20 V

(C) 10 V (D) 0 V

5. In the given figure, the current source is $1 \angle 0$. $R = 1\ \Omega$, $Z_C = -2j\ \Omega$, $Z_L = j2\ \Omega$. The Thevenin equivalent looking into the circuit across $X - Y$.



(A) $1 \angle 0$ V, $1 + j2\ \Omega$ (B) $1 \angle 45$ V, $1 - j2\ \Omega$

(C) $1 \angle 0$ V, $1\ \Omega$ (D) $1.414 \angle 45$ V, $1\ \Omega$

6. Form factor of the symmetrical square wave is _____.

(A) zero (B) 1

(C) 2 (D) ∞

7. Which of the following statements holds for the divergence of electric and magnetic flux densities?

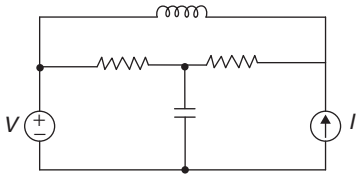
(A) Both are zero

(B) These are zero for static densities but non zero for time varying densities

(C) It is zero for the electric flux density.

(D) It is zero is for the magnetic flux density.

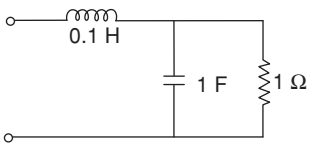
8. The number of chords in the graph of the given circuit will be



(A) 3 (B) 4

(C) 5 (D) 2

9. The resonant frequency for the given circuit will be



(A) 1 rad/sec (B) 2 rad/sec

(C) 3 rad/sec (D) 4 rad/sec

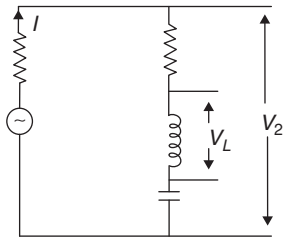
10. The resistance of a strip of copper of rectangular cross-section is 4 Ω. A metal of resistivity twice that of copper is coated on its upper surface to a thickness, equal to that of copper strip. The resistance of the composite strip will be_____.

(A) 12 Ω (B) $\frac{8}{3}$ Ω

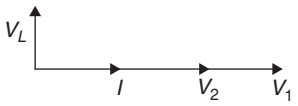
(C) $\frac{14}{5}$ Ω (D) $\frac{3}{8}$ Ω

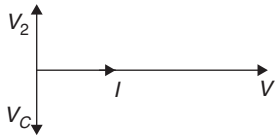
11. A lag network achieves the desired result through its
(A) Attenuation property at high frequencies
(B) Attenuation property at low frequencies
(C) Amplification property
(D) None of these

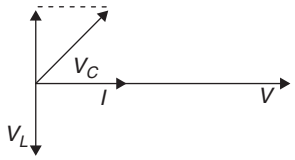
12. The circuit shown in the figure is energized by the sinusoidal voltage source at a frequency which causes resonance with current of I ampere.

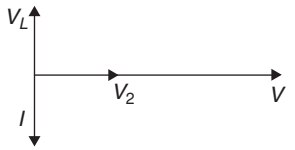


The phasor diagram which is applicable is

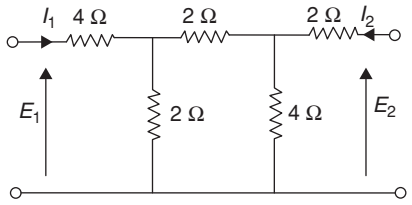
(A)


(B)


(C)


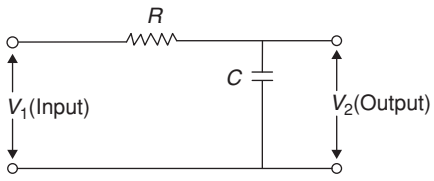
(D)


13. The h parameters of a two-port network are defined by: $\begin{bmatrix} E_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ E_2 \end{bmatrix}$. For the two port network shown in the figure, the value of h_{12} is given by



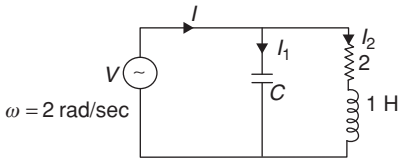
(A) 0.125 (B) 0.167
(C) 0.625 (D) 0.25

14.



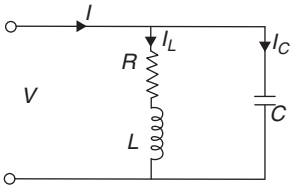
The two-port network shown in the above figure can be described as
(A) phase lead network, differentiating circuit or low-pass filter
(B) phase lag network, integrating circuit or low-pass filter
(C) phase lag network, differentiating circuit or high-pass filter
(D) phase lead network, integrating circuit or high-pass filter

15. For the given ac circuit, if the value of C is chosen such that V and I are in phase, then I_1 leads I_2 by an angle given by



(A) 0° (B) 45°
(C) 90° (D) 135°

16. For the given parallel resonant circuit, match list I with II and select the correct answer using the codes given below in the lists:



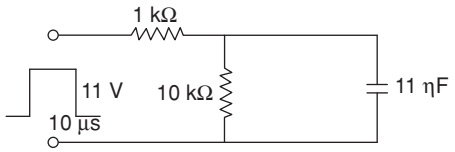
	List - I		List - II
a	I at resonance	1	$\frac{\omega}{R}$
b	I_L	2	In phase with voltage
c	Dynamic impedance	3	$\frac{L}{CR}$
		4	Lags the applied voltage

Codes:

	a	b	c		a	b	c
(A)	4	2	3	(B)	2	4	3
(C)	4	2	1	(D)	2	4	1

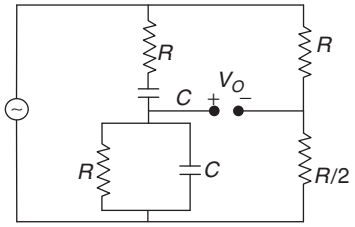
17. An 11 V pulse of 10 μs duration is applied to the circuit shown in the figure. Assuming that the capacitor is

completely discharged prior to applying the pulse, the peak value of the capacitor voltage is



(A) 11 V (B) 5.5 V
(C) 6.32 V (D) 0.86 V

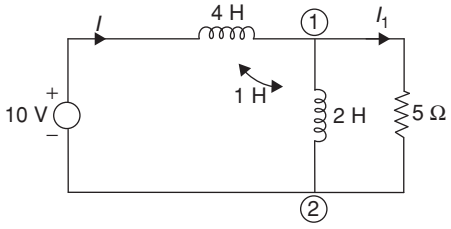
18. In the circuit shown in figure, V_o will be zero at a frequency of



(A) $\frac{1}{2\pi RC}$ Hz (B) $\frac{1}{4\pi RC}$ Hz

(C) $\frac{1}{\pi RC}$ Hz (D) $\frac{4}{3\pi RC}$ Hz

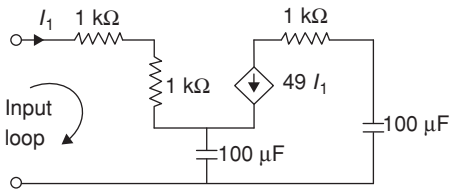
19. The tie set matrix corresponding to the coupled circuit shown below is



(A) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

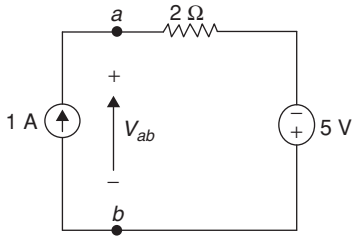
(C) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

20. The equivalent capacitance of the input loop of the circuit shown is _____.



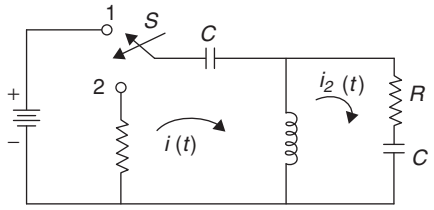
(A) 2 μF (B) 100 μF
(C) 200 μF (D) 4 μF

21. Assuming ideal elements in the circuit show below, V_{ab} will be



(A) 3 V (B) −3 V
(C) 5 V (D) 0 V

Common Data for Questions 22 and 23: For the circuit shown in the below figure:



22. At $t = 0^+$, the current is

(A) $\frac{-V}{2R}$ (B) $\frac{-V}{R}$

(C) $\frac{-V}{4R}$ (D) Zero

23. The equations for the loop currents after the switch is brought from position 1 to position 2 at $t = 0$; are

(A) $\begin{bmatrix} R+Ls+\frac{1}{s} & -Ls \\ -Ls & R+\frac{1}{Cs} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{V}{s} \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} R+Ls+\frac{1}{s} & -Ls \\ -Ls & R+\frac{1}{Cs} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-V}{s} \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} R+Ls+\frac{1}{s} & -Ls \\ -Ls & R+Ls+\frac{1}{Cs} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-V}{s} \\ 0 \end{bmatrix}$

(D) $\begin{bmatrix} R+Ls+\frac{1}{s} & -Ls \\ -Ls & R+Ls+\frac{1}{Cs} \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{V}{s} \\ 0 \end{bmatrix}$

Common Data for Questions 24 and 25: A two-port network shown below is excited by external dc source. The voltages and the currents are measured with voltmeters V_1, V_2 and ammeters A_1, A_2, (all assumed to be ideal), as indicated.

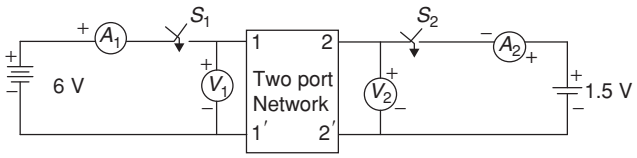
Under following switch conditions, the readings obtained are:

(i) S_1 – open, S_2 – closed

$A_1 = 0$ A, $V_1 = 4.5$ V, $V_2 = 1.5$ V, $A_2 = 1$ A

(ii) S_1 – closed, S_2 – open

$A_1 = 4$ A, $V_1 = 6$ V, $V_2 = 6$ V, $A_2 = 0$ A



24. The z-parameter matrix for this network is

(A) $\begin{bmatrix} 1.5 & 1.5 \\ 4.5 & 1.5 \end{bmatrix}$ (B) $\begin{bmatrix} 1.5 & 4.5 \\ 1.5 & 4.5 \end{bmatrix}$

(C) $\begin{bmatrix} 1.5 & 4.5 \\ 1.5 & 1.5 \end{bmatrix}$ (D) $\begin{bmatrix} 4.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}$

25. The h-parameter matrix for this network is

(A) $\begin{bmatrix} -3 & 3 \\ -1 & 0.67 \end{bmatrix}$ (B) $\begin{bmatrix} -3 & -1 \\ 3 & 0.67 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 3 \\ 1 & 0.67 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 1 \\ -3 & -0.67 \end{bmatrix}$

Answers Keys

1. A	**2.** A	**3.** D	**4.** B	**5.** C	**6.** B	**7.** D	**8.** D	**9.** C	**10.** B
11. A	**12.** A	**13.** D	**14.** B	**15.** D	**16.** B	**17.** D	**18.** A	**19.** B	**20.** A
21. B	**22.** A	**23.** C	**24.** C	**25.** A					

Electrical Machines

UNIT IV

Chapter 1:	Transformers	3.517
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Chapter 1

Transformers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Single-phase transformer
- Windings
- Bushings
- Breather
- Voltage transformation ratio
- Ideal two-winding transformer
- Phasor diagram of transformer under load
- Equivalent circuit
- Transformer tests
- Sumpner's test
- Losses and efficiency
- Voltage regulation of a transformer
- Auto-transformer
- Parallel operation of single-phase transformers

Introduction

Single-Phase Transformer

The transformer is a device that

1. Transfers electrical energy from one electrical circuit to another
2. Does so without a change in the frequency
3. Works on the principle of mutual induction
4. Has electric circuits that are linked by a common magnetic circuit

The energy transfer usually takes place with a change of voltage. If the energy transfer occurs at the same voltage, the purpose of transformer is merely to isolate the two electric circuits

- Transformer is a static device
- No rotating or moving parts
- Single-phase transformer has two circuits which are electrically isolated but magnetically coupled
- Transformer is an electromagnetic energy conversion device, since the energy received by the primary is first converted to magnetic energy and it is then reconverted to useful electrical energy in the secondary winding circuit
- Voltage and current levels change depending on transformation ratio
- Due to the absence of moving parts, transformers require very little maintenance and it has the highest possible efficiency out of all electrical machines
- Transformer is the main reason for the widespread popularity of AC systems over DC systems

Constructional Details

- Core: Sheet steel or silicon steel with 4% silicon is used
- The core is laminated and the laminations are insulated from each other by a coat of varnish or by an oxide layer on the surface to minimize eddy current losses
- The thickness of laminations varies from 0.35 mm for a frequency of 50 Hz to 0.5 mm for a frequency of 25 Hz
- The core provides a continuous magnetic path of low reluctance. The relative permeability of the core material is of the order of 1000
- Core stepping gives high space factor and also results in reduced length of mean turn and the consequent I^2R loss

Types of Transformers

1. **Core type:** In this type of transformers, the windings surround a considerable part of the core
2. **Shell type:** In this type of transformers, the core surrounds a considerable portion of the windings
3. **Spiral-core or wound-core (trade name—Spirakore):** In this type of transformers, the core is assembled of a continuous strip or ribbon of transformer steel wound in the form of a circular or elliptical cylinder

Advantages of Spiral—Core Construction

- High flux densities are used which reduces the weight/kVA
- Lower iron losses, lesser weight and size per kVA, lower cost of manufacture and a relatively more rigid core

Windings

Conventional transformer has two windings, which are made of high-grade copper or aluminium.

The winding in which electrical energy is fed from the AC supply mains is called primary winding and the other which delivers electrical energy is called secondary winding.

- Depending on voltage level, the windings can also be classified into LV and HV
- Low voltage winding is of higher cross section as it carries higher current compared to high voltage winding
- The winding is insulated from core and other parts of transformers

During the operation of transformer, two types of losses occur, namely core (iron) and Cu losses. These losses result in increase in temperature of both core and winding. To keep the temperature of core and winding within limits, it is necessary to radiate the heat produced due to losses in the transformer. Different cooling methods are used, depending upon the rating of the transformer.

Conservator

In an oil-filled transformer, due to variations in load and change in climatic conditions the oil expands or contracts. To allow room for expansion of the oil, conservator is used. In the absence of a conservator tank, high pressures may develop in transformer tank which may lead to bursting of tank.

Bushings

The purpose of bushings is to provide proper insulation to the output leads of the transformer tank. Porcelain bushings are used up to 33 kV. For higher voltages beyond 33 kV, condenser type or oil-filled type bushings are used.

Breather

Absorption of moisture and dust by the transformer oil will deteriorate the dielectric strength of transformer oil. To prevent moisture and dust from entering the transformer tank, breather is provided

E.m.f. Equation

- Transformer works on the principle of mutual induction. Voltage applied to the primary and the magnetic flux set up in the core assumed to be sinusoidal

Core flux $\phi = \phi_m \sin \omega t$

The primary induced e.m.f.

$$e_1 = -N_1 \frac{d\phi}{dt} = -N_1 \phi_m \omega \cos \omega t$$

$$= -N_1 \omega \phi_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

e_1 is maximum when $\sin\left(\omega t - \frac{\pi}{2}\right) = 1$

$$E_{1_{max}} = N_1 \omega \phi_m$$

The RMS value of induced e.m.f.

$$E_1 = \frac{1}{\sqrt{2}} E_{max} = \frac{1}{\sqrt{2}} N_1 \cdot 2\pi f \phi_m$$

$$= \sqrt{2}\, \pi N_1 f \phi_m = 4.44\, N_1 f \phi_m$$

Similarly $E_1 = \sqrt{2}\, \pi N_2 f \phi_m = 4.44\, N_2 f \phi_m$

$$\therefore \frac{E_1}{E_2} = \frac{N_1}{N_2}$$

- The voltage per turn of the primary and secondary windings is the same, since the same mutual flux is linking with both the windings
- E_1 and E_2 are in phase and lag behind the core flux by an angle of 90°
- In a loaded transformer, the primary draws a current to compensate for the secondary ampere turns so that core flux is maintained constant
- Core flux is constant and is independent of the load

If no load, primary ampere turns are neglected, then

$$N_1 I_1 = N_2 I_2$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} = \frac{V_2}{V_1}$$

Voltage Transformation Ratio (K)

$$\frac{E_2}{E_1} = \frac{N_2}{N_1} = K$$

(i) If $N_2 > N_1$, i.e., $K > 1$, then the transformer is called step-up transformer

(ii) If $N_2 < N_1$, i.e., $K < 1$, then the transformer is known as step down transformer

For an ideal transformer, input VA = output VA

$$V_1 I_1 = V_2 I_2 \quad \text{or} \quad \frac{I_2}{I_1} = \frac{V_1}{V_2} = \frac{1}{K}$$

Hence, currents are in the inverse ratio of the transformation ratio

Solved Examples

Example 1: If the applied voltage of a certain transformer is increased by 50% and the frequency is reduced to 50% (assuming that the magnetic circuit remains unsaturated), the maximum core flux density will. _____

Solution: Induced e.m.f. $E \propto B_m f$

neglecting the impedance drops

$$\frac{V_2}{V_1} = \frac{E_2}{E_1}; \quad \frac{V_2}{V_1} = \frac{B_{m_2} f_2}{B_{m_1} f_1}$$

$$B_{m_2} = \frac{V_2}{V_1} \times \frac{f_1}{f_2} \times B_{m_1}$$

$$= 1.5 \times \frac{1}{0.5} \times B_{m_1} = 3B_{m_1}$$

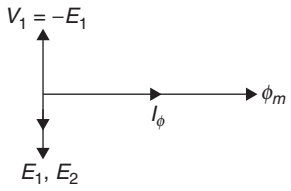
Ans: change to three times the original value.

Ideal Two-winding Transformer

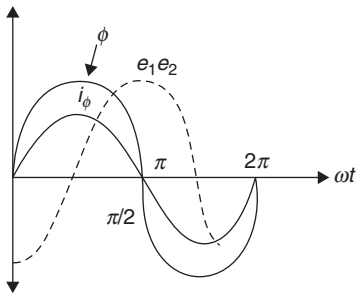
When the transformer is assumed to be ideal,

1. Winding resistances are negligible
2. No flux leakage. All the flux is confined to the magnetic core
3. The core losses are negligible
4. The core has constant permeability, i.e., the magnetization curve for the core is linear

(a) Phasor diagram



(b) Waveforms



When secondary of transformer is loaded, the load will draw secondary current I_2, which sets up N_2I_2, the secondary MMF and it opposes the main flux ϕ_m. The secondary MMF F_2 weakens the main flux ϕ_m. But any reduction in ϕ_m reduces E_1 and in case of ideal transformer $V_1 = -E_1$. Hence, an extra load component of current I_1' is drawn by primary to compensate the secondary ampere-turns to maintain constant core flux.

$$I_1'N_1 = N_2I_2 \Rightarrow \frac{I_1'}{I_2} = \frac{N_2}{N_1}$$

As I_ϕ is negligible

$$\overline{I_1} = \overline{I_1'} + \overline{I_\phi} \Rightarrow \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

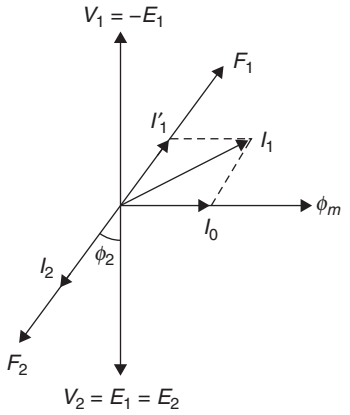
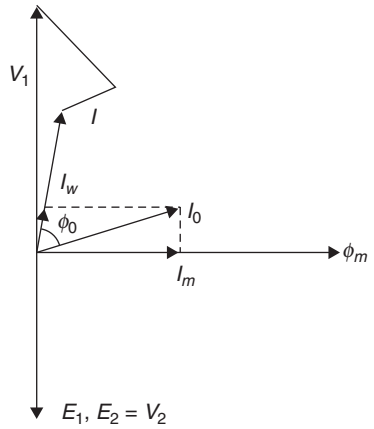


Figure 1 Phasor diagram of ideal transformer for inductive load

Phasor Diagram at no Load



The no-load primary current I_0 is called the exciting current of the transformer and can be resolved into two components.

1. One, in phase with V_1, known as active or working or iron loss component I_w because it mainly supplies the iron loss plus small quantity of primary Cu loss

$$I_w = I_0 \cos \phi_0$$

2. The other component is in quadrature with V_1 and is known as magnetizing component I_μ because its function is to sustain the alternating flux in the core. It is wattless.

$$I_\mu = I_0 \sin \phi_0$$

Even when the transformer is on no-load, the primary input current is not wholly reactive. The no-load primary current lags V_1 by an angle ϕ_0

$$W_0 = V_1 I_0 \cos \phi_0$$

This no-load power supplies the losses in transformer, i.e., a small amount of Cu loss and iron losses in the magnetic core. As it is principally the core-loss which is responsible for the shift in current vector from 90° position, angle ϕ_0 is known as hysteresis angle of advance

$$I_0 = \sqrt{I_w^{\,2} + I_\mu^{\,2}}$$

Phasor Diagram of Transformer Under Load

The transformer is said to be loaded, when its secondary circuit is completed through an impedance or load. The magnitude and phase of secondary current I_2 with respect to secondary terminals depends on the characteristics of the load, i.e., I_2 will be in phase, lags behind and lead the terminal voltage V_2, respectively, when load is resistive, inductive and capacitive. The net flux in the core remains constant irrespective of the load.

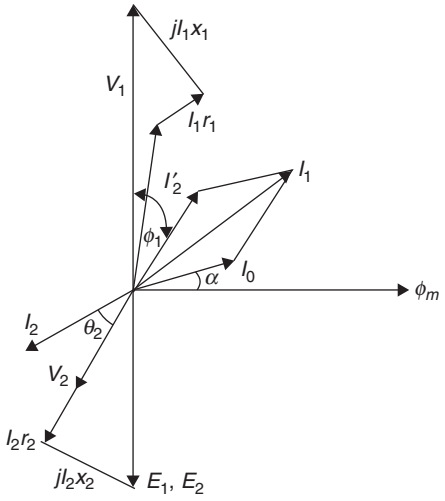


Figure 2 Phasor diagram for inductive load

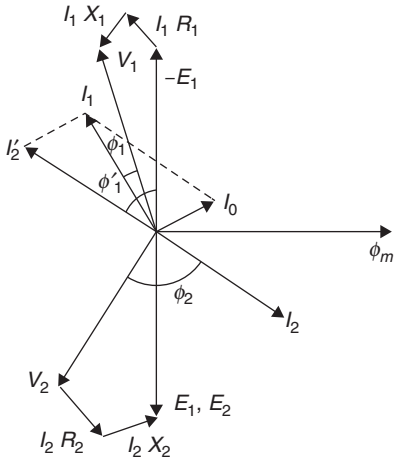


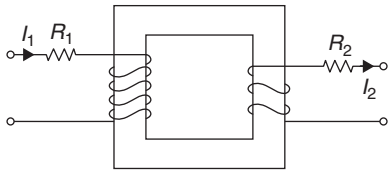
Figure 3 Phasor diagram for capacitive load

$$\overline{E_2} = \overline{V_2} + \overline{I_2}(r_2 + jx_2) = \overline{V_2} + \overline{I_2}\,Z_2$$

$$V_1 = \overline{V_1'} + \overline{I_1}\,(r_1 + jx_1) = \overline{V_1'} + \overline{I_1}\,Z_1$$

Impedance Transformation

1. Equivalent resistance



In the above figure, a transformer is shown whose primary and secondary windings have resistances of R_1 and R_2, respectively.

The copper loss in secondary is $I_2'R_2$. This loss is supplied by primary which takes a current I_1. Hence if R_2' is the equivalent resistance in primary which would have caused the same loss as R_2 in secondary, then

$$I_1^{\,2} R_2' = I_2^{\,2} R_2 \text{ or } R_2' = \left(\frac{I_2}{I_1}\right)^2 R_2$$

If we neglect no-load current I_0, then $\dfrac{I_2}{I_1} = \dfrac{1}{K}$

Hence $R_2' = \dfrac{R_2}{K^2}$

Similarly equivalent primary resistance as referred to secondary is

$$R_1' = K^2 R_1$$

Total or effective resistance of the transformer as referred to primary is

$$R_{01} = R_1 + R_2' = R_1 + \frac{R_2}{K^2}$$

Similarly total transformer resistance as referred to secondary is

$$R_{02} = R_2 + R_1' = R_2 + K^2 R_1$$

The leakage **reactance** can also be transferred from one winding to the other in the same way as resistance

$$X_2' = \frac{X_2}{K^2} \text{ and } X_1' = K^2 X_1$$

$$\text{and } X_{01} = X_1 + X_2' = X_1 + \frac{X_2}{K^2}$$

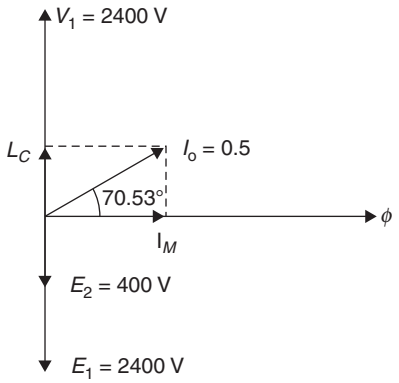
$$\text{and } X_{02} = X_2 + X_1' = X_2 + K^2 X_1$$

The total impedance of the transformer referred to primary and secondary is given by

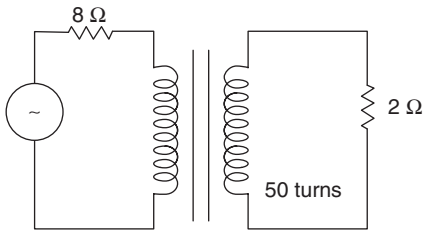
$$Z_{01} = \sqrt{R_{01}^{\,2} + X_{01}^{\,2}} \text{ and } Z_{02} = \sqrt{R_{02}^{\,2} + X_{02}^{\,2}}$$

Example 2: A 2400 V/400 V single-phase transformer takes a no-load current of 0.5 A and core loss = 400 W. Determine the values of the magnetizing and core loss components of no-load current. Draw the no load phasor diagram for the transformer.

Solution: $V_1 = 2400$; $V_2 = 400$ V and $I_o = 0.5$ A
Core loss = Iron loss = 400
$= V_1 I_o \cos \phi_o$
$\Rightarrow 400 = 2400 \times 0.5 \cos \phi_o$
$\phi_o = 70.53^\circ$
$\therefore$ Phasor



Example 3: If the secondary winding of the transformer of the circuit shown in the given figure has 50 turns, then for maximum power transfer to the 2-ohm resistor, the number of turns required in the primary winding will be



Solution: For maximum power transfer 2

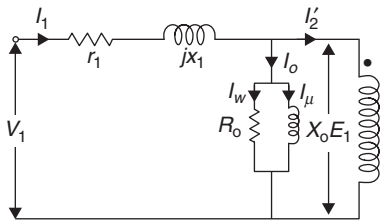
$$= 8 \cdot \left(\frac{N_2}{N_1}\right)^2 \quad \text{(or)} \quad \frac{N_2}{N_1} = \frac{1}{2}$$

Or $N_1 = 2N_2 = 2 \times 50 = 100$ turns

Equivalent Circuit

By making use of equivalent circuit, different performance indices such as efficiency, voltage regulation can be determined.

The equivalent circuit of a transformer having voltage transformation ratio $K = \frac{E_2}{E_1}$ is given in the following figure



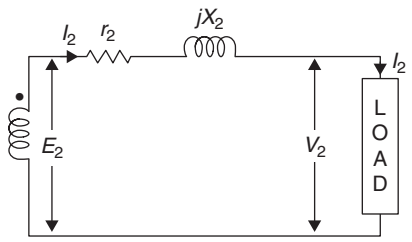


Figure 4(a)

To make calculations simpler, it is preferable to transfer voltage, current and impedance either to the primary or to the secondary. In that case we would have to work in one winding only which is more convenient.

The secondary circuit is shown in figure 4(a) and its equivalent primary values are shown in figure 4(b).

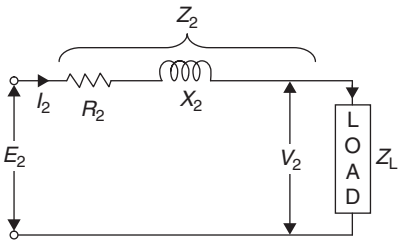
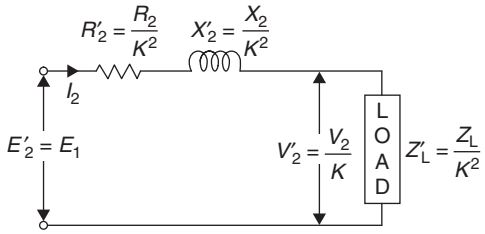
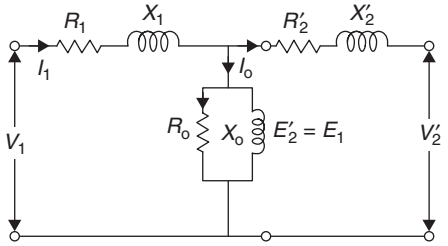


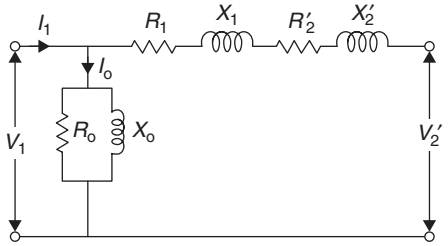
Figure 4(b)



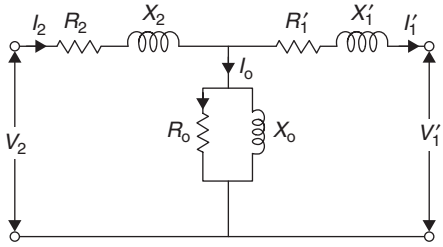
The total equivalent circuit of the transformer is obtained by adding in the primary impedance as shown in the following figure. This is known as exact equivalent circuit. A simplification can be made by transferring the existing circuit across the supply terminals as shown in below figure.



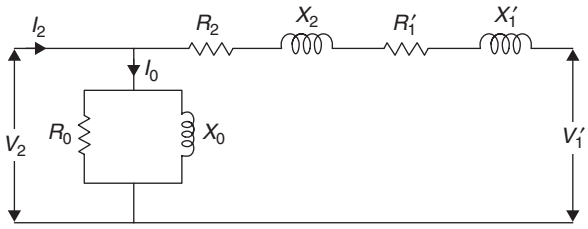
Approximate equivalent ckt referred to primary



In a similar manner, the equivalent circuit referred to the secondary can be obtained. Which is shown in the following figure?



Approximate equivalent circuit referred to the secondary is shown in figure.



Example 4: A 15 kVA, 2300/230 V, 50 Hz single-phase transformer gave the following test data.

Open-circuit test (H. V. side)

$$V_0 = 2300 \text{ V}, I_0 = 0.21 \text{ A}$$

$$W_0 = 50 \text{ W}$$

Short-circuit test (H. V. side)

$$V_{SC} = 47\text{V}, I_{SC} = 6.0 \text{ A}$$

$$W_{SC} = 160 \text{ W}$$

Obtain the equivalent circuit referred to high voltage side.

Solution: As both O.C. test and S.C. test carried out on high voltage side the results will directly lead to equivalent circuit referred to high voltage side.

Shunt branch elements

$$V_0 = 2300 \text{ V}, I_0 = 0.21 \text{ A}, W_0 = 50 \text{ W}$$

$$R_0 = \frac{V_0^2}{W_0} = \frac{2300^2}{50}$$

$$= 105.8 \times 10^3 \ \Omega$$

$$I_W = I_0 \cos\phi_0 = \frac{W_0}{V_0} = \frac{50}{2300}$$

$$= 0.02174 \text{ A}$$

$$I_\mu = \sqrt{I^2{}_0 - I^2{}_W}$$

$$= \sqrt{0.21^2 - (0.02174)^2} = 0.20887 \text{ A}$$

$$\therefore X_0 = \frac{V_0}{I_\mu} = \frac{2300}{0.20887} = 11.011 \times 10^3 \ \Omega$$

Series branch elements

$$V_{SC} = 47 \text{ V}, I_{SC} = 6.0 \text{ A},$$

$$W_{SC} = 160 \text{ W}$$

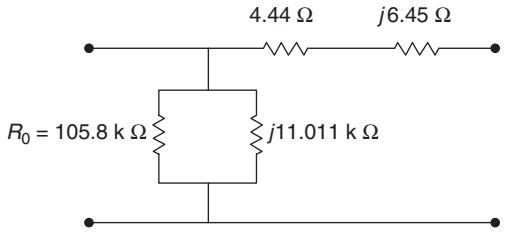
$$R = \frac{160}{6^2} = 4.44 \ \Omega$$

$$Z = \frac{47}{6.0} = 7.833 \ \Omega$$

$$X = \sqrt{Z^2 - R^2} = \sqrt{41.608}$$

$$= 6.45 \ \Omega$$

Equivalent circuit referred to h.v. side.



Example 5: A 3300/230 V, 50-kVA transformer is found to have impedance of 4% and a Cu. loss of 1.8% at full load. Find its percentage reactance and also the ohmic values of resistance, reactance and impedance as referred to primary. What would be the value of primary short-circuit current if primary voltage is assumed constant?

Solution: $\% X = \sqrt{(\% Z^2 - \% R^2)}$

$$= \sqrt{4^2 - 1.8^2} = 3.57\%$$

$(\therefore \%$ Cu loss $= \% R)$

$$I_1, \text{ full load} = \frac{50 \times 10^3}{3300} = 15.2 \text{ A}$$

Considering primary winding, we have $\% R = 1.8$

$$\% R = \frac{R_{01} \times I_1}{V_L} \times 100$$

$$R_{01} = \frac{\%R \times V_L}{I_1 \times 100} = 3.91$$

$$\text{Similarly } \% X = \frac{X_{01} \ I_1 \times 100}{V_1} = 3.57$$

$$\therefore X_{01} = \frac{3.57 \times 3300}{100 \times 15.2} = 7.76 \ \Omega$$

$$\text{Similarly } Z_{01} = \frac{4 \times 3300}{100 \times 15.2} = 8.7 \ \Omega$$

Short-circuit current

$$I_{SC} = \frac{V_1}{Z_{01}}$$

$$\text{Now } Z_{01} = \frac{V_1 \times \%Z}{100 \times I_1}$$

$$I_{SC} = \frac{V_1 \times 100 \times I_1}{V_1 \times \%Z} = \frac{100 \times I_1}{\%Z}$$

$\therefore$ Short-circuit current

$$= \frac{100 \times 15.2}{4} = 380 \text{ A}$$

Example 6: Calculate the regulation of a transformer in which the ohmic loss is 1% of the output and the reactance drop is 5% of the output voltage, when the power factor is 0.8 lead.

Solution: Per unit regulation at 0.8 leading
p.f. = 0.01 × 0.8 − 0.05 × 0.6
= –0.022 or –2.2%

Transformer Tests

The usual tests conducted on the transformer are

1. Open-circuit or no-load test
2. Short-circuit or impedance test
3. Sumpner's or back-to-back test

Open-circuit Test

The purpose of this test is to determine no load loss or core loss and no-load current I_0, which is helpful in finding shunt branch parameters of the equivalent circuit, i.e., R_0 and X_0.

Also this test is helpful in determining the magnetizing current I_μ and core loss component (working component) I_w.

- It is preferable to apply the rated voltage to the LV winding keeping the HV winding open. This is from the point of view of the safety of the testing personnel and also for reading the meters with great accuracy.

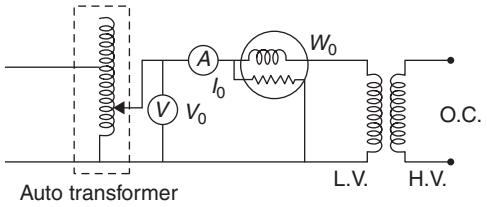


Figure 5 O.C. test circuit diagram

If W_0 is the no-load power input, (V_0 rated voltage), and I_0 the no-load current we have

$$\text{No-load p.f., } \cos\phi_0 = \frac{W_0}{V_0 I_0}.$$

$$\text{Active component of no-load current, } I_w = I_0 \cos\phi_0 = \frac{W_0}{V_0}$$

$$\text{The magnetizing component, } I_\mu = \sqrt{I_0^2 - I_w^2}$$

Shunt branch parameters of the equivalent circuit are

$$R_0 = \frac{V_0}{I_w} \text{ and } X_0 = \frac{V_0}{I_\mu}$$

Note: Since I_0 is only of the order of 5% of the rated current, the no-load Cu loss of the primary winding will be $\left(\frac{1}{20}\right)^2$ times the full-load copper loss. Hence copper loss can be neglected. So, the no load power input W_0 is equal to the iron or core loss P_i.

S.C. Test

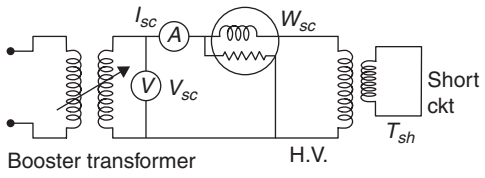


Figure 6 S.C. test circuit diagram

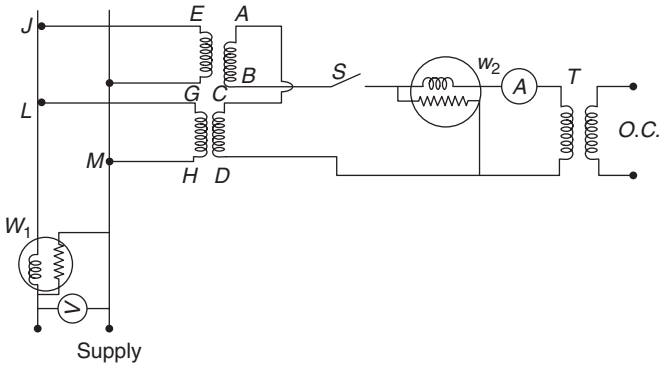
- Since the H.V. rated current is less, short-circuit test is conducted preferably on H.V. side keeping L.V. short circuited.
- For conducting short-circuit test only a small percentage (of the order 5% to 10%) of the rated voltage is sufficient to cause rated current to flow through the windings. As the voltage is less, the core flux density is very much less compared to normal value, so the iron loss which depends on flux density (voltage) can be neglected. The input is therefore equal to total copper loss.
- Through this test, the full-load Cu loss and the equivalent resistance and leakage reactance are determined.
- Let I_{sc} be the current, V_{sc} be the voltage and W_{sc} be the power input, respectively, on short circuit.

  Then, $R_{01} = \frac{W_{sc}}{I_{sc}^2}$ and $Z_{01} = \frac{V_{sc}}{I_{sc}}$ and $X_{01} = \sqrt{Z_{01}^2 - R_{01}^2}$

  If the H.V. winding is energized and L.V. is short circuited as is the usual case, the values of R and X obtained above are the values referred to the H.V. side.
- For drawing the equivalent circuit referred to H.V. or L.V. side, one set of parameters obtained from either O.C. side or S.C. side are to be referred to the other side.
- Limitations of O.C. and S.C. tests
  In the O.C. test, though the rated voltage is applied to one of the windings, the other winding carries no current, so that the effect of temperature rise is not taken care of. In the short-circuit test, though the windings are made to carry the rated currents, yet the voltage applied is only a small fraction of a rated voltage, with the result, the flux density is not at its usual value.
- When the transformer is operating under load conditions, both core loss and ohmic loss exists. But in O.C. and S.C. tests, each of them exist separately. Hence from O.C.

and S.C. tests the effect of temperature rises because the losses in the temperature are not taken care of, which is possible with Sumpner's or back-to-back test.

Sumpner's Test



This test is mainly performed to determine temperature rise and the drawbacks of the O.C. and S.C. tests are overcome. For conducting this test two identical transformers are required. The primaries are connected in parallel across the supply voltage and rated voltage of the winding is impressed. The secondaries are connected in series opposition, so that the resultant voltage acting around the closed circuit formed by the secondaries is zero. To circulate the current through the secondaries (and hence in primaries), a voltage is injected into the circuit.

By suitably adjusting the value of the injected voltage, both the secondaries can be made to carry the rated (or any desired value) of the current.

Before closing the switch 'S' to complete the local circuit formed by the secondaries, the correctness of the polarities is to be ensured. If zero reading of the voltmeter connected across the switch ensures that the polarities are correct.

- The wattmeter connected on the primary side reads twice the iron loss of each transformer. The watt meter connected in the secondary circuit reads twice the copper loss corresponding to the short-circuit current I_{sc} and V_{sc} equals twice the impedance drop of each transformer.

All-day Efficiency

The all-day efficiency is defined as the ratio of output power in kWh to the input in kWh during the day.

Transformers used for distribution have their primaries energized all 24 hours, although their secondaries supply little or no-load much of the time during the day except during the house lighting period. Hence, it is considered a good practice to design such transformers so that core losses are very low.

The Cu losses are relatively less important, because they depend on the load. The performance of distribution transformer is compared on the basis of energy consumed during a certain time period, usually a day of 24 hours.

$$\eta_{\text{all-day}} = \frac{\text{Output in kWh}}{\text{Input in kWh}} \text{ (for 24 hours)}$$

This efficiency is always less than the commercial efficiency of a transformer.

The Per Unit (p.u.) System

Per unit quantity

$$= \frac{\text{Actual value of the quantity in any unit}}{\text{Base value of the quantity in the same unit}}$$

Percentage Resistance, Reactance and Impedance

These quantities are usually measured by the voltage drop at full-load current expressed as a percentage of the normal voltage of the winding on which calculations are made.

1. Percentage resistance at full load

$$\%\, R = \frac{I_1 R_{01}}{V_1} \times 100$$

$$= \frac{I_1^2 R_{01}}{V_1 I_1} \times 100$$

$$= \frac{I_2^2 R_{02}}{V_2 I_2} \times 100 = \%\ \text{Cu loss at full load}$$

$$\%\, R = \%\ \text{Cu loss} = V_r$$

2. Percentage reactance at full load

$$\%\, X = \frac{I_1 X_{01}}{V_1} \times 100$$

$$= \frac{I_2 X_{02}}{V_2} \times 100 = V_x$$

3. Percentage impedance at full-load

$$\%\, Z = \frac{I_1 Z_{01}}{V_1} \times 100$$

$$= \frac{I_2 Z_{02}}{V_2} \times 100$$

4. $\%\, Z = \sqrt{\%\, R^2 + \%\, X^2}$

The ohmic values of resistance and reactance can be obtained from % values as given below.

$$R_{01} = \frac{\%\, R \times V_1}{100 \times I_1} = \frac{\%\ \text{Cu loss} \times V_1}{100 \times I_1}$$

Similarly

$$R_{02} = \frac{\%R \times V_2}{100 \times I_2} = \frac{\%\text{Cu loss} \times V_2}{100 \times I_2}$$

$$X_{01} = \frac{\%X \times V_1}{100 \times I_1} = \frac{V_x \times V_1}{100 \times I_1}$$

Similarly

$$X_{02} = \frac{\%X \times V_2}{100 \times I_2} = \frac{V_x \times V_2}{100 \times I_2}$$

Note: The percentage resistance and impedance are the same value whether referred to primary or secondary.

Example 7: A 3300/230 V, 50-kVA transformer is found to have impedance of 4% and a Cu loss of 1.8% at full load. Find its percentage reactance and also the ohmic values of resistance, reactance and impedance as referred to primary. What would be the value of primary short-circuit current if primary voltage is assumed constant?

Solution: $\%X = \sqrt{\left(\%Z^2 - \%R^2\right)}$

$$= \sqrt{4^2 - 1.8^2} = 3.57\%$$

$(\therefore \%$ Cu loss $= \%R)$

$$I_1, \text{ full load} = \frac{50 \times 10^3}{3300} = 15.2 \text{ A}$$

Considering primary winding, we have

$$\%R = 1.8$$

$$\%R = \frac{R_{01} \times I_1}{V_L} \times 100$$

$$R_{01} = \frac{\%R \times V_L}{I_1 \times 100} = 3.91$$

Similarly

$$\%X = \frac{X_{01} \ I_1 \times 100}{V_1} = 3.57$$

$$\therefore X_{01} = \frac{3.57 \times 3300}{100 \times 15.2} = 7.76\ \Omega$$

$$\text{Similarly } Z_{01} = \frac{4 \times 3300}{100 \times 15.2} = 8.7\ \Omega$$

Short-circuit current

$$I_{SC} = \frac{V_1}{Z_{01}}$$

Now $$Z_{01} = \frac{V_1 \times \%Z}{100 \times I_1}$$

$$I_{SC} = \frac{V_1 \times 100 \times I_1}{V_1 \times \%Z} = \frac{100 \times I_1}{\%Z}$$

$\therefore$ Short-circuit current

$$= \frac{100 \times 15.2}{4} = 380 \text{ A}$$

Losses and Efficiency

Transformer is a static device; hence there are no mechanical losses. Losses in the transformer are

1. Iron loss or core loss, i.e., hysteresis and eddy current losses, which occur in magnetic core.
2. Copper losses or I^2R loss which occurs in the windings.

The core loss of a transformer depends upon the frequency and the maximum flux density when the volume and the thickness of the core laminations are given. The core loss is made up of two parts.

1. Hysteresis loss, $W_h = PB_{\max}^{1.6} f$ as given by Steinmetz empirical relation and
2. Eddy current loss, $W_e = QB_{\max}^{\ 2} f^2$,
   where P and Q are constants.
   The total core loss is given by
   $$W_i = W_h + W_e = PB_{\max}^{1.6} f + QB_{\max}^{\ 2} f^2$$

Separation of Core Losses

- If we carry out two experiments using two different frequencies but the same maximum flux density, we should be able to find the constants P and Q and hence calculate hysteresis and eddy current losses separately.
- The flow of current through the windings gives rise to the copper losses, i.e., $I_1^2 r_1$ and $I_2^2 r_2$.
- The magnetic losses are present as long as primary is energized.
- The no-load current is only of the order of 5% of the rated or full-load current. Hence no load Cu loss can be neglected. So, the no-load input to a transformer is considered as Iron or core or magnetic loss. The iron loss is assumed to be the same under all operating conditions, i.e., from no load to full load. It is denoted by P_i.
- The Cu loss varies with the secondary (and hence primary) current. The copper loss corresponding to the rated value of current is called full-load Cu loss. It is denoted by P_{cu}.
- The efficiency (or commercial efficiency) of a transformer is given by

$$\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{V_2 I_2 \cos\phi_2}{V_2 I_2 \cos\phi_2 + P_i + I_2^{\ 2} R_{02}}$$

where P_i = iron loss

$I_2^{\ 2} R_{02}$ = ohmic loss or Cu loss.

$V_2 I_2$ = Output kVA and $\cos\phi_2$ = load pf.

Condition for Maximum Efficiency

Maximum efficiency occurs when variable losses (i.e., Cu losses) are equal to the fixed loss (i.e., core loss).

$$I_2^2 R_{02} = P_i$$

$$\text{or } I_2 = \sqrt{\frac{P_i}{R_{02}}}$$

$$= I_{FL} \sqrt{\frac{P_i}{\text{Full-load Cu loss}}}$$

kVA corresponding to maximum efficiency.

$$(\text{kVA})_{\max \eta} = \text{Full-load kVA} \sqrt{\frac{P_i}{I_{FL}^2 R_{02}}}$$

Example 8: A 50 kVA transformer has a core loss of 400 W and a full-load copper loss of 800 W. what is the load corresponding to maximum efficiency.

Solution: Iron loss $P_i = 400$ W
Full-load copper loss P_{Cu}, = 800 W
kVA corresponding to max efficiency

$$= \text{full-load kVA} \times \sqrt{\frac{P_i}{P_{CU}}}$$

$$= 50 \times \sqrt{\frac{400}{800}} = 35.35 \text{ kVA}$$

Example 9: In a transformer, the core loss is found to be 52 W at 40 Hz and 90 W at 60 Hz measured at same flux density. Calculate the hysteresis and eddy current losses at 50 Hz.

Solution: Since the flux density is the same in both cases, we can use the relation
Total core loss, $W_i = Af + Bf^2$ or $W_i/f = A + Bf$,

$$\therefore \quad \frac{52}{40} = A + 40\,B$$

And

$$\frac{90}{60} = A + 60\,B$$

$$\therefore \quad A = 0.9 \text{ and } B = 0.01$$

At 50 Hz,
The hysteresis loss $W_h = Af$

$$= 0.9 \times 50 = 45 \text{ W}$$

The eddy current loss

$$W_e = Bf^2 = 0.01 \times 50^2 = 25 \text{ W}$$

Common Data for Examples 10 to 12
A transformer with normal voltage impressed has a flux density of 1.4 wb/m^2 and a core loss comprising of 1000 W eddy current loss and 3000 W hysteresis loss. What do these losses become under the following conditions?

Example 10: Increasing the applied voltage by 10% at rated frequency

Solution: Hysteresis loss $W_h \propto B^{1.6}_{\max}$

$$= PB^{1.6}_{\max} f$$

Eddy current loss $W_e \propto B^2_{\max} f^2$

$$= QB^2_{\max} f^2$$

From the relation

$$E = 4.44 f N B_{\max} A.$$

We get $B_{\max} \propto E/f$
Putting this value of $B_{\max}$ in the above equations, we have

$$W_h = P\left(\frac{E}{f}\right)^2 f = PE^{1.6} f^{-0.6}$$

And

$$W_e = Q\left[\frac{E}{f}\right]^2 f^2 = QE^2$$

From the given data, we have

$$3000 = PE^{1.6} f^{-0.6} \quad (1)$$

And

$$1000 = QE^2 \quad (2)$$

where E and f are the normal values of primary voltage and frequency, respectively.

If the voltage is increased by 10% at rated frequency, the new hysteresis loss is

$$W_h = P(1.1E)^{1.6} f^{-0.6} \quad (3)$$

Dividing Eq. (3) by (1),

We get

$$\frac{W_h}{3000} = 1.1^{1.6}$$

$$W_h = 3000 \times 1.165 = 3495 \text{ W}$$

The new eddy current loss is

$$W_e = Q(1.1E)^2 \Rightarrow \therefore \frac{W_e}{1000} = 1.1^2$$

$$W_e = 1000 \times 1.21 = 1210 \text{ W}$$

Example 11: Reducing the frequency by 10% with normal voltage impressed.

Solution: Since $W_e = QE^2$, eddy current loss is unaffected with change in the supply frequency. The new hysteresis loss is,

$$W_h = PE^{1.6}(0.9) f^{-0.6} \quad (4)$$

From Eq. (1) and (4), we get

$$\frac{w_h}{3000} = 0.9^{-0.6}$$

$$W_h = 3000 \times 1.065 = 3196 \text{ W}$$

Example 12: Increasing both impressed voltage and frequency by 10 per cent

Solution: In this case, both E and f are increased by 10%. The new losses are as under;

$$W_h = P(1.1E)^{1.6} (1.1f)^{-0.6}$$

$$\frac{W_h}{3000} = 1.1^{1.6} \times 1.1^{-0.6}$$

$$= 1.165 \times 0.944$$

$$W_h = 3000 \times 1.165 \times 0.944$$

$$= 3299 \text{ W}$$

The eddy current loss, W_e depends only on applied voltage and is unaffected by changes in f,

$$W_e = Q(1.1)E^2$$

$$\therefore \quad \frac{W_e}{1000} = 1.1^2$$

$$= 1000 \times 1.21 = 1210 \text{ W}.$$

Voltage Regulation of a Transformer

When a transformer is supplied with a constant primary voltage, the secondary terminal voltage varies as the load current and p.f. of the load vary. The voltage regulation is defined as the change in secondary voltage, expressed as a percentage (or p.u.) of the secondary rated voltage, when load at a given power factor is reduced to zero, with primary applied voltage held constant.

The per unit voltage regulation =

$$\frac{(\text{No load secondary voltage} - \text{Secondary voltage at rated load})}{\text{Secondary voltage at rated load}}$$

$$\%\text{Regulation} = (\text{p.u. regulation} \times 100)$$

The magnitude of change in secondary voltage depends on the load power factor, load current total resistance and total reactance of the transformer.

$$\text{Voltage regulation} = \frac{E_2 - V_2}{E_2} \text{ in p.u.}$$

$$= \frac{E_2 - V_2}{E_2} \times 100 \text{ in percentage}$$

From the approximate equivalent circuit, the expression for voltage regulation can be given as follows.

$$E_2 - V_2 = I_2 R_{02} \cos\phi \pm I_2 X_{02} \sin\phi$$

[+ve and –ve signs are taken for lagging and leading p.f.'s, respectively]

where $\cos\phi$ is load power factor.

Therefore per unit voltage regulation

$$\varepsilon = \varepsilon_r \cos\phi \pm \varepsilon_x \sin\phi$$

where ε – p.u. voltage regulation

ε_r – p.u. resistance

ε_x – p.u. reactance

+ for lagging p.f. and

– for leading p.f.

- condition for maximum voltage regulation:

Since, $|I| R_{02} \cos\theta - |I| X_{02} \sin\theta = 0$

$$\phi = \tan^{-1}\left[\frac{X_{01}}{R_{01}}\right] = \tan^{-1}\left[\frac{X_{02}}{R_{02}}\right]$$

Maximum regulation occurs at a lagging p.f.

Condition for Zero Voltage Regulation

$$\phi = \tan^{-1}\left[\frac{R_{01}}{X_{01}}\right] = \tan^{-1}\left[\frac{R_{02}}{X_{02}}\right]$$

Zero voltage regulation occurs at a leading p.f.

Example 13: The equivalent resistance and reactance of a transformer are, respectively, 0.6 Ω and 0.8 Ω. At what power factor the voltage regulation will be zero?

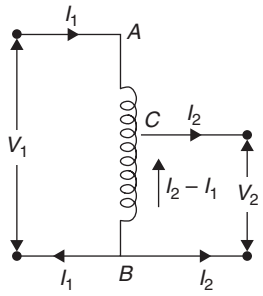
Solution: % Regulation is zero at a p.f. of

$$\phi = \tan^{-1}\left[\frac{R}{X}\right] \text{leading} \Rightarrow \phi = \tan^{-1}\left[\frac{0.6}{0.8}\right]$$

$$\cos\phi = \cos\left(\tan^{-1}\left(\frac{6}{8}\right)\right)_{\text{lead}}$$

Auto-transformer

It is a transformer with one winding only, part of this being common to both primary and secondary. The primary and secondary are connected electrically as well as coupled magnetically.



- There is a superimposition of the input and output currents in the part of the winding common to primary and secondary.
- As an ordinary transformer, the transformer ratio,

$$\frac{V_2}{V_1} = \frac{E_{BC}}{E_{AB}} = \frac{N_2}{N_1} = \frac{I_1}{I_2} = K$$

Power is transferred from the primary to the secondary both inductively and conductively.

$$\text{Power delivered to load } P = V_2 I_2$$

Power transformed or power transferred inductively $= V_2 I_2 (1 - K)$

$$\text{Power transferred conductively} = K V_2 I_2$$

- As compared to an ordinary two-winding transformer of same output, an auto-transformer uses less copper

because of one winding, has higher efficiency, smaller in size and cheaper.
- Since the HV and LV windings are electrically connected, a fault on the HV side may cause a high voltage to appear on LV side.
- Weight of Cu in auto-transformer

$= (1-K) \times$ weight of Cu in two-winding transformer

Saving $= K \times$ weight of Cu in two-winding transformer.

- $\dfrac{\text{kVA rating as an auto-transformer}}{\text{kVA rating as two-winding transformer}} = \dfrac{1}{1-K}$
- $\dfrac{\text{Full load losses as an auto-transformer}}{\text{Full load losses as a two-winding transformer}} = (1-K)$

$$\frac{\text{p.u. impedance drop in auto-transformer}}{\text{p.u. impedance drop in two-winding transformer}} = 1-K$$

$$\frac{\text{Voltage regulation as an auto-transformer}}{\text{Voltage regulation in two-winding transformer}} = 1-K$$

Applications

1. It is used as an auto starter for induction motor.
2. It is used to give small boost to a distribution line.
3. It is used in testing laboratories.

Conversion of Two-winding Transformer into Auto-transformer

Any two-winding transformer can be converted into an auto-transformer either step-down or step-up.

There are two possibilities of such conversion.

1. Connecting two windings with additive polarity to get a step-up auto-transformer.
2. Connecting two windings with subtractive polarity to get a step-down auto-transformer.

For example consider a 20-kVA, 2400/240 V two-winding transformer with its polarity markings shown in figure 7(a).

(a) Additive polarity

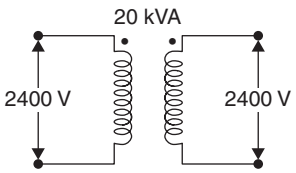


Figure 7(a)

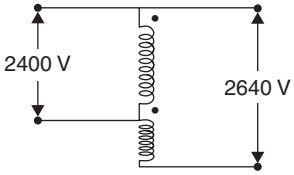


Figure 7(b)

Because of additive polarity

$V_2 = 2400 + 240 = 2640$ and

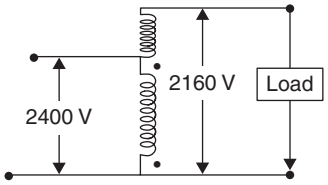
$V_1 = 2400$ V

There is a marked increase in the kVA rating of the auto-transformer.

kVA rating of auto-transformer

$$= \frac{V_2 I_2}{1000} = \frac{2640 \times 83.3}{1000} = 220 \text{ kVA}$$

(b) Subtractive polarity



Here $V_2 = 2400 - 240 = 2160$ V

In this case also there is a large increase in the kVA rating of auto-transformer.

kVA rating of auto-transformer

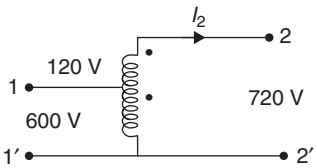
$$= \frac{V_2 I_2}{1000} = \frac{2160 \times 83.3}{1000} = 180 \text{ kVA.}$$

When two-winding transformer is converted to an auto-transformer some portion (or common portion) of winding is overloaded. The increase in kVA capacity is due to power transfer both by conduction and induction.

Example 14: A single-phase transformer has rating of 15 kVA, 600/120 V. It is recommended as an auto-transformer to supply at 720 V from a 600 V primary source. The maximum load it can supply is _____

Solution: Current rating of L.V. winding

$$= \frac{15 \times 10^3}{120} = 125 \text{ A}$$



Maximum load can be supplied, if the two windings are connected in additive polarity as shown in above figure.

$$\text{Output kVA} = \frac{720 \times 125}{1000} = 90 \text{ kVA}$$

Parallel Operation of Single-Phase Transformers

For supplying a load in excess of the rating of an existing transformer, a second transformer may be connected in parallel with it. If the demand is more than present supply, two or more transformers are placed in parallel instead of a single large unit, due to which the power system becomes more

reliable. Transformers are switched off or added as per load requirement.

There are certain definite conditions which must be satisfied in order to avoid any local circulating currents and to ensure that transformers share the common load in proportion to their kVA ratings. The conditions are

1. Primary windings of the transformers should be suitable for the supply system voltage and frequency
2. The transformers should be properly connected with regard to polarity.
3. The voltage ratings of both primaries and secondaries should be identical. In other words, the transformers should have the same turn ratio, i.e., transformation ratio.
4. The percentage impedance should be equal in magnitude and have the same X/R ratio in order to avoid circulating currents and operation at different power factors.
5. With transformers having different kVA ratings, the equivalent impedances should be inversely proportional to the individual kVA rating if circulating currents are to be avoided.

Of these conditions, condition (2) is absolutely essential (otherwise paralleling with incorrect polarities will result in dead short-circuit).

If condition (3) is not exactly satisfied, i.e., the two transformers have slightly different transformation or voltage ratios, even then parallel operation is possible. But due to inequality of induced e.m.f.'s in secondaries, there will be some circulating current between the secondaries even on no-load. When secondaries are loaded, this localized circulating current will tend to produce unequal loading condition. Hence, it may be impossible to take full kVA output from the parallel connected group without one of the transformers becoming overheated.

If condition (4) is not exactly satisfied, i.e., impedance triangles are not identical in shape and size, parallel operation will still be possible, but the power factors at which the transformers operate will be different from the power factor of common load. Therefore in this case, the two transformers will not share the load in proportion to their kVA ratings.

The currents carried by the two transformers are proportional to their ratings, provided the numerical impedances are inversely proportional to these ratings and their percentage impedances are identical.

If E_A and E_B are the no-load induced e.m.f.'s of the two transformers connected in parallel, 'I_A' and 'I_B' are the currents delivered and 'Z_A' and 'Z_B' are the ohmic impedances referred to the L.V side of the two transformers and Z_L is the Load impedance then

$$I_A = \frac{\left[E_A Z_B + (E_A - E_B)Z_L\right]}{(Z_A + Z_B)Z_L + Z_A Z_B} \quad I_B = \frac{\left[E_B Z_A + (E_A - E_B)Z_L\right]}{(Z_A + Z_B)Z_L + Z_A Z_B}$$

Let 'S', 'S_A' and 'S_B' be the total kVA, kVA delivered by the transformer 'A' and that delivered by the transformer 'B', respectively, then we have

$$S_A = S\frac{Z_B}{Z_A + Z_B} \quad S_B = S\frac{Z_A}{Z_A + Z_B}$$

Example 15: Two 1-ϕ transformers are connected in parallel at no-load. One has a turn ratio of 5000/440 and a rating of 200 kVA, the other has a ratio of 5000/480 and a rating of 350 kVA. The leakage reactance of each is 3.5%. What is the no-load circulation current expressed as a percentage of the nominal current of the 200 kVA transformer.

Solution: Assume that per turn voltage is 1V. the normal currents are

$$\frac{200\times10^3}{440} = 455 \text{ A}$$

And $$\frac{350\times10^3}{480} = 730 \text{ A}$$

Reactances seen from the secondary side are

$$\frac{3.5}{100}\times\frac{440}{455} = 0.034\ \Omega$$

$$\frac{3.5}{100}\times\frac{480}{730} = 0.023\ \Omega$$

The difference of induced voltages is 40 V. The circulating current is

$$I_C = \frac{40}{(0.034 + 0.023)} = 704 \text{ A}$$

= 1.55 times or 155% the normal current of 200 kVA unit.

Example 16: A 500-kVA single-phase transformer with 0.012 p.u resistance and 0.06 p.u. reactance, is connected in parallel with a 250 kVA single-phase transformer with 0.014 p.u. resistance and 0.045 p.u. reactance to share a load of 600 kVA at 0.8 p.f. lagging. Find the load shared by each transformer.

Solution: The p.u. resistance and reactance given above refer to different ratings. These should be adjusted to some basic kVA.

Let Base kVA = 500

$$Z_A \text{ p.u.} = 0.012 + j0.06$$
$$= 0.061\angle 78.69°$$

$$Z_B \text{ p.u.} = (0.014 + j0.045)\times\frac{500}{250}$$
$$= 0.094\angle 72.7°$$

Load shared by transformer A

$$S_A = \frac{Z_B}{Z_A + Z_B}\cdot S$$

$$= \frac{0.094\angle 72.7 \times 600\angle -36.87}{0.1552\angle 75.07}$$

$$= 364.37\angle -39.22°$$

364 kVA at a p.f. of 0.774 lag

Load shared by transformer B

$$S_B = \frac{Z_A \cdot S}{Z_A + Z_B}$$

$$= \frac{0.0612\angle 78.69 \times 600\angle -36.87°}{0.1552\angle 75.07}$$

$$= 236.60\angle -33.25$$

236.6 kVA at a p.f. of 0.836 lag.

Example 17: Two transformers A and B are connected in parallel to a load of 2 + j1.5 ohms. Their impedances in secondary terms are Z_A = (0.15 + j0.5) ohm and Z_B = (0.1 + j0.6 ohm). Their no-load terminal voltages are E_A = 207∠0° V and E_b = 205∠0° V. Find the power output and power factor of each transformer.

Solution: $I_A = \dfrac{E_A Z_B + (E_A - E_B) Z_L}{Z_A Z_B + Z_L (Z_A + Z_B)}$

$Z_A = (0.15 + j0.5)\ \Omega$

$Z_B = (0.1 + j0.6)\ \Omega;$

$Z_L = (2 + j1.5) = 2.5\angle 36.9°$

$$I_A = \frac{207(0.1 + j0.6) + (207 - 205)(2 + j1.5)}{(0.15 + j0.5) \times (0.1 + j0.6) + (2 + j1.5)(0.25 + j1.1)}$$

$$= \frac{24.7 + j127.2}{-1.435 + j2.715} = \frac{129.7\angle 79°}{3.07\angle 117.9°}$$

$$= 42.26\angle -38.9°$$

$$= (32.89 - j26.55)\ \text{A}$$

$$I_B = \frac{E_B Z_A - (E_A - E_B) Z_L}{Z_A Z_B + Z_L (Z_A + Z_B)}$$

$$= \frac{205(0.15 + j0.5) - 2(2 + j1.5)}{-1.435 + j2.715}$$

$$= \frac{103\angle 75°}{3.07\angle 117.9°} = 33.56\angle -42.9°$$

$$= (24.58 - j22.84)\ \text{A}$$

$V_2 = IZ_L = (I_A + I_B) Z_L$

$= (57.47 - j49.39)\ (2 + j1.5)$

$= 189 - j12.58 = 189.4\angle -3.9°$

Power factor angle of transformer A

$= -3.9° - (-38.9°) = 35°$

∴ p.f. of A = cos35° = 0.818 lag

p.f. of B = cos(–3.9°) – (–42.9°)

= 0.776 lag

Power output of transformer

A is $P_A = 189.4 \times 42.26 \times 0.818 = 6548$ W

Similarly $P_B = 189.4 \times 33.56 \times 0.776 = 4900$ W

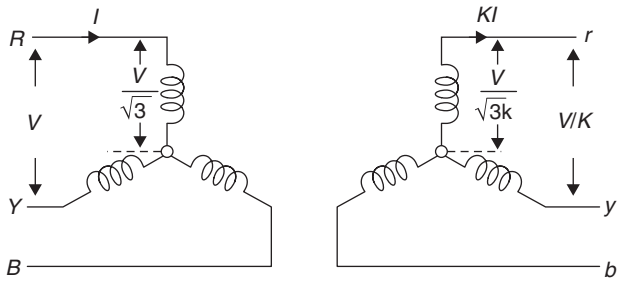
Three-phase Transformers

Three-phase transformation can be obtained by having a single, three-phase transformer or by suitable inter connection of three single-phase transformers.

As compared to a bank of single-phase transformers, the main advantages of a three-phase transformer are that it occupies less floor space for equal rating, weight less, costs about 15% less and further, that only one unit is to be handled and connected. However, if a fault occurs, all the loads connected to the three-phase transformer will be interrupted till the repair works are carried out. But, in the case of 3-phase bank of single-phase transformers if one transformer goes out of order, the system can still be run open-Δ at reduced capacity or the faulty transformer can be readily replaced by a single spare.

Three-phase Transformer Connections

Star/Star: 1:1



This connection is most economical for small, high-voltage transformers because the number of turns/phase and the amount of insulation required is minimum (as phase voltage is only $1/\sqrt{3}$ of line voltage). The ratio of line voltages on the primary and secondary is the same as the transformation ratio of each transformer.

There is a phase shift of 30° between the phase voltages and line voltages both on primary and secondary sides.

This connection works only if the load is balanced. With the unbalanced load, the neutral point shifts thereby making the three line-to-neutral voltages unequal. This difficulty of shifting (or floating) neutral can be obviated by connecting the primary neutral to neutral of generator. Star-Star transformers are operated with grounded neutrals.

With an isolated neutral, any unbalanced load causes the shifting of the neutral.

For delivering a sine wave of voltage, it is necessary to have a sine wave of flux in the core but on account of the characteristics of iron, a sine wave of flux requires a third harmonic component in the exciting current. At any instant the third harmonic component tends to flow towards or

away from neutral. If primary neutral is isolated, the third harmonic components of the magnetizing current do not have a path. Hence, the flux in the core cannot be a sine wave and so the phase voltages become non-sinusoidal, though the line voltages are sinusoidal.

Another way of avoiding this trouble of oscillating neutral is to provide each of the transformers with a third or tertiary winding of relatively low kVA rating. This tertiary winding is connected in Δ and provides a circuit in which the third harmonic component can flow.

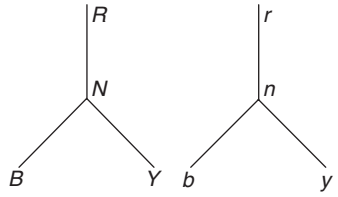
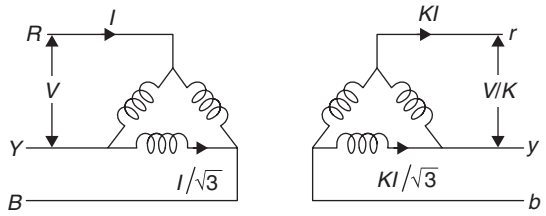


Figure 8 Phasor diagram

Delta/Delta



This connection is economical for large, low voltage transformers in which insulation problem is not so urgent, because it increases number of turns/phase.

The ratio of transformation between primary and secondary line voltage is exactly same as that of each transformer and there is no angular displacement between them. This connection has the following advantages:

1. In order that the output voltage be sinusoidal, it is necessary that the magnetizing current of the transformer must contain a third order harmonic component. In this case, the third harmonic component of the magnetizing current can flow in the Δ-connected transformer primaries without flowing in the line wires. The three phases are 120° apart, which is 3 × 120° = 360° with respect to the third harmonic, hence it merely circulates in the delta. Therefore the flux is sinusoidal, which results in sinusoidal voltages.
2. No difficulty is experienced from unbalanced loading as was the case in star-star connection. The three-phase voltages remain practically constant regardless of the load impedance.
3. An added advantage of this connection is that if one of the transformer phases becomes disabled, the system can continue to operate in open-delta or in *V–V* although with reduced available capacity. The reduced capacity is 58% and not 66.7% of the normal capacity.

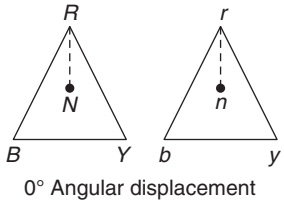
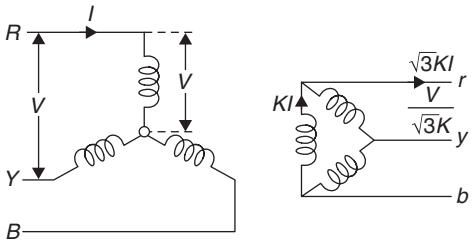


Figure 9 Phasor diagram

Star/Delta



The main use of this connection is at the substation end of the transmission line where the voltage is to be stepped down. The primary winding is star connected with grounded neutral.

The ratio between the secondary and primary line voltage is $1/\sqrt{3}$ times the transformation ratio of each transformer. There is a 30° shift between the primary and secondary voltages which means that a star-delta transformer bank cannot be paralleled with either a star-star or a delta-delta bank.

The third harmonic currents flow in the delta to provide a sinusoidal flux.

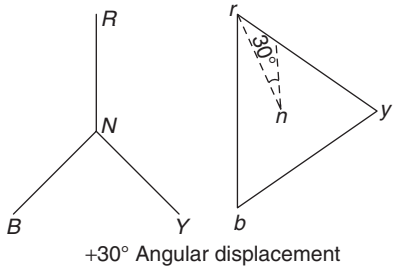
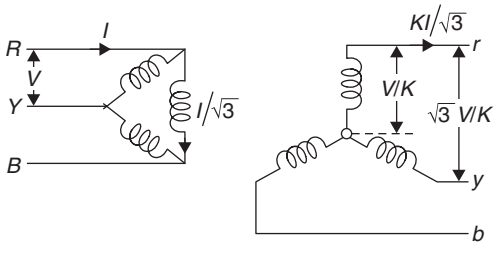


Figure 10 Phasor diagram

Delta/Star



This connection is generally employed where it is necessary to step up the voltage as for example, at the beginning of high tension transmission system. The neutral of the secondary is grounded for providing 3-phase, 4-wire service.

- This is the most popular connection as it can be used to serve both the 3-phase power equipment and single-phase lighting circuits.
- This connection is not open to the objection of a floating neutral and voltage distortion because the existence of a Δ-connection allows a path for the third-harmonic currents.
- The primary and secondary line voltages and line currents are out of phase with each other by 30°. Because of this 30° shift, it is impossible to parallel such a bank of transformers even though the voltage ratios are correctly adjusted. The ratio of secondary to primary voltage is $\sqrt{3}$ times the transformation ratio of each transformer.

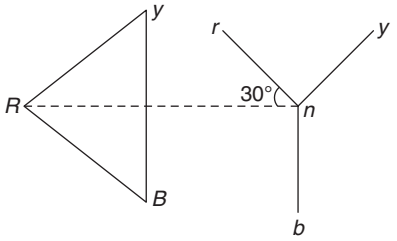


Figure 11 Phasor diagram

Example 18: The percentage impedance of a 100 kVA, 11 kV /400 V, delta/wye, 50 Hz transformer is 4.55 for the circulation of half the full-load current during short-circuit test, with low voltage terminals shorted, what should be the applied voltage on high voltage side?

Solution: Full-load primary current

$$I_1 = \frac{100\times10^3}{11\times10^3} = 9.09 \text{ A}$$

$$Z_{01} = \%\frac{Z\times V_1}{100\ I_1}$$

$$= \frac{4.5\times11\times10^3}{100\times9.09} = 55\ \Omega$$

For half full-load current during short-circuit test

$$Z_{01} = \%\frac{Z\times V_1}{100\times I_1/2}$$

$$Z_{01} = \frac{4.5\times11\times10^3}{100\times\frac{9.09}{2}} = 110\ \Omega$$

$$\text{Half full-load current} = \frac{9.09}{2} = 4.5$$

$$\text{Applied voltage} = 4.5\times55$$

$$= 247.5 \text{ V}$$

Open-Delta or V–V Connection

If one of the transformers of Δ–Δ is removed and 3-phase supply is connected to the primaries as shown in fig, then three equal 3-phase voltages will be available at the secondary terminals on no-load. This method of transforming 3-phase power by means of only two transformers is called the open-Δ or *V–V* connection.

The total load that can be carried by a *V–V* bank is not two-third of the capacity of a Δ–Δ bank but it is only 57.7% of it.

$$\frac{V-V\text{ capacity}}{\Delta-\Delta\text{ capacity}} = \frac{\sqrt{3}V_L I_S}{3V_L I_S}$$

$$= \frac{1}{\sqrt{3}} = 0.577 \text{ or } 58\%$$

i.e., only 86.6% of the rated capacity of the two remaining transformers is available. In other words, ratio of operating capacity to available capacity of an open Δ is 0.866 or utility factor is 0.866.

Each transformer will supply 57.7% of load and not 50% when operating in *V–V*.

If three transformers in a Δ–Δ bank are delivering their rated load and one transformer is removed, the overload on each of the two remaining transformers is 73.2% because

$$\frac{\text{Total load in } V-V}{\text{VA of transformer}} = \frac{\sqrt{3}V_L I_S}{V_L\ I_S} = \sqrt{3} = 1.732$$

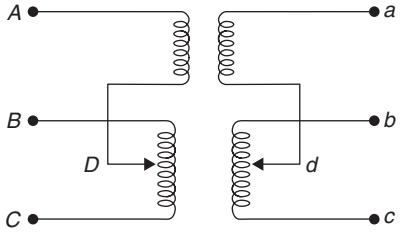
The disadvantages of this connection are

1. The average power factor is 86.6% of the balanced load power factor. One of two transformers operates at a p.f. of cos(30 + θ) and the other at cos(30 – θ).
2. The secondary terminal voltages tend to become unbalanced to a great extent when the load is increased, this happens even when the load is perfectly balanced.

SCOTT Connection or T-T Connection

This is a connection by which 3-phase to 3-phase transformation is accomplished using two transformers as shown in below figure.

This connection can also be used for 3-phase to 2-phase transformation.



If the two transformers are of identical voltage ratings, one of the transformers has centre taps both on the primary and

secondary windings and is known as the main transformer. The other has 0.866 tap and is known as teaser transformer.

- Alternatively teaser may be designed for 86.6% voltage of the main transformer.
- If the load power factor is $\cos\theta$ lag then
  (a) Teaser transformer operators at a lagging p.f. of $\cos\theta$
  (b) one half of the main transformer operates a p.f. of $\cos(30 + \theta)$ lag
- If V is the line voltage and I is the rated line current, the combined VA rating of the two transformers = VI (1 + 0.866) if both the teaser primary and secondary windings are designed for 0.866 times the voltage on the main transformer.

$$\text{Output} = \sqrt{3}\ VI$$

$$\text{Utilization factor} = \frac{\sqrt{3}VI}{(1+0.866)VI} = 0.928$$

If the two transformers are of equal voltage rating (and with taps at 86.6% points), 13.4% of the winding is unutilized.

$$\text{Then utility factor of the winding} = \frac{\sqrt{3}VI}{2\ VI} = 0.866$$

Example 19: A 5000 kVA, three-phase transformer 6.6/33-kV, Δ/Y, has a no-load loss of 15 kW and a full-load loss of 50 kW. The impedance drop at full load is 7%. Calculate the primary voltage when a load of 3,200 kW at 0.8 p.f. is delivered at 33 kV.

Solution: Full-load $I_2 = 5 \times 10^6/\sqrt{3} \times 33{,}000$
$= 87.5$ A

Impedance drop /phase

$= 7\%$ of $\left(33/\sqrt{3}\right)$ kV

$= 1330$ V

$$\therefore Z_{02} = \frac{1330}{87.5} = 15.3\ \Omega/\text{phase}$$

F. L. Cu loss = 50 – 15 = 35 kW

$$\therefore 3I^2_2\ R_{02} = 35{,}000$$

$$R_{02} = 35{,}000/3 \times (87.52)^2$$

$$= 1.53\ \Omega/\text{phase}$$

$$X_{02} = \sqrt{15.3^2 - 1.53^2} = 15.23\ \Omega$$

When the load is 3200 kW at 0.8 p.f.

$$I_2 = 3{,}200 / \sqrt{3} \times 33 \times 0.8 = 70\ \text{A}$$

Voltage drop = $70(1.53 \times 0.8 + 15.23 \times 0.6)$

$= 725$ V/phase

$$\therefore \%\ \text{regulation} = \frac{725 \times 100}{19{,}000} = 3.8\%$$

Primary voltage will have to be increased by 3.8%

∴ Primary voltage = 6.6 + 3.8% of 6.6

$= 6.58$ kV $= 6580$ V

Common data for Examples 20 and 21

A Δ–Δ bank consisting of three 30-kVA 2300/230-V transformer supplies a load of 60 kVA. If one transformer is removed, find for the resulting V–V connecting.

Example 20: (a) KVA load carried by each transformer (b) Per cent of rated load carried by each transformer

Solution: (a) $\dfrac{\text{Total kVA load in } V-V \text{ bank}}{\text{VA/transformer}} = \sqrt{3}$

$$\therefore \text{kVA load supplied by each of the two transformers} = \frac{60}{\sqrt{3}}$$

$$= 34.64\ \text{kVA}$$

Obviously, each transformer in V–V bank does not carry 50% of the original load but 57.7%

(b) Percentage of rated load

$$= \frac{\text{KVA load/ transformer}}{\text{kVA rating /transformer}}$$

$$= \frac{34.64}{30} = 115.5\%$$

Carried by each transformer obviously, in this case, each transformer is overloaded to the extent of 15.5 per cent.

Example 21: (a) Total kVA rating of the V–V bank (b) Ratio of the V–V bank to Δ–Δ bank transformer ratings (c) Percentage increase in load on each transformer

Solution: (a) kVA rating of the V–V bank = $(2 \times 30) \times 0.8666 = 51.96$ kVA

(b) $\dfrac{V-V \text{ rating}}{\Delta-\Delta \text{ rating}} = \dfrac{51.96}{90}$

$$= 0.577 = 57.7\%$$

As seen, the rating is reduced to 57.7% of the original rating.

(c) Load supplied by each transformer in Δ–Δ bank = 60/3 = 20 kVA

∴ Percentage increase in load supplied by each transformer

$$= \frac{\text{kVA load/transformer in } V-V \text{ bank}}{\text{kVA load/transforemer in } \Delta-\Delta \text{ bank}}$$

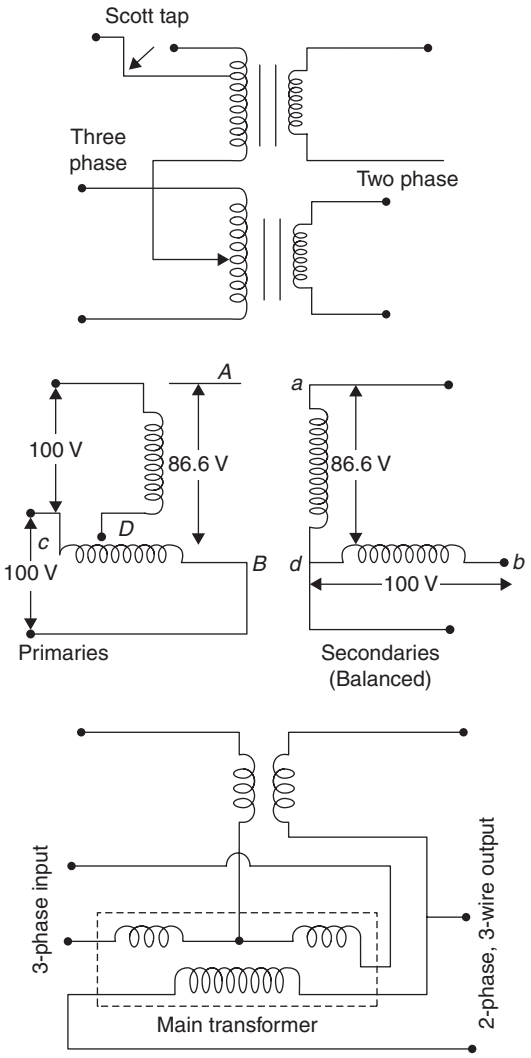
$$= \frac{34.64}{20} = 1.732 = 173.2\%$$

It is obvious that each transformer in the Δ–Δ bank supplying 60 kVA was running underloaded (20 vs. 30 kVA) but runs over loaded (34.64 vs. 30) in V–V connection.

Three-phase to Two-phase Conversion

This conversion is required (i) to supply 2-phase furnaces. (ii) to interconnect 2-phase systems with 3-phase systems. (iii) to supply a 3-phase apparatus from 2-phase supply source.

For this purpose, Scott connection as shown in following fig is employed.



- Teaser transformer primary has $\sqrt{3}/2$ times the turns of main primary. But volt /turn is the same. Their secondaries have the same turns which results in equal secondary terminal voltages.
- If main primary has N_1 turns and main secondary has N_2 turns then main transformation ratio is N_2/N_1. However the transformation ratio of teaser is

$$\frac{N_2}{\left(\sqrt{3}N_1/2\right)} = 1.15N_2/N_1 = 1.15\ \text{K}$$

- If the load is balanced on one side, it is balanced on the other side as well.
- Under balanced condition, main transformer rating is 15% greater than that of the teaser.
- The currents in either of the two halves of main primary are the vector sum of KI_{2M} and 0.58 KI_{2T} (or) $\left(\frac{1}{2}\ I_{1T}\right)$.

Parallel Operations of Three-phase Transformers

All the conditions which apply to the parallel operation of single-phase transformer also apply to the parallel operation of three-phase transformer but with following conditions.

1. The voltage ratio must refer to the terminal voltage of primary and secondary. It is obvious that this ratio may not be equal to the ratio of the number of turns per phase. For example if V_1, V_2 are the primary and secondary terminal voltages then for star/delta connection, this turns ratio is $V_2/(V_1/\sqrt{3})$
$$= \sqrt{3}\ V_2/V_1.$$
2. The phase displacement between primary and secondary voltages must be the same for all transformers which are to be connected for parallel operation.
3. The phase sequence must be the same.
4. All the three transformers in the three-phase transformer bank will be of the same construction either core or shell.

Exercises

Practice Problems I

Directions for questions 1 to 45: Select the correct alternative from the given choices.

1. In a transformer ratio of the copper losses at half load to that at full load will be

(A) 2 times (B) one fourth
(C) half (D) 4 times

2. The eddy current loss of a transformer at 120 V, 60 Hz is 150 W. The eddy current loss at 120 V, 30 Hz will be

(A) 150 W (B) 75 W
(C) 37.5 W (D) 300 W

3. The ratio of primary to secondary turns of a transformer is 4. The primary resistance is 1 Ω and the secondary resistance is 0.5 Ω. The resistance of the transformer referred to primary side is

(A) 9 Ω (B) $\frac{1}{9}\Omega$
(C) 3 Ω (D) $\frac{1}{3}\Omega$

4. At 50 Hz operation, a single-phase transformer has hysteresis and eddy current losses of 300 W and 150 W, respectively. Its core loss at 60 Hz will be

(A) 450 W (B) 576 W
(C) 648 W (D) 540 W

5. When the iron and full-load copper losses in a transformer are 900 W and 1600 W, respectively, the maximum efficiency occurs at _____ of full load.
(A) 25% (B) 50%
(C) 75% (D) 100%

6. A transformer has 600 primary turns and 120 secondary turns. If the load current is 10 A, its primary load component of current is _____
(A) 20 A (B) 10 A
(C) 2 A (D) 5 A

7. A transformer possesses a percentage resistance and percentage reactance of Z2% and 4%, respectively. Its voltage regulation at power factor of 0.6 leading and 0.6 lagging would be, respectively,
(A) –2% and –2% (B) 4.4% and –2%
(C) –2% and 4.4% (D) 4.4% and –4.4%

8. If the applied voltage of a certain transformer is increased by 100% and the frequency is reduced to 25% (assuming that the magnetic circuit remains unsaturated) the maximum flux density will change to _____ times the original value
(A) 2 (B) 4
(C) 6 (D) 8

9. A 50 kVA transformer has a core loss of 500 W and a full-load copper loss of 1000 W. Maximum efficiency occurs at _____ per cent of full load
(A) 50 (B) 37.5
(C) 70.7 (D) 100

10. A single-phase transformer has rating of 10 kVA, 400/100 V. It is recommended as an auto-transformer to supply at 500 V from a 400 V primary source. The maximum load which it can supply is
(A) 50 kVA (B) 25 kVA
(C) 100 kVA (D) 12.5 kVA

11. An auto-transformer having a transformation ratio of 0.5, supplies a load of 20 kW. The power transferred inductively from the primary to secondary is
(A) 20 kW (B) 10 kW
(C) 5 kW (D) zero

Common Data for Questions 12 and 13:
A single-phase transformer on full load has an impedance drop of 25 V and resistance drop of 15 V.

12. The reactance drop is ___ V
(A) 15 (B) 20
(C) 25 (D) 30

13. The p.f. at which regulation will be zero is _____
(A) zero (B) 0.6
(C) 0.8 (D) 0.707

Common Data for Questions 14 and 15:
The exciting current was found to be 4A when measured on the LV side of a 10 – kVA, 1000/100 V transformer. Its equivalent impedance referred to the HV side is $8.5 + j12.5$ (Assume that the transformer rating as the base.)

14. The exciting current in p.u. on the LV as well as HV side, respectively _____
(A) 0.04, 0.02 (B) 0.02, 0.04
(C) 0.04, 0.04 (D) 0.08, 0.08

15. The equivalent impedance in p.u. on the HV side is
(A) $0.105 + j0.085$ (B) $0.21 + j0.17$
(C) $0.085 + j0.105$ (D) None of these

16. Two transformers of the same type, using the same grade of iron and conductor materials, are designed to work at the same flux and current densities, but the linear dimensions, of one are two times those of the other in all respects. The ratio of kVA of the two transformers closely equals
(A) 2 (B) 4
(C) 8 (D) 16

17. Two transformers of different kVA ratings working in parallel share the load In reply to: proportion to their ratings when their
(A) ohmic values of the leakage impedances are inversely proportional to their ratings.
(B) ohmic values of the leakage magnetizing reactances are the same.
(C) per unit leakage impedances on the same kVA base are the same.
(D) per unit leakage impedances on their respective ratings are equal.

18. The percentage impedance of a 100 kVA, 11 kV/400 V, delta/wye, 50 Hz transformer is 5%. For the circulation of half the full-load current during short-circuit test, with low voltage terminals, shorted, the applied voltage on the high voltage side will be
(A) 245 V (B) 275 V
(C) 315 V (D) 550 V

19. The laws of electromagnetic induction are summarized in which of the following equation?
(A) $e = iR$ (B) $e = \frac{d\psi}{dt}$
(C) $e = -\frac{Ldi}{dt}$ (D) $e = \frac{Nd\phi}{dt}$

20. Keeping in view the requirement of parallel operation, which of the 3-phase connections given below are possible?
(A) star–star to delta–star
(B) delta–delta to star–delta
(C) delta–delta to delta– star
(D) delta–star to star–delta

21. If an AC voltage wave is corrupted with an arbitrary number of harmonics, then the overall voltage wave form differs from its fundamental frequency component in terms of
(A) only the peak values.
(B) only the average values.

(C) only the rms values.
(D) all the three measures (Peak, rms and average values).

22. A single-phase transformer is to be switched to the supply to have minimum inrush current. The switch should be closed at
(A) zero supply voltage.
(B) $\frac{1}{2}$ maximum supply voltage.
(C) $\frac{1}{\sqrt{2}}$ maximum supply voltage.
(D) maximum supply voltage.

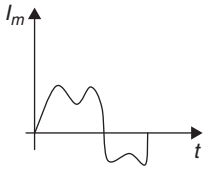
23. In the protection of transformers harmonic restraint is used to guard against
(A) switching.
(B) lightning.
(C) magnetizing in rush current.
(D) unbalanced operation.

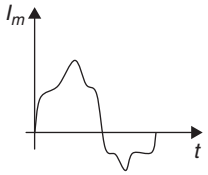
24. The efficiency of a 100 kVA transformer is 0.98 at full as well as at half full-load. For this transformer at full load, the copper loss
(A) is equal to core loss.
(B) is less than core loss.
(C) is more than core loss.
(D) None of the above

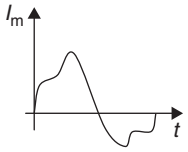
25. The core flux of a practical transformer with a resistive load
(A) increases linearly with load.
(B) increases as the square root of the load.
(C) is strictly constant with load changes.
(D) decreases with increased load.

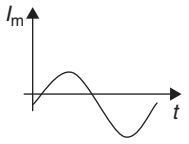
26. The single-phase, 50 Hz iron core transformer in the circuit has both the vertical arms of cross sectional area 20 cm^2 and both the horizontal arms of cross sectional area 10 cm^2. If the windings were wound instead on opposite horizontal arms, the mutual inductances will
(A) remain same.
(B) become one quarter.
(C) be halved.
(D) doubled.

27. A single-phase air core transformer, fed from a rated sinusoidal supply, is operating at no load. The steady state magnetizing current drawn by the transformers from the supply will have the wave form

(A) 

(B) 

(C) 

(D) 

28. In transformer, which of the following statements is valid?
(A) In an oc test, current is drawn at high power factor.
(B) In an oc test, copper losses are obtained, while in short-circuit test core losses are obtained.
(C) In a sc test, current is drawn at zero power factor.
(D) In an oc test, current is drawn at low power factor.

29. In a transformer, zero voltage regulation at full load is
(A) possible at unity power factor load.
(B) possible at leading power factor load.
(C) possible at lagging power factor load.
(D) Not possible.

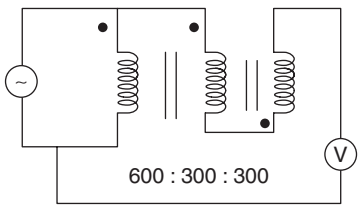
30. A 50 Hz transformer having equal hysteresis and eddy current losses at rated excitation is operated at 45 Hz at 90% of its rated voltage. Compared to rated operating point the core losses under this condition
(A) increase by 10%.
(B) reduce by 10%.
(C) reduce by 14.5%.
(D) remain unchanged.

31. The hysteresis loop of a magnetic material has an area of 10 cm^2 with the scales given as 1 cm = 2 AT and 1 cm = 50 mwb. At 50 Hz the total hysteresis loss is
(A) 10 W. (B) 25 W.
(C) 50 W. (D) 100 W.

32. The windings of a Q kVA, V_1/V_2 volt, three phase delta connected core type transformer are reconnected to work as a single-phase transformer. The maximum voltage and the power ratings are
(A) $\frac{V_1}{V_2}, Q/3$. (B) $\frac{\sqrt{3}V_1}{\sqrt{3}V_2}, 3Q$.
(C) $\frac{2V_1}{2V_2}, \frac{2Q}{3}$. (D) $\frac{\sqrt{3}V_1}{\sqrt{3}V_2}, Q$.

33. A 10 kVA 400 V/200V single-phase transformer with 5% impedance draws short-circuit line current of
(A) 50 A. (B) 100 A.
(C) 250 A. (D) 500 A.

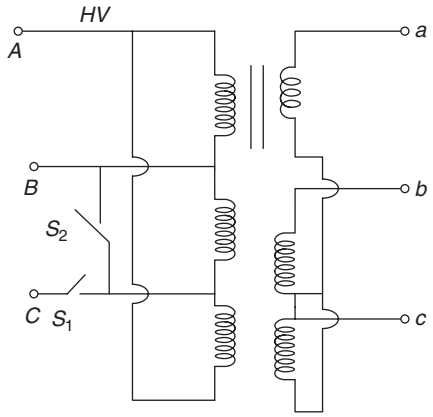
34. A 600 V / 300 V/ 300 V, 50 Hz three-winding transformer is connected as shown in figure. The reading of the voltmeter 'V' will be



(A) 0 V. (B) 300 V.
(C) 600 V. (D) 1200 V.

35. Figure shows Δ–Y connected 3-phase distribution transformer used to step down the voltage form 11 kV to 415 V line-to-line. It has two switches S_1 and S_2.

Under normal conditions S_1 is closed and S_2 is open. The magnitude of the voltage across LV terminals a and c with S_1 open and S_2 closed is

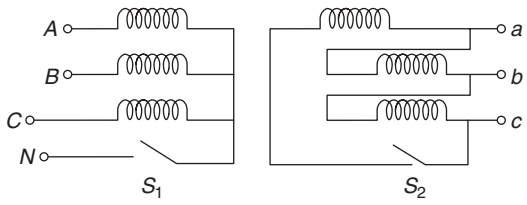


(A) 240 V. (B) 415 V.
(C) 830 V. (D) 0 V.

36. Two transformers are to be operated in parallel such that they share the load in proportion to their kVA ratings. The rating of the first transformer is 200 kVA and its p.u. leakage impedance is 0.05 p.u. If the second transformer is 100 kVA, its p.u. leakage impedance is
(A) 0.10. (B) 0.20.
(C) 0.05. (D) 0.025.

37. A single-phase 50 kVA, 250 V/500 V two-winding transformer has an efficiency of 95% at full load, unity power factor. It is reconfigured as a 500/750 V autotransformer, its efficiency at its new rated load at unity power factor will be
(A) 93.275%. (B) 95.752%.
(C) 98.276%. (D) 99.241%.

Common Data for Questions 38 and 39:

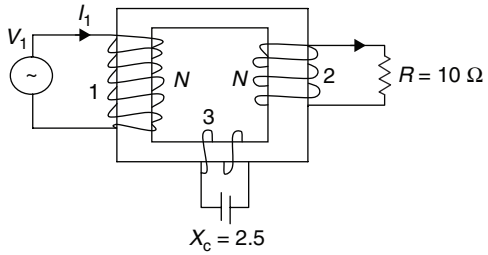


The star-delta transformer shown above is excited on the star side with balanced, 4-wire, 3-phase, sinusoidal voltage supply of rated magnitude. The transformer is under no load condition.

38. With both S_1 and S_2 open, the core flux waveform will be
(A) flat-topped with third harmonic.
(B) peaky with third harmonic.
(C) A sinusoid at fundamental frequency.
(D) flat-topped with fifth harmonic.

39. With S_2 closed and S_1 open, the current wave form in the delta winding will be
(A) flat-topped with fifth harmonic.
(B) a sinusoid with third harmonic.
(C) only third harmonic.
(D) flat-topped with third harmonic.

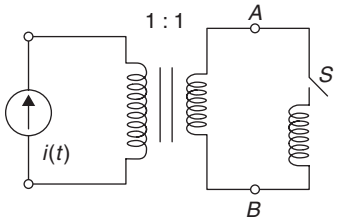
40. Figure shows an ideal three-winding transformer. The three windings 1, 2, and 3 of the transformer are wound as shown in fig. the turns ratio $N_1:N_2:N_3$ is 4:2:1, A resistor of 10 Ω is connected across winding-2. A capacitor of reactance 2.5 Ω is connected across winding – 3. Winding -1 is connected to a 200 V supply. If the supply voltage phasor $V_1 = 200 \angle 0°$, the supply current phasor I_1 is given by

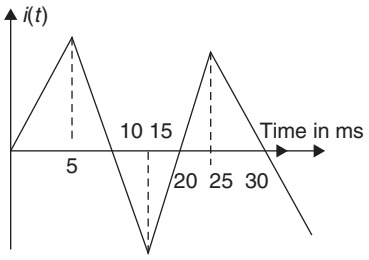


(A) $(-5 + j5)$ A. (B) $(5 + j5)$ A.
(C) $(5 - j5)$ A. (D) $(-5 - j5)$ A.

41. The resistance and reactance of a 100 kVA 11000/400 V, Δ–y distribution transformer are 0.02 and 0.07 p.u., respectively. The phase impedance of the transformer referred to the primary is
(A) $(0.55 + j1.925)\ \Omega$. (B) $(2.15 + j7.35)\ \Omega$.
(C) $(15.125 + j52.94)\ \Omega$. (D) $(72.6 + j254.1)\ \Omega$.

Common Data for Questions 42 and 43:
The circuit diagram shows a two-winding loss less transformer with no leakage flux, excited form a current source, $i(t)$, whose wave form is also shown. The transformer has a magnetizing inductance of $200/\pi$ mH.

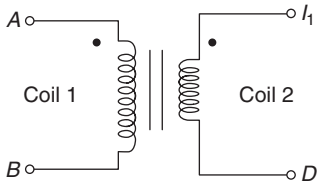




42. The peak voltage across A and B with S open is
(A) $200/\pi$ V. (B) 400 V.
(C) $\frac{2000}{\pi}$ V (D) $\frac{400}{\pi}$ V.

43. If the wave form of $i(t) = 10\sin(100\pi t)$A, the peak voltage across A and B with 'S' closed is
(A) 200 V. (B) $200 \times \sqrt{2}$ V.
(C) 400 V. (D) $400 \times \sqrt{2}$ V.

Common Data for Questions 44 and 45:



The figure above shows coils 1 and 2, with dot markings as shown, having 2000 and 3000 turns, respectively. Both the coils have a rated current of 25 A. Coil-1 is excited with single phase, 400 V, 50 Hz supply.

44. The coils are to be connected as a single phase, 400/1000 V auto-transformer to drive a load of 10 kVA. Which of the options given should be exercised to realize the required auto-transformer?
(A) connect A and C; common B.
(B) connect A and C; common D.
(C) connect A and D; common B.
(D) connect B and D; common C.

45. In the auto-transformer obtained in above question the current in each coil is
(A) coil-1 is 15 A and coil-2 is 10 A.
(B) coil-1 is 10 A and coil-2 is 25 A.
(C) coil-1 is 10 A and coil-2 is 15 A.
(D) coil-1 is 25 A and coil-2 is 10 A.

Practice Problems 2

Directions for questions 1 to 50: Select the correct alternative from the given choices.

1. The purpose of transformer oil is
(A) to provide lubrication
(B) to provide insulation and cooling
(C) to provide protection against short circuit
(D) to provide protection against lightning

2. A voltage $V = 400 \sin 314t$ is applied to a single-phase transformer on no load. If the no-load current of the transformer is $2\sin(314.16t - \angle 85°)$, then magnetization branch impedance will be approximately equal to
(A) $200 \angle -85° \ \Omega$ (B) $200 \angle 85° \ \Omega$
(C) $200 \angle 5° \ \Omega$ (D) $200 \angle -5° \ \Omega$

3. The efficiency of a 1000 kVA transformer is 95% at full load as well as at half the full- load both at u.p.f. The Cu loss at full load
(A) is less than core loss
(B) is equal to core loss
(C) is more than core loss
(D) none of these

4. The dominant harmonic in the magnetizing current of a transformer is
(A) 3rd harmonic (B) 5th harmonic
(C) 7th harmonic (D) 13th harmonic

5. A 60 Hz transformer having equal hysteresis and eddy current losses at rated excitation is operated at 48 Hz and 80% rated voltage. The core losses under this condition
(A) reduce by 20%
(B) reduce by 38%
(C) reduce by 28%
(D) remain unchanged

6. A 400 / 200 V 20 kVA two-winding transformer is reconnected as an auto-transformer. The maximum possible rating of auto-transformer could be
(A) 50 kVA (B) 15 kVA
(C) 40 kVA (D) 60 kVA

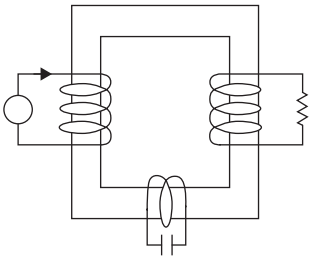
7. In a constant voltage transformer (CVT) output voltage remains constant because of
(A) saturation
(B) tapped windings
(C) capacitor across the terminals
(D) a DC voltage source across the terminals

8. Two transformers of different kVA rating working in parallel share the load in proportion to their rating when
(A) ohmic values of leakage impedances are inversely proportional to their ratings
(B) ohmic values of magnetizing reactances are same
(C) per unit leakage reactance on their respective kVA bases are equal
(D) per unit leakage impedance on same kVA base are equal

9. A transformer is subjected to a short circuit. The turns of the winding experience
(A) attractive force
(B) repulsive force
(C) no force
(D) can be attractive or repulsive

10. Maximum efficiency of a single-phase transformer is 90% at full load and u.p.f. Efficiency at half load and u.p.f. is
(A) 89.1% (B) 87.8%
(C) 90% (D) 86.8%

11.



$N_1:N_2:N_3 = 4:2:1$, $V_1 = 400 \angle 0$. The current phasor in N_1, I_1 is given by

(A) $-10 - 10j$ (B) $-10 + 10j$
(C) $10 - 10j$ (D) $10 + 10j$

12. Building steel core out of stampings reduces eddy current loss because it

(A) increases the effective length of eddy current paths thereby increasing effective resistance to flow of eddy currents
(B) reduces effective length of eddy current path, thereby reducing effective resistance to flow of eddy currents
(C) increases core resistivity
(D) increases core permeability

13. Assuming P_i is the iron loss and P_c is the copper loss, the maximum kVA delivered to the load corresponding to maximum efficiency is

(A) rated kVA $\times \sqrt{\frac{P_c}{P_i}}$ (B) rated kVA $\times \sqrt{\frac{P_i}{P_c}}$

(C) rated kVA $\times \left(\frac{P_c}{P_i}\right)^2$ (D) rated kVA $\times \left(\frac{P_i}{P_c}\right)^2$

14. A transformer of turns ratio 200/400, 50 Hz is to be excited from 40 Hz from 200 V side. For the exciting current to remain same, the applied voltage should be

(A) 300 V (B) 250 V
(C) 200 V (D) 60 V

15. The leakage reactance of a transformer can be found out using

(A) OC test only (B) SC test only
(C) both OC and SC (D) cannot be found

16. A transformer on no load, when switched on draws a current

(A) twice the steady-state magnetizing current provided core has no residual flux
(B) several times the steady-state magnetizing current independent of initial state of residual flux in the core
(C) Several times steady-state magnetizing current dependent on initial state of residual flux in core
(D) same as steady-state magnetizing current

17. If a two-winding 2/1 transformer is reconnected as auto-transformer, its kVA

(A) remains same (B) becomes 2 times
(C) becomes 3 times (D) becomes 1.5 times

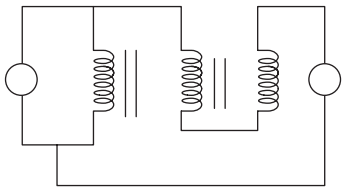
18. A transformer has 95% efficiency at full load and 0.85 p.f. lag. Efficiency at full load 0.85 pf lead will be

(A) > 95% (B) < 95%
(C) = 95% (D) 100%

19. A single-phase transformer is to be switched to have minimum inrush current. Switch should be closed at

(A) $\frac{1}{\sqrt{2}}$ times maximum value of V
(B) $\frac{1}{2}$ times maximum value of V
(C) at maximum value of V
(D) at zero value of V

20. A 400 V/200 V/200 V is connected as shown. Reading in voltmeter V is



(A) 0 V (B) 200 V
(C) 400 V (D) 800 V

21. Auto-transformer is used in transmission and distribution when

(A) operator is not available
(B) iron losses are to be reduced
(C) efficiency consideration can be ignored
(D) when transformation ratio is small

22. Which of the following 3-phase connection can possibly be operated in parallel

(A) $\Delta - \Delta$ to $\Delta - Y$ (B) $\Delta - \Delta$ to $Y - \Delta$
(C) $Y - Y$ to $\Delta - \Delta$ (D) $\Delta - Y$ to $Y - \Delta$

23. A 2400/240 V, 200 kVA single-phase transformer has a core loss of 1.8 kW at rated voltage. $R = 1.1\%$. Transformer efficiency at 0.9 p.f. and full load is

(A) 97.57% (B) 98.05%
(C) 97.82% (D) 96.56%

24. Hysteresis loss and eddy current loss of 340 W and 120 W, respectively, for a 220 V, 60 Hz single-phase transformer. If the transformer is operated from 230 V, 50 Hz supply mains then core loss assuming (Steinmetz) constant equal to 1.6 is

(A) 539.3 W (B) 408 W
(C) 349.4 W (D) 308 W

25. Full-load voltage regulation of a power transformer is zero when p.f. of load is

(A) unity and leading
(B) zero and leading
(C) zero and lagging
(D) unity and lagging

26. Three single-phase transformers each with a 10 kVA rating are connected in a closed delta arrangement. If one transformer is taken out, o/p capacity of system will be
(A) 20 kVA (B) 8.66 kVA
(C) 17.39 kVA (D) 10 kVA

27. A three-phase transformer has rating at 20 mVA, 220 kV (Y), 33 kV (Δ) with leakage reactance at 12%. The transformer reactance (in Ω) referred to each phase of L.V delta is
(A) 23.5 (B) 19.6
(C) 18.5 (D) 8

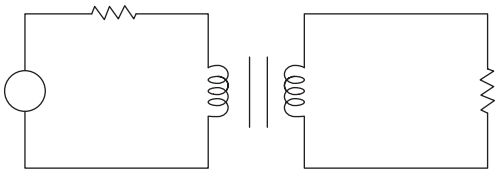
28. While performing the O.C. and S.C. tests on a transformer
(A) in O.C. LV is open, in S.C LV is short
(B) in O.C. hv is open in SC LV is short
(C) in O.C. hv is open and in S.C hv is short
(D) in O.C. lv is open and in S.C hv is short

29. Iron loss and full-load copper loss in a transformer are 40.5 kW and 50 kW. Efficiency is maximum at load fraction equal to
(A) 0.57 (B) 0.8
(C) 0.9 (D) 0.75

30. In a transformer, fed from a fundamental frequency voltage source, the source of harmonics is
(A) poor insulation
(B) iron loss
(C) overload
(D) saturation of core

31. In a 3 phase Δ–Y transformer phase displacement of secondary line voltages with primary line voltage will be
(A) 180° (B) zero
(C) 30° lead (D) 30° lag

32.

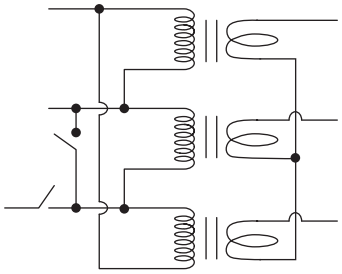


In the figure, transformer is ideal. For what number of turns in primary, maximum power is transferred.
(A) 40 (B) 80
(C) 160 (D) 320

33. A 5 kVA transformer has iron loss of 200 W and full-load copper loss of 300 W. At maximum efficiency, total loss is
(A) 200 W (B) 400 W
(C) 600 W (D) 800 W

34. A transformer has turns ratio 4:1. Reactance referred to secondary side is 0.05 p.u. Reactance referred to primary is
(A) 0.2 ohm (B) 0.05 ohm
(C) 0.2 p.u. (D) 0.05 p.u.

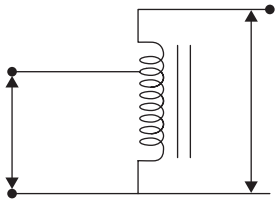
35.



Transformer is used to step down voltage from 11000 V to 415 V line to line. Normally S_1 is closed and S_2 is open. If S_1 is open and S_2 is closed, what is the voltage across a and c?
(A) 415 V (B) 240 V
(C) 480 V (D) 0 V

36. A 500 kVA, three-phase transformer has iron losses of 300 W and full-load copper loss of 600 W. Percentage load at which transformer is expected to have maximum efficiency?
(A) 50.0% (B) 70.7%
(C) 141.4% (D) 200%

37.



50 kVA 3300/230 V transformer is reconnected as an auto-transformer. The nominal rating of auto-transformer would be
(A) 50 kVA (B) 53.5 kVA
(C) 717.4 kVA (D) 767.4 kVA

38. A 100 kVA, 11000/400 V, Δ–Y distribution transformer has resistance and reactance 0.02 and 0.07 p.u., respectively, at the secondary side. The phase impedance at primary side is
(A) $24.2 + j\,84.7\ \Omega$ (B) $0.02 + j\,0.07\ \Omega$
(C) $41.91 + j146.70\ \Omega$ (D) $72.6 + j254.1\ \Omega$

39. The equivalent circuit of a transformer has leakage reactance X_1 and magnetizing reactance X_m. their magnitudes satisfy
(A) $X_1 << X_2' << X_m$
(B) $X_1 >> X_2' >> X_m$
(C) $X_1 \cong X_2 << X_m$
(D) $X_1 \cong X_2 >> X_m$

40. To introduce a phase difference of 30° between input and output line voltages, which of the 3 phase connections can be used
(A) Y–Y (B) Y–Δ
(C) Δ–Δ (D) Δ–zigzag

41. Which of the following statements are valid with respect to transformers?
(A) In O.C. Cu losses are obtained, while in S.C. core losses are obtained
(B) In O.C. current is drawn at high power factor
(C) In O.C. current is drawn at low p.f.
(D) In O.C. current is drawn at zero p.f.

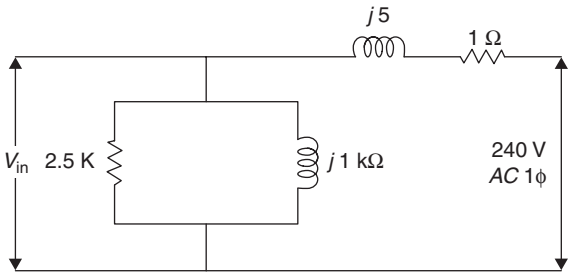
42. Two transformers are operated in parallel such that they share load in proportion to their kVA ratings. The rating of first transformer is 500 kVA and its p.u. leakage impedance is 0.05 p.u. If rating of second transformer is 250 kVA leakage impedance is
(A) 0.05 (B) 0.025
(C) 0.1 (D) 0.2

43. A bank of three identical transformers of 11 kV /230 V is used to provide 400 V from a 11 kV 3 phase substation. The kVA rating of the bank will be
(A) $250\sqrt{3}$ (B) 250
(C) 500 (D) 750

44. The frequency of power supply of a transformer designed for 60 Hz is changed to 50 Hz with voltage and current ratings remaining the same. The change in efficiency
(A) increases marginally
(B) increases by a factor 1.2
(C) remains unaltered
(D) decreases marginally

45. If the height-to-width ratio of the window core type transformer is increased then
(A) leakage reactance and % voltage regulation decreases
(B) leakage reactance and % voltage regulation increases
(C) leakage reactance will be unaffected
(D) efficiency will increase temperature rise decreases

Common Data for Questions 46 and 47:



46. Equivalent circuit of 240/110 V 1.2 kVA single-phase transformer referred to primary. Delivers full load at rated voltage and u.p.f. Efficiency will be
(A) 98% (B) 96%
(C) 94% (D) 92%

47. Regulation will be
(A) 1.546% (B) 2.546%
(C) 3.546% (D) 4.546%

48. A 250 kVA, 230/115 V 50 Hz transformer has $r_1 = 0.12\ \Omega$ $r_2 = 0.04\ \Omega$, $X_1 = 0.2\ \Omega$ $X_2 = 0.05\ \Omega$
At what loading will the primary induced emf be equal to primary terminal voltage when transformer is carrying full-load current? Neglect magnetizing current
(A) 15.57 kW (B) 16.57 kW
(C) 17.57 kW (D) 19.57 kW

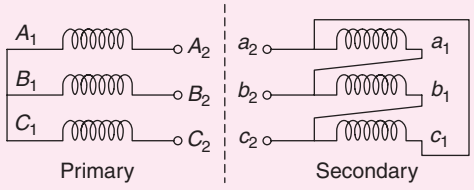
49. The leakage flux in a transformer depends on
(A) applied voltage (B) load current
(C) frequency (D) mutual flux

50. 15 kVA 600/120 V transformer is reconnected as an auto-transformer to supply 720 V from a 600 V source. Its new rating is
(A) 90 kVA (B) 18 kVA
(C) 15 kVA (D) 12 kVA

Previous Years' Questions

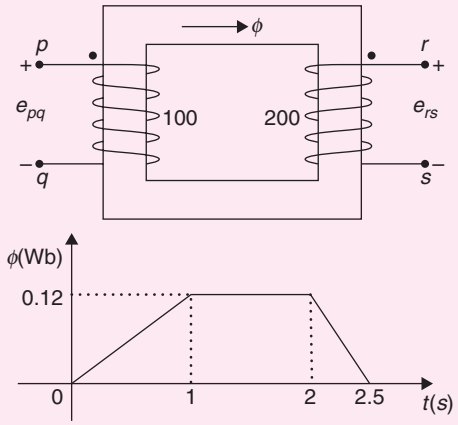
1. The equivalent circuit of a transformer has leakage reactance X_1, X'_2 and magnetizing reactance X_M. Their magnitudes satisfy **[2005]**
(A) $X_1 >> X'_2 >> X_M$ (B) $X_1 << X'_2 << X_M$
(C) $X_1 = X'_2 >> X_M$ (D) $X_1 = X'_2 << X_M$

2. Three single-phase transformers are connected to form a three-phase transformer bank. The transformers are connected in the following manner

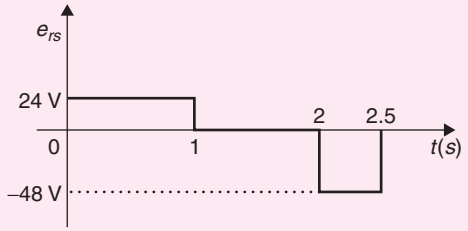


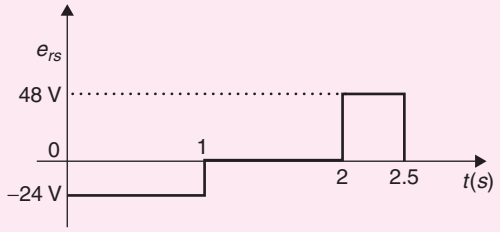
The transformer connection will be represented by **[2008]**
(A) Y d0 (B) Y d1
(C) Y d6 (D) Y d11

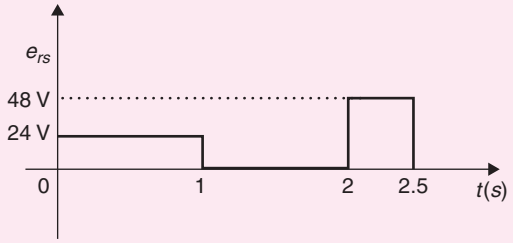
3. The core of a two-winding transformer is subjected to a magnetic flux variation as indicated in the figure. **[2008]**

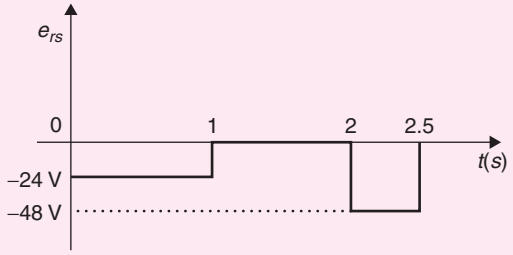


The induced emf (e_{rs}) in the secondary winding as a function of time will be of the form

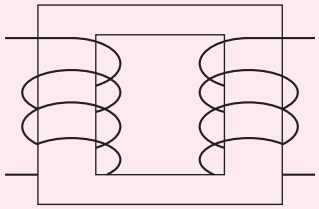
(A)


(B)


(C)


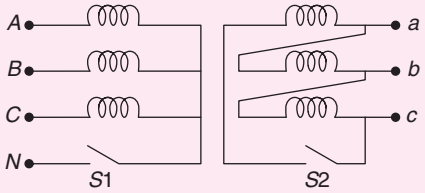
(D)


4. The single phase, 50 Hz, iron core transformer in the circuit has both the vertical arms of cross-sectional area 20 cm^2 and both the horizontal arms of cross sectional area 10 cm^2. If the two windings shown were wound instead on opposite horizontal arms, the mutual inductance will **[2009]**



(A) double
(B) remain same
(C) be halved
(D) become one quarter

Common Data for Questions 5 and 6



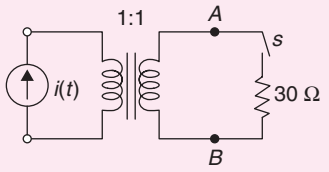
The star-delta transformer shown above is excited on the star side with a balanced, 4-wire, 3-phase, sinusoidal voltage supply of rated magnitude. The transformer is under no load condition.

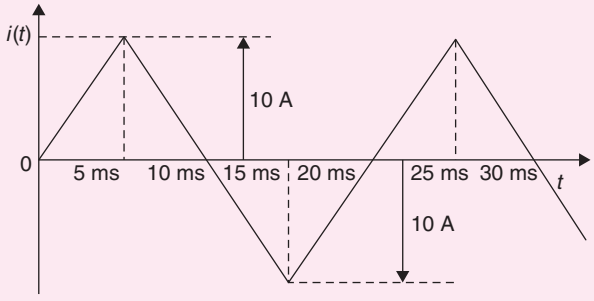
5. With both S1 and S2 open, the core flux waveform will be **[2009]**
(A) a sinusoid at fundamental frequency
(B) flat topped with third harmonic
(C) peaky with third harmonic
(D) none of these

6. With S2 closed and S1 open, the current waveform in the delta winding will be **[2009]**
(A) a sinusoid at fundamental frequency
(B) flat topped with third harmonic
(C) only third harmonic
(D) None of these

Common Data for Questions 7 and 8

The circuit diagram shows a two-winding, lossless transformer with no leakage flux, excited from a current source, $i(t)$, whose waveform is also shown. The transformer has a magnetizing inductance of $400/\pi$ mH.





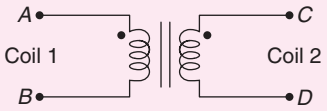
7. The peak voltage across A and B, with S open is **[2009]**

(A) $400/\pi$ V (B) 800 V
(C) $4000/\pi$ V (D) $800/\pi$ V

8. If the waveform of $i(t)$ is changed to $i(t) = 10\sin(100\pi t)$ A, the peak voltage across A and B with S closed is **[2009]**

(A) 400 V (B) 240 V
(C) 320 V (D) 160 V

Common Data for Questions 9 and 10:



The figure above shows coils 1 and 2, with dot markings as shown, having 4000 and 6000 turns, respectively. Both coils have a rated current of 25 A. Coil 1 is excited with single phase, 400 V, 50 Hz supply

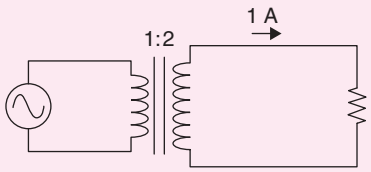
9. The coils are to be connected to obtain a single phase, 400/1000 V, auto-transformer to drive a load of 10 kVA. Which of the options given should be exercised to realize the required auto-transformer? **[2009]**

(A) Connect A and D; Common B
(B) Connect B and D; Common C
(C) Connect A and C; Common B
(D) Connect A and C; Common D

10. In the auto-transformer obtained in Question 57, the current in each coil is **[2009]**

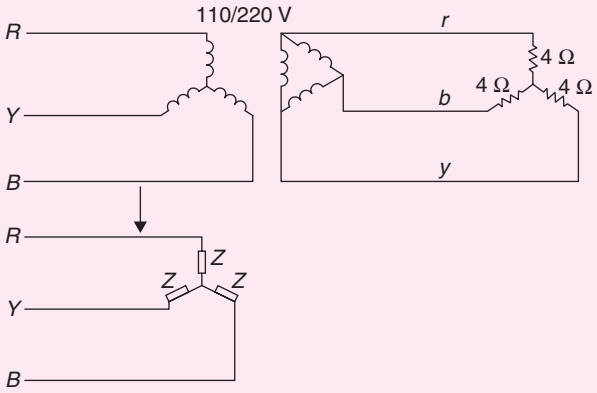
(A) Coil-1 is 25 A and Coil-2 is 10 A
(B) Coil-1 is 10 A and Coil-2 is 25 A
(C) Coil-1 is 10 A and Coil-2 is 15 A
(D) Coil-1 is 15 A and Coil-2 is 10 A

11. A single-phase transformer has a turns ratio of 1:2, and is connected to a purely resistive load as shown in the figure. The magnetizing current drawn is 1 A, and the secondary current is 1 A. If core losses and leakage reactances are neglected, the primary current is **[2010]**



(A) 1.41A (B) 2A
(C) 2.24 A (D) 3 A

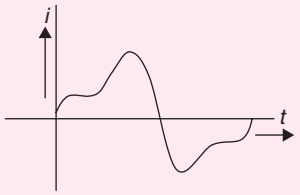
12. A balanced star-connected and purely resistive load is connected at the secondary of a star-delta transformer as shown in the figure. The line-to-line voltage rating of the transformer is 110 V/220 V. Neglecting the non-idealities of the transformer, the impedance 'Z' of the equivalent star-connected load, referred to the primary side of the transformer, is **[2010]**



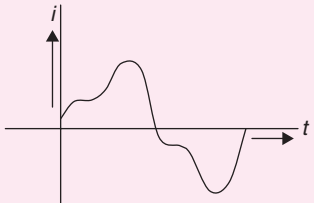
(A) $(3 + j0)\ \Omega$ (B) $(0.866 - j0.5)\ \Omega$
(C) $(0.866 + j0.5)\ \Omega$ (D) $(1 + j0)\ \Omega$

13. A single-phase air core transformer, fed from a rated sinusoidal supply, is operating at no load. The steady state magnetizing current drawn by the transformer from the supply will have the waveform. **[2011]**

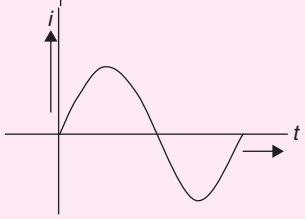
(A)



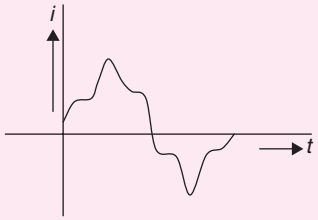
(B)



(C)



(D)



14. A single-phase 10 kVA, 50 Hz transformer with 1 kV primary winding draws 0.5 A and 55 W, at rated voltage and frequency, on no load. A second transformer has a core with all its linear dimensions $\sqrt{2}$ times the corresponding dimensions of the first transformer.

The core material and lamination thickness are the same in both transformers. The primary windings of both the transformers have the same number of turns. If a rated voltage of 2 kV at 50 Hz is applied to the primary of the second transformer, then the no-load current and power, respectively, are **[2012]**

(A) 0.7 A, 77.8 W (B) 0.7 A, 155.6 W
(C) 1 A, 110 W (D) 1 A, 220 W

15. A single-phase transformer has no-load loss of 64 W, as obtained from an open-circuit test. When a short-circuit test is performed on it with 90% of the rated currents flowing in its both LV and HV windings, the measured loss is 81 W. The transformer has maximum efficiency when operated at **[2013]**

(A) 50.0% of the rated current
(B) 64.0% of the rated current
(C) 80.0% of the rated current
(D) 88.8% of the rated current

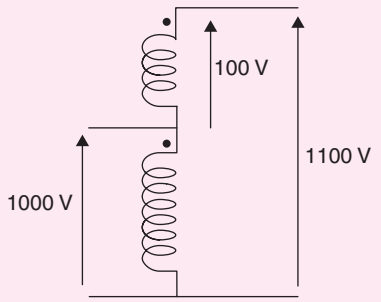
16. For a specified input voltage and frequency, if the equivalent radius of the core of a transformer is reduced by half, the factor by which the number of turns in the primary should change to maintain the same no-load current is **[2014]**

(A) 1/4 (B) 1/2
(C) 2 (D) 4

17. The core loss of a single phase, 230/115 V, 50 Hz power transformer is measured from 230 V side by feeding the primary (230 V side) from a variable voltage variable frequency source while keeping the secondary open circuited. The core loss is measured to be 1050 W for 230 V, 50 Hz input. The core loss is again measured to be 500 W for 138 V, 30 Hz input. The hysteresis and eddy current losses of the transformer for 230 V, 50 Hz input are, respectively, **[2014]**

(A) 508 W and 542 W (B) 468 W and 582 W
(C) 498 W and 552 W (D) 488 W and 562 W

18. A single-phase, 50 kVA, 1000 V/100 V two-winding transformer is connected as an auto-transformer as shown in the figure.

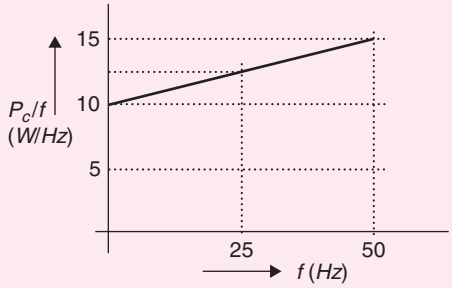


The kVA rating of the auto-transformer is _____. **[2014]**

19. For a single phase, two-winding transformer, the supply frequency and voltage are both increased by 10%. The percentage changes in the hysteresis loss and eddy current loss, respectively, are **[2014]**

(A) 10 and 21 (B) −10 and 21
(C) 21 and 10 (D) −21 and 10

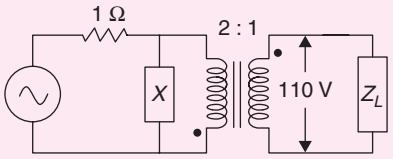
20. An open-circuit test is performed on 50 Hz transformer, using variable frequency source and keeping V/f ratio constant, to separate its eddy current and hysteresis losses. The variation of core loss/frequency as function of frequency is shown in the figure



The hysteresis and eddy current losses of the transformer at 25 Hz, respectively, are **[2014]**

(A) 250 W and 2.5 W
(B) 250 W and 62.5 W
(C) 312.5 W and 62.5 W
(D) 312.5 W and 250 W

21. The load shown in the figure absorbs 4 kW at a power factor of 0.89 lagging.



Assuming the transformer to be ideal, the value of the reactance X to improve the input power factor to unity is _____. **[2014]**

22. The parameters measured for a 220 V/110 V, 50 Hz, single-phase transformer are:

Self-inductance of primary winding = 45 mH

Self-inductance of secondary winding = 30 mH

Mutual inductance between primary and secondary windings = 20 mH

Using the above parameters, the leakage (L_{l1}, L_{l2}) and magnetizing (L_m) inductances as referred to primary side in the equivalent circuit, respectively, are **[2014]**

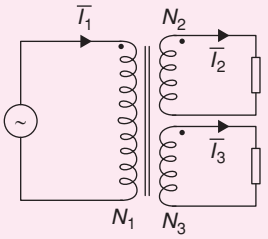
(A) 5 mH, 20 mH and 40 mH
(B) 5 mH, 80 mH and 40 mH
(C) 25 mH, 10 mH and 20 mH
(D) 45 mH, 30 mH and 20 mH

23. The primary mmf is least affected by the secondary terminal conditions in a **[2015]**
(A) power transformer
(B) potential transformer
(C) current transformer
(D) distribution transformer

24. A 200/400 V, 50 Hz, two-winding transformer is rated at 20 kVA. Its windings are connected as an auto transformer of rating 200/600 V. A resistive load of 12 Ω is connected to the high voltage (600 V) side of the auto-transformer. The value of equivalent load resistance (in Ohm) as seen from low voltage side is _____. **[2015]**

25. Two single-phase transformers T_1 and T_2 each rated at 500 kVA are operated in parallel. Percentage impedances of T_1 and T_2 are $(1 + j6)$ and $(0.8 + j4.8)$, respectively. To share a load of 1000 kVA at 0.8 lagging power factor, the contribution of T_2(in kVA) is _____. **[2015]**

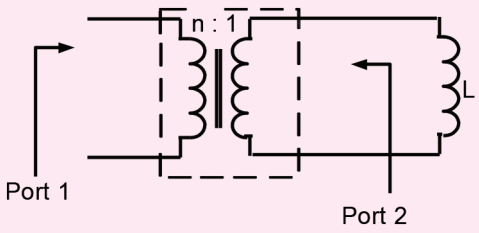
26. A three-winding transformer is connected to an AC voltage source as shown in the figure. The number of turns are as follows: $N_1 = 100$, $N_2 = 50$, $N_3 = 50$. If the magnetizing current is neglected, and the currents in two windings are $\overline{I}_2 = 2\angle 30°$ A and $\overline{I}_3 = 2\angle 150°$ A, then what is the value of the current $\overline{I}_1$ in Ampere? **[2015]**



(A) $1\angle 90°$ (B) $1\angle 270°$
(C) $4\angle 90°$ (D) $4\angle 270°$

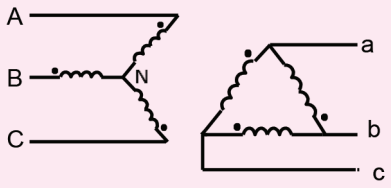
27. In a constant V/f induction motor drive, the slip at the maximum torque **[2016]**
(A) is directly proportional to the synchronous speed.
(B) remains constant with respect to the synchronous speed.
(C) has an inverse relation with the synchronous speed.
(D) has no relation with the synchronous speed.

28. If an ideal transformer has an inductive load element at port 2 as shown in the figure below, the equivalent inductance at port 1 is **[2016]**



(A) nL (B) n^2L
(C) $\frac{n}{L}$ (D) $\frac{n^2}{L}$

29. If the star side of the star-delta transformer shown in the figure is excited by a negative sequence voltage, then **[2016]**

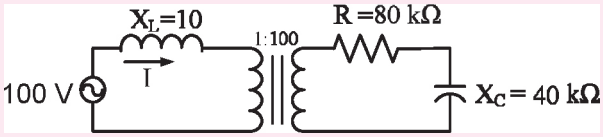


(A) V_{AB} leads V_{ab} by 60°
(B) V_{AB} lags V_{ab} by 60°
(C) V_{AB} leads V_{ab} by 30°
(D) V_{AB} lags V_{ab} by 30°

30. A single-phase 400 V, 50 Hz transformer has an iron loss of 5000 W at the rated condition. When operated at 200 V, 25 Hz, the iron loss is 2000 W. When operated at 416 V, 52 Hz, the value of the hysteretic loss divided by the eddy current loss is _____. **[2016]**

31. A single-phase, 22 kVA, 2200 V/ 220 V, 50 Hz, distribution transformer is to be connected as an auto-transformer to get an output voltage of 2420 V. Its maximum KVA rating as an auto- transformer is _____. **[2016]**
(A) 22 (B) 24.2
(C) 242 (D) 2420

32. The following figure shows the connection of an ideal transformer with primary to secondary turns ratio of 1: 100. The applied primary voltage is 100V (rms), 50Hz, AC. The rms value of the current I, in ampere, is _____. **[2016]**



33. A single-phase, 2kVA, 100/200V transformer is reconnected as an auto-transformer such that its kVA rating is maximum. The new rating, in kVA is _____. **[2016]**

34. Three single-phase transformers are connected to from a delta-star three-phase transformer of 110kV/11kV. The transformer supplies at 11kV a load of 8MW at 0.8p.f. lagging to a near by plant. Neglect the transformer losses. The ratio of phase currents in delta side to star side is _____. **[2016]**

(A) $1:10\sqrt{3}$ (B) $10\sqrt{3}:1$
(C) 1 : 10 (D) $\sqrt{3}:10$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** A	**4.** B	**5.** C	**6.** C	**7.** C	**8.** D	**9.** C	**10.** A
11. B	**12.** B	**13.** C	**14.** C	**15.** C	**16.** D	**17.** D	**18.** B	**19.** C	**20.** D
21. D	**22.** D	**23.** C	**24.** C	**25.** C	**26.** C	**27.** D	**28.** D	**29.** B	**30.** C
31. C	**32.** C	**33.** D	**34.** A	**35.** D	**36.** C	**37.** C	**38.** A	**39.** C	**40.** B
41. D	**42.** D	**43.** A	**44.** C	**45.** A					

Practice Problems 2

1. B	**2.** B	**3.** C	**4.** A	**5.** C	**6.** D	**7.** B	**8.** C	**9.** A	**10.** B
11. D	**12.** A	**13.** B	**14.** D	**15.** B	**16.** C	**17.** C	**18.** C	**19.** C	**20.** A
21. D	**22.** C	**23.** C	**24.** A	**25.** A	**26.** C	**27.** B	**28.** B	**29.** C	**30.** D
31. D	**32.** B	**33.** B	**34.** D	**35.** D	**36.** B	**37.** D	**38.** D	**39.** C	**40.** B
41. C	**42.** C	**43.** D	**44.** A	**45.** A	**46.** B	**47.** B	**48.** D	**49.** B	**50.** A

Previous Years' Questions

1. D	**2.** B	**3.** B	**4.** C	**5.** B	**6.** A	**7.** D	**8.** A	**9.** A	**10.** D
11. C	**12.** D	**13.** C	**14.** B	**15.** C	**16.** C	**17.** A	**18.** 550	**19.** A	**20.** B
21. 23.6	**22.** B	**23.** C	**24.** 1.3 Ω	**25.** 555.75 KVA		**26.** A	**27.** C	**28.** B	**29.** D
30. 1.44	**31.** C	**32.** 10	**33.** 6	**34.** A					

Chapter 2

Principles of Electro Mechanical Energy Conversion

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Single excited magnetic systems
- Electrical energy input
- Doubly excited magnetic systems
- MMF produced by polyphase machines
- Armature windings
- Distributed windings
- Pole-pitch
- Coil-pitch or coil-span
- Simplex and multiplex windings
- Closed-winding terms
- Simplex wave winding
- Multiplex lap and wave windings
- Open windings or AC armature windings

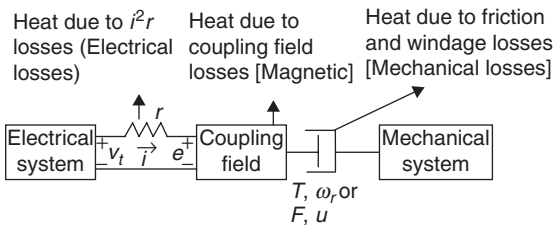
Introduction

Conversion of other forms of energy into electrical energy is a common practice owing to the advantage of electrical energy as it can be transmitted, utilized and controlled more easily, reliably and efficiently. Energy conversion devices are required at both ends of an electrical system, since energy is neither available and nor required in electrical form. An electromechanical energy conversion device is one which converts electrical energy into mechanical energy or vice versa. The operating principles of energy-conversion devices are similar, but their structural details differ depending upon their function. Energy conversion process is basically a reversible process. However in practice, the electromechanical energy conversion devices may be designed and constructed to suit one particular mode of conversion or the other. The electro-mechanical-energy conversion principles are developed with field energy as basis as the field energy concept is more general and broad-based, since it is applicable to all devices possessing rotational, linear or vibratory motions. The energy storing capacity of the magnetic field is much greater (about 25,000 times) than that of the electric field. In view of this fact, electromechanical energy conversion devices with magnetic field as the coupling medium between electrical and mechanical systems are more common in commercial applications.

Various electromechanical energy conversion devices may be categorized as under:

1. Low energy conversion devices such as telephone receivers, loud-speakers, microphones, low signal transducers, etc.
2. The second category consists of force, or torque-producing devices with limited mechanical motion such as electromagnets, relays, moving-iron instruments, moving-coil instruments, actuators, etc.
3. The third category includes bulk energy conversion and utilization devices such as generators and motors. Motors and generators are continuous energy conversion devices.

An electromechanical energy conversion device consists of three essential parts, namely, electrical system, coupling field and mechanical as shown in following figure. The coupling field may be magnetic or electric.



General representation of electromechanical conversion device.

When energy is converted from one form to another, the law of conservation of energy can be invoked. According to this law,

energy can neither be created nor be destroyed, it can merely be converted from one form into another.

In an energy conversion device, out of the total input energy, some energy is converted into the required form, some energy is stored and the rest is dissipated. The energy balance equation includes these four terms. For a motor, it can be written as

$$\begin{pmatrix}\text{Total electrical}\\ \text{energy input}\end{pmatrix} = (\text{Mechanical energy output}) + (\text{Total energy stored}) + (\text{Total energy dissipated}) \quad (1)$$

For generator action

$$\begin{pmatrix}\text{Total mechanical}\\ \text{energy input}\end{pmatrix} = (\text{Electrical energy output}) + (\text{Total energy stored}) + (\text{Total energy dissipated}) \quad (2)$$

The energy balance equation, i.e., Eq. (1) can be written in more specific terms as

$W_{ei} = W_{mo} + (W_{es} + W_{ms})$
+ (Ohmic energy losses)
+ (Coupling field energy losses)
+ (Energy losses in mechanical system)

The subscripts *e*, *m*, *I*, *s* and *o* stand for electrical, mechanical input, stored and output respectively.

The various forms of energies involved in Eq. (1) for an electromechanical energy conversion device are given below:

1. Total electrical energy input from the supply mains is W_{ei}.
2. The mechanical energy output is W_{mo}.
3. Total energy stored in any device = Energy stored in magnetic field, W_{es} + Energy stored in mechanical system, W_{ms}.
4. Total energy dissipated = Energy dissipated in electric circuit as ohmic losses + Energy dissipated as magnetic core loss (hysteresis and eddy-current losses) + Energy dissipated in mechanical system (friction and windage losses etc.).

Associated the losses with the corresponding systems Eq. (1) can be rewritten as

$$\begin{pmatrix}\text{Electric energy input}\\ \text{– ohmic losses}\end{pmatrix} = \begin{pmatrix}\text{Mechanical energy}\\ \text{output + energy}\\ \text{stored in mechanical}\\ \text{system + mechanical}\\ \text{losses (friction \& windage)}\end{pmatrix} + \begin{pmatrix}\text{Energy stored in the}\\ \text{coupling field +}\\ \text{coupling filed}\\ \text{energy losses}\end{pmatrix} \quad (2a)$$

or

$$W_{elec} = W_{mech} + W_{fld}$$

where

W_{elec} = Net energy input to the coupling field.
W_{mech} = The total energy converted to mechanical form.

[Mechanical energy output + Energy stored in the mechanical system + friction and windage losses] and W_{fld} = Total energy absorbed by the coupling field [which is equal to the sum of both the stored field energy w_{es} and the coupling field energy losses]

$$W_{elec} = W_{mech} + W_{fld} \quad (2b)$$

The electrical losses (ohmic and coupling – field losses) as well as mechanical losses (friction and windage losses), though always present, play no basic role in the energy conversion process.

Equation (2b) can be written in differential form as

$$dW_{elec} = dW_{mech} + dW_{fld} \quad (3)$$

where

dW_{elec} = differential electrical energy input to coupling field
dW_{mech} = differential mechanical energy output
dW_{fld} = differential change in energy stored in the coupling field.

From Eq. (1), the differential electrical energy input in time d*t* is

$$dW_{ei} = v_t i\, dt$$

ohmic loss in resistance *r* in time *dt* is $i^2 r\, dt$.

∴ Differential electrical energy input or net energy input to the coupling field,

$$dW_{elec} = dW_{ei} - \text{ohmic loss} = (V_t - ir)\, i\, dt = ei\, dt \quad (4)$$

Equation (3) now becomes

$$ei\, dt = dW_{mech} + dW_{fld} \quad (5)$$

Energy balance Eq. (5) is obtained by applying law of conservation of energy to the motoring mode. This Eq. (5) along with faradays laws of electromagnetic induction forms the fundamental basis for the analysis of energy-conversion devices.

The coupling field is the link between electrical and mechanical systems.

In order that a moving member can rotate, or move, with respect to the stationary member, an air gap must exist in between the stator and rotator. The energy stored in the coupling field acts as interaction medium between electrical and mechanical systems.

This energy stored in the coupling field must produce action and reaction on electrical and mechanical systems for the conversion of energy from electrical to mechanical

(motoring mode) or from mechanical to electrical (generating mode).

If the output is mechanical, the coupling field must react with the electrical system in order to absorb electrical energy from it. In a motor, this reaction is the back emf or counter emf 'e'. The coupling field extracts energy proportional to $e.i$ from the electrical system, converts and delivers energy proportional to $T.\ \omega_r$ (or F.u) to the mechanical system.

If the output is electrical, the coupling field must react with the mechanical system so as to absorb mechanical energy from it. In a generator this reaction is counter-torque T (or reaction torque), opposite to the applied mechanical torque of the prime mover.

Thus, the reaction of the coupling field on the electrical or mechanical system is essential for the process of electromechanical energy conversion.

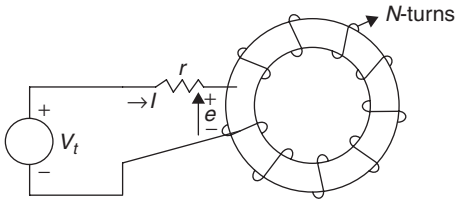
The induced emf 'e' and torque, T are called electromechanical coupling terms as they are associated with the coupling field.

Electromechanical energy conversion devices are slow-moving because of the inertia associated with mechanical components. Therefore the coupling field must be slowly varying and as such the nature of this field is quasi-static. Electromagnetic radiation from the coupling field is almost negligible.

Single Excited Magnetic Systems

Electrical Energy Input

Consider a simple magnetic system of a toroid, excited by a single coil as shown in the figure.



Applying KVL for the electrical circuit, the instantaneous voltage equation can be obtained as $V_t = ir + e$

where e is the reaction emf (or back emf) taken as a voltage drop in the direction of current.

$$\therefore e = \frac{d\psi}{dt} \text{ and } V_t = ir + \frac{d\psi}{dt}$$

Where ψ denotes the instantaneous flux linkages with the circuit, multiplying both sides of Eq. (2) we have

$$V_t\, idt = i^2\, r\, dt + id\psi$$

or $$(V_t - ir)idt = id\psi$$

or $$e\ idt = id\psi$$

$$dW_{elec} = v'_t\, i\, dt = dW_{ei} - \text{ohmic loss} = (v_t - ir)\, i\, dt = eidt.$$

If the toroidal core is made of good magnetic material (ferro-magnetic material), most of the flux would be confined to the core. Assuming that flux ϕ links all the N turns of the coil, we have flux linkages, $\psi = N\phi$

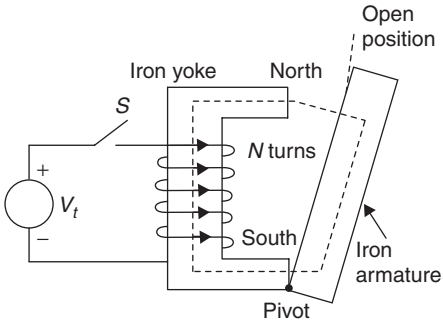
$$dW_{elec} = id\ \psi$$
$$= iN\, d\phi = F\, d\phi \qquad (6)$$

where ϕ is the instantaneous value of the coil flux and $F = iN$ is the instantaneous coil mmf. From Eq. (6) it is evident that the flux linkages of the system must change for the toroid to extract energy from the supply system.

These changing flux linkages cause the generation of reaction emf e (or back emf or counter emf). The flow of current against the reaction emf e causes the extraction of energy from the electrical system or in other words e equals the extracted power from electrical system.

Magnetic Field Energy Stored

Consider a simple magnetic relay shown in following figure.



Initially the switch 'S' and the armature are in open position. When switch 'S' is closed, current i flows through the coil, setting up a flux. The magnetic field thus produced, creates north and south poles on yoke and armature respectively. As a result of the magnetic field there is established a magnetic force that acts on the armature tending to shorten the air-gap. If the armature is not allowed to move, the mechanical work done is zero. i.e. $dW_{mech} = 0$. Therefore from the energy balance equation,

$$dW_{elec} = 0 + dW_{fld}$$

The above relation infers that when the movable part of any physical system is kept fixed, the entire electrical energy input is stored in the magnetic field.

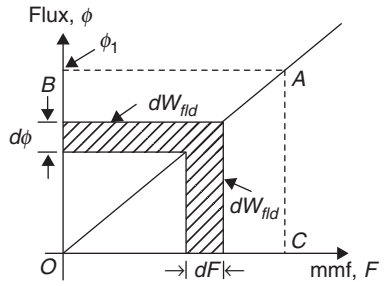
$$\therefore \qquad dw_{fld} = dW_{elec} = id\psi = F \cdot d\phi$$

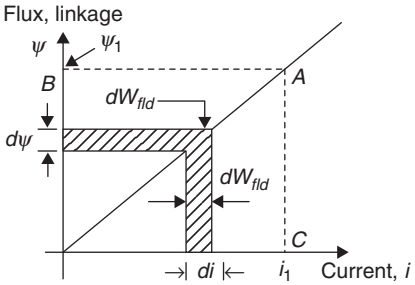
If the initial flux is zero, the magnetic field energy stored W_{fld}, in establishing a flux ϕ, or flux linkage ψ, is given by

$$W_{fld} = \int_0^{\psi_1} i \cdot d\psi = \int_0^{\phi} F \cdot d\phi$$

In the above equation, i must be expressed in terms of ψ and F in terms of ϕ.

When the armature is held open, then most of the mmf is consumed in the air gap and magnetic saturation may not occur. Then ψ and ϕ vary linearly with i and F, respectively.





For figure (a)

$$W_{fld}\int_0^{\phi_1} dW_{fld} = \int_0^{\phi_1} F\cdot d\phi = \text{area OABO}$$

For figure (b),

$$W_{fld} = \int_0^{\psi_1} dW_{fld} = \int_0^{\psi_1} id\psi = \text{area OABO}$$

and area OACO $= \int dW_{fld}$

$$= \int_0^{F_i} \phi\, dF = \int_0^{i_i} \psi di$$

This area OACO is called the co-energy W'_{fld}. Co-energy has no physical significance, it is however useful in calculating the magnetic forces.

With no magnetic saturation

$$\text{Area OABO} = \text{Area OACO or } W_{fld} = W'_{fld}$$

and $W_{fld} + W'_{fld} = \text{Area OCABO}$

$= \text{Area OABO} + \text{Area OACO}$

$= \phi_1 F_1 = \psi_1 i_1$

In general for a linear magnetic circuit,

$$W_{fld} = W'_{fld} = \frac{1}{2}\psi i = \frac{1}{2}F\phi$$

The magnetic stored energy and co-energy expressed in terms of reluctance and permeance is as follows.

$$\text{mmf} = (\text{flux})(\text{Reluctance})$$

$$= \phi S$$

$$= \frac{\phi}{\text{Permeance, } \Delta}$$

$$W_{fld} = W'_{fld} = \frac{1}{2}\phi^2 S = \frac{1}{2}\frac{\phi^2}{\Delta}$$

$$W_{fld} = W'_{fld} = \frac{1}{2}F^2\Delta = \frac{1}{2}\frac{F^2}{S}$$

In terms of self-inductance, $L = \frac{\psi}{i}$

$$W_{fld} = W'_{fld} = \frac{1}{2}Li^2 = \frac{1}{2}\frac{\psi^2}{L}$$

The stored magnetic energy W_{fld} and co-energy W'_{fld} can be written as follows

$$W_{fld} = W'_{fld} = \frac{1}{2}F\phi = \frac{1}{2}\psi^i$$

$$= \frac{1}{2}\phi^2 S = \frac{1}{2}\frac{\phi^2}{\Delta} = \frac{1}{2}\frac{F^2}{S}$$

$$= \frac{1}{2}F^2\Delta = \frac{1}{2}Li^2 = \frac{1}{2}\frac{\psi^2}{L}.$$

The magnetic stored energy density is given by

$$W_{fld} = \frac{1}{2}HB = \frac{1}{2}\mu H^2 = \frac{1}{2}\frac{B^2}{\mu} \text{ J/m}^3$$

For linear magnetic circuit

$$W'_{fld} = W_{fld}$$

Note: Here W'_{fld} is the co-energy density.

The field-energy approach serves as the physical basis for the generalized theory of electrical machines.

During the moz of armature the electromagnetic force or torque developed in any physical system acts in such a direction as to tend to

1. Decrease the magnetic stored energy at constant ψ or ϕ.
2. Increase both field stored and co-energy at constant current or mmf.
3. Decrease the reluctance.
4. Increase the permeance and inductance.
5. Decrease current i at constant flux linkages ψ or increase flux linkages ψ at constant i.

Examples of singly excited electromagnetic systems.

1. Moving iron instruments
2. Relays
3. Electromagnets
4. Reluctance motor, etc.

$$\text{Magnetic force, } f_e = \frac{-1}{2}\psi\frac{\partial i}{\partial x}(i, x)$$

If ψ is expressed in terms of i and x

$$f_e = \frac{1}{2} i \frac{\partial \psi}{\partial x}(i, x)$$

where 'x' is the displacement of the armature in the direction of magnetic force.

The electromagnetic torque,

$$T_e = \frac{-1}{2} \phi^2 \frac{dS}{d\theta} = \frac{1}{2} I^2 \frac{dL}{d\theta}$$

$$= \frac{-1}{2} \psi \frac{\partial i}{\partial \theta}(\psi, \theta)$$

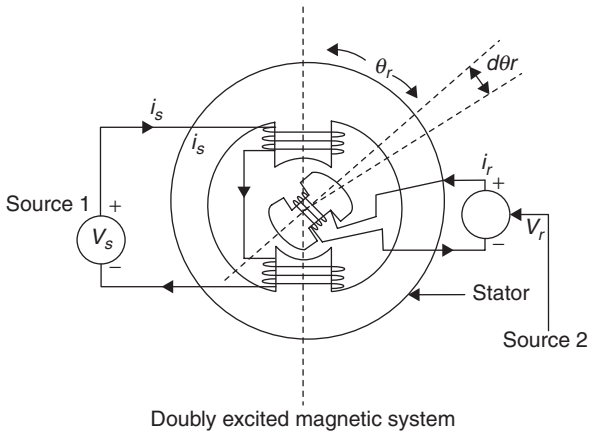
$$= \frac{1}{2} i \frac{\partial \psi}{\partial \theta}(i, \theta)$$

Doubly Excited Magnetic Systems

Most of the electromagnetic energy conversion devices belong to doubly excited or multiply excited magnetic systems. A doubly excited magnetic system is one which has two independent sources of excitations.

For example, Synchronous machines, DC shunt machines, loud speakers, tachometers, etc.

The following figure shows a simple model of doubly excited magnetic system.



Doubly excited magnetic system

The stator with N_s turns is energized from source 1 and the rotor with N_r turns is excited from source 2. The mmf's (or rmf's) produced by both the stator and rotor windings are in the same direction and the electromagnetic torque, T_e is in the counter clockwise direction as shown in fig. The magnetic saturation and hysteresis are neglected in the following developments. The differential electrical energy input dW_{elec} for the doubly excited system from two energy sources 1 and 2 is

$$dW_{elec} = i_s \, d\psi_s + i_r \, d\psi_r \qquad (7)$$

where ψ_s and ψ_r are the instantaneous total flux linkages of stator and rotor windings respectively. Since the magnetic saturation is neglected, ψ_s and ψ_r can be expressed as

$$\psi_s = L_s \, i_s + M_{sr} \, i_r$$

$$\psi_r = L_r \, i_r + M_{rs} \, i_s$$

where L_s and L_r are the self-inductances of the stator and rotor windings, respectively, and

$M_{sr} = M_{rs}$ = mutual inductance between the stator and rotor windings.

Initially the space angle between stator and rotor axes is θ_r and both the currents i_s and i_r are assumed to be zero. On switching the stator and rotor windings to their respective energy sources, the currents rise from zero to i_s and i_r, respectively. If the rotor is not allowed to move, then $dW_{mech} = 0$.

$$\therefore \qquad dW_{elec} = 0 + dW_{fld}$$

Thus, with rotor held fixed, all the energy supplied by sources is stored in the magnetic field.

From Eq. (7)

$$dW_{fld} = dW_{elec} = i_s \, d\psi_s + i_r d\psi_r$$

$$= i_s \, d(L_s \, i_s + M_{sr} \, i_r) + i_r \, d(L_r \, i_r + M_{rs} \, i_s) \qquad (8)$$

L_s, L_r and $M_{sr}(=M_{rs})$ are given by

$$L_s = \frac{N_s^{\,2}}{S_s}, L_r = \frac{N_r^{\,2}}{S_r}$$

$$M_{sr} = M_{rs} = \frac{N_s N_r}{S_{sr}}$$

where S_s = reluctance seen by the stator flux
S_r = reluctance seen by the rotor flux
S_{sr} = reluctance seen by the resultant of stator and rotor fluxes.

Since the rotor is not allowed to move, the reluctances and therefore the inductances are constant.

$$\therefore \qquad dL_s = dL_r = dM_{sr} = dM_{rs} = 0$$

From Eq. (3), we can have

$$dW_{fld} = i_s \, L_s \, di_s + i_s \, M_{sr} \, di_r + i_r \, L_r \, di_r + i_r \, M_{sr} \, di_s$$

$$= i_s \, L_s \, di_s + i_r \, L_r \, di_r + M_{sr} \, d(i_s \, i_r)$$

Therefore, the magnetic field energy stored in establishing the currents from zero to i_s and i_r is given by

$$W_{fld} = L_s \int_0^{i_s} i_s \, d\, i_s + L_r \int_0^{i_r} i_r \, di_r + M_{sr} \int_0^{i_s} d\left(i_s \, i_r\right)$$

$$= \frac{1}{2} i_s^{\,2} \, L_s + \frac{1}{2} i_r^{\,2} \, L_r + M_{rs} \, i_s \, i_r$$

For obtaining the magnetic torque T_e, assume the rotor to move through a virtual displacement $d\theta_r$ in the direction of T_e. With rotor movement, reluctances S_s, S_r, S_{sr} and therefore L_s, L_r, M_{sr} must vary.

Therefore, the differential electrical energy input dW_{elec} during virtual displacement $d\theta_r$ is given by

$$dW_{elec} = i_s\, d[L_s\, i_s + M_{sr}\, i_r] + i_r\, d[L_r\, i_r + M_{sr}\, i_s]$$
$$= i_s\, L_s\, di_s + i_s{}^2 dL_s + i_s\, M_{sr}\, di_r + i_s\, i_r\, dM_{sr}$$
$$+ i_r\, L_r\, di_r + i_r{}^2\, dL_r + i_r\, M_{sr}\, di_s + i_r\, i_s\, dM_{sr}$$

and the differential magnetic energy stored dW_{fld}, during the virtual displacement $d\theta_r$ is given by

$$dW_{fld} = \frac{1}{2} i_s{}^2 dL_s + L_s i_s\, d\, i_s + \frac{1}{2} i_r{}^2\, dL_r + L_r\, i_r di$$
$$+ M_{sr}\, i_s\, d\, i_r + M_{sr}\, i_r\, d + i_s\, i_r\, dM_{sr} + i_s i_r dM_{sr}$$

The differential mechanical work done dW_{mech} during the virtual displacement $d\theta_r$ is given by

$$dW_{mech} = T_e\, d\theta_r$$

From the energy balance equation,

$$dW_{elec} = dW_{fld} + dW_{mech}$$

substituting the value of dW_{elec} dW_{mech} and dW_{mech} and on simplification, we get

$$T_e = \frac{1}{2} i_s{}^2 \frac{dL_s}{d\theta_r} + \frac{1}{2} i_r{}^2 \frac{dL_r}{d\theta_r} + i_s i_r \frac{d\, M_{sr}}{d\theta_r}$$

It can be seen from the above equation of T_e, that the differential changes di_s and di_r do not contribute to the production of magnetic torque T_e (In other words, T_e is independent of di_s and di_r)

Therefore torque depends on

1. Instantaneous values of currents i_s and i_r and
2. The angular rate of change of inductances,

If W_{fld} is differentiated with respect to the space angle θ_r (with constant i_s and i_r) then the electromagnetic torque T_e is given by

$$T_e = \frac{\partial W_{fld}}{\partial \theta_r}(i_s, i_r, \theta_r)$$

i.e., torque can be obtained from the space derivative of field energy expression, when it is expressed in terms of i_s, i_r and θ_r.

Integration of dW_{elec}, with constant currents, gives

$$W_{elec} = \int dw_{elec}$$
$$= i_s{}^2\, L_s + i_r{}^2\, L_r + 2 i_s\, i_r\, M_{sr}$$

For a linear magnetic circuit, $W_{fld} = W'_{fld}$

$$\therefore T_e = \frac{\partial W'_{fld}}{\partial \theta_r}(i_s, i_r, \theta)$$

If the electrical energy input takes place at constant currents, then half of it is converted to mechanical energy, and the remaining half is stored in the magnetic field at constant currents.

In other words, the magnetic energy stored at constant currents is equal to the mechanical work done.

The linear magnetic force f_e, for a doubly excited system can be obtained as follows.

The differential electrical energy input dW_{elec}, from two energy sources is

$$dW_{elec} = i_1\, d\psi_1 + i_2\, d\psi_2$$

Where

$$\psi_1 = L_1\, i_1 + M_{12}\, i_2$$
$$\psi_2 = L_2\, i_2 + M_{21}\, i_1$$

Also $M_{12} = M_{21}$

$$f_e = \frac{1}{2} i_1^2 \frac{dL_1}{dx} + \frac{1}{2} i_2^2 \frac{dL_2}{dx} + i_1\, i_2 \frac{dM_{12}}{dx}$$

$$f_e = \frac{\partial W_{fld}}{\partial x}(i_1, i_2, x)$$

$$= \frac{\partial W'_{fld}}{\partial x}(i_1, i_2, x)$$

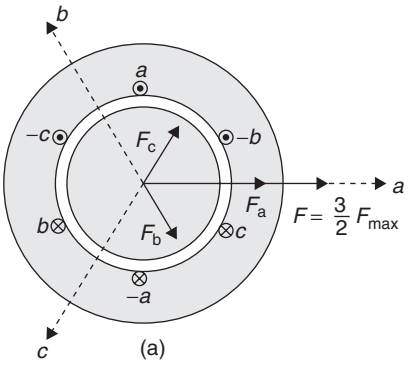
MMF Produced by Polyphase Machines

The production of a rotating mmf can be shown graphically. Consider the state of affairs at $t = 0$, the moment when the phase – a current is at its maximum value I_m. The mmf of phase a then has its maximum value F_{max}, as shown by the vector $F_a = F_{max}$ drawn along the magnetic axis of phase a in the two – pole machine shown schematically in figure (a). At this moment, currents i_b and i_c are both $I_m/2$ in the negative direction, as shown by the dots and crosses in figure (a) indicating the actual instantaneous directions. The corresponding mmf's of phases b and c are shown by the vectors F_b and F_c, both of magnitude $F_{max}/2$ drawn in the negative direction along the magnetic axes of phases b and c, respectively. The resultant, obtained by adding the individual contributions of the three phases, is a vector of magnitude $F = \frac{3}{2} F_{max}$ centred on the axis of phase α. It represents a sinusoidal space wave with its positive peak centred on the axis of phase α and having an amplitude $\frac{3}{2}$ times that of the phase–α contribution alone.

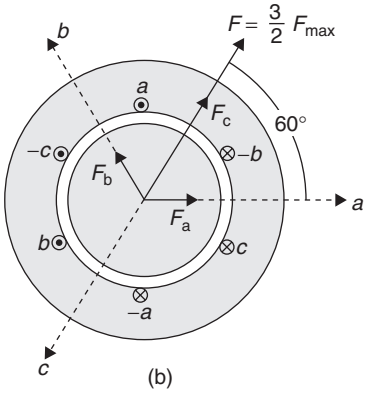
At a later time $\omega_e t = \pi/3$ the currents in phases α and b are a positive half maximum, and that in phase c is a negative maximum. The individual mmf components and their resultant are now shown in Figure (b). The resultant has the same amplitude as at $t = 0$, but it has now rotated counterclockwise 60 electrical degrees in space. Similarly, at $\omega_e t = 2\pi/3$ (when the phase-b current is a positive maximum and the phase–α and phase–c currents are a negative half maximum) the same resultant mmf distribution is again obtained, but it has rotated counterclockwise 60 electrical degrees still father and is now aligned with the magnetic axis of phase b figure c. As time passes, then, the resultant mmf wave retains its sinusoidal form and amplitude but

rotates progressively around the air gap; the net result can be seen to be an mmf wave of constant amplitude rotating at a uniform angular velocity.

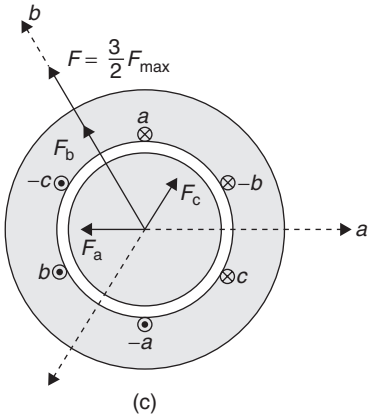
In one cycle the resultant mmf must be back in the position of Figure (a). The mmf wave therefore makes one revolution per electrical cycle in a two-pole machine. In a multipole machine the mmf wave trave*ls* one pole-pair per electrical cycle and hence one revolution in pole/2 electrical cycles.



(a)



(b)



(c)

Solved Examples

Example 1: A coil of 1000 turns on a core would create a flux of 2 m Wb when carrying a current of 1 A. Calculate the energy stored in the magnetic field.

Solution: Flux linkages $\Psi = N\phi$

$= 1000 \times 2 \times 10^{-3}$

$= 2$ Weber-turns

Energy stored in the magnetic field

$$= \frac{1}{2}\psi i$$

$$= \frac{1}{2} \times 2 \times 1 = 1 \text{ J}$$

Example 2: One 800-turn flat coil with an area of 5×10^{-2} m^2 is rotating in a magnetic field of flux density 60×10^{-3} Wb/m^2

Solution: emf induced, $e = N \phi \omega \sin\theta$

Here $\theta = 90°$.

$\phi = 5 \times 10^{-2} \times 60 \times 10^{-3}$

$= 3 \times 10^{-3}$ Wb

$$\omega = \frac{2\pi \times 1500}{60} = 50\pi \text{ rad/s}$$

And $N = 800$ turns

$e = 800 \times 3 \times 10^{-3} \times 50\pi$

$= 377$ V

Armature Windings

There are two types of windings that are used in rotating electrical machines, namely

1. Concentrated windings
2. Distributing windings

Concentrated Winding

All the turns of the winding are wound together in series to form one multi-turn coil. In concentrated windings, all the turns have the same magnetic axis.

Example: Field winding of salient-pole synchronous machines and DC machines, primary and secondary windings of a transformer.

Distributed Windings

All the winding turns are arranged in several full-pitch or fractional (short) – pitch coils which are housed in the slots spread around the air-gap periphery to form phase or commutator winding.

Example: Stator and rotor windings of induction machines, armature windings of both synchronous and DC machines.

The field (or exciting) windings produces the working flux. The armature winding is one in which working emf is induced by the working flux.

The armature windings, in general are classified as

1. Closed windings and
2. Open windings

The closed windings are used only for commutator machines, such as DC machines and AC commutator (e.g.: universal motor) machines. The open windings are used

only for AC machines like synchronous machines, induction machines etc.

In closed windings, there is a closed path, i.e., if one starts from any point on the winding and traverses it, one again reaches the starting point from where one had started, whereas the open windings terminate at a suitable number of slip-rings or terminals.

Note: Three phase delta connected winding used for AC machines forms a closed circuit. This delta can, however, be open-circuited and reconnected in star if desired. The closed circuit in a delta connected winding is a result of interconnection of the armature windings of different phases. The closed windings used for commutator machines can, in no case, be open-circuited.

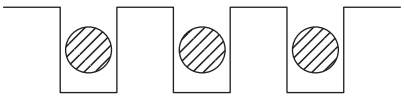
Some of the terms common to both the open and closed armature windings are described below.

1. **Conductor:** The active length of wire which takes part in energy-conversion process.
2. **Turn:** One turn consists of two conductors.
3. **Coil:** A coil may consist of any number of turns. E.g. a coil having one turn is called one-turn coil. A multi-turn coil is one which has more than one-turn.
4. **Coil-sides:** One coil with any number of turns has two coil-sides.
   No. of coil sides = 2 × No. of coils
   Each coil-side will have several conductors.

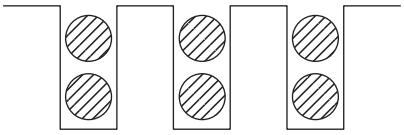
Single-layer and Double-layer Windings

In a single-layer winding, one coil-side occupies the total slot area.

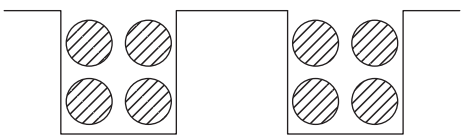
In case the slot contains even number of coil-sides (may be 2, 4, 6, etc.) the winding is referred to as a two-layer winding.



Single layer



Double layer winding (2 coil sides per slot)



Double layer winding (4 coil sides per slot)

The one occupying the upper portion of the slot is called top layer and the other bottom layer.

If one coil-side of a coil occupies the top position in a slot, the other occupies the bottom position in a slot which is one coil-pitch apart.

Advantages of double-layer windings:

1. Easier to house the winding in slots during repairs.
2. Leakage reactance is low which results in better performance.
3. Better induced emf waveform in case of generators.
4. In a single-layer winding, variety of coils that are of different sizes and shapes are used. In double-layer windings, identical diamond-shaped coils are used. So they are more economical.

Pole-pitch

The term 'pitch' indicates a particular method of measurement in terms of coil sides, teeth etc. A pole pitch is defined as the peripheral distance between identical points on two adjacent poles. Pole pitch is always equal to 180° electrical.

Coil-pitch or Coil-span

The distance between the two coil-sides of a coil measured in terms of teeth, slots or electrical degrees.

Chorded-coil

A coil with full-pitch will have the coil-span equal to the pole-pitch. If the coil-pitch is less than pole-pitch, then it is called chorded, short-pitch or fractional-pitch coil.

If there are S slots (or teeth) and P poles, then

$$\text{Pole pitch} = \frac{S}{P} \text{ slots per pole}$$

$$\text{For a full-pitched coil, coil pitch} = \frac{S}{P}$$

Incase coil-pitch $< \frac{S}{P}$ it results in chorded, short pitched or fractional-pitch winding. The coil-pitch is rarely greater than pole pitch. In such a case, for a full-pitched coil $\frac{S}{P}$ must be a positive integer.

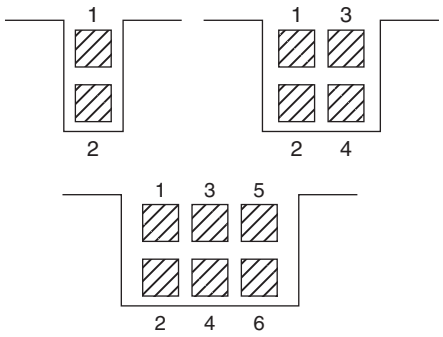
Simplex and Multiplex Windings

In case higher current ratings are to be obtained with given size of a conductor, multiplex windings are used.

In some cases, the number of parallel circuits may not be adequate enough from design considerations. For example in an 8-pole commutator machine, simplex wave winding gives only 2 parallel paths. In case the parallel paths required are in between 2 and 8 or more than 8, then multiplex windings are employed.

Closed-winding Terms

In commutator machines having closed windings, the coil sides for convenience are numbered as shown in the following figure.



The top coil-sides are numbered 1, 3, 5…. and bottom coil sides are numbered 2, 4, 6….

Back Pitch (Y_b)

The distance between the top and bottom coil sides of one coil, measured at the back of armature (or measured at the rear side of the commutator) is called back pitch.

Front Pitch (Y_f)

The distance between the two coil-sides connected to the same commutator segment, is called front pitch.

For convenience, Y_b and Y_f are expressed in terms of coil sides.

Winding Pitch (Y_w)

The distance between the two consecutive and similar top, or bottom coil sides as the winding progresses is called the winding pitch. It is expressed in terms of coil-sides. The winding pitch Y_w is always even, because it is equal to either the difference or the addition of two odd numbers Y_b and Y_f.

For simplex lap winding

$$Y_w = Y_b - Y_f \text{ (progressive)}$$
$$Y_w = Y_f - Y_b \text{ (retrogressive)}$$

For simplex wave winding

$$Y_w = Y_b + Y_f$$

Commutator Pitch (Y_c)

The distance between the two commutator segments, to which ends of one coil are joined, is called the commutator pitch. Y_c is always expressed in terms of commutator segments.

The armature windings are of two types, namely (i) Lap winding (ii) Wave winding.

In lap winding, the two coil-ends of a coil are connected to the two adjacent commutator segments. In other words, for simplex lap winding each commutator segment has two coil-ends connected to it one coil-end is from the top coil-side of the one coil and the other coil-end is from the bottom coil-side of the adjacent coil.

In simplex wave winding, the two coil-ends of a coil are bent in opposite directions and connected to the commutator segments which are approximately two pole-pitches (i.e., 360° electrical) apart. In wave winding also, each commutator segment has two coil-ends connected to it one from the top coil-side and the other from bottom coil-side.

If lap coils are traversed, the movement is forward and backward alternately, whereas for wave winding, the movement is forward (or backward for retrogressive) only. In both lap and wave windings, all the coils in one parallel path are connected in series.

Simplex Lap Windings

Let C be the number of armature coil*s* and P the number of poles. Since each coil has two coil-sides, total number of coil-sides is $2C$. The back pitch Y_b, almost equal to pole pitch, must be odd

$$\therefore \qquad Y_b = \text{coil-sides per pole} \pm K \text{ or } Y_b = \frac{2C}{P} \pm K$$

where K is a number (integer or fraction) added to $2C/P$ to make Y_b an odd integer. Hence, $2C/P$ is equal to pole pitch in terms of coil-sides per pole.

For the progressive simplex lap winding

$$Y_w = Y_b - Y_f = +2 \text{ and } Y_c = \pm 1$$

For simplex retrogressive lap winding

$$Y_w = Y_b - Y_f = -2 \text{ and } Y_c = -1$$

Split Coils

In the process of winding design, a situation may arise where same coils with their top coil-sides together in one slot, may not have their bottom coil-sides together in another slot. Such cases arise only when there are more than two coil-sides per slot. These coils with their top coil-sides in one slot and bottom coil-sides in two different slots are called split coils. The split coils are helpful in commutation but the labour charges increase significantly.

Simplex Wave Winding

In simplex wave winding, two ends of a coil are connected to the commutator segments which are approximately 360° electrical apart.

The number of commutator segments is equal to the number of coils.

For a P-pole machine there are $P/2$ commutator pitches,

$$\frac{P}{2} Y_c = C \pm 1 \text{ or } Y_c = \frac{C \pm 1}{P/2}$$

where C is the number of coils or commentator segments.

Winding pitch for a simplex wave winding is

$$Y_w = \frac{2C \pm 2}{P/2} = 2Y_C$$

Note: +sign for progressive and –sign for retrogressive windings.

Wave winding is also known as series winding. The cause of designating it as wave windings is that during the traverse of a parallel path, one moves through the winding in a wave like progression.

Dummy Coils

For simplex wave winding the equation $Y_c = \frac{C \pm 1}{P/2}$ must be satisfied. If this relation is not fulfilled, then two ends of one (or more) coil may be cut off, insulated and not connected to the commutator segments. This unused and inactive coil, though placed in the armature slots for mechanical balance is called a dummy or dead coil. In order to have symmetrical armature winding, a dummy coil should be avoided as far as possible. However, it may be used in low-rating commutator machines where armature core with existing number of slots have to be used in order to avoid extra expenditure on new punching tools etc.

Note: If dummy coils are to be avoided, then $\frac{S}{P/2}$ should not be an integer.

Multiplex Lap and Wave Windings

$Y_c = \pm m$ for lap winding and $Y_c = \frac{C \pm m}{P/2}$ for wave winding

Where m is multiplicity of winding.

In the above expressions for Y_c positive sign is used for progressive winding and negative sign for retrogressive winding.

The number of armature parallel paths, A. In lap winding, $A = mp$ and in wave winding $A = 2$ m.

Multiplex wave windings are rarely used in DC machines.

Double layer windings are used in DC machines invariably.

Example 3: A synchronous generator with a synchronous reactance of 1.3 p.u. is connected to an infinite bus whose voltage is 1 p.u., through an equivalent reactance of 0.2 p.u., for maximum output of 1.2 pu, the alternator emf must be

(A) 1.5 p.u. (B) 1.56 p.u.
(C) 1.8 p.u. (D) 2.5 p.u.

Solution: (C)

$$P = \frac{EV_t}{X} \sin\delta$$

$$P_{max} = \frac{EV_t}{X}$$

$$\Rightarrow E = \frac{X P_{max}}{V_t}$$

$$\Rightarrow \frac{(0.2+1.3)\times 1.2}{1} = 1.8 \text{ p.u.}$$

Equalizer Rings

In order to avoid overloading and sparking at the brushes which is caused by unequal emf's of parallel paths it is essential to take some steps so that circulating currents are not handled by the brushes. An equalizer ring is a low resistance conductor wire which connects together the points in the armature winding which should be at same potential whose function is to cause the armature currents to flow within the armature itself, without letting them pass through the brushes. It is used to provide equalizer rings at the back of the armature, i.e., at the other side of the commutator.

Note: Equalizer rings are not required for simplex wave windings even if there is magnetic unbalance because the coil-sides in two parallel paths are distributed under all the poles. Therefore both the parallel circuits affected equally ad there is no need of equalizer rings in simplex wave windings.

Advantages of Wave Windings Over Lap Windings

1. A wave winding required only two brushes (or two sets of brushes) whereas lap winding requires brushes equal to the number of poles.
   A wave winding may be fitted with as many brushes as the number of poles. In such a case, if one or more brushes develop poor contact with the commutator, the commutator machine continues operating satisfactorily whereas this is not possible in case of lap-connected machines.
2. Equalizer rings are required for lap-connected machines for obtaining better commutation, whereas wave windings do not require any equalizer ring.
3. Lap-connected machines require more brushes and equalizer rings hence they are costly compared to wave-connected machines.

Lap windings are used on low-voltage (medium power (50 to 500 kW) and high power (above 500 kW) machines. Wave windings are used on high voltage low-power (less than 50 kW) and medium power machines.

Wave windings are preferred because of the fact that they do not require equalizer-rings and are less expensive. However for armature currents more than 400 A, lap winding is the only choice. It is because for armature currents above 400 A, current per path in wave winding would be more than 200 A and such a high current leads to poor commutation in wave-wound machines.

In addition to Lap and wave windings, there is another type of winding called composite winding or frog-leg winding which is a combination of simplex lap and multiplex wave windings and requires no equalizer rings. This type of winding is used mainly on high-power commutator machines.

Open Windings or AC Armature Windings

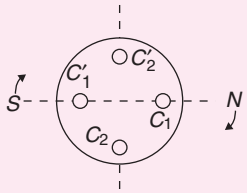
AC machines do not require a commutator and their armature windings are open at one or more points. In commutator machines, the turns per coil are limited for obtaining better commutation. But in machines without commutator, there is no such restriction on the choice of number of turns per coil.

Closed or commutator windings are always double layer windings whereas AC armature windings may be single layer or double layer. Single layer windings are used for small AC machines, whereas double layer windings are used for machines above 5 kW.

Previous Years' Questions

1. A rotating electrical machine having its self-inductances of both the stator and the rotor windings, independent of the rotor position will be definitely not develop **[2004]**
(A) starting torque
(B) synchronizing torque
(C) hysteresis torque
(D) reluctance torque

2. Two magnetic poles revolve around a stationary armature carrying two coils ($c_1 - c'_1$, $c_2 - c'_2$) as shown in Figure. Consider the instant when the poles are in a position as shown. Identify the correct statement regarding the polarity of the induced emf at this instant in coil sides c_1 and c_2. **[2005]**



(A) ⊙ in c_1, no emf in c_2
(B) ⊗ in c_1, no emf in c_2
(C) ⊙ in c_2, no emf in c_1
(D) ⊗ in c_2, no emf in c_1

3. Distributed winding and short chording employed in AC machines will result in **[2008]**
(A) increase in emf and reduction in harmonics
(B) reduction in emf and increase in harmonics
(C) increase in both emf and harmonics
(D) reduction in both emf and harmonics.

Answer Keys

Previous Years' Questions

1. D **2.** A **3.** D

Chapter 3

DC Machines

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Armature windings
- Types of generators
- E.M.F. equation of a DC generator
- Performance equations of DC generators
- Losses in a DC generator
- Armature reaction
- Demagnetizing ampere-turns per pole
- Compensating windings
- Commutation
- Equalizing connections
- Characteristics of DC generator
- Parallel operation of DC generators
- DC motors
- Motor characteristics

Fundamentals of DC Machines

Energy can neither be created nor be destroyed. We can change its forms, using appropriate energy -conversion processes. Change of the form of energy requires an energy conversion device.

- DC generator and DC motor are electromechanical energy conversion devices,
- A DC generator converts mechanical energy into electrical energy. It requires a prime mover such as a turbine diesel engine, etc.
- In a DC generator, the energy conversion is based on the principle of the production of dynamically induced e.m.f. This process can be explained by Faraday's laws of electromagnetic induction.
- The two basic essential parts of an electrical generator are
  1. A magnetic field, and
  2. A conductor or conductors which can so move as to cut the flux.
- Whenever there is relative motion between the conductor and the magnetic field, an emf is induced in the conductors which causes a current to flow if the conductor circuit is closed.
- In a DC generator, relative motion is produced by rotation of the armature. The armature winding is on the rotor and field winding is on the stator.
- In normal DC machines stator core is not laminated, armature core is laminated to reduce eddy current losses.

Constructional Details

The major parts of a DC machine are

1. Field system
2. Armature
3. Commutator
4. Brush and brush gear.

Constructionally there is no difference between a DC generator and DC motor.

1. The Field System

The purpose of the field system is to provide a uniform magnetic field within which armature rotates. The magnetic field is generated by electromagnets rather than permanent magnets on account of their greater magnetic strength and field strength regulation (i.e., field flux can be varied according to the requirement). The field system consists of four parts:

1. **Yoke:** This provides mechanical support for the poles and acts as a protecting cover for the whole machine and it carries the magnetic flux produced by the pole.

2. **Pole core:** The pole-core offers low reluctance path for magnetic flux. The pole core is of smaller cross-section than the pole shoe.
3. **Pole shoe:** The pole shoes serve two purposes.
   (a) They spread out the flux in the air-gap and also, being of larger cross-section, reduce the reluctance of the magnetic path.
   (b) They support the exciting coils (or field coils)
4. **Magnetizing coils (or) field winding:** These coils electromagnetize the poles which produce the necessary flux. The field coils which consist of copper wire or strip, are former wound for the correct dimension. Then, the former is removed and wound coil is put into place over the pole core.

2. Armature Core

It houses the armature conductors or coils and causes them to rotate and hence cut the magnetic flux of field magnets.

In addition to this, its most important function is to provide a path of very low reluctance to the flux through the armature from a N-pole to a S-pole. Armature core is made of high permeability material such as silicon steel. The use of high grade steel is made to keep hysteresis loss low. The core is laminated to reduce eddy current losses. These laminations are coated with insulating varnish.

3. Commutator

The function of commutator is to facilitate collection of current from the armature conductors. Inherently the voltage induced in the armature winding is alternating. Commutator acts as a mechanical rectifier to convert induced alternating voltages into DC voltage at brush terminals.

[In a DC Motor commutator acts as a mechanical inverter to convert DC applied voltage to AC Voltage in the armature winding].

- Commutator is of cylindrical structure and is built up of wedge-shaped hard-drawn copper segments. These segments are insulated from each other by thin layers of mica.

4. Brushes and Brush Gear

The function of brushes is to collect current from the commutator and supply it to the external-load circuit. Brushes are usually made of carbon or graphite. These brushes are housed in brush-holders. The brushes are made to bear down on the commutator by a spring to avoid any air-gap which may give rise to sparking.

Armature Windings

Two types of windings are mostly used for DC machine. They are lap winding and wave winding. For a given number of poles and armature conductors, the wave winding gives more emf than lap winding. Conversely for the same emf lap winding would require more no. of conductors which will result in higher winding cost. Wave windings are suitable for low capacity generators with a voltage rating of 500–600 V. Lap windings are used for large capacity generators because lap winding have more parallel paths.

- Another advantage with the wave winding is that it requires no equalizing connections, where in a lap-wound machine they are necessary.
- Lap winding is suitable for comparatively low voltage but high current machines whereas wave winding is used for comparatively high voltage, low-current machines.

Types of Generators

DC generators are broadly classified as

1. Separately excited generators: Field is energized from an independent external source of DC current.
2. Self-excited generators – field current is provided by the generators themselves. Due to residual magnetism, there is always present some flux in the poles. When the armature is rotated some emf and hence some induced current is produced which is partly or fully passed through the field coils there by strengthening the residual pole flux.

- For the process of self-excitation to sustain,

1. There must be residual flux in the field poles.
2. The field winding flux must aid residual flux.

There are 3 types of self-excited generators named according to the manner in which their field coils are connected to the armature.

1. **Shunt wound:** Field winding is connected directly across the armature terminals.
2. **Series wound:** Field winding is connected in series with the armature winding.
3. **Compound wound:** It is a combination of a few series and few shunt windings.

There are two types of connections in a compound machine. They are (1) Short shunt (2) Long shunt.

In a compound generator, the shunt field is stronger, than the series field. When series field aids the shunt field, generator is said to be cumulatively compounded. On the other hand if series field opposes shunt field the generator is said to be differentially compounded.

- By reversing the connections of the series field winding, cumulative connection can be converted to differential connection and vice versa.

EMF Equation of a DC Generator

Let ϕ = Flux / pole in Wb
Z = Total number of armature conductors
P = No. of generator poles
A = No. of armature parallel paths
N = Armature speed in rpm
E = emf induced in any parallel path in armature,

$$\text{The induced E.M.F.E.} = \frac{\phi ZN}{60} \times \frac{P}{A}$$

- The number of parallel paths A, depends upon the type of the armature winding, i.e., whether wave or lap-wound,

For wave-wound armature $A = 2m$, where m is multiplicity of the winding.

$m = 1$ for simplex, $m = 2$ for duplex, $m = 3$ for triplex and so on for a lap-wound armature.

$A = Pm$, where m is the multiplicity of the winding

For a given DC machine Z, P and A are constants.

Hence putting $K_a = \frac{ZP}{A}$

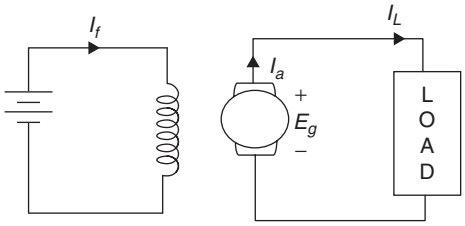
We get $E_g = K_a \phi N$

The induced emf can be increased by

1. Increasing the flux / pole ϕ by increasing the field current (up to saturation point).
2. Increasing the speed of the prime mover.

Performance Equations of DC Generators

Separately Excited

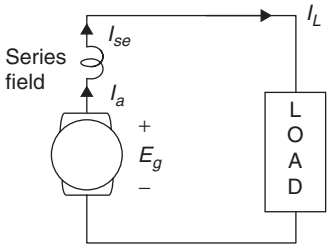


$$I_a = I_L = I$$
$$V = E_g - I_a r_a$$

Power developed = $E_g I_a$

Power delivered to load = $V.I_L$

Series Wound

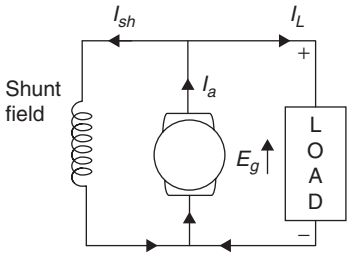


$$I_a = I_{se} = I_L = I$$
$$V = E_g - I(r_a + r_{se})$$

Power developed = $E_g I_a$

Power delivered = VI

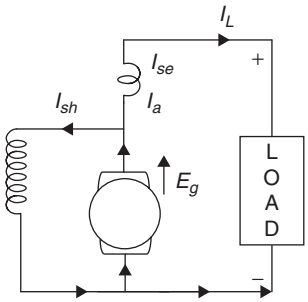
Shunt Wound



$$I_{sh} = \frac{V}{R_{sh}}$$
$$I_a = I_L + I_{sh}$$
$$V = E_g - I_a r_a$$

Power developed = $E_g I_a$

Short Shunt Compound

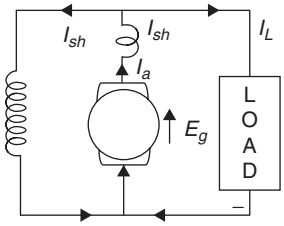


$$I_{se} = I_L$$
$$I_{sh} = \frac{V + I_{se} r_{se}}{R_{sh}} \text{ suffixes}$$
$$I_a = I_L + I_{sh}$$
$$V = E_g - I_a r_a - I_{se} r_{se}$$

Power developed = $E_g . I_a$

Power delivered = VI_L

Long Shunt Compound



$$I_{sh} = \frac{V}{R_{sh}}$$
$$I_a = I_L + I_{sh}$$
$$V = E_g - I_a r_a - I_{se} r_{se}$$

Power developed = $E_g I_a$

Power delivered = VI_L

Solved Examples

Example 1: A 4-pole lap-connected DC generator with 480 conductors has an armature resistance 0.06 Ω. If the conductors are reconnected to form a wave winding, other things remaining unchanged what is the value of armature resistance?

Solution: $\frac{Z_{lap}}{Z_{wave}} = \frac{4}{P^2}$ or

$$Z_{wave} = \frac{P^2}{4} \times Z_{Lap}$$

$$= \frac{P^2}{4} \times 0.06 = 0.24\ \Omega$$

Example 2: Calculate commutator pitch for two-circuit simplex-wave winding having 4-poles and 129 commutator segments.

Solution: Commutator pitch for simplex wave winding is given by

$$Y_C = \frac{C \pm 1}{P/2} = \frac{129 \pm 1}{4/2} = \frac{128 \text{ or } 130}{2}$$

The commutator pitch may be 64 or 65.

Example 3: A 4-pole generator with 24 conductors has a two layer lap winding. The pole pitch is ____.

(A) 24 (B) 12 (C) 6 (D) 4

Solution: (C)

Pole pitch $= Y_p = \frac{Z}{P}$

$Z = 24; P = 4$

$$Y_p = \frac{Z}{p} = \frac{24}{4} = 6$$

Example 4: An 8-pole, lap-wound armature running at 350 rpm is required to generate 260 V. The useful flux/pole is 0.05 Wb. If the armature has 120 slots, the no. of conductors/slot is _____

(A) 6 (B) 8 (C) 10 (D) 4

Solution: (B)

$$E_g = \frac{\phi ZNP}{60A} \Rightarrow 260 = \frac{0.05 \times Z \times 350 \times 8}{60 \times 8}$$

$$\Rightarrow Z = 890$$

$$\therefore \text{No of conductors/slot} = \frac{890}{120} = 7.14$$

Conductors/slot should be an integer

Hence, conductors/slot = 8

Example 5: A 4-pole lap-wound generator is run at 900 rpm. The reactance voltage if self inductance is 153 × 10^{-6} H, armature current is 108 A, no of segments = 55, width of brush = 1.74 segments is _____

(A) 3.195 V (B) 3.915 V
(C) 5.391 V (D) 5.915 V

Solution: (B)

$$I_c = \frac{I_a}{A} = \frac{108}{4} = 27\ \text{A}$$

Time taken by one commutator segment to cross brush axis

$$= \frac{60}{900 \times 55}$$

Time taken for 1.74 segments $= \frac{60 \times 1.74}{900 \times 55} = 2.11 \times 10^{-3}$ sec

$$\therefore \text{Reactance voltage} = E = L \cdot \frac{2I_c}{t_c}$$

$$\therefore E = \frac{153 \times 10^6 \times 2 \times 27}{2.11 \times 10^{-3}} = 3.915\ \text{V}$$

Losses in a DC Generator

The various losses occurring in a generator can be subdivided as follows.

Copper Losses

1. Armature copper loss = $I^2_a R_a$.
   where R_a = resistance of armature and inter poles and series field winding etc.
   This is about 30% to 40% of full-load losses.
2. Field copper loss:
   In case of shunt generators, it is practically constant and $I^2_{sh} R_{sh}$. In the case of series generator, it is = $I^2_{se} R_{se}$, where R_{se} is resistance of the series field winding.
3. The loss due to brush contact resistance. It is usually included in the armature copper loss.

Magnetic Losses (Iron or Core Losses)

Hysteresis Loss

This loss is due to the reversal of magnetization of the armature core. Every portion of the rotating core undergoes one complete cycle of magnetic reversal after passing under one pair of poles.

For normal flux densities (i.e., up to 1.5 Wb/m^2), hysteresis loss is given by Steinmetz formula.

$$W_h = \eta B_{\max}^{1.6} \times f\ V\ W.$$

where η = Steinmetz hysteresis coefficient
$B_{\max}$ = Maximum flux density
f = Frequency of magnetic reversals
V = Volume of the core in m^3

The hysteresis loss is reduced by choosing magnetic materials which have a low hysteresis coefficient such as silicon steel.

Eddy Current Loss

When the armature core rotates, it also Cuts The Magnetic Flux. Hence an emf is induced in the magnetic core which

sets up large current in the body of the core due to its small resistance. These currents circulate in the core and find no other path, hence they are known as eddy current. The power loss due to the flow of this current is known as eddy current loss. In order to reduce this loss and consequent heating of the core to a small value, the core is built up of thin laminations which are insulated from each other by a thin coating of varnish. These laminations are stacked and riveted at right angles to the path of the eddy current.

Eddy current loss is given by the following relation

$$W_e = KB^2_{max} f^2 t^2 V W.$$

Where

B_{max} = Maximum flux density
F = Frequency of magnetic reversals
T = Thickness of each lamination
V = Volume of the core.

The iron losses are practically constant for shunt and compound wound generators, because in their case, field current is approximately constant, the iron losses are about 20% to 30% of full-load losses.

Mechanical Losses

These losses consist of

1. Friction loss at bearings and commutators.
2. Air friction or windage loss of rotating armature. These are about 10% to 20% of full load losses.

Stray Losses

Magnetic and mechanical losses are collectively known as stray losses. These are also known as rotational losses as they exist only when the rotor is rotating.

Constant or Standing Losses

Field Cu loss for shunt and compound generators is constant. Hence, stray losses and shunt Cu loss are constant in their case. These losses are together known as standing or constant losses W_C.

Example 6: A 10 kW, 230 V, DC shunt generator operation are 80% efficiency. Input power and total losses are

(A) 5 kW, 2.5 kw (B) 12.5 kW, 1 kW
(C) 12.5 kW, 2.5 kW (D) 5 kW, 1 kW

Solution: (C)

Output = 10 kW

$$\therefore \quad \frac{80}{100} = \frac{10}{\text{input}}$$

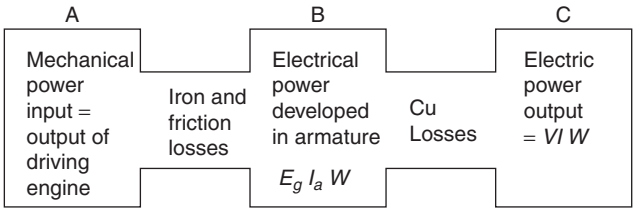
$$\Rightarrow \quad 0.8x = 10$$

$$\Rightarrow \quad x = \frac{10}{0.8} = 12.5 \text{ kW}$$

$\therefore$ Losses = input–output = 2.5 kW.

Power Stages in Generator

Various power stages in generator are as shown below



Following are the three generator efficiencies

1. Mechanical efficiency

$$\eta_M = \frac{B}{A} = \frac{\text{Total power generated}}{\text{Mech power input}}$$

2. Electrical efficiency

$$\eta_E = \frac{C}{B} = \frac{\text{Electric power output}}{\text{Electrical power developed}}$$

3. Overall or commercial efficiency

$$\eta_c = \frac{C}{A} = \frac{\text{Electric power output}}{\text{Mechanical power input}}$$

Condition for Maximum Efficiency

Generator efficiency is maximum, when

Variable loss = Constant loss

Armature Reaction

The effect of the magnetic field set up by armature current on the distribution of flux under main poles of a generator is called armature reaction.

The armature magnetic field has two effects.

1. It demagnetizes or weakens the main flux and
2. It cross magnetizes or distorts it.

- The first effect leads to reduced generated voltage and the second to the sparking at the brushes.
- The flux is strengthened at the trailing pole tips but weakened at the leading pole tips.
- Because of armature reaction there is a shift in the magnetic neutral axis (M. N. A.) in the direction of rotation of the armature.
- Armature reaction gives rise to cross magnetization and de-magnetization.

Demagnetizing Ampere-turns Per Pole

The armature demagnetizing ampere-turns are neutralized by adding extra ampere-turns to the main field winding.

Demagnetizing ampere-turns pole,

$$AT_d / \text{pole} = ZI \times \frac{\theta_m}{360^\circ}$$

where

Z = Total number of armature conductors

I = Current in each armature conductor (or) Current in a parallel path

$I_{a/2}$ for simplex wave winding

$I_{a/p}$ for simple lap winding

θ_m = Forward lead in mechanical or geometrical or angular degrees

Cross-magnetizing Ampere-turns Per Pole

Cross magnetizing ampere-turns per pole,

$$AT_C/\text{pole} = ZI\left[\frac{1}{2P} - \frac{\theta_m}{360°}\right]$$

To reduce the effects of armature reaction the following methods are normally used.

1. Provision of inter poles.
2. Provision of compensating windings.
3. Chamfering of the poles.

Compensating Windings

In large direct current machines which are subjected to large fluctuations in load (i.e., rolling mill motors and turbo-generators, etc.) compensating windings are used. Their function is to neutralize the cross-magnetizing effect of armature reaction.

In the absence of compensating winding, with sudden changes in load, the flux will be suddenly shifting backward and forward which will induce an emf in the armature coils. It may be so high as to strike an arc between the consecutive commutator segments which may lead to flash over around the whole commutator there by resulting in a short circuit of armature.

- These windings are embedded in slots in the pole shoes and are connected in series with armature in such a way that the current in them flows in opposite direction to that following in the armature conductors directly below the pole shoes.

Number of Compensating Windings

No. of armature amp-turns/pole for compensating winding

$$= \frac{ZI}{2P} \times \frac{\text{Pole arec}}{\text{Pole pitch}} = 0.7 \times \frac{ZI}{2P} \text{ (approximately)}$$

Commutation

The process by which the current in the short-circuited armature coil is reversed while it crosses the magnetic neutral axis is called commutation.

During the commutation period the brush spans and hence short circuits that particular coil under going reversal of current through it. The self-induced emf in the coil undergoing commutation is known as reactance voltage.

Reactance voltage = (Coefficient of self-inductance) × Rate of change of current

Time of commutation or commutation period,

$$T_c = \frac{W_b - W_m}{v} \text{ second}$$

where

W_b = Brush width in cm

W_m = Width of mica insulation in cm

v = Peripheral velocity of commutator segments in cm/sec

Reactance voltage

$$L \times \frac{2I}{Tc}$$

If commutation is linear = 1.11 $L \times \frac{2I}{T_c}$

If commutation is sinusoidal

Methods of Improving Commutation

There are two practical ways of improving commutation, i.e., making current reversal in the short-circuited coil as sparkless as possible.

Resistance Commutation

In this method, low resistance copper brushes are replaced by comparatively high resistance carbon brushes

EMF-commutation

In this method, arrangement is made to neutralize the reactance voltage by producing a reversing emf in the short-circuited coil under commutation. The reversing emf may be produced in two ways.

(a) Either by giving the brushes a forward lead, sufficient enough to bring the short – circuited coil under the influence of next pole of opposite polarity.
Shifting of the brushes is not a solution for reducing the effect of armature reaction. Though the cross-magnetization effect may be reduced somewhat, de-magnetization is also introduced.

(b) By providing inter poles.
Inter poles: These are small poles fixed to the yoke and spaced in between the main poles. They are wound with comparatively few heavy gauge Cu wire turns and are connected in series with the armature so that they carry full armature current.

Their polarity in the case of a generator is the same as that of the main pole a head in the direction of rotation. The function of inter poles is two-fold:

1. As their polarity is the same as that of the main pole ahead, they induce an emf in the coil (under

commutation) which helps the reversal of current. The emf induced by inter-poles is known as commutating or reversing emf The commutating emf neutralizes the reactance emf there by making commutation sparkless. As inter poles carry armature current, their commutating emf is proportional to the armature current. This ensures automatic neutralization of reactance voltage which is also due to armature current.

2. Another function of inter poles is to neutralize the cross-magnetizing effect of armature reaction. Cancellation of cross magnetization is automatic and for all loads because both are produced by the same armature current.

Equalizing Connections

It is characteristic of lap-winding that all conductors in any parallel path lie under one pair of poles. If fluxes from all poles are exactly the same, then the emf induced in each parallel path is same and each path carries same current. But some inequalities in flux inevitably occur due to which there is always a slight imbalance of emf in the various parallel paths.

The function of equalizer rings is to avoid unequal distribution of current at the brushes there by helping to get sparkless commutation.

$$\text{Number of equalizer rings} = \frac{\text{No. of conductors}}{\text{No. of pair of poles}}$$

Equalizer rings are not used in wave-wound armature, because there is no imbalance in the emf's of the two parallel paths, this is due to the characteristic of wave-winding that armature conductors in either parallel path are not confined under one pair of poles as in lap winding but are distributed under all poles.

Characteristics of a DC Generator

There are three important characteristics or curves of a DC generator.

1. **Open-circuit Characteristic:** O.C.C. (or No-load saturation characteristic).
   It shows the relation between the generated emf E_g and field current I_f at a given fixed speed.
   It is also known as magnetic characteristic.
2. **External Characteristic:** It gives relation between the terminal voltage V and the load current I_L.
3. **Internal or total characteristic:** This characteristic gives relation between the emf, E actually induced in the armature (after allowing for the demagnetizing effect of armature reaction) and the armature current I_a.

Characteristic Curves

Separately Excited Generator

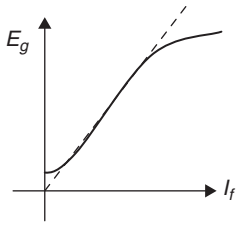
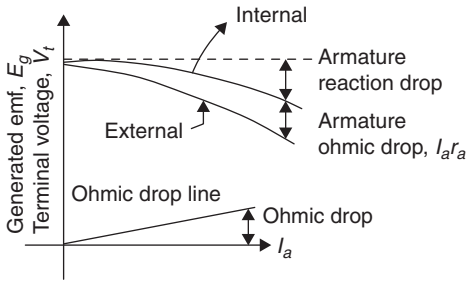


Figure 1 No-load characteristic (For all types of generators)



Series-wound Generators

For a series generator, as the field windings are in series with the armature, the field current (which is same as armature current) is zero, hence we do not have O.C.C. for a series wound generator. But it can be drawn from O.C.C. by considering extra current needed to neutralize the effect of armature reaction.

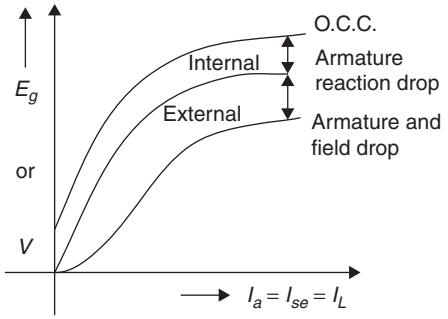
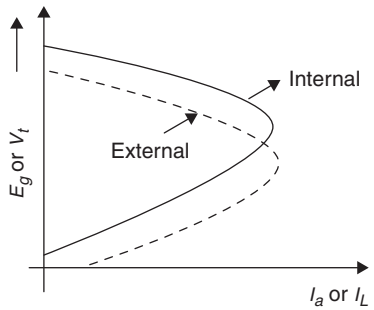
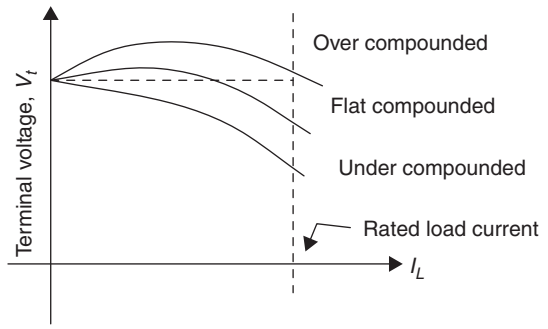


Figure 2 Series generator has rising voltage characteristic

Shunt-wound Generator



Compound Generator



Condition for build-up of a self excited (shunt) generator:

The conditions required to be fulfilled for the build up of a shunt generator are as follows.

1. There must be some residual magnetism in the generator poles.
2. For the given direction of rotation the shunt field coils should be correctly connected to the armature i.e., the shunt field current produces a flux in the direction of residual flux.
3. If excited on open circuit, its shunt field resistance should be less than the critical resistance.
4. If excited on load, then its shunt field resistance should be more than certain minimum value of resistance whose value is given by internal characteristic.

- For a series wound generator the external (or load) circuit resistance should be less than the critical resistance and for a shunt generator load circuit resistance must be greater than critical load resistance.

A DC generator may fail to build up voltage because of failure of any of the above given cases.

Parallel Operation of DC Generators

Parallel operations is extremely desirable for the following reasons

1. Continuity of service
2. Efficiency
3. Maintenance and repair
4. Additions to plant

Condition to be Satisfied

- Polarity of the incoming generator should be same as that of existing generator.
- The no-load induced emf s are to be the same in order to avoid circulating currents on no load.

Shunt generators are best suited for parallel operation because of the drooping nature of their external characteristics

Because of the rising nature of the external characteristics of series generators, their parallel operations will pose some problems, which can be overcome by

1. Cross-connecting the fields.
2. Providing equalizing bar.

Example 7: Two DC generators having rectilinear external characteristics operate in parallel. One machine has the terminal voltage of 270 V on no load and 220 V at the load current of 30 A. The other has a voltage of 280 V at no load and 220 V at the load current of 30 A. Calculate the load shared and bus voltage of each machine when the total load current is 50 A.

Solution: Generator 1: Slope of external characteristic or voltage drop/ampere

$$= \frac{270-220}{30} = \frac{50}{30} = 5/3 \text{ V/A}.$$

Generator 2: Slope of external characteristic or voltage drop/ampere

$$= \frac{280-220}{30} = \frac{60}{30} = 2 \text{ V/A}$$

Let V be the common bus voltage

I_1 = Load current shared by generator 1

I_1 = Load current shared by generator 2.

Then

$$V = 270 - \frac{5}{3} I_1 \text{ for generator 1}$$

$$V = 280 - 2I_2 \text{ for generator 2.}$$

Since the terminal voltage is same

$$\therefore\ 270 - \frac{5}{3} I_1 = 280 - 2I_2$$

$$\Rightarrow \frac{-5}{3} I_1 + 2I_2 = 10 \qquad (1)$$

Also $I_1 + I_2 = 50$ (2)

Solving Equations (1) and (2)

We get,

$I_1 = 24.54$ A and $I_2 = 25.46$ A

and bus bar voltage $V = 229$ V.

DC Motors

An electric motor is a machine which converts electrical energy into mechanical energy. It works on the principle that when a current carrying conductor is placed in a magnetic field, it experiences a mechanical force. Whose direction is given by Fleming's left-hand rule and hence the conductor moves in the direction of force.

The magnitude of force is given by

$F = BIL \sin\theta$ Newton

where

B = Flux density in Wb/m^2

I = Current flowing in the conductor

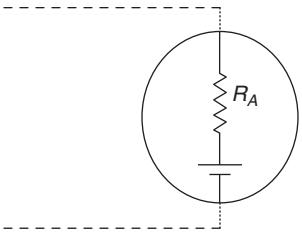
L = Length of the conductor

θ = Angle made by the tangent drawn at conductor with field direction

Significance of the Back emf

As the armature of a DC motor rotates, all the necessary conditions for an emf to be induced in a conductor are satisfied, namely a conductor, a magnetic field and the relative motion between the conductor and magnetic field. Hence an emf is induced in the armature, the direction of which can be determined by Flemings right hand rule. This induced emf is in opposition to the applied voltage hence it is known as back or counter emf.

The applied voltage has to overcome the back emf in addition to the armature drop. The equivalent circuit of armature is shown in below figure.



The voltage equation of a DC motor is

$$V = E_b + I_a R_a$$

Where E_b is the back emf given by

$$E_b = \frac{\phi ZN}{60} \times \frac{P}{A}$$

From the voltage equation, on multiplying both the side by I_a, we get

$$VI_a = E_b I_a + I_a^2 R_a$$

where

VI_a = Electrical input to the armature.

$E_b I_a$ = Electrical equivalent of mechanical power developed in the armature.

$I^2_a R_a$ = Cu loss in the armature.

Example 8: DC motor gives maximum power when

(A) $V = 2E_b$ (B) $V = E_b/2$

(C) $V = E_b/\sqrt{2}$ (D) $V = E_b$

Solution: (A)

$$V = E_b + I_a R_a$$

$$\therefore \quad P_m = E_b I_a = VI_a - I_a^2 R_a$$

It is maximum when $\frac{dP_m}{dI_a} = 0$

$$V - 2I_a R_a = 0 \Rightarrow I_a R_a = \frac{V}{2}$$

$\therefore$ Back emf: $E_b = V - I_a R_a$

$$= V - \frac{V}{2} = \frac{V}{2}$$

$$\therefore \quad E_b = \frac{V}{2}$$

Condition for Maximum Power

The gross mechanical power developed by a motor is $P_m = VI_a - I^2_a R_a$.

Differentiating both sides w.r.t i_a and equating the result to zero, we get,

$$\frac{dP_m}{dI_a} = V - 2I_a R_a = 0$$

$$\therefore \quad I_a R_a = \frac{V}{2}$$

As $V = E_b + I_a R_a$ and $I_a R_a = V/2$

$$\therefore \quad E_b = V/2$$

Thus gross mechanical power developed by a motor is maximum when back emf is equal to half the applied voltage. This condition is however, not realized in practice, because in that case current would be much beyond the normal current of the motor. Moreover, half the input would be wasted in the form of heat and taking other losses into consideration, the motor efficiency will be below 50 percent.

Condition for Maximum Efficiency

Variable loss = Constant loss
(same as that of generator)

Armature Torque (T_a) or Torque developed (T_d):

Let T_a be the torque developed by the armature of a motor running at N rps. If T_a is in Nm, then

Power developed = $T_a \times 2\pi NW$

$$T_a \times 2\pi N = E_b I_a$$

$$T_a = \frac{E_b I_a}{2\pi N} = \frac{\phi ZN\left(\frac{P}{A}\right) \times I_a}{2\pi N}$$

$$= \frac{1}{2\pi}\phi Z\left[\frac{P}{A}\right] I_a \text{ Nm}$$

$$T_a = 0.159\phi Z I_a \left[\frac{P}{A}\right] \text{ Nm}$$

From the above equation for the torque, we find that $T_a \propto \phi I_a$

1. For series motor, ϕ is directly proportional to I_a (before saturation) because field windings carry full armature current

   $\therefore \quad T_a \propto I_a^2$

2. For shunt motors, ϕ is practically constant hence $T_a \propto I_a$

Alternate expression for torque developed.
If N is in rpm then,

$$T_a \times \frac{2\pi N}{60} = E_b I_a$$

$$T_a = \frac{60 E_b I_a}{2\pi N} = \frac{60}{2\pi}\frac{E_b I_a}{N} = 9.55\frac{E_b I_a}{N} \text{ Nm}$$

Shaft Torque

The torque which is available for doing useful work is known as shaft torque, T_{sh}.

$$T_{sh} = \frac{\text{Output in watts}}{2\pi N/60} \; N \text{ in rpm}$$

$$= \frac{60}{2\pi}\frac{\text{Output}}{N} = 9.55\frac{\text{Output}}{N}$$

The difference $(T_a - T_{sh})$ is known as lost torque and is due to iron and friction losses of the motor.

Speed of a DC Motor

From the voltage equation of a DC motor, we get

$$E_b = V - I_a R_a$$

Or $$\frac{\phi ZN}{60}\frac{P}{A} = V - I_a r_a$$

$$\therefore \quad N = \frac{V - I_a r_a}{\phi} \times \frac{60A}{ZP} \text{ rpm}$$

Or $$N = \frac{E_b}{\phi} \times \left(\frac{60A}{ZP}\right)$$

For a given DC motor A, Z and P are fixed.

$$\therefore \quad N = K\frac{E_b}{\phi}$$

where K is a constant

$$N \propto \frac{E_b}{\phi} \text{ Or } \frac{N_2}{N_1} = \frac{E_{b2}}{E_{b1}} \times \frac{\phi_1}{\phi_2}$$

1. For series motor,
   $\phi \propto I_a$ (prior to saturation of magnetic poles)

   $$\therefore \quad \frac{N_2}{N_1} = \frac{E_{b2}}{E_{b1}} \times \frac{I_{a1}}{I_{a2}}$$

2. For shunt motor,

   $$\frac{N_2}{N_1} = \frac{E_{b2}}{E_{b1}} \times \frac{\phi_1}{\phi_2}$$

   If flux is constant i.e., $\phi_2 = \phi_1$

   Then $$\frac{N_2}{N_1} = \frac{E_{b2}}{E_{b1}}$$

Speed Regulation

The speed regulation is defined as the change in speed when the load on the motor is reduced from rated value to zero, expressed as percent of the rated load speed.

$$\%\text{Speed regulation} = \frac{\text{No load speed} - \text{Full load speed}}{\text{Full load speed}} \times 100$$

$$\text{P. U. speed regulation} = \frac{\text{No load speed} - \text{Full load speed}}{\text{Full load speed}}$$

Example 9: Determine the full-load speed shunt field, current.

Solution: $I_{sh} = \frac{230}{230} = 1$ A

$I_{a1} = 3 - 1 = 2$ A

$E_{b1} = 230 - 0.5 \times 2 = 229$ V

$I_{a2} = 23 - 1 = 22$ A

$E_{b2} = 230 - 22 \times 0.5 = 219$ V

$$N_2 = \frac{E_{b2}}{E_{b1}} \times \frac{\phi_1}{\phi_2} \times N_1$$

$$= \frac{219}{229} \times \frac{\phi_1}{0.98\phi_1} \times 1000$$

$(\because \phi_2 = 0.98\phi_1) = 975.84$ rpm

Example 10: Determine the full load torque

Solution:

Power developed $= E_{b2} I_{a2} = 219 \times 22$

$= 4.818$ kW

Angular speed, $\omega = \frac{2 \times 3.14 \times 975.84}{60}$

$= 102.14$ rad/sec

$$\text{Torque} = \frac{4818}{102.14} = 47.17 \text{ Nm.}$$

Common Data for Examples 11 and 12
A 220 V DC shunt motor takes 20 A at rated voltage and runs at 1000 rpm. Its field circuit resistance is 100 Ω and armature circuit resistance is 0.1 Ω. Compute the value of additional resistance required in the armature circuit to reduce the speed to 800 rpm when

Example 11: The load torque is proportional to the speed

Solution: Given the condition torque, $T \propto$ speed, N

$$\therefore \quad \frac{T_1}{T_2} = \frac{N_1}{N_2}$$

Shunt field current, $I_{sh} = \frac{220}{100} = 2.2$ A

$$I_{a1} = I_L - I_{sh} = 20 - 2.2 = 17.8 \text{ A}$$

$$E_{b1} = V - I_{a1} R_a = 220 - 17.8 \times 0.1$$

$$= 218.22 \text{ V}$$

$$E_{b2} = E_{b1} \times \frac{N_2}{N_1}$$

$$218.22 \times \frac{800}{1000} = 174.57 \text{ V}$$

Also $T \propto \phi I_a$, where ϕ is constant

$$\therefore \quad I_{a2} = \frac{I_{a1} N_2}{N_1} = \frac{17.8 \times 800}{1000}$$

Total armature circuit resistance

$$R = \frac{V - E_b}{I_a} = \frac{220 - 174.57}{14.24} = 3.19 \ \Omega$$

$$R_{ext} = 3.19 - 0.1 = 3.09 \ \Omega$$

Example 12: The load torque varies as the square of the speed

Solution: Given that $T \propto N^2$

$$\therefore \quad I_{a2} = I_{a1} \times \left(\frac{N_2}{N_1} \right)^2$$

$$= 17.8 \times \left(\frac{800}{1000} \right)^2 = 11.4 \text{ A}$$

$$R = \frac{220 - 174.57}{11.40} = 3.988 \ \Omega$$

$$R_{ext} = 3.988 - 0.1 = 3.89 \ \Omega$$

Example 13: The full-load speed regulation is 8% when the speed at full load is 800 RPM. The no-load speed is _____ RPM.

(A) 832 (B) 864
(C) 736 (D) 768

Solution: (B)

$$\text{Speed regulation} = \frac{N_0 - N}{N}$$

N_0 = No-load speed

N = Speed in load

$$\frac{8}{100} = \frac{N_0 - 800}{800} \Rightarrow 64 = N_0 - 800$$

$$N_0 = 864 \text{ rpm}$$

Example 14: A 230 V shunt motor develops a torque of 48 Nm at an armature current of 12 A. The torque produced when the armature current is 15 A, is

(A) 48 Nm (B) 36 Nm
(C) 60 Nm (D) 72 Nm

Solution: (C)

$$T \propto \phi I_a$$

For a shunt motor, ϕ is constant

$$T \propto I_a$$

$$\frac{T_2}{T_2} = \frac{I_{a2}}{I_{a2}} \Rightarrow T_2 = T_1 \times \frac{I_{a2}}{I_{a1}}$$

$$= 48 \times \frac{15}{12} = 60 \text{ Nm}$$

Example 15: A DC motor develops a torque of 120 Nm at 20 rps. At 30 rps it will develop a torque of _____ Nm.

(A) 160 (B) 120
(C) 80 (D) 40

Solution: (C)

$$T \propto \frac{1}{N}$$

$$N_1 T_1 = N_2 T_2$$

$$T_2 = \frac{N_1 T_1}{N_2} = \frac{20 \times 120}{30} = 80 \text{ Nm}$$

Example 16: A DC motor connected to a 230 V supply has an armature resistance of 0.15 Ω. The value of armature current when the back emf E_b = 200 V is ____ A.

(A) 50 (B) 100
(C) 400 (D) 200

Solution: (D)

$$E_b = V - I_a R_a$$

$$I_a = \frac{V - E_b}{R_b}$$

$$= \frac{230 - 200}{0.15} = \frac{30}{0.15} = 200 \text{ A}$$

Motor Characteristics

The characteristic curves of DC motors can be broadly classified as

1. Electrical characteristics: 'T' vs. 'I_a' and 'N' vs. 'I_a'.
2. Mechanical characteristic: 'N' vs. 'T'.

Characteristics of Shunt Motors

1. T_a / I_a characteristic

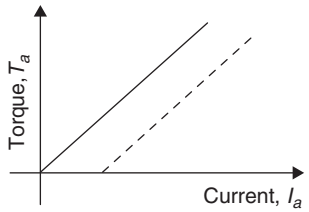
$$T = K \phi I_a$$

where K is a constant

In the case of shunt motors, the flux is assumed to be practically constant (though at heavy loads, ϕ decreases somewhat due to increased armature reaction)

$$\therefore T \propto I_a$$

The torque varies linearly with armature current.

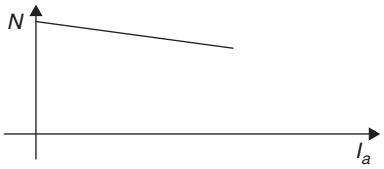


2. N / I_a characteristic

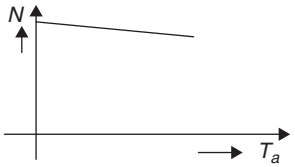
Variations of speed can be obtained from the relation

$$N \propto \frac{E_b}{\phi} \text{ or } \frac{V - I_a r_a}{\phi}$$

Since the flux is practically constant for a shunt motor, the speed decreases linearly with armature current. The speed curve is slightly dropping as shown in the figure but for all practical purposes, shunt motor is taken as a constant – speed motor.



3. N / T_a characteristic

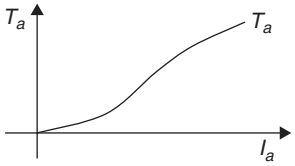


Characteristics of Series Motors

1. T_a / I_a characteristic

$$\text{Torque } T_a \propto \phi I_a$$

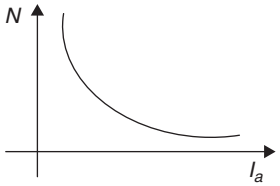
In the case of the series motors field windings also carry the armature current, $\phi \propto I_a$ up to the point of magnetic saturation. At light loads, I_a and hence ϕ are small. But as I_a increases, T_a increases as the square of the current, i.e, $T_a \propto I_a^2$. Hence T_a / I_a curve is a parabola as shown in below figure. After saturation ϕ is almost independent of I_a hence $T_a \propto I_a$ only. So the characteristic becomes a straight line



2. N / I_a characteristic

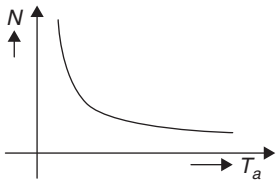
$$N \propto \frac{E_b}{\phi}$$

Change in E_b for various load currents is small and hence may be neglected with increased $I_a \cdot \phi$ also increases hence the speed varies inversely as armature current as shown in the following figure.



- The series motor is a variable speed motor.
- On no-load and at light loads, a DC series motor has a tendency to run at dangerously high value of speed.

3. N / T_a or Mechanical characteristic.



When the speed is high, the torque is low and vice versa.

Characteristic of Compound Motors

These motors have both series and shunt windings. So the characteristics of such motors lie in between those of shunt and series motors. The characteristics approach those of a shunt or a series motor, depending upon the relative field strengths of the shunt and the series field windings.

If series excitation helps the shunt excitation, then the motor is said to be cumulatively compounded. If on the other hand, series field opposes the shunt field, then the motor is said to be differentially compounded.

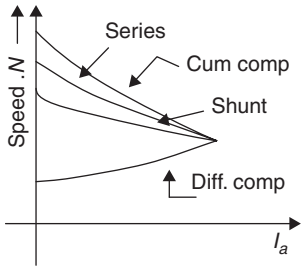
Cumulative Compound Motors

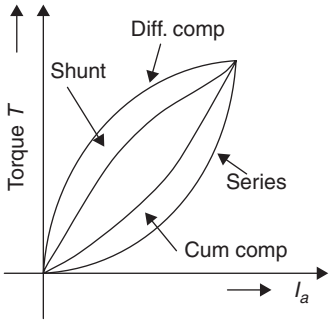
These machines are used where series characteristics are required and where, in addition, the load is likely to be removed totally as in some types of coal cutting machines or for driving heavy machine tools which have to take sudden cuts quite often. Such machines are used to drive electric shovels, metal stamping, machines, reciprocating pumps, hoists and compressors etc.

Differential Compound Motors

Since series field opposes the shunt field, the flux is decreased as load is applied to the motor. This results in the motor speed remaining almost constant or even increasing with the load ($\therefore N \propto E_b/\phi$). Due to this reason there is a decrease in the rate at which the motor torque increases with load. Such motors are not in common use. One of the biggest drawback of such a motor is that due to weakening of flux with increase in load, there is a tendency towards

speed instability and motor running away unless designed properly.





Type of motor	Characteristics	Applications
Shunt	Approximately constant speed, adjustable speed	For driving constant speed line shafting. Lathes, centrifugal pumps, machine tools, blowers and fans, reciprocating pumps
Series	Variable speed, Adjustable varying speed, high starting torque	For traction work, i.e., electric locomotives rapid transit systems, trolley, cars, cranes and hoists, conveyors
Cumulative compound	Variable speed, adjustable varying speed, high starting torque	For intermittent high torque loads, for shears and punches, elevators, conveyors, heavy planers, rolling mills, ice machines, printing presses, air compressors.

Speed Control of DC Motors

The speed of a DC motor is given by the relation

$$N = \frac{V - I_a r_a}{Z\phi}\left[\frac{A}{P}\right] = K\,\frac{V - I_a r_a}{Z\phi}\ \text{rps}$$

where K is a constant.

Sometimes, it may become necessary to vary the speed of a DC motor at a given load condition. There are various methods of controlling the speed of a DC motor.

1. By varying the applied voltage
2. By varying the resistance in the armature circuit
3. By varying flux/pole

Example 17: The value of torque produced by armature of 4-pole motor having 774 conductors, two paths in parallel, 24 mWb flux per pole, when the total armature current is 50 A, is ___.

(A) 350 Nm (B) 373 Nm
(C) 295.35 Nm (D) 420 Nm

Solution: (C)

$$T_a = 0.159 \cdot \phi I_a\left(\frac{P}{A}\right)$$

$$= 0.159 \times 774 \times 24 \times 10^{-3} \times 50 \times \frac{4}{2}$$

$$= 295.35\ \text{Nm}$$

Common Data for Example 18

A 230 V DC shunt motor takes a current of 40 A and runs at 1100 RPM. If armature and shunt field resistances are 0.25 Ω and 230 Ω respectively.

Example 18: The torque developed by the motor is _____.

(A) 78.9 Nm (B) 74.66 Nm
(C) 98.7 Nm (D) 87.9 Nm

Solution: (B)

$$E_b = 220.25\ \text{V}$$

$$T_a = \frac{E_b I_a}{N} \times 9.55$$

$$= \frac{220.25 \times 39}{1100} \times 9.55$$

$$\therefore \quad T_a = 74.66\ \text{Nm}$$

Common Data for Examples 19 and 20:

The counter emf of a shunt motor is 227 V, the field resistance is 160 Ω and the field current is 1.5 A.

Example 19: If the line current is 39.5 A, the armature resistance is _____

(A) 0.342 Ω (B) 0.243 Ω
(C) 0.423 A (D) 0.432 Ω

Solution: (A)

Line current = $I_L = 39.5$ A

$$I_a = I_L - I_{sh} = 39.5 - 1.5 = 38\ \text{A}$$

$$V = E_b + I_a R_a \Rightarrow R_a = \frac{V - E_b}{I_a}$$

$$\therefore \quad R_a = \frac{240 - 227}{38} = 0.342\ \Omega$$

$$[\because\ V = I_{sh} R_{sh} = 1.5 \times 160 = 240\ \text{V}]$$

Example 20: The starting current of the motor is _____.
(A) 250 A (B) 107.57 A
(C) 255.5 A (D) 701.75 A

Solution: (D)
When the motor is starting, $E_b = 0$

$$I_a = \frac{V}{R_a} = \frac{240}{0.342} = 701.75 \text{ A}$$

Common Data for Examples 21 and 22:
A 450 V series motor runs at 500 rpm taking a current of 40 A. Total resistance of the armature and field circuits is 0.8 Ω, assuming flux is proportional to field current and if the load is reduced so that motor draws a current of 30 A.

Example 21: The percentage change in torque is
(A) 38.75% (B) 41.75%
(C) 39.75% (D) 43.75%

Solution: (D)

$$\phi \propto I_a$$

$$\Rightarrow \quad T \propto \phi I_a$$

$$T \propto I_a^2 \quad [\because I_a = I_{sc} \text{ in series motor}]$$

$$\therefore \quad T_1 \propto 40^2$$

$$T^2 \propto 30^2$$

$$\frac{T_2}{T_1} = \frac{9}{16}$$

$$\% \text{ change in torque} = \frac{T_1 - T_2}{T_1} \times 100$$

$$= \frac{7}{16} \times 100 = 43.75\%$$

Example 22: Speed is _____ rpm
(A) 627.94 (B) 697.4
(C) 679.42 (D) 649.72

Solution: (C)

$$E_{b1} = 450 - (40 \times 0.8) = 450 - 32 = 418 \text{ V}$$

$$E_{b2} = 450 - (30 \times 0.8)$$

$$= 450 - 24 = 426 \text{ V}$$

$$E_b \propto \phi N$$

$$\propto I_a N$$

$$\Rightarrow \quad \frac{E_{b2}}{E_{b1}} = \frac{I_{a2} N_2}{I_{a1} N_1}$$

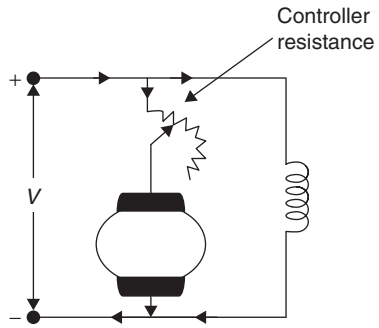
$$\therefore \quad N_2 = N_1 \frac{E_{b2}}{E_{b1}} \times \frac{I_{a1}}{I_{a2}}$$

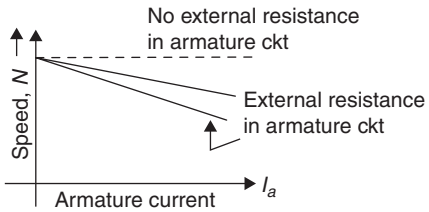
$$= 500 \times \frac{426}{418} \times \frac{40}{30} = 679.42 \text{ rpm}$$

Speed Control of Shunt Motors

Armature or Rheostatic Control Method

This method is used when speeds below the no-load speed are required. As the supply voltage is constant, the voltage across the armature is varied by inserting and additional resistance in series with the armature circuit. Different speeds can be obtained by varying the additional series resistance.





Let I_{a1} = Armature current with no external resistance

I_{a2} = Armature current with external resistance in the armature ckt

N_1, N_2 = Corresponding speeds

V = Supply voltage

Then $N_1 \propto E_{b1}$

Or $\propto V - I_{a1} R_a$

Suppose the external resistance in series with the armature ckt be r, and $r + r_a = R_t$.

$$N_2 \propto E_{b2}$$

$$\propto V - I_a R_t$$

$$\therefore \quad \frac{N_2}{N_1} = \frac{E_{b2}}{E_{b1}}$$

If N_0 be the no-load speed (and I_a r_a drop at no-load neglected)

Then $$\frac{N}{N_0} = \frac{V - I_a R_2}{V} = \left(1 - \frac{I_a R_t}{V}\right)$$

It is seen that for a given resistance R_2, speed is a linear function of armature current

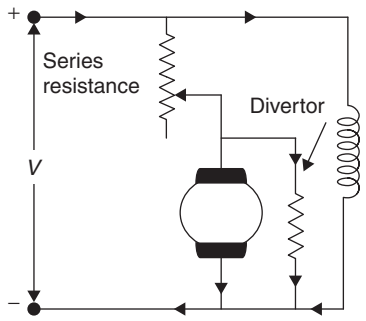
At $N = 0$

$I_a = \frac{V}{R_t}$ is the maximum current and is called stalling current

The main disadvantage of this method lies in the fact that a considerable amount of power is lost in the external resistance.

This method is expensive and unsuitable for rapidly changing loads because for a given value of R_t, speed will change with load.

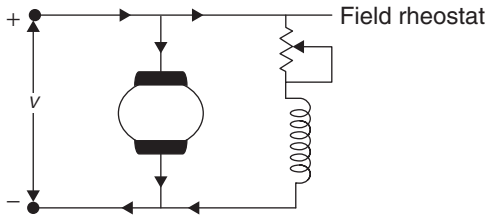
- A more stable operation can be obtained by using a divertor across the armature in addition to armature control resistance as shown in below figure.



Now, the changes in armature current (due to changes in the load torque) will not be so effective in changing the p.d., across the armature. (hence the armature speed).

Flux (or) Field Control

As $N \propto \frac{1}{\phi}$, by decreasing the flux, the speed can be increased and vice versa. The flux of a DC shunt motor can be changed by changing I_{sh} with help of a shunt field rheostat as shown in below figure.

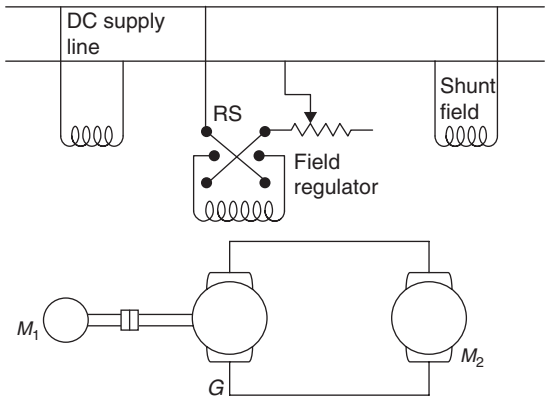


The external resistance connected in series with the shunt field winding is called a regulator.

Voltage Control Method

1. **Multiple voltage control:** In this method, the shunt field of the motor is connected permanently to a fixed exciting voltage, but the armature is supplied with different voltages by connecting it across one of several different voltages by means of suitable switchgear. The armature speed will be proportional to their different voltages. The intermediate speeds can be obtained by adjusting the shunt field regulator. This method is not much used.

2. **Ward Leonard system:**



This system is used where a usually wide (up to 10:1) and very sensitive speed control is required as for coillery winders, electric excavators, elevators and the main drives in steel mills and blooming and paper mills.

With the help of Ward Leonard set speed in both directions can be varied.

M_1 is the main motor whose speed is to be controlled. This motor is separately excited. It is fed from a variable voltage generator G, which is driven by an auxiliary motor M_2 which may be another DC or AC motor. By applying a variable voltage across armature terminals of M_1, any desired speed can be obtained. This variable voltage is supplied by motor-generator set, i.e., $M_2 - G$. By controlling the speed of auxiliary motor M_2, magnitude of voltage can be controlled. The polarity of voltage is changed by changing the direction of field current of generator G with the help of reversing switch RS.

This system has advantages of smooth variation in speed and the change of the sense of rotation.

The disadvantages of this system are

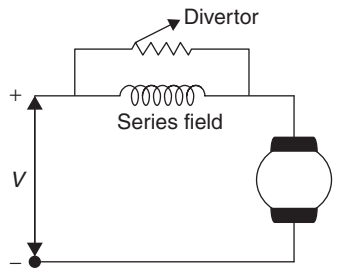
(a) It requires an auxiliary motor to drive the variable voltage generator which involves high capital outlay.
(b) Low efficiency especially at light loads.

Speed Control of Series Motors

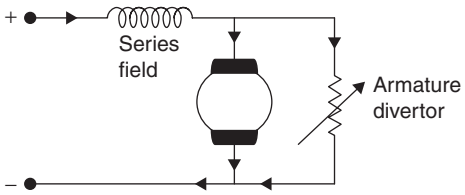
Flux Control Method

A series motor speed can be controlled by changing the flux with the help of any one of the following methods.

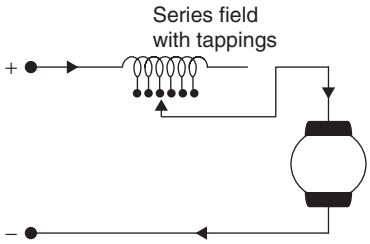
(a) **Field divertors:** The series field winding is shunted by a variable resistance known as field divertor. Any desired amount of current can be passed through the diverter by adjusting its resistance. Hence speed is increased by decreasing flux.



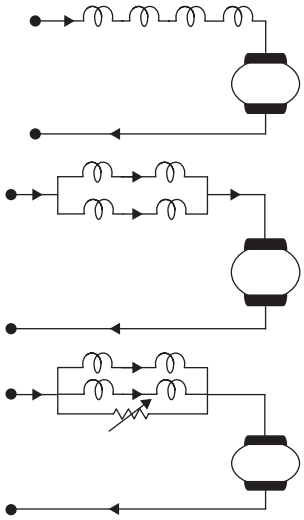
(b) **Armature Divertor:** A divertor across the armature can be used for given speeds lower than the normal speed. For a given constant-load torque, if I_a is reduced due to armature divertor, then ϕ must ($\therefore T_a \propto \phi I_a$) an increase in current drawn from the supply $\left(N \propto \frac{1}{\phi}\right)$, and hence the speed is reduced.



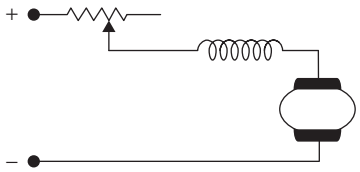
(c) **Tapped field control:** The number of series field turns in the circuit can be changed as per the requirement. With full field, the motor runs at its minimum speed which can be raised in steps by cutting out some of the series turns.



(d) **Paralleling field coils:** In this method, used for fan motors, several speeds can be obtained by regrouping the field coils as shown in below figure.



Variable Resistance in Series with Motor

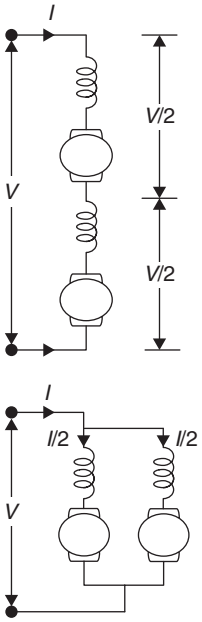


By increasing the resistance in series with the armature as shown in the figure, the voltage applied across the terminals can be decreased. With reduced voltage across the armature, the speed is reduced.

In this method, as full motor current passes through this series resistance there is a considerable loss of power in it.

Series Parallel Control

In this system of speed control, which is widely used in electric traction, two or more similar mechanically coupled series motors are employed. At low speeds, the motors are joined in series and for high speeds, are joined in parallel



When in parallel,

$$\text{Speed} \propto \frac{E_b}{\phi} \propto \frac{V}{I/2} \propto \frac{2V}{I}$$

As torque $\propto \phi I \propto I^2$

$$\therefore \quad T \propto \left(\frac{I}{2}\right)^2 \propto \frac{I^2}{4}.$$

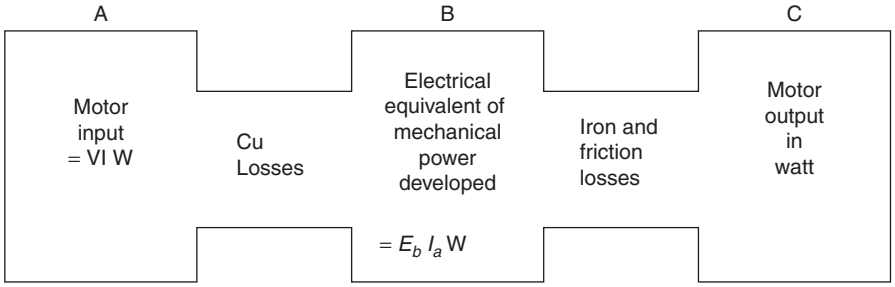
When in series

$$\text{Speed} \propto \frac{E_b}{\phi} \propto \frac{(V/2)}{I} \propto \frac{V}{2I}$$

$$T \propto \phi I \propto I^2$$

This speed is one-fourth of the speed of the motors when in parallel and the torque is four times that produced by two motors when in parallel.

Power Stages in a DC Motor



When the motor is at rest there is, as yet, obviously no back emf developed in the armature. If full supply voltage is applied across the stationary armature, it will draw a very large current because armature resistance is relatively small. This excessive current will blow out the fuses and prior to that it will damage the commutator, and brushes etc

- To limit the starting current to a safe predetermined value, every DC motor is to be provided with a starter or starting resistance. The starting resistance is gradually cut out as the motor gains speed and develops the back emf which then regulates its speed.
- There are two types of starters for DC shunt motors. They are as follows:
  (i) Three-point starter
  (ii) Four-point starter.
- Along with the limiting resistance protective devices are also mounted on the three-point starter of DC shunt motor, they are N.V.R. (no volt release) and the O.L.R. (over load relay)

$A - B$ = Copper losses and $B - C$ = Iron and friction losses.

$$\text{Overall or commercial efficiency } \eta_c = \frac{C}{A}$$

$$\text{Electrical efficiency } \eta_e = \frac{B}{A}$$

$$\text{Mechanical efficiency } \eta_m = \frac{C}{B}$$

Example 23: A 440 V shunt motor has an armature resistance of 0.8 Ω. The back emf when giving an output of 7.46 kW at 80% efficiency is _____ V.

(A) 440 (B) 400

(C) 220 (D) 425

Solution: (C)

$$\text{Input power} = \frac{7.46\times1000}{0.8}$$

$$= 9325 \text{ W}$$

$$\text{Input current} = \frac{9325}{440} = 21.193 \text{ A}$$

$$I_f = \frac{440}{220} = 2{,}2 \text{ A}$$

$$I_a = 21.193 - 2.2 = 18.99 \text{ A}$$

$$E_b = V - I_a R_a$$
$$= 440 - 18.99 \times 0.8$$
$$= 424.8 \text{ V}$$

Starters for DC Motors

The current drawn by a motor armature is given by the relation

$$I_a = \frac{(V - E_b)}{R_a}$$

- The main drawback of the three-point starter is maloperation of the N.V.R. which makes it unsuitable for use with variable speed motors, if the speed is controlled by the field flux control method.

Electric Braking

A motor comes to rest by applying Brake to the motor. There are generally two breaking phenomenon (i) electro mechanical braking (or) friction braking (ii) electrical braking. In these electro mechanical Breaking drawback is sudden application of braking force.

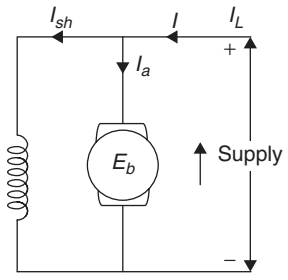
Electrical braking are generally three types.

1. Dynamic (or) rheostatic braking
2. Plugging (or) counter current braking
3. Regenerative braking

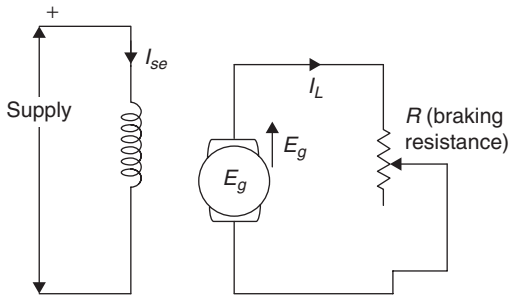
Dynamic (or) Rheostatic Braking

In this method the armature of DC motor is disconnected from the supply and is connected across a Braking (variable) resistance. The different DC motor connection as shown below.

DC Shunt Motor



(i) Under normal condition

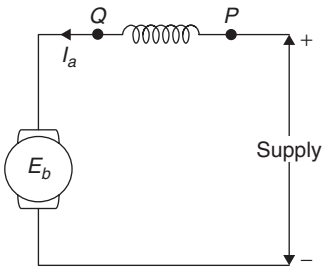


(ii) Under braking condition

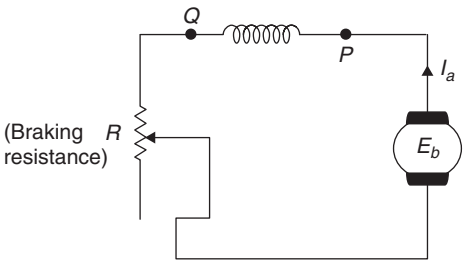
$$\text{Braking torque } T_B = \frac{1}{2\pi}\phi Z I_a \left(\frac{P}{A}\right) \text{Nm}$$

DC Series Motor

In this motor, the field connection also reversed during braking to make the current flowing through the field winding in the same direction. The connection of the motor under normal condition and braking condition is as shown below.



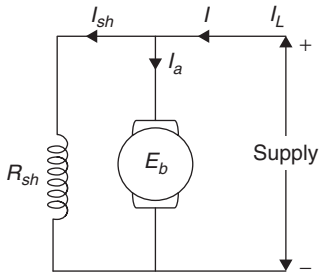
(i) Under normal condition



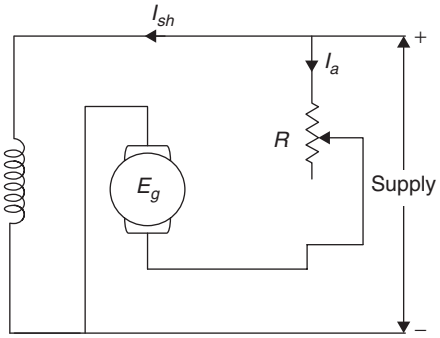
(ii) Under braking condition

Plugging (or) Counter Current Braking

In these method, the armature terminals is reversed without changing the supply terminals and a variable resistance is connected in series with the armature terminals to limit the armature current. The circuit connection of DC motors is as shown below.

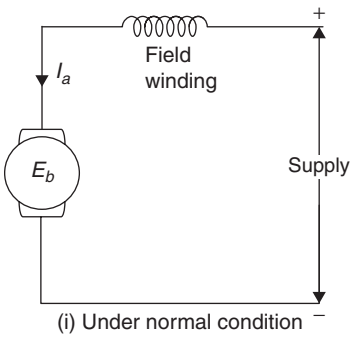


(i) Under normal condition

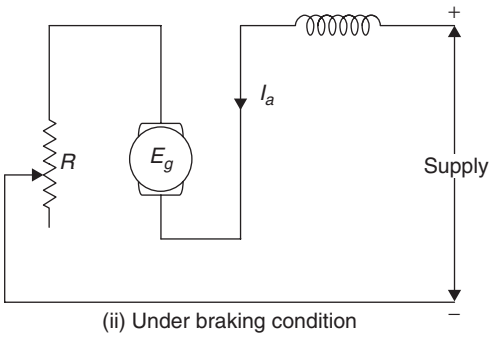


(ii) Under braking condition

Figure 3 DC Shunt motor



(i) Under normal condition



(ii) Under braking condition

Figure 4 DC Series motor

Regenerative Braking

Regenerative braking takes place when E_b becomes greater than supply voltage. Regenerative braking can be easily applied to DC shunt motors without changing any circuit connections. But In DC series motor it is not possible without changing circuit connection because back emf more than the supply voltage and the current flowing through the field winding is in reverse direction. So this method is applied to DC series motor by special arrangements only.

Testing of DC Machines

DC machines are to be tested for determining the efficiency and other performance indices.

There are three types of testing DC machines. They are

1. Direct testing
2. Indirect testing
3. Regenerative or Back-to-Back method.

Brake Test

It is a direct method and consists of applying a brake to a water-cooled pulley mounted on the motor shaft.

Swinburne's Test

It is a simple indirect test in which losses are measured separately and from their knowledge, efficiency at any desired load can be predetermined. However this test is applicable to those machines which have practically flux, i.e., shunt and compound machines.

Advantages

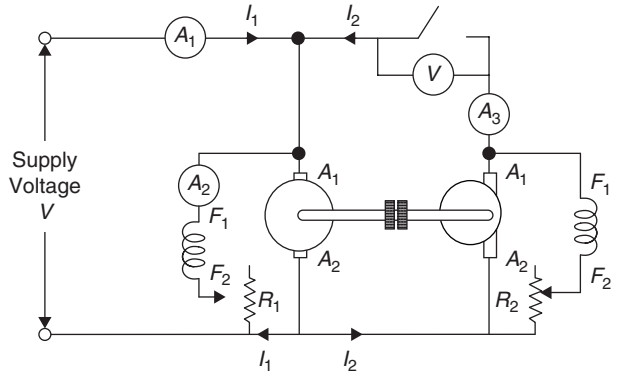
1. It is simple and economical because power required to test a large machine is small, i.e., only no-load input power.
2. The efficiency can be predetermined at any load because constant losses are known.

Disadvantages

1. No account is taken of the change in iron losses from no load to full load. At full-load due to armature, reaction flux is distorted which increases the iron losses in some cases as much as 50%.
2. As the test is on no load, it is impossible to know the effects of the temperature rise and commutation at full load.

Regenerative or Hopkinson's Test (Back-to-Back test)

It is performed on a pair of identical shunt machines, the two machines are mechanically coupled and field excitations are so adjusted that one of them runs as a motor and the other as a generator. By this method full-load test can be carried out, without wasting their outputs and hence the name regenerative test. The mechanical output of motor drives the generator and the electrical output of generator is used in supplying the greater part of input to the motor. The power supplied in this test is to meet the losses in two machines.



If η is efficiency of both motor and generator

Input to motor = $V(I_1 + I_2)$

Output of motor = $\eta \cdot V(I_1 + I_2)$

Output of generator = $\eta \cdot \eta V(I_1 + I_2) = \eta^2 V(I_1 + I_2)$ but output of generator is $I_2.V$

$$\therefore \quad \eta^2 V(I_1 + I_2) = I_2 \cdot V$$

$$\eta = \sqrt{\frac{I_2}{I_1 + I_2}}$$

Merits of Hopkinson's Test

1. Power required for the test is small as compared to the full-load powers of the two machines.
2. As machines are being tested under full load conditions, the temperature rise and the commutation qualities of the machines can be observed.
3. Because of full-load conditions, any change in iron loss due to flux distortion at full load, is being taken into account.

*The only disadvantage is with regard to the availability of two identical machines.

Retardation or Running Down Test

This method is applicable to shunt motors and generators and is used for finding stray losses. Then knowing the armature and shunt Cu losses at a given load current, efficiency can be calculated.

The machine under test is speeded up slightly beyond its normal speed and then supply is cut off from the armature while keeping the field excited.

Consequently the armature slows down and its kinetic energy is used to meet the rotational losses if friction, windage and iron losses.

Field Test for Series Motor

This test is applicable to two similar series motors. Series motors mainly used for traction work are easily available in pairs. The two machines are coupled mechanically. One machine runs normally as a motor and drives generator whose output is wasted in a variable load R.

Iron and frictional losses of two machines are made equal

1. By joining the series field winding of the generator in the motor armature circuit so that both machines are equally excited and
2. By running them at equal speed.

Field test is not a regenerative method although the two machines are mechanically coupled, because the generator output is wasted instead of being fed back into the motor as in Hopkinson's test.

EXERCISES

Practice Problems I

Directions for questions 1 to 29: Select the correct alternative from the given choices.

1. DC motor gives maximum power when
(A) $V = 2E_b$ (B) $V = E_b/2$
(C) $V = E_b/\sqrt{2}$ (D) $V = E_b$

2. The full-load speed regulation is 8% when the speed at full load is 800 rpm. The no-load speed is _____ rpm.
(A) 832 (B) 864
(C) 736 (D) 768

3. A 4-pole generator with 24 conductors has a two-layer lap winding. The pole pitch is _____.
(A) 24 (B) 12
(C) 6 (D) 4

4. A DC motor develops a torque of 120 Nm at 20 rps. At 30 rps, it will develop a torque of _____ Nm.
(A) 160 (B) 120
(C) 80 (D) 40

5. A DC motor connected to a 230 V supply has an armature resistance of 0.15 Ω. The value of armature current when the back emf E_b = 200 V is _____ A.
(A) 50 (B) 100
(C) 400 (D) 200

6. A 230 V shunt motor develops a torque of 48 Nm at an armature current of 12 A. The torque produced when the armature current is 15 A, is
(A) 48 Nm (B) 36 Nm
(C) 60 Nm (D) 72 Nm

7. An 8-pole, lap-wound armature running at 350 rpm is required to generate 260 V. The useful flux/pole is 0.05 Wb. If the armature has 120 slots, the no. of conductors/slot is _____
(A) 6 (B) 8
(C) 10 (D) 4

8. The value of torque produced by armature of 4-pole motor having 774 conductors, two paths in parallel, 24 mWb flux per pole, when the total armature current is 50 A, is _____.
(A) 350 Nm (B) 373 Nm
(C) 295.35 Nm (D) 420 Nm

9. A 440 V shunt motor has an armature resistance of 0.8 Ω. The back emf when giving an output of 7.46 kW at 80% efficiency is _____ V.
(A) 440 (B) 400
(C) 220 (D) 425

Common Data for Question 10:
A 230 V DC shunt motor takes a current of 40 A and runs at 1100 rpm. If armature and shunt field resistances are 0.25 Ω and 230 Ω respectively.

10. The torque developed by the motor is _____.
(A) 78.9 Nm (B) 74.66 Nm
(C) 98.7 Nm (D) 87.9 Nm

Common Data for Questions 11 and 12:
The counter emf of a shunt motor is 227 V, the field resistance is 160 Ω and the field current is 1.5 A.

11. If the line current is 39.5 A, the armature resistance is _____
(A) 0.342 Ω (B) 0.243 Ω
(C) 0.423 A (D) 0.432 Ω

12. The starting current of the motor is _____.
(A) 250 A (B) 107.57 A
(C) 255.5 A (D) 701.75 A

Common Data for Question 13:
A 250 V DC shunt motor takes a total current of 20 A. Resistance of shunt field winding is 200 Ω and that of armature is 0.3 Ω.

13. A 4-pole lap-wound generator is run at 900 rpm. The reactance voltage if self inductance is 153×10^{-6} H, armature current is 108 A, no of segments = 55, width of brush = 1.74 segments is _____
(A) 3.195 V (B) 3.915 V
(C) 5.391 V (D) 5.915 V

Common Data for Questions 14 and 15:
A 450 V series motor runs at 500 rpm taking a current of 40 A. Total resistance of the armature and field circuits is 0.8 Ω, assuming flux is proportional to field current and that the load is reduced so that motor draws a current of 30 A.

14. The percentage change in torque is
(A) 38.75% (B) 41.75%
(C) 39.75% (D) 43.75%

15. Speed is _____ rpm
(A) 627.94
(B) 697.4
(C) 679.42
(D) 649.72

16. A cumulative compounded long shunt motor is driving a load at rated torque and rated speed. If the series field is shunted by a resistance of the series field, keeping the torque constant then
(A) the armature current increases
(B) the armature current decreases
(C) the series field current remains the same
(D) the motor fails to run and comes to a standstill

17. A differentially compounded DC motor with interpoles and with brushes on the neutral axis is to be driven as a generator in the direction with the same polarity of the terminal voltage. It will then
(A) be a cumulatively compounded generator but interpole coil connections are to be reversed
(B) be a differentially compounded generator but the interpole coil connections are to be reversed
(C) be a cumulatively compounded generator without reversing the interpole coil connections
(D) be a differentially compounded generator without reversing the interpole coil connections

18. In case of an armature controlled separately excited DC motor drive with closed loop speed control, an inner current loop is useful because it
(A) limits the peak current of the motor to the permissible value.
(B) helps in improving energy efficiency of the drive.
(C) limits the speed of the motor to a safe value.
(D) reduces the steady state speed error.

19. The compensating winding in a DC machine
(A) is located on interpoles for improving commutation
(B) is a separate winding in armature slots for compensation of the armature reaction
(C) is located on pole shoes for avoiding the flashover at the commutator surface
(D) is located on pole shoes to avoid the sparking at the brushes

20. The DC motor which can provide zero speed regulation at full load without any controller is
(A) shunt
(B) series
(C) differential compound
(D) cumulative compound

21. In relation to DC machines, match the following and choose the correct combination:

List-I (performance variables)	List-II (proportional to)
P. Armature emf (E)	1. Flux (ϕ), speed (ω), armature current (I_a)
Q. Developed power (P)	2. I_a only
R. Developed torque (T)	3. ϕ and I_a only
	4. ϕ and ω only
	5. I_a and ω only

(A) P – 4, Q – 3, R – 1
(B) P – 2, Q – 3, R – 4
(C) P – 3, Q – 4, R – 1
(D) P – 2, Q – 5, R – 4

22. A DC series motor driving an electric train faces a constant power load. It is running at rated speed and rated voltage. If the speed has to be brought down to 0.5 p.u., the supply voltage has to be approximately brought down to
(A) 0.25 p.u. (B) 0.5 p.u.
(C) 0.707 p.u. (D) 0.8 p.u.

23. An electric motor whose torque-speed characteristic in the form of a rectangular hyperbola will have
(A) constant speed
(B) constant torque
(C) constant output power
(D) none of the above

24. The armature resistance of permanent magnet DC motor is 0.5 Ω at no load the motor draws 1.2 A from a supply voltage of 25 V and runs at 1500 rpm. The efficiency of the motor while it is operating on load at 1500 rpm drawing a current of 3 A from the same source will be
(A) 48.12%. (B) 54.96%.
(C) 62.28%. (D) 74.56%.

25. A 220 V DC series motor takes 30 A when giving its rated output at 1500 rpm. Its resistance is 0.25 Ω. The value of resistance which must be added to obtain rated torque at 1000 rpm is
(A) 1.26 Ω. (B) 2.36 Ω.
(C) 3.26 Ω. (D) 4.26 Ω.

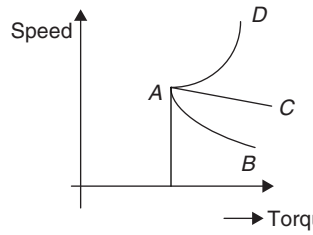
26. A 4-pole lap-wound DC generator has a developed power of P W and voltage of E volt. Two adjacent brushes of the machine are removed as they are worn out. If the machine operates with the remaining brushes, the developed voltage and power that can be obtained from the machine are

(A) $\frac{E}{4}, \frac{P}{4}$ (B) $\frac{E}{2}, \frac{P}{2}$

(C) $E, \frac{P}{4}$ (D) $E\,\frac{P}{2}$

27. An electric motor with 'constant output power' will have a torque speed characteristic in the form of a
(A) straight line parallel to speed axis.
(B) straight line through the origin.
(C) straight line parallel to torque axis.
(D) rectangular hyperbola.

28. A DC series motor fed from rated supply voltage is overloaded and its magnetic circuit is saturated. The torque-speed characteristic of this motor will be approximately represented by which curve of figure.



(A) curve A (B) curve B
(C) curve C (D) curve D

29. A permanent magnet DC commutator motor has a no-load speed of 6000 rpm when connected to 120 V DC supply. The armature resistance is 2.5 Ω and other loses may be neglected. Supply voltage of 60 V developing a torque of 0.5 Nm, is
(A) 2326 rpm (B) 2673 rpm
(C) 2836 rpm (D) 5346 rpm

Practice Problems 2

Directions for questions 1 to 50: Select the correct alternative from the given choices.

1. The core losses of a certain magnetic material, operated at constant flux density are 2200 and 3500 W at frequencies 50 Hz and 75 Hz respectively. Determine the hysteresis and eddy current loss at a frequency of 100 Hz
(A) 1067 W, 3867 W
(B) 3867 W, 1067 W
(C) 2167 W, 1867 W
(D) 1867 W, 2167 W

2. A 4-pole lap-wound DC armature with 600 conductors draws 60 A from the mains. Its armature reaction per pole is
(A) 775 AT peak sinusoidal
(B) 625 AT peak triangular
(C) 500 AT peak triangular
(D) 500 AT peak sinusoidal

3. In a DC motor torque developed in (Nm) is

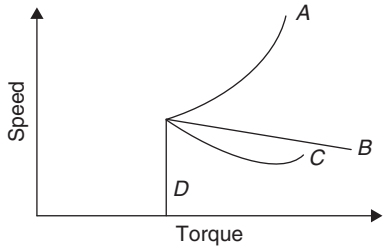
(A) $\frac{\omega_n}{E_b I_a}$ in the direction of ω_n (speed in rad/s)

(B) $\frac{\omega_n}{E_b I_a}$ opposite to the direction of ω_n

(C) $\frac{E_b I_a}{\omega_n}$ in the direction of ω_n

(D) $\frac{E_b I_a}{\omega_n}$ opposite to the direction of ω_n

4. A DC series motor is overloaded and its magnetic circuit is saturated. The torque speed characteristic of this motor will be



(A) A (B) B
(C) C (D) D

5. A 6-pole DC generator with 36 coils has a two layer lap winding. What can be the pole pitch?
(A) 6 (B) 12
(C) 24 (D) 48

6. A separately excited DC generator has ratings of 240 V, 2500 rpm and 10 A. R_a = 10 Ω. Armature and field winding is excited with rated DC voltage. If armature draws 5 A, torque developed is
(A) 1.63 Nm (B) 3.63 Nm
(C) 2.225 Nm (D) 0.25 Nm

7. A 240 V DC series motor, running at 1500 rpm takes 20 A when giving its rated output. Total series resistance = 0.5 Ω. To obtain rated torque at 1000 rpm, value of resistance that must be added is
(A) 3.7 Ω (B) 5.7 Ω
(C) 2.2 Ω (D) 3.2 Ω

8. A permanent magnet DC motor when connected to a 100 V DC supply runs at a no load speed of 4000 rpm. R_a = 2 Ω and other losses may be neglected. When supplied with a voltage of 50 V and developing a torque of 0.5 Nm speed of motor is
(A) 2000 rpm (B) 1832 rpm
(C) 2836 rpm (D) 4346 rpm

9. A cumulatively compounded DC motor runs at 1500 rpm at no load. At full load the combined drop in the armature and field is 8% and the flux increases by 12%. Neglecting magnetic saturation, full-load speed is equal to
(A) 1632 rpm (B) 1432 rpm
(C) 1232 rpm (D) 1032 rpm

10. The prime mover of an over compounded DC generator, supplying power to infinite bus is cut off. Which among the following is true?
(A) Machine stops running
(B) Machine runs as cumulatively compounded motor in reverse direction
(C) Machine runs in reverse direction as a differentially compounded motor
(D) Machine runs in same direction as a differentially compounded motor

11. The armature resistance of a 4-pole lap connected DC generator with 480 conductors is 0.08 Ω. The conductors are reconnected to form a wave winding, other things remaining the same. The new value of armature resistance will be
(A) 0.02 Ω (B) 0.08 Ω
(C) 0.64 Ω (D) 0.32 Ω

12. A DC series motor with 2-poles runs at 500 rpm with its two field coils connected in series. Now if the field coils are connected in parallel. The new speed will be (Assume magnetic circuit is unsaturated and torque is constant)
(A) 250 rpm (B) 500 rpm
(C) $250\sqrt{2}$ rpm (D) 1000 rpm

13. A DC generator on load has its brushes on geometric neutral axis. The magnetic neutral axis gets shifted in the direction of rotation. If a 90° lead is given to the brushes,
(A) The magnetic neutral axis coincides with geometric neutral axis
(B) Magnetic neutral axis is shifted forward by 90°
(C) Magnetic neutral axis is shifted forward by more than 90°
(D) Magnetic neutral axis is shifted forward by less than 90°

14. A 230 V DC series motor has $R_a = 0.4\ \Omega$ and $R_f = 0.2\ \Omega$. It draws a line current of 20 A and runs at a speed of 1500 rpm. Assuming that flux at 10 A line current is 65% of flux at 20 A, what is the speed of the motor at a line current of 10 A and 230 V?
(A) 1913 rpm (B) 2371 rpm
(C) 2331 rpm (D) 2531 rpm

15. The output of a 220 V DC shunt motor with an input of 15 kW is 13 kW. $R_f = 100\ \Omega$ and $R_a = 0.08\ \Omega$. The efficiency is maximum when armature copper losses are
(A) 1670 W (B) 2000 W
(C) 3000 W (D) 2670 W

16. A full pitch coil in a 4-pole machine has a mechanical angle span of
(A) 30° (B) 45° (C) 90° (D) 180°

17. The armature circuit resistance and field circuit resistance of a 200 V DC shunt motor are 0.2 Ω, and 200 Ω respectively. It takes 4 A at no load. The efficiency when motor takes an input current of 20 A is
(A) 51.73% (B) 81.17%
(C) 78.2% (D) 73.62%

Common Data Questions 18 and 19:
A 18 kW, 200 V DC shunt motor has an armature resistance of 0.3 Ω and field resistance of 100 Ω. Speed at rated load is 1500 rpm and full-load efficiency is 75%

18. To limit starting current to 300% of rated current, starting resistance is
(A) 0.12 Ω (B) 0.26 Ω
(C) 0.46 Ω (D) 1 Ω

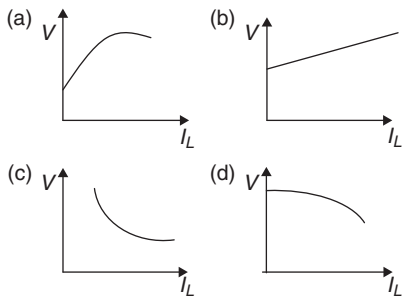
19. If the starting resistance in Q18 remains as it is on full load, the speed at full load will then be
(A) 1134 rpm (B) 1434 rpm
(C) 1500 rpm (D) 1534 rpm

20. A 4-pole, 230 V, 100 kW DC shunt generator has 600 lap-wound conductors. Two-degree brush lead is given from geometric neutral. The cross and demagnetizing turns per pole will respectively be
(A) 7790 A and 362 A
(B) 362 A and 7790 A
(C) 6680 A and 532 A
(D) 532 A and 6650 A

21. The armature of a 6-pole lap-wound DC machine has 600 conductors. Ratio of pole arc to pole pitch is 0.7. The compensating winding carries full armature current. The no. of conductors on each pole of this compensating winding will be
(A) 6 (B) 12 (C) 24 (D) 30

22. At maximum power output, the efficiency of machine is
(A) 0% (B) 75% (C) 50% (D) 100%

23. The terminal voltage vs. load current characteristics of a DC shunt generator is depicted by which among the following curves



24. In a DC generator, polarity of interpole is
(A) always south pole
(B) always north pole
(C) same as the polarity of main pole that follows in the direction of rotation
(D) same as the polarity of main pole that precedes in the direction of rotation

25. The external characteristics of a DC generator is
(A) Magnetisation characteristics + Armature reaction
(B) Magnetisation characteristics – Armature reaction
(C) Magnetisation characteristics – Armature reaction – ohmic drop
(D) Magnetisation characteristics + Armature reaction – ohmic drop

26. Lap-wound DC generators use equalizer rings to
(A) avoid unequal distribution of current at brushes giving sparkless commutation
(B) to provide mechanical balance
(C) to counter armature reaction by producing equalizing voltages in widening
(D) to reduce the load as the bearings

27. The torque developed by a DC series motor at 8 A is 25 Nm. If the current is increased to 16 A, the torque developed is
(A) 50 Nm (B) 25 Nm
(C) 100 Nm (D) 200 Nm

28. Armature reaction in a DC machine is
(A) in same direction as main poles
(B) to make an angle with main pole axis depending on load current
(C) in direct opposition to main poles
(D) to make an angle of 90° with main pole axis

29. Open slots are used in DC machine armature as
(A) it reduces coil reactance and aids in commutation
(B) it increases induced emf per coil
(C) it is easy to place the winding in the slots
(D) it reduces armature voltage drop

Common Data Questions 30 and 31:
A separately excited DC motor runs at 1200 rpm at no load and 250 V applied to armature. The field is given rated excitation. When it delivers a load of 8 Nm, speed is 1100 rpm. Rotational loses and armature reaction can be neglected.

30. Armature resistance of motor is
(A) 3 Ω (B) 5.2 Ω
(C) 4.4 Ω (D) 7.7 Ω

31. For motor to deliver a torque of 4 Nm at 1100 rpm armature voltage to be applied is
(A) 200 V (B) 239.6 V
(C) 250 V (D) 225.2 V

32. A motor has zero speed regulation at full load without the use of any controller. The motor is
(A) DC series motor
(B) DC shunt motor
(C) DC cumulatively compounded motor
(D) DC differentially compounded motor

33. A DC machine supplies 25 A at 250 V as a generator $R_a = 0.2\ \Omega$. The machine operated as a motor with 10% increased flux but terminal voltage and current remaining same, then ratio of motor speed to generator speed is
(A) 0.87 (B) 0.95
(C) 0.96 (D) 1.06

34. With regard to a DC machine, the function of compensating winding and interpole is
(A) neutralize armature reaction and produce residual flux, respectively
(B) neutralize armature reaction and improve commutation, respectively
(C) improve commutation and produce residual flux, respectively
(D) improve commutation and neutralize armature reaction, respectively

35. In relation to DC machines, match the following and choose correct combination

P. Developed Power (P)	1. Flux (ϕ), speed (ω) and armature current (I_a)
Q. Developed Torque (T)	2. I_a and ω only
R. Armature EMF (E)	3. ϕ and ω only
	4. ϕ and I_a only
	5. I_a only

(A) P – 1, Q – 4, R – 2
(B) P – 2, Q – 5, R – 3
(C) P – 1, Q – 4, R – 3
(D) P – 2, Q – 5, R – 1

36. A 30 kW DC shunt motor is drawing rated current at a given speed. When driven
(1) at half-rated speed by armature voltage control
(2) 2 times rated speed by field control Output power delivered by the motor are approximately
(A) 15 kW in (1) 45 kW in (2)
(B) 15 kW in (1) 30 kW in (2)
(C) 30 kW in (1) 45 kW in (2)
(D) 30 kW in (1) 30 kW in (2)

37. A DC series motor driving an electric train faces a constant power load at rated speed and rated voltage. For the speed to be 0.5 p.u., the supply voltage should be
(A) 0.8 p.u. (B) 0.5 p.u.
(C) 0.7 p.u. (D) 0.6 p.u.

38. Match the following in Group I and Group II with regard to a DC motor

Group I	Group II
P. Field Control	1. Below base speed
Q. Armature Control	2. Above base speed
	3. Above base torque
	4. Below base torque

(A) P – 1, Q – 4 (B) P – 2, Q – 3
(C) P – 1, Q – 3 (D) P – 2, Q – 1

Common Data Questions 39 and 40:
A 7.5 kW, 200 V, 50 A, 1500 rpm DC series motor has following full-load losses expressed in percentage of motor input
Losses in armature resistance and brush = 2.6%
Losses in field circuit = 2.4 %
Friction and windage losses (Rotational loss) = 2.2%
Assume rotational loss to be constant.

39. When motor draws one half rated current at rated voltage Speed in rpm is
(A) 2057 rpm
(B) 2889 rpm
(C) 3079 rpm
(D) 1500 rpm

40. Shaft power output will be
(A) 2925 W
(B) 3925 W
(C) 4925 W
(D) 5925 W

41. In an electromechanical energy conversion system, the developed electromagnetic force/torque at in a direction that tends to
(A) decrease stored energy at constant flux
(B) increase stored energy at constant mmf
(C) decrease stored energy at constant mmf
(D) decrease co-energy at constant mmf

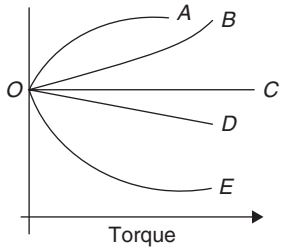
42. The hysteresis loop of a magnetic material has an area of 10 cm^2. The scales are marked as 1 cm = 4AT and 1 cm = 100 mWb Total hysteresis loss at 50 Hz is
(A) 50 W (B) 100 W
(C) 200 W (D) 400 W

43. A DC shunt motor is started with an open circuited field. Then
(A) The motor picks up fast and acquires full speed while drawing large current
(B) Motor does not pick up speed but draws a large current
(C) The motor does not pick up speed but draws only a small current
(D) The motor picks up speed fast and acquires full speed while drawing only a small current

44. In a DC series motor with linear magnetization and negligible armature resistance, the motor speed is
(A) inversely proportional to T
(B) directly proportional to $T\sqrt{T}$
(C) directly proportional to $\sqrt{T}$
(D) inversely proportional to $\sqrt{T}$

45. A DC shunt generator coupled to a prime move of constant speed and fixed field resistance is shouted across the terminals. Then short-circuit current is
(A) several times the rated current
(B) equal to full-load rated current
(C) less than maximum rated current
(D) None of the above

46. The speed torque characteristics of a cumulatively compounded DC motor is



(A) Curve OA (B) Curve OB
(C) Curve OD (D) Curve OE

47. The supply terminals of a DC shunt motor are reversed, then
(A) it will run as a DC generator
(B) it will continue to run in same direction
(C) it will reverse its direction
(D) it will stop

48. In Ward Leonard system of speed control of DC motor, the lower speed limit is restricted by
(A) core losses in the motor
(B) residual magnetism of generator
(C) mechanical losses of motor and generator together
(D) All the above

49. A DC machine used as a generator has an efficiency of 80% when output voltage and currents are 200 V and 8 A. If machine is used as a motor and takes 8 A from 200 V supply then efficiency will be
(A) 80% (B) Less than 80%
(C) More than 80% (D) None of these

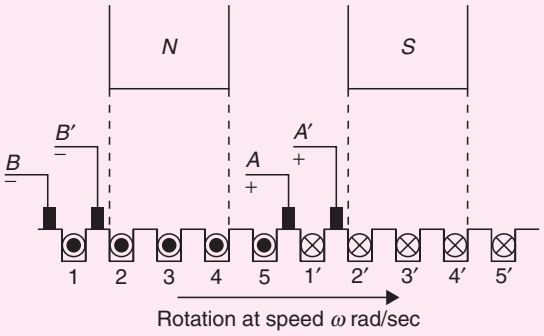
50. A DC shunt motor is running at 1500 rpm with rated voltage applied to its terminals. Neglecting the saturation of magnetic circuit, if applied voltage is reduced to $\frac{2}{3}$ of original value, motor will run at
(A) 1005 rpm (B) 750 rpm
(C) 1500 rpm (D) 2250 rpm

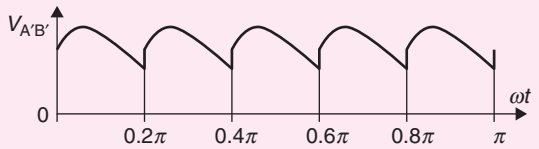
Previous Years' Questions

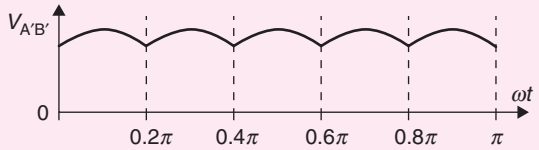
Data for Question 1:
A 240 V, DC shunt motor draws 15 A while supplying the rated load at a speed of 80 rad/s. The armature resistance is 0.5 Ω and the field winding resistance is 80 Ω

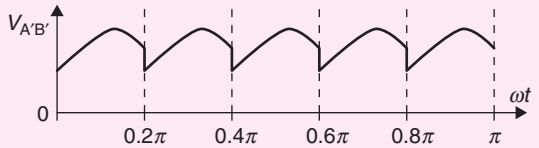
1. The external resistance to be added in the armature circuit to limit the armature current to 125% of its rated value is **[2008]**
(A) 31.1 Ω (B) 31.9 Ω
(C) 15.1 Ω (D) 15.9 Ω

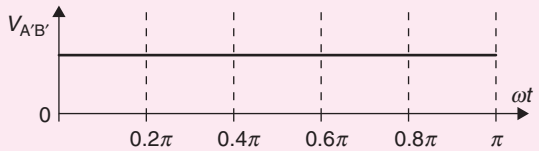
2. Figure shows the extended view of a 2-pole DC machine with 10 armature conductors. Normal brush positions are shown by A and B, placed at the interpolar axis. If the brushes are now shifted, in the direction of rotation, to A' and B' as shown, the voltage waveform $V_{A'B'}$. will resemble **[2009]**



(A)


(B)


(C)


(D)


3. The armature resistance of the motor is, **[2010]**
(A) 2 Ω (B) 3.4 Ω
(C) 4.4 Ω (D) 7.7 Ω

4. For the motor to deliver a torque of 2.5 Nm at 1400 rpm, the armature voltage to applied is **[2010]**
(A) 125.5 V (B) 193.3 V
(C) 200 V (D) 241.7 V

5. A 4-point starter is used to start and control the speed of a **[2011]**
(A) DC shunt motor with armature resistance control
(B) DC shunt motor with field weakening control
(C) DC series motor
(D) DC compound motor

6. A 220 V, DC shunt motor is operating at a speed of 1440 rpm. The armature resistance is 1.0 Ω and armature current is 10 A. If the excitation of the machine is reduced by 10%, the extra resistance to be put in the armature circuit to maintain the same speed and torque will be **[2011]**
(A) 1.79 Ω (B) 2.1 Ω
(C) 3.1 Ω (D) 18.9 Ω

7. A 220 V, 15 kW, 1000 rpm shunt motor with armature resistance of 0.25 Ω, has a rated line current of 68 A and a rated field current of 2.2 A. The change in field flux required to obtain a speed of 1600 rpm while drawing a line current of 52.8 A and a field of 1.8 A is **[2012]**
(A) 18.18 % increase (B) 18.18 % decrease
(C) 36.36 % increase (D) 36.36 % decrease

8. A 15 kW, 230 V DC shunt motor has armature circuit resistance of 0.4 Ω and field circuit resistance of 230 Ω. At no load and rated voltage, the motor runs at 1400 rpm and the line current drawn by the motor is 5 A. At full load, the motor draws a line current of 70 A. Neglect armature reaction. The full load speed of the motor in rpm is _____. **[2014]**

9. A 250 V DC shunt machine has armature circuit resistance of 0.6 Ω and field circuit resistance of 125 Ω. The machine is connected to 250 V supply mains. The motor is operated as a generator and then as a motor separately. The line current of the machine in both the cases is 50 A. The ratio of the speed as a generator to the speed as a motor is _____. **[2014]**

10. The no-load speed of a 230 V separately excited DC motor is 1400 rpm. The armature resistance drop and the brush drop are neglected. The field current is kept constant at rated value. The torque of the motor in Nm for an armature current of 8 A is _____. **[2014]**

11. A separately excited 300 V DC shunt motor under no load runs at 900 rpm drawing an armature current of 2 A. The armature resistance is 0.5 Ω and leakage inductance is 0.01 H. When loaded, the armature current is 15 A. then the speed in rpm is _____. **[2014]**

12. A separately excited DC generator has an armature resistance of 0.1 | and negligible armature inductance. At rated field current and rated rotor speed, its

open-circuited voltage is 200 V. When this generator is operated at half the rated speed, with half the rated field current, an uncharged 1000 μF capacitor is suddenly connected across the armature terminals. Assume that the speed remains unchanged during the transient. At the time (in microsecond) after the capacitor is connected will the voltage across it reach 25 V? **[2015]**

(A) 62.25 (B) 69.3
(C) 73.25 (D) 77.3

13. A separately excited DC motor runs at 1000 rpm on no load when its armature terminals are connected to a 200 V DC source and the rated voltage is applied to the field winding. The armature resistance of this motor is 1 . The no-load armature current is negligible. With the motor developing its full load torque, the armature voltage is set so that the rotor speed is 500 rpm. When the load torque is reduce to 50% of the full load value under the same armature voltage conditions, the speed rises to 520 rpm. Neglecting the rotational losses, the full load armature current (in Ampere) is _____. **[2015]**

14. A DC motor has the following specifications: 10 hp, 37.5 A, 230 V; flux/pole = 0.01 Wb, number of poles = 4, number of conductors = 666, number of parallel paths = 2. Armature resistance = 0.267 . The armature reaction is negligible and rotational losses are 600 W. The motor operates from a 230 V DC supply. If the motor runs at 1000 rpm, the output torque produced (in Nm) is _____. **[2015]**

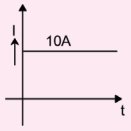
15. A shunt-connected DC motor operates at its rated terminal voltage. Its no-load speed is 200 radian/second. At its torque of 500 Nm, its speed is 180 radian/second. The motor is used to directly drive a load whose load torque T_L depends on its rotational speed ω_r (in radian/second), such that $T_L = 2.78 \times \omega_r$. Neglecting rotational losses, the steady-state speed (in radian/second) of the motor, when it drives this load, is _____. **[2015]**

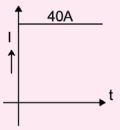
16. A 4-pole, separately excited, wave wound DC machine with negligible armature resistance is rated for 230 V and 5 kW at a speed of 1200 rpm. If the same armature coils are reconnected to form a lap winding, what is the rated voltage (in volts) and power (in kW) respectively at 1200 rpm of the reconnected machine if the field circuit is left unchanged?

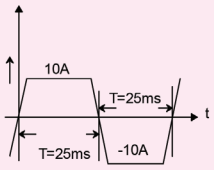
(A) 230 and 5 (B) 115 and 5
(C) 115 and 2.5 (D) 230 and 2.5

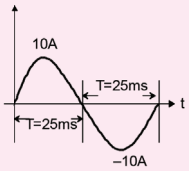
17. With an armature voltage of 100 V and rated field winding voltage, the speed of a separately excited DC motor driving a fan is 1000 rpm, and its armature current s 10 A. The armature resistance is 1 Ω. The load torque of the fan load is proportional to the square of the rotor speed. Neglecting rotational losses, the value of the armature voltage (in Volt) which will reduce the rotor speed to 500 rpm is _____. **[2015]**

18. A 4-pole, lap-connected, separately excited dc motor is drawing a steady current of 40 A while running at 600 rpm. A good approximation for the wave shape of the current in an armature conductor of the motor is given by **[2016]**

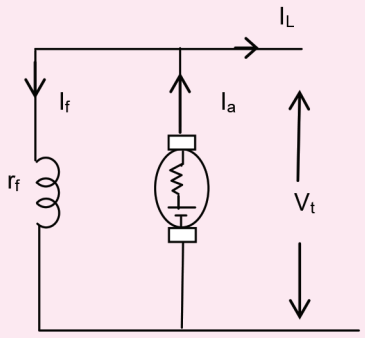
(A)


(B)


(C)


(D)


19. A DC shunt generator delivers 45 A at a terminal voltage of 220 V. The armature and the shunt field resistances are 0.01 Ω and 44 Ω respectively. The stray losses are 375 W. The percentage efficiency of the DC generator is _____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. A	**2.** B	**3.** C	**4.** C	**5.** D	**6.** C	**7.** B	**8.** C	**9.** C	**10.** B
11. A	**12.** D	**13.** B	**14.** D	**15.** C	**16.** A	**17.** C	**18.** A	**19.** C	**20.** C
21. A	**22.** C	**23.** C	**24.** C	**25.** B	**26.** C	**27.** D	**28.** C	**29.** B	

Practice Problems 2

1. B	**2.** A	**3.** C	**4.** B	**5.** A	**6.** B	**7.** D	**8.** B	**9.** C	**10.** D
11. D	**12.** D	**13.** C	**14.** B	**15.** A	**16.** C	**17.** C	**18.** B	**19.** A	**20.** A
21. B	**22.** C	**23.** D	**24.** C	**25.** D	**26.** A	**27.** C	**28.** D	**29.** A	**30.** B
31. B	**32.** D	**33.** A	**34.** B	**35.** C	**36.** B	**37.** C	**38.** D	**39.** C	**40.** A
41. A	**42.** C	**43.** B	**44.** D	**45.** C	**46.** D	**47.** B	**48.** B	**49.** A	**50.** C

Previous Years' Questions

1. A	**2.** A	**3.** B	**4.** B	**5.** B	**6.** A	**7.** D	**8.** 1240.63	**9.** 1.27	**10.** 12.5
11. 880	**12.** B	**13.** 8	**14.** 57.78	**15.** 177 to 183		**16.** B	**17.** 47.5 V	**18.** C	**19.** 86.84

Chapter 4

Induction Motors

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Principle of operation
- Rotor current and power factor
- Power stages in an induction motor
- Synchronous watt
- Rotor torque and breakdown torque
- Equivalent circuit of the rotor
- Starting of induction motors
- Double squirrel cage motor
- Speed control of induction motor
- Revolving field theory of single phase induction motor
- Capacitor split phase motors
- Shaded pole single-phase motor
- Repulsion start induction motor

Introduction

Three-phase induction motors (IMs) are the widely used AC motors due to their low cost, simple and rugged construction, high reliability, high efficiency. IMs have good speed regulation, less maintenance, simple starting arrangement, reasonable overload capacity. Conversion of electrical power into mechanical power takes place in the rotating part of an electric motor. In DC motors, the electric power is conducted directly to the armature (i.e., rotating part) through brushes and commutator. Hence in this sense, a DC motor can be called a conduction motor. However, in AC motors, the rotor does not receive electric power by conduction but by induction. This is why such motors are known as IMs. An IM can be treated as a rotating transformer. Unlike a transformer due to the presence of air gap, IM has more magnetizing currents, almost lagging 90° the applied voltage and therefore p.f. is low: The no-load current of IM is about 30% to 50% of full-load current, whereas in transformer no-load current is only 2% to 6% of full-load current.

Constructional Details

The IMs essentially consists of two parts:

(a) A stationary part called stator
(b) The revolving part, called rotor

Stator

The stator of an IM is made up of a number of stampings, which are slotted on the inner periphery to receive the windings. The stator carries a 3-phase winding which may be star or delta connected and is fed from a 3-phase supply. It is wound for a definite number of poles, the exact number of poles being determined by the requirements of speed.

The number poles P, produced in the rotating field is $P = 2n$, where n is the number of stator slots/pole/phase.

Rotor

Rotor is made up of thin laminations of the same material as stator. These laminations are slotted on the outer periphery. There are two types of rotors.

1. Squirrel-cage rotor: Motors employing this type of rotor are known as squirrel-cage induction motors (SCIMs)
2. Phase wound (or) wound rotor: The motors employing this type of rotor are variously known as 'phase-wound' motors or 'wound' motors or 'slip-ring' motors

Slip-ring Rotor

- Has slotted armature
- Provided with 3-phase double-layer distributed winding housed in the rotor slots

- The rotor windings are connected in star
- The open ends are connected to slip rings
- Brushes resting on the slip rings are further externally connected to a 3-phase star-connected rheostat. This makes possible
  (a) to increase the starting torque and decrease the starting current and
  (b) to control the speed of the motor
- A cage motor has a higher efficiency than a slip-ring IM
- A slip-ring IM has a high starting torque and a low starting current
- When 3-phase windings displaced in space by 120°, are fed by three-phase currents displaced in time by 120°, they produce a resultant magnetic flux, which rotates in space. This type of field is known as rotating magnetic field (R.M.F.)
- The resultant flux is of constant value = $\frac{3}{2}\phi_m$, i.e., 1.5 times the maximum values of the flux due to any phase
- The resultant flux rotates around the stator at synchronous speed given by

$$N_s = \frac{120f}{P}$$

- The direction of R.M.F. depends upon the phase sequence

Principle of Operation

- When the 3-phase stator windings, are fed by a 3-phase supply, then a magnetic flux of constant magnitude but rotating at synchronous speed, is set up. The flux passes through the air-gap, sweeps past the rotor surface and so cuts the rotor conductors which, as yet, are stationary. Due to the relative speed between the rotating flux and the rotor conductors, an emf is induced in rotor conductors. Since the rotor bars or conductors form a closed circuit, a current is established by the induced emf. The rotor currents generate rotor flux which in turn interacts with the stator flux. The interaction of stator and rotor fluxes produces a torque which makes the rotor to rotate in the direction of R.M.F.
- A 3-phase IM is self-starting. An IM runs at a speed slightly less than the synchronous speed. Hence it is known as an asynchronous motor. It cannot run at synchronous speed.

Slip

The difference between the synchronous speed N_s and the actual speed N of the rotor is known as slip. It is usually expressed as a percentage of the synchronous speed N_s.

$$\text{Slip } s = \frac{N_s - N}{N_s}$$

$N_s - N$ is called slip speed.

The slip speed expresses the speed of the rotor relative to the field.

Frequency of rotor current or emf

$$\text{Rotor frequency } f_2 = sf_1$$

where s = slip and f_1 = supply frequency

When rotor is at standstill, $N = 0$, and $s = 1.0$ and $f_2 = f_1$

- Air-gap flux is rotating synchronously, relative to the stator (or stationary space) but at slip speed relative to the rotor.

Solved Examples

Example 1: A voltmeter gives 120 oscillation per minute when connected to the rotor of an IM.

The stator frequency is 50 Hz. Calculate the slip of the motor?

Solution:

$$f' = sf$$

$$\text{The rotor circuit frequency } f' = \frac{120}{60}$$

$$= 2 \text{ Hz}$$

$$s = \frac{f'}{f} = \frac{2}{50} = 0.04$$

$$\% \text{ slip} = 4\%$$

Rotor Current and Power Factor

If R_2 and X_2 are the resistance and reactance at standstill per phase of rotor and E_2 be the standstill induced emf per phase

$$I_2 = \frac{E_2}{Z_2} = \frac{E_2}{\sqrt{R_2^2 + X_2^2}} \text{ at standstill}$$

$$\text{Rotor power factor} = \frac{R_2}{Z_2} = \frac{R_2}{\sqrt{R_2^2 + X_2^2}}$$

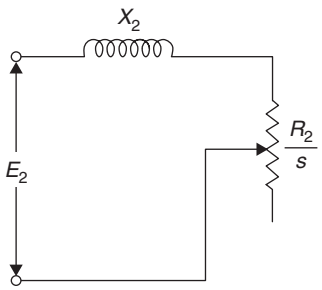
When motor running with slip s, then the rotor current/phase

$$I_2 = \frac{sE_2}{Z_2} = \frac{sE_2}{\sqrt{R_2^2 + (sX_2)^2}} \text{ or } \frac{E_2}{\sqrt{\left(\frac{r_2}{s}\right)^2 + X_2^2}}$$

Now the rotor power factor

$$\cos\phi_2 = \frac{R_2}{\sqrt{R_2^2 + (sX_2)^2}} = \frac{\frac{R_2}{s}}{\sqrt{\left(\frac{R_2}{s}\right)^2 + X_2^2}}$$

This equation gives the rotor equivalent circuit as shown below:



Relation Between Torque and Rotor Power Factor

In case of an IM, the torque is proportional to the product of flux per stator pole and the rotor current. However there is one more factor that has to be taken into account, i.e., the power factor of the rotor.

$$\therefore \quad T_2 \propto \phi I_2 \cos\phi_2 \quad \text{or} \quad T = k\phi I_2 \cos\phi_2$$

where I_2 = Rotor current at standstill
ϕ_2 = Angle between rotor emf and rotor current
k = A constant

Denoting rotor emf at standstill by E_2, we have that

$$E_2 \propto \phi$$

$$\therefore \quad T \propto E_2 I_2 \cos\phi_2 \quad \text{or} \quad T = k_1 E_2 I_2 \cos\phi_2$$

where k_1 is another constant.

Example 2: The rotor of a 4-pole, 50-Hz slip-ring IM has a resistance of 0.20 Ω per phase and runs at 1440 rpm at full load. Calculate the external resistance per phase which must be added to lower the speed to 1300 rpm, the torque being the same as before

Solution: The motor torque is given by

$$T = \frac{KsR_2}{R_2^2 + (sX_2)^2}$$

Since, X_2 is not given,

$$T = \frac{KsR_2}{R_2^2} = \frac{Ks}{R_2}$$

In the first case $T_1 = Ks_1/1\ \Omega$

In the second case, $T_2 = Ks_2/(R_2 + r)$,

where r is the external resistance per phase, added to the rotor circuit

Since $T_1 = T_2$

$$\therefore \quad Ks_1/R_2 = Ks_2/(R_2 + r)$$

Or $(R_2 + r)/R_2 = s_2/s_1$

Now, $N_S = 120 \times 5/4 = 1500$ rpm.

$N_1 = 1440$ rpm, $N_2 = 1300$ rpm.

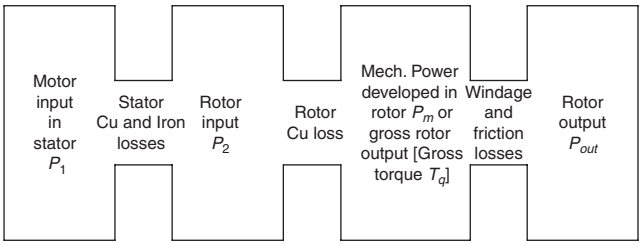
$$\therefore \quad s_1 = \frac{1500 - 1440}{1500} = 0.04$$

$$s_2 = \frac{1500 - 1300}{1500} = 0.133$$

$$\therefore \quad \frac{0.2 + r}{0.2} = \frac{0.133}{0.04}$$

$$\therefore \quad r = 0.465\ \Omega$$

Power Stages in an IM



Stator iron loss depends on the supply frequency and the flux density in the iron core. It is practically constant. The iron loss of the rotor is, however, negligible because frequency of rotor currents under normal running conditions is always small.

Torque Developed by an IM

An IM develops gross torque T_g due to gross rotor output P_m. Its value can be expressed either in terms of rotor input P_2 or rotor gross output P_m as given below:

$$T_g = \frac{P_2}{\omega_s} = \frac{P_2}{2\pi N_s}$$

or $$T_g = \frac{P_m}{\omega} = \frac{P_m}{2\pi N} \quad (N \text{ and } N_s \text{ are in rps})$$

in terms of rotor output.

The shaft torque is due to output power P_{out} which is less than P_m because of rotor friction and windage losses.

$$\therefore \quad T_{sh} = \frac{P_{out}}{\omega} = \frac{P_{out}}{2\pi N}$$

The difference between T_g and T_{sh} equals the torque lost due to friction and windage loss in the motor.

Some Important Relations

$$\frac{\text{Rotor Cu loss}}{\text{Rotor input}} = s$$

Rotor gross output = $(1 - s)$ Rotor input

$$\frac{\text{Rotor Cu loss}}{\text{Rotor gross output}} = \frac{s}{1 - s}$$

$$\text{Rotor efficiency, } \frac{P_m}{P_2} = \frac{N}{N_s}$$

If some power P_2 is delivered to a rotor, then a part sP_2 is lost in the rotor itself as copper loss and the remaining $(1-s)P_2$ appears as gross mechanical power P_m (Including friction and windage losses)

$$\therefore \qquad P_2 : P_m : I^2R :: 1 : (1-s) : s$$

IM Torque Equation

The gross torque T_g developed by an IM is given by

$$T_g = \frac{P_2}{2\pi N_s} - N_s \text{ in rps}$$

$$= \frac{60P_2}{2\pi N_s} = \frac{9.55P_2}{N_s} - N_s \text{ in rpm}$$

Now $P_2 =$ rotor Cu loss/s

$$= 3I_2^2R_2/s$$

$$I_2 = \frac{sE_2}{\sqrt{R_2^2 + (sX_2)^2}} = \frac{sKE_1}{\sqrt{R_2^2 + (sX_2)^2}}$$

where K is rotor/stator turn ratio per phase

$$\therefore \qquad P_2 = 3 \times \frac{s^2E_2^2R_2}{R_2^2 + (sX_2)^2} \times \frac{1}{s}$$

$$\therefore \qquad T_g = \frac{P_2}{2\pi N_s} = \frac{3}{2\pi N_s} \times \frac{sE_2^2R_2}{R_2^2 + (sX_2)^2}$$

Synchronous Watt (T_{SW})

Synchronous watt is that torque which, at the synchronous speed of the machine under consideration, would develop a power of 1 watt.

$$\text{Rotor input} = T_{SW} \times 2\pi N_s$$

$$\therefore \qquad T_{SW} = \frac{\text{Rotor input}, P_2}{2\pi \times N_s}$$

$$= \frac{1}{\omega_s} \cdot \frac{N_s}{N} \cdot P_g = \frac{1}{\omega_s} \cdot \frac{N_s}{N} . P_m$$

Synchronous wattage of an IM equals the power transferred across the air-gap to the rotor.

$\therefore$ Torque in synchronous watt

$$= \text{Rotor input} = \frac{\text{Rotor Cu loss}}{s}$$

$$= \frac{\text{Gross output power}, P_m}{(1-s)}$$

Starting torque, T_{st}:

$$T_{st} = \frac{3}{2\pi N_s} \cdot \frac{E_2^2R_2}{R_2^2 + X_2^2}$$

Example 3: A 3-ϕ, 50 Hz IM has a full load speed of 1440 rpm. What is slip and rotor frequency of the IM?

(A) 6%, 4 Hz (B) 10%, 46 Hz
(C) 8%, 6 Hz (D) 4%, 2 Hz

Solution: (D)

$$N_s = \frac{120 \times 50}{4} = 1500 \text{ rpm}$$

$$N = N_s(1-S) \Rightarrow S = 4\%$$

$$\Rightarrow \qquad f_r = S \times f_s \quad \Rightarrow \quad f_r = 2 \text{ Hz}$$

Example 4: An 8-pole, 50 Hz, 3-ϕ slip-ring IM has effective rotor resistance of 0.8 Ω/ph. Stalling speed is 650 rpm. The resistance required in the rotor phase to obtain the maximum torque at starting?

(A) 0.42 Ω (B) 0.52 Ω
(C) 0.48 Ω (D) 0.58 Ω

Solution: (B)

$$N_s = 750 \text{ rpm}$$

$$S_b = 13.33\%$$

$$\Rightarrow S_b = \frac{R^2}{X^2} = 0.6 \ \Omega$$

$$\frac{T_{st}}{T_{\max}} = \frac{2a}{1+a^2}, \ T_{st} = T_{\max}$$

Here $a = 1$

$$\Rightarrow a = \frac{R_2 + r}{X_2}$$

$$\Rightarrow r = .52 \ \Omega$$

Example 5: The rotor power output of 3-ϕ IM is 15 kW and the corresponding slip is 4%. The rotor ohmic loss will be

(A) 600 W (B) 625 W
(C) 650 W (D) 700 W

Solution: (B)

$$\text{Rotor loss} = P_g \times \left(\frac{S}{1-S}\right)$$

$$= 15 \times \frac{0.04}{1-0.04} = 625 \text{ W}$$

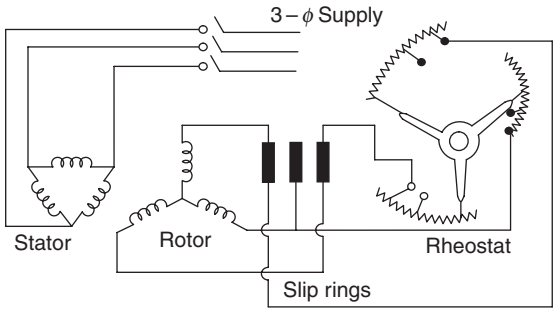
Starting Torque of a SCIM

The resistance of a SCIM is fixed and small compared to its reactance which is very large at start because at standstill, frequency of the rotor currents equals the supply frequency. Hence, the starting current I_2 of the rotor though very large in magnitude, lags by an angle nearly equal to 90°, which results in poor starting torque. It is roughly 1.5 times the full-load torque, although the starting current is 5 to 7 times the full-load current. Hence SCIMs are not useful where the motor has to start against heavy loads.

Starting Torque of a Slip-ring Motor

The starting torque of such a motor is increased by improving its power factor by adding external resistance in the rotor circuit from the star-connected rheostat, the rheostat being progressively cut out as the motor gathers speed. Addition of external resistance, however, increases the rotor impedance and so reduces the rotor current. At first, the effect of improved power factor predominates the current-decreasing effect of impedance. Hence, starting torque is increased. But after a certain point, the effect of increased impedance predominates the effect of improved power factor so the torque starts decreasing.

Starting Torque of a Slip-ring Motor Starting Arrangement



Condition for Maximum Starting Torque

Starting torque is maximum when rotor resistance equals rotor reactance.

i.e., $R_2 = X_2$

Effect of change in supply voltage on starting torque

$$T_{st} = \frac{K_1 E_2^{\,2} R_2}{R_2^{\,2} + X_2^{\,2}}$$

Now $E_2 \propto$ supply voltage V

$$\therefore \quad T_{st} = \frac{KV^2 R_2}{R_2^{\,2} + X_2^{\,2}},$$

where k is yet another constant.

$$\therefore \quad T_{st} \propto v_2$$

The torque is very sensitive to any changes in supply voltage

Torque under running conditions:

$$T = \frac{3}{2\pi N_s} \cdot \frac{sE_2^{\,2} R_2}{R_2^{\,2} + (sX_2)^2}$$

Condition for Maximum Torque (or Breakdown Torque) under Running Conditions

Breakdown torque of a motor is the maximum torque; a motor will develop under increasing load conduction without an abrupt drop in speed and power. It is also sometimes known as pullout torque.

The torque under running conditions is maximum at that value of slip 's', which makes rotor reactance per phase equal to rotor resistance per phase.

Slip corresponding to maximum torque is 's' $= \frac{R_2}{X_2}$

$$T_{\max} = \frac{3}{2\pi N_s} \cdot \frac{E_2^{\,2}}{2X_2} \text{ Nm}$$

1. The maximum torque is independent of rotor resistance.
2. The speed or slip at which maximum torque occurs is determined by the rotor resistance. Torque becomes maximum when rotor reactance equals its resistance. Hence, by varying rotor resistance (possible only with slip-ring motors) maximum torque can be made to occur at any desired slip.
3. Maximum torque varies inversely as standstill reactance. Hence, it should be kept as small as possible.
4. Maximum torque varies directly as the square of the applied voltage.
5. For obtaining maximum torque at starting (s = 1), rotor resistance must be equal to rotor reactance.

Rotor Torque and Breakdown Torque (Pull Out Torque)

The rotor torque T at any slip s can be expressed in terms of the maximum (or breakdown) torque T_b as

$$T = T_b \left[\frac{2}{\left(\frac{s_b}{s}\right) + \left(\frac{s}{s_b}\right)} \right]$$

where T_b = Breakdown or maximum torque

s_b = Breakdown or pull-out slip

Torque/speed or Torque/Slip Curves

$$T \propto \phi I_2 \cos \phi_2 \text{ or}$$

$$T = K\phi \frac{sE_2 R_2}{R_2^{\,2} + (sX_2)^2}$$

At normal speeds, close to synchronous speed, the term sX_2 is small and hence negligible compared to R_2.

$$\therefore \quad T \propto \frac{s}{R_2} \quad \text{or}$$

$T \propto s$ if R_2 is constant. Hence, for low values of slip (or close to N_s) the torque/slip (torque/speed) curve is approximately a straight line. As slip increases (i.e., motor speed falls due to increasing load on the motor), the torque also increases and becomes maximum when $s = \frac{R_2}{X_2}$.

As the slip further increases (i.e., motor speed falls) with further increase in motor load, then R_2 becomes negligible as compared to sX_2.

Therefore, for large values of slip

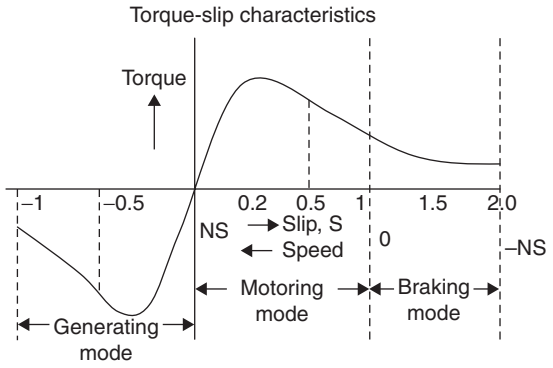
$$T \propto \frac{s}{(sX_2)^2} \propto \frac{1}{s}$$

Hence, the torque/slip curve is a rectangular hyperbola. Beyond the point of maximum torque, any further increase in motor load results in decrease of torque developed by motor. The result is that motor slows down and operation of the motor lies between the values of $s = 0$ and that corresponding to maximum torque.

The maximum torque does not depend on R_2, but the exact location of T_{max} is dependent on it. Greater the R_2, greater is the value of slip at which the maximum torque occurs.

Torque–slip (or) Torque–speed Characteristics

Torque–slip (or) Torque–speed characteristics of three-phase IM under different modes as shown below.



When speed varies from 0 to N_S and slip varies from 0 to 1 then the IM operates under motoring mode. If the slip goes to negative value and speed greater than synchronous speed. Then the Induction machine operates under generating mode. Similarly if the speed is negative and slip is greater than unity then the motor operates under Braking mode.

Effect of Change in Supply Voltage on Torque and Speed

$$T = \frac{K\phi s E_2 R_2}{R_2^2 + (sX_2)^2}$$

As $E_2 \propto \phi \propto V$, where V is the supply voltage

$$\therefore \quad T \propto sV^2$$

Torque at any speed is proportional to the square of the applied voltage. Changes in supply voltage affect both starting torque and running torque.

If supply voltage V decreases, then torque T also decreases. Hence, for maintaining the same torque, slip increases, i.e., speed falls.

Let V change to V', s to s' and T to T'

$$\text{then} \quad \frac{T}{T'} = \frac{sV^2}{s'V'^2}$$

Example 6: The power input to a 415 V, 50 Hz, 6-pole, 3-ϕ IM running at 975 rpm is 40 kW. The stator losses are 1 kW and friction and windage losses are 2 kW. The efficiency of the motor is

(A) 92.5% (B) 91%
(C) 90.06% (D) 88%

Solution: (C)

$$\Rightarrow \quad S = \frac{N_S - N}{N_S} = 0.025$$

Rotor Cu loss = $S \times$ Rotor Input
= $0.025 \times (40 - 1)$
Total loss = 3975 W
Output = 36,025 W

$$\% \text{ Efficiency} = \frac{\text{Output}}{\text{Input}} = 90.06\%$$

$$= \frac{36025}{4000} \times 100 = 90.06\%$$

Important Relations

(a) Ratio of full-load torque and maximum torque

$$\frac{T_{fl}}{T_{max}} = \frac{2as_f}{a^2 + s_f^2},$$

where $a = \frac{R_2}{X_2}$

s_f – full-load slip

In fact $a = s_m$, slip corresponding to maximum torque.

In that case the ratio $\frac{T_{fl}}{T_{max}}$ is given by

$$\frac{T_{fl}}{T_{max}} = \frac{2s_m s_f^2}{s_m + s_f^2}$$

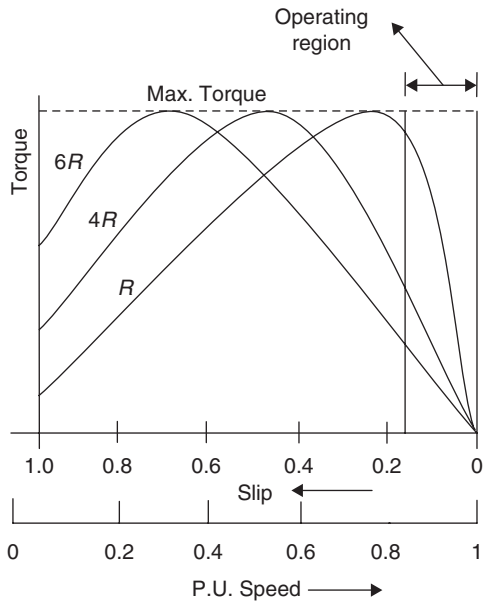
In general,

$$\frac{\text{Operating torque at any slips}}{\text{Maximum torque}} = \frac{2as}{a + s^2}$$

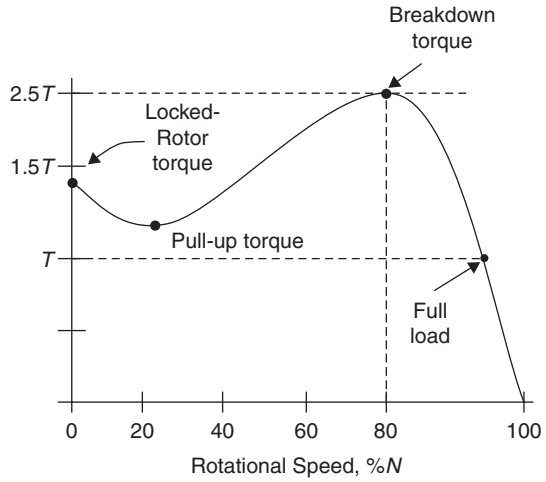
(b) Ratio of staring torque and maximum torque

$$\frac{T_{st}}{T_{max}} = \frac{2a}{1 + a^2}$$

where $a = \frac{R_2}{X_2}$



Torque/Speed Curve



Example 7: A 12-pole 3-phase, 600-V 50-Hz, star-connected, IM has rotor-resistance and standstill reactance of 0.02 and 0.4 Ω per phase, respectively. Calculate:

(a) Speed corresponding to maximum torque.

(b) Ratio of full-load torque to maximum torque, if the speed is 495 rpm.

Solution: $N_S = \dfrac{120 \times 50}{12} = 500$ rpm

For $r = 0.02$ and $x = 0.4$ Ω, the slip corresponding to maximum torque is related is

$$S_{MT} = a = r_2/x_2 = \frac{0.02}{0.4} = 0.05$$

(a) Corresponding speed

$= 500\ (1 - S_{MT}) = 500\ (1 - 0.05)$

$= 475$ rpm

(b) Full-load speed = 495 rpm

$$S_f = 0.01$$

$$\frac{\text{Full-load torque}}{\text{Maximum torque}} = \frac{2a s_f}{a^2 + s_f^{\ 2}}$$

$$= \frac{2 \times 0.05 \times 0.01}{0.05^2 + 0.01^2} = 0.38$$

Example 8: An 8-pole 50-Hz, 3-phase, slip-ring IM has effective rotor resistance of 0.08 Ω/ph. Stalling speed is 650 rpm. How much resistance must be inserted in the rotor phase to obtain the maximum torque at starting? Ignore magnetizing current and stator leakage impedance.

Solution: The stalling speed corresponding to stalling (or maximum torque) and to maximum slip under running conditions.

$$N_S = 120 \times 50/8 = 750 \text{ rpm}$$

Stalling speed is = 650 rpm

s_b = slip corresponding to max torque.

$$s_b = \frac{750 - 650}{750} = \frac{2}{15}$$

$= 0.133$ or 13.33%

$$s_b = R_2/X_2$$

$\therefore \quad X_2 = 0.08 \times 15/2 = 0.6\ \Omega$

$$\frac{T_{st}}{T_{\max}} = \frac{2a}{1 + a^2}$$

Since $T_{st} = T_{\max}$

$\therefore \quad 1 = \dfrac{2a}{1 + a^2}$ or $a = 1$

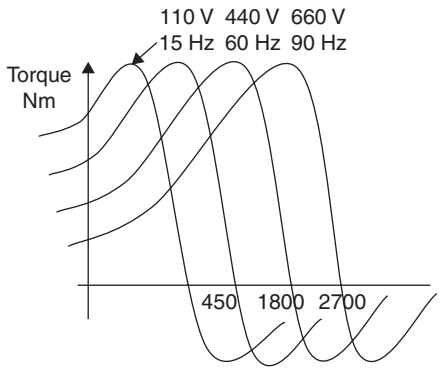
Let r be the external resistance per phase added to the rotor circuit, then

$$A = \frac{R_2 + r}{x_2} \quad \text{or} \quad 1 = \frac{0.08 + r}{0.6}$$

$\therefore \quad r = 0.52\ \Omega$/phase

Shape of Torque/Speed Curve

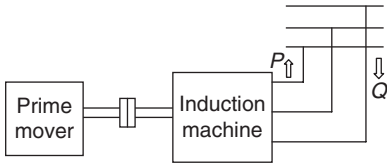
For a SCIM shape of its torque/speed curve depends on the voltage and frequency applied to its stator. In practice, supply voltage and frequency are varied in the same proportion in order to maintain a constant flux in the air gap. Under those conditions, shape of the torque/speed curve remains the same but its position along the X-axis (i.e., speed axis) shifts with frequency.



Above fig shows torque/speed curve of an 11 kW, 440 V, 60 Hz, 3-phase SCIM. Since the shape of the torque/speed curve remains same at all frequencies, it follows that torque developed by an SCIM is the same whenever slip-speed is the same.

Induction Generator

When run faster than its synchronous speed, an IM runs as a generator called an induction generator. To run the IM faster than its synchronous speed a prime mover is needed.



For creating its own magnetic field, it absorbs reactive power 'Q' from the line and delivers active power 'P' to the 3-phase line. The active power is directly proportional to the slip above the synchronous speed.

Equivalent Circuit of the Rotor

When motor is loaded, the rotor current I_2 is given by

$$I_2 = \frac{sE_2}{\sqrt{R_2^2 + (sX_2)^2}}$$

From the above relation it appears that the rotor circuit which actually consists of a fixed resistance R_2 and a variable reactance sX_2 (proportional to slip) connected across $E_r = sE_2$ (figure 1(a)) can be looked upon as equivalent to a rotor circuit having a fixed reactance X_2 connected in series with a variable resistance $\frac{R_2}{s}$ (inversely proportional to slip) and supplied with constant voltage E_2 as shown in fig (b).

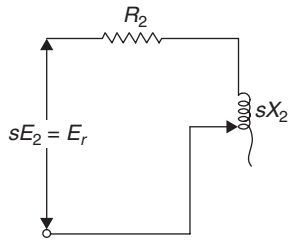


Figure 1(a)

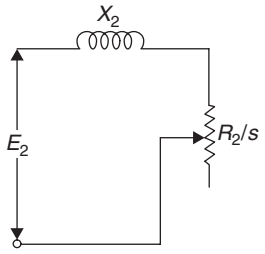


Figure 1(b)

Now, the reactance $\frac{R_2}{s} = R_2 + R_2\left(\frac{1}{s} - 1\right)$

It consists of two parts:

1. The first part R_2 is the rotor resistance itself and represents the rotor Cu loss.
2. The second part is $R_2\left(\frac{1}{s} - 1\right)$. This is known as the load resistance R_L and is the electrical equivalent of the mechanical load on the motor

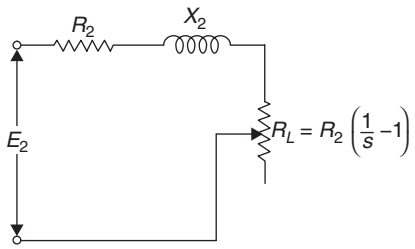
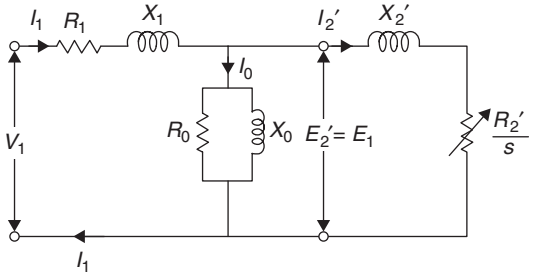
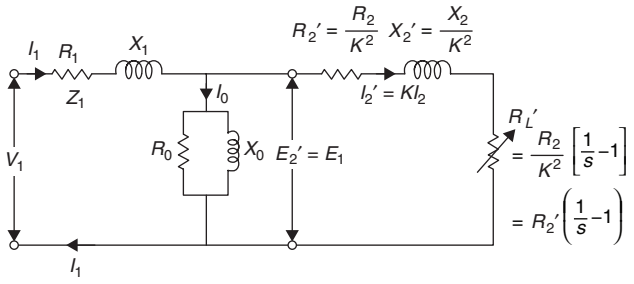


Figure 1(c)

Equivalent Circuit of an IM

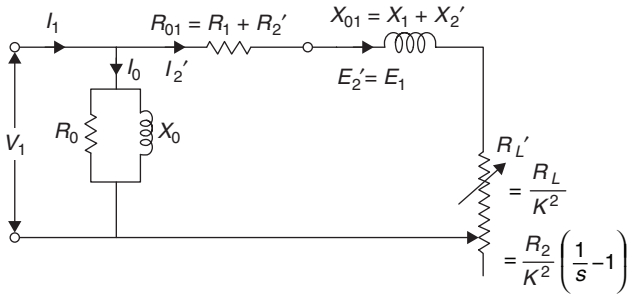
The equivalent circuit of an IM where all values have been referred to primary, i.e., stator is shown in the below figure.





Here, 'K' refers to the transformation ratio of the given machine. Since IMs can be visualized as three-phase transformers with a rotating secondary; 'K' is the effective turns ratio.

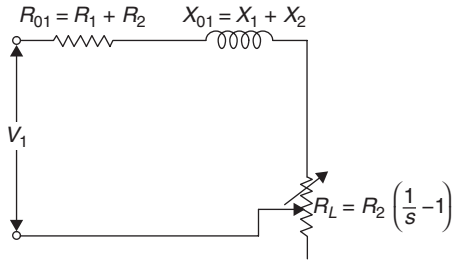
The exciting circuit may be transferred to left, to make calculations simple, thereby we get approximate equivalent circuit of the IM.



Condition for Maximum Power Output

Approximate equivalent circuit of an IM with the following assumptions is shown in figure.

1. Exciting circuit is omitted, i.e., I_0 is neglected and
2. K is assumed unity



With the above assumptions, the power output is maximum when the equivalent load resistance is equal to the standstill leakage impedance of the motor.

i.e., $R_L = Z_{01}$

Corresponding slip, $s = \dfrac{R_2}{R_2 + Z_{01}}$

Maximum gross power output,

$$P_{gmax} = \frac{3V_1^2}{2\left(R_{01} + Z_{01}\right)}$$

where V_1 is voltage/phase of the motor and K has been taken as unity.

Starting of IMS

An IM at rest is like a transformer with a short-circuited secondary. Therefore, if normal supply voltage is applied to the stationary motor, a very large initial current, i.e., 5 to 7 times the full-load current is taken by the primary and develop only 1.5 to 2.5 times their full-load torque.

- The wound-rotor IMs are started by introducing external resistance across the slip-rings which also improves the starting torque.
- SCIMs are started by applying a reduced voltage at starting and then increasing to the full line voltage as the motor picks up speed.

Direct-switching or Line Starting of IMs

- No effort is made to reduce the starting current. Motors of small capacity can be started by direct-switching.

$$\frac{T_{st}}{T_{fl}} = \left(\frac{I_{sc}}{I_{fl}}\right)^2 S_{fl}$$

Starting of SCIMs

A reduction in the stator applied voltage can be accomplished in three ways

(a) Stator resistance (or) reactor starting
(b) Auto-transformer starting
(c) Star-delta starting

(a) Stator resistance or reactor starting:

- A resistor or a reactor is inserted in between motor and supply terminals. Their purpose is to drop some voltage and hence reduce the initial current drawn by the motor.
- If the stator voltage is reduced to the fraction 'x' (of the rated voltage) the short-circuit current also reduced to the fraction 'x'

$$\text{i.e., } I_{st} = \frac{xV_1}{Z_1} = xI_{sc}$$

$$\frac{T_{st}}{T_{fl}} = \left[\frac{I_{st}}{I_{fl}}\right]^2 \cdot s_{fl}$$

$$= \left(\frac{xI_{sc}}{I_{fl}}\right)^2 \cdot s_{fl}$$

$$= x^2\left(\frac{I_{sc}}{I_{fl}}\right)^2 \cdot s_{fl}$$

Starting Torque with Reactor Starting

$$\text{Starting torque with direct switching} = \left(\frac{xV_1}{V_1}\right)^2 = x^2$$

Auto-transformer Starting

The reduction in the stator impressed voltage is accomplished by means of auto-transformer

$$I_{st} = \frac{xV_1}{Z_1} = xI_{sc}$$

Neglecting the no-load current of the auto-transformer, primary VA = secondary VA.

i.e., $I_L V_1 = xV_1 I_{st}$, where I_L is line (primary) current

$$\therefore \quad I_{st} = \frac{I_L}{x} = xI_{sc}$$

$$\therefore \quad I_L = x^2 I_{sc}$$

$$\frac{T_{st}}{T_{fl}} = \left[\frac{X\ I_{sc}}{I_{fl}}\right]^2 \cdot s_{fl} = x^2\left[\frac{I_{sc}}{I_{fl}}\right]^2 \cdot s_{fl}$$

Star-delta Starting

This method of starting is used for motors designed to operate normally in delta. Initially, the stator winding is connected in star, so that the voltage applied to the stator is $\left(\frac{1}{\sqrt{3}}\right)$ times the voltage. After the motor attains a speed nearly equal to the steady-state speed, the stator winding are connected in delta.

Let V_L be the line voltage with stator winding star-connected, then starting current/phase

$$= \left(\frac{V_L}{\sqrt{3}}\right)\frac{1}{Z_{sc}} = (I_L)_Y$$

with stator connected in delta the starting current/phase

$$(I_{st})_D = V_L\left(\frac{1}{Z_{sc}}\right)$$

and the line current

$$(I_{st})_{L\Delta} = \sqrt{3}(I_{st})_\Delta$$

$$\therefore \quad (I_L)_Y = (I_{st})_Y = \frac{V_L}{\sqrt{3}Z_{sc}} = \frac{1}{\sqrt{3}}(I_{st})_\Delta$$

$$\therefore \quad \frac{\text{Starting line current with star-connected stator}}{\text{Starting line current with delta-connected stator}} = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$$

$$(I_{st})_Y = \frac{1}{\sqrt{3}}(I_{sc})_D$$

$$\frac{\text{Starting torque with startor}}{\text{Starting torque in direct switching}}$$

$$= \frac{\left(\frac{V_L}{\sqrt{3}}\right)^2}{V_L^2} = \frac{1}{3}$$

$$\frac{\text{Starting torque with star} - \Delta \text{ starter}, T_{st}}{\text{Full load torque with stator winding in } \Delta, (T_{FL})_\Delta}$$

$$= \frac{\frac{1}{\omega_s}(I_{st})_Y^2 \frac{r_2}{1}}{\frac{1}{\omega_s}(I_{fl})_\Delta^2 \frac{r_2}{s_{fl}}}$$

$$= \frac{1}{3}\left[\frac{(I_{sc})_\Delta}{(I_{fl})_\Delta}\right]^2 \cdot s_{fl}$$

Common Data for Examples 9 to 11:
An SCIM has a starting current of 6 times the full-load current at a slip of 0.04. Calculate the line current and starting torque in PU of full-load values of the following methods.

Example 9: Direct switching

Solution: In Direct switching method,
Line current = Starting current

Starting current , $I_{st} = 6I_{fL}$

$$\frac{T_{st}}{T_{fl}} = \left(\frac{I_{st}}{I_{fl}}\right)^2 \times s_{fl} = (6)^2 \times 0.04$$

$$= 1.44 \text{ PU}$$

Example 10: Auto-transformer starting with motor current limited to 2.0 PU

Solution: Line current $= x \cdot I_{st} = x^2 I_s = \frac{1}{9}6I_{fl} = \frac{2}{3}I_{fl}$

$$I_{SC} = 6I_{fl}$$

$$\therefore \quad x = \frac{I_{st}}{I_{sc}} = \frac{1}{3}$$

$$\frac{T_{st}}{T_{fl}} = \left(\frac{xI_{sc}}{I_{fl}}\right)^2 \times s_{fl} = x^2\left(\frac{I_{sc}}{I_{fl}}\right)^2 \times s_{fl}$$

$$= \frac{1}{9} \times 36 \times 0.04$$

$$= 0.16 \text{ PU}$$

Example 11: Star-delta starting.

During starting period the motor is star-connected

Solution: I_{st} per phase $= \frac{1}{\sqrt{3}} I_{sc}$ per phase

$$= \frac{1}{\sqrt{3}} \times 6I_{FL} = 3.45 \text{ PU}$$

$$\frac{T_{st}}{T_{fl}} = \left[\frac{I_{st}}{I_{fl}}\right]^2 s_{fl}$$

$$= \left[\frac{I_{SC}}{\sqrt{3} I_F}\right]^2 s_{fl}$$

$$= \frac{1}{3}\left[\frac{I_{SC}}{I_{FL}}\right]^2 s_{fl}$$

$$= \frac{1}{3} \times (6)^2 \times 0.04 = 0.48 \text{ PU}$$

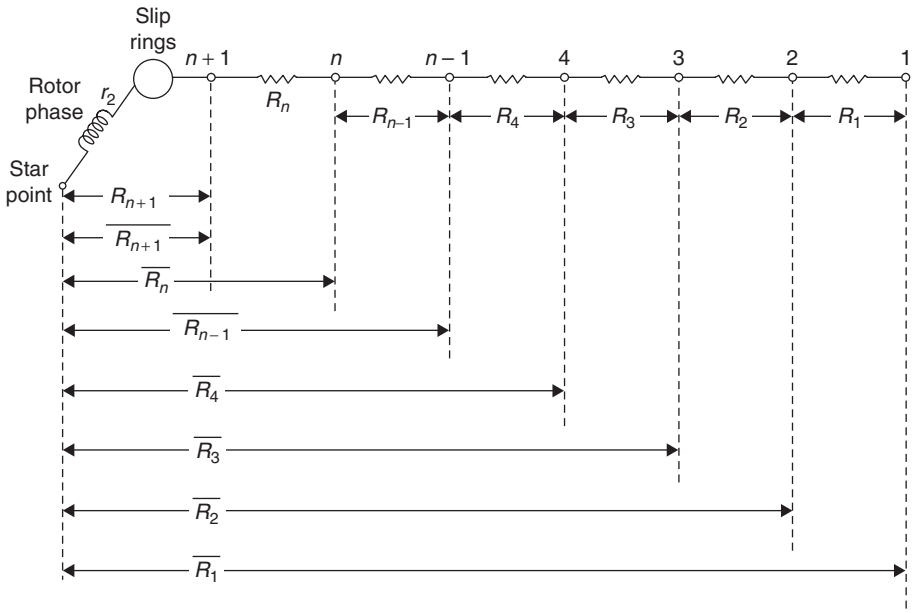
Starting of Wound-rotor Motors

All the starting methods that are used for SCIM are applicable to wound rotor IMs also. However, in a wound rotor IM, the starting current is limited for a safe value by increasing the impedance of the rotor circuit at the time of starting. This is possible since an external resistance can be included in the rotor circuit at the time of starting the motor. It is gradually cut out as the motor comes up to speed.

Rotor resistance starting offers the following advantages

1. It is simple and cheap
2. It limits starting current to a safe value
3. It can increase the starting torque even to a value equal to the pull-out torque
4. The rotor p.f. (and hence the starting p.f.) is improved

Calculation of Starter Resistance Steps



Assumptions

1. During the start-up period, the load torque remains constant
2. No-load current of the motor is neglected
3. The input current varies between a maximum of $I_{1\max}$ and a minimum of $I_{1\min}$

$$\frac{\bar{R}_2}{\bar{R}_1} = \frac{\bar{R}_3}{\bar{R}_2} = \ldots\ldots = \frac{\bar{R}_n}{\bar{R}_{n-1}}$$

$$= \frac{\bar{R}_{n+1}}{\bar{R}_n} = \frac{r_2}{\bar{R}_n} = \gamma \text{ (say)}$$

Then $\bar{R}_2 = \bar{R}_1\gamma, \bar{R}_3 = \bar{R}_2\gamma = \bar{R}_1\gamma^2; \ldots \bar{R}_{n+1} = \bar{R}_1\gamma^n = r_2$

$\gamma = \sqrt[n]{s_m}$, where s_m is the slip under normal operating conditions.

$$R_1 = \bar{R}_1 - \bar{R}_2 = (1-\gamma)\bar{R}_1$$

$$R_2 = \bar{R}_2 - \bar{R}_3 = (1-\gamma)\bar{R}_2$$

$$= (1-\gamma)\,\gamma\,\bar{R}_1 = \gamma R_1$$

$$R_3 = \gamma^2 R_1; \ldots\ldots R_n = \gamma^{n-1} R_1$$

Crawling

This problem is found, particularly in SCIM. They sometimes exhibit a tendency to run stably at speeds as low as one-seventh of their synchronous speed, N_s. This phenomenon is known as crawling of an IM.

This is due to 7th harmonic currents in the three stator windings which have a phase difference of $7 \times 120° = 2 \times 360° + 120° = 120°$.

They set up a forward speed equal to $\frac{1}{7}$ th of the synchronous speed of the fundamental torque.

Cogging or Magnetic Locking

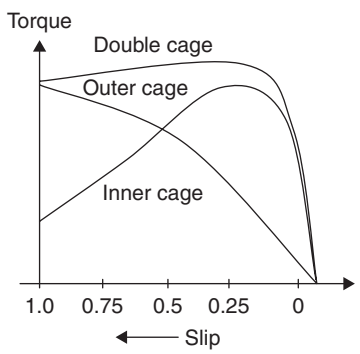
The rotor of a SCIM sometimes refuses to start at all, particularly when the voltage is low

This happens when the number of stator teeth S_1 is equal to number of rotor teeth S_2 and is due to the magnetic locking between the stator and rotor teeth. This is because of the minimum reluctance of magnetic path when the stator and rotor teeth face each other rather than when the teeth of one element are opposite to the slots on the other. It is in such positions of minimum reluctance that the rotor tends to remain fixed and thus cause serious trouble during starting. Cogging of SCIMs can be easily overcome by making the number of rotor slots prime to the number of stator slots.

Double SCIM

The main disadvantage of SCIM is its poor starting torque, which can be overcome by having a cage of high resistance. The motor has two independent cages on the same rotor, one inside the other.

The outer cage has high resistance and low ratio of reactance to resistance whereas the inner cage has low resistance but, being situated deep in the rotor has a large ratio of reactance-to-resistance. Hence, the outer cage develops maximum torque at starting, while the inner cage does so at about 15% slip.



Speed Control of IM

The speed control techniques of an IM can be mainly classified as

1. Control from stator side
   (a) by changing the applied voltage or line voltage control
   (b) by changing the applied frequency
   (c) by changing the number of stator poles
2. Control from rotor side
   (d) rotor rheostat control
   (e) by operating two motors in concatenation or cascade
   (f) by injecting an emf in the rotor circuit (secondary foreign voltage control)

Voltage control, rotor-resistance control and secondary foreign voltage control methods are commonly known as slip control methods.

Changing the Applied Voltage

This method is the cheapest and the easiest, but is rarely used because

1. A large change in voltage is required for a relatively small change in speed.
2. This large change in voltage will result in a large change in the flux density thereby seriously disturbing the magnetic conditions of the motor.

Changing the Applied Frequency

This method is also used very rarely.

$$N_s = \frac{120 f}{P}$$

Changing the Number of Stator Poles

Change of number of poles is achieved by having two or more entirely independent stator windings in the same slots. Each winding gives a different number of poles and hence different speeds. This method is easily applicable to SCIMs because the squirrel-cage rotor adopts itself to any reasonable number of stator poles.

Rotor Rheostat Control

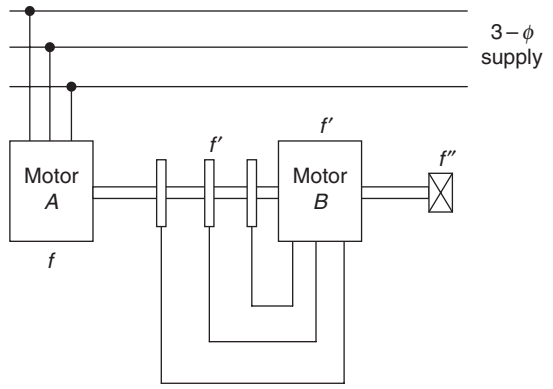
This method is applicable to slip-ring motors alone. The motor speed is reduced by introducing an external resistance in the rotor circuit.

Drawbacks of Rotor Rheostat Control

1. With increase in rotor resistance, I^2R losses also increase which decreases the operating efficiency of the motor. The losses are directly proportional to the reduction in speed.
2. Double dependence of speed, not only on R_2 but on load as well.

Cascade or Concatenation or Tandem Connection

In this method, two motors are used and are ordinarily mounted on the same shaft, so that both run at the same speed.



The stator winding of the main motor A is connected to the mains, while that of the auxiliary motors B is fed from rotor circuit of motor A.

There are at least three ways in which the combination may be run.

1. Main motor A may be run separately from the supply. In that case, the synchronous speed is $N_{sa} = \dfrac{120f}{P_a}$, where P_a = number of stator poles of motor A.
2. Auxiliary motor B may be run separately from the mains (with motor A being disconnected). In that case, synchronous speed is $N_{sb} = \dfrac{120f}{P_b}$, where P_b = Number of stator poles of motor B.
3. The combination may be connected in cumulative cascade, i.e., in such a way that the phase rotation of the stator fields of both motors is in the same direction. The synchronous speed of the cascaded set, in this case, is $N_{sc} = \dfrac{120f}{(P_a + P_b)}$.
4. The fourth possible connection is the differential cascade. In this method, the phase rotation of stator field of motor B is opposite to that of motor A. This reversal of phase rotation of stator of motor B is obtained by interchanging any of its two leads. The synchronous speed of the set is $N_{sc} = \dfrac{120f}{(P_a \sim P_b)}$ (for $P_a \neq P_b$).

   As differentially cascaded set has a very small or zero starting torque, this method is rarely used. Moreover, the above expression for synchronous speed becomes meaningless for $P_a = P_b$.
5. V/f control.

   We know that Torque (τ)

   $$= \frac{\text{power}}{\omega} = \frac{\eta\sqrt{3}V\,1\cos\phi}{N}$$

   (N is the speed)

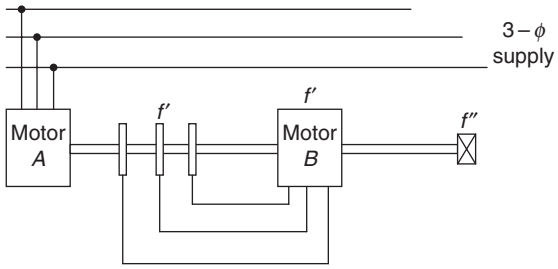
   But; $N = \dfrac{120f}{P}(1-S)$

   Now, if voltage and frequency are constant, torque stays constant and if torque stays constant and V/f = constant; slip remains the same.

Example 12: A 4-pole IM and a 6-pole IM are connected in cumulative cascade. The frequency in the secondary circuit of the 6-pole motor is observed to be 2.0 Hz. Determine the slip in each machine and combined speed of the set. (supply frequency = 50 Hz)

Solution: Speed of cascaded set

$$N_{SC} = 120 \times 50/(4 + 6) = 600 \text{ rpm}$$



$$s = \frac{f''}{f} = \frac{2}{50} = 0.04$$

Let N be the actual speed of concatenated set,

$$\therefore \quad 0.04 = \frac{600 - N}{600} \text{ or } N = 576 \text{ rpm}$$

$$N_{sa} = 120 \times 50/4 = 1500 \text{ rpm}$$

$$s_a = \frac{1500 - 576}{1500}$$

$$= 0.616 \text{ or } 61.6\%$$

$$f' = s_a f$$

$$= 0.616 \times 50$$

$$= 30.8 \text{ Hz}$$

N' = Synchronous speed of 6-pole motor with frequency f'

$$= 120 \times 30.8/6 = 616 \text{ rpm}$$

$$s_b = \frac{N' - N}{N} = \frac{616 - 576}{616}$$

$$= 0.064$$

or 6.4%

Injecting an emf in the Rotor Circuit

In this method, speed of an IM is controlled by injecting a voltage in the rotor circuit. The injected voltage must have the same frequency as the slip frequency. There is no restriction to phase of the injected emf In this method,

1. Any speed, within the working range can be obtained and smooth speed control is possible.
2. By adjusting the phase and magnitude of the injected emf, the power factor of the system can be improved.

Revolving Field Theory of Single-Phase IM

A single-phase IM has a stator winding and a squirrel-cage rotor. When an alternating quantity given, it can be represented two oppositely rotating vectors of half magnitude.

$$F = 1/2\ F_m \cos(\theta - \omega t) + 1/2\ F_m \cos(\theta + \omega t)$$

In the above equation first one is the forward rotating field with half of the maximum value of the pulsating mmf along the axis of the winding then the slip of the rotor with respect to the forward rotating field.

$$s_f = \frac{(N - N)}{N_s} = s$$

The second term is the backward rotating field with half the magnitude. The slip corresponding one is $s_b = 2 - s$

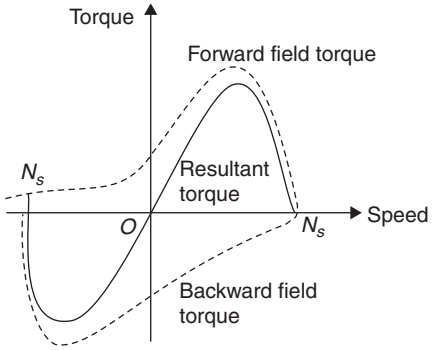


Figure 2 Torque speed characteristics of single winding single-phase motor

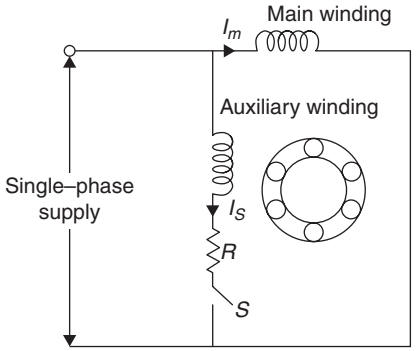
Single-phase Induction Motors

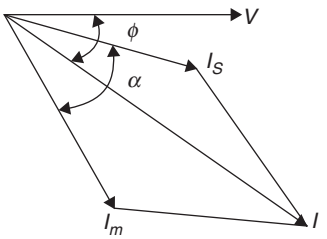
A single-phase IM is not self-starting. To make the motor self-starting, it is temporarily converted into a two-phase motor during starting period. For this purpose, in addition to the main winding, the stator of a single-phase motor is provided with an extra winding, known as starting (or auxiliary winding). The currents flowing through these windings should have a phase difference of 90° so that the motor behaves like a two-phase motor. These two currents produce a rotating magnetic field and hence make the motor self-starting.

The following are the methods by which necessary phase-difference between the two currents can be obtained.

Split Phase Machine

The main winding has low resistance but high reactance whereas the starting winding has a high resistance but low reactance. Hence the current I_s drawn by the starting winding lags behind the applied voltage V by a small angle whereas current I_m taken by the main winding lags behind V by a very large angle.





The phase angle between 'I'$_s$ and 'I'$_m$ is made as large as possible (i.e., near to 90°) because the starting torque of a split-phase motor is proportional to sin α. A centrifugal switch 'S' is connected in series with the starting winding and is located inside the motor to automatically isolate the starting winding from the supply when the motor speed has reached 70% to 80% of steady state speed.

The value of resistance of auxiliary winding, r_a for the production of maximum torque at starting is

$$r_a = \left(\frac{x_a}{x_m}\right)(r_m + Z_m)$$

or $$r_a = \left(\frac{N_a}{N_m}\right)^2 (r_m + z_m)$$

($\because$ reactance $\propto$ square of number of turns)

Capacitor Split-phase Motors

In these motors, the phase displacement between the main winding current I_m and auxiliary winding current I_a is produced by connecting a capacitor in series with the auxiliary winding.

The starting torque $\propto I_a.I_m.\sin\beta$

where β = angle between I_a and I_m. By choosing suitable value of capacitance, the angle β can be made 90° at standstill, which results in highest possible starting torque but best compromise with the running torque is to make β slightly less than 90°. This type of starting gives the highest starting torque.

Capacitor split-phase motors can be further classified as

1. Capacitor start induction run
2. Capacitor start capacitor run

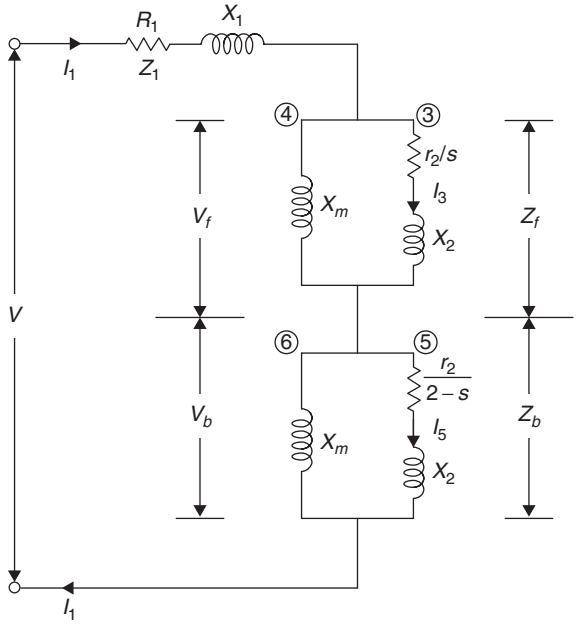
The value of capacitive reactance to be inserted in series with the auxiliary winding to get maximum starting torque is

$$X_c = X_a + \frac{r_a \cdot r_m}{Z_m + X_m}$$

$\therefore$ The value of capacitor, $C = \left(\frac{1}{\omega X_c}\right)$ farad

Equivalent Circuit of a Single-phase IM

A single-phase IM may be viewed as consisting of two motors, having a common stator winding, but with their respective rotors revolving in opposite directions. The equivalent circuit of such a motor based on double field revolving theory is shown in fig.



The stator impedance is $Z = R_1 + jX_1$. The impedance of each rotor is $(r_2 + jx_2)$, where r_2 and x_2 represent half the actual rotor values in stator terms (i.e., x_2 stands for half the standstill reactance of the rotor, as referred to stator) Since iron loss has been neglected, the magnetizing branch is shown consisting of magnetizing reactance only. Each rotor has been assigned half the magnetizing reactance (i.e., x_m represents half the actual reactance). The impedance of the 'forward running' rotor is

$$Z_f = \frac{jx_m\left(\frac{r_2}{s} + jx_2\right)}{\frac{r_2}{s} + j\left(x_m + x_2\right)}$$

and it runs with a slip of s.

The impedance of backward running rotor is

$$Z_b = \frac{jx_m\left(\frac{r_2}{2-s} + jx_2\right)}{\frac{r_2}{2-s} + j\left(x_m + x_2\right)}$$

and it runs with a slip of $(2 - s)$. Under standstill conditions $V_f = V_b$, but under running conditions V_f is almost 90% to 95% of applied voltage.

The forward torque in synchronous watt is

$$T_f = \frac{I_3^2 r_2}{s}.$$

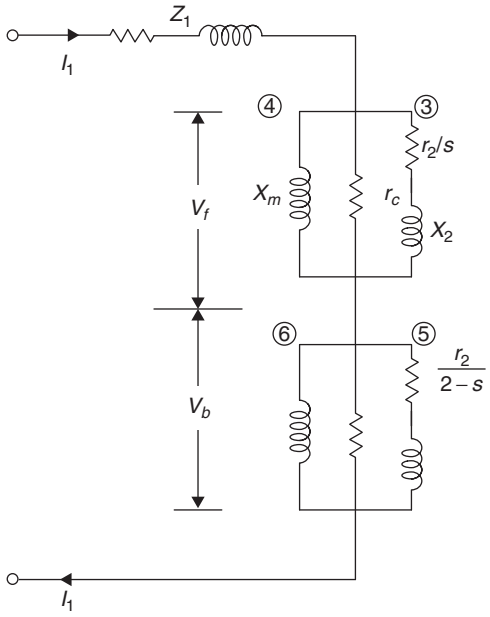
Similarly backward torque is

$$T_b = \frac{I_5^2 r_2}{2-s}$$

The total torque is $T = T_f - T_b$

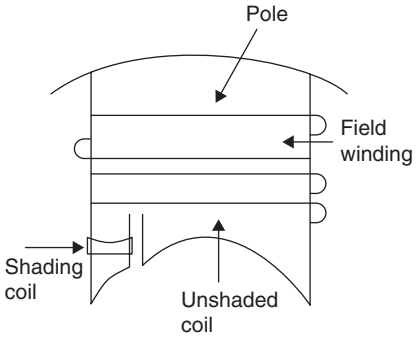
Equivalent Circuit with Core Loss

The core loss can be represented by an equivalent resistance which may be connected in parallel as shown in fig.

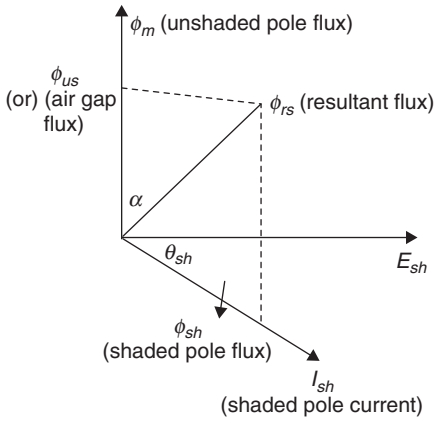


Shaded-pole Single-phase Motor

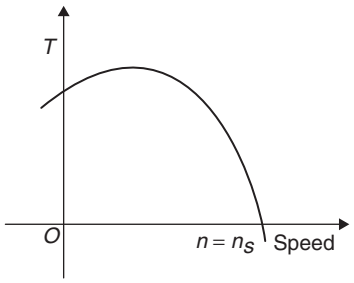
In this type motor, the small portion of each pole is covered with a short-circuited copper coil known as shading coil (or) shaded portion of the pole, The other part of the pole is known as unshaded part as shown in the figure. When an alternating current passed through the coil, the motor rotates in the direction of unshaded portion to shaded portion.



Below phasor diagram shows the flux and induced emf of the shaded-pole IM.



The Torque speed characteristics of the shaded-pole motor show below:



Repulsion Start IM

In Repulsion motors the rotor and stator windings are inductively coupled. In these motors the stator has a distributed winding and the rotor is similar to the DC motor armature, i.e., the windings are connected to commutator. The Repulsion start IM start as a repulsion motor by the transformer action and the motor run as an IM by short-circuiting the commutator segment using a centrifugal switch.

Reluctance Motor

The reluctance motor is just similar to unexcited salient-pole synchronous machine, In these motor the stator produces a rotating field and the reluctance of the magnetic path offered by the rotor to the rotating field is a function of the space angle. Then the rotor aligns in the minimum reluctance path with respect to the synchronous rotating flux. The motor is self-starting by the induction principle by providing short-circuited copper bars in the rotor.

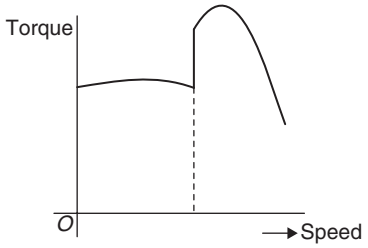


Figure 3 Reluctance motor torque–speed characteristics

EXERCISES

Practice Problems I

Directions for questions 1 to 27: Select the correct alternative from the given choices.

1. A 3-ϕ, 50 Hz IM has a full load speed of 1440 rpm. What is slip and rotor frequency of the IM?
(A) 6%, 4 Hz (B) 10%, 46 Hz
(C) 8%, 6 Hz (D) 4%, 2 Hz

2. An 8-pole, 50 Hz, 3-ϕ slip-ring IM has effective rotor resistance of 0.8 Ω/ph. Stalling speed is 650 rpm. The resistance required in the rotor phase to obtain the maximum torque at starting?
(A) 0.42 Ω (B) 0.52 Ω
(C) 0.48 Ω (D) 0.58 Ω

3. The rotor power output of 3-ϕ IM is 15 kW and the corresponding slip is 4%. The rotor ohmic loss will be
(A) 600 W (B) 625 W
(C) 650 W (D) 700 W

4. In case of 3-ϕ IM, shaft power is 2700 W and mechanical losses are 180 W. At a slip of 4%, the rotor ohmic losses are
(A) 115.2 W (B) 120 W
(C) 108 W (D) 105 W

5. A 3-ϕ IM having a star-connected rotor has an induced emf of 80 V between slip-rings at stand still on open circuit. Find the current and power factor when the slip-rings are connected to 3 Ω/phase rheostat?

(A) 8.85 A and 0.748 Lag
(B) 8.16 A and 0.707 Lag
(C) 9.5 A and 0.8 Lag
(D) 9.5 A and 0.6 Lag

6. A 3-ϕ IM is driving the full-load torque which is independent of speed. If the line voltage drops to 85% of the rated value, the increase in motor Cu losses is _____
(A) 123% (B) 33%
(C) 38.4% (D) 33.6%

7. If the supply voltage of a 3-ϕ SCIM is reduced by 20% then the maximum torque will
(A) increase by 20%
(B) decrease by 20%
(C) increase by 36%
(D) decrease by 36%

8. A cascaded set of IM, consists of two motors A and B with 4 poles and 6 poles, respectively. The motor A is connected to a 50 Hz supply. What is the speed of the set _____?
(A) 3000 rpm (B) 600 rpm
(C) 1500 rpm (D) 1000 rpm

9. A 3-ϕ IM has a full-load slip of 3% at normal voltage. Which one of the following will be the value of the slip of the motor if it develops the same torque theoretically while operating at 110% of its normal voltage?
(A) 2.48% (B) 0.248
(C) 0.483% (D) 4.83%

10. A 6-pole, 3-ϕ, 60 Hz IM runs at 1000 rpm, developing maximum torque. Rotor resistance per phase is 1.2 Ω. Neglecting stator impedance, then for developing maximum torque, the external resistance to be connected in series with the each rotor phase will be
(A) 7.2 Ω (B) 6 Ω
(C) 1.44 Ω (D) 1.2 Ω

Common Data for Questions 11 and 12:
A 400 V, 4-pole 3-ϕ 50 Hz, IM has a rotor resistance and reactance per phase of 0.02 Ω and 0.2 Ω, respectively. Stator to rotor turns ratio is 4.

11. The T_{max} and corresponding slip of motor.
(A) 320 Nm, 10% (B) 320 Nm, 20%
(C) 159.23 Nm, 10% (D) 159.23 Nm, 20%

12. The full-load slip and power output, if the maximum torque is twice the full-load torque, are _____?
(A) .027, 12.16 kW (B) 0.373, 12.16 kW
(C) 0.27, 16.18 kW (D) 0.373, 14.3 kW

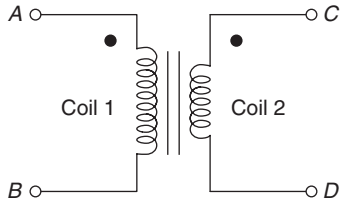
Common Data for Questions 13 and 14:
A 3-ϕ, star-connected 400 V, 50 Hz, 4-pole IM has the following per phase constants in ohm referred to stator and
$Nr = 1440$ rpm.
$R_1 = .15\ \Omega, x_1 = .45\ \Omega, r_2 = 0.12\ \Omega$
$x_2 = 0.45\ \Omega, x_m = 28.5\ \Omega$

13. The stator current I_1 is _____?
(A) 71.06∠–21.47° (B) 122.97∠–21.47°
(C) 71.06∠–36.47° (D) 122.97∠–36.47°

14. The mechanical power developed is _____
(A) 87.73 kW (B) 6.45 kW
(C) 41.73 kW (D) 100.49 kW

15. The power input to a 415 V, 50 Hz, 6-pole, 3-ϕ IM running at 975 rpm is 40 kW. The stator losses are 1 kW and friction and windage losses are 2 kW. The efficiency of the motor is
(A) 92.5% (B) 91%
(C) 90.06% (D) 88%

Common Data for Question 16:



The figure above shows coils 1 and 2, with dot markings as shown, having 2000 and 3000 turns, respectively. Both the coils have a rated current of 25 A. Coil-1 is excited with single phase, 400 V, 50 Hz supply.

16. In the auto-transformer obtained in above question the current in each coil is
(A) coil-1 is 15 A and coil-2 is 10 A.
(B) coil-1 is 10 A and coil-2 is 25 A.
(C) coil-1 is 10 A and coil-2 is 15 A.
(D) coil-1 is 25 A and coil-2 is 10 A.

17. A three-phase slip-ring IM is fed from the rotor side with stator winding short-circuited. The frequency of the currents flowing in the short-circuited stator winding is
(A) zero.
(B) supply frequency.
(C) slip frequency.
(D) frequency corresponding to rotor speed.

18. Skew is used in IMs in order to reduce torque due to
(A) space harmonics.
(B) slot harmonics.
(C) time harmonics.
(D) reverse rotating fields.

19. An IM is fed from a balanced three – phase supply at rated voltage and frequency through a bank of three single-phase transformers connected in delta – delta. One unit of the bank develops fault and is removed. Then
(A) single phasing will not occur but the motor terminal voltages will become unbalanced and the machine can be loaded to the extent of 57.7% of its rating.
(B) the machine can be loaded to the extent of 57.7% of its rating with balanced supply at its terminals.

(C) the machine can be loaded to the extent of 66.66% with balanced supply at its terminals.
(D) single phasing will occur and the machine fails to start.

20. A 3-phase, delta-connected IM fed from unbalanced supply will have
(A) zero sequence currents.
(B) negative sequence component current.
(C) less heating of the rotor.
(D) All of these.

21. In case of a split-phase motor, the phase shift between currents in the two windings is around
(A) 120°. (B) 90°.
(C) 80°. (D) 30°.

22. In an IM, if the air-gap is increased
(A) speed will reduce.
(B) breakdown torque will reduce.
(C) efficiency will improve.
(D) power factor will be lowered.

23. A three-phase SCIM has a full load efficiency of 0.85 and a maximum efficiency of 0.9. It is operated at a slip of 0.65 by applying a reduced voltage. The efficiency of the motor at this operating point is
(A) is less than 0.5.
(B) greater than 0.6.
(C) in the range of 0.85 ± 0.05.
(D) None of the above

24. A single-phase IM with identical main and auxiliary windings can be made self-starting by connecting
(A) a capacitor in series with the main winding.
(B) a capacitor in series with the auxiliary winding.
(C) a capacitor across the supply terminals.
(D) the main and auxiliary windings in series.

25. Which of the following starting method for an IM is inferior in view of the poor starting torque per ampere of the line current drawn?
(A) Star-Delta method
(B) Direct on line starting
(C) Auto-transformer method of starting
(D) Series inductor method of starting

26. The type of single-phase IM having the highest power factor at full load is
(A) Split-phase type
(B) Shaded-pole type
(C) Capacitor-start type
(D) Capacitor-run type

27. A plot of input power versus voltage was drawn from the no – load test readings (at different supply voltages) of a 3-phase IM. This curve was extrapolated to intersect the Y-axis. The intersection point yields
(A) stray load loss.
(B) stator copper loss.
(C) core loss.
(D) friction and windage loss.

Practice Problems 2

Directions for questions 1 to 50: Select the correction alternative from the given choices.

1. The speed of the rotor field of an IM is (s is the slip and N_s, the synchronous speed)
(A) $s \times N_s$ in the same direction as stator field
(B) N_s in the same direction as stator field
(C) $s \times N_s$ in the opposite direction as stator field
(D) N_s in the opposite direction as that of stator field

2. The rotor of a three-phase IM draws a current which is lagging at a power factor of 0.707. The spatial displacement between stator and rotor magnetic field is
(A) 45° (B) 90°
(C) 135° (D) 180°

3. A synchronous motor and an IM are connected to a common bus. For the power factor to be unity at the bus,
(A) synchronous motor should be over excited
(B) synchronous motor should be under excited
(C) the speed of synchronous motor and synchronous speed of IM should be same
(D) synchronous motor is normally excited

4. The starting current of an IM is 6 times the full-load current. If the full-load slip is 3%, ratio of starting torque to full-load torque is
(A) 1.2 (B) 1.08
(C) 0.8 (D) 0.6

5. The rotor power output of a 3-ϕ **IM** is 12 kW at a slip of 4%. The rotor copper losses would be
(A) 400 W (B) 500 W
(C) 600 W (D) 700 W

6. The rotor resistance per phase of a 4-pole 3-phase, 50 Hz IM is 1.4 Ω. It develops maximum torque at 1250 rpm. For developing maximum torque at starting, the resistance to be connected in series with each rotor phase will be
(A) 8.4 ohm (B) 1.4 ohm
(C) 7.2 ohm (D) 7 ohm

7. A 3-ϕ IM operating at rated voltage has a full-load slip of 4%. It is now operated at 110% of rated voltage. What should be the value of slip theoretically, if it develops the same torque?
(A) 3.30% (B) 0.330%
(C) 1.483% (D) 4.83%

8. A 3-ϕ supply is given to the rotor of a 3-ϕ slip-ring IM with short-circuited stator. The frequency of current in the stator is
(A) supply frequency
(B) frequency corresponding to rotor speed
(C) slip frequency
(D) zero

9. A 3-ϕ IM running at 4% slip takes an input power of 10 kW. If stator resistance and core losses are assumed negligible, then the torque developed in synchronous watts is
(A) 12 kW (B) 9 kW
(C) 10 kW (D) 0.4 kW

10. A 3-ϕ, 6-pole, 50 Hz IM is delivering 3500 W at a slip of 0.04. Stray losses are found to be 100 W. The gross torque is
(A) 3750 synchronous watts
(B) 35.81 Nm
(C) Either A or B
(D) 41.58 kgm

11. A 3-ϕ 4-pole 400 V 50 Hz Y-connected IM has rotor resistance of 0.1 Ω and stand still reactance of 0.3 Ω. The effective stator to rotor turns ratio is 4. If the motor is to drive a constant-load torque of 100 Nm, the minimum resistance to be added in series in rotor circuit for the motor to start on load
(A) 0.01 Ω (B) 0.11 Ω
(C) 0.02 Ω (D) 0.22 Ω

12. A 3-ϕ IM has a rotor resistance of 0.02 Ω/phase. If the resistance is increased to 0.04 Ω /phase, the maximum torque
(A) increase by 50%
(B) becomes two times
(C) remains unaltered
(D) will reduce to half

13. When the stator of a 3-ϕ IM is excited with a 3-ϕ balanced supply, the rotor of the IM runs in same direction as the rotating stator magnetic field. The law that is obeyed here is
(A) Faraday's law of electromagnetic induction
(B) Lenz's law
(C) Fleming's left hand rule
(D) Newton's law of motion

14. An SCIM having a rated slip of 2.5% on full load has a starting torque of 0.9 full-load torque, then starting current is
(A) equal to full-load current
(B) twice full-load current
(C) four times full-load current
(D) six times full-load current

15. Maintaining the voltage to frequency ratio of an IM constant, the frequency of voltage applied is increased. Assuming stator impedance is zero, maximum torque
(A) increases and occurs at large values of slip
(B) remains constant and occurs at smaller values of slip
(C) remains constant and occurs at larger values of slip
(D) increases and occurs at smaller values of slip

16. In an SCIM, blocked rotor test is done determines its equivalent
(A) to determine shunt resistance and reactance as seen from rotor
(B) to determine series resistance and reactance as seen from rotor
(C) to determine shunt resistance and reactance as seen from the stator
(D) to determine series resistance and reactance as seen from the stator

17. An ac IM is used for a speed control application is driven from an inverter. Name plate details of motor V : 415, V_{ph} : 3, f : 50 Hz, N : 1440 rpm. Keeping v/f constant, inverter output frequency is set to 40 Hz and motor is operated at half rated slip. The running speed of motor is
(A) 1200 rpm (B) 1140 rpm
(C) 1176 rpm (D) 1340 rpm

18. Both frequency and voltage of a 3-ϕ IM, driving a constant-load torque are halved. Ignore stator impedance and core losses. In this condition,
(A) Flux in the air gap remains constant
(A) Stator current remains constant
(C) The per unit slip remains constant
(D) Difference between synchronous speed and actual speed remains constant
Among the above, correct statements are
(A) A, B, D (B) B, C, D
(C) A and D (D) All the above are correct

19. Only the main winding of a 1-ϕ IM is excited. The forward and backward rotating fields
(A) are equal
(B) are zero
(C) forward rotating field is more than backward rotating field
(D) forward rotating field is less than backward rotating field

20. The main winding impedance and auxiliary winding impedance of a single-phase 4-pole, capacitor-start IM has the following stand still values.

Z_m (main) = $3 + j4$ Ω

Z_a (aux) = $12 + j10$ Ω

Value of capacitor required to produce 90° phase difference between currents in main winding and auxiliary winding
(A) 167.53 μF (B) 197.53 μF
(C) 267.53 μF (D) 287.53 μF

21. A 400 V, 10 kW, 6-pole 50 Hz Y-connected IM has full-load slip of 5%. The output torque of machine at full load is
(A) 2.9 Nm (B) 92.5 Nm
(C) 100.51 Nm (D) 248.73 Nm

22. The single-phase IM that has the greatest power factor at full load is
(A) split-phase type
(B) shaded-pole type
(C) capacitor-start type
(D) capacitor-run type

23. A 3-ϕ IM runs in anticlockwise direction when supplied with a phase sequence of R – B – Y. For a clockwise rotation, the phase sequence should be
(A) Y – R – B (B) B – Y – R
(C) B – R – Y (D) either (A) or (B)

24. For an IM ratio of rotor power output to air gap power is
(A) $(1-s)^2$ (B) $(1-s)$
(C) $\sqrt{1-s}$ (D) $\left(1-\sqrt{s}\right)$

25. The voltage applied to a 3-ϕ IM, running at no load is reduced to one half from the rated value. Then the speed and stator current
(A) decreases and increases, respectively
(B) both decreases
(C) both remain practically constant
(D) remain same and decreases, respectively

26. A 3-ϕ IM is started DOL at the rated voltage. If starting current is 4 times full-load current, and full-load slip is 5% ratio of starting torque to full-load torque is
(A) 0.2 (B) 0.4
(C) 0.8 (D) 1

27. The rotor resistance of a 1-ϕ IM, driving a fan load is made high in order to achieve
(A) high efficiency
(B) low starting torque
(C) quick acceleration
(D) reduced size

28. The equivalent circuit parameters of a 3-phase 6-pole 400 V, 50 Hz Y-connected IM are as follows.
$r_1 = 0.9\ \Omega,\ r_2' = 0.7\ \Omega, x_1 = x_2' = 1.4\ \Omega, x_m = 40\ \Omega$
The starting torque when motor is started direct on line
(A) 105.06 Nm (B) 195.06 Nm
(C) 235.06 Nm (D) 285.06 Nm

29. The no load and blocked rotor test data for a 3-ϕ, 10 kW, 400 V, 4-pole, 50 Hz star-connected IM drawing 20 A on full load is as given

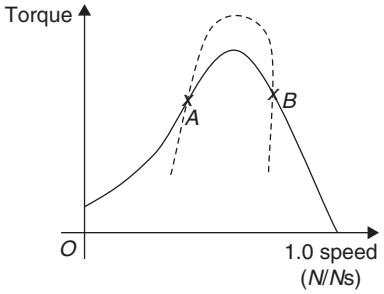
No load test:	400 V	5 A	1120 W
Blocked rotor:	70 V	16 A	692 W

Estimate the full load efficiency of motor
(A) 77% (B) 82%
(C) 83.4% (D) 86%

30. A 400 V, 50 Hz, 25 hp, 3-ϕ IM is drawing 25 A current at 0.8 p.f. lag. Stator and rotor Cu losses are 10 kW and 700 W, respectively. Friction and windage losses are 950 W and core loss is 1100 W. The air gap power of the motor will be
(A) 10.806 kW (B) 12.85 kW
(C) 16.32 kW (D) 11.95 kW

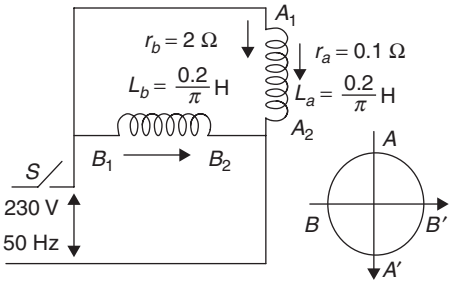
31. A 3-ϕ, 400 V, 50 Hz, 4-pole, 1400 rpm, Y-connected SCIM is controlled from a 3-ϕ voltage source inverter with constant v/f control. The motor has the following parameters referred to the stator. $R_r' = 1.0\ \Omega$, $X_s = X_r' = 20\ \Omega$. Neglecting stator resistance and rotational losses, the stator line to line voltage and frequency to obtain maximum torque at starting will be
(A) 323 V, 40.3 Hz
(B) 266.6 V, 33.3 Hz
(C) 100 V, 12.5 Hz
(D) 200 V, 25 Hz

32. Figure below shows the speed torque curves of a 3-ϕ SCIM and its load. Points A and B represents the operating conditions. Which among them is stable



(A) A (B) B
(C) both are stable (D) none are stable

33. AA' and BB' are the axes of the main winding A_1A_2 and auxiliary winding B_1B_2, respectively, of a 230 V, 50 Hz 1-ϕ IM. The direction of axes shows the direction of flux when currents in winding are as shown in the figure. When switch S is closed,



(A) motor rotates clockwise
(B) motor rotates anticlockwise
(C) motor rotates momentarily and comes to a halt
(D) motor does not rotate

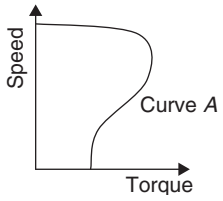
34. In a shaded-pole IM, the rotor runs from
(A) shaded portion to the unshaded portion of the pole while flux in shaded pole leads that of unshaded pole

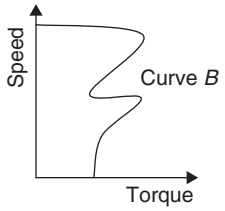
(B) shaded portion to the unshaded portion of the pole while flux in the shaded pole lags that of unshaded pole
(C) unshaded portion to shaded portion and flux in the unshaded pole leads that of shaded pole
(D) unshaded portion to the shaded portion, and flux in unshaded pole lags that of shaded pole

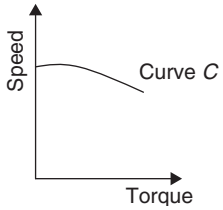
35. A 3-ϕ 50 Hz IM has a full load speed of 960 rpm. For this motor, the speed of stator field with respect to revolving rotor field is
(A) 1000 rpm (B) 0 rpm
(C) 40 rpm (D) 960 rpm

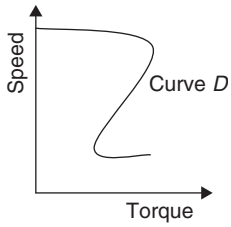
36. Two slip-ring IMs having number of poles 6 and 2, respectively, are connected in cascade. If supply frequency is 50 Hz, speed of the main motor is
(A) 1500 rpm (B) 3000 rpm
(C) 750 rpm (D) 1000 rpm

37. The torque speed characteristics of a repulsion start induction run motor is

(A) 

(B) 

(C) 

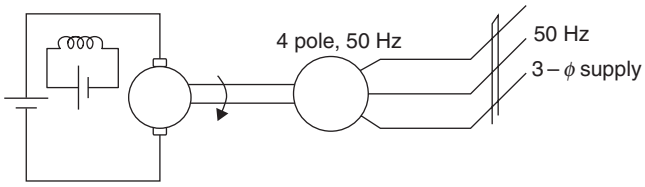
(D) 

(A) Curve A (B) Curve B
(C) Curve C (D) Curve D

38. The power output of an induction machine is minimum when
(A) when equivalent resistance is equal to equivalent reactance of the motor
(B) equivalent resistance equal to slip multiplied by equivalent reactance
(C) when slip is equal to zero
(D) when slip is equal to one

39. The main winding and auxiliary winding current of a 2 winding single-phase IM are I_m = 15 A and I_a = 7.5 A at stand still. The auxiliary winding current leads the main winding current by α = 45° electrical. The two windings are in space quadrature and the effective number of turns are N_m = 150 and N_a = 175. Amplitude of forward rotating stator mmf is
(A) 1508 AT (B) 1237 AT
(C) 1045 AT (D) 653 AT

40. A separately excited **DC** machine is coupled to a 50 Hz, 3-phase 4-pole induction machine as shown in the figure. The **DC** machine is energized first and machines rotate at 1600 rpm. The induction machine is also connected to a 50 Hz 3-ϕ source, phase sequence being consistent with the direction of rotation. In steady state,



(A) both machine acts as generators
(B) **DC** machine acts as generator and induction machine motor
(C) **DC** machine acts as motor and induction machine generator
(D) both machines acts as motor

41. Which type of motor is most suitable for a computer printer drive?
(A) Reluctance motor
(B) Hysteresis motor
(C) Shaded-pole motor
(D) Stepper motor

42. By adding resistance in the rotor circuit of a slip-ring IM, the starting current and torque, compared to DOL
(A) both are reduced
(B) both are increased
(C) starting current is reduced but starting torque is increased
(D) starting current is increased and starting torque is reduced

43. An IM is connected to a 3-ϕ supply constituting three 1-ϕ transformers connected in Delta–Delta configuration. If one transformer is removed, then
(A) single phasing occurs and machine does not start
(B) single phasing does not occur and the terminal voltages become unbalanced and machine can be loaded to 57.7% of its rated capacity
(C) machine can be loaded to 57.7% of rated capacity with balanced supply at the terminals
(D) machine can be loaded to 66.67% of its rated power with balanced supply at terminals

44. The skewing of rotor slots in a 3-ϕ IM reduces
(A) harmonic torques and noise but increases torque
(B) harmonic torques and noise but increases starting torque
(C) noise but increases breakdown torque and harmonic torque
(D) harmonic torque, noise and breakdown torque and starting torque

45. If the air gap of an IM is increased, then
(A) pull out torque is reduced
(B) power factor is lowered

(C) efficiency is improved
(D) speed is reduced

Common Data for Questions 46 and 47:
The starting current of a 3-ϕ SCIM is 8 times the full-load current and the full-load slip is 5%.

46. If an auto transformer is used to reduce voltage at starting, the auto transformer ratio to provide 2 PU starting torque
(A) 79% (B) 57.7%
(C) 70.7% (D) 82%

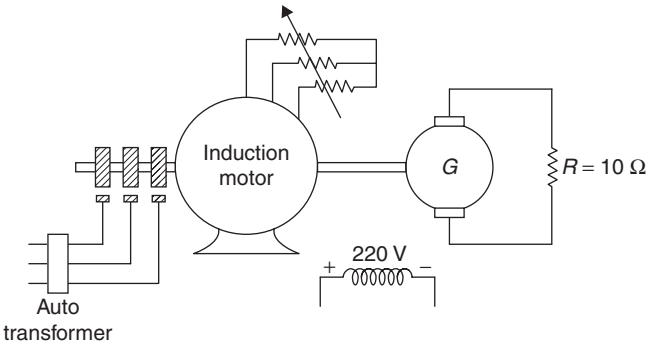
47. If a starting torque of 0.5 PU is required, then starting current in PU should be
(A) 2.16 (B) 4.16
(C) 3.75 (D) 3.16

Common Data for Questions 48 and 49:

A slip-ring IM has the following name plate details

V : 400 V, f : 50 Hz, Poles 6, V_{ph} : 3

It is fed from rotor side through an auto transformer and stator is connected to a variable resistance.



The motor is coupled to a 220 V, separately excited **DC** generator feeding power to fixed resistance of 20 Ω. By adjusting the variable resistance, motor is run at 960 rpm. Two watt meters used to measure input power gives the following readings → $W_1 = 2000$ W, $W_2 = -400$ W.

48. The speed of rotation of stator magnetic field with respect to rotor structure is
(A) 40 rpm in direction of rotation
(B) 40 rpm opposite to direction of rotation
(C) 1000 rpm in direction of rotation
(D) 1000 rpm opposite to direction of rotation

49. Neglecting all losses of both the machines, generator power output and current through resistance
(A) 64 W, 1.79 A
(B) 1236 W, 7.86 A
(C) 1536 W, 8.76 A
(D) 1880 W, 9.7 A

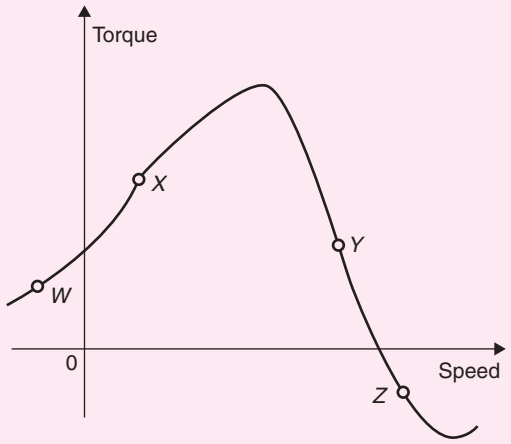
Common Data for Question 50:
Consists of two statements: one is Assertion (A) and the other is Reason (R). You have to examine these two statements and select the answer using the code given below.
(A) Both A and R are individually correct and R is the correct explanation of A
(B) Both A and R are individually correct, but R is not the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

50. A: The maximum value of developed torque in a 3-ϕ IM remains a constant for various speeds in the sub synchronous region in the v/f control.
R: The ratio of voltage and frequency over the considered speed range is maintained a constant and hence flux in the machine is almost constant.

Previous Years' Questions

1. A 400 V, 15 kW, 4-pole, 50 Hz, Y-connected IM has full-load slip of 4%. The output torque of the machine at full load is **[2004]**
(A) 1.66 Nm (B) 95.50 Nm
(C) 99.47 Nm (D) 624.73 Nm

2. On the torque/speed curve of IM shown in the below figure, four points of operation are market as W, X, Y and Z. Which one of them represents the operation at a slip greater than 1? **[2005]**



(A) W (B) X
(C) Y (D) Z

3. For an IM, operating at a slip s, the ratio of gross power output to air gap power is equal to: **[2005]**

(A) $(1-s)^2$ (B) $(1-s)$

(C) $\sqrt{(1-s)}$ (D) $(1-\sqrt{s})$

4. Under no load condition, if the applied voltage to an IM is reduced from the rated voltage to half the rated value, **[2005]**

(A) the speed decreases and the stator current increases

(B) both the speed and the stator current decrease

(C) the speed and the stator current remain practically constant

(D) there is negligible change in the speed but the stator current decreases

5. A three-phase cage IM is started by direct-on-line (DOL) switching at the rated voltage. If the starting current drawn is 6 times the full-load current, and the full-load slip is 4%, the ratio of the starting developed torque to the full-load torque is approximately equal to **[2005]**

(A) 0.24 (B) 1.44

(C) 2.40 (D) 6.00

6. In a single-phase IM driving a fan load, the reason for having a high resistance rotor is to achieve **[2005]**

(A) low starting torque

(B) quick acceleration

(C) high efficiency

(D) reduced size

7. Determine the correctness of otherwise of the following **Assertion [a]** and the **Reason [r].**

Assertion: Under *V*/*f* control of IM, the maximum value of the developed torque remains constant over a wide range of speed in the sub-synchronous region.

Reason: The magnetic flux is maintained almost constant at the rated value by keeping the ratio *V*/*f* constant over the considered speed range. **[2005]**

(A) Both [a] and [r] are true and [r] is the correct reason for [a]

(B) Both [a] and [r] are true but [r] is not the correct reason for [a]

(C) Both [a] and [r] are false

(D) [a] is true but [r] is false

8. For a single-phase capacitor start IM which of the following statements is valid? **[2006]**

(A) The capacitor is used for power factor improvement

(B) The direction of rotation can be changed by reversing the main winding terminals

(C) The direction of rotation cannot be changed

(D) The direction of rotation can be changed by interchanging the supply terminals

9. The speed of a 4-pole IM is controlled by varying the supply frequency while maintaining the ratio of supply frequency (*V*/*f*) constant. At rated frequency of 50 Hz and rated voltage of 400 V its speed is 1440 rpm. Find the speed at 30 Hz, if the load torque is constant. **[2006]**

(A) 882 rpm (B) 864 rpm

(C) 840 rpm (D) 828 rpm

10. A 3-phase, 4-pole, 400 V, 50 Hz, star-connected IM has following circuit parameters $r_1 = 1.0\ \Omega$, $r_2' = 0.5\ \Omega$, $x_1 = x_2' = 1.2\ \Omega$, $x_m = 35\ \Omega$ The starting torque when the motor is started direct-on-line is (use approximate equivalent circuit model) **[2006]**

(A) 63.6 Nm (B) 74.3 Nm

(C) 190.8 Nm (D) 222.9 Nm

11. A 3-phase, 10 kW, 400 V, 4-pole, 50 Hz, star-connected IM draws 20 A on full load. Its no load and blocked rotor test data are given below:

No Load test: 400 V 6A 1002 W

Blocked Rotor test: 90 V 15 A 762 W

Neglecting copper loss in no Load test and core loss in Blocked Rotor test, estimate motor's full load efficiency. **[2006]**

(A) 76% (B) 81%

(C) 82.4% (D) 85%

12. A three-phase SCIM has a starting torque of 150% and a maximum torque of 300% with respect to rated torque at rated voltage and rate frequency. Neglect the stator resistance and rotational losses. The value of slip for maximum torque is **[2007]**

(A) 13.48% (B) 16.24%

(C) 18.92% (D) 26.79%

Common Data for Questions 13 to 15:

A three-phase SCIM has a starting current of seven times the full-load current and full-load slip of 5%

13. If an autotransformer is used for reduced voltage starting to provide 1.5 per unit starting torque, the autotransformer ratio (%) should be **[2007]**

(A) 57.77% (B) 72.56%

(C) 78.25% (D) 81.33%

14. If a star-delta starter is used to start this IM, the per unit starting torque will be **[2007]**

(A) 0.607 (B) 0.816

(C) 1.225 (D) 1.616

15. If a starting torque of 0.5 per unit is required then. The per unit starting current should be **[2007]**

(A) 4.65 (B) 3.75

(C) 3.16 (D) 2.13

16. A 230 V, 50 Hz, 4-pole, single-phase IM is rotating in the clockwise (forward) direction at a speed of 1425 rpm. If the rotor resistance at standstill is 7.8 Ω

then the effective rotor resistance in the backward branch of the equivalent circuit will be **[2008]**

(A) 2 Ω (B) 4 Ω
(C) 78 Ω (D) 156 Ω

17. A 400 V, 50 Hz, 30 hp, three-phase IM is drawing 50 A current at 0.8 power factor lagging. The stator and rotor copper losses are 1.5 kW and 900 W, respectively. The friction and windage losses are 1050 W and the core losses are 1200 W. The air-gap power of the motor will be **[2008]**

(A) 23.06 kW (B) 24.11 kW
(C) 25.01 kW (D) 26.21 kW

18. A 400 V, 50 Hz, 4-pole, 1400 rpm, star-connected SCIM has the following parameters referred to the stator.

$$R_r' = 1.0\ \Omega, X_s = X_r' = 1.5\ \Omega.$$

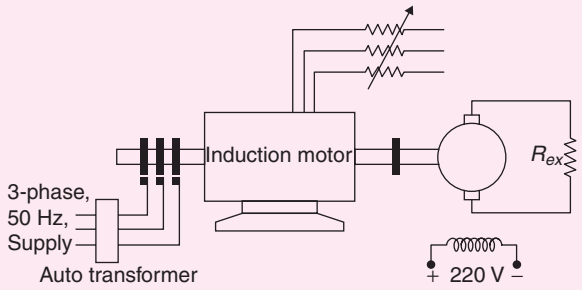
Neglect stator resistance and core and rotational losses of the motor.

The motor is controlled from a 3-phase voltage source inverter with constant V/f control. The stator line-to-line voltage (rms) and frequency to obtain the maximum torque at starting will be **[2008]**

(A) 20.6 V, 2.7Hz
(B) 133.3 V, 16.7 Hz
(C) 266.6 V, 33.3 Hz
(D) 323.3 V, 40.3 Hz

Common Data for Questions 19 and 20:

A 3-phase, 440 V, 50 Hz, 4-pole, slip-ring IM is fed from the rotor side through an auto-transformer and the stator is connected to a variable resistance as shown in the figure.



The motor is coupled to a 220 V, separately excited DC generator feeding power to fixed resistance of 10 Ω. Two-wattmeter method is used to measure the input power to IM. The variable resistance is adjusted such that the motor runs at 1410 rpm and the following readings were recorded.

$$W_1 = 1800\ \text{W}, W_2 = -200\ \text{W}$$

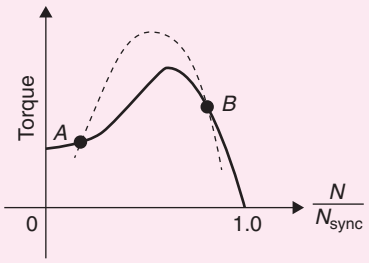
19. The speed of rotation of stator magnetic field with respect to rotor structure will be **[2008]**

(A) 90 rpm in the direction of rotation
(B) 90 rpm in the opposite direction of rotation
(C) 1500 rpm in the direction of rotation
(D) 1500 rpm in the opposite direction rotation.

20. Neglecting all losses of both the machines, the DC generator power output and the current through resistance (R_{ex}) will, respectively, be **[2008]**

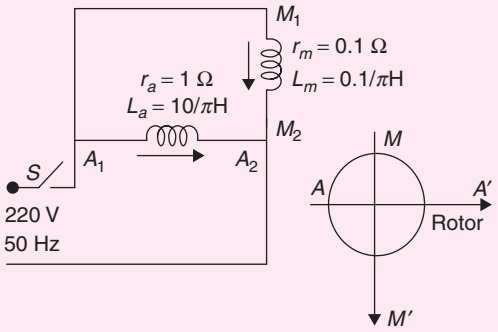
(A) 96 W, 3.10A (B) 120W, 3.46 A
(C) 1504 W, 12.26 A (D) 1880 W, 13.71 A

21. A 3-phase SCIM supplied from a balanced 3-phase source drives a mechanical load. The torque-speed characteristics of the motor (solid curve) and of the load (dotted curve) are shown. Of the two equilibrium points A and B, which of the following options correctly describes the stability of A and B? **[2009]**



(A) A is stable B is unstable
(B) A is unstable B is stable
(C) Both are stable
(D) Both are unstable

22. A 220 V, 50 Hz, single-phase IM has the following connection diagram and winding orientations shown. MM' is the axis of the main stator winding (M_1M_2) and AA' is that of the auxiliary winding (A_1A_2). Directions of the winding axes indicate direction of flux when currents in the windings are in the directions shown. Parameters of each winding are indicated. When switch S is closed, the motor **[2009]**



(A) rotates clockwise
(B) rotates anticlockwise
(C) does not rotate
(D) rotates momentarily and comes to a halt

23. A balanced three-phase voltage is applied to a star-connected IM, the phase to neutral voltage being V. The stator resistance, rotor resistance referred to the stator, stator leakage reactance, rotor leakage reactance referred to the stator, and the magnetizing

reactance are denoted by r_s, r_r, x_s, x_r and X_m, respectively. The magnitude of the starting current of the motor is given by **[2010]**

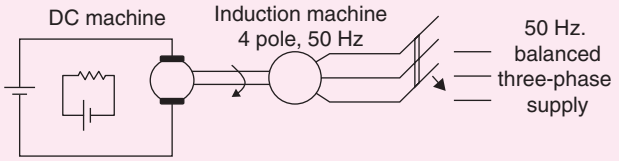
(A) $\dfrac{V}{\sqrt{(r_s+r_r)^2+(x_s+x_r)^2}}$

(B) $\dfrac{V}{\sqrt{r_s^2+(x_s+X_m)^2}}$

(C) $\dfrac{V}{\sqrt{(r_s+r_r)^2+(X_m+x_r)^2}}$

(D) $\dfrac{V}{\sqrt{r_s^2+(X_m+x_r)^2}}$

24. A separately excited DC machine is coupled to a 50 Hz, three-phase, 4-pole induction machine as shown in the figure. The DC machine is energized first and the machines rotate at 1600 rpm. Subsequently the induction machine is also connected to a 50 Hz, three-phase source, the phase sequence being consistent with the direction of rotation. In steady state, **[2010]**



(A) Both machines act as generators
(B) The DC machine acts as a generator, and the induction machine acts as a motor
(C) The DC machine acts as a motor, and the induction machine acts as a generator
(D) Both machines act as motors

25. A three-phase 440 V, 6-pole, 50 Hz, SCIM is running at a slip of 5%. The speed of stator magnetic field with respect to rotor magnetic field and speed of rotor with respect to stator magnetic field are. **[2011]**

(A) zero, −50 rpm (B) zero, 950 rpm
(C) 1000 rpm, −5 rpm (D) 1000 rpm, 950 rpm

26. The slip of an IM normally does not depend on **[2012]**

(A) rotor speed
(B) synchronous speed
(C) shaft torque
(D) core-loss component

27. The locked rotor current in a 3-phase, star-connected 15 kW, 4-pole, 230 V, 50 Hz IM at rated conditions is 50 A. Neglecting losses and magnetizing current, the approximate locked rotor line current drawn when the motor is connected to a 236 V, 57 Hz supply is **[2012]**

(A) 58.5 A (B) 45.0 A
(C) 42.7 A (D) 55.6 A

28. Leakage flux in an IM is **[2013]**

(A) flux that leaks through the machine
(B) flux that links both stator and rotor windings
(C) flux that links none of the windings
(D) flux that links the stator winding or the rotor winding but not both

29. A 4-pole IM, supplied by a slightly unbalanced three-phase 50 Hz source, is rotating at 1440 rpm. The electrical frequency in Hz of the induced negative sequence current in the rotor is **[2013]**

(A) 100 (B) 98
(C) 52 (D) 48

30. An 8-pole, 3-phase, 50 Hz IM is operating at a speed of 700 rpm. The frequency of the rotor current of the motor in Hz is _____. **[2014]**

31. A 3-phase, 50 Hz, 6-pole IM has a rotor resistance of 0.1 Ω and reactance of 0.92 Ω. Neglect the voltage drop in stator and assume that the rotor resistance is constant. Given that the full-load slip is 3%, the ratio of maximum torque to full-load torque is **[2014]**

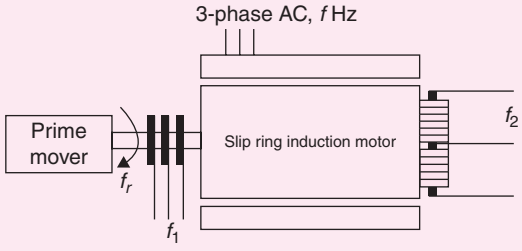
(A) 1.567 (B) 1.712
(C) 1.948 (D) 2.134

32. A three-phase, 4-pole, self-excited induction generator is feeding power to a load at a frequency f_1. If the load is partially removed, the frequency becomes f_2. If the speed of the generator is maintained at 1500 rpm in both the cases, then **[2014]**

(A) $f_1, f_2 > 50$ Hz and $f_1 > f_2$
(B) $f_1 < 50$ Hz and $f_2 > 50$ Hz
(C) $f_1, f_2 < 50$ Hz and $f_2 > f_1$
(D) $f_1 > 50$ Hz and $f_2 < 50$ Hz

33. A single-phase IM draws 12 MW power at 0.6 lagging power. A capacitor is connected in parallel to the motor to improve the power factor of the combination of motor and capacitor to 0.8 lagging. Assuming that the real and reactive power drawn by the motor remains same as before, the reactive power delivered by the capacitor in MVAR is _____ **[2014]**

34. A three-phase slip-ring IM, provided with a commutator winding, is shown in the figure. The motor rotates in clockwise direction when the rotor windings are closed.

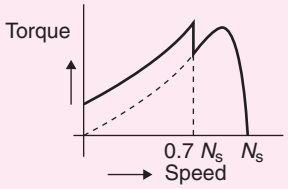


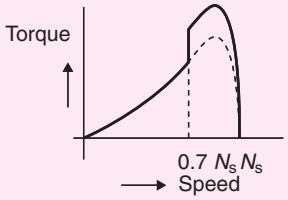
If the rotor winding is open-circuited and the system is made to run at rotational speed f_r with the help of prime-mover in anticlockwise direction, then the frequency of

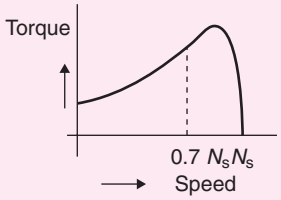
voltage across slip rings is f_1 and frequency of voltage across commutation brushes is f_2. The values of f_1 and f_2, respectively, are **[2014]**

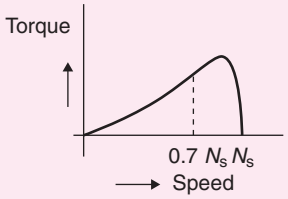
(A) $f + f_r$ and f (B) $f - f_r$ and f
(C) $f - f_r$ and $f + f_r$ (D) $f + f_r$ and $f - f_r$

35. A single-phase IM is provided with capacitor and centrifugal switch in series with auxiliary winding. The switch is expected to operate at a speed of 0.7 Ns, but due to malfunctioning the switch fails to operate. The torque-speed characteristic of the motor is represented by **[2014]**

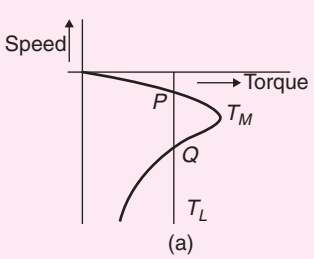
(A)


(B)


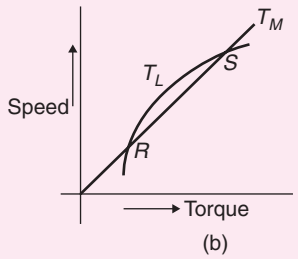
(C)


(D)


36. The torque-speed characteristics of motor T_M and load (T_L) for two cases are shown in the figures (a) and (b). The load torque is equal to motor torque at points P, Q, R and S



(a)



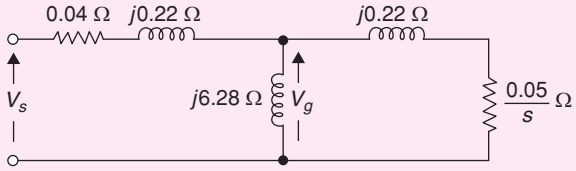
(b)

The stable operating points are **[2014]**

(A) P and R (B) P and S
(C) Q and R (D) Q and S

37. The figure shows the per-phase equivalent circuit of a two-pole three-phase induction motor operating at 50 Hz. The "air-gap" voltage, V_g across the magnetizing inductance, is 210 V rms, and the slip, s, is 0.05. The torque (in Nm) produced by the motor is _____.

[2015]



38. A 220 V, 3-phase, 4-pole, 50 Hz induction motor of wound rotor type is supplied at rated voltage and frequency. The stator resistance, magnetizing reactance, and core loss are negligible. The maximum torque produced by the rotor is 225% of full load torque and it occurs at 15% slip. The actual rotor resistance is 0.03 Ω/phase. The value of external resistance (in Ohm) which must be inserted in a rotor phase if the maximum torque is to occur at start is _____.

[2015]

39. The direction of rotation of a single-phase capacitor run induction motor is reversed by **[2016]**

(A) Interchanging the terminals of the AC supply
(B) Interchanging the terminals of the capacitor
(C) Interchanging the terminals of the auxiliary winding
(D) Interchanging the terminals of both the windings.

40. The starting line current of a 415V, 3-phase, delta connected induction motor is 120A, when the rated voltage is applied to its stator winding. The starting line current at a reduced voltage of 110V, in ampere is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** B	**3.** B	**4.** B	**5.** B	**6.** C	**7.** D	**8.** B	**9.** A	**10.** B
11. C	**12.** A	**13.** A	**14.** C	**15.** C	**16.** A	**17.** C	**18.** B	**19.** B	**20.** B
21. D	**22.** D	**23.** D	**24.** B	**25.** C	**26.** D	**27.** D			

Practice Problems 2

1. B	**2.** C	**3.** A	**4.** B	**5.** B	**6.** D	**7.** A	**8.** C	**9.** C	**10.** C
11. B	**12.** C	**13.** B	**14.** D	**15.** B	**16.** D	**17.** C	**18.** A	**19.** C	**20.** A
21. C	**22.** D	**23.** C	**24.** B	**25.** B	**26.** C	**27.** C	**28.** C	**29.** B	**30.** B
31. C	**32.** B	**33.** D	**34.** C	**35.** B	**36.** C	**37.** D	**38.** C	**39.** A	**40.** C
41. D	**42.** C	**43.** C	**44.** D	**45.** B	**46.** A	**47.** D	**48.** A	**49.** C	**50.** A

Previous Years' Questions

1. C	**2.** A	**3.** B	**4.** D	**5.** B	**6.** B	**7.** A	**8.** B	**9.** C	**10.** A
11. B	**12.** D	**13.** C	**14.** B	**15.** C	**16.** A	**17.** C	**18.** B	**19.** D	**20.** C
21. A	**22.** B	**23.** A	**24.** C	**25.** A	**26.** D	**27.** B	**28.** D	**29.** B	**30.** 3.35
31. C	**32.** C	**33.** 7	**34.** A	**35.** C	**36.** B	**37.** 400 to 403		**38.** 0.16 to 0.18	
39. C	**40.** 31.8								

Chapter 5

Synchronous Machines

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Armature windings
- Damper windings
- EMF equation
- Flux and MMF phasors in synchronous machines
- Operation of a salient pole synchronous machine
- Slip test
- Parallel operation of alternators
- Synchronizing power and synchronizing torque
- Synchronous motors
- Servomotor
- Stepper motor

Introduction

A synchronous machine is a doubly-excited AC machine, because its field winding is energized from a DC source and its armature winding is connected to an AC source. When operating as a motor, the synchronous machine absorbs active power from an AC source. During its operation as a generator, a synchronous machine delivers AC power. However, the field winding of a synchronous machine always absorbs power from a DC source.

Since a synchronous generator delivers AC output, it is also known as an alternator. Alternators operate on the same fundamental principles of electromagnetic induction as DC generators. In large synchronous machines, the field winding is always provided on the rotor because of the following advantages:

1. The output current can be led directly from fixed terminals on the stator to the load circuit.
2. Insulation of stationary armature winding for high AC voltage is easier compared to rotating armature.
3. The sliding contacts i.e. slip rings are transferred to the low-voltage, low power DC field circuit which can, therefore, be easily insulated.
4. The armature conductors can be made easily braced to prevent any deformation which could be produced by the high centrifugal forces or the mechanical stresses set up as a result of short-circuited currents.

- Small AC generators and of low voltage rating are usually made of rotating armature. In such generators the required magnetic field is produced by DC electromagnet placed on the stator.

  In large alternators placed on the excitation is usually provided from a small DC shunt or compound generator mounted on the shaft of alternator itself.

- Synchronous machines are of two types depending upon the geometrical structure of the rotor, viz., (i) salient pole or projected pole and (ii) cylindrical-rotor, round-rotor or non-salient pole type.
- The cylindrical construction is used for two or four-pole steam-turbine generators. That is why these are called turbo-generators. A cylindrical-rotor synchronous machine is characterized by long core length and small diameter so as to limit the centrifugal forces developed in the high-speed rotor.
- The salient-pole construction is the most suitable for multipolar slow-speed water-turbine generators. That is why salient-pole synchronous generators are called hydro-generators. A salient-pole synchronous machine has small core length and large diameter so that large number of field poles can be accommodated on the rotor periphery.

The frequency of the induced emf is given by f = No. of cycles/revolution × No. of revolutions/second

$$= \frac{P}{2} \times \frac{N}{60} = \frac{PN}{120} \text{ Hz}$$

Armature Windings

The armature windings in DC machines are closed winding but in case of alternator, windings are open, in the sense that there is no closed path for the armature currents in the winding itself. Hence there are no parallel paths for an AC winding.

The two types of armature windings commonly used for 3-phase alternators are

1. Single-layer winding
2. Double-layer winding

There are two types of interconnection of the 3-phase windings. One *Y* and other Δ. If similar terminals are joined, we get a star configuration and for delta connection dissimilar ends are connected together.

Damper Windings

Damper windings consist of low-resistance bars embedded in slots in the pole-faces of salient-pole machines. The projecting ends of the bars are short-circuited at both ends by strips of the same material as used for bars. Damper windings are not used on turbo-generators. But the solid-steel rotor cores of such machines provide path for eddy currents, especially in the quadrature axis, where the iron may form an equivalent circuit, thus producing the same effects as those of damper bars. These damper bars are useful in preventing the hunting (monetary speed fluctuations) in generators and are needed in synchronous motors to provide the starting torque. Under normal running conditions, damper bars do not carry any current because rotor runs at synchronous speed.

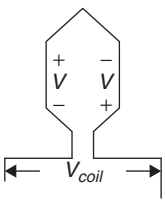
Short-pitch winding: Pitch factor/chording factor.

Short-pitched coils are used owing to the following advantages.

1. Saving of copper due to reduced length of end connections.
2. Waveform of generated emf is improved as the distorting harmonics are reduced or totally eliminated.
3. Due to the elimination of high frequency harmonics, eddy currents and hysteresis losses are reduced thereby increasing the efficiency.

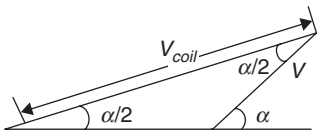
Disadvantages

Use of short-pitched coil results in decrease of induce emf. In full pitched coils, emf's induced by two coils sides are exactly 180° out of phase:



Hence $V_{coil} = 2$ V

But, in case of short-pitched coils, this angle is $180 - \alpha$ ∴ (say)



$$\therefore \qquad \text{Hence } V_{coil} = 2V\cos\frac{\alpha}{2}$$

$$\text{Pitch factor} = \frac{\text{Resultant emf of short pitched coil}}{\text{Resultant emf of full coil}}$$

$$= \frac{2V\cos\alpha/2}{2V} = \cos\alpha/2 \quad (\alpha \text{ in electrical degrees})$$

Distribution or Breadth Factor or Winding Factor or Spread Factor

In practice, not all coil slots are used. The winding is neither fully distributed in all slots, nor fully concentrated under on pole. This is indicated by distribution factor:

The distribution factor (K_d) is defined as

$$K_d = \frac{\text{e.m.f with distributed winding}}{\text{e.m.f with concentrated winding}}$$

(or)

$$\frac{\text{Vector sum of coil e.m.f. s}}{\text{Arithmetic sum of coil e.m.f. s}}$$

$$K_d = \frac{\sin m\beta/2}{m\sin\beta/2}$$

where β is slot angle

$$\beta = \frac{180°}{\text{No. of slots/pole}}$$

m = No. of slots/pole/phase

$m\beta$ is phase spread angle.

Solved Examples

Example 1: A 3-phase, 16-pole alternator has 144 slots. What is the distribution factor?

Solution: Distribution factor, $K_d = \dfrac{\sin(m\beta/2)}{m\sin\beta/2}$

n = no. of slots per pole

m = no. of slots per pole/pole

β = slot angle

$$n = \frac{144}{16} = 9; \qquad m = \frac{144}{3\times16} = 3$$

$$\beta = \frac{180}{9} = 20°$$

$$K_d = \frac{\sin(3\times20/2)}{3\sin\dfrac{20}{2}} = 0.96$$

Example 2: What is the distribution factor for a uniformly distributed 3-phase winding having a phase spread of 60°?

Solution: For a uniformly distributed winding, distribution factor is given by

$$k_d = \frac{\sin \sigma/2}{\sigma/2}$$

where σ = Phase spread in rad.

For $\sigma = 60°$

$$k_d = \frac{\sin 30°}{30 \times \frac{\pi}{180}} = 0.955$$

EMF Equation

The induced emf (RMS value) of an alternator is given by

$$E/\text{Ph} = 4.44 K_p K_d f \phi T \ldots\ldots V$$

where K_p = Pitch factor

K_d = Distribution factor

ϕ = Flux per pole in webers

T = Number of turns per phase

Effect of Harmonics on Pitch and Distribution Factors

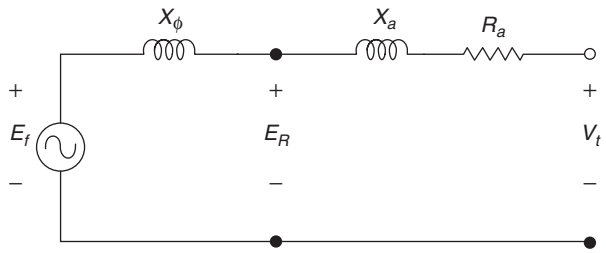
For an n^{th}-order harmonic

$$K_c = \cos \frac{n\alpha}{2} \text{ and}$$

$$K_d = \frac{\sin nm\beta/2}{m \sin n\beta/2}$$

Flux and MMF Phasors in Synchronous Machines

Synchronous machine equivalent circuit:



where $X_s = X_\Phi + X_a$

X_a = Armature Leakage Reactance

X_Φ = Reactance of air gap

R_a = Armature Resistance

E_f = emf generated

V_t = Terminal voltage

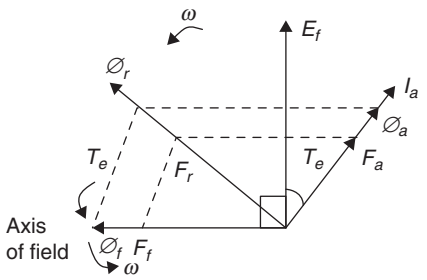


Figure 1 Axis of phase *a*

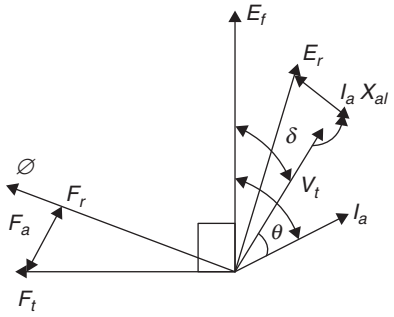
Combined space and time phasor diagram of a cylindrical rotor alternator for a lagging p.f. load.

In an alternator

1. If I_a lags the excitation emf, E_f by 90° the nature of F_a is demagnetizing.
2. If I_a leads the excitation emf, E_f by 90°, the nature of armature F_a is magnetizing.

Phasor Diagram of a Cylindrical Rotor Alternator

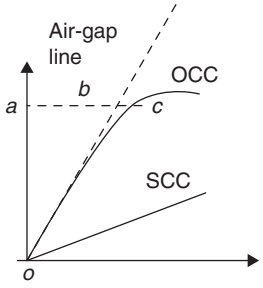
The phasor diagram is drawn assuming that the per phase quantities, i.e. V_t the terminal voltage, I_a the armature current, r_a the armature resistance, X_{al} the armature leakage reactance and cos θ power factor, are known.



Characteristics of an Alternator

Open-circuit Characteristic (OCC)

For obtaining OCC, the alternator is driven at constant rated speed and the open-circuit terminal voltage is noted as the field current is increased from zero. Thus the OCC is a graph between the field current I_f or field mmf F_f and the induced voltage. It is just like the B-H curve.



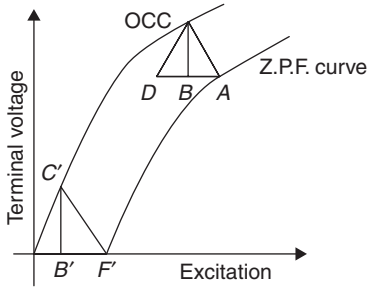
Short-circuit Characteristics (SCCs)

SCC is obtained by short-circuiting the armature (i.e. stator) windings through a low-resistance ammeter. The field current is gradually increased from zero, until the short-circuit armature current has not reached its maximum safe value, equal to 125% to 150% of the rated current. During this test, the speed which is not necessarily synchronous is kept constant.

Zero Power-factor Characteristic and Potier Triangle

Zero power factor characteristic (z.p.f.c.) of an alternator is a plot between the armature terminal voltage and its field current for constant values of armature current and speed. z.p.f.c. in conjunction with o.c.c is useful in obtaining the armature leakage reactance X_{al} and armature reaction mmf F_a. For an alternator z.p.f.c. is obtained as follows:

1. The synchronous machine is run at rated synchronous speed by the prime mover.
2. A purely inductive load (z.p.f. load) is connected across the armature terminals and field current is increased till full-load armature current is flowing.
3. The load is varied in steps and the field current at each step is adjusted to maintain armature current. The plot of armature terminal voltage and field current recorded at each step gives the zero-power factor characteristic at full-load armature current.



The triangle ABC shown above is known as Potier triangle. If the armature resistance is assumed zero and the armature current is kept constant, then the size of Potier triangle ABC remains constant and can be shifted parallel to itself with its corner remaining on the OCC and its corner A, tracing the z.p.f.c. Thus the z.p.f.c has the same shape as the O.C.C. and is shifted vertically down-ward by an amount equal to $F_a X_{al}$ (i.e. leakage reactance drop) and horizontally to the right by an amount equal to the armature reaction mmf F_a or the field current equivalent to the armature reaction mmf, i.e. BC = $I_a X_{al}$ and AB = F_a. For determining X_{al} and F_a experimentally, entire z.p.f.c. is not necessary, but only two points A and F′ are sufficient. The point A (PA = rated voltage) is obtained actually loading the over-excited alternator by an under-excited synchronous motor, so that rated armature current flows in the alternator. The point F' on the z.p.f.c. corresponds to the zero terminal voltage and can, therefore, be obtained by performing short-circuit test. So here OF′ is the field current required to circulate short-circuit current equal to the armature current at which point A is determined in the zero power factor test.

Now draw a horizontal line AD, parallel and equal to F′O. Through point D, draw a straight line parallel to the air-gap line, intersecting the O.C.C. at C. Draw CB perpendicular to AD. Then from the Potier triangle ABC

$$\text{BC} = I_a X_{al} \qquad \text{and} \qquad \text{AB} = F_a$$

Since I_a at which point A is obtained is known, X_{al} can be calculated.

Voltage Regulation of an Alternator

It is defined as the 'rise in voltage when full-load is removed (field excitation and speed remaining the same)', expressed as a percentage (or p.u.) of the rated voltage.

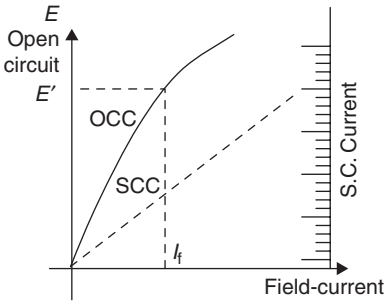
$$\therefore \text{ Voltage regulation} = \frac{E_f - V_t}{V_t} \text{ in p.u.}$$

$$= \frac{E_f - V_t}{V_t} \times 100 \text{ in percentage.}$$

The following are the methods to determine voltage regulation.

Synchronous Impedance or EMF Method

In this method, OCC and SCC of given alternator are obtained and are drawn on a common field current base.



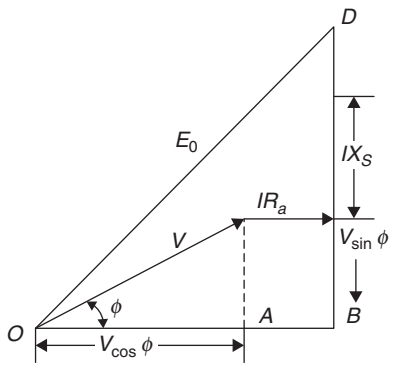
Consider a field current I_f. The OC voltage corresponding to this field current is E_1. When the armature winding is short-circuited, whole of this voltage E_1 is used to circulate the armature short-circuit current I_1, against the synchronous impedance Z_s

$$E_1 = I_1 Z_s; \quad \therefore Z_s = \frac{E_1 (\text{open-circuit})}{I_1 (\text{short-circuit})}$$

After determining R_a, X_s can be calculated as

$$X_s = \sqrt{Z_s^2 - R_a^2}$$

Knowing R_a and X_s, the phasor diagram can be drawn for any load and any power factor.



$$OD = E_0$$

$$\therefore \quad E = \sqrt{OB^2 + BD^2}$$

$$E_0 = \sqrt{(V\cos\phi + IR_a)^2 + (V\sin\phi + IX_s)^2}$$

$$\text{and \% regulation} = \frac{E_0 - V}{V} \times 100$$

For leading p.f, ϕ is taken –ve and +ve for lagging p.f.

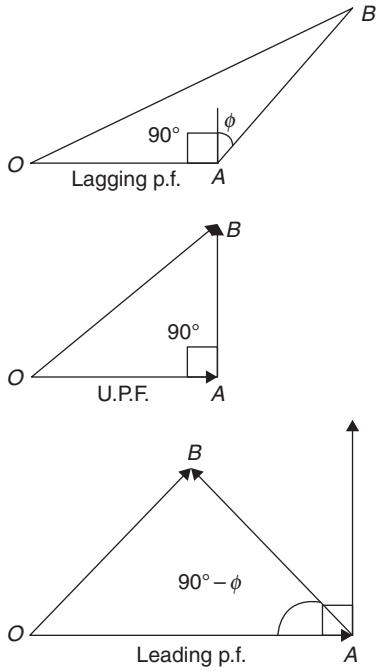
The synchronous impedance Z_s calculated by this method is higher than actual value. Hence the value of regulation so obtained is always more than that found from an actual test. That is why it is called pessimistic method.

Ampere-turn or MMF Method

- In this method, OC and SC data are used.
- The voltage is attributed entirely to armature reaction (i.e. armature leakage reaction drop, which is rather small, is clubbed with armature reaction). In other words, it is assumed that the change in terminal voltage on load is entirely due to armature reaction. Normally R_a is neglected.
- Since R_a is neglected and is small for the low voltage applied on short circuit, the p.f. may be assumed to be zero lagging.
- Therefore, the field mmf is used to overcome the demagnetizing effect of armature reaction.
- The excitation required to overcome the demagnetizing effect of armature reaction is determined on the unsaturated portion of the OCC.
- This method is optimistic since the regulation calculated by this method is less than the actual value.

Determination of Voltage Regulation, by MMF Method

1. The field current I_{f1} corresponding to rated voltage is obtained from the OCC.
2. The field current I_{f2}, to circulate rated armature current I_a, is obtained from the SCC.
3. $I_f \cos\phi$ is the load p.f., I_{f2} is added vectorially at an angle of $(90 \pm \phi)$ with I_{f1} to get the resultant field current I_f as shown in following figures.



$$OA = I_{f1},\ AB = I_{f2}$$

$$OB = I_f,\ \phi = \text{Power factor angle}$$

The emf E, corresponding to I_f is read from the OCC

$$\text{\% Regulation} = \frac{E - V}{V} \times 100$$

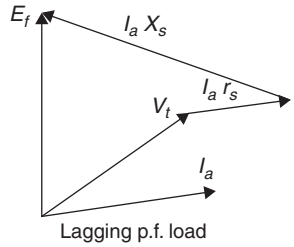
Zero Power Factor or Potier Triangle Method

In the EMF and MMF methods, the effects of armature leakage reactance and armature reaction are clubbed. ZPF method is based on the separation of armature leakage reactance drop and the armature reaction effects. Hence the regulation obtained by this method is more accurate. In addition to no-load curve, full-load ZPF curve is also required.

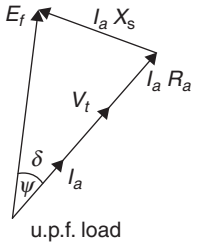
New A.S.A. (American Standard Association) Method

This method is essentially a modification of the MMF method and gives satisfactory results both for cylindrical and salient-pole machines. New A.S.A. methods require O.C.C. and z.p.f.c.

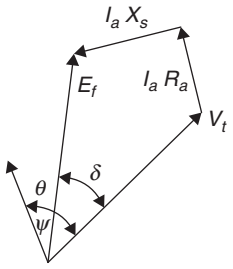
Voltage Phasor Diagram for an Alternator



Lagging p.f. load



u.p.f. load



Leading p.f. load

For lagging p.f. load

$$E = \sqrt{(V_t \cos\theta + I_a r_a)^2 + (V_t \sin\theta + I_a X_s)^2}$$

For u.p.f. load

$$E = \sqrt{(V_t + I_a r_a)^2 + (I_a X_s)^2}$$

For leading p.f. load

$$E = \sqrt{(V_t \cos\theta + I_a r_a)^2 + (V_t \sin\theta - I_a X_s)^2}$$

Example 3: A three-phase, Y-connected, 1000 kVA, 2000 V, 50 Hz alternator has the following readings:

I_F (A)	10	20	25	30	40	50
OC (V_{LL})	800	1500	1760	2000	2350	2600
SCI$_a$ (A)		200	250	300		
OC V_P	462	866	1016	1155	1357	1501

$R_a = 0.2\,\Omega$ /phase

Find the full-load voltage regulation at 0.8 p.f. leading.

Solution: Full-load phase voltage $= \dfrac{2000}{\sqrt{3}} = 1155$ V

$$\text{KVA} = \frac{\sqrt{3}\, V_{LL} I_{fl}}{1000}$$

$$\Rightarrow 1000 \times 1000 = \sqrt{3} \times 2000 \times i_{fl}$$

$$\Rightarrow I_{fc} = 288.7 = I_a$$

at 0.8 leading

$$E = V_p + I_a R_a = 1155 + (288.7 \times 0.2 \angle \cos^{-1}(0.8))$$
$$= 1155 + 46.2 + j34.44 = 1201.7 \angle 1.6°$$
$$I_{FI} = I_F \angle(90 + \alpha) = 32 \angle(90 + 1.6°)$$

(Ampere-turns method)

$$= -0.89 + j\,31.98 \text{ A}$$
$$I_{f2} = 29 \angle(180 + 36.87°) = 29 \angle 216.87° \text{A}$$
$$I_f = I_{f1} + I_{f2} = 28.15 \angle 131.18°$$

at this $I_{f'}$ $E = 1098$ V

$$\Rightarrow V_R = \frac{10998 - 1155}{1155}$$

$$= -5.02\%$$

Example 4: A 3-phase alternator delivers power to a balanced 3-phase load of power factor 0.8 lagging. It is observed that the open-circuit emf phasor leads the corresponding terminal voltage phasor by 15°. Neglecting the effect of harmonics, what is the angle between the axis of the main field mmf and the axis of the armature mmf?

Solution: Angle between E_f and $\phi_{ar} = \delta + \phi$

$$= 15 + \cos^{-1}(0.8) = 51.86°$$

[where δ is load angle and ϕ is power factor angle]

Example 5: A 3-phase, 400 V, delta-connected alternator has a synchronous impedance of $(0 + j20)$ ohm per phase. If it delivers a load of 12 kVA at zero power factor leading, then what is the value of percentage voltage regulation.

Solution: Load $= 12 \times 10^3$ VA $= \sqrt{3} \times V_L I_L$

$$\text{Line current } I_L = \frac{12 \times 10^3}{\sqrt{3} \times 400}$$

$$\text{Phase current} = \frac{I_L}{\sqrt{3}} = \frac{12 \times 10^3}{3 \times 400} = 10 \text{ A}$$

$$\% \text{ Regulation} = \frac{IR\cos\phi - I X \sin\phi}{V}$$

$$\because \theta = 90°$$

$$\% \text{ Regulation} = \frac{-I X \sin 90}{V}$$

$$= \frac{10 \times 20}{400} = -50\%$$

Example 6: A single-phase alternator has a synchronous reactance of 2 ohm and negligible resistance. If it supplies 15 A to a purely capacitive load at 200 V, then what is the value of generated emf?

Solution: Generated emf $E_f = \overline{V}_t + \overline{I}_a X_s$

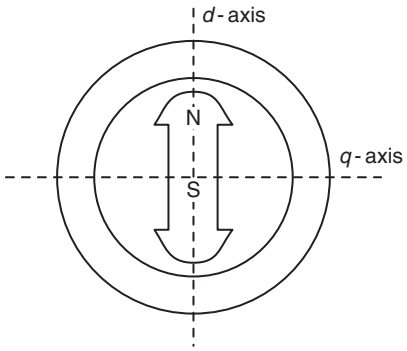
$= 200 + (j15)(j2)$

$= 170$ V

Operation of a Salient-pole Synchronous Machine

Non-salient-pole synchronous machines have a uniform air-gap, because of which its reactance remains the same irrespective of the spatial position of rotor. However a synchronous machine with salient or projected poles has non-uniform air gap due to which its reactance varies with rotor position. A salient-pole machine possesses two-axes of geometric symmetry.

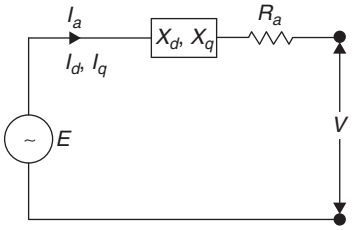
1. Fixed-pole axis called direct axis or *d*-axis
2. Axis passing through the centre of the inter polar space, called the quadratic axis or *q*-axis.



To analyse the operation of a salient-pole machine two-reaction theory is used, according to which the armature mmf, F_a is resolved into two sinusoidal components, one F_{ad} along the *d*-axis and the other F_{aq} along the *q*-axis. So two mmf act on the *d*-axis, i.e. field mmf and armature mmf (F_{ad}) whereas only one mmf armature mmf (F_{aq}) acts on the *q*-axis. The magnetic reluctance is low along the poles and high between the poles.

According to two-reaction theory,

1. armature current I_a can be resolved into two components, i.e. I_d perpendicular to E_0 and I_q along E as shown in figure.



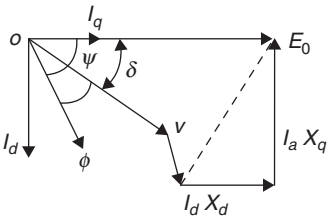
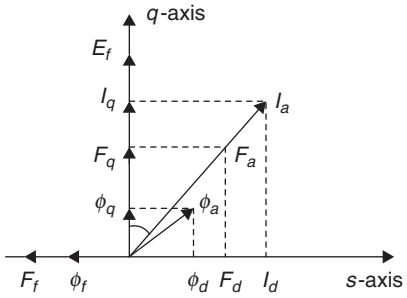


Figure 2 Phasor diagram for a salient-pole alternator

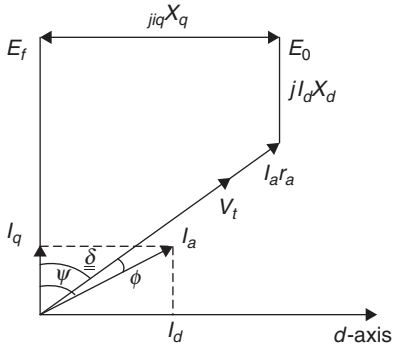
2. armature reactance has two components, i.e. *q*-axis armature reactance X_{ad} associated with I_d and *d*-axis armature reactance X_{aq} associated with I_q.
   If we include the armature leakage reactance X_l, which is the same on both axes, we get
   $X_d = X_{ad} + X_l$
   and $X_q = X_{aq} + X_l$
   Since reluctance is higher on the q-axis, because of the large air-gap, we have
   $X_{aq} < X_{ad}$, $X_q < X_d$ or $X_d > X_q$

Two-reaction Theory

In a cylindrical rotor synchronous machine, the permeance offered to a mmf wave is Independent on rotor position of the wave axis with respect to field-pole axis. But in salient-pole synchronous the permeance offered is different because the rotor alignment is different therefore the mmf vector also different. The mmf vector and induced emf in salient-pole synchronous machine are as shown below. The reactance of direct and quadrature axis are X_d and X_q, respectively, generally $X_d > X_q$.



The phasor diagram of the generated voltage and terminal voltage of salient-pole synchronous machine is as shown in the following figure.



The direct and quadrature axis currents are

$$I_d = I_a \sin\Psi$$

$$I_q = I_a \cos\Psi$$

The generated emf in salient-pole synchronous machine is

$$E_f = V_t + I_a R_a + jX_d I_d + jX_q I_q$$

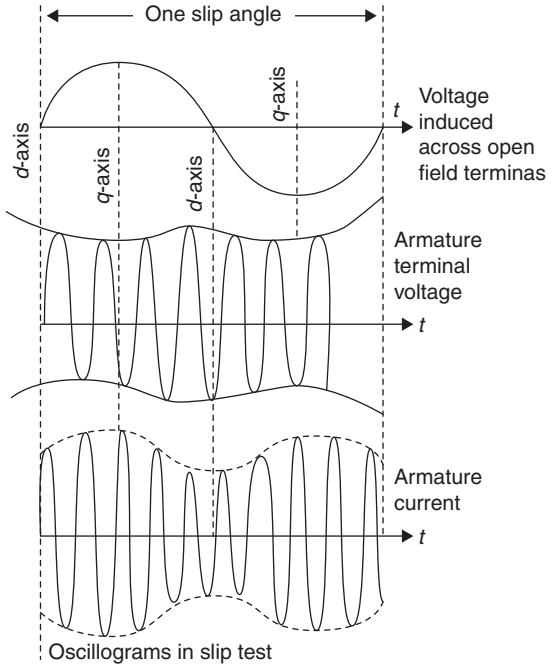
Slip Test

From this test X_d and X_q can be determined. The synchronous machine is driven by a separate prime-mover (or motor) at a speed slightly different from synchronous speed. The field winding is left open, and positive sequence balanced voltages of reduced magnitude (around 25% of rated value) and of rated frequency are impressed across the armature terminals. Under these conditions, the relative velocity between the field poles and the rotating armature mmf wave is equal to the difference between synchronous speed and the rotor speed, i.e. slip speed. A small AC voltage across the open field winding indicates that the field poles and rotating mmf wave are revolving in the same direction - and that is what is required in slip test. If field poles revolve in a direction opposite to the rotating mmf wave, negative sequence reactance would be measured.

The field winding is open-circuited and rotor is rotated at synchronous speed N_s. As the excitation is absent, armature would draw heavy current from terminals. Now, the armature current would vary form minimum (d-axis) and maximum (at the q-axis position).

It would vary at the rate of twice the corresponding slip frequency.

The voltage at the terminal would vary from minimum at (q-axis) to maximum (at d-axis). Oscillations of voltage and current are shown in following figure.



$$= \frac{[\text{Maximum armature terminal voltage per phase}]}{[\text{Minimum armature current per phase}]}$$

and

$$X_q = \frac{[\text{Minimum armature terminal voltage per phase}]}{[\text{Maximum armature current per phase}]}$$

Power Developed by a Synchronous Generator

The per phase power developed by an alternator is given by

$$P_d = \frac{E_0 V}{X_d}\sin\delta + \frac{1}{2}V^2\left[\frac{1}{X_q} - \frac{1}{X_d}\right]\sin 2\delta$$

$$= \frac{E_0 V}{X_d}\sin\delta + \frac{V^2\left(X_d - X_q\right)}{2X_d X_q}\sin 2\delta$$

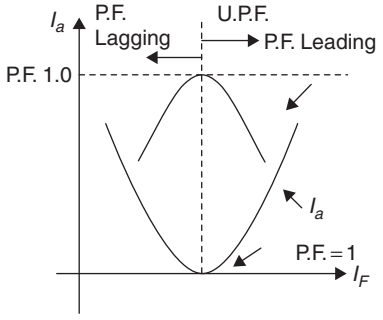
The above expression for P_d consists of two components, the first term represents the power due to field excitation and the second term gives the reluctance power, i.e. power due to saliency.

If $X_d = X_q$, i.e. the machine has a cylindrical rotor, then the reluctance power is zero and the power is given by the first term only. If on the other hand, the field has not excited, i.e. $E_0 = 0$, then the first term in the above expression becomes zero and the power developed is given by second term. The value of δ is positive for a generator and negative for a motor.

Power-factor Control of Alternators

An under excited alternator operates at a leading power factor and absorbs reactive power from infinite bus.

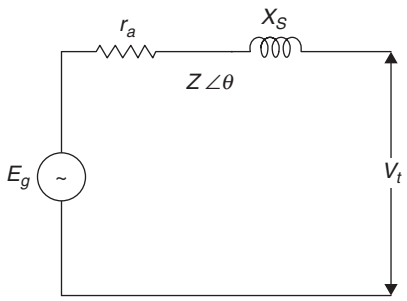
The over excited alternator operates at a lagging p.f. and delivers reactive power to infinite bus.



Effect of field current on an alternator connected to infinite bus is given by the V-curve and inverted V-curve shown in above figure.

Power Angle Characteristics of Synchronous Machine

The power flow in a cylindrical rotor synchronous generator is analysed by



In the above circuit E_g represent the generated voltage in generator and V_t represent the terminal voltage across the generator terminals after subtracting the armature and friction drop from the generated voltage. Therefore the power input (i.e., power developed) in generator

$$P_{in} = \frac{E_g V_t}{Z_s}\sin(\delta - \alpha) + \frac{E_g^2}{Z_s^2} r_a$$

δ – is the angle between generated voltage and terminal voltage.

$$\therefore \quad \theta = \text{impedance angle} = \tan^{-1}\left(\frac{X}{R}\right)$$

$$\alpha = 90 - \theta$$

Maximum power input occurs at $\frac{dP_{in}}{d\delta} = 0$

$$\frac{E_g V_t}{Z_s}\cos(\delta - \alpha) = 0$$

$$\delta = 90 + \alpha = 180 - \theta$$

$\therefore$ Maximum input power is $P_{in}(\max) = \frac{E_g V_t}{Z_s} + \frac{E_g^2}{Z_s^2} r_a$

Power output equation of generator

$$P_{out} = \frac{E_g V_t}{Z_s}\sin(\delta + \alpha) - \frac{V_t^2}{Z_s^2} r_a$$

Maximum real power output obtained at $\frac{dP_{out}}{d\delta} = 0$

$$\frac{E_g V_t}{Z_s}\cos(\delta + \alpha) = 0$$

$$\delta + \alpha = 90^\circ$$

$$\delta = 90^\circ - \alpha = \theta$$

$\therefore$ The corresponding maximum output power is

$$P_{out(\max)} = \frac{E_g V_t}{Z_s} + \frac{V_t^2}{Z_s^2} r_a$$

Similarly Reactive power output at generator terminals

$$Q_{out} = V_t\left[\frac{E_g}{Z_s}\sin(\theta - \delta) - \frac{V_t}{Z_s}\sin\theta\right] \text{(or)}$$

$$Q_{out} = \frac{V_t E_g}{Z_s}\cos(\delta - \alpha) - \frac{V_t^2}{Z_s} X_s$$

Maximum output reactive power obtained at

$$\frac{dQ_{out}}{d\delta} = \frac{V_t E_g}{Z_s}\sin(\delta + \alpha) = 0$$

$$\Rightarrow \quad \delta + \alpha = 0$$

$$\therefore Q_{out(\max)} = \frac{V_t}{Z_s}\left[E_g - \frac{V_t}{Z_s}\cdot X_s\right]$$

$$= \frac{V_t E_g}{Z_s} = \frac{V_t^2}{Z_s} X_s$$

If $r_a = 0$

$$Q_{out} = \frac{V_t E_g}{X_s}\cos\delta - \frac{V_t^2}{X_s}$$

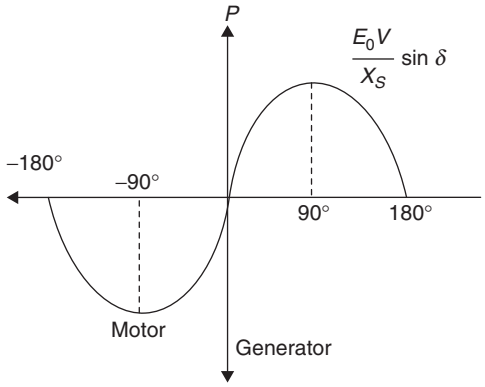
$$= \frac{V_t}{X_s}[E_g \cos\delta - V_t]$$

1. when $E_g \cos\delta > V_t$, then Q_{out} positive. Therefore the generator delivers the reactive power.
2. when $E_g \cos\delta = V_t$, then Q_{out} is zero. Therefore the generator neither absorbs nor delivers reactive power.
3. when $E_g \cos\delta < V_t$, then Q_{out} is negative therefore the generator absorbs the reactive power from bus bar.

For cylindrical rotor synchronous machine

$$P = \frac{E_0 V}{X_s}\sin\delta$$

The power vs. load angle curve has sinusoidal shape and is called power angle characteristic.



The Power-angle Curves of Synchronous Generator

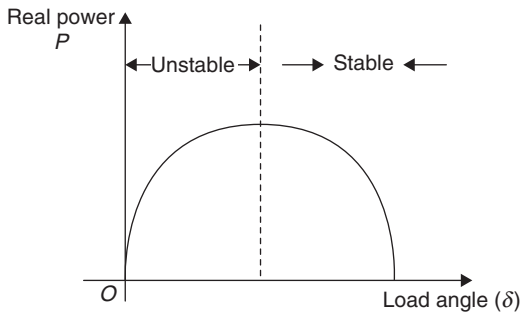


Figure 3 Real power versus load angle characteristic

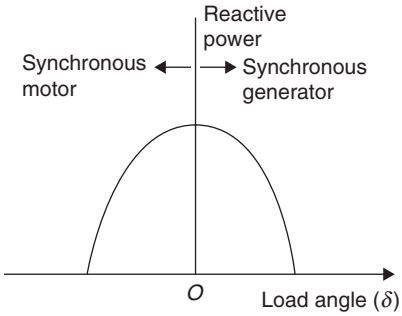
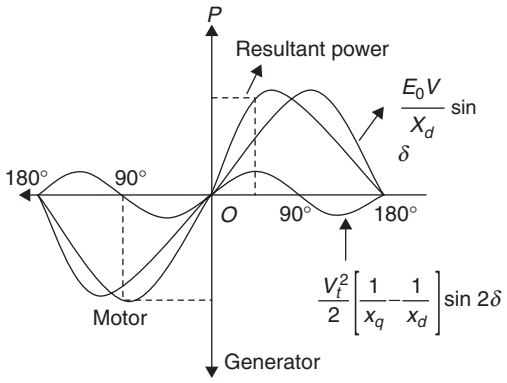


Figure 4 Reactive power versus load angle characteristic

For salient-pole synchronous machine

$$P = \underbrace{\frac{E_0 V}{X_d}\sin\delta}_{\text{Fundamental component}} + \underbrace{\frac{1}{2}V^2\left[\frac{1}{X_d}-\frac{1}{X_q}\right]\sin 2\delta}_{\text{Second harmonic component}}$$



Common data for Example 7:
A 400 V, 4-pole 3-ϕ 50 Hz, induction motor has a rotor resistance and reactance per phase of 0.02 Ω and 0.2 Ω, respectively. Stator-to-rotor turns ratio is 4.

Example 7: A synchronous generator with a synchronous reactance of 1.2 p.u. is connected to an infinite bus whose voltage is 1 p.u., through an equivalent reactance of 0.3 p.u. What should be the value of emf to give a maximum output of 1.2 p.u.?

Solution: $P = \dfrac{E_f V_t}{x}\sin\delta$

$$P_{max} = \frac{E_f V_t}{X}$$

$$E_f = \frac{X P_{max}}{V_t} = \frac{(1.2+0.3)\times 1.2}{1.0} = 1.8 \text{ p.u.}$$

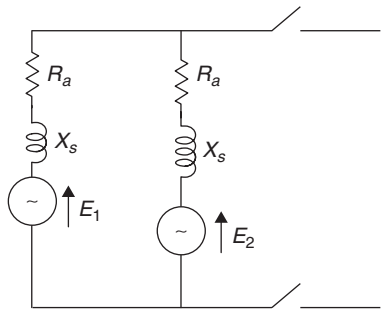
Parallel Operation of Alternators

Connecting an alternator in parallel with another alternator or with common bus-bars is known as synchronizing. Conditions to be satisfied for synchronization of alternators are:

1. The terminal voltage of the incoming alternator must be the same as bus-bar voltage
2. The speed of the incoming machine must be such that its frequency $\left(f = \dfrac{NP}{120}\right)$ equals bus-bar frequency
3. The phase of the alternators must be same as that of bus-bar voltage, it means that the synchronizing switch must be closed at (or very near) the instant the two voltages have correct phase relationship

Synchronizing Current and Synchronizing Power

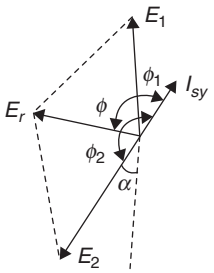
Once synchronized properly, two alternators continue to run in synchronism. Any tendency on the part of one to drop out of synchronism is immediately counteracted by the production of a synchronizing torque, which brings it back to synchronism.



When in exact synchronism, the two alternators have equal terminal voltages and are in exact phase opposition.



But now suppose that E_2 falls back by a phase angle of α electrical degrees as shown in following figure.



Neglecting the resistance R_a, then $\theta = 90°$ and

$$E_r = 2E\sin \alpha/2$$

since α is small (α is in electrical radians)

$$I_{sy} = \frac{E_r}{2X_s} = \frac{\alpha E}{2X_s}$$

From the above phasor diagram, it can be seen that machine no. 1 is generating and supplying synchronizing power whereas machine no. 2 acts a motor and receives the synchronizing power. Hence I_{sy} is generating current with respect to machine no. 1 and motoring current with respect to machine no. 2

$$P_{sy} = E_1 I_{sy}\cos \phi_1 = EI_{sy}\cos (90 - \theta)$$
$$= EI_{sy} \sin \theta = EI_{sy} (\because R_a \text{ is neglected and } \theta = 90°)$$

Substituting the value of I_{sy}

$$P_{sy} = E.\alpha E/2X_s$$
$$= \alpha E^2/2X_s \text{ per phase}$$

$$\left(\text{More accurately, } P_{sy} = \frac{\alpha E^2 \sin\theta}{2X_s}\right)$$

This is the value of the synchronizing power when two alternators are connected in parallel and are on no-load.

Alternators Connected to Infinite Bus-bars

In this case impedance (or reactance) of only one alternator has to be considered

$$\therefore \quad I_{sy} = \frac{\alpha E}{X_s}$$

$$P_{sy} = \frac{\alpha E^2}{X_s}$$

$$\left[\text{or } P_{sy} = \frac{\alpha E^2 \sin\theta}{X_s}\right]$$

Synchronizing torque (T_{sy}) can be calculated from the value of synchronizing power by using relation

$$\omega \times T_{sy} = P_{sy}$$

$$\text{i.e. } T_{sy} = \frac{P_{sy}}{\omega}$$

Synchronizing Power and Synchronizing Torque

The rate at which synchronous power, 'P' varies with δ is called the synchronizing-power coefficient P_{sy}. It is also known as stiffness of coupling, rigidity factor or stability factor.

The coefficient P_{sy} is equal to the slope of the power angle curve.

For a cylindrical rotor machine

$$P_{sy} = \frac{dP}{d\delta} = \frac{EV}{X_s}\cos\delta$$

$$P_{sy} = \frac{dP}{d\delta} = \frac{EV}{X_d}\cos\delta + V^2\left[\frac{1}{X_q} - \frac{1}{X_d}\right]\cos 2\delta$$

From the above equations it is obvious that P_{sy} is inversely proportional to X_s or X_d and is directly proportional to E, the excitation voltage.

Hence, an over excited synchronous machine is more rigidly coupled than the one which is under excited. Large air gap decreases the value of X_s or X_d, thus a synchronous machine with longer air gap is more stiffer than the one with smaller air-gap units of synchronizing power coefficient are watt per electrical radian.

The variation of synchronous power with the change of load angle is called the synchronizing power. It exists only during the transient state, i.e. whenever there is a sudden disturbance in load (or steady-state operating conditions). Once the steady-state is reached, the synchronizing power reduces to zero.

The synchronizing power flows from or to the bus in order to maintain the relative velocity between interacting stator and rotor field, zero, once the equality is reached, the synchronizing power vanishes.

Hunting

The operation of a synchronous machine is satisfactory, if the mechanical speed of the rotor is equal to the stator field speed i.e. if the relative speed between the rotor and stator field is zero. Any departure from these conditions gives rise to synchronizing forces which tend to maintain this equality. With a sudden change in the load, the rotor swings or oscillates first to one side and then to other side of the new equilibrium position of load angle. This phenomenon involving the oscillations of the rotor about its final equilibrium position, is called hunting. During the rotor oscillation or hunting, the orientation of phasor, E_f changes relative to terminal voltage, V_t and because of this reason, hunting is also called phase-swinging.

Common data for Example 8:

A 3-ϕ, star-connected 400 V, 50 Hz, 4-pole induction motor has the following per phase constants in ohm referred to stator and $N_r = 1440$ rpm.

$R_1 = .15\ \Omega$, $x_1 = .45\ \Omega$, $r_2 = 0.12\ \Omega$

$x_2 = 0.45\ \Omega$, $x_m = 28.5\ \Omega$

Example 8: A 3000 kVA, 6-pole star-connected alternator runs at 1000 rpm in parallel with other machines on 3300 V bus-bars. The synchronous reactance is 25%. Calculate the synchronizing power for 2° (mech) displacement and the corresponding synchronizing torque.

Solution: Alternator is working in parallel with many alternators, hence it may be considered to be connected to infinite bus-bars.

$$\text{Voltage/phase} = 3300/\sqrt{3} = 1905 \text{ V}$$

$$\text{Full-load current, } I = 3 \times 10^6/\sqrt{3} \times 3300 = 525 \text{ A}$$

Now, $IX_S = 25\%$ of 1905

$\therefore$ $X_S = 0.25 \times 1905/525$

$= 0.9075\ \Omega$

$P_{Sy} = 3 \times \alpha E^2/\text{Xs}$

$\alpha = 2°$ mech $\alpha_{elect} = 2 \times \frac{6}{2} = 6°$

$$\alpha = 6 \times \frac{\pi}{180} = \frac{\pi}{30} \text{ elect rad}$$

$\therefore$ $$P_{SY} = \frac{3 \times \pi \times 1905^2}{30 \times 0.9075 \times 1000}$$

$= 314.2$ kW

$$T_{SY} = \frac{60 \cdot P_{SY}}{2\pi N_S} = 9.55 \frac{P_{SY}}{N_S}$$

$$= 9.55 \times \frac{314.2 \times 10^3}{1000}$$

$= 3000$ Nm

Load Sharing of Alternators Connected in Parallel

(a) Effect of change in excitation: The change in excitation merely changes the p.f. at which the load is delivered without affecting the load so long as the input to prime mover remains unchanged, i.e. kVA component of its output is changed (indirect proportion with excitation) but not its kW output.

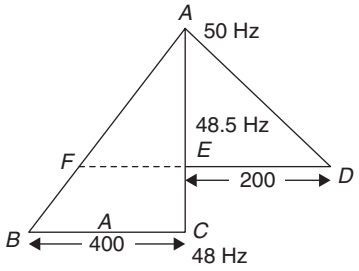
(b) Effect of change in the power input to prime mover: The amount of load taken up by an alternator running, in parallel with other machines, directly depends upon its driving torque or in other words upon the power input (or the angular advance of its rotor). The increase in the power input to prime mover does not disturb the division of reactive power.

Example 9: An alternator has a per unit impedance of 0.9 p.u. to a base of 20 MVA, 33 kV, find the p.u. impedance to the base of 50 MVA and 11 kV.

Solution: $$Z_{new} = Z_{old} \frac{MVA_{new}}{MVA_{old}} \times \left(\frac{kV_{old}}{kV_{new}}\right)^2$$

$$= 0.9 \times \frac{50}{20} \times \left(\frac{33}{11}\right)^2 = 20.25$$

Example 10: An alternator of 400 kW is driven by a prime mover of speed regulation 4% and another alternator of 200 kW is driven by a prime mover of speed regulation 3%. Find the total load they can take, without over loading either of the machines



Solution: For getting maximum load, DE is extended to cut AB at F.

Max load = DF

$$\frac{FE}{BC} = \frac{AE}{AC}$$

$$\frac{FE}{400} = \frac{(50 - 48.5)}{(50 - 48)}$$

$$FE = \frac{400 \times 1.5}{2} = 300 \text{ kW}$$

Maximum load FD = FE + ED

$= 300 + 200 = 500$ kW

Short-circuit Ratio (SCR)

- $$\text{SCR} = \frac{\text{Field current required to produce rated voltage on open circuit}}{\left(\begin{array}{l}\text{Field current required to produce rated} \\ \text{current on 3-}\phi\text{ short circuit}\end{array}\right)}$$
- A lower value of SCR means a greater change in field current to maintain a constant terminal voltage and lower value of steady-state stability limit.
- Lesser the SCR, lesser is the size, weight and cost of machine.
- If there were no magnetic saturation, SCR is the reciprocal of the p.u. value of saturated synchronous reactance.
- Modern alternators have SCR between 0.5 and 1.5.

Synchronous Motors

A synchronous motor construction is identical with an alternator. It has following features.

1. It runs either at synchronous speed or not at all, i.e. while running it maintains a constant speed.

$$N_s = \frac{120f}{P}$$

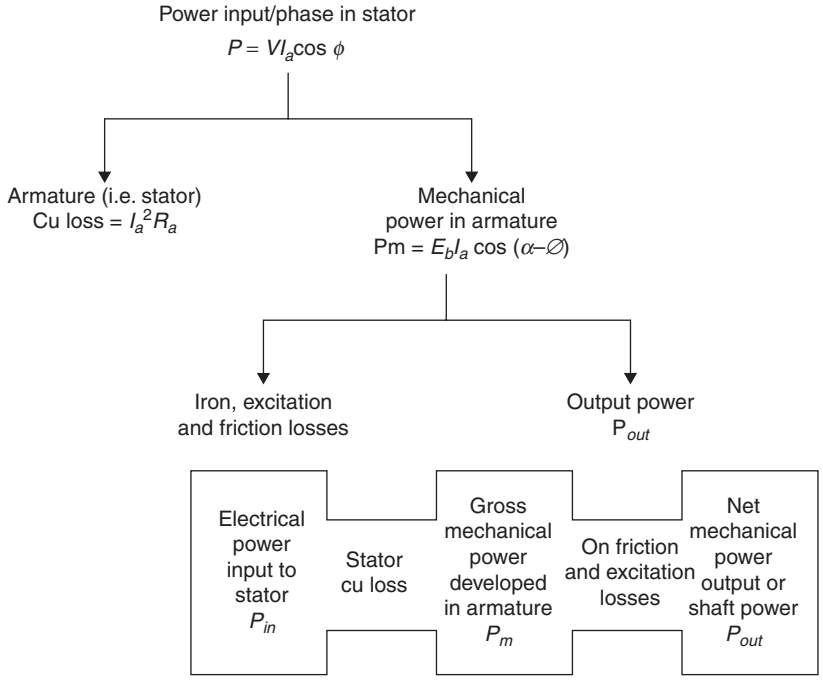
2. It is not inherently self-starting. It has to be run up to synchronous (or near synchronous) speed by some means (known as starting methods) before it is connected to the supply.
3. A synchronous motor can be operated under a wide range of power factors both lagging and leading. A range of zpf lag to zpf lead through unity can be obtained by adjusting the excitation. Hence it can be used for power factor improvement purposes, in addition to supplying driving torque to drive loads.

Principle of Operation

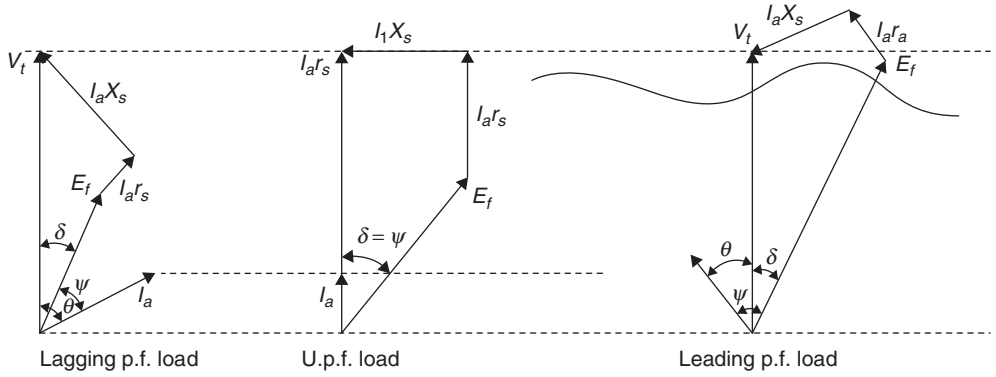
A synchronous motor works on the principle of magnetic locking between rotating air-gap flux and rotor poles.

Method of Starting

The rotor (which is as yet unexcited) is speeded up to synchronous/near synchronous speed by some arrangement and then field circuit is excited. The moment this synchronously (or near) rotating rotor is excited, and it is magnetically locked into position with the stator poles. A synchronous motor may be started with the help of auxiliary motor connected to the same shaft or by means of induction torque obtained by damper bars.

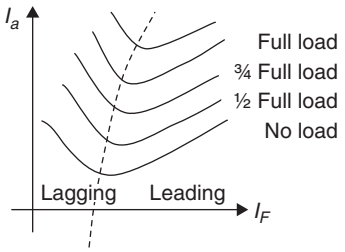


Synchronous Motor Phasor Diagram

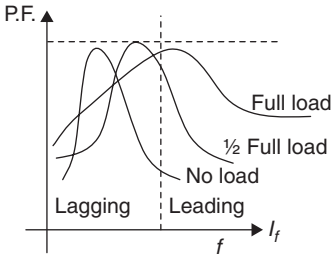


V- and Inverted V-curves

The V-curves of a synchronous motor show how armature current varies with its field current when motor input is kept constant and are so called because of their shape. Minimum armature current corresponds to unity power factor.



The inverted *V*-curves of a synchronous motor show how power factor varies with excitation (or I_f) when motor input is constant. The curve for p.f. looks like inverted '*V*' curve.



In case of synchronous motor

1. If armature current I_a lags the excitation emf, E_f 90°, the nature of armature MMF or armature reaction MMF F_a is magnetizing.
2. If armature current I_a leads E_f by 90°, the nature of armature MMF, or armature reaction MMF F_a is demagnetizing.

Effect of Increased Load with Constant Excitation

1. As load on the motor increases the armature current, I_a increases regardless of level of excitation (i.e. normal, under, or over-excited conditions)
2. For under-and over-excited motors p.f. tend to approach unity with increase in load
3. Both with under-and over-excited, change in p.f. is greater than in I_a with increase in load
4. For normal excited case, when load is increased, change in I_a is greater than in p.f. which tends to become increasingly lagging.

Effect of Changing Excitation on Constant Load

The variations in the excitation of a synchronous motor running with a given load produce variations in its load angle only. When excitation is decreased the load angle increases and vice-versa.

Example 11: A 3-phase, induction motor draws 1000 kVA at a p.f. of 0.8 lag. A synchronous condenser is connected in parallel to draw an additional 750 kVA at 0.6 p.f. lead. What is the p.f. of the total load supplied by the mains?

Solution: Reactive power drawn by induction motor

$$Q_m = 1000 \sin 36.86$$

$$j\ 600 \text{ kVAR lagging}$$

Reactive power supplied by condenser

$$Q_C = 750 \sin(\cos^{-1} 0.6)$$

$$= -J\ 600 \text{ kVA lead}$$

Total reactive power $Q = Q_m + Q_c$

$$Q = j600 + -j600 = 0$$

Power factor of total load

$$\cos\left[\tan^{-1}\left[\frac{Q}{P}\right]\right] = 1$$

Example 12: A 2300 V, 3-phase synchronous motor driving a pump is provided with a line ammeter and a field rheostat. When the rheostat is adjusted such that the AC line current is minimum, the ammeter reads 8.8 A. What is the power being delivered to the pump, neglecting losses? How should the rheostat be adjusted so that the motor operates at 0.8 leading power factor? What is the reactive power supplied by the motor at this new power factor?

Solution: Line current will be minimum at unity power factor

$$\text{Power delivered} = \sqrt{3} \times 2300 \times 8.8 \times 1 = 35.05 \text{ kW}$$

For operating motor at 0.8 leading power factor field current should be increased.

As the load is unaltered $I \cos\phi$ remains same.

So $I_{a2} \cos\phi_2 = I_{a1} \cos\phi_1$

$$I_{a2} = \frac{8.8 \times 1}{0.8} = 11 \text{ A}$$

Leading kVAR supplied

$$= \frac{\sqrt{3} \times 2300 \times 11 \times 0.6}{1000} = 26.3$$

Power Developed by a Synchronous Motor

$$P_m = \frac{E_b V}{Z_s} \cos(\theta - \alpha) - \frac{E_b^2}{Z_s} \cos\theta$$

where α = Load angle

θ = Internal or Machine angle

Since R_a is generally negligible, $Z_s = X_s$ so that $\theta \cong 90°$. Hence, mechanical power developed,

$$P_m = \frac{E_b V}{X_s} \cos(90° - \alpha) = \frac{E_b V}{X_s} \sin\alpha$$

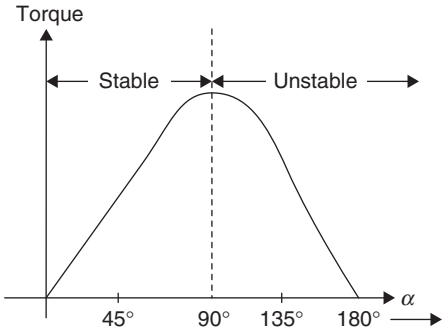
Condition for Maximum Power Developed

Maximum power can be developed when $\theta = \alpha$

$\therefore$ Value of maximum power $(P_m)_{max}$

$$= \frac{E_b V}{Z_s} - \frac{E_b^2}{Z_s}\cos\alpha \text{ or } (P_m)_{max} = \frac{E_b V}{Z_s} - \frac{E_b^2}{Z_s}\cos\theta$$

The above expression shows that maximum power and hence torque ($\because$ speed is constant) depends on V and E_b. Maximum value of θ (and hence α) is 90°. The variation of torque (or power) with load angle α is shown in following figure.



If R_a is neglected then $Z_s \cong X_s$ and $\theta = 90°$.

$\therefore$ $\cos\theta = 0$

$$(P_m)_{max} = \frac{E_b V}{X_s}.$$

This corresponds to pull-out or maximum torque.

Special Motors

Servomotor

The motors that are used in automatic control systems are called servomotors. When the objective of the system is to control the position of an object, then the system is called servomechanism. The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft. The suitability of a motor for a particular application depends on the characteristics of the system, the purpose of the system and its operating conditions.

A servomotor should have the following features.

1. Linear relationship between the speed and electric control signal
2. Steady-state stability
3. Wide range of speed control
4. Linearity of mechanical characteristics throughout the entire speed range
5. Low mechanical and electrical inertia and
6. Fast response.

A two-phase induction motor with certain special design features can be used as servomotors, they are

1. The rotor of servomotor is built with high resistance, so that its $\frac{X}{R}$ (Inductive reactance/resistance) ratio is small which results in linear speed-torque characteristics.
2. The excitation voltage applied to two stator windings should have a phase difference of 90°.

Stepper Motor

A stepper motor transforms electrical pulses into equal increments of rotor shaft motion called steps. A one-to-one correspondence exists between the electrical pulses and the motor steps. The number of teeth or poles on the rotor and the number of poles on the stator determine the size of the step.

$$\text{Step angle} = \frac{360}{\begin{bmatrix}\text{No. of rotor poles}\\ \text{or teeth}\end{bmatrix} \times \begin{bmatrix}\text{No. of stator pole}\\ \text{pairs}\end{bmatrix}}$$

The frequency and sequence of pulses given to the stator windings is controlled by switching devices. Stepper motors are used in computer peripherals, X-Y plotters, robots and machine tools.

Exercises

Practice Problems I

Directions for questions 1 to 26: Select the correct alternative from the given choices.

1. Per unit synchronous impedance of synchronous machine is 1.10. Its SCR is _____?
(A) 1.20 (B) 0.417
(C) 0.909 (D) 0.833

2. A synchronous generator with a synchronous reactance of 1.3 p.u. is connected to an infinite bus whose voltage is 1 p.u., through an equivalent reactance of 0.2 p.u., for maximum output of 1.2 p.u., the alternator emf must be
(A) 1.5 p.u. (B) 1.56 p.u.
(C) 1.8 p.u. (D) 2.5 p.u.

3. A single-phase alternator has a synchronous reactance of 2 Ω and negligible resistance. If it supplies 10 A to a purely capacitive load at 200 V, then the generated emf will be _____
(A) 240 V (B) 220 V
(C) 200 V (D) 180 V

4. An alternator has 18 slots/pole and the first coil lies in slots 1 and 16. What is the pitch factor for 5th harmonic _____?
(A) 0.96 (B) 0.259
(C) 0.289 (D) 0.829

5. An alternator on open circuit generates 400 V and 60 Hz. When the field current is 3.6 A. Neglecting saturation,

determine the open-circuit emf. When the frequency is 40 Hz and the field current is 3 A _____
(A) 160 V (B) 200 V
(C) 220 V (D) 240 V

6. A salient-pole alternator has X_d = 1.0 p.u., X_q = 0.6 p.u. and r = 0. If this alternator delivers a power of 0.9 p.u. to an infinite bus at rated voltage and at p.f. 0.8 lag. Its load angle is _____
(A) 42.048° (B) 21.024°
(C) 60.234° (D) 36.06°

7. A synchronous generator is running over excited with E_f = 1.40 p.u. This machine with a synchronous reactance of 1.20 p.u. is delivering a synchronous power of 0.5 p.u. to the bus. If the prime mover torque is increased by 1%, by how much will the P and Q change?
(A) P increases by 1% but the Q decreases by 0.475%
(B) P decreases by 1% but the Q increases by 0.475%
(C) P and Q both increases by 1%
(D) P and Q both decreases by 1%

8. A 3-phase, 415 V, 6-pole, 50 Hz, star-connected synchronous motor develops a torque of 220 Nm and the stator winding has a synchronous reactance of 2 ohm/phase. Its load angle is _____. Its emf is 520 V (L – L)
(A) 24.66° (B) 36.80°
(C) 18.40° (D) 12.33°

9. A 4-pole, 3-phase, 50 Hz, star-connected alternator has 60 slots with 4 conductors per slot. Coils are short-pitched by 3 slots. If the phase spread is 60°, find the line voltage induced for a flux per pole of 0.943 Wb distributed in space.
(A) 7.613 kV (B) 5.234 kV
(C) 13.185 kV (D) 12.42 kV

10. A 3-ϕ, star-connected alternator supplies a load of 10 MW of P.F of 0.85 lag and at 11 kV, terminal voltages, its resistance is 0.1 ohm per/phase and synchronous reactance 0.66 ohm per phase. The line value of emf generated is _____
(A) 6.625 kV (B) 13.21 kV
(C) 7.22 kV (D) 11.4863 kV

11. A 3-phase star-connected alternator is rated at 1600 kVA, 13.5 kV. The armature resistance drop and synchronous reactance drops are 3% and 10%, respectively. Find the regulation of alternator on full load at a power factor of 0.8 leading?
(A) –2.1% (B) –3.1%
(C) +3.1% (D) 2.1%

Common Data for Questions 12 and 13:
A 3-ϕ star-connected 400 V synchronous motor takes a power input of 6 kW at rated voltage. Its synchronous reactance is 10 Ω per phase and resistance is negligible. If the excitation voltage is adjusted equal to the rated voltage of 400 V then

12. The load angle is _____
(A) 20.0° (B) 24.02°
(C) 22.02° (D) 29.82°

13. The armature current is _____
(A) 8.82 A (B) 7.86 A
(C) 6.43 A (D) 4.4 A

Common Data for Questions 14 and 15:
A 500 kVA, 11 KV, 3-ϕ star-connected alternator has the following data:
Friction and windage losses = 1000 W
core loss = 1500 W
Field winding resistance at 75°C is 180 Ω
Effective armature resistance per phase = 4 Ω
Field copper loss = 1000 W
For the above data

14. The load at which maximum efficiency occurs is _____.
(A) 65% of full load
(B) 58% of full load
(C) 70%of full load
(D) 80% of full load

15. The maximum efficiency at 0.8 p.f lagging is _____.
(A) 96. 85% (B) 86.89%
(C) 78.79% (D) 70.23%

16. List-I lists different applications and List-II lists motors for these applications. Match the application with the most suitable motor.

	List-I		List-II
A.	Escalator	1.	Single-phase induction motor
B.	Food mixer	2.	Three-phase induction motor
C.	Cassette tape recorder	3.	Permanent magnet DC motor
D.	Domestic water pump	4.	Universal motor
		5.	DC series motor
		6.	Stepper motor

	A	B	C	D
(A)	1	4	3	2
(B)	1	4	2	3
(C)	5	4	3	6
(D)	1	4	3	5

17. A rotating electrical machine having its self-inductance of both the stator and the rotor windings independent of the rotor position will be definitely not develop
(A) synchronizing torque.
(B) starting torque.
(C) reluctance torque.
(D) hysteresis torque.

18. A synchronous motor on load draws a current at a leading power factor angle ϕ. If the internal power factor angle which is the phase angle between the excitation emf and the current in time phasor diagram, is Ψ, then the air gap excitation m. m. f. lags the armature mmf by

(A) $\pi/2 + \Psi$ (B) Ψ

(C) $\Psi - \phi$. (D) $\frac{\pi}{2} - \Psi$

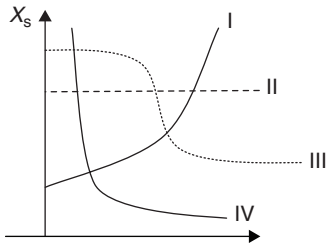
19. The developed electromagnetic force and/or torque in electro-mechanical energy conversion system act in a direction that tends

(A) to decrease the co-energy at constant mmf.
(B) to decrease the stored energy at constant mmf.
(C) to decrease the stored energy at constant flux.
(D) to increase the stored energy at constant mmf.

20. The following motor definitely has a permanent magnet rotor

(A) Brushless DC motor.
(B) DC commutator motor.
(C) Universal motor.
(D) Stepper motor.

21. In the figure the characteristic that corresponds to the variation of synchronous reactance of synchronous motor with field current is



(A) curve I (B) curve II
(C) curve III (D) curve IV

22. X_d, X_d' and X_d'' are steady-state *d*-axis synchronous reactance, transient *d*-axis reactance and sub-transient d-axis reactance of synchronous machine, respectively. Which of the following relation is true?

(A) $X_d'' > X_d' > X_d$

(B) $X_d' > X_d'' > X_d$

(C) $X_d > X_d' > X_d''$

(D) $X_d > X_d'' > X_d'$

23. An isolated engine driven synchronous generator is feeding a partly inductive load. A capacitor is now connected across the load to completely compensate the inductive current. For this operating condition

(A) the field current has to be increased and fuel input left unaltered.
(B) the field current has to be reduced and fuel input left unaltered.
(C) the field current and fuel input have to be reduced.
(D) the field current and fuel input have to be increased.

24. In relation to the synchronous machines, which one of the following statements is false?

(A) the damper bars help the synchronous motor self-start.
(B) in salient-pole machines, the quadrature axis synchronous reactance is less than the direct axis synchronous reactance.
(C) the V curve of a synchronous motor represents the variation in the armature current with field excitation, at a given output power.
(D) the short-circuit ratio is the ratio of the field current required to produce the rated voltage on open circuit to the rated armature current.

25. A three-phase synchronous motor connected to AC mains is running at full-load unity power factor. If the load is reduced by half, with excitation unaltered, its new power factor will be

(A) leading
(B) lagging
(C) unity
(D) dependent on machine parameters

26. A 4-pole, 3-phase, double layer winding is housed in a 36 slot stator for an AC machine with a phase spread of 60°. The coil span is 7 slot pitches Number of slots in which top and bottom layers belonging to same phases is

(A) 12 (B) 14
(C) 24 (D) 28

Practice Problems 2

Directions for questions 1 to 100: Select the correct alternative from the given choices.

1. Two similar alternators are working in parallel. They have identical field excitations and supply a common load. Now the steam supply to one of the alternator is increased keeping the field excitation constant, its active power component

(A) remains the same, but the reactive power component will increase
(B) decreases while reactive power increases
(C) increases and reactive power also increases
(D) increases while reactive power contribution remains unchanged

2. Two mechanically coupled alternators deliver power at 50 Hz and 60 Hz, respectively. The highest speed of the alternator is

(A) 3000 rpm (B) 3600 rpm
(C) 500 rpm (D) 600 rpm

3. The phase groups of machines A and B are 60° and 120°, respectively. If the breadth factor of machine A is x times that of B, then value of x is
(A) 1 (B) <1
(C) >1 (D) None of these

4. For the same power rating an alternator at lower voltage will be
(A) larger in size (B) smaller in size
(C) less noisy (D) more efficient

5. When an alternator, designed for operation at 60 Hz is operated at 50 Hz
(A) kVA rating increases by 1.2
(B) operating voltage reduces by $\frac{5}{6}$
(C) operating voltage increases by 1.2
(D) operating voltage reduces by $\left(\frac{5}{6}\right)^2$

6. When a balanced 3-ϕ distributed type armature winding is carrying 3-ϕ balanced currents, the strength of rotating magnetic field is
(A) three times the amplitude of each constituent pulsating magnetic field
(B) equal to the amplitude of each constituent pulsating magnetic field
(C) half the amplitude of each pulsating magnetic field
(D) one and a half times the amplitude of each constituent pulsating magnetic field

7. The armature of a 1-ϕ alternator is completely wound with T single turn coils distributed uniformly, the induced voltage in each turn is 2 V (RMS). The emf of the whole winding is
(A) 2 T V (B) 1.27 T V
(C) 1.11 T V (D) 1.414 T V

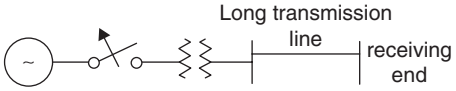
8. A 3-ϕ, 4-pole alternator has 48 stator slots carrying the 3-ϕ distributed winding. Each coil of the winding is short-circuited by one slot pitch. The winding factor is
(A) $\frac{1}{16}\cot 7.5°$ (B) $\frac{1}{8}\cot 7.5$
(C) $\frac{1}{16\sin 7.5°}$ (D) $\cos 7.5°$

9. A 50 kW, 3-ϕ Y-connected, 50 Hz, 415 V cylindrical rotor synchronous generator operates at rated excitation with 0.8 p.f. leading. The motor efficiency excluding field and stator losses is 90% and $X_s = 2.5\ \Omega$. The power angle of the machine is approximately
(A) 27° (B) 54°
(C) –27° (D) –54°

10. A stationary alternator is connected across line bus bars. It
(A) will get short-circuited
(B) will run as a synchronous motor
(C) will decrease bus bar voltage through momentarily
(D) will distinct generated emf's of other alternators connected in parallel

11. Two alternators P and Q operate in parallel and supply a load of 20 MW at 0.8 p.f. lag. By adjusting the steam input of P its power output is adjusted to 12 MW and by changing its excitation its p.f. is adjusted to 0.9 lag. Find p.f. of alternator Q
(A) 0.557 lag (B) 0.657 lag
(C) 0.557 lead (D) 0.657 lead

12. A 50 Hz alternator is initially connected to a long lossless transmission line which is open circuited at the receiving end. With field voltage held constant, generator is disconnected from transmission line. Which of the following can be said about steady-state terminal voltage and field current of the generator?



(A) magnitude of terminal voltage decreases and magnitude of field current does not change
(B) magnitude of terminal voltage increases and field current does not change
(C) magnitude of terminal voltage increases and field current increases
(D) magnitude of terminal voltage does not change and field current decreases

13. A 250 MW, 21 kV, 50 Hz, 3-ϕ, 4-pole alternator having a rated p.f. = 0.8 has moment of inertia 30×10^3 kg m^2. The inertia constant H will be
(A) 1.18 s (B) 1.91 s
(C) 2.68 s (D) 4.82 s

14. A 80 kVA, 400 V, Y-connected synchronous machine generates rated open-circuit voltage of 400 V at a field current of 16 A. The short-circuit armature current at a field current of 10 A is equal to rated armature current. The per unit saturated synchronous reactance is
(A) 0.571 (B) 1.25
(C) 0.625 (D) 1.63

15. In relation to synchronous machines, which one of the following statements is false?
(A) In salient-pole machines, direct axis synchronous reactance is greater than quadrature axis synchronous reactance
(B) Damper winding helps synchronous motor self-start
(C) Short-circuit ratio is the ratio of field current required to produce, the rated voltage on O.C. to rated armature current
(D) The V curve of a synchronous motor represent variation in armature current with field excitation at a given output power

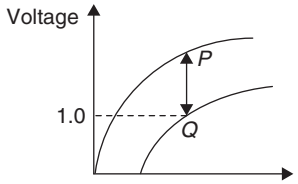
16. A 50 Hz, 6-pole, 100 MVA, 22 kV turbo generator is delivering rated MVA at 0.8 p.f. Suddenly a fault occurs reducing electric output by 30%. Neglect losses and assume constant power input to the shaft. The accelerating torque of generator in KNm at time of fault is
(A) 534.76 (B) 356.27
(C) 296.35 (D) 178.13

17. Two 3-ϕ, Y-connected alternators are to be paralleled to a set of common bus bars. The armature has a per phase synchronous reactance of 1.5 Ω and negligible armature resistance. The line voltage of first machine is adjusted to 3300 V and of second machine is 3200 V. The machine voltages are in phase at the instant they are paralleled. Under this condition, the synchronizing current per phase will be
(A) 19.1 A (B) 33.08 A
(C) 38.2 A (D) 66.16 A

18. A 4-pole, 3-ϕ, double-layer winding is housed in a 36 slot stator for an **AC** machine with 60° phase speed. Coil span is 7 slot pitches. No. of slots in which top and bottom layers belong to different phases is
(A) 24 (B) 18
(C) 12 (D) 0

19. A standalone engine drives synchronous generator is feeding a partly inductive load. A capacitor is now connected across the load to completely nullify the inductive current. For this operating condition
(A) field current and fuel input has to be reduced
(B) field current and fuel input has to be increased
(C) field current has to be increased and fuel input left unaltered
(D) field current has to be reduced and fuel input left unaltered

20. Curves X and Y denote O.C. and full-load ZPF characteristics of a synchronous generator. Q is a point on ZPF characteristics at 1 p.u. voltage. The vertical distance PQ gives voltage drop across.



(A) synchronous reactance
(B) magnetizing reactance
(C) Potier reactance
(D) leakage reactance

21. A round rotor generator with internal voltage $E_1 = 2$ p.u. and $X = 0.9$ p.u. is connected to a round rotor synchronous motor with internal voltage. $E_2 = 1.2$ p.u. and $x = 1.1$ p.u. The reactance of line connecting generator and motor is 0.5 p.u. When generator supplies 0.4 p.u. power, rotor angle difference between machines is
(A) 30° (B) 24°
(C) 18° (D) 45°

22. The phase sequence of a 3-ϕ alternator can be reversed by
(A) reversing the field current and doubling the number of poles
(B) doubling the number of poles without reversing the field current
(C) reversing the field current keeping direction of rotation same
(D) reversing the direction of rotation keeping the field current same

23. A 5000 kVA, 10,000 V 1500 rpm, 50 Hz alternator is connected with infinite bus bar. Its synchronous reactance is 20%. The no-load synchronizing torque when there is a displacement of 0.5° mechanical is
(A) 672 kW (B) 772 kW
(C) 872 kW (D) 972 kW

24. The terminals of a 3-ϕ unloaded alternator is shorted. The role of damper windings here is to
(A) establish flux through the direct axis magnetic circuit of the machine
(B) repel the armature flux and confine it to leakage flux path in the air gap
(C) allow only partial linkage of armature flux with main field winding
(D) confine the armature flux to completely link the damper winding

25. Synchronous impedance method is used to calculate the regulation of a synchronous generator. This is
(A) lower than actual because of saturation of magnetic circuit
(B) higher than actual due to saturation of magnetic circuit
(C) nearly accurate as generator is normally operated with unsaturated magnetic circuit
(D) nearly accurate as it takes into account the magnetic saturation

26. The armature current of an alternator lags behind the excitation emf by an angle ϕ. The electrical angle between field axis and the axis of armature reaction field will be
(A) ϕ (B) 90°
(C) $90° - \phi$ (D) $90° + \phi$

27. In an alternator, the stator field,
(A) leads the rotor field and electromagnetic torque is developed in a direction opposite to direction of rotation of rotor
(B) leads the rotor field and electromagnetic torque is developed in the direction of rotation of rotor
(C) lags the rotor field and electromagnetic torque is developed opposite to direction of rotation of rotor
(D) lags the rotor field and electromagnetic torque is developed in same direction of rotation of rotor

28. A 50 Hz 4-pole turbo generator rated 100 MVA, 11 kV has been given a mechanical input of 80 MW suddenly for an electrical load of 40 MW, then speed of rotor after 10 cycles will be, if machine inertia constant is 8 μJ/MVA
(A) 1515 rpm (B) 1468 rpm
(C) 496 rpm (D) None of these

29. The maximum electrical output of a synchronous generator is
(A) $\frac{X_s}{V_t C_f}$ (B) $\frac{V_t^2}{X_s}$
(C) $\frac{C_f^{\ 2}}{X_s}$ (D) $\frac{V_t C_f}{X_s}$

30. An alternator is synchronized with the infinite bus bar. If the input to prime mover is cut off with field supply remaining constant
(A) it gets burnt
(B) it behaves as an induction motor but run in reverse direction
(C) it behaves as a synchronous motor and will rotate in same direction
(D) it behaves as a synchronous motor and rotate in reverse direction

31. High frequency on aircraft alternators is used to
(A) free system from external disturbance
(B) compensate for high speed
(C) compensate for high altitudes
(D) reduce bulk

Common Data for Questions 32 and 33:
A 4-pole, 50 Hz, synchronous generator has 36 slots in which a double layer winding is housed. Each coil has 10 turns and is short pitched by an angle of 30° electrical. The fundamental flux per pole is 0.025 Wb.

32. The line to line induced emf for a 3-ϕ Y connection is approximately
(A) 969 V (B) 869 V
(C) 1369 V (D) 1069 V

33. The line to line induced emf for a 2-ϕ connection is approximately
(A) 873 V (B) 973 V
(C) 1269 V (D) 1069 V

34. The fifth-harmonic component of phase emf for the 3-ϕ *Y* connection
(A) 0 (B) 269
(C) 281 (D) 808

Common Data for Questions 35 and 36:
A 500 kVA, 13.2 kV, 3-ϕ Y-connected cylindrical-pole synchronous generator has a synchronous reactance of 80 Ω. Neglect armature resistance and consider operation at full load and u.p.f.

35. The induced emf (line to line) is close to
(A) 11.24 kV (B) 13.54 kV
(C) 16.24 kV (D) 14.24 kV

36. The power or torque angle is close to
(A) 13.9° (B) 16.3°
(C) 12.9° (D) 30.3°

37. Distribution factor K_d is also known as
(A) Length factor
(B) Breadth factor
(C) Winding Factor
(D) None of the above

38. Which losses are reduced, by having laminated poles and pole shoes?
(A) Eddy current losses
(B) Hysteresis losses
(C) Iron losses
(D) Copper losses

39. Two 3-phase star-connected alternators, whose armature has a per phase synchronous reactance of 0.85 Ω and negligible armature resistance, are to be paralleled to a set of common bus bars. The line voltage of the first machine is adjusted to 330 V and that of second machine is adjusted to 320 V the synchronizing current per phase would be
(A) 16.97 A (B) 15.88 A
(C) 3.39 A (D) 1.697 A

40. Which among the following is not correct about distributed winding?
(A) Improves the waveform of induced emf
(B) Provides better utilization of core
(C) Enhances armature reaction
(D) Facilitates cooling

41. A 3.3 kV single-phase alternator delivering a current of 100 A at 0.8 p.f. lagging, produces a full-load current of 100 A on short circuit by a field excitation of 5 A and produces an emf of 250 V on open circuit by the same field current. The armature resistance is 0.65 Ω. The percentage regulation of the alternator is
(A) –6% (B) –8.9%
(C) +8.9% (D) +6%

42. For a uniformly distributed 3-phase winding the value of distribution factor would be
(A) 0.955 (B) 0.8
(C) 0.755 (D) 0.675

43. A 6.6 MVA, 3-phase star-connected, 6.6 kV, 2-pole turbo generator has a synchronous impedance of (0.2 + *j*1.4)Ω per phase. The generator losses are as given below
(a) S.C load loss = 110 kW
(b) O.C core loss at rated voltage = 45 kW
(c) Friction and windage losses = 25 kW
(d) Field winding resistance = 3 Ω
(e) Field current = 100 A

Neglecting the change in field current compute the efficiency at rated load 0.8 p.f. leading
(A) 94.65% (B) 95.46%
(C) 96% (D) 92%

44. The power angle at which a synchronous generator will develop maximum power will be
(A) $\delta = \theta$ and $\theta = \tan^{-1}\left(\frac{Xs}{\text{Re}}\right)$
(B) $\delta = \theta$ and $\theta = \tan^{-1}\left(\frac{\text{Re}}{Xs}\right)$
(C) $\delta = \theta$ only
(D) $\delta = \theta$ and $\theta = \tan^{-1}(\text{Xs})$

45. A 380 V, 100 kVA, 0.8 p.f. leading delta-connected, 60 Hz synchronous machine has a synchronous reactance of 2 Ω and negligible armature resistance. The friction and windage losses are 1 kW and the core loss is 1.6 kW. Calculate the power supplied to the load if the load is drawing a line current of 21 A at 0.8 p.f. leading
(A) 9 kW (B) 9.5 kW
(C) 8.5 kW (D) 11.785 kW

46. Given below are the test results that were conducted on an alternator for a field excitation of 20 A. Test Results
S.C. test: Armature current = 200 A
O.C. test: Terminal voltage = 1000 V
The magnitude of internal voltage drop within the machine for a load current of 25 A is
(A) 1000 V (B) 500 V
(C) 250 V (D) 125 V

47. A synchronous machine with large air gap has higher synchronizing power due to
(A) Larger air gap offers large reluctance to the path of flux produce by armature mmf
(B) Large air gap reduces the effect of armature reaction
(C) Large air gap results in small value of synchronous reactance
(D) All of the above

48. The methods employed to reduce hunting in alternators are by
(A) Employing heavy flywheels
(B) Putting dash-pots on the engine governors
(C) Using squirrel cage windings on the surface of the rotor
(D) All of the above

49. Synchronous generator with a synchronous reactance of 1.2 p.u. is connected to an infinite bus whose voltage is 1 p.u. through an equivalent reactance of 0.3 p.u. What should be the value of emf to give a maximum output of 1.2 p.u.?
(A) 0.8 p.u. (B) 1.8 p.u.
(C) 2.8 p.u. (D) 3.65 p.u.

50. Power angle δ depends upon which of the following factors:
(i) Supply voltage
(ii) Armature current
(iii) Power factor
(A) (i) and (ii) (B) (ii) and (iii)
(C) (i) and (iii) (D) (i), (ii) and (iii)

51. The increase in field current causes an increase in armature current in a synchronous motor, fed from an infinite bus and is delivering half full-load. Then the motor
(A) delivers reactive power to the bus an absorb active power from the bus
(B) Absorbs reactive power from the bus and delivers active power to the bus
(C) Absorbs active power and reactive power from the bus
(D) deliver reactive power and active power to the bus

52. The field current of a synchronous motor operating at 0.8 p.f. lag is continuously increased. Then
(A) the armature current increases up to a certain value of field current, and thereafter decreases
(B) the armature current decreases up to a certain value of field current and then increases
(C) the power factor increases up to a certain value of field current and then decreases
(D) the power factor decreases up to a certain value of field current and then increases

53. During hunting of synchronous motor
(A) negative phase sequence currents are generated
(B) harmonics are developed in the armature circuit
(C) damper bar develops torque
(D) field excitation increases

54. A 11 kV synchronous motor is connected to a load taking 0.8 p.f. lagging current. The armature reaction is
(A) cross-magnetizing (B) demagnetizing
(C) non-effective (D) magnetizing

55. A 3-ϕ synchronous motor connected to **AC** mains is running at full load and unity p.f. If the shaft load is reduced by half, with field current held constant, new power factor will be
(A) unity
(B) leading
(C) lagging
(D) dependent on m/c parameters

56. During hunting of a synchronous motor
(A) negative phase sequence current are generated
(B) harmonics are developed in armature circuit
(C) damper bar develops torque
(D) field excitation increases

57. A synchronous condenser is used to compensate a 2400 kVA 0.65 p.f. load so that the p.f. becomes unity. The kVA rating of condenser is

(A) 27 kVAr (B) 1560 kVAr
(C) 1825 kVAr (D) 840 kVAr

58. If the excitation of a salient-pole synchronous motor running under normal excitation is reduced to zero, then
(A) it becomes an induction motor
(B) it becomes a reluctance motor
(C) it remains a synchronous motor
(D) motor stops

59. Inverted *V* curves for a synchronous motor show variation of
(A) power factor and supply voltage when motor is hunting
(B) field current and supply voltage when excitation is constant
(C) power factor with **DC** field current when load on motor is constant
(D) None of these

60. A 3-ϕ synchronous motor is connected to an infinite bus. It is operating at half load with normal excitation. When load on the machine is suddenly increased
(A) speed decreases and then becomes synchronous
(B) speed first increases and then becomes synchronous
(C) speed will fluctuate around synchronous speed and then becomes
(D) speed remains unchanged

61. Damper windings are provided
(A) on pole faces
(B) on armature
(C) on rotor shaft
(D) on stator frame

62. The torque angle in a synchronous machine is the angle between
(A) rotating stator flux and rotor poles
(B) magnetizing current and back emf
(C) supply voltage and back emf
(D) none of the above

Common Data for Questions 63 and 64:
A synchronous motor connected to an infinite bus at 1 p.u. voltage draws 0.8 p.u. current at u.p.f. Its synchronous reactance is 0.9 p.u. and resistance is negligible.

63. The excitation voltage (E) and load angle δ will be
(A) 0.616 and 35.7 lag
(B) 0.616 and 35.7 lead
(C) 1.23 and 35.7 lag
(D) 1.23 and 35.7 lead

64. Keeping the excitation voltage same, the load on the motor is increased such that motor current increases by 20%. The operating power factor becomes
(A) 0.812 lag (B) 0.812 lead
(C) 0.749 lag (D) 0.749 lead

65. A synchronous motor runs at Ns rpm at full load. The speed of the synchronous motor at half full-load will be
(A) Half the rated rpm
(B) $1/4^{th}$ the rated rpm
(C) Remains the same
(D) Double the rated rpm

66. A 3-phase synchronous motor driving a constant load torque draws power from infinite bus bar at leading power factor. With increase in excitation the power angle and power factor will
(A) Both power angle and p.f. decreases
(B) Power angle increases, p.f. decreases
(C) Power angle decreases, p.f. increases
(D) Both power angle and p.f. would increase

67. What should be the nature of excitation for a synchronous motor to operate as synchronous condenser?
(A) Under excited
(B) Over Excited
(C) Normally excited
(D) cannot be determined

68. The maximum value of torque that a motor can develop without losing synchronism is called
(A) Starting torque (B) Running torque
(C) Pull in torque (D) Pull out torque

69. For synchronous motors v-curves are plotted between the values of
(A) Armature current and power factor
(B) Armature current and Field current
(C) Field current and power factor
(D) Terminal voltage and Field current

70. A 1000 hp, 3.75 kV, 3-phase, star-connected synchronous motor has a synchronous impedance of $(5 + j12)$ Ω per phase. It is excited to develop an O.C emf of 2.5 kV. The ratio of current due to supply voltage and current due to induced emf is
(A) 1:2 (B) 2:3
(C) 3:2 (D) 2:1

71. Which of the following is false with regards to applications of a synchronous motor?
(A) Power factor correction equipment
(B) Voltage regulation equipments
(C) Constant speed drives
(D) Variable load drives

72. A synchronous motor is operated on no-load u.p.f. What should be done to the field current for the power factor to be leading?
(A) should not be changed
(B) should be increased
(C) should be decreased
(D) none of the above

73. If the excitation of a synchronous motor on infinite mains at no-load is increased, the current drawn by it will be of

(A) Unity p.f.
(B) Zero p.f. lagging
(C) It will not draw any current
(D) Zero p.f. leading

74. What would happen to load angle of a synchronous motor which is supplied power from a common bus bar and the bus voltage and frequency are reduced while field current and load torque are held constant?
(A) Load angle would decrease
(B) Load angle would increase
(C) Load angle would remain the same
(D) the motor will lose synchronism

75. If the phase angle between the excitation emf and current of a synchronous motor is given by θ_1. The synchronous motor on load draws a current at lagging p.f. θ_2. Then the armature mmf leads the excitation mmf by
(A) θ_2 (B) θ_1
(C) $\theta_1 - \theta_2$ (D) $\theta_1 + \theta_2$

76. What may be the causes if the synchronous motor fails to attain synchronous speed?
(A) Low supply voltage (B) Excessive load
(C) Field excited (D) All of the above

77. In what ways magnitude of armature current and power factor of a synchronous motor would vary if it is connected to an infinite bus, supplying constant torque, operating at u.p.f.
(A) Armature current will increase and p.f. will decrease
(B) Armature current will decrease and p.f. will increase
(C) Both armature current and p.f. will decrease
(D) Both armature current and p.f. will increase

78. A 440 V, 3-ϕ, 50 Hz circuit takes 10 A at 0.8 lagging p.f. A synchronous motor is used to raise the p.f. to unity. If the motor is driving a mechanical load of 10 kW and the motor has an efficiency of 80%, calculate the kVa input given to the motor and its p.f.
(A) 13.63, 0.64 lead
(B) 12.75, 0.875 lead
(C) 15.231, 0.75 lead
(D) 13.3, 0.94 lead

79. Which among the following is INCORRECT regarding a servomotor?
(A) They have non-linear relationship between the speed and electric control signal
(B) It has steady-state stability
(C) Wide range of speed control
(D) Low mechanical and electrical inertia

80. When a synchronous motor operates unexcited it is a
(A) AC series motor
(B) Universal motor
(C) Reluctance motor
(D) Repulsion motor

81. The maximum value of power developed in the synchronous motor is dependent on which of the following factors
(A) Maximum value of δ
(B) Rotor excitation
(C) Supply voltage
(D) All of the above

82. The purpose of damper winding in a synchronous motor is to
(A) reduce noise
(B) minimize circulating currents
(C) provide starting torque
(D) prevent hunting and provide starting torque.

83. Back emf set up in the stator of a synchronous motor will depend on
(A) Rotor speed, excitation and load angle
(B) Rotor excitation only
(C) Rotor speed only
(D) Rotor excitation and rotor speed

84. If the field windings of a synchronous motor are shorted on themselves and the motor is switched on then it will
(A) not start
(B) start and continue to run as an induction motor
(C) start as an induction motor and then run as synchronous motor
(D) Burn immediately

85. A 2300 V, 3-phase synchronous motor driving a pump is provided with a line ammeter and a field rheostat. When the rheostat is adjusted such that the AC line current is minimum, the ammeter reads 8.8 A. What is the reactive power supplied by the motor, if the motor was operating at 0.8 p.f. leading?
(A) 26.3 kVAR
(B) 36.3 kVAR
(C) 62.3 kVAR
(D) None of the above

86. A 3-ϕ star-connected 440 V, 20 A, 50 Hz, 500 rpm, develops an internal power of 6.7 kW. Calculate the effective resistance of the motor winding when operated at rated current and 0.8 leading p.f.?
(A) 1.5 Ω (B) 2.5 Ω
(C) 3.5 Ω (D) 4.5 Ω

87. A 3-ϕ, 2.3 kV, star-connected synchronous motor has an effective resistance and synchronous reactance of 0.45 Ω and 4.5 Ω, respectively. If 'θ' was the internal angle of the motor then what would happen to the value of θ if the value of R was increased by 0.9 Ω
(A) remain the same (B) 2θ
(C) 3θ (D) 4θ

88. A salient-pole synchronous motor has X_q = 0.35 p.u. and X_d = 0.55 p.u.. If the excitation is adjusted to 1.2 p.u. and the motor is connected to a bus bar of

1.0 p.u. voltage. The maximum power that the motor can deliver without loss of synchronism will be
(A) 0.958 p.u. (B) 1.58 p.u.
(C) 2.58 p.u. (D) 0.85 8 p.u.

89. The stepping angle of a stepper motor be increased by
(A) Decreasing the number of phases
(B) Reducing the number of poles
(C) Depends on the type of motor
(D) Both (A) and (B)

90. The number of stacks for a stepper motor of 6 poles and a stepping angle of 15° would be
(A) Single (B) Double
(C) Three (D) Four

91. Which of the following statements are true regarding servo motors
(i) The motor output torque is proportional to the voltage applied
(ii) The direction of torque developed depends on the instantaneous polarity of control voltage
(iii) The motor output torque is inversely proportional to the voltage applied
(iv) The direction of torque developed is independent of the polarity of control voltage
(A) (i) and (ii) (B) (ii) and (iii)
(C) (i) and (iv) (D) (iii) and (iv)

92. In a 3-phase synchronous motor the air gap flux will
(A) lag behind the armature mmf
(B) lead the armature mmf
(C) in phase with armature mmf
(D) either lead or lag based on load

93. For a 3-ϕ synchronous motor if
E_0 was O.C. emf/phase
V was Terminal voltage/phase
θ was the internal angle with lagging p.f. The relationship among the quantities will be
(A) $E_0 = \left[V^2 + E_R^{\,2} + 2VE_R \cos(\theta + \phi)\right]^{\frac{1}{2}}$
(B) $E_0 = \left[V^2 + E_R^{\,2} - 2VE_R \cos(\theta + \phi)\right]^{\frac{1}{2}}$
(C) $E_0 = \left[V^2 + E_R^{\,2} + 2VE_R \cos(\theta - \phi)\right]^{\frac{1}{2}}$
(D) $E_0 = \left[V^2 + E_R^{\,2} - 2VE_R \cos(\theta - \phi)\right]^{\frac{1}{2}}$

94. A 440 V, 1-ϕ synchronous motor gives a net output mechanical power of 6.5 kW and operates at 0.8 lagging p.f. If the effective resistance is 0.9 Ω and mechanical losses are 350 W then the estimated armature current is
(A) 15A (B) 25A
(C) 20 A (D) 30 A

95. The 'pull-out torque' of a synchronous motor is x times the 'full-load torque'. x can vary from
(A) 0.5 to 1.5
(B) 1.5 to 2.5
(C) 1.25 to 3.5
(D) It is a constant value 0.6

96. A 440 V, 50 Hz, 3-phase circuit takes a 10 A at lagging p.f. of 0.8 A synchronous motor is used to raise the p.f to unity. Calculate the efficiency of the motor when driving a mechanical load of 8.5 kW, supplied with an input of 13.63 kVa at a power factor of 0.6472
(A) 60% (B) 66%
(C) 71.25% (D) 76.66%

97. A 4-pole, 3-phase, 50 Hz, star-connected synchronous motor has a negligible armature resistance and armature reactance of 8 Ω/phase. The excitation is such as to give an E_{OC} of 13.2 kV. If the motor is connected to a 11.5 kV, 50 Hz supply. The maximum load that the motor can supply during synchronism is
(A) 18.97 kW
(B) 18.97 MW
(C) 17.89 kW
(D) 17.89 MW

98. Which among the following is the expression for maximum mechanical power developed by a synchronous motor if the synchronous impedance was purely reactive and armature resistance is negligible
(A) $(P_{\text{mech}})_{\max} = 3\left[\frac{EV}{Zs} - \frac{E^2}{Zs}\cos\theta\right]$
(B) $(P_{\text{mech}})_{\max} = \frac{3EV}{Zs}$
(C) $(P_{\text{mech}})_{\max} = 3\left[\frac{EV}{Zs} - \frac{E^2}{Z_s}\right]$
(D) None of the above

Common Data for Questions 99 and 100:
A 3-ϕ star-connected 400 V synchronous motor takes a power input of 6 kW at rated voltage. Its synchronous reactance is 10 Ω per phase and resistance is negligible. If the excitation voltage is adjusted equal to the rated voltage of 400 V, then

99. The load angle is _____
(A) 20° (B) 24.02°
(C) 22.02° (D) 29.82°

100. The armature current is _____
(A) 8.82 A (B) 7.80 A
(C) 6.43 A (D) 4.4 A

Previous Years' Questions

1. For a 1.8°, 2-phase bipolar stepper motor, the stepping rate is 100 steps/second. The rotational speed of the motor in rpm is **[2004]**
(A) 15 (B) 30
(C) 60 (D) 90

2. A 400 V, 50 kVA, 0.8 p.f. leading Δ-connected, 50 Hz synchronous machine has a synchronous reactance of 2 Ω and negligible armature resistance. The friction and windage losses are 2 kW and the core loss is 0.8 kW. The shaft is supplying 9 kW load at a power factor of 0.8 leading. The line current drawn is **[2004]**
(A) 12.29 A (B) 16.24 A
(C) 21.29 A (D) 36.88 A

3. A 500 MW 3-phase Y-connected synchronous generator has a rated voltage of 21.5 kV at 0.85 p.f. The line current when operating at full-load rated conditions will be **[2004]**
(A) 13.43 kA (B) 15.79 kA
(C) 23.25 kA (D) 27.36 kA

Common Data for Questions 4a and 4b:
A 1000 kVA, 6.6 kV, 3-phase star-connected cylindrical-pole synchronous generator has a synchronous reactance of 20 Ω. Neglect the armature resistance and consider operation at full load and unity power factor.

4. (a) The induced emf (line-to-line) is close to **[2005]**
(A) 5.5 kV (B) 7.2 kV
(C) 9.6 kV (D) 12.5 kV

(b) The power (or torque) angle to close to
(A) 13.9° (B) 18.3°
(C) 24.6° (D) 33.0°

5. In relation to the synchronous machines, which one of the following statements is false? **[2005]**
(A) In salient-pole machines, the direct-axis synchronous reactance is greater than the quadrature-axis synchronous reactance
(B) The damper bars help the synchronous motor self-start
(C) Short-circuit ratio is the ratio of the field current required to produce the rated voltage on open circuit to the rated armature current
(D) The V-curve of a synchronous motor represents the variation in the armature current with field excitation, at a given output power

6. A synchronous generator is feeding a zero power factor (lagging) load at rated current. The armature reaction is **[2006]**
(A) magnetizing
(B) demagnetizing
(C) cross-magnetizing
(D) ineffective

7. A 3-phase, 400 V, 5 kW, star-connected synchronous motor having an internal reactance of 10 Ω is operating at 50% load, unity p.f. Now, the excitation is increased by 1%. What will be the new load in per cent, if the power factor is to be kept same? Neglect all losses and consider linear magnetic circuit. **[2006]**
(A) 67.9% (B) 56.9%
(C) 51% (D) 50%

Common Data for Questions 8 to 10:
A 4-pole, 50 Hz, synchronous generator has 48 slots in which a double layer winding is housed. Each coil has 10 turns and is short pitched by an angle to 36° electrical. The fundamental flux per pole is 0.025 Wb.

8. The line-to-line induced emf (in volts), for a three-phase star connection is approximately **[2006]**
(A) 808 (B) 888
(C) 1400 (D) 1538

9. The line-to-line induced emf (in volts), for a three-phase connection is approximately **[2006]**
(A) 1143 (B) 1332
(C) 1617 (D) 1791

10. The fifth harmonic component of phase emf (in volts), for a three-phase star connection is **[2006]**
(A) 0 (B) 269
(C) 281 (D) 808

11. A three-phase synchronous motor connected to AC mains is running at full load and unity power factor. If its shaft load is reduced by half, with field current held constant, its new power factor will be **[2007]**
(A) unity
(B) leading
(C) lagging
(D) dependent on machine parameters

12. A 100 kVA, 415 V (line), star-connected synchronous machine generates rated open-circuit voltage of 415 V at a field current of 15 A. The short-circuit armature current at a field current of 10 A is equal to the rated armature current. The per unit saturated synchronous reactance is **[2007]**
(A) 1.731 (B) 1.5
(C) 0.666 (D) 0.577

13. A three-phase, three-stack, variable reluctance step motor has 20 poles on each rotor and stator stack. The step angle of this step motor is **[2007]**
(A) 3° (B) 6°
(C) 9° (D) 18°

14. In a stepper motor, the detent torque means **[2008]**
(A) minimum of the static torque with the phase winding excited.

(B) maximum of the static torque with the phase winding excited.
(C) minimum of the static torque with the phase winding unexcited.
(D) maximum of the static torque with the phase winding unexcited.

Common Data for Questions 15 and 16:
A synchronous motor is connected to an infinite bus at 1.0 p.u. voltage and draws 0.6 p.u. current at unity power factor. Its synchronous reactance is 1.0 p.u. and resistance is negligible.

15. The excitation voltage (E) and load angle (δ) will, respectively, be **[2008]**
(A) 0.8 p.u. and 36.86° lag
(B) 0.8 p.u. and 36.86° lead
(C) 1.17 p.u. and 30.96° lead
(D) 1.17 p.u. and 30.96° lag

16. Keeping the excitation voltage same, load on the motor is increased such that the motor current increases by 20%. The operating power factor will become **[2008]**
(A) 0.995 lagging (B) 0.995 leading
(C) 0.791 lagging (D) 0.848 leading

17. A field excitation of 20 A in a certain alternator results in an armature current of 400 A in short circuit and a terminal voltage of 2000 V on open circuit. The magnitude of the internal voltage drop within the machine at a load current of 200 A is **[2009]**
(A) 1 V (B) 10 V
(C) 100 V (D) 1000 V

18. The direct axis and quadrature axis reactances of a salient-pole alternator are 1.2 p.u. and 1.0, respectively. The armature resistance is negligible. If this alternator is delivering rated kVA at u.p.f. and at rated voltage then its power angle is. **[2011]**
(A) 30° (B) 45°
(C) 60° (D) 90°

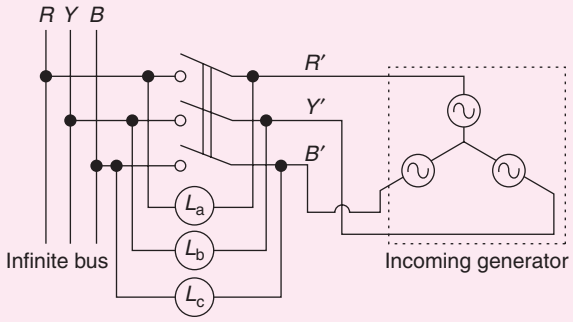
19. A star-connected 400 V, 50 Hz, 4-pole synchronous machine gave the following open-and short-circuit test results:

Open-circuit test: V_{oc} = 400 V (RMS, line-to-line) at field current, I_f = 2.3 A

Short-circuit test: I_{sc} = 10 A (RMS, phase) at field current, I_f = 1.5 A

The value of per phase synchronous impedance in Ω at rated voltage is _____. **[2014]**

20. A three-phase synchronous generator is to be connected to the infinite bus. The lamps are connected as shown in the figure for the synchronization. The phase sequence of bus voltage is R-Y-B and that of incoming generator voltage is R'-Y'-B'.



It was found that the lamps are becoming dark in the sequence L_a-L_b-L_c. It means that the phase sequence of incoming generator is **[2014]**
(A) opposite to infinite bus and its frequency is more than infinite bus
(B) opposite to infinite bus but its frequency is less than infinite bus
(C) same as infinite bus and its frequency is more than infinite bus
(D) same as infinite bus and its frequency is less than infinite bus

21. A 20-pole alternator is having 180 identical stator slots with 6 conductors in each slot. All the coils of a phase are in series. If the coils are connected to realize single-phase winding, the generated voltage is V_1. If the coils are reconnected to realize three-phase star-connected winding, the generated phase voltage is V_2. Assuming full pitch, single-layer winding, the ratio V_1/V_2 is **[2014]**
(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{2}$
(C) $\sqrt{3}$ (D) 2

22. In a synchronous machine, hunting is predominantly damped by **[2014]**
(A) mechanical losses in the rotor
(B) iron losses in the rotor
(C) copper losses in the stator
(D) copper losses in the rotor

23. A non-salient-pole synchronous generator having synchronous reactance of 0.8 p.u. is supplying 1 p.u. power to a unity power factor load at a terminal voltage of 1.1 p.u. Neglecting the armature resistance, the angle of the voltage behind the synchronous reactance with respect to the angle of the terminal voltage in degrees is _____. **[2014]**

24. A three-phase, salient-pole synchronous motor is connected to an infinite bus. It is operated at no load at normal excitation. The field excitation of the motor is first reduced to zero and then increased in the reverse direction gradually. Then the armature current.
(A) increases continuously
(B) first increases and then decreases steeply

(C) first decreases and then increases steeply
(D) remains constant

25. A three-phase, 11 kV, 50 Hz, 2 pole, stars connected, cylindrical rotor synchronous motor is connected to an 11 kV, 50 Hz source, Its synchronous reactance is 50 Ω per phase, and its stator resistance is negligible. The motor has a constant field excitation. At a particular load torque, its stator current is 100 A at unity power factor. If the load torque is increased so that the stator current is 120 A, then the load angle (in degrees) at this load is _____. **[2015]**

26. Consider a system consisting of a synchronous generator working at a lagging power factor, a synchronous motor working at an overexcited condition and a directly grid-connected induction generator. Consider capacitive VAr to be a source and inductive VAr to be a sink of reactive power. Which one of the following statements is TRUE? **[2016]**

(A) Synchronous motor and synchronous generator are sources and induction generator is a sink of reactive power.
(B) Synchronous motor and induction generator are sources and synchronous generator is a sink of reactive power.
(C) Synchronous motor is a source and induction generator and synchronous generator are sinks of reactive power.
(D) All are sources of reactive power.

27. A three-phase, 50 Hz salient-pole synchronous motor has a per-phase direct-axis reactance (X_d) of 0.8 pu and a per-phase quadrature-axis reactance (X_q) of 0.6 pu. Resistance of the machine is negligible. It is drawing full-load current at 0.8 pf (leading). When the terminal voltage is 1 pu, per-phase induced voltage, in pu, is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. C	**2.** C	**3.** D	**4.** B	**5.** B	**6.** B	**7.** A	**8.** D	**9.** C	**10.** D
11. B	**12.** C	**13.** A	**14.** A	**15.** C	**16.** A	**17.** C	**18.** D	**19.** C	**20.** A
21. C	**22.** C	**23.** B	**24.** D	**25.** B	**26.** A				

Practice Problems 2

1. D	**2.** D	**3.** B	**4.** A	**5.** B	**6.** D	**7.** B	**8.** B	**9.** C	**10.** A
11. B	**12.** D	**13.** A	**14.** C	**15.** C	**16.** A	**17.** A	**18.** C	**19.** D	**20.** D
21. A	**22.** D	**23.** C	**24.** B	**25.** B	**26.** C	**27.** C	**28.** A	**29.** D	**30.** C
31. D	**32.** D	**33.** A	**34.** A	**35.** B	**36.** C	**37.** B	**38.** A	**39.** C	**40.** C
41. D	**42.** A	**43.** C	**44.** A	**45.** C	**46.** D	**47.** D	**48.** D	**49.** B	**50.** D
51. A	**52.** B	**53.** C	**54.** D	**55.** B	**56.** C	**57.** C	**58.** B	**59.** C	**60.** C
61. A	**62.** A	**63.** D	**64.** D	**65.** C	**66.** A	**67.** B	**68.** D	**69.** B	**70.** C
71. D	**72.** B	**73.** D	**74.** B	**75.** C	**76.** B	**77.** A	**78.** D	**79.** A	**80.** C
81. D	**82.** D	**83.** B	**84.** C	**85.** A	**86.** D	**87.** C	**88.** A	**89.** D	**90.** D
91. A	**92.** A	**93.** B	**94.** C	**95.** C	**96.** B	**97.** B	**98.** B	**99.** C	**100.** A

Previous Years' Questions

1. B	**2.** C	**3.** B	**4.** (a) B (b) C		**5.** C	**6.** B	**7.** A	**8.** C	**9.** C
10. A	**11.** B	**12.** C	**13.** B	**14.** D	**15.** D	**16.** A	**17.** D	**18.** B	**19.** 15.06
20. A	**21.** D	**22.** D	**23.** 33.61	**24.** B	**25.** –48 to –46°		**26.** A	**27.** 1.606	

TEST
Electrical Machines

Ref: ELE1431435 **Time: 60 min.**

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. An auto transformer supplies a load of 3 KW at 150 V at a unity power factor.

If the applied voltage is 230 V. The power transferred to the load inductively is

(A) 1.043 KW (B) 1.956 KW
(C) 3 KW (D) 1.5 KW

2. Neglecting the effects of armature reaction, if the excitation of a synchronous motor running with a constant load is decreased from its normal value it leads to

(A) increase in p.f.
(B) increase in I_a
(C) decrease in p.f.
(D) increase in I_a and decrease in p.f.

3. No load test on a 3-phase induction motor was conducted at different supply voltage and a plot of input power versus voltage was drawn. This curve is extrapolated to intersect the Y axis. This intersection point yields

(A) friction and windage loss
(B) core loss
(C) stator copper loss
(D) stray load loss

4. In a 3-phase delta/star transformer, the phase displacement of secondary line voltage with respect to corresponding primary line voltages will be _____.

(A) zero (B) 30°lag
(C) 30° lead (D) 180°

5. If a 230 V d.c. series motor is connected to a 230 V a.c. supply,

(A) the motor will vibrate violently.
(B) the motor will run with less efficiency and more sparking.
(C) the motor will not run.
(D) the fuse will be blown.

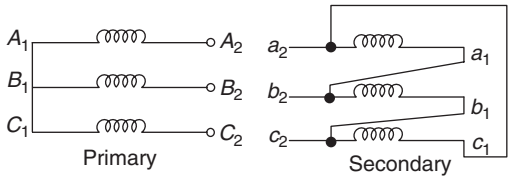
6. A 3-phase 20-pole, synchronous generator has 180 stator slots with single-layer full pitch coils. There are 6 conductors per slot and all coils per phase are connected in series. The rotor is driven at 300 rpm and the flux pole (sinusoidally distributed) is 25 mWb. The voltage induced per phase is nearest to

(A) 500 V
(B) 1000 V
(C) 1500 V
(D) 2000 V

7. The phase sequence of an alternator reverses if

(A) The field current is reversed keeping the direction of rotation same
(B) The field current remains the same but the direction of rotation is reversed
(C) Field current is reversed and number of poles doubled
(D) Number of poles doubled and field current remain same

8. Three single phase transformer are connected to form a 3 phase transformer bank. The transformer are connected as in figure. The transformer connection will be represented by



(A) Yd0 (B) Yd1 (C) Yd6 (D) Yd11

9. The primary current in a current transformer is dictated by

(A) The secondary burden
(B) The core of the transformer
(C) The load current
(D) None of these

10. A 100 MVA synchronous generator operates on full load UPF at a frequency of 50 Hz. The load is suddenly reduced to 50 MW. Due to time lag in governor system, the system valve begins to close after 0.4 seconds. The frequency at the end of 0.4 seconds is _____. (Given H = 5 KW – seconds/KVA of generator capacity)

(A) 51 Hz (B) 52 Hz (C) 53 Hz (D) 49 Hz

11. Which of the following conditions are necessary for parallel operation of alternators?

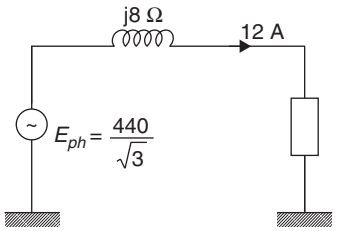
1. Equal terminal voltage
2. Same frequency
3. Same phase sequence
4. Same KVA rating
5. Same $\frac{X}{R}$ ratio

(A) 1, 2 and 3 (B) 2, 3 and 5
(C) 1, 2 and 4 (D) 2, 3 and 4

12. A 400 V, 20 kW, 4 pole, 50 Hz Y-connected induction motor has full load slip of 3.4%. The output torque of the machine at full load is

(A) 250 Nm (B) 141 Nm (C) 120 Nm (D) 131.8 Nm

13. The per phase synchronous reactance of a star connected 440 V, 50 Hz alternator is 8 Ω. It supplies a balanced capacitive load current of 12 A, as shown in figure. If zero voltage regulation is desirable, the load power factor should be

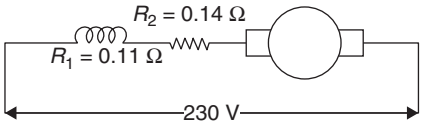


(A) 0.923 (B) 0.783 (C) 0.85 (D) 0.65

14. If the applied voltage of a certain transformer is increased by 50% and the frequency is reduced to 50% (assuming that the magnetic circuit remains unsaturated), the maximum core flux density will _____.
(A) change to three times the original value
(B) change to 1.5 times the original value
(C) change to 0.5 times as the original value
(D) remain the same as the original value.

15. A 20 kVA 400V/200V single phase transformer with a percentage resistance of 4% and percentage reactance of 8% is supplying a current of 80 A to a resistive load. The value of load voltage is
(A) 193.6 V (B) 387.2 V
(C) 190.4 V (D) 197.8 V

16. A DC series motor is rated 230 V, 1000 rpm, 80 A. The series field resistance is 0.11 Ω and the armature resistance is 0.14 Ω. If the flux at an armature current of 20 A is 0.4 times of that under rated condition, calculate the speed at the reduced armature current of 20 A.

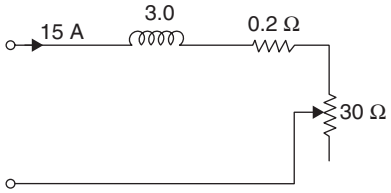


(A) 2168 rpm
(B) 876 rmp
(C) 2678 rpm
(D) 1980 rpm

17. A separately excited dc machine, having an armature resistance of 2 ohms was working on a 220 V supply and drawing 10 A armature current from the source. When the supply voltage suddenly dropped to 200 V, assuming that the field circuit source voltage remained unaffected, how will the armature current of the machine react to the change?
(A) It will initially rise to 11 A and then settle down to 10 A.
(B) It will fall momentarily to 9.09 A and then slowly attain 10 A.
(C) It will reduce to zero first and then settle back to 10 A.
(D) It will remain unaffected by the change and continue to be 10 A.

18. A 50 KW synchronous motor is tested by driving it by another motor. When the excitation is not switched on, diving motor takes 800 W. When the armature is short circuited and rated armature current of 10 A is passed through it, the driving motor requires 2500 W. On open circuiting the armature with rated excitation, driving motor takes 1800 W. Neglecting the losses in the driving motor, the efficiency of the synchronous motor is
(A) 87.71%
(B) 91.83%
(C) 83.05%
(D) 85.32%

19. A 3 ϕ, 50 Hz, 6 pole induction motor has an equivalent diagram as shown in the figure. The torque developed in the motor is



(A) 113.4 Nm
(B) 92.8 Nm
(C) 185.6 Nm
(D) 193.4 Nm

20. Two transformers having equal voltage ratio are connected in parallel. They have equal impedance and equal $\frac{X}{R}$ ratio. The ratio of full load KVA delivered to the sum of their individual KVA ratings is
(A) 0.5 (B) 1 (C) 0.6 (D) 0.75

21. A 400 V, 50 Hz, 100 KVA, 0.9 pf leading, delta connected synchronous machine has synchronous reactance of 3 Ω. The armature resistance can be neglected. The core loss of the machine is 1.2 KW and the friction and windage losses are 2.8 KW. If the shaft is supplying 20 KW at 0.9 leading, then the line current drawn is
(A) 36.88 A
(B) 52.16 A
(C) 18.44 A
(D) 26.08 A

22. A 200 V dc series motor runs at 500 rpm when taking a current of 25 A. The resistance of the armature is 0.5 Ω and that of field is 0.3 Ω. If the current remains constant, the resistance necessary to reduce the speed to 250 rpm is
(A) 1.8 Ω (B) 3.6 Ω (C) 5.4 Ω (D) 7.2 Ω

23. A 3 – ϕ squirrel cage induction motor has a starting torque of 140% and a maximum torque of 280% with respect to rated torque at rated voltage and rated frequency. Neglect the stator resistance and rotor losses. The value of slip for maximum torque is
(A) 300%
(B) 26.79%
(C) 2.67%
(D) 267.9%

Common Data for Questions 24 and 25:

A 250 V separately excited DC motor has armature resistance of 2.5 ohm. When driving a load at 600 rpm with constant torque, the armature takes 20 A. The motor is controlled by a DC chopper operating with a frequency of 400 Hz and an input voltage of 250 V D.C.

24. What should be the value of duty ratio, if it is desired to reduce the speed from 600 rpm to 400 rpm?
(A) 0.63 (B) 0.55
(C) 0.73 (D) 0.85

25. The motor speed at rated current and a duty ratio of 0.5, if the motor is regenerating is?
(A) 225 rpm (B) 525 rpm
(C) 325 rpm (D) 425 rpm

Answer Keys

1. A	**2.** D	**3.** A	**4.** C	**5.** B	**6.** B	**7.** B	**8.** B	**9.** C	**10.** A
11. A	**12.** D	**13.** A	**14.** A	**15.** A	**16.** C	**17.** C	**18.** B	**19.** D	**20.** B
21. A	**22.** B	**23.** B	**24.** C	**25.** B					

Power Electronics and Drives

UNIT V

Chapter 1: Power Semiconductor Devices 3.645

Chapter 2: AC to DC Converters (or) Rectifiers 3.667

Chapter 3: DC–DC Converters (or) Choppers 3.694

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Chapter 1

Power Semiconductor Devices

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Power scale
- Drive applications
- Power devices
- Power diode
- Power transistor
- Bipolar junction transistor
- BJT darlington pair
- Voltage and current rating
- Safe operating region
- Thyristor

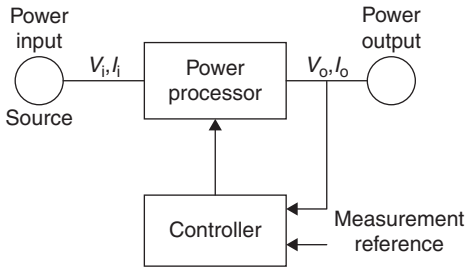
Introduction

To convert and control the flow of electric power by power semiconductor devices where these devices operate as switches by supplying voltages and currents in a form that is optimally suited for user loads.

Power Scale

Milli watts (mW) → Megawatts (MW) → Giga watts (GW)

Block Diagram



Power electronic devices are used to convert electric energy from source to load with the following properties.

- Highest efficiency
- Highest availability
- Highest reliability
- Lowest cost
- Smallest size
- Least weight.

Static Applications

Static applications of power electronics involve non-rotating or moving mechanical components.

Examples:

- DC Power supply
- Un interruptible power supply
- Power generation and transmission (HVDC)
- Electroplating
- Welding
- Heating
- Cooling
- Electronic Ballast

Drive Applications

Drive applications of power electronics intimately contains moving rotating components such as motors.

Examples:

- Electric trains
- Electric vehicles
- Air conditioning system
- Pumps
- Compressor
- Conveyer Belt (Factory Automation)

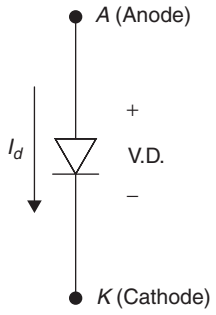
Power Devices

- Power Diodes (or Rectifiers).
- Bipolar Transistors—1948 and Power BJT—Bipolar Junction Transistor—1960.
- Thyristor or SCR (SILICON-CONTROLLED Rectifier)—1957.
- POWER MOSFETs (Metal oxide semiconductor field effect transistor)—1970.
- IGBT (Insulated gate bipolar transistor—1990 a hybrid between MOSFET and BJT).

Typical Power Devices

- Two-terminal devices—Pin Diodes (> 300 V)—Schottky Diodes (< 300 V), No reverse recovery loss.
- Three terminal Devices—BJT (not used much in power converters, high-voltage blocking capability).
- MOSFET (Commonly used for voltages < 300 V, very fast devices).
- IGBT (for voltages > 300 V) a hybrid of BJTs and MOSFETs.
- Thyristors—GTO, IGCT, ETO, MCT etc. used for high-voltage applications.

Power Diode



- It has two terminals, namely cathode and an anode.
- The power circuit controls the ON and OFF states.
- Power diodes are similar to p–n junction diodes but more complex in structure.
  They are of three types:
  - General purpose diodes
  - High-speed (fast recovery) diodes
  - Schottky diodes
- Diode blocks voltage in reverse direction and allows current in forward direction.
- They start conduction once the voltage in the forward direction goes beyond a certain value and a relatively small forward voltage drops across it.
- The diode is said to be reverse biased if the cathode potential is positive with respect to the anode, and a small amount of reverse current flows known as leakage current (mA or mA).
- A diode can be treated as an ideal switch.

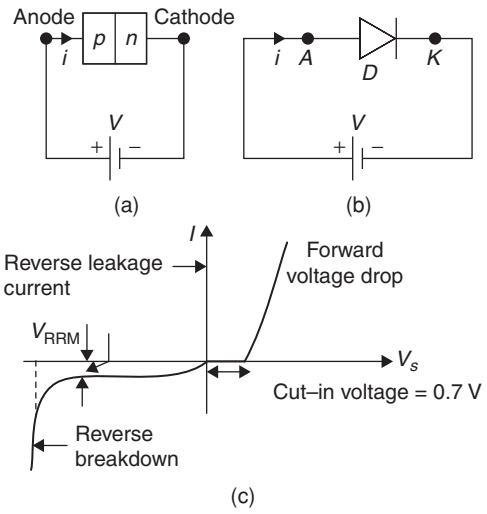


Figure 1 (a) p–n Junction; (b) Diode symbol; (c) *V.I.* Characteristics

- For silicon diode, the threshold voltage is around 0.7 V and the forward drop is around 0.8 V to 1 V.
- When the leakage current increases slowly with the reverse voltage until breakdown or avalanche voltage is reached, breakdown of diode takes place. At this point, diode is turn – on in the reverse direction.
- Peak reverse repetitive voltage (V_{RRM}): The reverse avalanche breakdown of a diode is avoided by operating the diode below specified peak repetitive reverse voltage beyond this voltage power diode is burn – out.
- Manufactures indicate the value of peak inverse voltage (PIV) or VRRM.
- Diode continues to conduct in reverse direction, because of the presence of stored charges in the two layers.
- The time for which the reverse current flows is known as reverse recovery time

$$t_{rr} = t_a + t_b$$

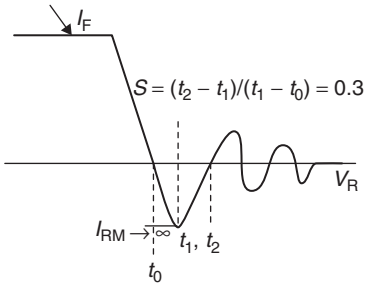
t_a – Time between zero crossing of forward current and I_{RM} (Peak reverse current).

t_b – Measure from the instant I_{RM} to the instant where 0.25 I_{RM} decays.

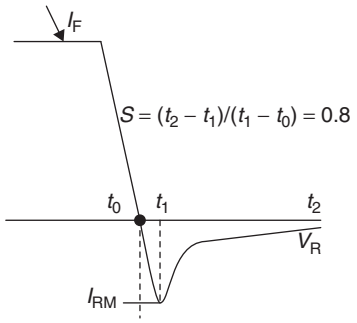
Softness Factor or S-factor

$$S = \frac{t_b}{t_a}$$

- The measure of voltage transients which occurs during diode recovery is called softness factor



Soft Recovery



Solved Examples

Example 1: The diode with a faster operation is
(A) Vacuum diode
(B) Zener diode
(C) Schottky diode
(D) Normal p–n diode

Solution: (C)

Example 2: Reverse recovery of a diode is delayed due to
(A) Metallic effects
(B) Different doping levels of two layers
(C) Higher voltage drop of the device
(D) During forward conduction, charge storage around the junction

Solution: (D)

Example 3: The rate of fall of diode current for a diode is 40 A/μs and reverse recovery time is 2.5 μs. The Peak reverse current (assuming the softness factor as 0.5) is given by
(A) 100 A (B) 75 A
(C) 50 A (D) 80 A

Solution: (A)

$$Q_{rr} = \frac{1}{2}\left(\frac{di}{dt}\right) t_{rr}^{\,2}$$

$$= \frac{1}{2} \times 40 \times (2.5 \times 10^{-6})^2$$

$$= 125\ \mu c.$$

$$I_{rr} = \sqrt{2 \times Q_{rr} \times \left(\frac{di}{dt}\right)}$$

$$= \sqrt{2 \times 125 \times 10^{-6} \times 40 \times 10^{6}}$$

$$= 100\ A$$

Example 4: The correct pairs for the following listed in Table I to Table II are

TABLE I	TABLE II
(1) Schottky Diodes	(P) High-current–Low-voltage applications
(2) General purpose diodes	(Q) Low-speed applications
(3) Fast recovery diodes	(R) Less reverse recovery time (Less than 5 μs)

(A) 1 – Q 2 – P 3 – R
(B) 1 – P 2 – Q 3 – R
(C) 1 – R 2 – Q 3 – P
(D) 1 – P 2 – R 3 – Q

Solution: (B)

Power Transistor

- Can be turned 'ON' and 'OFF' by relatively very small control signals.
- Operated in saturation and cut-off modes only.
- Have controlled turn-on and turn-off characteristics.
- No 'linear region' operation is allowed due to excessive power loss.
- Generally, do not operate in latched mode.

They are classified into:

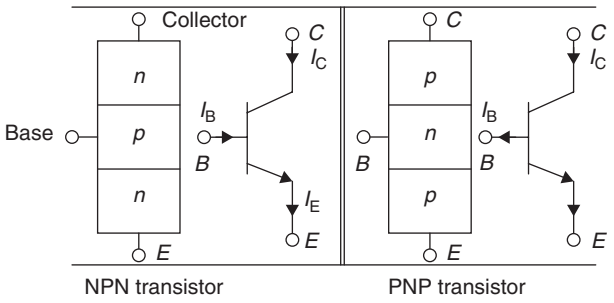
- Bipolar junction transistors (BJT).
- Metal oxide silicon field effect transistor (MOSFET).
- Insulated gate bipolar transistors (IGBT).
- Static induction transistor (SIT).

Emerging New Devices

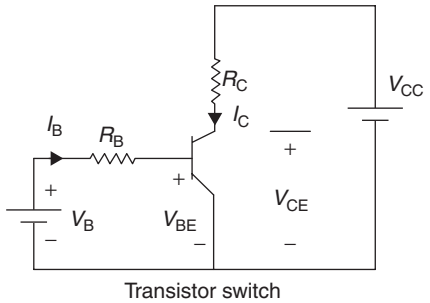
Gate-controlled thyristors (GCTs).

Bipolar Junction Transistor (BJT)

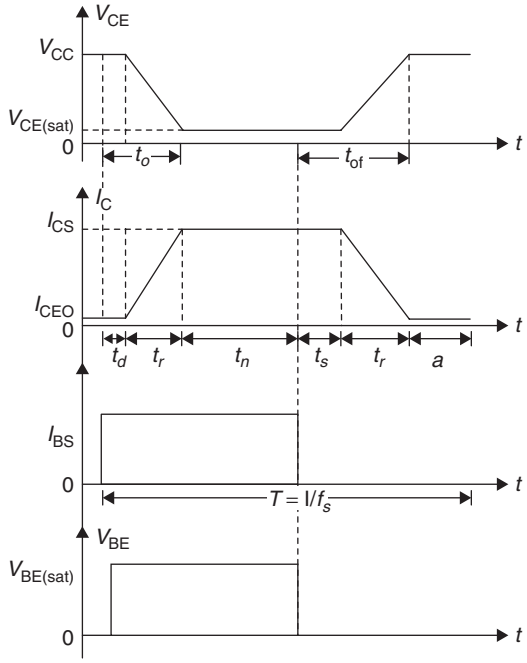
- It is a three layer, two junction, NPN or PNP device.
- It has three terminals, namely emitter, base and collector.
- Current flows in the device is due to movement of both holes and electrons.
- It is a current controlled device.



Symbol



NPN Transistor Circuit



Switching Waveform for NPN Transistor

$t_r \rightarrow$ rise time
$t_d \rightarrow$ delay time
$t_n \rightarrow$ conduction time
$t_s \rightarrow$ storage time
$t_f \rightarrow$ fall time

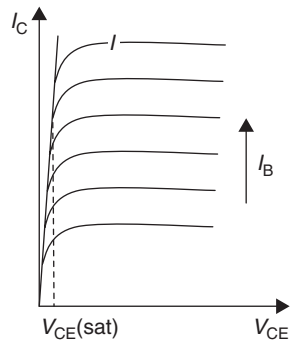


Figure 2 *V–I* characteristics

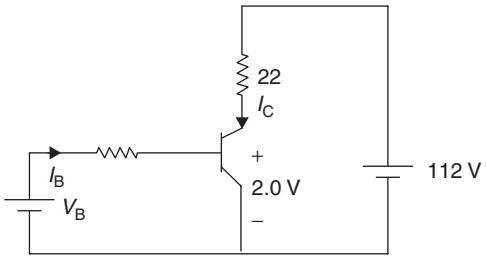
Ratings: voltage: $V_{CE} < 100$ current: $I_C < 400$ A
Switching frequency up to 5 kHz
Low on-state voltage, V_{CE} (sat): 2–3 V
Low current gain ($\beta < 10$)
Need high base current to obtain reasonable I_C

- Common emitter arrangement is more common in switching applications (N–P–N) transistors.
- For $I_B = 0$, V_{CE} is increased a small leakage (collector) current exists.

$$\text{DC current gain} = \frac{I_C}{I_B} \quad \text{and} \quad \frac{I_C}{I_E} = \alpha$$

$$I_E = I_B + I_C$$
$$I_B = I_E - I_C$$
$$I_B = I_E - \alpha I_E$$
$$I_B = I_E(1-\alpha)$$
$$\therefore \ \beta = \frac{I_C}{I_\beta} = \frac{\alpha}{(1-\alpha)}$$
$$\alpha = \frac{\beta}{\beta+1} \text{ or } \alpha = \beta(1-\alpha)$$
$$\beta = \frac{\alpha}{(1-\alpha)} \text{ or } \beta = \alpha\ (1+\beta)$$

Example 5:



The BJT in the figure above is specified to have forward current gain in the range of 4 to 20. The value of base current at saturation for minimum value of current gain is

(A) 1.A (B) 1.25 A
(C) 1.5 A (D) 2 A

Solution: (B)

$$I_{C(sat)} = \frac{112-2}{22} = 5 \text{ A}$$
$$\beta_{min} = 4$$
$$\frac{I_C}{I_B} = \beta \Rightarrow I_B = \frac{I_C}{\beta} = 1.25 \text{ A}$$

- The collector voltage (V_C) must be greater and positive with respect to the emitter voltage, to allow current to flow through the transistor between the collector emitter junctions.
- There is a voltage drop about 0.7 V between the base and the emitter terminal for silicon devices.
- The base voltage of NPN must be greater than this 0.7 V. otherwise the transistor will not conduct with the base current given as

$$I_B = \frac{V_B - V_{BE}}{R_B}$$

- A forward biased p–n junction exhibits two parallel capacitances
  - Depletion capacitance
  - Diffusion capacitance
- Reverse junction has only depletion capacitance.

- These capacitances influence the turn – ON and OFF behaviour of the transistor during transients.
- Due to internal capacitance the transistor does not turn on instantly.
- Delay Time (T_d): The time required for the base current to rise to I_{B_1}, as the input voltage V_B rises from 0 to V_1 before which any collector current flows.

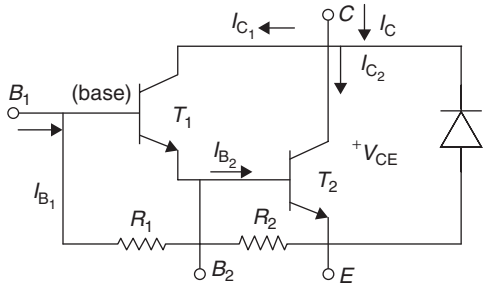
Rise Time (T_r)

The time in which the collector current rises to steady-state value is called Rise Time

- Storage Time (T_s): When input voltage is reversed from V_1 to V_2 and the base current is also changed to $I_1 - I_2$, the collector current does not change for a time t_s, called storage time.
  - Turn-ON Time
    $$T_{ON} = t_d + t_r$$
  - Turn-OFF Time
    $$T_{OFF} = t_s + t_f$$

BJT Darlington Pair

- Normally used when higher current gain is required.
- Whereas power transistor is used for low to medium power applications, with low current gain.
- Power transistor requires a continuous base drive during on–state conditions but does not require forced commutation circuitry.



- A Darlington pair consists of a cascade connection of two transistor stages and can have a large DC common emitter current gain.

  In the above diagram bypass diode has been fabricated on the same chip.
- The shunt resistors R_1 and R_2 reduce the collector leakage current and help establish bias voltage across base emitter junctions. The intermediate base B_2 can be used for fast turning off T_2.
- Base power requirement is low but it is at a cost of reduced switching frequency and higher conduction drop

$$\beta = I_C / I_{B_1} = (I_{C_1} + I_{C_2}) / I_{B_1}$$

$$= \frac{I_{C_1}}{I_{B_1}} + \frac{I_{C_2}}{I_{B_1}}$$

$$= \beta_1 + \left[\frac{I_{C_2}}{I_{B_2}}\right]\left[\frac{I_{B_2}}{I_{B_1}}\right]$$

$$= \beta_1 + \beta_2 \left[\frac{I_{B_1} + I_{C_1}}{I_{B_1}}\right] = \beta_1 + \beta_2 (1 + \beta_1)$$

$$\beta = \beta_1 + \beta_2 + \beta_1 \beta_2$$

Voltage and Current Rating

- Voltage rating is defined as collector to emitter breakdown voltage at a relatively higher value of collector current with base open.
- Collector current of a device are specified as I_{DC} continuous DC current $I_{Cmax} \rightarrow$ peak current which are limited on the basis of maximum junction temperature.
- Current Gain: The DC current gain of a transistor is a strong function of collector current it is influenced by the junction temperature and collector voltage.
- The gain is low at low collector current, it then increases with collector current, reaches peak value and then decreases again.

Safe Operating Region

- The collector current I_C and voltage must always lie within this area for reliable operation of the transistor. For DC as well as single pulse operation I_C and V_{CE} are taken as logarithmic scale.

Forward-biased Safe Operating Area (FBSOA)

When base–emitter junction is forward biased to turn on the transistor

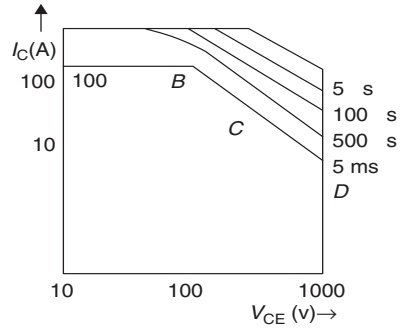


Figure 3 Forward-biased safe operating area

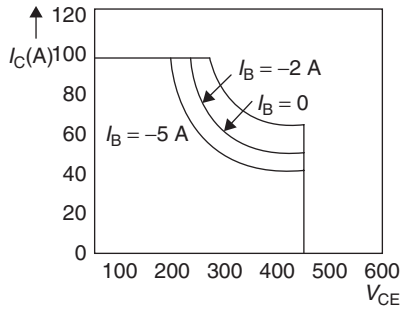


Figure 4 Reverse-block safe operating area

Thyristor (SCR)

- A thyristor is a four-layered semiconductor that is often used or handling large amounts of power.
- It can be turned ON or OFF; it can regulate power using something called phase angle control.
- It has the same characteristics as thyratron tube, it is a solid state device.
- It is a three-junction PNPN semiconductor switching device and has three terminals anode, cathode and gate.

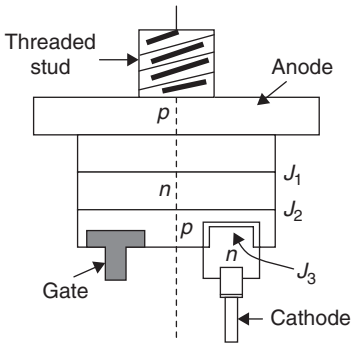


Figure 5 Schematic of thyristor

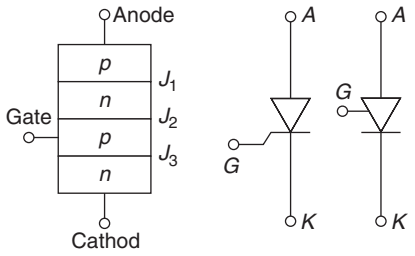


Figure 6 Circuit symbols of thyristor

- The threaded portion is meant for tightening the thyristor to the frame with the help of a nut.

Operation

- If the forward breakover voltage (V_{BO}) is exceeded the SCR 'Self-triggers' into conducting state.
- The presence of gate current will reduce V_{BO}.
- Normal conditions for thyristor to turn on: The device is in forward-blocking state (V_{AK} is positive).
- Once conducting the anode current is latched. V_{AK} collapses to normal forward voltage drop, typically 1.5–3 V.
- In reverse-biased mode the SCR behaves like a diode.

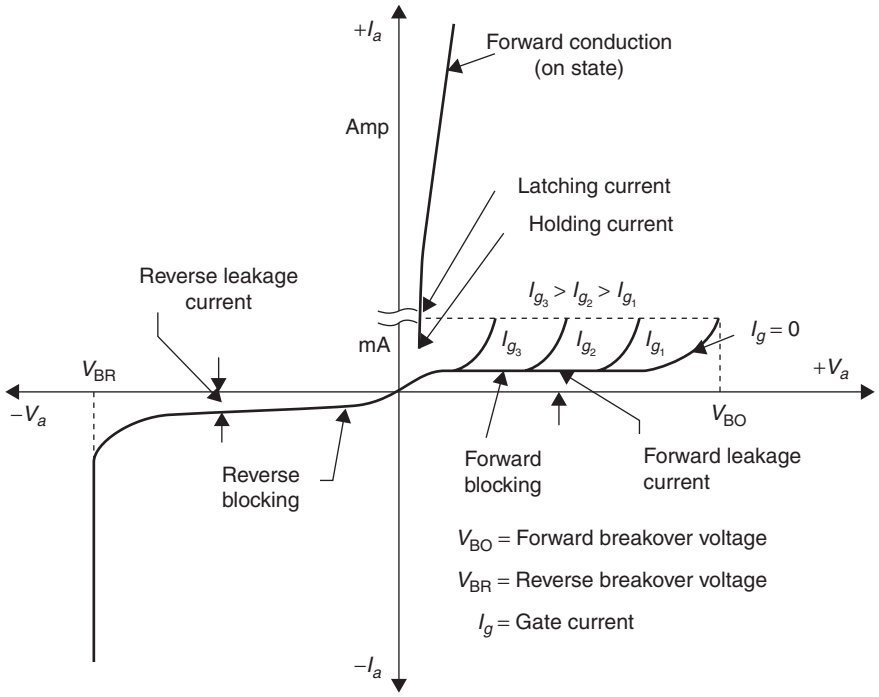


Figure 7 *V* – *I* Characteristics

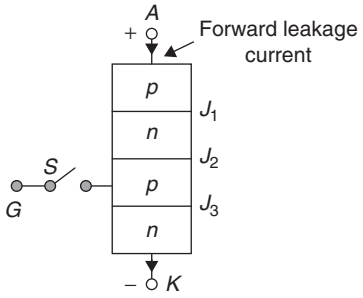
Forward Conduction Mode

- A thyristor is brought to the forward conduction mode by exceeding forward breakover voltage or by applying a gate pulse between gate and cathode.
- In this mode thyristor behaves as a closed switch and the voltage across the thyristor is 1 to 2 V depending on the rating of SCR.
- This small voltage drop depends upon the ohmic drop in the four layers.

Forward-blocking Mode

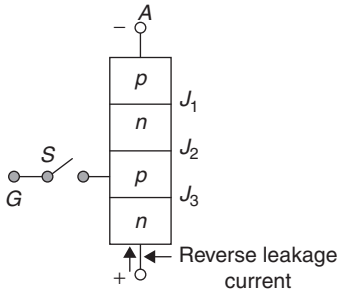
- In this mode anode is made positive with respect to cathode as shown with gate circuit open. J_1, J_3 are forward biased and J_2 is reverse biased.

- In this mode a small leakage current results and increasing the voltage in forward direction, J_2 will have an avalanche breakdown at a 'forward peak over voltage' V_{BO}. For a voltage less than V_{BO}, thyristor acts as open switch as it offers a high impedance.



Reverse-blocking Mode

When cathode is made positive with respect to anode with the switch *S* open as shown in the figure, thyristor is reverse biased.



- Junctions J_1, J_3 are seen to be reverse biased and J_2 forward biased. A small leakage current of a few milli amperes flows.
- This is the reverse-blocking mode called OFF-state of the thyristor. In this mode, increasing the voltage in reverse direction, avalanche breakdown occurs at J_1 and J_3 at a certain breakdown voltage called reverse breakdown voltage.
- At this voltage, a large reverse current results which can lead to thyristor damage. Hence maximum working voltage should not exceed V_{BR}. In this mode thyristor offers a high impedance, thus acts as an open switch.

Example 6: An asymmetrical thyristor has
(A) Anti parallel connection of a diode and thyristor.
(B) No reverse-blocking capability and small forward-blocking capability.
(C) Small forward-blocking capability and large reverse-blocking capability.
(D) Small reverse-blocking capability and large forward-blocking capability.

Solution: (D)

Example 7: Reverse conducting thyristors would possess which among the following features
(A) High survival voltage
(B) Large current handling capacity
(C) Faster turn off
(D) All of the above

Solution: (D)

Thyristor Turn-ON Methods

Forward Voltage Triggering

- When anode to cathode voltage exceeds forward breakover voltage thyristor turns ON.
- Junction J_2 has an avalanche break down, allows free movement of carriers across three junctions and large forward anode current results.
- Forward V_{BO} and reverse V_{BO} are of the same magnitude and are temperature dependent but practically V_{BR} is slightly more than V_{BO}.
- V_{BR} is taken as final voltage rating of a device.
- After avalanche breakdown, if the anode voltage is reduced below V_{BO}, SCR will continue conduction of current. The SCR is then turned off only by reducing the anode current below a certain value called holding current (I_H).

Gate Triggering

- When there is a positive gate voltage between gate and cathode, charges are inducted into the inner p-layer and V_{BO} is reduced, since cathode n-layer is more heavily doped than gate p-layer.
- As the thyristor is forward-biased, some of these electrons reach J_2.
- As a result, width of depletion layer shrinks, which causes J_2 to breakdown at an applied voltage lower than forward V_{BO}.
- Increasing the magnitude of gate current will turn on the thyristor at a lower forward voltage.

Once SCR is in conduction, reverse biased J_2 no longer exists and the conduction of current from anode to cathode remains unaffected. However if gate current is reduced to zero before the anode current attains a value called latching current the thyristor will turn-off again.

Switching Characteristics

Switching Characteristics During Turn-ON

A forward-biased thyristor is turned ON when a positive gate voltage is applied across gate and cathode, and it takes some time from forward OFF state to forward ON-state known as transition time. This transition time is divided into three intervals:

1. **Delay time (t_d):** It is defined as the time during which anode voltage falls from V_a to 0.9 V_a, where V_a is the initial value of anode voltage.

(or)

It can be defined as the time during which anode current rises from forward leakage current to $0.1I_a$, where, I_a is the final value of anode current.

2. **Rise time (t_r):** It is the time required for the forward-blocking OFF-state voltage to fall from 0.9 to 0.1 of its initial value. This time is inversely proportional to the magnitude of gate current and its build-up rate. (t_r) can be minimized if high and steep current pulses are applied to the gate.

   For series RL circuit $\frac{di}{dt}$ is slow therefore, t_r is more and for RC circuit $\frac{di}{dt}$ is higher, causing t_r to be less.

   Due to high anode voltage V_a and large anode current, turn-on losses are the highest during rise time.

3. **Spread Time (t_p):** It is the time required for the forward-blocking voltage to fall from 0.1 of its initial value to the ON-state voltage drop (1 to 1.5 V).

   After the spread time anode current attains steady-state value and voltage drop across SCR is equal to the ON-state voltage drop of the order of 1 to 1.5 V.

   Total turn-on time is typically of the order 1 to 4 μS depending upon the anode circuit parameters and gate signal wave shapes.

Switching Characteristics During Turn-OFF

- The process involving change of state of an SCR from the conduction state to forward-blocking state is called commutation process or turn-off process.
- If forward voltage is applied to the SCR at the moment, its anode current falls to zero the device is still in conduction since the carriers and electrons in the four layers still favour conduction. Therefore, a thyristor is reverse biased for a finite period after the anode current has reached zero.

Turn-off time (t_q): It is defined as the time between the instant anode current becomes zero and the instant SCR regains the forward-blocking capability.

Example 8: Which among the following statements regarding a Thyristor is/are true?

1. Thyristor can be turned off by a negative gate pulse.
2. Thyristor would remain in its blocking state if the forward anode current is reduced below its holding current.
3. Due to the regenerative effect in a thyristor during turn on, it can turn on with a small gate current and latch into conduction.

(A) 1 and 2 only
(B) 2 and 3 only
(C) 1, 2 and 3
(D) 1 and 3 only

Solution: (B)

Thyristor Protection

1. $\frac{di}{dt}$ **protection:** If the rate of rise of anode current is higher as compared to the spread velocity of carriers, local hot spots will be formed near the gate terminal due to high current density which can destroy the thyristor.

   The rate of rise of anode current at the time of turn-ON should be kept low. This can be achieved using a small inductor in series with the anode circuit.

2. $\frac{dV}{dt}$ **protection:** When anode is made positive with respect to cathode, junction J_2 is reverse-biased and having the characteristics of a capacitor due to stored charges across the junction.

   Entire voltage across the thyristor develops across the J_2 junction and if the charge stored is Q then

$$\text{Charging current, } i = \frac{dQ}{dt} = \frac{d}{dt}(C_j V_a)$$

$$= C_j \frac{dV_a}{dt} + V_a \frac{dC_j}{dt}$$

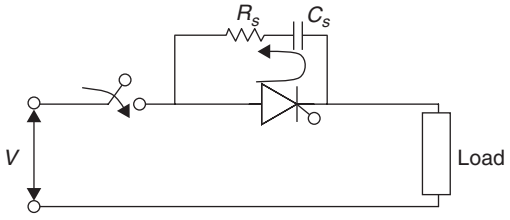
   C_j, the capacitance of the junction J_2 is a constant

$$i = \frac{C_j \, dV_a}{dt}$$

   If the rate of rise of forward voltage $\frac{dV_a}{dt}$ is high then resultant high charging can turn-ON the SCR even when gate signal is zero.

   This can be avoided by using a snubber circuit in parallel with the device.

   Snubber circuit consists of a series combination of resistance R_s and capacitance C_s.



   Before SCR is being fired by gate pulse, C_s charges to full voltage V_s.

   When SCR is turned ON, capacitor discharges through SCR and sends a current equal to V_s/ (resistance of the local path formed by C_s and SCR).

   As this resistance is quite low, the turn-ON $\frac{di}{dt}$ will be large enough to damage SCR. Therefore to limit the magnitude of discharge current, a resistance R_s is inserted in series with C_s.

Thyristor can be turned off by applying negative gate current. It can only be turned off if I_a goes negative (reverse).

- This happens when negative portion of the sine-wave occurs (natural commutation).
- Another method is forced commutation, where the anode current is 'diverted' to another circuitry.

Latching Current

Latching current is that minimum value of anode current which must be attained during turn on process to maintain conduction even after removal of gate signal.

- The gate pulse width should be chosen such that the anode current rises above the latching current.

Holding Current

The minimum value of anode current below which it must fall for turning off the thyristor

- Generally latching current is more than the holding current (2 or 3 times more than the holding current).

Types of Thyristors Phase-controlled

- Rectifying line frequency voltage and current for AC and DC motor drives.
- Large voltage (up to 7 kv) and current (up to 4 kA) capability.
- Low-on-state voltage drop (1.5 to 3 V).

Inverter Grade

- Used in inverter and chopper.
- Quite fast. Can be turned – on using 'forced-commutation' method.

Light Activated

- Similar to phase controlled, but triggered by pulse of light.
- Normally very high power ratings.

Example 9: In a $\frac{dV}{dt}$ protection employed for a thyristor protection it is observed that junction capacitance of the thyristor is 20 pF. The latching current of the thyristor is 10 mA. The value of additional capacitance required to limit $\frac{dV}{dt}$ to 200 V/μs is

(A) 50 pF (B) 20 pF
(C) 30 pF (D) 40 pF

Solution: (C)

$$\text{Latching current } I_L = c\frac{dV}{dt}$$

$$C = \frac{I_L}{\left(\frac{dV}{dt}\right)} = \frac{10\times10^{-3}}{(200\times10^{6})} = 50\times10^{-12} = 50 \text{ pF}$$

Additional capacitance = 30 pF

TRIAC

- Dual polarity thyristors.
- A TRIAC or bidirectional triode thyristor can conduct in both directions and is used in AC phase control.
- If two SCRs are joined in back-to-back parallel fashion two Schottky as if diodes were joined, we have a new device called TRIAC.

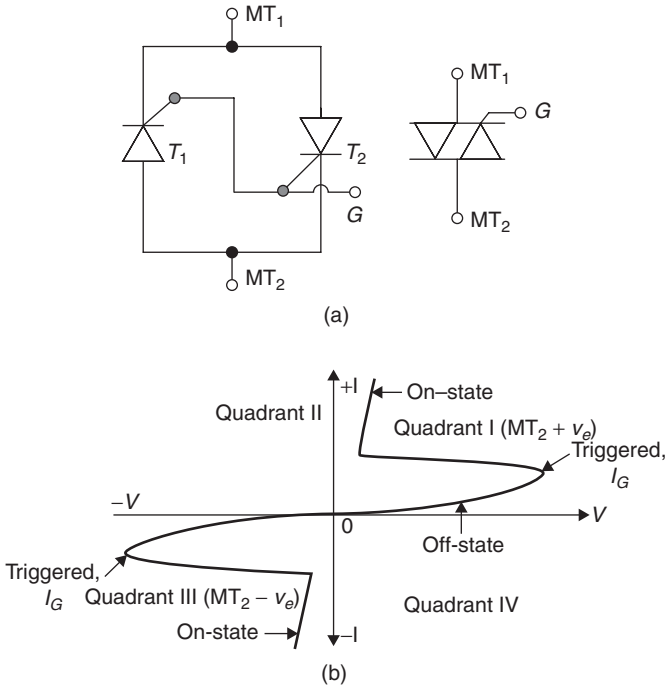


Figure 8 (a) Equivalent of TRIAC; (b) *V–I* characteristics

- The terminals of a TRIAC cannot be designated as anode and cathode because it is a bidirectional device.
- If terminal MT_2 is positive with respect to terminal MT_1, the TRIAC can be turned ON by applying a positive gate signal between gate G and terminal MT_1.
- If terminal MT_2 is negative with respect to terminal MT_1, it is turned ON by applying a negative gate signal between gate G and terminal MT_1.
- It is not necessary to have both polarities of gate signals and a TRIAC can be turned ON with either a positive or a negative gate signal.
- In practice, the sensitivities may vary from one quadrant to the another and TRIACs are normally operated in quadrant I^+ (positive gate voltage and gate current) and quadrant III^- (negative gate voltage and gate current).
- Individual SCRs are more flexible to use in advanced control systems; these are more commonly seen in circuits like motor drives.
- TRIACs are usually seen in simple low power applications like household dimmer switches.
- The TRIAC is easy to use and provides vast advantages over the use of two thyristors for many low power applications. Where higher power needed, two thyristors placed in is anti-parallel are almost always used.

Example 10: An incorrect statement regarding the disadvantages of a Triac is

(A) Triac is highly sensitive towards gate current.

(B) Triac protection requires a well-designed RC snubber circuit.

(C) Triac usage is difficult with inductive load due to lower rating of reapplied

$$\left(\frac{dV}{dt}\right)$$

(D) Storage effect of the minority carrier results in longer turn OFF time.

Solution: (A)

Gate current sensitivity of Triac is poorer compared to other Power semiconductor devices.

GTO's (Gate-turn-OFF Thyristor)

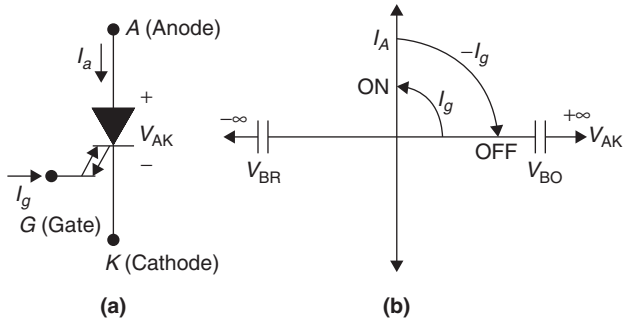


Figure 9 (a) Symbol; (b) Characteristics *V–I*

- The gate turn off thyristor is a PNPN device which can be turned on like an ordinary thyristor by a pulse of positive gate current.
- It can be easily turned off by a negative gate pulse of approximate amplitude.
- As no forced commutation circuitry is required for GTOs, these devices are compact and less costly.
- The negative gate current required to turn OFF GTO is quite a large percentage (20 to 30%) of anode current 800 A.
- GTO requires a negative current pulse of 200 A peak for turning it OFF.
- A Thyristor – transistor like behaviour of GTO can be explained by the two transistor analogy.
- If the transistor T_2 with current gain is designed to be near unity in the on state, and if negative gate current can completely divert the P–N–P collector T_{C_1} out of the gate, the device can be turned off successfully.
- Turn-off gain is defined as the ratio of anode current prior to turn OFF to the negative gate current required for turn-OFF. It is very low, typically between 3 and 5.

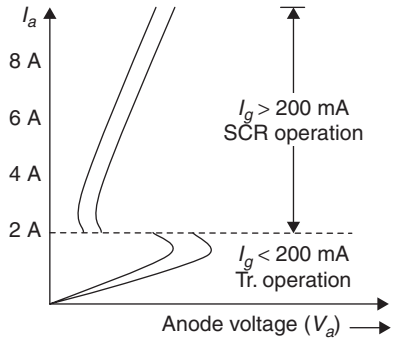


Figure 10 Static. *V–I* characteristics

Switching Characteristics

Turn-ON

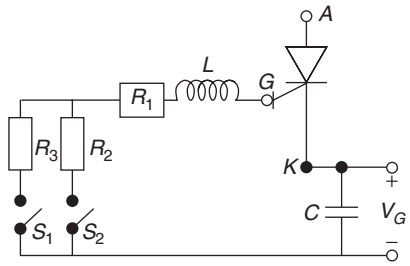
- A large initial gate trigger pulse is required to turn-on GTO since it has highly interdigitated gate structure with no regenerative gate.
- The rate of rise of $\frac{di}{dt}$ gate current affects the device turn-ON losses. A longer period is required, if the anode current $\frac{di}{dt}$ is low such that I_{GM} is maintained until a sufficient level of anode current is established.

On State

- Once the GTO is turned ON forward gate current must be continued for the whole of the conduction period to ensure the device remains in conduction.
- Otherwise the device cannot remain in conduction during the ON state period. The on state gate current should be at least 1% of the turn-ON pulse to ensure that the gate does not unlatch.

Turn-OFF

A turn-OFF circuit arrangement of a GTO is shown in the figure.



- Since a GTO requires a large turn-OFF current, a charged capacitor C is normally used to provide the required turn-off current. L limits the turn-OFF $\frac{di}{dt}$ of the gate current

through the circuit formed but R_1, R_2, S_1 and L. The gate circuit supply voltage V_{GS} should be selected to give the required value of V_{GQ}.

- After the fall of the tail current to zero the gate should ideally be kept reverse biased and turn-OFF period begins.
- The reverse bias ensures maximum blocking capability, reverse bias can be obtained either by keeping S_1 closed during the whole OFF state period (or) by using a higher impedance circuit S_2 and R_3 provided a minimum negative voltage exits, it must also sink, the gate leakage current.
- GTOs are mostly used in voltage source converters in which a fast recovery antiparallel diode is required across each GTO. Thus GTOs normally do not need reverse voltage capabilities, such GTOs are known as asymmetric GTOs.
- Controllable peak on site current I_{TGQ} is the peak value of on state current that can be turned off by gate control.
- Once a GTO is turned OFF, the load current I_L, which is diverted through and charges the snubber capacitor determines the reapplied $\frac{di}{dt}$.

$$\frac{dV}{dt} = \frac{I_L}{C_s}$$

$C_s \rightarrow$ Snubber capacitance

- Gate drive design is very difficult due to very large reverse gate current at turn off

Ratings: Highest power ratings switch:
Voltage: $V_{AK} < 5$ kV
Current: $I_a < 5$ kA
Frequency < 5 kHz

MOSFETs (Metal Oxide Silicon Field Effect Transistor)

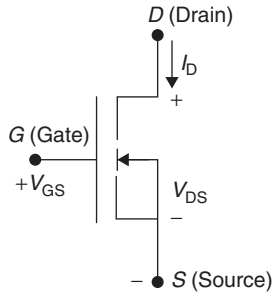


Figure 11 **MOSFET:** Symbol (n-channel)

Power MOSFET

- A power MOSFET is a voltage controlled device and requires only a small amount of input current.
- Switching speed is very high and switching times are of the order of nanoseconds.
- Two types of MOSFETs are
  1. Depletion MOSFET
  2. Enhancement MOSFET
- The three terminals are called gate, drain and source.
- With a positive voltage applied to the gate with respect to source (n-channel), it introduces an n-channel and permits electron current to flow from the source to the drain with applied voltage V_{DS}.
- Because of the SiO_2 isolation, the gate circuit impedance is extremely high, typically in the range of 10^9 W. This feature permits power MOSFET to drive directly from CMOS or TTL Logic.
- The devices have an integrated reverse rectifier, which permits free-wheeling current of the same magnitude as that of the main device.
- The MOSFET is used for low-power, high-frequency applications. Also DMOS structure is used for high-voltage devices.

n-channel Depletion-type MOSFET

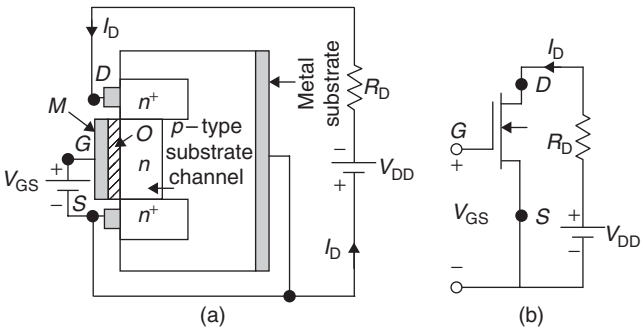


Figure 12 (a) Basic structure; (b) Symbol

p-channel Depletion-type MOSFET

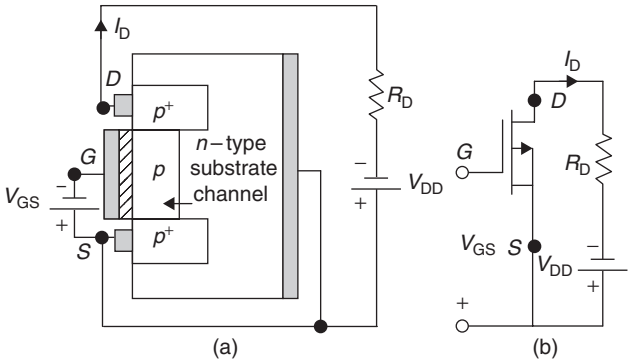
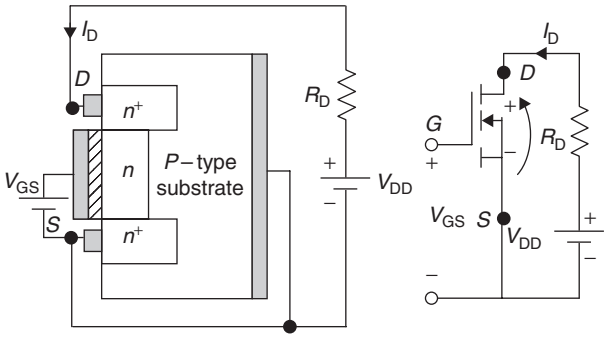
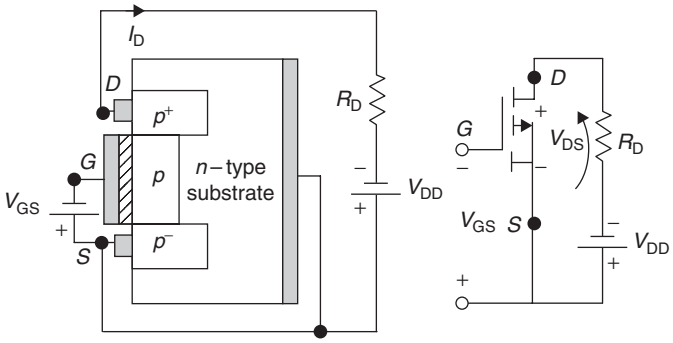


Figure 13 (a) Basic structure; (b) Symbol

n-channel Enhancement-type MOSFET

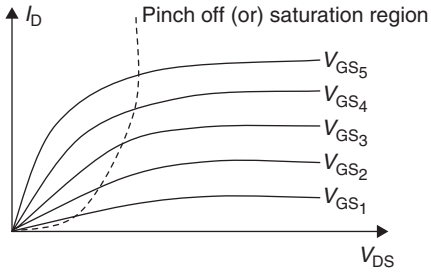


p-channel Enhancement-type MOSFET



Steady-State Characteristics

Drain-source steady-state characteristics are as shown below



The gate circuit has a threshold voltage, typically 2 to 4 V below which the drain current is very small. With V_{GS} beyond the threshold value $I_D - V_{DS}$ characteristics have two distinct regions, a constant resistance [$R_{DS(ON)}$] region and a constant current region.

- The [$R_{DS(ON)}$] of a MOSFET is a key parameter and determines the conduction voltage drop. For a device the [$R_{DS(ON)}$] increases with the voltage rating and has positive temperature coefficient.

Merits and Demerits when Compared to BJT

- Power MOSFET has lower switching losses but its ON-state resistance and conduction losses are more.
- At higher frequencies MOSFET is preferred and at lower frequencies BJT is preferred (30 kHz)
- MOSFET has positive temperature coefficient of resistance whereas BJT has negative temperature coefficient of resistance.
- In MOSFET secondary breakdown does not occur, because it has positive temperature coefficient of resistance.
- Power MOSFET of higher voltage rating has more conduction loss.

Ratings

Voltage $V_{DS} < 500$ V,
Current $I_{DS} < 300$ A,
Frequency $f > 100$ kHz

- For some low power devices (few hundred watts) may go up to MHz range.
- Turning ON and OFF is very simple.
- Gate drive circuit is simple.
- Power MOSFET is basically a low-voltage device. High-voltage device are available up to 600 V but with limited current.
- Can be paralleled quite easily for higher current capability.
- Internal (dynamic) resistance between drain and source during on state R_{DS} (ON), limits the power handling capability of MOSFET.
- High losses especially for high-voltage device due to R_{DS} (ON).
- Dominate in high-frequency application (>100 kHz) Biggest applications is in switched-mode power supplies.

Example 11: A power MOSFET would act as an amplifier in its
(A) Both linear and saturation regions
(B) Linear region only
(C) Saturation region only
(D) At the boundary of these regions.

Solution: (C)
Explained from the operating characteristics of MOSFET.

Example 12: Two parallel connected MOSFET'S carry a total current of 40 A. The drain source voltage and current storing series resistances of MOSFET 1 are $V_{DS_1} = 5$ V and $R_s = 1\ \Omega$ and that of MOSFET 2 are $V_{DS_2} = 6$ V and $R_{S_2} = 1\ \Omega$. The drain current of each transistor is
(A) 20 A, 20 A
(B) 15 A, 25 A
(C) 20.5 A, 19.5 A
(D) None of the above

Solution: (C)

$$I_T = I_{D_1} + I_{D_2}$$

$$V_{DS_1} + I_{D_1}R_{S_1} = V_{DS_2} + I_{D_2}R_{S_2}$$

[Due to Parallel connection]

$$= V_{DS_2} + (I_T - I_{D_1})R_{S_2}$$

$$I_{D_1} = \frac{V_{DS_2} - V_{DS_1} + I_T R_{S_2}}{R_{S_1} + R_{S_2}}$$

$$= \frac{6-5+40\times1}{1+1} = 20.5$$

$$I_{DQ} = 40 - 20.5 = 19.5 \text{ A}$$

IGBTS (Insulated Gate Bipolar Transistor)

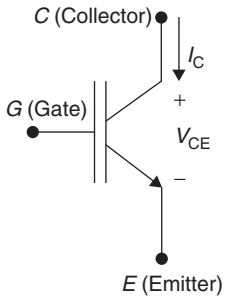
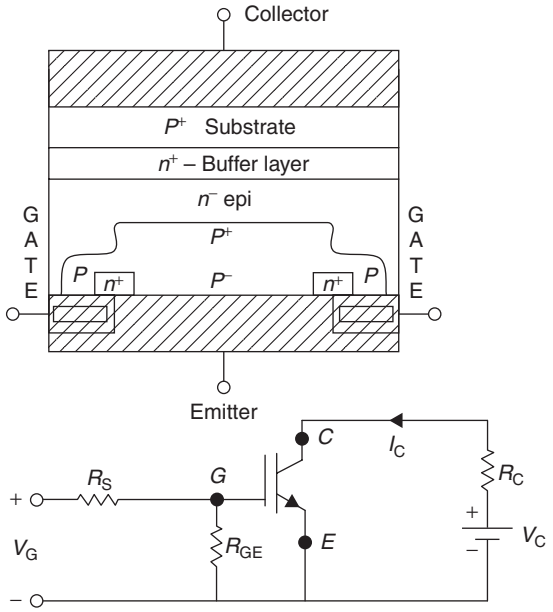


Figure 14 IGBT: Symbol

- Combination of BJT and MOSFET characteristics.
- Gate behaviour similar to MOSFET—easy to turn on and off.
- Low losses like BJT due to low on—state collector–emitter voltage (2–3 V).
- IGBT has high input impedance like a MOSFET and low ON state power loss as in a BJT.
- IGBT is virtually constructed in the same manner as a power MOSFET however there is a major difference in the substrate.



Symbol and Circuit for an IGBT

Ratings

Voltage: $V_{CE} < 3.3$ kV

Current $I_C < 1.2$ kA

Currently available

Latest: HVIGBT 4.5 kV/1.2 kA switching frequency up to 100 kHz.

Typical applications: 20–50 kHz.

- IGBTs are widely used in medium-power applications such as DC and AC motor devices, UPS Systems, power supplies and drives.

Comparison of IGBT with MOSFET

1. The terminals used in MOSFET are gate, source, drain, but in IGBT the terminals are represented by gate, emitter and collector.
2. The input impedance of both MOSFET and IGBT is high.

Insulated Gate-commutated Thyristor (IGCT)

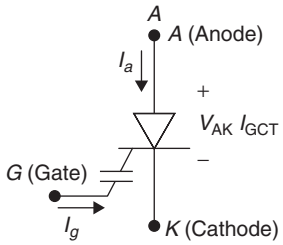


Figure 15 IGCT: Symbol

- Conducts like normal thyristor (latching) but can be turned off using gate signal, similar to IGBT turn off; 20 V is sufficient.
- Power switch is integrated with the gate drive unit.

Ratings

Voltage: $V_{AK} < 6.5$ kV

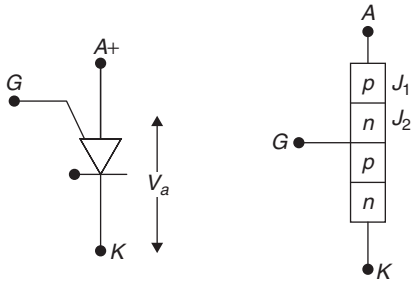
Current: $I_a < 4$ kA

Frequency < 1 kHz

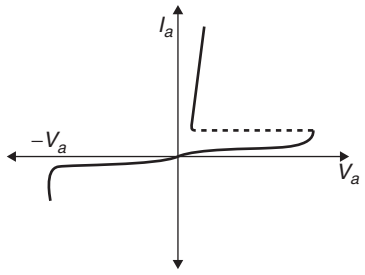
- Currently 10 kV device is being developed.
- Very low on-state voltage 2.7 V for 4 kA device.

Other Members of Thyristor Family

Programmable Unijunction Transistor (PUT)



- It is an ordinary unijunction transistor.
- V–I characteristics is almost similar to an ordinary UJT.



- It is a four layer P–N–P–N device with a gate connected directly to the sandwiched N-type layer.
- The term 'programmable' is applied because the inter base resistance R_{BB}, the intrinsic standoff ratio and peak point voltage V_p as defined in UJT can be programmed to any desired values through external base Resistors R_{B_1} and R_{B_2} and the supply voltage V_{BB}.
- Because of its superiority over ordinary unijunction transistors it is popularly used for relaxation oscillators, time delay, logic and SCR trigger circuits.
- Its largest rating is about 200 V and 1 A.

Silicon Unilateral Switch (SUS)

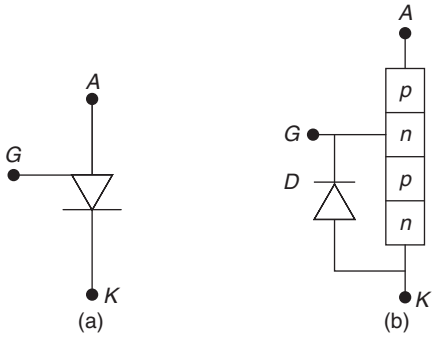
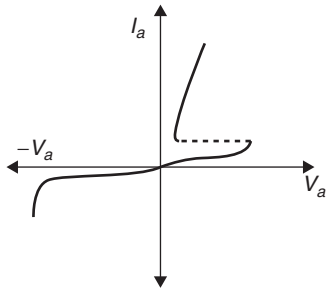


Figure 16 (a) Symbol; (b) Structure

- It is a type of thyristor used as a breakover device that conducts in only one direction; it has a third terminal that is used to alter the breakover voltage if connected to a zener diode



- It is an inbuilt low-voltage avalanche diode between gate and cathode.
- Mainly used in timing, logic and trigger circuits.
- Ratings are about 20 V and 0.5 A.

Silicon-controlled Switch (SCS)

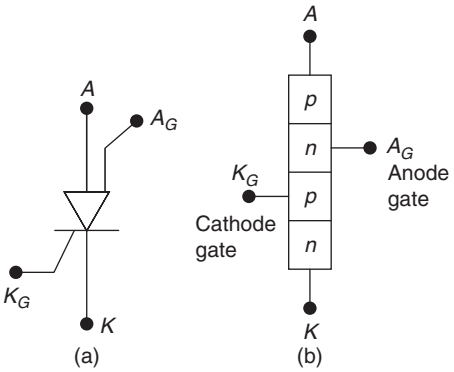


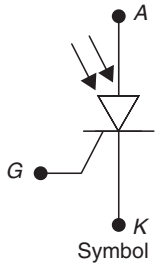
Figure 17 (a) Symbol; (b) Structure

- Like SCR, it is a unilateral, four layer three-junction P–N–P–N Silicon device with four electrodes, namely cathode, cathode gate G_1, anode gate G_2 and the anode A.
- Low-power device.
- Handles currents in milli amperes.
- A positive pulse at gate KG turns on the device.
- (Just like an SCR) and a negative pulse at AG turns it off (just like a GTO).

Ratings

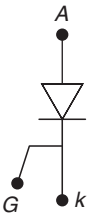
- 100 V and 200 mA.
- Can be operated like an OR gate.
- Used in timing, logic and triggering circuits, pulse generators, voltage sensors and oscillators.

Light-Activated Thyristor (LASCR)



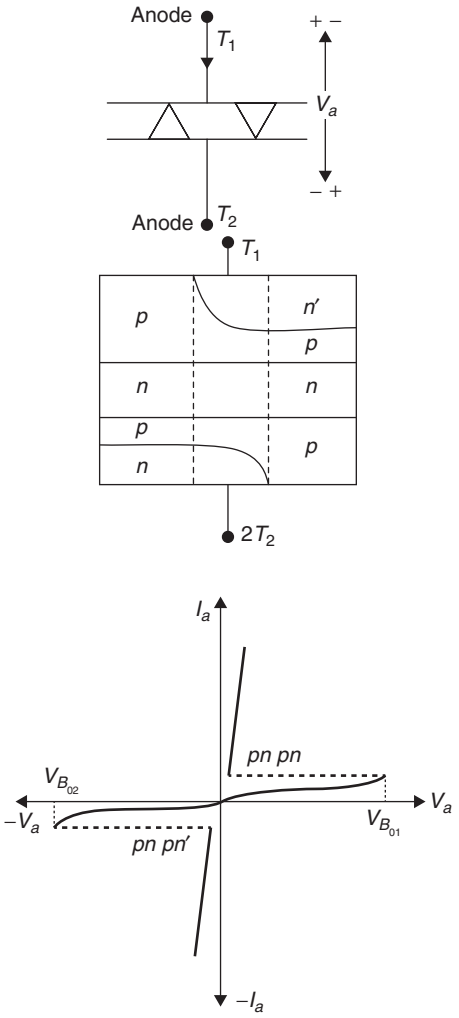
- It is an SCR whose state is controlled by the light falling upon a silicon semiconductor layer of the device.
- They are used in high-voltage high-current applications static reactive-power compensation, optical light controls, relays, phase control, motor control and a variety of computer application.
- The maximum currents 3.5 kA, and voltage 6 kV with on state voltage drop of about 2 V and with light – triggering requirements of 5 mW.
- Interesting application of an LASCR is in the AND & OR circuits.

Static Induction Thyristors (SITHS)



- Static induction thyristor is a thyristor with a buried gate structure in which the gate electrodes are placed in n-base region. Since they are normally on-state, gate electrodes must be negatively biased to hold-off-state.
- The V–I characteristics of a SITH are similar to those of an SCR.
- The ON state voltage drop is low.
- They are available up to about 2500 V, 500 A ratings.
- Used for medium-power converter with frequency range beyond that used for GTOs.

The DIAC (Bidirectional Thyristor Diode)



V–I Characteristics

- Diac is a two-electrode device; it can conduct in either direction.
- Diac stands for ‘diode for AC’. The four layers are pn pn and pn pn′.
- It has symmetrical breakdown characteristics.
- Its leads are interchangeable. When conducting it acts like an open switch.
- It is sometimes called a gate less triac.
- Diac is mainly used for triggering triacs.
- A triac requires either positive or negative gate pulse for turning it on. This is provided by a Diac.

Asymmetrical Thyristor (ASCR)

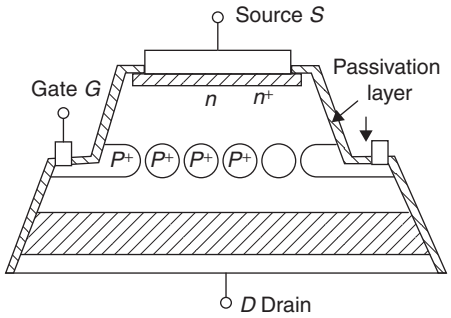
- Specially fabricated to have limited reverse voltage capability, by which it permits reduction in turn on time, turn off time and on-state voltage drop in SCR.
- The fast turn off ASCRs Permit high-frequency operation (20 kHz) in turn minimises the size weight and cost of commutating components with improved efficiency.

Reverse Conducting Thyristor (RCT)

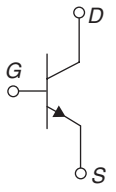
- It is a special case asymmetrical thyristor with a monolithically integrated anti parallel diode on the same silicon chip.

Static Induction Transistor (SIT)

- A SIT is a high-power, high-frequency device. It is essentially the solid state version of the triode vacuum tube. It is a vertical structure device with short multi channels. Hence it is not subject to area limitations and is suitable for high speed high power operations.
- A SIT has a short channel length, low gate series resistance, low gate source capacitance and small thermal resistance.
- A SIT normally is an ON device and a negative gate voltage holds it OFF.



Cross section

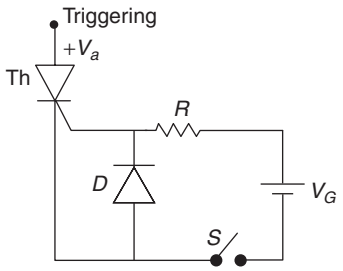


- Since SITs have a higher current and switching speed, it is most suitable for high-power and high-frequency applications.

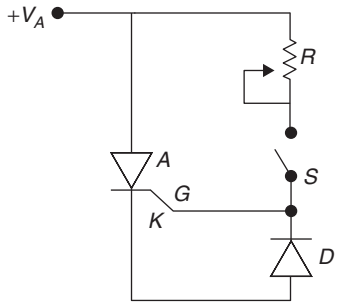
Thyristor-triggering Circuits

- Generally the thyristor gate circuits are within the low and high resistance values. It is a pn junction. The minimum level of voltage and current required to turn on the thyristor is a function of junction temperature.

Direct Current



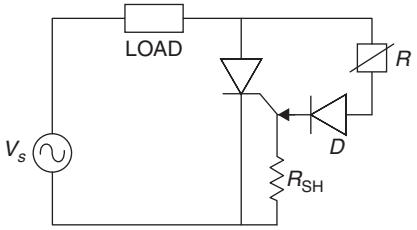
- A separate DC source, or the current tapped off from the main DC supply to the gate are used.



- When the thyristor conducts the forward voltage drop goes down about 1.5 V, then the positive voltage applied to the gate goes below the break over voltage.
- When the current is interrupted, then the reverse connected diode D prevents a large reverse bias at the gate-cathode terminals.

AC Signal Triggering

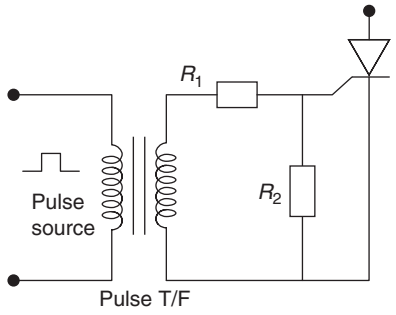
- Also known as phase control triggering.
- The firing instant can be manipulated to occur at any particular phase between 0 and p of the half cycle of an AC signal.



The firing angle is adjusted by variable resistance.

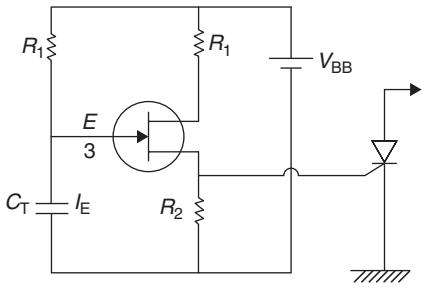
- The gate is not triggered when the variable resistance is high.
- When resistance is decreased, then I_G increases beyond latching current and SCR triggers.
- There is a delay by an angle a when a signal is applied to gate which causes the output load current to decrease.
- When the resistance is fully decreased to zero, a full positive angle of 180° is applied to the gate which causes large current to flow in the load.
- The drawback of the RC circuit is that R and C values have to be adjusted for different source voltages.
- This can be overcome by AC phase shift control vcircuit with a centre-tapped transformer.

Pulse Triggering



- To trigger a thyristor a pulse of current equal to forward break over voltage is required.
- Anode current should be greater than latching current.
- Reduces the power dissipation in the gate circuit.
- Conduction b can occur for only 90°.
- It has many defects as the turn-on angle ° may vary from cycle to cycle due to temperature variations.
- The turn on is slow.

UJT Trigger Circuit



- Due to simplicity low power consumption, high effective power gain, compactness and because of its combination of economy, UJT is mostly used for triggering SCRs.
- When voltage across capacitor after discharge fall below valley voltage V_V, UJT turns off.
- When $V_{CT} = V_P > \eta V_{BB} + V_r$, then I_E flows.

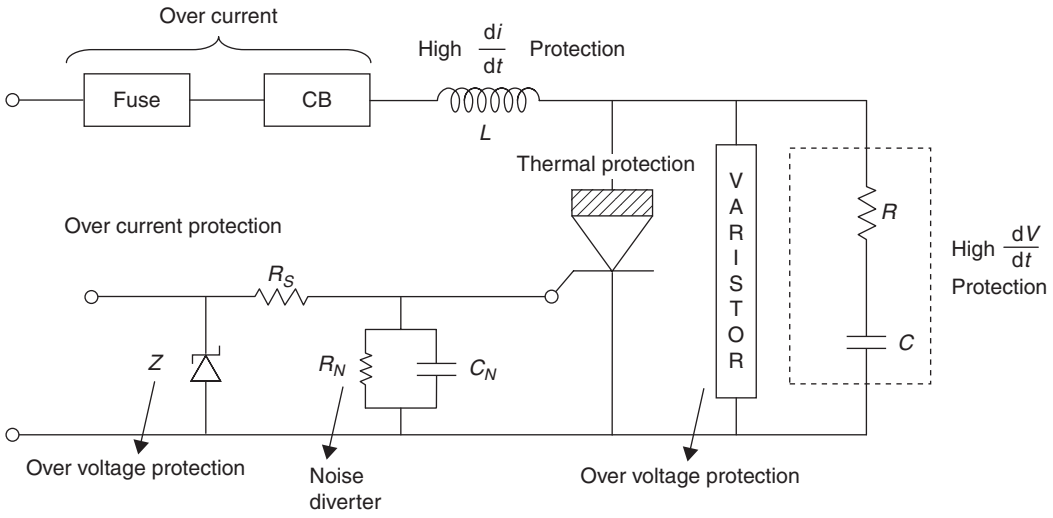
- For turning ON and OFF the UJT, the line of $I_E R_T$ should fall within the valley and peak point.

Protection of SCR

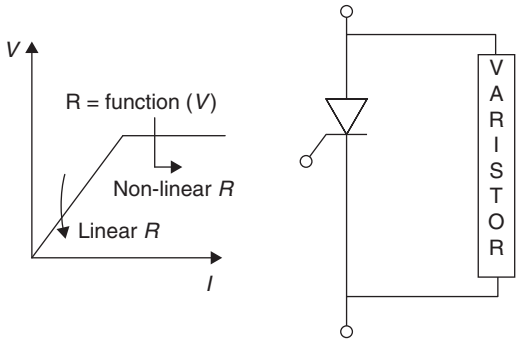
Generally protection is required against main circuit and Gate circuit.

Protection Circuit

1. **Main Circuit:**
   - Overvoltage
   - Overcurrent
   - Thermal
   - *dv*/*dt*
   - *di*/*dt*
2. **Gate Circuit:**
   - Overvoltage
   - Overcurrent
   - Noise
3. **Main Circuit Protection:**
   Overvoltage Protection: Overvoltages exist due to switching and lightning surges.



To reduce this VARISTORS are used. VARISTORS are non-linear resistor.



At higher voltages slope is less and VARISTORS operating with lesser resistance.

Overcurrent Protection

- They are fault currents or normal overload currents.
- Circuit breakers and fuses connected in series are used to protect.

Thermal Protection

- With the rise in temperature, insulation losses, which result in leakage current.
- Heat sinks made of aluminium discs are used with ventilating ducts.
- Coolants and hallow conductors are used.

dv/dt Protection

- When *dv*/*dt* is high without application of gate signal, then turn on of SCR takes place.
- Capacitors are used for *dv*/*dt* protection which delay the build-up of voltage.

di/dt Protection

- Series inductors reduces *di*/*dt* rate.

Gate Protection

Overcurrent protection: Connect the Resistor R_s to limit gate current with in permissible value.

Overvoltage protection: Voltage clampers are used for over voltage protection. Zener diode under reverse biased mode operates as voltage regulator.

Protection against noise: Noise signal produces the disturbance in the gate circuit due to the voltage transients of the power supply or due to the interference with the nearby communication lines. These noise signals may false turn-ON the SCR so that the gate loses control over itself.

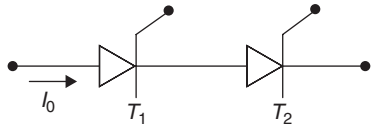
An RC parallel circuit is to be connected across the gate so that it will divert the noise signal without entering into the gate terminal.

$\downarrow\downarrow X_C = \dfrac{1}{\omega C \uparrow\uparrow}$, RC noise diverter, diverts the high-frequency noise signal and dissipates across R.

Series and Parallel Operation of SCRs

Series Operation

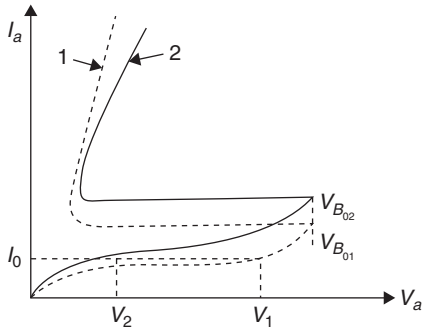
- It improves voltage rating



De-rating factor = 1 – String efficiency

String efficiency =

$$\frac{\text{Voltage (series) OR Current (parallel) rating of string}}{\text{No. of SCR in the string} \times \text{Voltage OR Current rating of each SCR}}$$



$$\eta_s = \frac{V_1 + V_2}{2V_1}$$

- Since the characteristics are not matching in forward-blocking mode, the efficiency is less than 1. To improve the efficiency this mismatch should be reduced.
- The SCR which offers more leakage resistance will block more voltage.
- With less leakage current blocks more voltage.
- Voltage distribution across the SCRs is proportional to the leakage resistance offered by each SCR.
- To make voltage distribution equal, leakage resistances should be made equal

Static equalizing resistance

$$R_s = \frac{\eta V_{bm} - V_s}{(n-1)\Delta I_b}$$

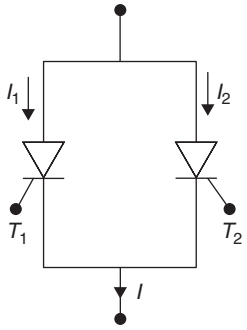
Dynamic equalizing capacitance

$$C = \frac{(n-1)\Delta Q}{nV_{bm} - V_s}$$

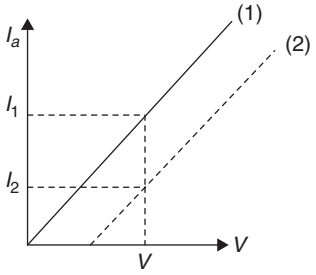
- An R in series with C acts as the snubber circuit for dV/dt protection as well as for dynamic equalization.

Parallel Operation

- Applied for SCRs with higher current ratings



$$M_S = \frac{I_1 + I_2}{2I_1}$$



- All SCRs should be mounted on heat sink in symmetrical positions.
- It is not compulsory that all SCRs should be mounted on the same heat sink to maintain constant temperature.
- A magnetic coupling is employed to equalize currents in parallel connected SCRs.

EXERCISES

Practice Problems 1

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Which among the following, electronic switch employed in a power electronic converter which is uncontrollable is
(A) MOSFET (B) Diode
(C) Thyristor (D) BJT

2. An NPN transistor has a DC current gain (β) of 200. The base current required to switch resistance load of 3 mA would be
(A) 40 μA (B) 20 μ
(C) 30 μA (D) 10 μA

3. A power control switching is done using a BJT by basing it in the saturation region (or) cut off region. In the ON state for BJT
(A) Both emitter base and collector base junctions are reverse biased
(B) Emitter base junction is forward biased and collector base junction is reverse biased
(C) Both emitter base and collector base junctions are forward biased
(D) Emitter base junction is reverse biased and collector base junction is forward biased

4. A thyristor is categorized under a semi-controlled device because
(A) It conducts only during one half cycle of an AC wave.
(B) It can be switched ON but not OFF with a gate pulse
(C) It can be switched OFF but not ON with a gate pulse
(D) It conducts only during one half cycle of an alternating voltage wave.

5. The typical ratio of holding current to latching current in a 25 A thyristor is
(A) 5 (B) 2
(C) 1 (D) 0.5

6. The main switching element used in a switched mode power supply operating at 30 kHz to 90 kHz range preferably is
(A) UJT
(B) TRIAC
(C) Thyristor
(D) MOSFET

7. The ON-state condition of a MOSFET switch is equal to
(A) Capacitor (B) Battery
(C) Inductor (D) Resistor

8. For a power MOSFET, the characteristic curve drawn for conduction loss versus device current would follow the path of
(A) An exponentially decaying curve
(B) A parabola
(C) A rectangular hyperbola
(D) A straight line

9 The only non-current triggered, power semiconductor device among the following is
(A) Triac (B) Thyristor
(C) MOSFET (D) GTO

10. Combined characteristics of BJT and MOSFET, incorporated in a modern power semiconductor device would be
(A) An MCT (B) An FCT
(C) An IGBT (D) GTO

11. Which among the following parameters regarding an IGBT is true?
(1) High input impedance
(2) Combined attributes of BJT and MOSFET
(3) Inferior switching speed compared to MOSFET
(4) Low Forward Voltage Drop
(A) 1, 2, 3 and 4 (B) 1, 2 and 3
(C) 1, 2, and 4 (D) 3 and 4

12. Under steady-state conditions, between a MOSFET and BJT
(A) MOSFET requires lower control power
(B) BJT requires lower control power
(C) Both requires same range of control power
(D) MOSFET has an ON state voltage drop greater than that of BJT

13. In a thyristor, reverse anode to cathode voltage and forward gate current will cause
(A) Low forward current
(B) Large reverse current
(C) Large forward current
(D) Low reverse current

14. A TRIAC can be triggered by a gate pulse of _____ polarity
(A) Positive
(B) Negative
(C) Positive or Negative
(C) None

15. The Triac can be used only in
(A) Inverter
(B) Rectifier
(C) Multiquadrant chopper
(D) Cyclo converter

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. A GTO thyristor
(A) Can be turned off by a positive current pulse at the gate.
(B) Can be turned off by a negative current pulse at the gate.
(C) Requires a special turn off thyristor.
(D) Can be turned off by removal of gate pulse.

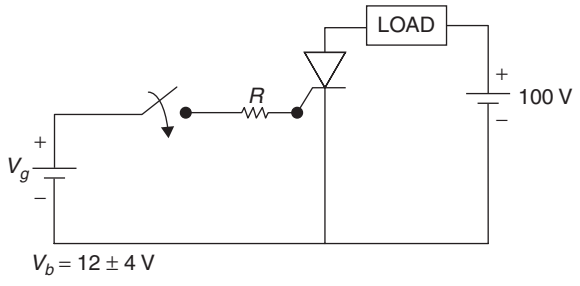
2. A gate turn-off thyristor (GTO) has the capacity to
(A) Amplify the gate current
(B) Turn-off when positive current pulse is given at the gate
(C) Turn-off when a gate pulse is given at the gate even though it is reverse biased
(D) Turn-off when a negative current pulse is given at the gate

3. A power semiconductor may undergo damage due to
(A) High di/dt (B) High $\frac{dv}{dt}$
(C) High gate current (D) Low $\frac{dv}{dt}$

4. A satisfactory turn off using a commutation circuit employed to turn off a Thyristor is obtained when
(A) Device turn off time > circuit turn off time
(B) Device turn off < circuit turn off time
(C) Device turn off time > circuit time constant
(D) Device turn off time < circuit time constant.

5. In synchronized UJT triggering of an SCR, voltage V_C across capacitor reaches UJT threshold voltage thrice in each half cycle so that there are three firing pulses during each half cycle. The firing angle of the SCR can be controlled
(A) Once in each half cycle
(B) Thrice in each half cycle
(C) Thrice in a full cycle
(D) Twice in a half cycle

6. The triggering circuit of a thyristor is shown in figure. The thyristor requires a gate current of 10 mA, for guaranteed turn-on. The value of R required for the thyristor to turn-on reliably under all conditions of V_g variation



(A) 10 kΩ (B) 1.6 kΩ
(C) 1.2 kΩ (D) 0.8 kΩ

7. Which of the following statements are true about power electronic systems?
(i) Tendency to generate harmonics in the supply system as well as in the load circuit
(ii) Fast dynamic response as compared to electromechanical converter systems
(iii) Low input power factor under certain operating conditions
(iv) Less efficiency as compared to electromechanical converter systems
(A) (i), (ii) and (iii)
(B) (i) and (iv)
(C) (ii) and (iv)
(D) (i) and (iii)

8. An IGBT has three terminals called
(A) Collector, emitter and base
(B) Drain, source and gate
(C) Drain, source and base
(D) Collector, emitter and gate

9. High-frequency operation of a circuit is limited by
(A) On-state loss in the device
(B) Off-state loss in the device
(C) Switching losses in the device
(D) Both (A) and (B)

10. For an SCR, dV/dt protection is achieved through the use of
(A) RL in parallel with SCR
(B) RC in series with SCR
(C) RL in series with SCR
(D) RC across SCR

11. Secondary breakdown occurs in
(A) MOSFET but not in BJT
(B) Both MOSFET and BJT
(C) BJT but not in MOSFET
(D) None of the above

12. Choose the correct statement
(A) MOSFET has positive temperature coefficient (TC) whereas BJT has negative TC
(B) Both MOSFET and BJT have positive TC
(C) Both MOSFET and BJT have negative TC
(D) MOSFET has negative TC whereas BJT has positive TC

13. Which of the following statements are true?
(1) Power MOSFET is a majority carrier device
(2) IGBT is a bipolar device
(3) BJT is a majority carrier device
(4) MCT is a unipolar device
(A) 1, 3 (B) 2, 3
(C) 2, 4 (D) 1, 2

14. The correct sequence of the given devices in the increasing order of their speed of operation is

[PMOSFET—Power MOSFET PBJT—Power BJT]

(A) SCR, IGBT, PMOSFET, PBJT
(B) SCR, PBJT, PMOSFET, IGBT
(C) PMOSFET, IGBT, PBJT, SCR
(D) SCR, PBJT, IGBT, PMOSFET

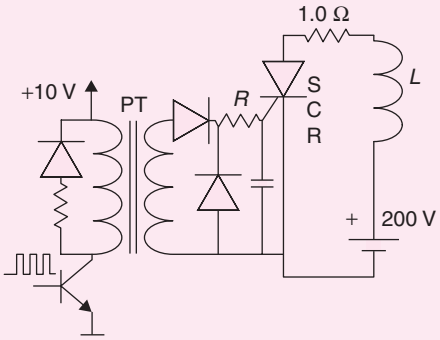
15. For a string voltage of 3300 V, there are six series connected SCRs, each of 600 V rating. Then the de-rating factor is

(A) 9.17%
(B) 91.7%
(C) 8.3%
(D) 10.9%

Previous Years' Questions

Common Data for Questions 1 and 2:

A 1:1 pulse transformer (PT) is used to trigger the SCR in the adjacent figure. The SCR is rated at 1.5 kV, 250 A with $I_L = 250$ mA, $I_H = 150$ mA, and $I_{G\max} = 150$ mA, $I_{G\min} = 100$ mA. The SCR is connected to an inductive load, where $L = 150$ mH in series with a small resistance and the supply voltage is 200 V DC. The forward drop of all transistors/diodes and gate-cathode junction during ON state is 1.0 V.



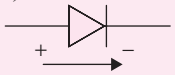
1. The resistance R should be **[2007]**

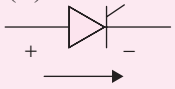
(A) 4.7 kΩ (B) 470 Ω
(C) 47 Ω (D) 4.7 Ω

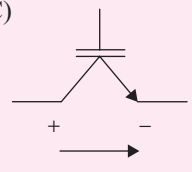
2. The minimum approximate volt-second rating of the pulse transformer suitable for triggering the SCR should be: (volt-second rating is the maximum of product of the voltage and the width of the pulse that may be applied) **[2007]**

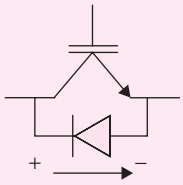
(A) 2000 μV-s (B) 200 μV-s
(C) 20 μV-s (D) 2.0 μV-s

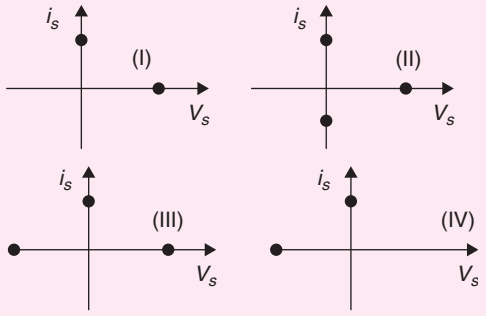
3. Match the switch arrangements on the top row to the steady-state V–I characteristics on the lower row. The steady-state operating points are shown by large black dots **[2009]**

(A)


(B)


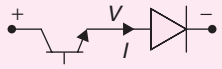
(C)


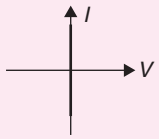
(D)


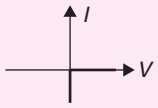


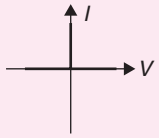
(A) A – I B – II C – III D – IV
(B) A – II B – IV C – I D – III
(C) A – IV B – III C – I D – II
(D) A – IV B – III C – II D – I

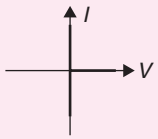
4. Figure shows a composite switch consisting of a power transistor (BJT) in series with a diode. Assuming that the transistor switch and the diode are ideal, the I–V characteristic of the composite switch is **[2010]**



(a)


(b)


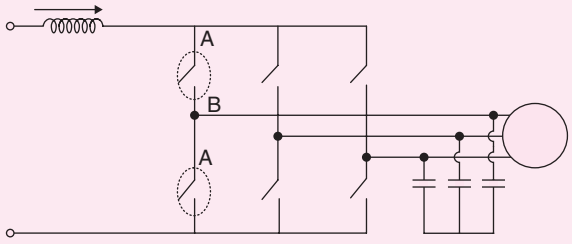
(c)


(d)


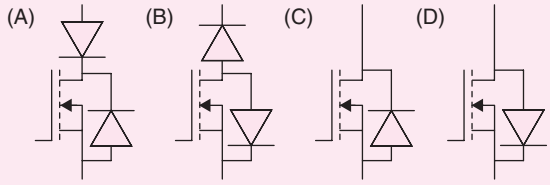
5. Circuit turn-off time of an SCR is defined as the time **[2011]**

(A) Taken by the SCR to turn off
(B) Tequired for the SCR current to become zero
(C) For which the SCR is reverse-biased by the commutation circuit.
(D) For which the SCR is reverse-biased to reduce its current below the holding current.

6. A three-phase current source inverter used for the speed control of an induction motor is realized using MOSFET switches as shown below. Switches S_1 to S_6 are identical switches. **[2011]**



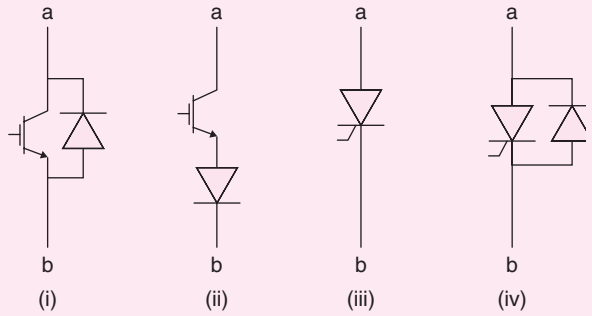
The proper configuration for realizing switches S_1 to S_6 is



7. The typical ratio of latching current to holding current in a 20 A thyristor is **[2012]**

(A) 5.0 (B) 2.0
(C) 1.0 (D) 0.5

8. Figure shows four electronic switches (i), (ii), (iii) and (iv). Which of the switches can block voltages of either polarity (applied between terminals 'a' and 'b') when the active device is in OFF state? **[2014]**

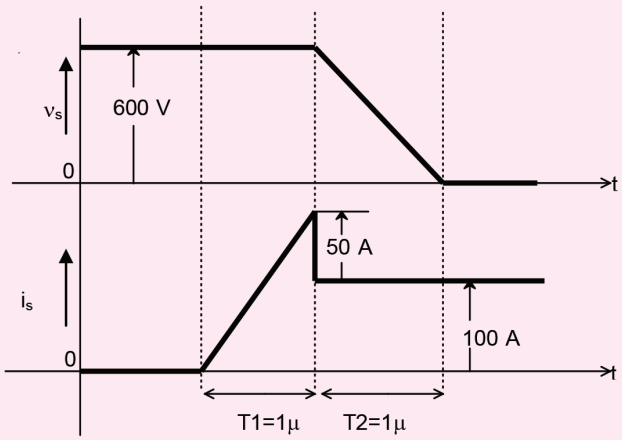


(A) (i), (ii) and (iii) (B) (ii), (iii) and (iv)
(C) (ii) and (iii) (D) (i) and (iv)

9. A single-phase thyristor-bridge rectifier is fed from a 230 V, 50 Hz, single-phase AC mains. If it is delivering a constant DC current of 10 A, at firing angle of 30°, then value of the power factor at AC mains is **[2016]**

(A) 0.87 (B) 0.9
(C) 0.78 (D) 0.45

10. The voltage (V_s) across and the current (I_s) through a semiconductor switch during a turn-ON transition are shown in figure. The energy sissipated during the turn-ON transition, in mJ is_____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. D	**2.** C	**3.** C	**4.** B	**5.** D	**6.** D	**7.** A	**8.** B	**9.** C	**10.** C
11. A	**12.** A	**13.** D	**14.** C	**15.** C					

Practice Problems 2

1. B	**2.** D	**3.** A	**4.** B	**5.** A	**6.** D	**7.** A	**8.** D	**9.** C	**10.** D
11. A	**12.** A	**13.** D	**14.** D	**15.** B					

Previous Years' Questions

1. C	**2.** A	**3.** C	**4.** C	**5.** C	**6.** A	**7.** B	**8.** C	**9.** C	**10.** 75

Chapter 2

AC to DC Converters (or) Rectifiers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

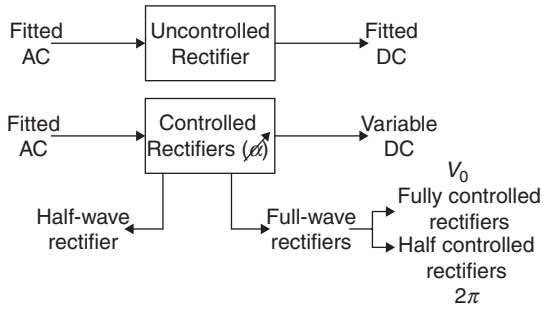
- Single phase half–wave rectifiers: with R load
- Half wave rectifier with R.L. load
- Half wave rectifier with RLC load
- Three phase controlled half-wave converter
- Gating sequence
- Single phase full controlled full wave rectifier
- Single phase dual converter
- Gating sequence
- Effect of source inductance

Earlier, DC power was obtained from motor-generator sets but nowadays phase-controlled AC to DC converters employing thyristors are extensively used for changing constant AC input voltage to controlled DC output voltage.

The members of thyristor family such as SCR and triac are more suitable for power central application, because they can be switched ON and OFF by means of simple gate control circuit.

The variation in load power at constant input voltage which can be obtained by changing the firing angle α is called as phase control.

Many industrial applications make use of controlled DC power.



Examples:

- Steel–rolling mills, paper mills, printing presser and textile mills employing DC drives.
- Magnet power supplies.
- Electromechanical and electro metallurgical process.
- High voltage DC transmission.
- Traction system working on DC.
- SCR power controllers provide a relatively economical means of power central. SCR power controllers cast less and are more efficient than saturable core reactors and variable transformers. Compared to contactors, SCR power controllers offer a much fine degree of central and do not suffer, from the maintenance problems of mechanical devices.
- The DC current is unidirectional, but the DC voltage may have either polarity.
- With one polarity, the flow of power is from the AC source to the DC load is called rectification.
- With a reversal of the DC voltage by the load, the flow of power is from the DC source to the AC supply, this process is called Inversion.
- If the half of the SCR's are replaced by diodes, then it is called half-controlled or semi-converter circuit.

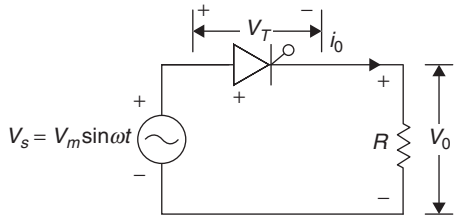
Therefore, the circuit that converts an AC signal into a unidirectional signal is called rectifier. Depending upon the type of input supply, the rectifiers are classified into

1. Single phase and
2. Three phase

Each type can be subdivided into

1. **Semi-converter:** One quadrant converter and polarity of its output voltage and current is the same.
2. **Full converter:** Two quadrant converter and the polarity of its output voltage can either be positive or negative.
3. **Dual converter:** Four quadrant converter and both output voltages and currents can be either positive or negative.

Single-phase Half-wave Rectifiers: With R Load



It consists of single thyristor feeding DC power to a resistive load.

So, $i_S = i_T = i_0$

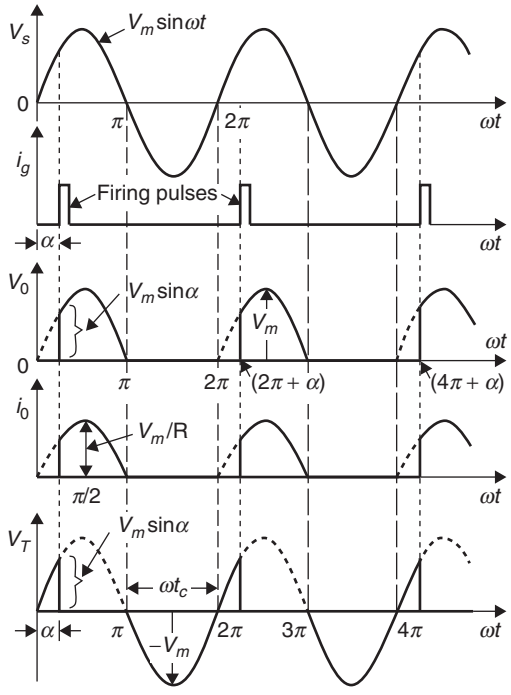
At $\omega t = \alpha$

$$V_0 = V_m \sin\alpha$$

$$I_0 = \frac{V_m \sin\alpha}{R}$$

Principle of operation of single-phase half-wave rectifier with R load

- During positive half-cycle of the supply voltage, the SCR is forward-biased. And will conduct if a trigger pulse is applied to gate.
- Then the load current flows, output voltage V_0 is same as the input voltage.
- At $t = \pi$, current falls naturally to zero since the SCR is reverse-biased.
- During negative half cycle, SCR blocks the flow of current and no voltage is applied to load.
- ωt_0 to $+2\pi$, again the SCR starts conduct.
- From ωt_0 to 2π it is called firing angle or delay angle and ωt_0 to π is called conduction angle (θ).



Current waveform is same. As V_0 but magnitude is differed depending upon the value of R.

Line commutation takes place

Turn-off time = $t_C = \pi/\omega$

Average output voltage

$$V_0 = \frac{1}{2\pi}\int_{\alpha}^{\pi} V_m \sin\omega t d(\omega t)$$

$$V_0 = \frac{V_m}{2\pi}(1+\cos\alpha)$$

RMS value of output voltage

$$V_{RMS} = \sqrt{\left(\frac{1}{2\pi}\int_{\alpha}^{\pi} V_m^2 \sin^2\omega t d(\omega t)\right)}$$

$$V_{RMS} = \frac{V_m}{2\sqrt{\pi}}\left[\pi - \alpha + (\sin 2\alpha)/2\right]^{1/2}$$

Average output current

$$I_0 = V_0/R$$

RMS output current

$$I_{RMS} = \frac{V_{RMS}}{R}$$

Expression for input power factor:

Source Side

$$V_S = V_m \sin\omega t$$

$$V_{sr} = \frac{V_m}{\sqrt{2}}$$

$$(V_0)_{avg} = 0\ \text{V}$$

$$(I_S)_{avg} = \frac{V_0}{R}\frac{V_m}{2\pi R}[1+\cos\alpha]$$

$$(I_0)_{avg} = (I_S)_{avg}$$

$$P_{IN} = V_{sr} I_{sr} \cos\phi_s$$

Control Output

$$V_{or} = \left\{\frac{V_m^2}{2\pi}\left[\frac{\pi-\alpha}{2} + \frac{\sin 2\alpha}{4}\right]\right\}^{1/2}$$

$$(V_0)_{avg} = \frac{V_m}{2\pi}[1+\cos\alpha]$$

$$P_{OUT} = V_{or} I_{or} \cos\theta$$

$$P_{IN} = P_{OUT}$$

$$V_{sr} I_{sr} \cos\phi_S = V_{or} \times I_{or} \times 1$$

$$\text{Source P.F.} = \cos\phi_s = \frac{V_{or}}{V_{sr}}$$

Power Factor (P.F.) of input supply

$$= \frac{\text{Power supplied to load}}{\text{Source volt ampere}}$$

$$\text{P.F.} = \frac{V^2_{\text{RMS}}/R}{V_S\ I_S} = \frac{V^2_{\text{RMS}}/R}{V_S\ I_S}$$

$$\left[\because I_S = I_{\text{RMS}} = \frac{V_{\text{RMS}}}{R}\right]$$

$$\text{P.F.} = \frac{V_{\text{RMS}}}{V_S}$$

when $\alpha = 0$

Then $V_0 = \frac{V_m}{\pi} = 0.318\ V_m.$

$$I_0 = \frac{0.318\ V_m}{R}$$

$$V_{\text{RMS}} = \frac{V_m}{2} = 0.5\ V_m\ I_{\text{RMS}}$$

$$= \frac{V_{\text{RMS}}}{R} = \frac{0.5\ V_m}{R}$$

$$P_{DC} = (0.318 V_m)^2/R$$

$$P_{AC} = (0.5 V_m)^2/R$$

(A) $\eta = \frac{P_{DC}}{P_{AC}} = \frac{(0.318\ V_m)^2/R}{(0.5\ V_m)^2/R} = 40.5\%$

(B) F.F. $= \frac{0.5\ V_m}{0.318\ V_m} = 157\% = 1.57\%$

(C) R.F. $= \sqrt{1.57^2 - 1} = 1.21$ or 121%

(D) $(V_S)_{\text{RMS}} = \left[\frac{1}{T}\int_0^T V_m \sin \omega t^2 dt\right]^{1/2}$

$$= \frac{V_m}{\sqrt{2}} = 0.707\ V_m.$$

$(I_S)_{\text{RMS}} = 0.5\ V_m/R$

(E) Peak Inverse Voltage (PIV)

$$\text{PIV} = V_m$$

(F) $I_{s(\text{peak})} = \frac{V_m}{R} = I_S = 2$

(G) Input P.F. for a resistive load

$$\text{P.F.} = \frac{P_{AC}}{VA} = \frac{(0.5)^2}{(0.707)\times 0.5}$$

$$= 0.707$$

$V_S \rightarrow$ Transformer secondary voltage

$I_S \rightarrow$ Transformer secondary current

$$\text{V–A rating} = V_S I_S = 0.707\ V_m \times \frac{0.5\ V_m}{R}$$

$$\text{TUF} = \frac{P_{AC}}{(V_S\ I_S)} = \frac{(0.318)^2}{(0.707)\times 0.5} = 0.286$$

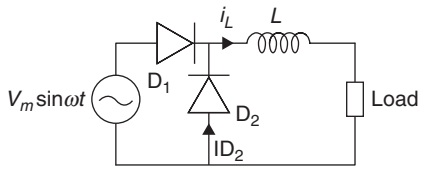
Note:

- The performance of the converter is degraded at higher range of α.
- Varying the delay angle α from 0 to π can vary the average output voltage from V_m/π to 0.

Input transformer can carry DC current, thereby causing a magnetic saturation problem.

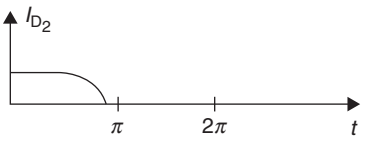
Solved Examples

Example1:

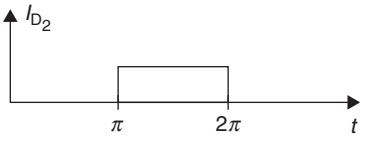


The figure above is that of a single-phase half wave uncontrolled converter fed from the output of a two winding isolating transformer. The current waveform through diode D_2 will be [assume the load current to be constant].

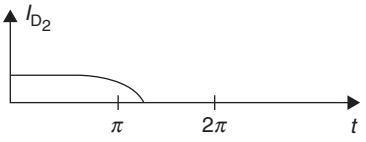
(A)



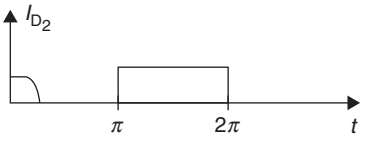
(B)



(C)



(D)



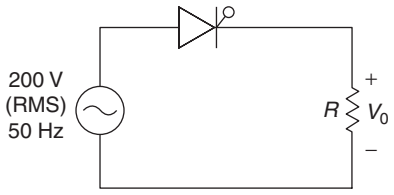
Solution: (B)

Diode D_2 conducts only at negative half cycles, i.e., from π to 2π.

Choice (A) and (C) gives I_{D_2} during positive cycle hence in correct.

Choice (D) has a small current I_{D_2} in the positive half cycle hence wrong. The correct option is choice (B).

Example 2: In the phase-controlled half-wave converter shown above, the thyristor is fired in every positive half cycle of the input voltage, at an angle α. The firing angle for the peak value of the instantaneous output voltage of 200 V would be close to



(A) 90° (B) 135° (C) 45° (D) 78.23°

Solution: (B)

For $\alpha < 90°$ $\quad V_m = V_p$

For $\alpha > 90°$ $\quad V_m \sin\alpha = V_p$

$$V_{RMS} = 200 \text{ V}$$

$$\Rightarrow \quad V_m = 200\sqrt{2}$$

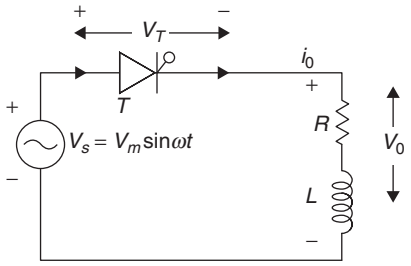
$$200\sqrt{2}\sin\alpha = 200$$

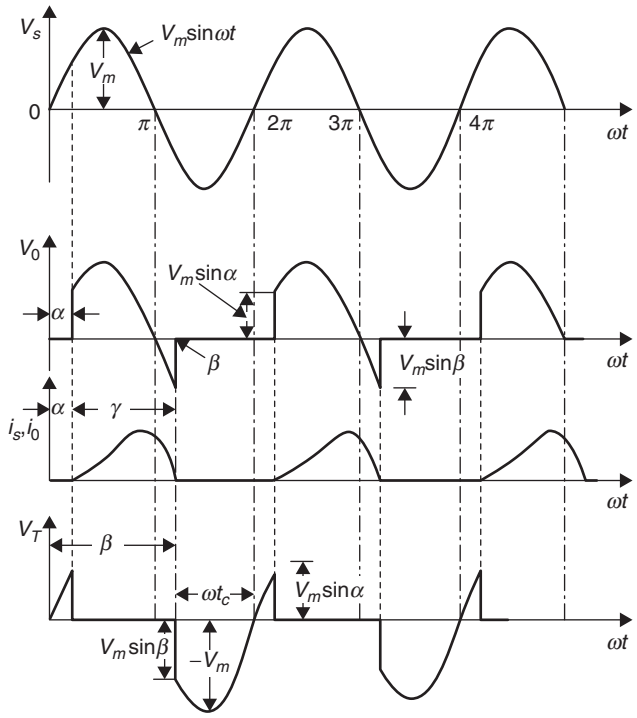
$$\sin\alpha = \frac{1}{\sqrt{2}} \{\text{Since } \alpha > 90\}$$

$$\alpha = 45°$$

$$\alpha = 90 + 45 = 135°$$

Half-wave Rectifier with R.L. Load





Principle of Operation

Circuit turn-off time

$$T_C = \frac{(2\pi - \beta)}{\omega}$$

Average output voltage

$$V_0 = \frac{1}{2\pi}\int_{\alpha}^{\beta} V_m \sin\omega t \, d(\omega t)$$

$$V_0 = \frac{V_m}{2\pi}(\cos\alpha - \cos\beta)$$

RMS output voltage

$$V_{RMS} = \sqrt{\left(\frac{1}{2\pi}\right)\int_{\alpha}^{\beta} V_m^{\,2} \sin^2\omega t \, d(\omega t)}$$

$$V_{RMS} = \frac{V_m}{2\sqrt{\pi}}\left[\beta - \alpha + \frac{1}{2}\left[\sin(2\alpha) - \sin 2\beta\right]\right]^{1/2}$$

Average output current

$$I_0 = V_0/R$$

RMS value of output current,

$$I_{RMS} = \frac{V_{RMS}}{R}$$

β is called extinction angle.

It depends on the strength of the inductor minimum value of β is π.

α is called conduction angle.

The power from $\alpha \to \beta$ is less than the power form $\alpha \to \pi$ because some energy is returned by the inductor to source.

$\Rightarrow$ By using free-wheeling diode, we can avoid this.

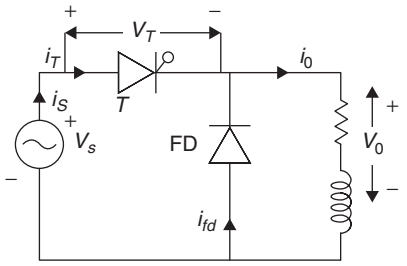
Power factor of supply,

$$\text{P.F.} = \frac{V_{RMS}}{V_S}$$

Principle of Operation

- If the SCR is triggered, at a firing angle of ∞, the load current increases slowly, since the inductance in the load forces the current to lag the voltage.
- The voltage across the load is positive, inductor, stores energy.
- In the negative, region, SCR is reverse biased, energy stored in the inductor is returned and maintains a forward decaying current through the load.
- Current flows until β even though SCR turns off.
- Due to which the average output voltage becomes less than that of purely resistive load.

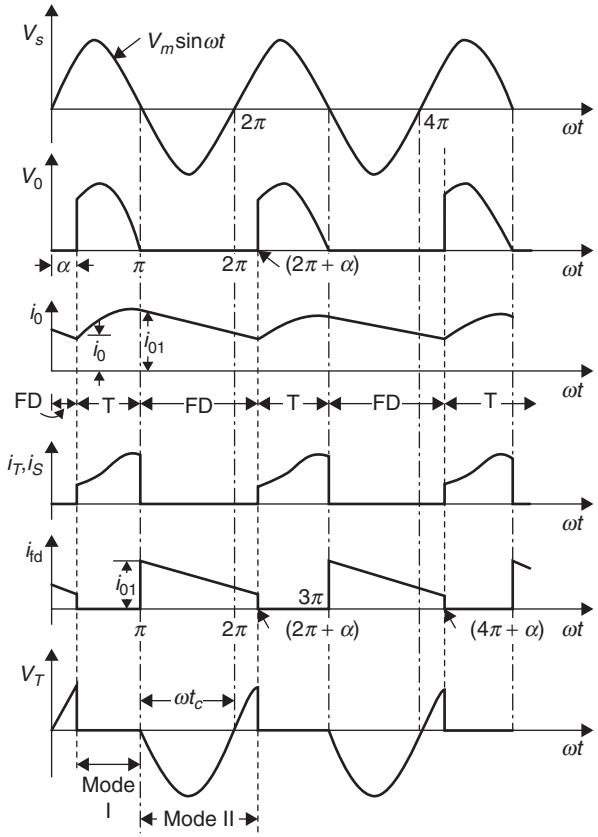
Half-wave Rectifier with R.L. Load and Free-wheeling Diode



Principle of Operation

- To cut off the negative portion of the instantaneous output voltage and smooth the output current ripple, a free-wheeling diode is used.
- When the load voltage tends to reverse, the FWD becomes forward biased and turns on. The SCR then becomes reverse biased and turns off.
- The current that was flowing form source to load now freewheels between the load and diode.

Current continuous to flow even through SCR turns off because of Inductor energy.



Circuit turn-off time

$$T_c = \pi/\omega$$

Average output voltage

$$V_0 = \frac{1}{2\pi}\int_{\alpha}^{\pi} V_m \sin\omega t\, d\omega t$$

$$V_0 = \frac{V_m}{2\pi}[1+\cos\alpha]$$

RMS output voltage

$$V_{RMS} = \frac{V_m}{2\sqrt{\pi}}\left[\pi-\alpha+\frac{1}{2}(\sin 2\alpha)\right]^{1/2}$$

Average output current

$$I_0 = V_0/R$$

RMS value of output current

$$I_{RMS} = \frac{V_{RMS}}{R}$$

Power factor of supply

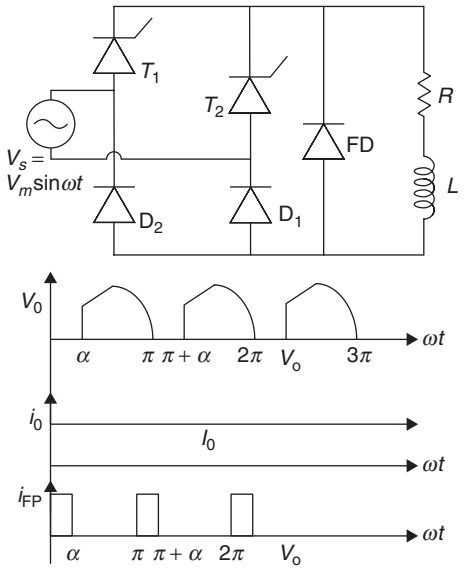
$$\text{P.F.} = \frac{V_{RMS}}{V_s}$$

$\Rightarrow$ Output current is continuous and power factor is improved resulting better load performance.

Example 3: An RL load is connected across a single-phase half-controlled bridge rectifier which is operated at firing angle α and continuous load current. The period for which the freewheeling diode conducts is

(A) $\left(1-\frac{\alpha}{\pi}\right)$ (B) $\frac{1}{2}$ (C) $\frac{\alpha}{2\pi}$ (D) $\frac{\alpha}{\pi}$

Solution: (B)

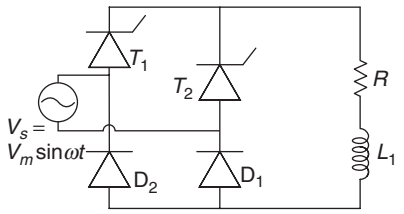


Conduction of Freewheeling diode in a cycle = 2α

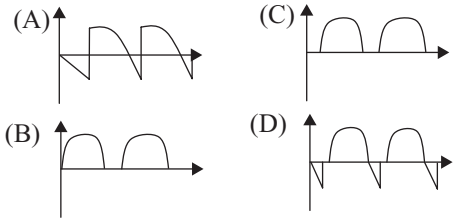
Fraction of conduction time of freewheeling diode in cycle = $\frac{2\alpha}{2\pi}$

$\therefore$ $t = \frac{\alpha}{\pi}$

Example 4:



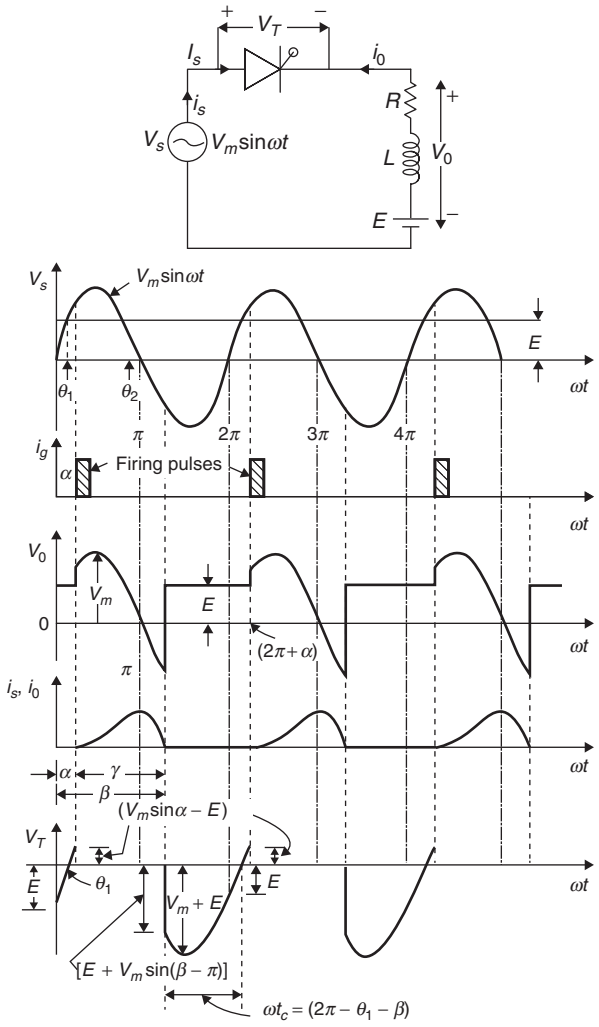
The output DC voltage wave shape of a single-phase-controlled semi-converter shown in the figure, operating at $\alpha = 30°$, will be as shown in the following figure.



Solution: (C)

In a semi-converter, the output voltage will never be negative and only will conduct after firing.

Half-wave Rectifier with RLE Load



- Firing angles ranges from θ_1 to θ_2.
- β is greater than θ_2 but can be less than π circuit turn-off time.

$$t_c = \frac{2\pi + \theta_1 - \beta}{\omega}$$

Average output voltage

$$V_0 = E + I_0 R$$

Average output current

$$I_0 = \frac{1}{2\pi}\int V_m (\sin\omega t - E)/R \, d\omega t$$

$$I_0 = \frac{1}{2\pi R}\left[V_m[\cos\alpha - \cos\beta] - E(\beta - \alpha)\right]$$

Supply power factor

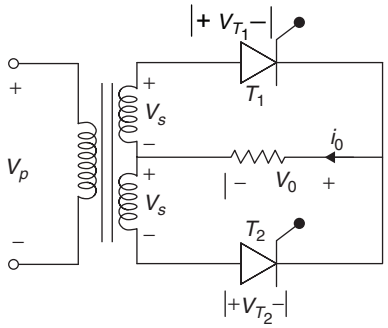
$$P_f = \frac{I^2_{RMS} R + I_0 E}{V_s \; I_{RMS}}$$

$$\theta_1 = \sin^{-1}(E/V_m)$$

$$\theta_2 = 180° - \theta_1$$

Single-phase Full wave Mid-point Type Rectifiers

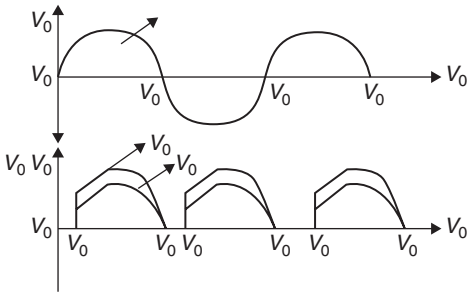
- A full-wave rectifier circuit with a centre tapped transformer is shown below.



- Each half of the transformer with its associated diode acts as a half-wave rectifier and the output of a full-wave rectifier is shown.
- Because there is no DC current flowing through the transformer, there is no DC saturation problem of transformer core.

⇒ It is assumed that the number of turns from primary to each secondary is same.

(R-load) Mid-point Converter:



Average output voltage

$$V_0 = \frac{1}{\pi}\int_{\alpha}^{\pi} V_m \sin(\omega t)\, d\omega t$$

$$(V_0)_{avg} = \frac{V_m}{\pi}[1+\cos\alpha]$$

$$(I_0)_{avg} = \frac{(V_0)_{avg}}{R} = \frac{V_m}{\pi R}[1+\cos\alpha]$$

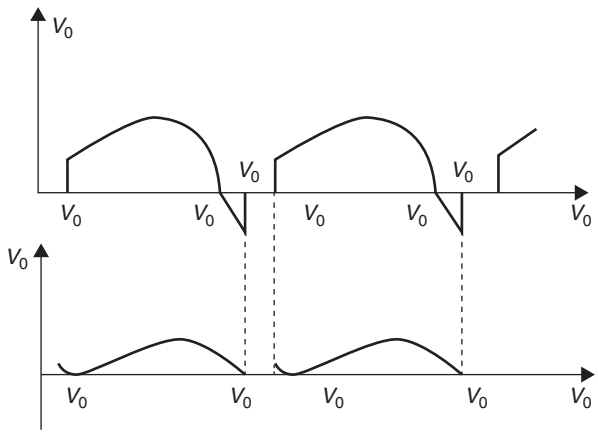
$$V_{RMS} = \left[\frac{1}{\pi}\int_{\alpha}^{\pi} V_m^2 \sin^2 \omega t\, d\omega t\right]^{1/2}$$

$$V_{RMS} = \left[\frac{V_m^2}{\pi}\left[\frac{\pi - a}{2} + \frac{\sin 2\alpha}{+}\right]\right]^{1/2}$$

RI-load

Circuit turn-off time

$$t_c = \frac{2\pi - \beta}{\omega}$$



Conduction angle = $\beta - \alpha$
Average output voltage,

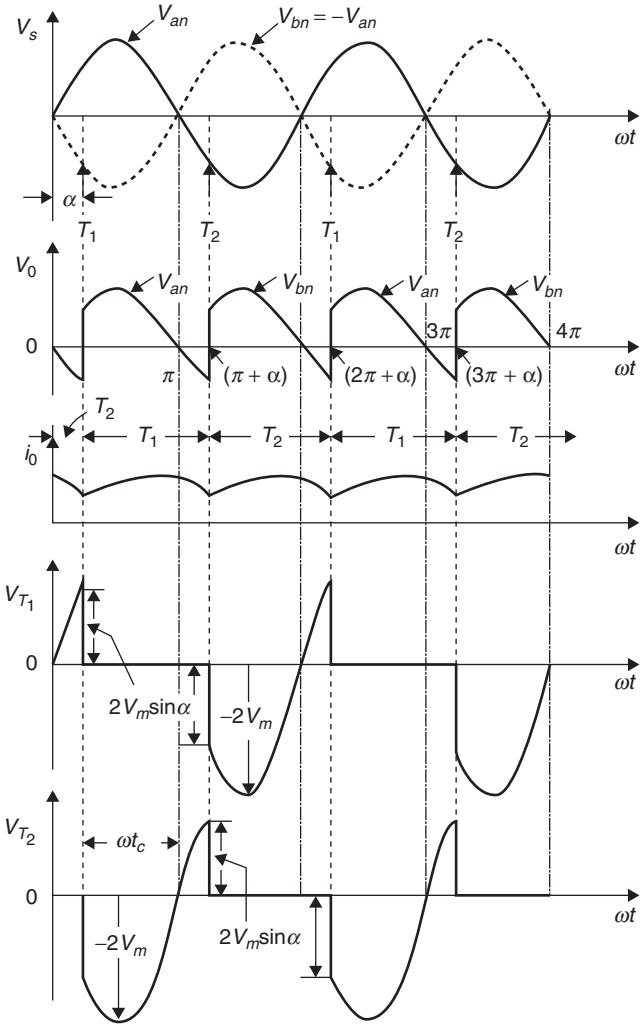
$$V_0 = \frac{1}{\pi}\int_{\alpha}^{\beta} V_m \sin\omega t\, d\omega t$$

$$V_0 = \frac{V_m}{\pi}[\cos\alpha - \cos\beta]$$

$$V_{or} = \left[\frac{1}{\pi}\int_{\alpha}^{\beta} V_m^2 \sin^2 \omega t\, d\omega t\right]^{\frac{1}{2}}$$

$$V_{or} = \left\{\frac{V_m^2}{\pi}\left[\frac{\beta - \alpha}{2} + \frac{\sin 2\alpha - \sin 2\beta}{4}\right]\right\}^{\frac{1}{2}}$$

Continuous Conduction Load



Circuit turn-off time

$$T_c = \frac{\pi - \alpha}{\omega}$$

Average output voltage

$$V_0 = \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} V_m \sin\omega t d(\omega t)$$

$$V_0 = \frac{2V_m}{\pi}\cos\alpha$$

For R-load $\quad V_0 = \frac{V_m}{\pi}[1+\cos\alpha]$

In this method, thyristor is subjected twice to the supply voltage $2V_m \sin\alpha$, which is not desirable. Also thyristor are under rated.

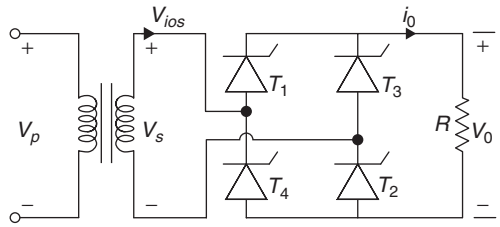
For both mid-point and bridge converters output voltage equations and waveforms are same except their peak inverse voltages.

$$(PIV)_{mid\text{-}point} = 2\,(PIV)_{Bridge}$$

For same rating SCRs used,

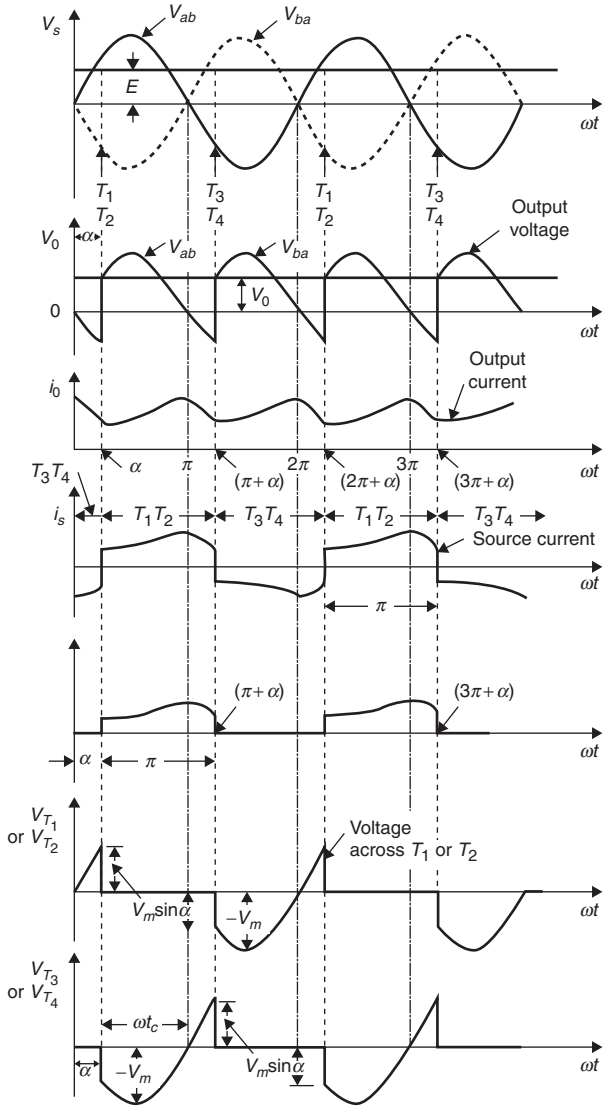
$$(\text{Power-handling capacity})_{\text{Bridge}} = 2\,(\text{Power-handling capacitor})_{\text{mid-point}}$$

Single-phase Full-wave Bridge-type Rectifier



- During positive half-cycle of input voltage, the power is supplied to the load through diodes D_1 and D_2.
- During negative half-cycle diodes D_3 and D_4 conduct. The output voltage is shown in figure below. The PIV of a diode is only V_m.

Continuous Conduction (Highly Inductive)



R-Load

0 to α:

T_1 T_2, T_3, $T_4 \rightarrow$ OFF

$i_0 = 0$, $i_s = 0$, $V_T(1, 2, 3, 4) = 0$, $V_0 = 0$.

α to π

T_1, $T_2 \rightarrow$ ON, T_3, $T_4 \rightarrow$ OFF

$V_0 = V_s$, $i_0 = $ is, $V_T(1, 2) = 0$, $V_T(3, 4) = -V_s$

PIV $= V_m$

Average output voltage:

$$V_0 = \frac{1}{\pi}\int_{\alpha}^{\pi} V_m \sin\omega t\, d\omega t$$

$$V_0 = \frac{V_m}{\pi}(1+\cos\alpha)\, V_0$$

$$I_0 = \frac{V_m}{\pi R}(1+\cos\alpha)\, I_0$$

RMS Output Voltage

$$V_{or} = \left[\frac{1}{\pi}\int_{\alpha}^{\pi} V_m^2 \sin^2\omega t\, d\omega t\right]^{1/2}$$

$$= \left(\frac{V_m^2}{\pi}\left(\frac{\pi-\alpha}{2} + \frac{\sin 2\alpha}{4}\right)\right)^{1/2}$$

Circuit turn-off time,

$$t_c = \frac{\pi-\alpha}{\omega}.$$

RL Load (Discontinuous Condition)

$$(V_0)_{\text{avg}} = \frac{1}{\pi}\int_{\alpha}^{\beta} V_n \sin\omega t\, d\omega t = \frac{V_m}{\pi}(\cos\alpha - \cos\beta).$$

$$V_{or} = \left\{\frac{V_m^2}{\pi}\left[\frac{\beta-\alpha}{2} + \frac{\sin 2\alpha - \sin 2\beta}{4}\right]\right\}^{1/2}$$

Circuit turn-off time,

$$t_c = \frac{2\pi-\beta}{\omega}$$

RL and RLE Continuous Conduction Load

$$(V_0)_{\text{avg}} = \frac{2V_m}{\pi}\cos\alpha$$

$$V_{or} = V_m/\sqrt{2}$$

Circuit turn-off time, $t_c = \dfrac{\pi}{\omega}$.

$\Rightarrow$ The thyristors can be subjected to a maximum voltage of V_m only not $2V_m$ as in midpoint type converter.

$\Rightarrow$ SCR are used to full rated voltage.
$\Rightarrow$ Power loss and voltage drop is more.
$\Rightarrow \alpha > 90°$, output voltage is negative, and can be used for regenerative braking applications.

Performance Parameter

For a full-wave DC rectifier
Average output voltage is

$$V_{DC} = \frac{2V_m}{\pi} = 0.6366\ V_m$$

Average load current is

$$I_{DC} = \frac{V_{DC}}{R} = \frac{0.6336\ V_m}{R}$$

$$P_{DC} = (0.6366\ V_m)^2/R$$

RMS value of output voltage and current is

$$V_{RMS} = \left[\frac{2}{T}\int_0^{T/2}(V_m \sin\omega t)d\omega t\right]^{1/2}$$

$$= \frac{V_m}{\sqrt{2}} = 0.707\ V_m$$

$$I_{RMS} = \frac{V_{RMS}}{R} = \frac{0.707\ V_m}{R}$$

$$P_{AC} = (0.707 V_m)^2/R$$

(a) Efficiency $\eta = \dfrac{P_{DC}}{P_{AC}} = \dfrac{(0.6366\ V_m)^2}{(0.707\ V_m)^2} = 81\%$

(b) Form Factor FF $= \dfrac{V_{RMS}}{V_{avg}} = \dfrac{0.707 V_m}{0.6366\ V_m}$

FF = 1.11

(c) Ripple Factor RF $= \sqrt{1.11^2 - 1} = 0.482$

RF = 48.2%

(d) $V_s = V_m/\sqrt{2} = 0.707\ V_m$

$I_s = 0.5\ V_m/R$

VA rating of transformer $= 2V_sI_s$

$= 2 \times 0.707\ V_m \times 0.5\ V_m/R$

$$\text{TUF} = \frac{P_{DC}^2}{VA} = \frac{0.6360^2}{2\times0.707\times0.5} = 0.5732$$

TUF = 57.32 %

(e) PIV = $2\ V_m$

(f) $(I_s)_{peak} = V_m/R$ and $I_s = \dfrac{0.707\ V_m}{R}$

C.F. of input current is $\dfrac{(I_s)_{peak}}{I_s}$

$$\text{C.F.} = \frac{1}{0.707} = \sqrt{2}$$

(g) The input P.F. for a resistive load

$$\text{P.F.} = \frac{P_{AC}}{VA} = \frac{0.707^2}{2\times0.707\times0.5} = 0.707$$

Note:

- $\dfrac{1}{\text{TUF}} = \dfrac{1}{0.5732} = 1.75$ signifies that the input transformer, if present, must be 1.75 times larger than when it is used to deliver power from a pure AC sinusoidal voltage.
- It has R.F. = 48.2% and a rectification of 81%.
- The performance of a full-wave rectifier is significantly improved compared with that of a half-wave rectifier.

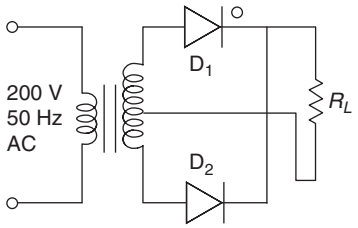
Example 5: If a ripple free current is assumed to be flowing through a highly inductive load, which is supplied from a single-phase diode bridge rectifier. The input side current wave form will be

(A) square (B) sinusoidal
(C) triangular (D) constant

Solution: (A)

Ripple free output current will have constant magnitude and thereby the supply current will take a square-wave form.

Example 6: The figure shown below is a full-wave rectifiers fed from a centre tapped transformer.



The diodes D_1 and D_2 would have a peak reverse voltage of

(A) 30 V (B) 60 V
(C) $30\sqrt{2}$ V (D) $60\sqrt{2}$ V

Solution: (B)

Peak reverse voltage across the diode $= 2V_m$
V_m = Peak value of each secondary voltage and the secondary of centre tapped transformer

$$V_s = 30(\text{RMS})$$

$$V_{s(max)} = 30\sqrt{2}\ \text{V} = V_m$$

Peak reverse voltage $= 2\ V_m = 2 \times 30 \times \sqrt{2}$

$$= 60\sqrt{2}\ \text{V}$$

Example 7: A single-phase fully controlled thyristor bridge converter is fed from a 230 V, 50 Hz supply is operating at a firing angle of 30° and constant DC output current of 10 A. The displacement factor at input is [assume overlap angle of 10°]

(A) 0.7 (B) 0.6 (C) 0.8 (D) 0.85

Solution: (A)

Displacement factor D.F. $= \dfrac{V_0 I_0}{V_s I_s}$

$$V_0 = \frac{2V_m}{\pi}\cos\alpha - \frac{\omega L_s I_0}{\pi}$$

$$I_0 = \frac{V_m}{\omega L_s}\left[\cos\alpha - \cos(\alpha+\mu)\right]$$

$$\omega = 2\pi f = 2\pi \times 50 = 314.16$$

$$L_s = \frac{V_m}{\omega I_0}\left[\cos\alpha - \cos(\alpha+\mu)\right]$$

$$= \frac{230\times\sqrt{2}}{314.16\times 10}\left[\cos 30 - \cos(30+10)\right]$$

$$L_s = 0.0103$$

$$V_0 = \frac{2\times 230\times\sqrt{2}\cos 30}{\pi} - \frac{314.16\times 0.0103\times 10}{\pi}$$

$$= \frac{1}{\pi}[531.024] = 169 \text{ V}$$

$$\text{D.F.} = \frac{169\times 10}{230\times 10} = 0.73$$

Performance Analysis 1–ϕ Full Converter

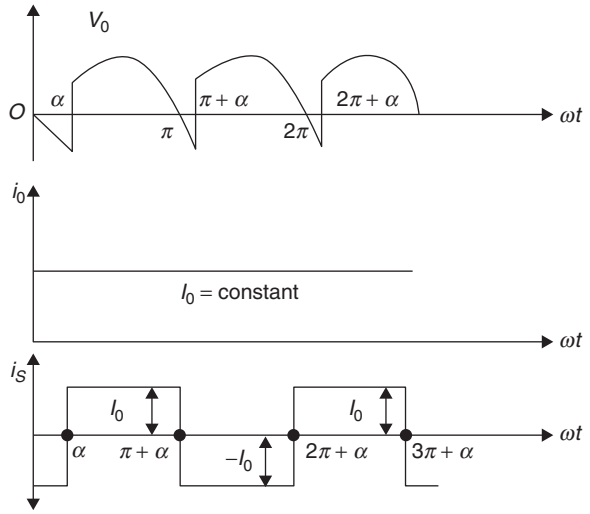
Single-phase fully controlled rectifier in continuous conduction mode draws a constant load current, I_0.

$$(V_0)_{avg} = \frac{2V_m}{\pi}\cos\alpha$$

$$V_{or} = V_{sr} = \frac{V_m}{\sqrt{2}}$$

$$I_{sr} = I_0 = I_{or}$$

$$I_{s(avg)} = 0$$



Source current is not a sinusoidal wave.

Applying Fourier series analysis to it to carry out the harmonic distortion analysis,

$$i_s = \sum_{n=1,3,5}^{\infty} \frac{4I_0}{n\pi}\sin(n\omega t + \phi_n)$$

where $\phi_n = -n\alpha$

RMS value of n^{th} harmonic current, $I_{sn(RMS)} = \dfrac{4I_0}{n\pi\times\sqrt{2}}$

$$= \frac{2\sqrt{2}I_0}{n\pi}$$

(i) Distortion Factor $= \dfrac{\text{Fundamental RMS}}{\text{Overall RMS}} = \dfrac{(I_{s1})_{RMS}}{I_{sr}}$

(ii) $\dfrac{\frac{2\sqrt{2}I_0}{\pi}}{I_0} = \dfrac{2\sqrt{2}}{\pi}$

(iii) Fundamental Displacement Factor = $\cos\phi_1 = \cos(-\alpha)$ = $\cos\alpha$

(iv) Source Power Factor = (Distortion Factor) (Fundamental Displacement Factor)

$$\frac{2\sqrt{2}}{\pi}\cos\alpha$$

(iv) Total Harmonic Distortion (THD)

$$= \frac{(\text{Useless harmonics})_{RMS}}{(\text{Fundamental})_{RMS}}$$

$$\text{THD} = \sqrt{\left(\frac{I_{sr}}{I_{s1}}\right)^2_{RMS} - 1}$$

$$\text{THD} = \sqrt{\frac{\pi^2}{8} - 1} = 48.34\%$$

(v) From Factor

$$= \frac{\text{RMS output voltage}}{\text{Average output voltage}} = \frac{V_m/\sqrt{2}}{\frac{2V_m}{\pi}\cos\alpha}$$

$$= \frac{1}{\frac{2\sqrt{2}}{\pi}\cos\alpha} = \frac{1}{\text{Source Power Factor}}$$

(vi) Ripple Factor (voltage ripple factor)

$$= \frac{\text{Ripple output voltage}}{\text{Average output voltage}}$$

$$= \sqrt{\frac{V_{or}^2 - V_0^2}{V_0^2}}$$

$$= \sqrt{(\text{From factor})^2 - 1}$$

$$= \sqrt{\frac{\pi^2}{8\cos^2\alpha} - 1}$$

(vii) Source Active Power $= V_{sr} I_{sr} \cos\phi_s$

$$= \frac{V_m}{\sqrt{2}} \times I_0 \times \frac{2\sqrt{2}}{\pi} \cos\alpha$$

$$= \frac{2V_m}{\pi} \cos\alpha \cdot I_0 = (V_0)_{avg} (I_0)_{avg}$$

$$P_{IN} = P_{OUT}$$

(viii) Source Reactive Power $= V_{sr} I_{sr} \sin\phi_s$

$$= \frac{V_m}{\sqrt{2}} \times I_0 \times \frac{2\sqrt{2}}{\pi} \sin\alpha = \frac{2V_m}{\pi} \cos\alpha \cdot I_0 \cdot \tan\alpha$$

$$= V_0 I_0 \tan\alpha$$

Single-phase Semi-converter

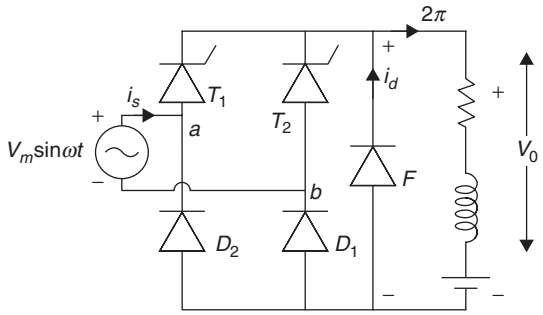


Figure 1 Symmetrical semi converter

For R-Loads

Symmetrical: Circuit turn-off time, $t_c = \dfrac{\pi - \alpha}{\omega}$

Asymmetrical: Circuit turn-off time, $t_c = \dfrac{\pi}{\omega}$

$$(V_0)_{avg} = \frac{1}{\pi} \int_{\alpha}^{\pi} V_m \sin\omega t \, d\omega t = \frac{V_m}{\pi} [1 + \cos\alpha]$$

Output RMS voltage, V_{RMS}

$$= \left[\frac{1}{\pi} \int_{\alpha}^{\pi} V_m^2 \sin^2 \omega t \, d\omega t \right]^{1/2}$$

$$V_{RMS} = \frac{V_m^2}{\pi} \left[\frac{\pi - \alpha}{2} + \frac{\sin 2\alpha}{4} \right]^{1/2}$$

For Continuous Conduction Loads (RL, RLE)

On the assumption of constant load current, I_0

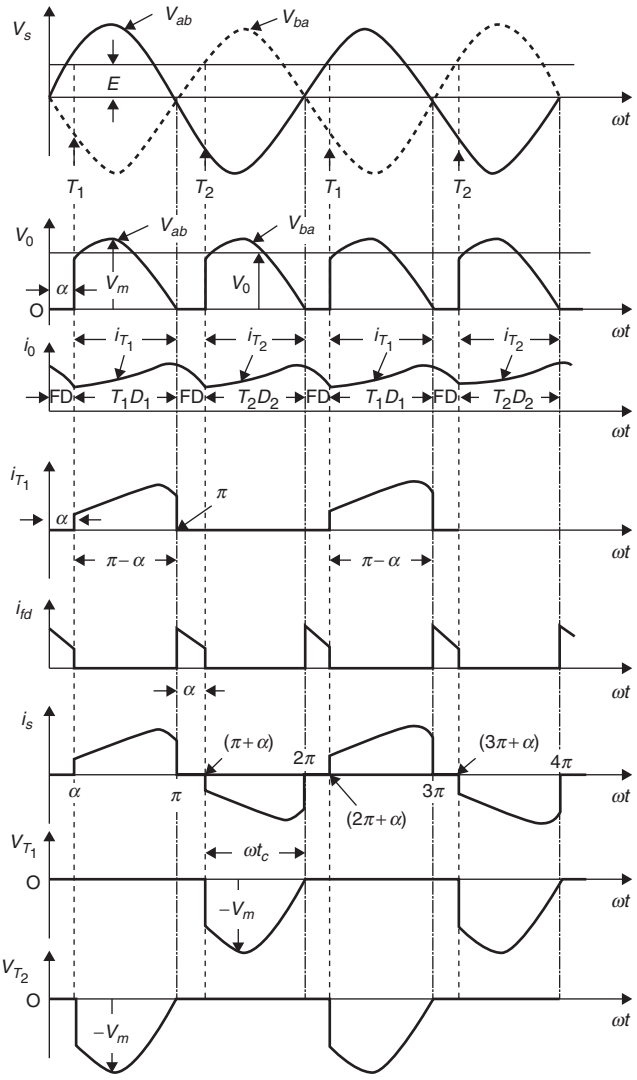
$$(I_{T_1})_{avg} = I_0 \left(\frac{\pi - \alpha}{2\pi} \right)$$

$$(I_{T_1})_{RMS} = I_0 \left(\frac{\pi - \alpha}{2\pi} \right)^{\frac{1}{2}}$$

$$V_0 = \frac{V_m}{\pi} [1 + \cos\alpha]$$

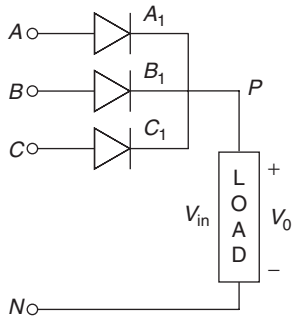
$$V_{or} = \left\{ \frac{V_m^2}{\pi} \left[\frac{\pi - \alpha}{2} + \frac{\sin 2\alpha}{4} \right] \right\}^{\frac{1}{2}}$$

$$I_{sr} = I_0 \left[\frac{\pi - \alpha}{\pi} \right]^{\frac{1}{2}}$$

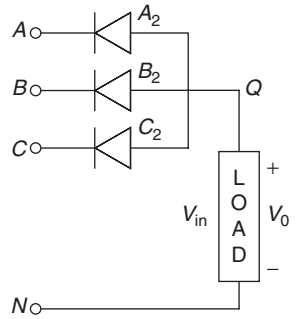


No negative voltage is obtained in this case. Since only two thyristors are used cost is less.

Three-phase Uncontrolled Half-pave Rectifier



Common Cathode Configuration



Common Anode Configuration

- In common cathode configuration, the diode is subjected to maximum positive voltage will be in conduction. Load is subjected only to phase voltages.
- The diode which is subjected to maximum negative voltage will be in conduction in common anode configuration.

Average output voltage

$$V_0 = \frac{1}{2\pi/3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} V_m \sin\omega t d(\omega t)$$

$$V_0 = \frac{3\sqrt{3}}{2\pi} V_{mp}$$

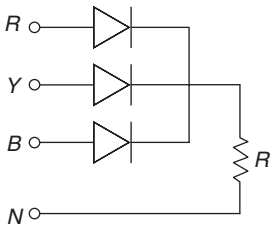
$$V_0 = \frac{3\sqrt{6}}{2\pi} V_{ph}.$$

- If diodes are replaced with thyristors,

$$V_0 = \frac{3\sqrt{6}}{2\pi} V_{ph}\cos\alpha$$

- This type of converter is rarely used in industry because it introduces DC component in the supply current.

Example 8: The ripple frequency of a 3-phase half-wave rectifier, which is connected to a 380 V (line to line) 200 Hz symmetrical 3-∅, 4wire supply, as shown below, would be



(A) 200 Hz (B) 800 Hz (C) 400 Hz (D) 600 Hz

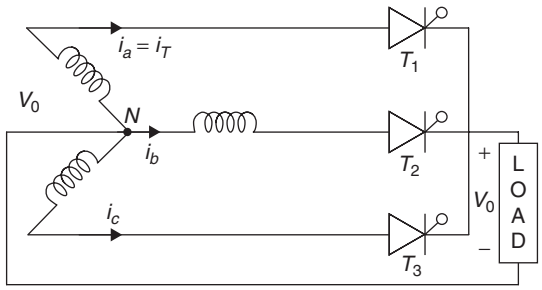
Solution: (D)

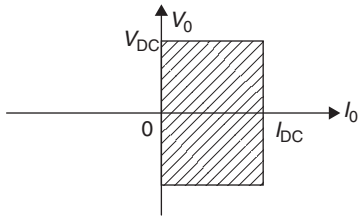
$V_l = 380$ V
$f = 200$ Hz

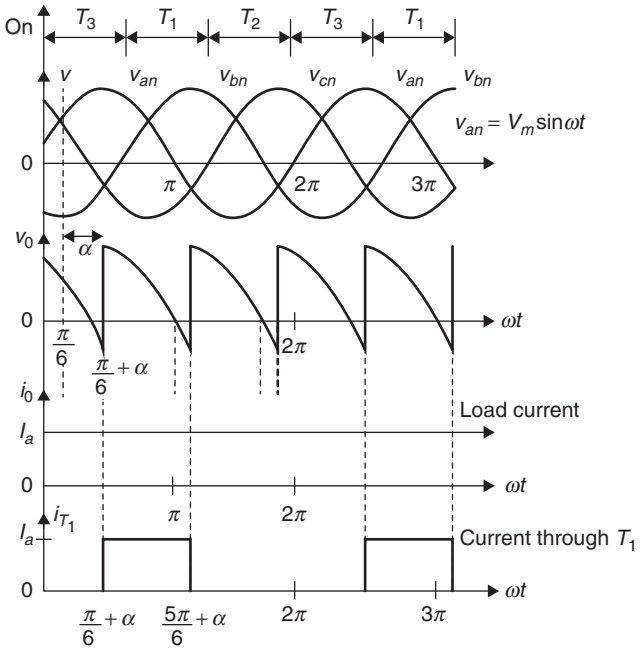
In a 3-phase rectifier one cycle of supply voltage, the output voltage has three pulses. The ripple frequency = $3f$

$= 3 \times 200 = 600$ Hz

Three-phase Controlled Half-wave Converter







$$V_{an} = V_m \sin \omega t$$

$$V_{bn} = V_m \sin\left(\omega t - \frac{2\pi}{3}\right)$$

$$V_{cn} = V_m \sin\left(\omega t - \frac{4\pi}{3}\right)$$

- When T_1 is fired at $\omega t = \pi/6 + \alpha$, the phase voltage V_{an} appears across the load until thyristor T_2 is fired at $\omega t = 5\pi/6 + \alpha$.
- When T_2 is fired, T_1 is reverse-biased, because $V_{ab} = (V_{an} - V_{bn})$ is negative and T_1 is turned off. The phase voltage appears across the load until T_3 is fired at $\omega t = \frac{3\pi}{2} + \alpha$.
- When T_3 is fired, T_2 is turned off and V_{cn} appears across the load until T_1 is fired again at the beginning of next cycle

$$V_{DC} = \frac{3}{2\pi} \int_{\left(\frac{\pi}{6}+\alpha\right)}^{\left(\frac{5\pi}{6}+\alpha\right)} V_m \sin \omega t \, d\omega t$$

$$V_{DC} = \frac{3\sqrt{3}\, V_m}{2\pi} \cos\alpha$$

Maximum average output voltage that occurs at $\alpha = 0$

$$V_{DM} = \frac{3\sqrt{3}\, V_m}{2\pi}$$

Normalized average output voltage

$$V_n = \frac{V_{DC}}{V_{DM}} = \cos\alpha$$

RMS value of output voltage

$$V_{RMS} = \left[\frac{3}{2\pi}\left(\int_{\frac{\pi}{6}+\alpha}^{\frac{5\pi}{6}+\alpha} V_m^2 \sin^2 \omega t \; (d\omega t)\right)\right]^{\frac{1}{2}}$$

$$V_{RMS} = \sqrt{3} V_m \left(\frac{1}{6} + \frac{\sqrt{3}}{8\pi}\cos 2\alpha\right)^{\frac{1}{2}}$$

For R-load continuous conduction $\left(\alpha < \frac{\pi}{6}\right)$:

$$V_0 = \frac{3\sqrt{3} V_m}{2\pi} \cos\alpha$$

For R-load discontinuous conduction $\left(\alpha \geq \frac{\pi}{6}\right)$

$$V_0 = \frac{3V_m}{2\pi}\left[1 + \cos\left(\alpha + \frac{\pi}{6}\right)\right]$$

Gating Sequence

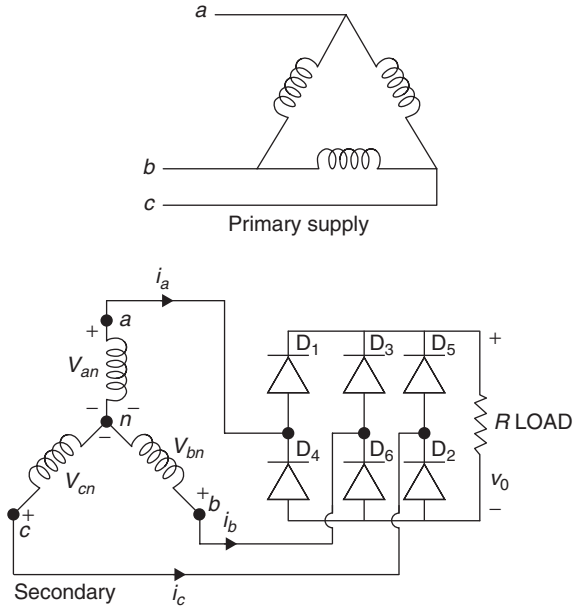
1. Generate a pulse signal at positive zero crossing of the phase voltage V_{an}. Delay the pulse by desired angle $(\alpha + \pi/6)$ and apply it to gate and cathode terminals of T_1 through a gate isolating circuit.
2. Generate two more pulses of delay angles $\alpha + 5\pi/6$ and $\alpha + 9\pi/6$ for gating T_2 and T_3 through gate isolating circuits.

Note:

- The frequency of the output ripple is three times the supply frequency.
- For $\alpha > \pi/6$ with a resistive load, the current is discontinuous.
- This converter is not normally used in practical applications.

Three-phase Uncontrolled Full-wave Rectifier

- It is commonly used in high power applications. It can operate with or without transformer and gives a six-pulse ripples on the output voltage. The diodes are numbered in the order of conduction sequences and each one conducts for 120°.
- The conduction sequence for diode is D_1–D_2, D_3–D_2, D_3–D_4, D_5–D_6 and D_1–D_6.
- The pair of diodes, connected between that pair of supply lines having the highest amount of instantaneous line to line voltage will conduct.



If V_{an}, V_{bn}, V_{cn} are the instantaneous values of phase voltages, then they can be expressed in terms of peak voltage as

$$V_{an} = V_m \sin\omega t$$

$$V_{bn} = V_m \sin\left(\omega t - \frac{2\pi}{3}\right)$$

$$V_{cn} = V_m \sin\left(\omega t - \frac{4\pi}{3}\right)$$

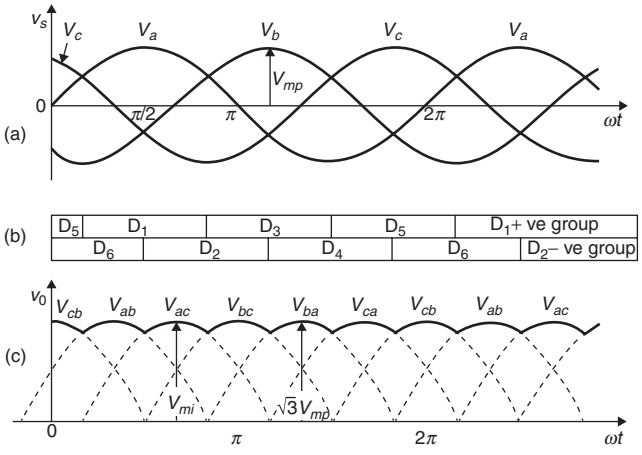
Since line-to-line voltages lead the phase voltages by 30°, the instantaneous value of line-to-line voltages can be described by

$$V_{ab} = \sqrt{3}V_{ml} \sin\left(\omega t + \frac{\pi}{6}\right)$$

$$V_{bc} = \sqrt{3}V_{ml} \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$V_{ca} = \sqrt{3}V_{ml} \sin\left(\omega t - \frac{7\pi}{6}\right)$$

$D_1\ D_3\ D_5 \rightarrow$ Common Cathode configuration
$D_4\ D_6\ D_2 \rightarrow$ Common Anode configuration
The load is connected between two common points.



- The diode which is subjected to maximum positive voltage and the diode which is subjected to maximum negative voltage will be in conduction simultaneously.

Performance Parameters

Average value of output voltage

$$V_{DC} = \frac{2}{2\pi/6}\int_0^{\pi/6} \sqrt{3}V_m \cos\omega t\ d(\omega t) = \frac{3\sqrt{3}V_m}{\pi}$$

$$V_{DC} = 1.654 V_m$$

$$I_{DC} = 1.654 V_m/R$$

$$P_{DC} = \frac{(1.654 V_m)^2}{R}$$

RMS value of output voltage

$$V_{RMS} = \left[\frac{2}{2\pi/6}\int_0^{\frac{\pi}{6}} 3V_m^2 \cos^2 \omega t\, d(\omega t)\right]^{\frac{1}{2}}$$

$$= \left(\frac{3}{2} + \frac{9\sqrt{3}}{4\pi}\right)^{\frac{1}{2}} V_m R$$

$$V_{RMS} = 1.6554\ V_m$$

$$I_{RMS} = 1.6554\ V_m/R$$

$$P_{AC} = (1.6554\ V_m)^2/R$$

$$I_{RMS} = \left[\frac{4}{2\pi}\int_0^{\frac{\pi}{6}} I_m^2 \cos^2 \omega t\, d(\omega t)\right]^{\frac{1}{2}}$$

$$= I_m\left[\frac{1}{\pi}\left(\frac{\pi}{6} + 1/2 \sin\frac{2\pi}{6}\right)\right]^{\frac{1}{2}}$$

$I_{RMS} = 0.5515\ I_m$. If load is purely resistive.

where $I_m = \sqrt{3}\,V_m/R$

RMS value of secondary current

$$I_s = \left[\frac{8}{2\pi}\int_0^{\frac{\pi}{6}} I_m^2 \cos^2 \omega t\, d(\omega t)\right]^{\frac{1}{2}}$$

$$I_m\left[\frac{2}{\pi}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{2\pi}{6}\right)\right]^{\frac{1}{2}}$$

$$I_s = 0.7804\ I_m$$

(a) $\eta = \dfrac{P_{DC}}{P_{AC}} = \dfrac{(1.654 V_m)^2}{(1.6554 V_m)^2}$

$\eta = 99.83\%$

(b) F.F. $= \dfrac{1.6554}{1.654} = 1.0008$

F.F. $= 100.08\%$

(c) R.F. $= \sqrt{1.0008^2 - 1} = 0.04$

R.F. $= 4\%$

(d) $V_s = 0.707\ V_m$

$$I_s = 0.7804\ I_m = 0.7804 \times \frac{\sqrt{3}\ V_m}{R}$$

$$\text{T.U.F.} = \frac{1.654^2}{3 \times \sqrt{3} \times 0.707 \times 0.7804} = 0.9542$$

(e) Peak line to neutral voltage

$$V_m = \frac{280.7}{1.654} = 169.7$$

$$\text{PIV} = \sqrt{3}\,V_m = \sqrt{3} \times 169.7$$

PIV = 293.9 V

(f) Average current through each diode is

$$I_d = \left[\frac{4}{2\pi}\int_0^{\frac{\pi}{6}} I_m \cos\omega t\ d(\omega t)\right]$$

$$= I_m \times \frac{2}{\pi}\sin\frac{\pi}{6}$$

$$I_d = 0.3183\ I_m$$

Note:

- A three-phase bride rectifier has considerably improved performances compared with those of single-phase rectifiers
- If diode is replaced by thyristor

$$V_0 = \frac{3V_{ml}\cos\alpha}{\pi}$$

For R-load continuous conduction $\left(\alpha < \frac{\pi}{3}\right)$

$$V_0 = \frac{3V_{ml}}{\pi}\cos\alpha$$

For R-load discontinuous conduction $\left(\alpha \geq \frac{\pi}{3}\right)$

$$V_0 = \frac{3V_{ml}}{\pi}\left[1+\cos\left(\alpha+\frac{\pi}{3}\right)\right]$$

Example 9: A three-phase 230 V (L–L) AC source is supplying power to a purely resistive load through an AC to DC diode bridge rectifier. The average DC voltage across the load will be

(A) 300 V (B) 294.20 V (C) 310.61 V (D) 330.5 V

Solution: (C)

Average DC voltage across the load $= \frac{3V_{ml}}{\pi}$

$V_{ml} \rightarrow$ Peak value of L–L voltage

$$V_{ml} = \sqrt{2}\times 230$$

$$\text{Average DC voltage} = \frac{2\times\sqrt{2}\times 230}{\pi}$$

$$= 310.609$$

Example 10: With an increase of overlap angle in a three-phase-controlled bridge rectifier, the output DC voltage would

(A) Not change (B) Increase
(C) Depend on load (D) Decrease

Solution: (D)

Example 11: The ratio of peak to peak voltage ripple to peak output DC voltage at the output of a fully controlled natural commutated 3-phase bridge rectifier operating at a firing angle of $\alpha = 30°$ is

(A) 0.866 (B) 0.5 (C) 0.73 (D) 0.134

Solution: (B)

$$V_{p.pr} = V_m - V_m\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$\frac{V_{p\text{-}pr}}{V_m} = 1-\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$= 1 - \cos(30+30) = 0.5$$

Example 12: A three-phase diode bridge rectifier is connected to a 200 V (RMS) 50 Hz, 3-∅ supply. The peak instantaneous output voltage across a purely resistive load connected across the output of a rectifier would be

(A) $200\sqrt{2}\ V$ (B) $200\ V$

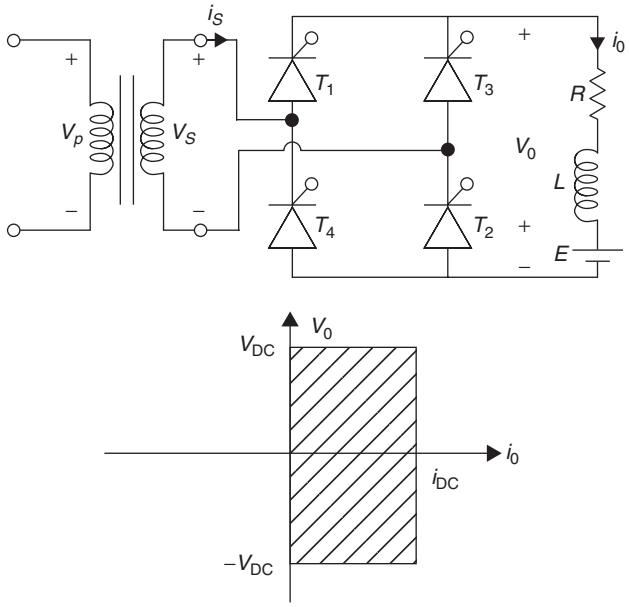
(C) $200\times\sqrt{\frac{2}{3}}\ V$ (D) $\frac{200}{\sqrt{3}}\ V$

Solution: (A)

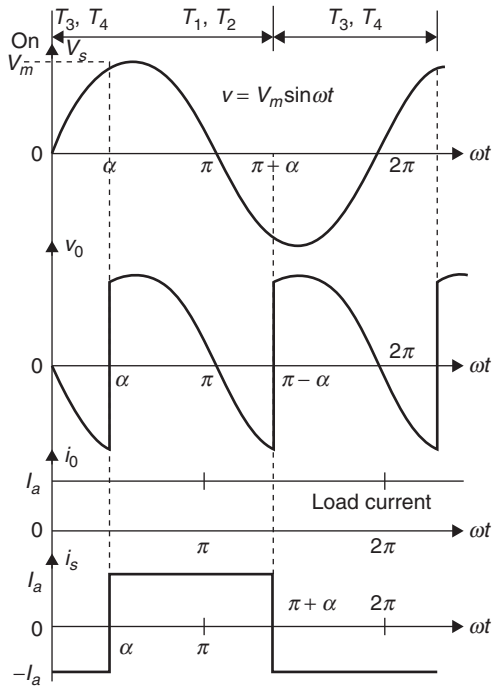
Peak instantaneous value of output voltage (resistive load) $= \sqrt{2}V_{ml}$

$$= 200\sqrt{2}$$

Single-phase Full-controlled Full-wave Rectifier



- During positive half-cycle, thyristors T_1 and T_2 are forward biased and when these two thyristors are fired simultaneously at $\omega t = \alpha$, the load is connected to the input supply through T_1 and T_2. Due to inductive load, T_1 and T_2 continue to conduct beyond $\omega t = \pi$ even though the input voltage is already negative.
- During negative half cycle of the input voltage, T_3 and T_4 are forward biased and firing of T_3 and T_4 applies the supply voltage across thyristors T_1 and T_2 as reverse blocking voltage. T_1 and T_2 are turned off due to line or natural commutation and load current is transferred from T_1 and T_2 to T_3 and T_4.
- The diagram below shows the waveforms for input–output voltages and currents.



- During the period α to π the input voltage V_s and input current are positive and power flows from supply to load. The converter is said to be operated in rectification mode.
- During the period form π to $\pi + \alpha$, the input voltage V_s is negative and input current is positive and reverse power flows from the load to the supply. The converter is said to be in inversion mode.
- Depending on the value of α, the average output voltage could be either positive or negative and it provides two-quadrant operation.

Average output voltage, V_{DC}

$$= \frac{2}{\pi}\int_{\alpha}^{\pi+\alpha} V_m \sin\omega t \, d(\omega t)$$

$$V_{DC} = \frac{2V_m}{\pi}\cos\alpha$$

V_{DC} can be varied from $2V_m/\pi$ to $-2V_m/\pi$ by varying α from 0 to π.

Maximum average output voltage

$$V_{dm} = 2V_m/\pi$$

Normalized average output voltage

$$V_n = \frac{V_{DC}}{V_{DM}} = \cos\alpha$$

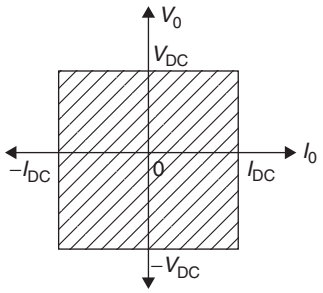
$$V_{RMS} = \left[\frac{2}{2\pi\alpha}\int_{\alpha}^{\pi+\alpha} V_m^2 \sin^2\omega t \, d(\omega t)\right]^{\frac{\pi}{2}}$$

$$V_{RMS} = \frac{V_m}{\sqrt{2}} = V_s$$

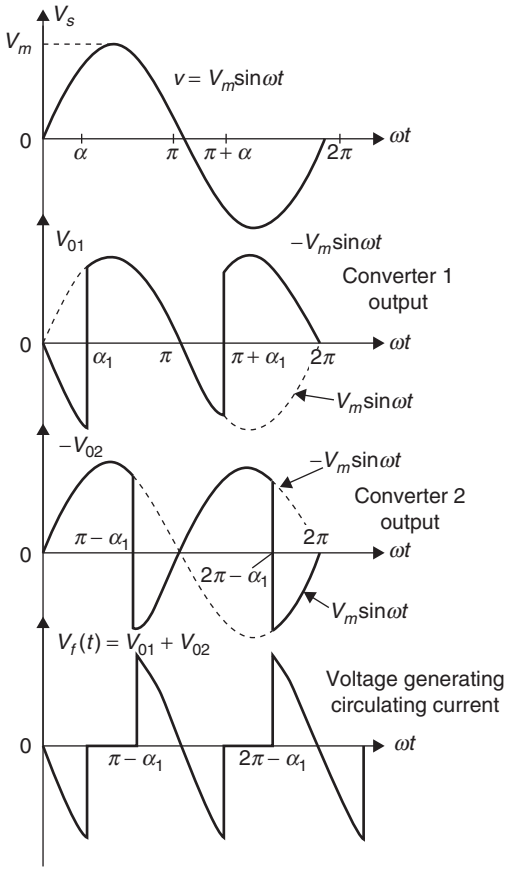
Note:

- Varying the delay angle α from 0 to π can vary the average output voltage from $2V_m/\pi$ to $-2V_m/\pi$ provided the load is highly inductive and its current is continuous.
- For a purely resistive load, the delay angle α can be varied from 0 to $\pi/2$ producing an output voltage ranging from $2V_m/\pi$ to 0.
- The full converter can operate in two quadrants for a highly inductive load and in one quadrant only for a purely resistive load.

Single-phase Dual Converter



- If two full converters are connected back to back as shown in the above figure, both the output voltage and load current flow can be reversed.
- The system provides a four-quadrant operation and is called a dual converter.
- If α_1 and α_2 are the delay angles of converters 1 and 2, respectively, the corresponding average output voltage are V_{DC1} and V_{DC2}. The delay angles are controlled such that one converter operates as inverter. But both the converters produce same average output voltage.



- The above diagram shows the output waveforms of two converters where two average output voltages are the same.

The average output voltages are

$$V_{DC1} = \frac{2V_m}{\pi}\cos\alpha_1$$

$$V_{DC2} = \frac{2V_m}{\pi}\cos\alpha_2$$

Since one converter is rectifying and the other is inverting.

$$V_{DC1} = -V_{DC2} \text{ (or) } \cos\alpha_2 = -\cos\alpha_1 = \cos(\pi - \alpha_1)$$
$$\alpha_2 = \pi - \alpha$$

- Since the instantaneous output voltages of the two converters are out of phase, there can be instantaneous voltage difference and this can result in circulating current between two converters. This circulating current cannot flow through the load and is normally limited by a circulating current reactor L_r.
- Circulating current can be expressed as

$$i_r = \frac{1}{\omega L_r}\int_{\pi-\alpha_1}^{\omega t} V_r\, d(\omega t) = \frac{1}{\omega L_r}\int_{\pi-\alpha_1}^{\omega t}(V_{01}+V_{02})\,d\omega t)$$

$$= \frac{V_m}{\omega L_r}\int_{2\pi-\alpha_1}^{\omega t}\sin\omega t\; d(\omega t) - \int_{2\pi-\alpha_1}^{\omega t} -\sin\omega t\; d(\omega t)$$

$$= \frac{2V_m}{\omega L_r}(\cos\alpha_1 - \cos\alpha_2)$$

$$i_r > 0 \text{ for } 0 \le \alpha_1 \le \frac{\pi}{2}$$

$$i_r < 0 \text{ for } \frac{\pi}{2} \le \alpha_1 \le \pi$$

- For $\alpha_1 = 0$, converter 1 operates.
- For $\alpha_1 = \pi$, converter 2 operates.
- For $0 \le \alpha_1 \le \pi/2$, converter 1 supplies a positive load current $+i_0$ and thus circulating current can only be positive.
- For $\pi/2 < \alpha_1 < \pi$, converter 2 supplies a negative load current $-i_0$ and thus only a negative circulating current can flow.
- At $\alpha_1 = \pi/2$, converter 1 supplies positive circulating current during first half cycle and converter 2 supplies negative circulating current during the second half cycle.
- The instantaneous circulating current depends on the delay angle. For $\alpha_1 = 0$ its magnitude is maximum when $\omega t = n\pi$ ($n = 1, 3, 5\ldots$.) and minimum when $\omega t = n\pi$ ($n = 0, 2, 4,\ldots$..).

Gating Sequence

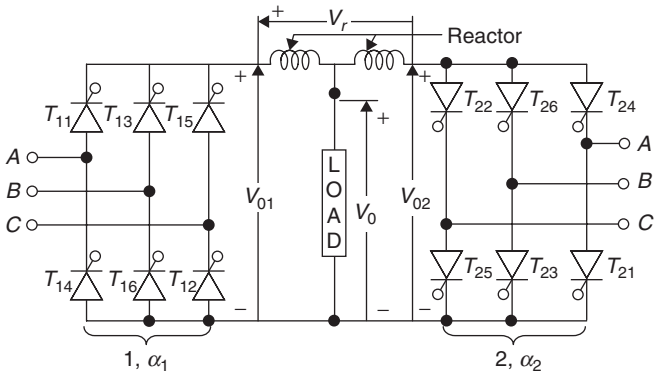
1. Triggering the positive converter with a delay angle of $\alpha_1 = \alpha$.
2. Triggering the negative converter with a delay angle of $\alpha_2 = \pi - \alpha$ through gate-isolating circuit.

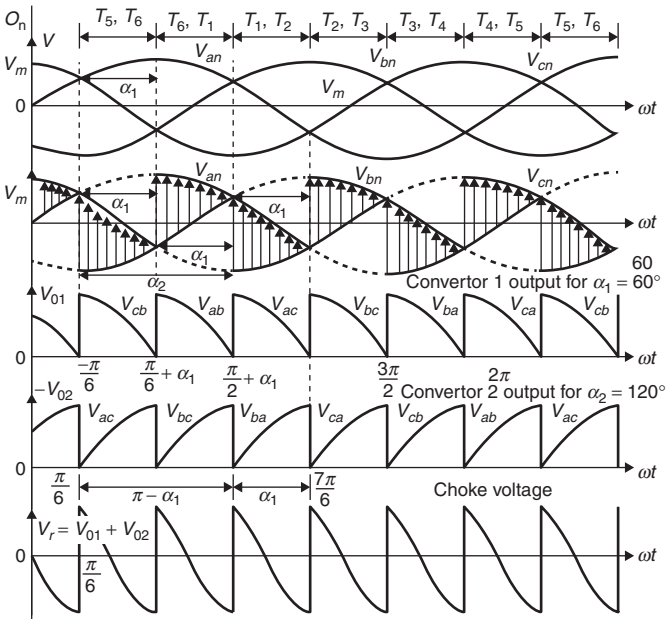
Note:

- Dual converter consists of two, full converters: one converter producing output voltage and another converter producing negative output voltage.
- Varying the delay angle α from 0 to π can vary the average output voltage from $2V_m/\pi$ to $-2V_m/\pi$ provided the load is highly inductive and its current is continuous.
- For a highly inductive load, the dual converter can operate in four quadrants. The current can flow in and out of the load. A DC inductor is needed to reduce the circulating current.

Three-phase Dual Converter

- They are used in applications up to 2000 kW level, variable speed drives where a four-quadrant operation is generally required.





- The above diagram shows the three-phase dual converter where two, three-phase converters are connected back to back.
- The principle of operation is similar to that of a three-phase full converter.

- During interval $(\pi/6 + \alpha_1) \le \omega t \le (\pi/2 + \alpha_1)$, the line to line voltage V_{ab} appears across the output of converter 1 and V_{bc} appears across converter 2.

If L–N voltages are defined as $V_{an} = V_m \sin\omega t$, $V_{bn} = V_m \sin(\omega t - 2\pi/3)$, $V_{cn} = V_m \sin(\omega t + 2\pi/3)$

Corresponding L–L voltages are

$$V_{ab} = V_{an} - V_{bn} = \sqrt{3}\, V_m \sin\left(\omega t + \frac{\pi}{6}\right)$$

$$V_{bc} = \sqrt{3}\, V_m \sin\left(\omega t - \frac{\pi}{2}\right),$$

$$V_{ca} = \sqrt{3}\, V_m \sin\left(\omega t + \frac{5\pi}{6}\right)$$

If V_{01} and V_{02} are output voltages of convertrs 1 and 2, respectively, the instantaneous voltage across the inductor during interval $(\pi/6 + \alpha_1) \le \omega t \le (\pi/2 + \alpha_1)$ is

$$V_r = V_{01} + V_{02} = V_{ab} - V_{bc}$$
$$V_r = \sqrt{3}\, V_m \cos(\omega - \pi/6)$$

Circulating current

$$i_r(t) = \frac{1}{\omega L_r} \int_{\frac{\pi}{6}+\alpha_1}^{\omega t} 3V_m \cos\left(\omega t - \frac{\pi}{6}\right) d\omega t$$

$$i_r(t) = \frac{3V_m}{\omega L_r}\left[\sin\left(\omega t - \frac{\pi}{6}\right) - \sin\alpha_1\right]$$

- The circulating current depends on delay angle α_1 and on inductance L_r. This current becomes maximum when $\omega t = 2\pi/3$ and $\alpha_1 = 0$. Even without any external load the converters would be continuously running due to circulating current as a result of ripple voltage across the inductor.
- This allows smooth reversal of load current during the change over from one quadrant to another and provides fast dynamic responses especially for electrical motor drives.

Gating Sequence

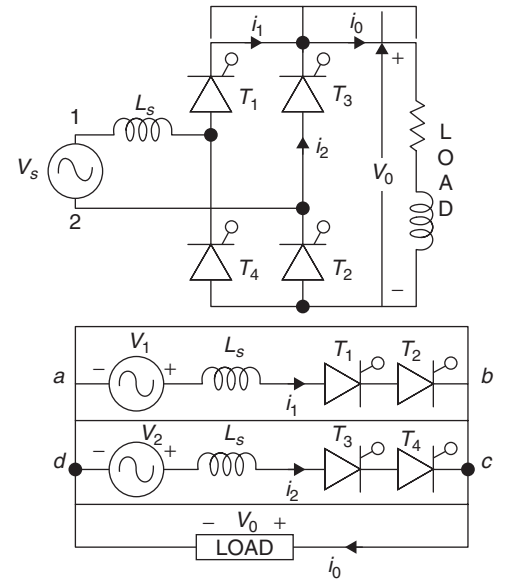
- Trigger the positive converter with a delay angle of $\alpha_1 = \alpha$.
- Trigger the negative converter with a delay angle of $\alpha_2 = \pi - \alpha$ through gate-isolating circuits.

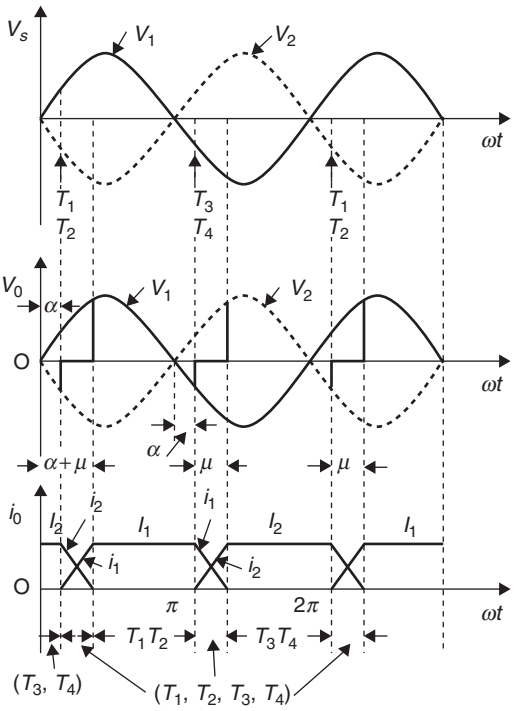
Note:

- Three-phase dual converters are used for high-power applications up to 2000 kW.
- For a highly inductive load, the dual converter can operate in 4 quadrants. The current can flow in and out of the load.
- A DC inductor is needed to reduce the circulating current.

Effect of Source Inductance

In a full-wave rectifier, when conduction shifts from T_1 T_2 to T_3 T_4 all the four thyristors are in conduction which short circuits the source. T_0 limit the short. Circuit current, L_s is included in the circuit.





By KVL $V_1 - L_s \frac{di_1}{dt} = V_2 - L_s \frac{di_2}{dt}$

$$V_1 - V_2 = L_s\left[\frac{di_1}{dt} - \frac{di_2}{dt}\right]$$

$$\frac{di_1}{dt} - \frac{di_2}{dt} = \frac{V_1 - V_2}{L_s}$$

$$= \frac{2V_m \sin\omega t}{L_s}$$

$$\frac{di_1}{dt} - \frac{di_2}{dt} = \frac{2V_m \sin\omega t}{L_s}$$

During the overlapping angle $I_0 = i_1 + i_2$

$$\frac{di_1}{dt} + \frac{di_2}{dt} = \frac{di_0}{dt} = 0$$

Adding above two equations,

$$\frac{di_1}{dt} = \frac{V_m}{L}\sin\omega t$$

$$\int_0^{I_0} di_1 = \int_{\alpha}^{\alpha+4} \frac{V_m}{L}\sin(\omega t)\,d(\omega t)$$

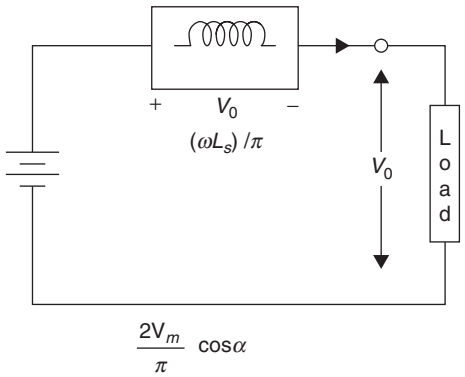
$$I_0 = \frac{V_m}{wL}\left[\cos\alpha - \cos(\alpha+\mu)\right] I$$

$$V_0 = 1/\pi \int_{(\alpha+\mu)}^{\pi+\alpha} V_m \sin\omega t\, d\omega t$$

$$V_0 = \frac{V_m}{\pi}\left(\cos(\alpha+\mu) + \cos\alpha\right)$$

But $\cos(\alpha+\mu) = \dfrac{-L_w}{V_m} I_0 + \cos\alpha$

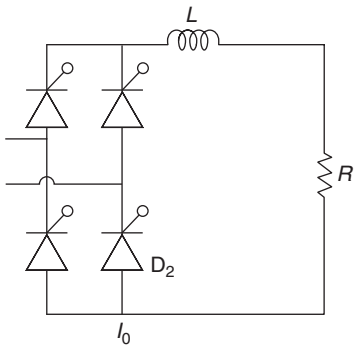
$$\therefore \quad V_0 = \frac{2V_m}{\pi}\cos\alpha - \frac{\omega L_s}{\pi} I_0$$



The inductance of source results in a lesser value of voltage. For a 3-ϕ converter,

$$V_0 = \frac{3\sqrt{6}}{\pi} V_{ph}\cos\alpha - \frac{3\omega L_s L_0}{\pi}$$

Example 13: The figure shows a fully controlled thyristor converter fed from a single-phase source.



The DC output voltage of the converter is 400 V when $\alpha = 0°$. At what value of α will the DC output voltage of the converter be 200 V (assuming continuous conduction)?

(A) 30° (B) 45° (C) 60° (D) 90°

Solution: (C)

Given a full-wave rectifier feeding inductive load and assuming continuous conduction

$$V_0 = \frac{2V_m}{\pi}\cos\alpha$$

$$V_0 = 400, \alpha = 0°$$

$$\frac{2V_m}{\pi} = 400 \text{ For } V_0 = 200\ V\alpha = ?$$

$$\frac{2V_m}{\pi}\cos\alpha = 200$$

$$\cos\alpha = \frac{200}{400} \text{ and } \alpha = 60°$$

Example 14: The displacement power factor of a single fully controlled rectifier feeding constant DC current into the load and operating at a firing angle of 18° would be

(A) 0.95 (B) 0.86 (C) 0.77 (D) 0.5

Solution: (A)

Displacement factor of rectifier

$$= \cos\alpha = \cos 18° = 0.95$$

Example 15: A thyristarized three-phase fully controlled converter feed a load that drawn constant DC current. The converter must be fed an input line current which has

(A) An equivalent average value to that of the DC load current.
(B) An equivalent peak value to that of the DC load current.
(C) An equivalent RMS value to that of the load current.
(D) None of the above.

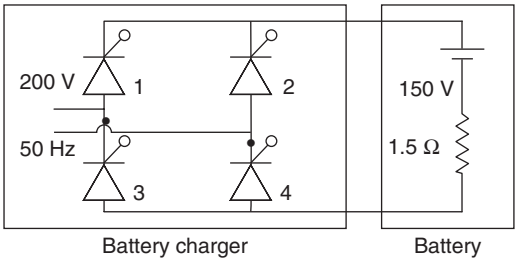
Solution: (B)

EXERCISES

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1.



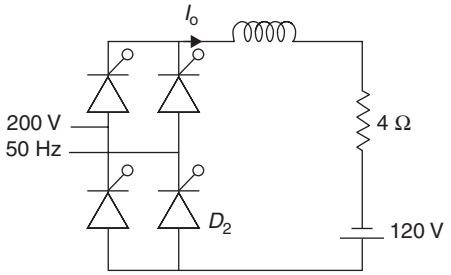
In the figure shown above if the SCRs are triggered by a constant DC signal. The average charging current if the SCR gets open circuited would be

(A) 11.9 A (B) 15.9 A
(C) 12.65 A (D) 18.73 A

2. A constant and ripple free load current is supplied by a single-phase full bridge converter. The input power factor for a triggering angle of 60° would be

(A) 0.65 (B) 0.45
(C) 0.75 (D) 0.55

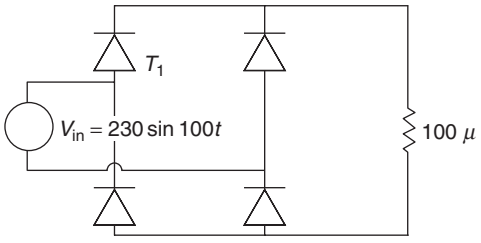
3. A single excited DC motor employs a single-phase fully controlled bridge converter for electrical braking. The equivalent circuit is as shown below



The triggering angle of the bridge for a load current of 5 A would be [Assume the load inductance is sufficient to ensure continuous and ripple free load current]

(A) 56° (B) 129°
(C) 124° (D) 51°

4.



The power dissipated in the load resistor of 100 connected across a single-phase diode bridge rectifier as shown below approximately is

(A) $\frac{3680}{\pi}$ (B) $\frac{230}{\pi}$
(C) 920 W (D) 265 W

5. A 230 V, 50 Hz supply is feeding a 3-phase fully controlled bridge converter with a free-wheeling diode and operating at a firing angle of 30°. Due to high load inductance, the load current is assumed to be constant at 8 A. If '*K*' is the ratio of the input power factor to input displacement factor, the value of *K* is

(A) 0.5 (B) 0.55
(C) 0.65 (D) 0.45

6. A DC supply of 230 V, 50 Hz is fed from a synchronous generator set of 200 V through a 3-∅ fully controlled bridge converter. The load current is maintained constant at 10 A by the large inductance connected to the DC circuit. If the internal of the generator is 0.4 then each thyristor will be reverse biased for a period of

(A) 129° (B) 55°
(C) 125° (D) 60°

7. A constant DC load current of 120 A is fed by a three-phase fully controlled converter. Each thyristor of the converter would carry an RMS current of

(A) 40 A (B) 80 A
(C) 97.98 A (D) 69.28 A

8. The circulating current in a dual converter

(A) Flows only if there is no inter connecting inductor.
(B) Improves response time and allows smooth reversal of load current.
(C) Reduces response time and does not allow smooth reversal of load current.
(D) Increases response time and allows smooth reversal of load current.

9. If θ_1 and θ_2 are triggering angles of an AC to DC circulating current, dual converter, the converter is operated with the following relationship between θ_1 and θ_2

(A) $\theta_2 + \theta_1 = 90°$
(B) $\theta_2 - \theta_1 = 180°$
(C) $\theta_2 + \theta_1 = 360°$
(D) None of the above

10. Which among the following statements are true regarding zero current switching (ZCS) and zero voltage switching (ZVS) type resonant converters?

(1) ZCS can eliminate switching losses at turn off and reduce switching losses at turn ON.
(2) ZVS eliminates the capacitive turn on loss.

(3) ZCS operates with a constant OFF time control.
(4) ZVS operates with a constant ON time control.
(A) 3 and 4 only (B) 1 and 2 only
(C) 2 and 3 only (D) All 1, 2, 3 and 4

11. A 380 V, 50 Hz, 3-∅, balanced power supply is feeding a six-pulse thyristor rectifier bridge. The lowest harmonic component in the AC source line current [assuming a constant DC output current] is
(A) 50 Hz (B) 100 Hz
(C) 150 Hz (D) 250 Hz

12. A full-wave bridge converter (single-phase half-controlled) feeds an inductive load. A common DC bus is connected to two SCR's in the converter. The converter should be equipped with a freewheel diode because
(A) A high AC current will flow through one SCR if a gate pulse to the other SCR is missed.
(B) Free-wheeling action is not provided by the converter for high triggering angles.
(C) Shorting of the AC triggering pulses due to free-wheeling action of the converter shall be compensated.
(D) Inherent free-wheeling is not provided by the converter.

13. A half-wave uncontrolled AC–DC converter is employed as a battery charger to charge a 12 V battery. The converter is connected to a single-phase 120 V, 50 Hz supply mains through a 4:1 stepdown transformer. If the battery is connected to the converter through a current-limiting resistance of 5.7 , the charging current is
(A) 1.4 A (B) 0.5 A
(C) 3.4 A (D) 2.22 A

14. A two-pulse midpoint converter feeds a constant ripple free current to a load and the current is I_d at all firing angles. The parameters are such that it has an overlap angle of 45° at 0° triggering angle. The overlap angle, at $\alpha = 45°$ would be
(A) 65.35° (B) 0°
(C) 20.35° (D) 12.97°

15. Which among the following are true regarding the effects of free-wheeling diode in a single pulse converter feeding RL load
(A) Improved power factor due to reduced reactive power requirements
(B) Reduced ripple content in the load current
(C) Continuous conduction at smaller value of α
(D) All of the above

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. In a single-phase half-wave circuit with RL load, and a free-wheeling diode across the load, extinction angle β is more than π. For a firing angle α, SCR and free-wheeling diode would conduct, for
(A) $\beta - \alpha, \pi - \alpha$ (B) $\beta - \alpha, \beta$
(C) $\pi - \alpha, \beta - \pi$ (D) $\beta - \pi, \pi - \alpha$

2. A single-phase half-wave controlled rectifier has 300 sin (314 t) V as the input voltage and R as the load. For a firing angle of 60° for the SCR, the average output voltage is
(A) $\frac{900}{4\pi}$ V (B) $\frac{600}{\pi}$ V
(C) $\frac{200}{\pi}$ V (D) $\frac{300}{\pi}$ V

3. In a single-phase one-pulse circuit with RL load, and a freewheeling diode across the load, extinction angle is less than π. For a firing angle α, the SCR and free-wheeling diode would conduct for
(A) $\pi - \alpha, \alpha$ (B) $\beta - \alpha, \alpha$
(C) $\beta - \alpha, \pi - \alpha$ (D) $\beta - \alpha, 0°$

4. A single-phase two pulse bridge converter has an average output voltage and power output of 400 V and 6 kW, respectively. The SCRs used in the two-pulse bridge converter are now re-employed to form a single-phase two-pulse mid-point converter. This new controlled converter would give an average output voltage and power output of
(A) 400 V, 6 kW (B) 200 V, 6 kW
(C) 200 V, 1.5 kW (D) 200 V, 3 kW

5. In a single-phase full converter, for discontinuous load current and extinction angle $\beta > \pi$, each SCR conducts for
(A) $\pi + \alpha$ (B) π
(C) $\beta - \alpha$ (D) $\beta + \alpha$

6. Commutation overlap in the phase-controlled AC to DC converter is due to
(A) Load inductance
(B) Harmonic content of load current
(C) Switching operation in the converter
(D) Source inductance

7. A single-phase fully controlled line commutated AC to DC converter operates as an inverter, when
(A) $0 \le \alpha \le 90°$
(B) $90° \le \alpha \le 180°$
(C) It supplies to a back emf load
(D) $90° \le \alpha \le 180°$ and there is a suitable DC source in the load circuit

8. A single-phase full-bridge converter feeds an inductive load. R_L = 15.53 Ω and it has a large inductance providing constant and ripple free DC current. Input to

converter is from a 230 V, 50 Hz single-phase source. If $\alpha = 60°$, the average value of output current is
(A) 10 A
(B) 8.165 A
(C) 5.774 A
(D) 3.33 A

9. The overlap angle of a phase-controlled converter would increase on increasing the
(1) Supply voltage
(2) Supply frequency
(3) Load current
(4) Source inductance of these statements
(A) 1, 2 and 3 are correct
(B) 2, 3 and 4 are correct
(C) 1, 2 and 4 are correct
(D) 1, 3 and 4 are correct

10. A converter which can operate both in 3-pulse and 6-pulse modes is a
(A) 1-phase full-converter
(B) 3-phase half-wave converter
(C) 3-phase semi-converter
(D) 3-phase full-converter

11. The quality of output AC voltage of a cyclo converter is improved with a/an
(A) Increase in output voltage at reduced frequency
(B) Increase in output voltage at increased frequency
(C) Decrease in output voltage at reduced frequency
(D) Decrease in output voltage at increased frequency

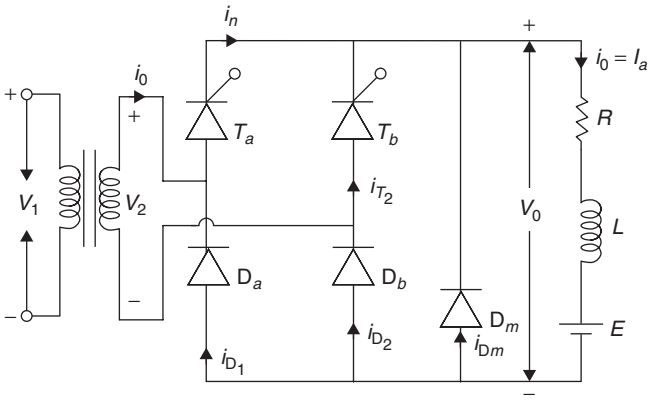
12. A separately-excited DC motor is required to be controlled from a 3-phase source for two quadrant operation. The most preferred converter would be
(A) Half-controlled converter
(B) Fully controlled converter
(C) Dual converter
(D) Half-wave converter

13. The ratio of form factor of a full-wave rectifier to that of a half-wave rectifier would be

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) $\frac{1}{\sqrt{3}}$

Common Data Questions 14 and 15
If the semi-converter given below is operated from a 230 V, 50 Hz supply. The load current with an average value of I_a is continuous. If the thyristors are operated at a delay angle of 30° and transformers had a turns ratio of 1.

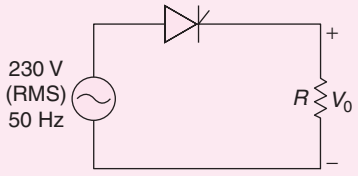


14. The harmonic factor of input current is
(A) 0.32 (B) 0.23
(C) 0.427 (D) 0.11

15. Input power factor would be
(A) 0.86
(B) 0.92
(C) 0.77
(D) 0.65

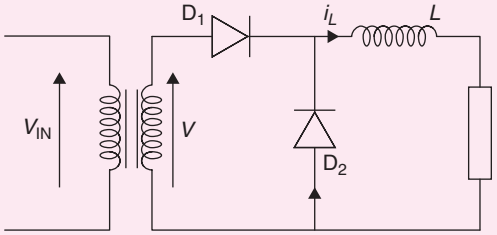
Previous Years' Questions

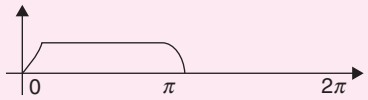
1. Consider a phase-controlled converter shown in figure shown below. The thyristor is fired at an angle α in every positive half cycle of the input voltage. If the peak value of the instantaneous output voltage equals 230 V, the firing angle α is close to **[2005]**

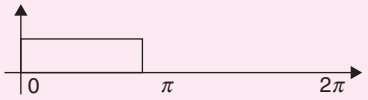


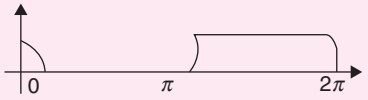
(A) 45° (B) 135° (C) 90° (D) 83.6°

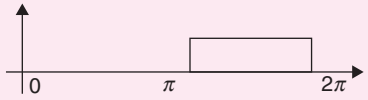
2. A single-phase half-wave uncontrolled converter circuit is shown in figure. A 2-winding transformer is used at the input for isolation. Assuming the load current to be constant and $v = V_m \sin \omega t$, the current waveform through diode D_2 will be **[2006]**



(A)


(B)


(C)


(D)


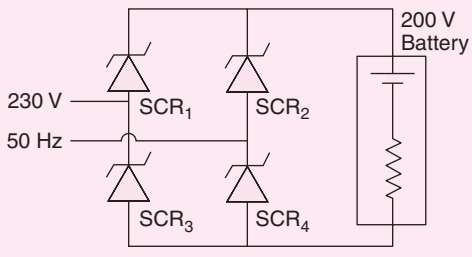
3. A 3-phase fully controlled bridge converter with free-wheeling diode is fed from 400 V, 50 Hz AC source and is operating at a firing angle of 60°. The load current is assumed constant at 10 A due to high load inductance. The input displacement factor (IDF) and the input power factor (PF) of the converter will be **[2006]**

(A) IDF = 0.867 IPF = 0.828
(B) IDF = 0.867 IPF = 0.552
(C) IDF = 0.5 IPF = 0.478
(D) IDF = 0.5 IPF = 0.318

4. A solar cell of 350 V is feeding power to an AC supply of 440 V, 50 Hz through a 3-phase fully controlled bridge converter. A large inductance is connected in the DC circuit to maintain the DC current at 20 A. If the solar cell resistance is 0.5 Ω, then each thyristor will be reverse biased for a period of **[2006]**

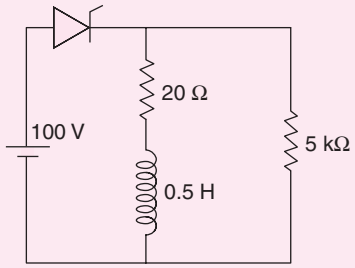
(A) 125° (B) 120° (C) 60° (D) 55°

5. A single-phase bridge converter is used to charge a battery of 200 V having an internal resistance of 0.2 Ω as shown in figure. The SCRs are triggered by a constant DC signal. If SCR 2 gets open circuited, then what will be the average charging current? **[2006]**



(A) 23.8 A (B) 15 A
(C) 11.9 A (D) 3.54 A

6. An SCR having a turn ON time of 5 μsec, latching current of 50 mA and holding current of 40 mA is triggered by a short duration pulse and is used in the circuit shown in figure. The minimum pulse width required to turn the SCR ON will be **[2006]**



(A) 251 μ sec (B) 150 μ sec
(C) 100 μ sec (D) 5 μ sec

7. A single-phase fully controlled thyristor bridge AC–DC converter is operating at a firing angle of 25° and an overlap angle 10° with constant DC output current of 20 A. The fundamental power factor (displacement factor) at input AC mains is **[2007]**

(A) 0.78 (B) 0.827
(C) 0.866 (D) 0.9

8. A three-phase, fully controlled thyristor bridge converter is used as line commutated inverter to feed 50 kW power at 420 V DC to a three-phase, 415 V (line), 50 Hz AC mains. Consider DC link current to be constant. The RMS current of the thyristor is **[2007]**

(A) 119.05 (B) 79.37 A
(C) 68.73 A (D) 39.68 A

9. A single-phase full-wave half-controlled bridge converter feeds an inductive load. The two SCRs in the converter are connected to a common DC bus. The converter has to have a freewheeling diode **[2007]**

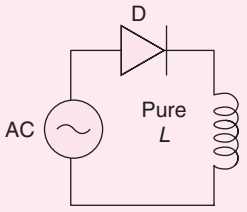
(A) because the converter inherently does not provide for free-wheeling
(B) because the converter does not provide for free-wheeling for high values of triggering angles
(C) or else the free-wheeling action of the converter will cause shorting of the AC supply
(D) or else if a gate pulse to one of the SCRs is missed, it will subsequently cause a high load current in the other SCR

10. A single-phase, 230 V, 50 Hz AC mains fed step down transformer (4:1) is supplying power to a half-wave uncontrolled AC–DC converter used for charging a battery (12 V DC) with the series current

limiting resistor being 19.04 Ω. The charging current is **[2007]**

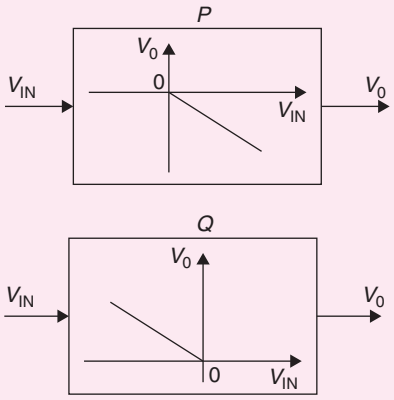
(A) 2.43 A (B) 1.65 A
(C) 1.22 A (D) 1.0 A

11. In the circuit of adjacent figure the diode connects the AC source to a pure inductance L.
The diode conducts for **[2007]**



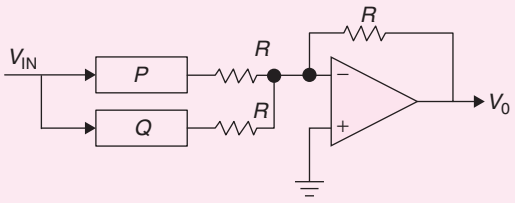
(A) 90° (B) 180° (C) 270° (D) 360°

12. The block diagrams of two types of half-wave rectifiers are shown in the figure. The transfer characteristics of the rectifiers are also shown within the block. **[2008]**

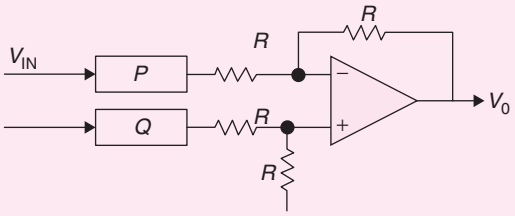


It is desired to make full-wave rectifier using above two half-wave rectifiers. The resultant circuit will be

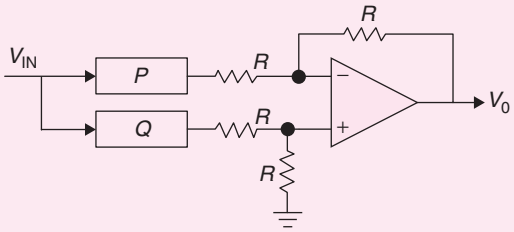
(A)



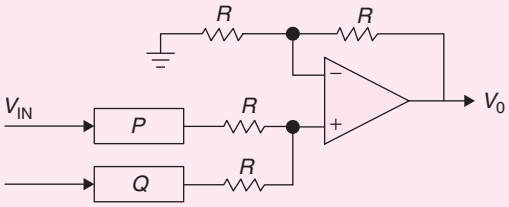
(B)



(C)



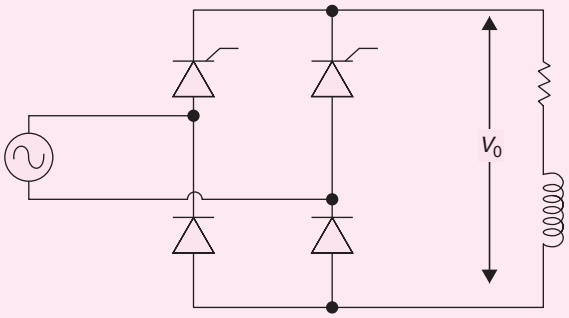
(D)



13. A single-phase fully controlled bridge converter supplies a load drawing constant and ripple free load current. If the triggering angle is 30°, the input power factor will be **[2008]**

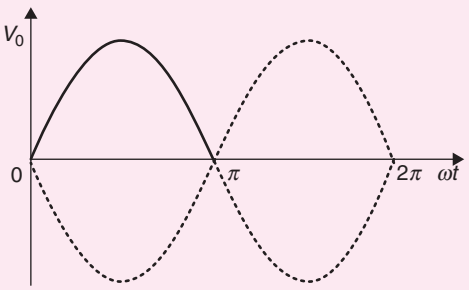
(A) 0.65 (B) 0.78 (C) 0.85 (D) 0.866

14. A single-phase half-controlled converter shown in the figure is feeding power to highly inductive load. The converter is operating at a firing angle of 60°. **[2008]**

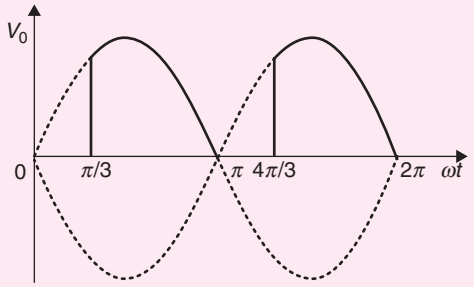


If the firing pulses are suddenly removed, the steady-state voltage (V_0) waveform of the converter will become

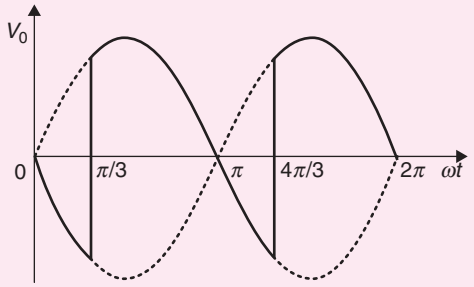
(A)



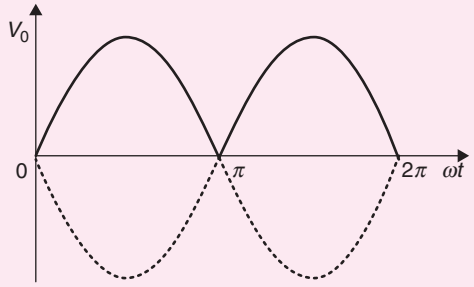
(B)



(C)



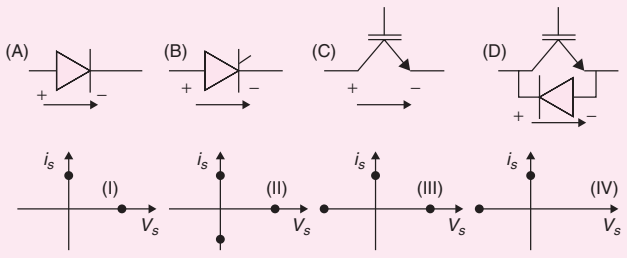
(D)



15. A three-phase fully controlled bridge converter is feeding a load drawing a constant and ripple free load current of 10 A at a firing angle of 30°. The approximate total harmonic distortion (% THD) and the RMS value of fundamental component of the input current will, respectively, be **[2008]**

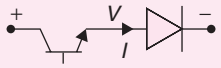
(A) 31% and 6.8 A (B) 31% and 7.8 A
(C) 66% and 6.8 A (D) 66% and 7.8 A

16. Match the switch arrangements on the top row to the steady-state V–I characteristics on the lower row. The steady-state operating points are shown by large black dots. **[2009]**

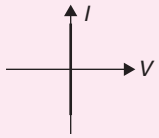


(A) A – I B – II C – III D – IV
(B) A – II B – IV C – I D – III
(C) A – IV B – III C – I D – II
(D) A – IV B – III C – II D – I

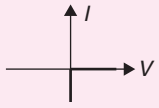
17. Figure shows a composite switch consisting of a power transistor (BJT) in series with a diode. Assuming that the transistor switch and the diode are ideal, the I–V characteristic of the composite switch is **[2010]**



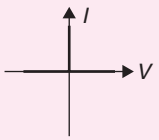
(A)



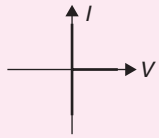
(B)



(C)



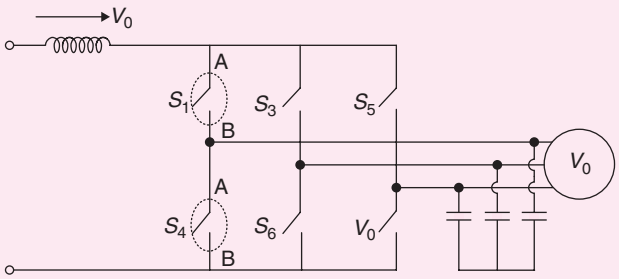
(D)



18. Circuit turn-off time of an SCR is defined as the time **[2011]**

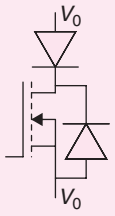
(A) taken by the SCR to turn off
(B) required for the SCR current to become zero
(C) for which the SCR is reverse biased by the commutation circuit.
(D) for which the SCR is reverse biased to reduce its current below the holding current

19. A three-phase current source inverter used for the speed control of an induction motor is realized using MOSFET switches as shown below. Switches S_1 to S_6 are identical switches. **[2011]**

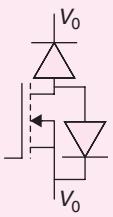


The proper configuration for realizing switches S_1 to S_6 is

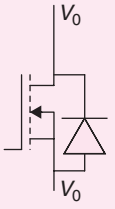
(A)



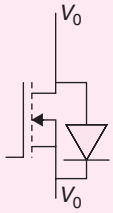
(B)



(C)

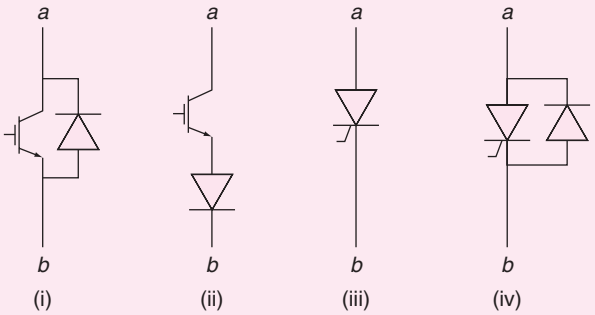


(D)



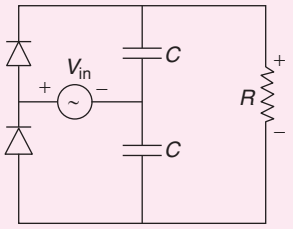
20. The typical ratio of latching current to holding current in a 20 A thyristor is **[2012]**
(A) 5.0 (B) 2.0 (C) 1.0 (D) 0.5

21. Figure shows four electronic switches (i), (ii), (iii) and (iv). Which of the switches can block voltages of either polarity (applied between terminals 'a' and 'b') when the active device is in OFF state? **[2014]**



(A) (i), (ii) and (iii) (B) (ii), (iii) and (iv)
(C) (ii) and (iii) (D) (i) and (iv)

22. In the following circuit, the input voltage V_{in} is 100 sin(100 πt). For 100 πRC = 50, the average voltage across R(in Volts) under steady-state is nearest to **[2015]**



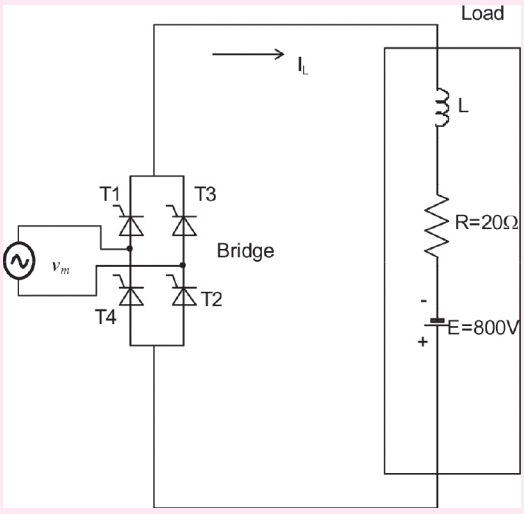
(A) 100 (B) 31.8
(C) 200 (D) 63.6

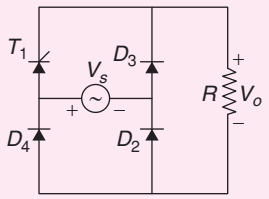
23. In the given rectifier, the delay angle of the thyristor T_1 measured from the positive going zero crossing of V_s is 30°. If the input voltage V_s is 100 sin(100πt) V, the average voltage across R(in Volt) under steady-state is _____. **[2015]**

24. A single-phase full-bridge voltage source inverter (VSI) is fed from a 300 V battery. A pulse of 120° duration is used to trigger the appropriate devices in each half-cycle. The rms value of the fundamental component of the output voltage, in volts, is **[2016]**
(A) 234 (B) 245
(C) 300 (D) 331

25. A three-phase diode bridge rectifier is feeding a constant DC current of 100A to a highly inductive load. If three - phase, 415V, 50Hz AC source is supplying to this bridge rectifier then the rms value of the current in each diode, in ampere, is _____. **[2016]**

26. A full-bridge converter supplying an RLE load is shown in figure. The firing angle of the bridge converter is 120°. The supply voltage $v_m(t)$ = 200πSin(100πt) V, R = 20Ω, E = 800V. The inductor L is large enough to make the output current I_L a smooth dc current. Switches are lossless. The real power feedback to the source, in kW, is _____. **[2016]**





Answer Keys

Exercises

Practice Problems 1

1. D	**2.** B	**3.** C	**4.** D	**5.** B	**6.** A	**7.** D	**8.** B	**9.** D	**10.** B
11. C	**12.** C	**13.** A	**14.** C	**15.** D					

Practice Problems 2

1. C	**2.** A	**3.** D	**4.** D	**5.** C	**6.** A	**7.** B	**8.** D	**9.** B	**10.** C
11. C	**12.** B	**13.** A	**14.** A	**15.** B					

Previous Years' Questions

1. B	**2.** D	**3.** C	**4.** D	**5.** C	**6.** B	**7.** A	**8.** C	**9.** C	**10.** D
11. D	**12.** B	**13.** B	**14.** B	**15.** B	**16.** C	**17.** C	**18.** C	**19.** A	**20.** B
21. C	**22.** C	**23.** 61 to 62		**24.** A	**25.** 57.735	**26.** 6			

Chapter 3

DC–DC Converters (or) Choppers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Principle of operation and control
- Time ratio control
- Current limit control
- Step up chopper
- Step down chopper
- Types of chopper circuits
- Second quadrant (type-B) chopper
- Voltage commutated chopper
- Current commutated chopper
- Peak commutating current
- Load commutated chopper
- Multiphase chopper

Principle of Operation and Control

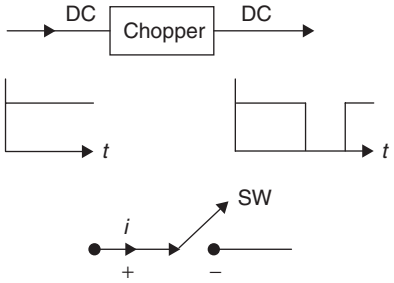
In industries, several applications need DC supply, it will be better if it is a variable DC voltage source.

- This conversion of fixed DC voltage to an adjustable DC output voltage by using semiconductor devices are called chopping examples of such DC systems are trolley buses, battery operated vehicles, subway cars, battery charging, etc.

Principle of Chopper

Chopper is a static device that converts fixed DC input voltage to a variable DC output voltage directly.

- It is a high-speed on/off semiconductor switch (connects source to the load and disconnects the load from the source of a fast speed).
- The device used for a chopper circuit can be force commutated thyristor, power BJT, Power MOSFET, GTO or IGBT.



- It is a DC equivalent of an AC transformer.
- It can be used to step up and step down the fixed DC input voltage.
- Step–down DC chopper is most commonly used.

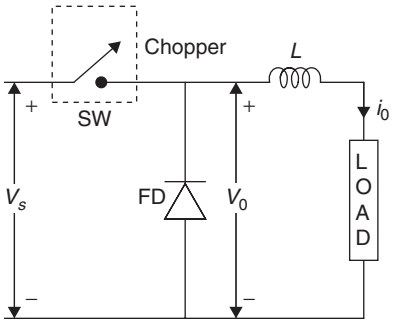
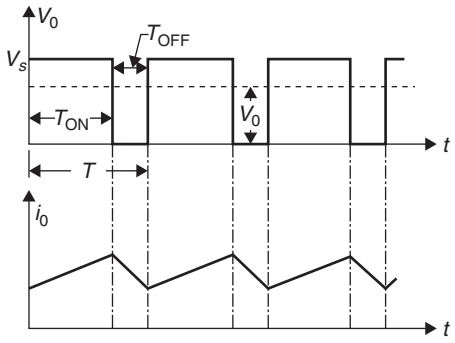


Figure 1 Basic chopper circuit



Output voltage and current waveforms

When chopper is ON, T_{ON} load voltage is equal to source voltage vs. during off period T_{OFF} load current flows through the free-wheeling diode FD. Load terminals are short-circuited by FD and V_0 is zero.

The load power can be controlled by varying the ON and OFF times.

The ratio of T_{ON} to $(T_{ON} + T_{OFF})$ is known as duty cycle.

There are three ways of obtaining the variable ON-time to OFF time for voltage control.

1. T_{ON} constant and T varying (Frequency modulation)
2. T constant and T_{ON} varying (Pulse width modulation)
3. T_{ON} and T are both varying and for all three

Average load voltage

$$V_0 = \frac{T_{ON}}{T_{ON} + T_{OFF}} \times V_s$$

$$= \left(\frac{T_{ON}}{T}\right) V_s$$

where T_{ON} = ON – time

T_{OFF} = OFF – time

$T = T_{ON} + T_{OFF}$ = chopping period

and $\alpha = \frac{T_{ON}}{T}$ = duty cycle

then $V_0 = \alpha V$s

$V_0 = T_{ON} f \cdot V_s$

$f = 1/T$ = chopping frequency

- Therefore, load voltage can be controlled by varying duty cycle α.

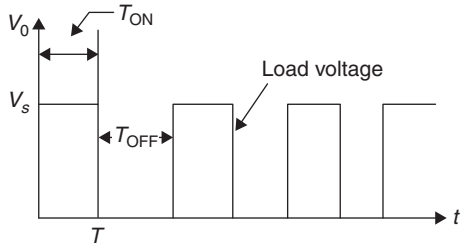
Control Strategies

Average output voltage V_0 can be varied through α by opening and closing the semiconductor switch periodically. The various control strategies for varying duty cyclo α are as follows:

1. Time ratio control
2. Current limit control

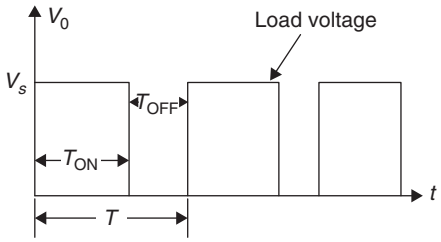
Time Ratio Control

1. **Constant frequency system:**



In this method, T_{ON} is varied but chopping frequency $f(T)$ is kept constant Also called as pulse-width modulation scheme.

2. **Variable frequency system**



In this method, chopping frequency f is varied and either

- ON-time T_{ON} is kept constant
- OFF-time T_{OFF} is kept constant

This method is also called as frequency modulation scheme.

- Frequency modulation scheme has some disadvantages compared to pulse-width modulation scheme.
  Chopping frequency has to be varied over a wide range for the control of output voltage and so the design of filter is quite difficult.
- For the control of α, frequency variation is wide. Due to which interference with signalling and telephone lines in frequency modulation.
- The large time T_{OFF} in frequency modulation make the load current discontinuous which is undesirable. Therefore pulse width modulation scheme is better than variable frequency scheme.
- The only limitation in PWM is T_{ON} cannot be reduced to near zero for most of the commutation circuits used in choppers. So, low range of a control is not possible in PWM.

Current Limit Control

In this control strategy, the ON and OFF of chopper circuit is guided by the previous set value of load current. These two set values are maximum load current I_{max} and minimum Road current I_{min}. When load current reaches the upper limit I_{max}, chopper is switched OFF. Now load current free wheels and begins to decay exponentially. When it falls to lower limit I_{min} chopper is switched on and load current begins to rise. So the load current fluctuates between I_{max} and I_{min}. Therefore it cannot be discontinuous. current limit control involves feedback loop, the trigger circuitry for the chopper is therefore more complex.

Chopper system offers smooth control, high efficiency, fast response and regeneration.

There are two types of choppers

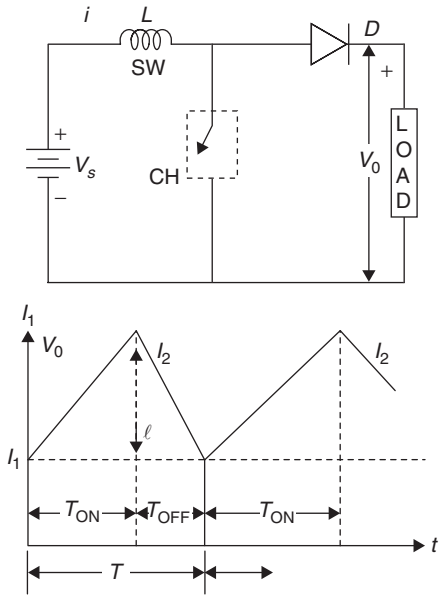
1. Step-down chopper
2. Step-up chopper

Step-up Chopper

Average output voltage V_0 is greater than input voltage.

$V_0 > V_s$ is called step-up chopper and $V_0 < V_s$ is called step-down chopper.

Working Principle



When the switch is ON, the current circulates in the loop and inductor stores energy.

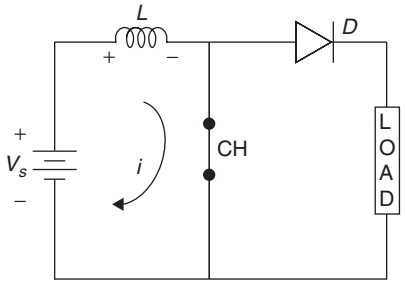
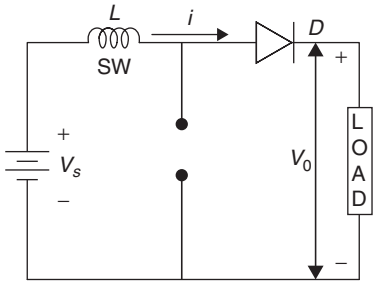


Figure 2 L' stores energy

When the switch is opened the total input voltage is now added up with $L\frac{di}{dt}$.



$L\ di/dt$ is added to V_s

$$\therefore \quad V_{IN} = V_s + L\frac{di}{dt}$$

Therefore, average output voltage V_0 is greater than the input voltage V_s, assuming the output current is linear energy input to conductor from the source, during period T_{ON}, W_{IN} = Avg. volt across Lx Avg. current the inductor x time

$$W_{IN} = V_s\left[\frac{I_1 + I_2}{2}\right]T_{ON}$$

Energy released by the inductor to the load during period T_{OFF}.

$$W_{OFF} = \text{Average voltage across} \times \text{Average Current} \times \text{Time}.$$

$$= (V_0 - V_s)\left(\frac{I_1 + I_2}{2}\right) \times T_{OFF}$$

For a lossless system

$$V_s\left[\frac{I_1 + I_2}{2}\right]T_{ON} = (V_0 - V_S)\left[\frac{I_1 + I_2}{2}\right]T_{OFF}$$

$$V_0 = V_S\left[\frac{T}{T - T_{ON}}\right]$$

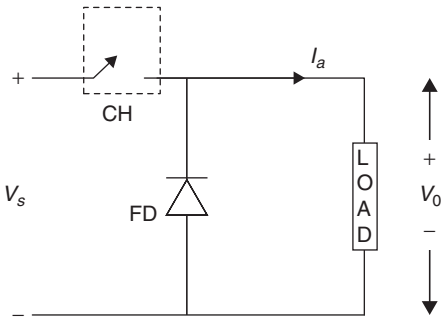
$$T = T_{ON} + T_{OFF}$$

$$V_0 = \frac{V_s}{1 - \infty}$$

where $\alpha = T_{ON}/T$

From the above equation the average voltage across the load can be stepped up by varying the duty cycle.

Step-down Chopper



If $V_0 < V_s$, it is called step-down chopper

$$V_0 = V_s \times \left(\frac{T_{ON}}{T'}\right) = V_S \times f \times T_{ON}$$

$$V_0 = \alpha V_s$$

α-duty cycle of chopper. The average output voltage is less than the input voltage.

RMS output voltage, $V_{RMS} = \left[\frac{1}{T}\int_0^{T_{ON}} V_s^2\, dt\right]^{1/2}$

$$V_{RMS} = \sqrt{\alpha}\, V_s$$

Effective Source Resistance (R_{eff})

$$R_{eff} = \frac{(V_s)_{avg}}{(I_s)_{avg}} = \frac{V_s}{\alpha \cdot \frac{V_s}{R}} = \frac{R}{(\alpha)}$$

It can be used in static rotor resistance control of an induction motor.

Solved Examples

Example 1: If V_{OUT} and V_{IN} are the magnitudes of output voltage and DC input voltages, respectively, for a step-down chopper. If the chopper is operated in a steady state with a constant duty ratio α and continuous conduction mode, then the ratio $\frac{V_{OUT}}{V_{IN}}$ is given by

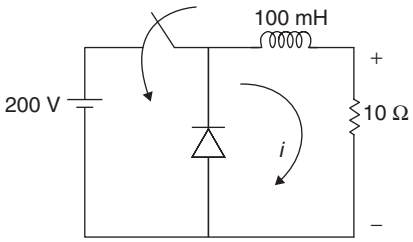
(A) $\left(\frac{\alpha}{1-\alpha}\right)$ (B) $\left(\frac{1}{1-\alpha}\right)$ (C) α (D) $\left(\frac{\alpha-1}{\alpha}\right)$

Solution: (C)
For a step-down chopper

$$V_{OUT}=\alpha V_{IN} \quad \frac{V_{OUT}}{V_{IN}}=\alpha$$

Example 2: The peak-to-peak ripple in load current in a step-down chopper (as shown below) switched at 2 kHz with a duty ratio of 0.75 is close to

(A) 0.125 A (B) 0.25 A (C) 0.5 A (D) 1 A



Solution: (B)
Maximum ripple current

$$\Delta I_{max}=\frac{V_s}{4fL}=\frac{200}{4\times2000\times0.1}$$
$$=0.25\text{ A}$$

Example 3: If the desired output voltage of a step DC–DC chopper with an input DC voltage of 230 V is 280 V. The conducting time for the thyristor chopper operating at a T_{OFF} of 50 μs would be

(A) 33.33 μs (B) 66.66 μs
(C) 50 μs (D) 100 μs

Solution: (A)

$$V_0=\frac{V_s}{1-\alpha}$$
$$380=230\left(\frac{1}{1-\alpha}\right)$$
$$(1-\alpha)=0.6$$
$$\alpha=0.4=\frac{T_{ON}}{T_{ON}+T_{OFF}}$$
$$1+\frac{T_{OFF}}{T_{ON}}=\frac{1}{0.4}=2.5$$
$$T_{ON}=\frac{50}{(2.5-1)}=33.33\,\mu s$$

Example 4: The duty cycle of a boost regulator which has an input voltage of 4 V and an average output voltage of 8 V is

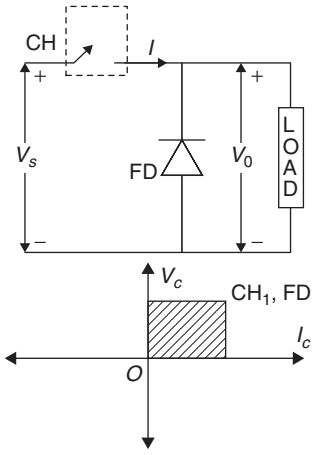
(A) 1/2 (B) 3/2 (C) 2/3 (D) 5/2

Solution: (A)
For a boost regulator

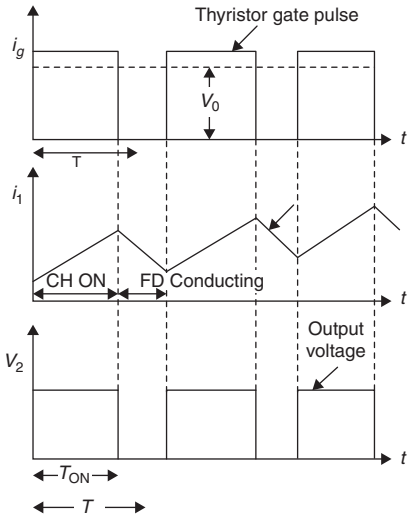
$$V_0=\frac{V_s}{1-\alpha}\Rightarrow 1-\alpha=\frac{4}{8}=\frac{1}{2}$$
$$\alpha=(1-1/2)=(1/2)$$

Types of Chopper Circuits

First-quadrant (Type-A) Chopper

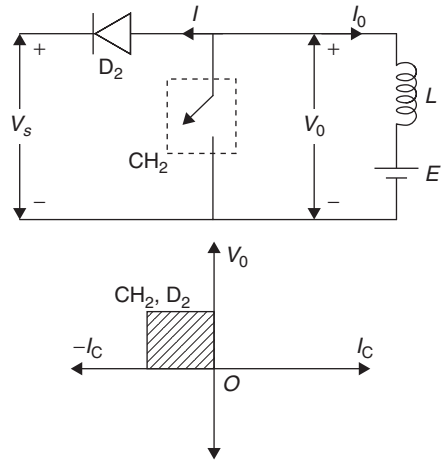


- When chopper CH_1 is ON, $V_0 = V_s$ and current i_0 flows in the direction as shown.
- When CH_1 is OFF $V_0 = 0$ but i_0 in the load continues flowing in the same direction through FD (free-wheeling diode).
- Hence average values of both load voltages and currents, i.e. V_0 and I_0 are always positive as shown by the shaded area in the I quadrant.
- The average output voltage V_0 is always less than input DC voltage V_s and hence it is also known as step-down chopper.
- The power flow in Type-A chopper is always from source to load.

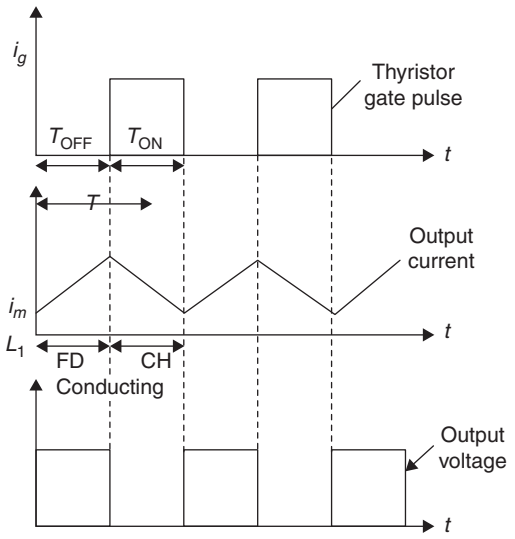


It is a step-down chopper in which the average value of output voltage and current is always positive, A free-wheeling diode is connected across the load to prevent the output current being negative. Its region of operation is in the first quadrant only, so it is also called first–quadrant chopper.

Second-quadrant (Type-B) Chopper

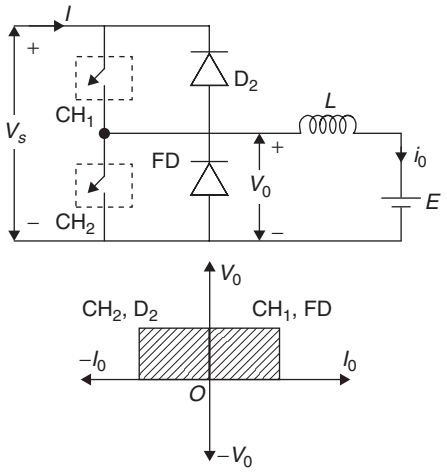


- When CH_2 is ON, $V_0 = 0$ but load voltage E drives current through L and CH_2. Inductance stores energy during T_{ON} of CH_2.
- When CH_2 is OFF $V_0 = \left(E + L\frac{di}{dt}\right)$ exceeds source voltage V_s: then D_2 becomes forward biased and begins conduction thus allowing power to flow to the source.
- CH_2 may be ON or OFF, current I_0 flows out of the load, current i_0 is therefore treated as negative. Since V_0 is always positive and I_0 is negative, power flow is always from load to source.
- As load voltage $V_0 = \left(E + L\frac{di}{dt}\right)$ is more than source voltage V_s, type-B chopper is also called step-up chopper.

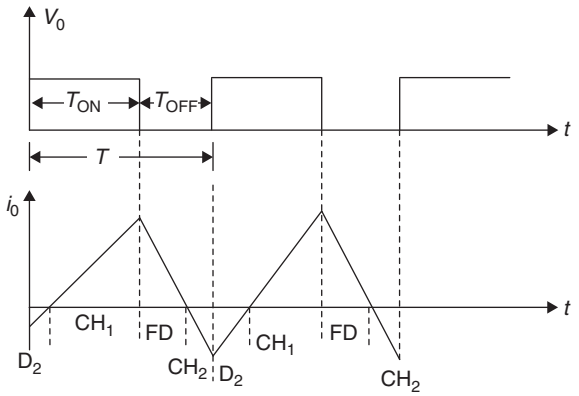


In this type the average output voltage is positive but the average output current is negative. Therefore the class B chopper operates in the second quadrant. Power flows from the load to the source. It is a step-up chopper. It is widely used for regenerative breaking of DC Motor.

Two-quadrant Type-A (Type-C) Chopper



- This chopper is obtained by connecting type-A and type-B choppers in parallel.
- V_0 is always positive due to the presence of FD across the load.
- When CH_2 is ON (or) FD conducts $V_0 = 0$, and in case CH_1 is ON or diode D_2 conducts and $V_0 = V_s$.
- Load current i_0 can however reverse its direction. i_0 is positive when CH_1 is ON and FD conducts. I_0 is negative when CH_2 is ON or D_2 conducts.
- CH_1 and FD operate together as type-A chopper in the I quadrant CH_2 and D_2 operate together as type-B in the second quadrant.
- Average load voltage is always positive but average load current may be positive or negative. Hence the power flow may be from source to load (I quadrant operation) or from load to source (II quadrant operation).
- Choppers CH_1 and CH_2 should not be ON at the same time as this would lead to a direct short circuit of supply lines.
- Such choppers are used for motoring and regenerative braking of DC motors.

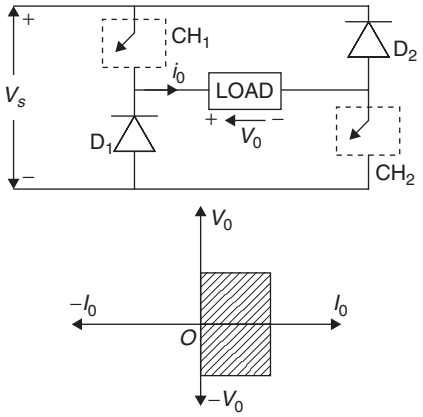


Class C chopper is a combination of class A and class B choppers. For first-quadrant operation, CH_1 is ON or D_1 conducts. For second quadrant, CH_2 is ON or D_1 conducts. When CH_1 is ON, the load current is positive. The output voltage is equal to 'V' and the load receives power from the source.

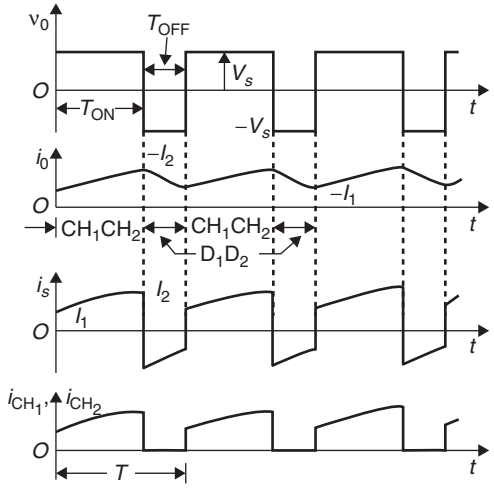
When CH_1 is turned OFF, energy stored in inductance, L forces current to flow through the diode D_2 and the output voltage in zero. Current continuous to flow in positive direction.

$\Rightarrow$ When CH_2 is triggered, the voltage E forces current to flow in opposite direction through L and CH_2. The output voltage is zero turning OFF CH_2, the energy stored in the inductance drives current through diode D_1 and the supply output voltage is V_1 the input current becomes negative and power flows from load to source.

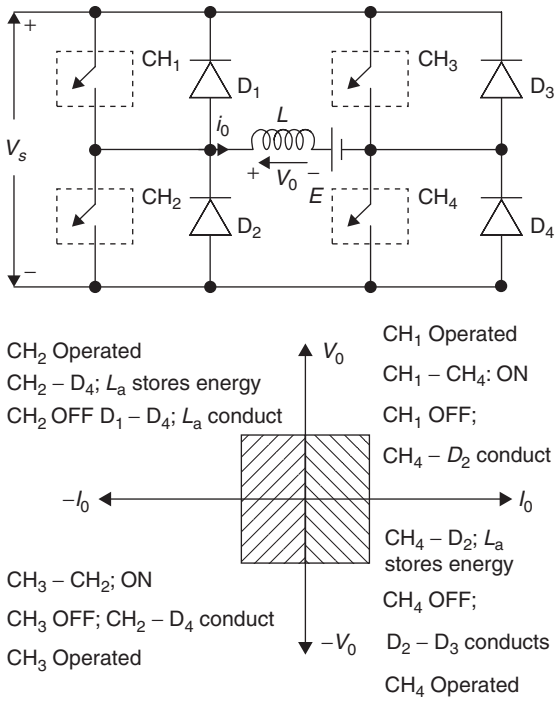
Two-quadrant Type-B (Type-D) Chopper



- Type-D chopper is obtained by connecting Type-A and Type-B chopper in differential mode, i.e., load is connected in between them.
- When both CH_1 and CH_2 are ON $V_0 = V_s$.
- When both choppers are off D_1 and D_2 conducts $V_0 = -V_s$.
- Average output voltage V_0 is positive when their $T_{ON} < T_{OFF}$ and V_0 is negative when their $T_{ON} > T_{OFF}$.
- The direction of load current is always positive because choppers and diodes can conduct current only in the direction of arrows. As V_0 is reversible, power flow is reversible.



Four-quadrant Type-E Chopper



First Quadrant

- CH_4 is kept ON, CH_3 is kept OFF and CH_1 is operated. With CH_1, CH_4 ON $V_0 = V_s$ and i_0 begins to flow. Here both V_0 and i_0 are positive resulting first quadrant operation. When CH_1 is turned OFF positive current freewheels through CH_4 and D_2.

Second Quadrant

- CH_2 is operated and CH_1, CH_3, CH_4 are kept OFF. With CH_2 ON, reverse (negative) current flows through L, CH_2, D and E. Inductance L stores energy during the time CH_2 is ON. When CH_2 is turned OFF, current is fed back to source through diodes D_1, D_4. Here, $E + L\frac{di}{dt}$ is more than the source voltage V_s.
- As V_0 is positive and I_0 is negative, it is second-quadrant operation of chopper.
- Power is fed back from load to source.

Third Quadrant

CH_1 is kept off, CH_2 is kept ON and CH_3 is operated. Polarity of load emf E must be reversed for this quadrant working when CH_3 ON, load gets connected to V_s so that both V_0 and I_o are negative leading to third-quadrant operation.

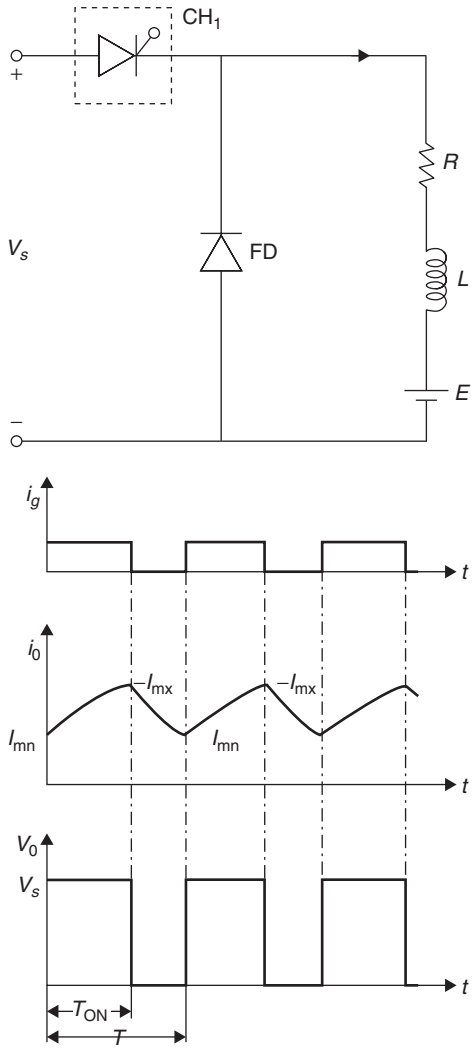
- When CH_3 is turned off, negative current freewheels through CH_2, D_4. In this manner, V_0 and i_0 can be controlled in the third quadrant.
- Here, the chopper operates as step-down chopper.

Fourth Quadrant

CH_4 is operated and other devices are kept OFF. Load emf E must have its polarity reversed, for operation in IV Quadrant. With CH_4 ON positive current flows through CH_4, D_2, L and E.

- L stores energy during the time CH_4 is ON. When CH_4 is turned OFF current is fed back to the source through diodes D_2, D_3. Here, load voltage is negative, but load current is positive leading to the chopper operation in IV quadrant.
- Power is fed back from load to source. Chopper here operates as step-up chopper.

Steady-state Analysis of Type-A Chopper



For type A-chopper with RLE load, for $0 < t\ T_{ON}$

$$V_s = Ri + \frac{Ldi}{dt} + E$$

$$V_0 = V_s$$

$$RI(s) + L[SI(s) - I_{mn}] = \frac{V_s - F}{s}$$

$$I(s) + \frac{V_s - E}{R}\left[\frac{1}{S} - \frac{1}{(S + R/L)}\right] + \frac{I_{mn}}{S + (R/L)}$$

$$I(t) - \frac{\frac{V_s - E}{R}}{I_{mn}\ e^{-R/L \times t}}\left[1 - e^{-(R/L)L}\right] +$$

$$I(t) - \left[\frac{V_s - E}{R}\right]\left[1 - e^{-(R/L)t}\right] + I_{mn}\ e^{-R/Lt}$$

When $t = T_{ON}$, then $i(t) = I_{mx}$

$t = T$; $i(t) = I_{mn}$

Let $\frac{L}{R} = T_a$ (Time constant)

$$I_{max} = \frac{V_s}{R}\left[\frac{1 - e^{-T_{ON}/T_a}}{(1 - e^{-T/T_a}}\right] - \frac{E}{R}$$

$$I_{min} = V_s/R\frac{\left[e^{T_{ON}/T_a} - 1\right]}{\left[e^{T/T_a} - 1\right]} - \frac{E}{R}$$

Steady-state ripple current

$$\Delta I = I_{max} - I_{min}$$

$$\Delta I = \frac{V_s}{R}\left[\frac{\left(1 - e^{T_{ON}/T_a}\right)\left(1 - e^{-(T - T_{ON})/T_a}\right)}{1 - e^{-T/T_a}}\right]$$

$\Rightarrow$ The ripple current given by the equation is independent of load converter emf E with $T_{ON} = \alpha T$ and $(T - T_{ON}) = (1 - \alpha)T$ and duty cyclo $\delta = \frac{T_{ON}}{T}$.

$$\Delta I = \frac{V_s}{R}\left[\frac{\left(1 - e^{\delta x}\right)\left(1 - e^{-(1-\delta)x}\right)}{1 - e^{-x}}\right]$$

For maximum value of ripple current,

$$\delta = 0.5 = 1/2$$

From which we can get

$$\Delta I_{max} = \frac{V_s}{4fL} \text{ at } \alpha = 0.5$$

Fourier Analysis of Output Voltage

$$V_0 = V_0 + \sum_{n=1}^{\infty} V_n$$

where V_n = value of n^{th} harmonic voltage

$$= \frac{2V_s}{n\pi}\sin(n\pi\alpha)\ \sin(n\omega t + \theta n)$$

$$V_0 = \alpha V_s;\ \alpha = T_{ON}/T$$

$$\theta_n = \tan^{-1}\left[\frac{\sin(2\pi n\alpha)}{1 - \cos(2\pi n\alpha)}\right]$$

$$\text{Ripple Factor} = \frac{\text{AC ripple voltage}}{\text{DC voltage}}$$

$$\text{R.F.} = \sqrt{\frac{\alpha V_s^2 - \alpha^2 V_s^2}{\alpha V_s}}$$

$$= \frac{V_s \times \sqrt{\alpha - \alpha^2}}{\alpha V_s}$$

$$\text{R.F.} = \sqrt{(1/\alpha) - 1}$$

RMS value of output voltage

$$= \sqrt{1/T \int_0^{T_{ON}} V_s^2 dt}$$

$$= \sqrt{1/T} V_s^2 \cdot T_{ON}$$

$$(V_0)_{RMS} = \sqrt{\alpha} V_s$$

Example 5: The duty ratio of a chopper, operating at a fixed frequency and feeding an RL load is increases from 0.3 to 0.7. The ripple in the load current

(A) Increases with duty ratio
(B) Does not change
(C) Would decrease, reach minimum at 50% duty ratio and then increase
(D) Would increase, reach maximum at 50% ratio and then decrease

Solution: (D)

Example 6: A transistor chopper operates at 0.5 duty cycle and at a frequency of 2 kHz. The chopper is supplied from a fixed voltage DC source and feeds a fixed RL load and a freewheeling diode. If it is desired to reduce the ripple content of load current, without changing the value of average DC current through the load. The chopper should be operated

(A) With constant duty cycle and increased frequency
(B) By decreasing only duty cycle
(C) By increasing the value of frequency and duty cycle α in equal proportion
(D) By decreasing only chopper frequency

Solution: (A)

Average DC current through the load $I_0 = \frac{V_0}{R}$ and $V_0 = \alpha V_s$

If α changes, average value of I_0 changes which has to be kept constant.

Hence, α has to be constant.

$$\text{Ripple } \Delta I = \frac{V_s}{4fL}$$

Ripple can be reduced by increasing the chopper frequency that is constant average DC load current and reduced ripple content can be obtained by constant α and increasing chopper frequency.

Example 7: A DC motor armature with a back emf E. is fed from a step-down chopper which is connected to supply of V. When the average output DC current of the chopper was observed in a CRO the following were noted in every chopping cycle.

(A) The current increases for the time t_1.
(B) The current falls to zero over t_2.
(C) The current remains zero for time t_3.
(D) The average DC voltage across the freewheeling diode is

(A) $\frac{V_s\ t_r}{(t_r + t_f + t_0)}$ (B) $\frac{V_s\ t_r + E_b\ t_f}{(t_r + t_f + t_0)}$

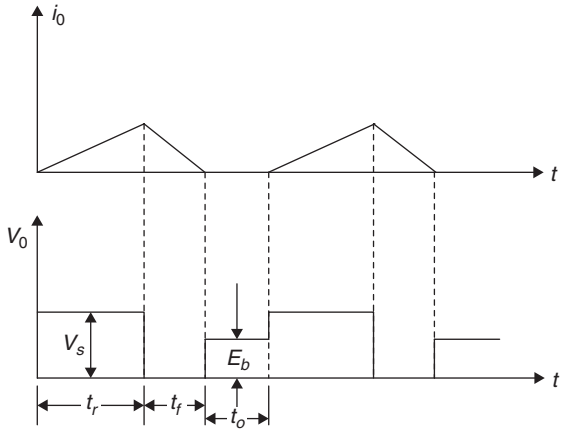
(C) $\frac{V_s\ t_r + E_b\ t_0}{(t_r + t_f + t_0)}$ (D) $\frac{V_s\ t_r + E_b\ (t_f + t_0)}{(t_r + t_f + t_0)}$

Solution: (C)

For step-down chopper for R_L load

$$V_t = V_0 = \alpha V_s = \frac{T_{ON}}{T} V_s$$

For RLE load, the output waveform is



$$T = t_r + t_f + t_0$$

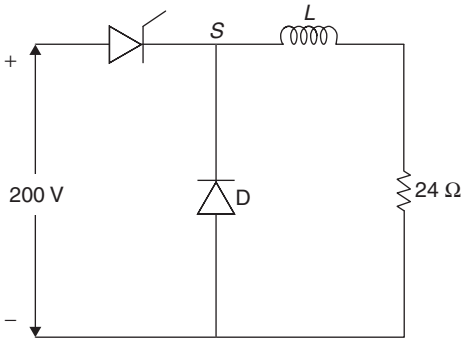
The terminal voltage

Exist only for the periods t_r and t_0 remaining time zero

$\therefore$ The average voltage

$$= \frac{V_s t_r + E_b\ t_0}{t}$$

Example 8: A chopper connected to a supply of 200 V DC input feeding an RL load is shown below.



The duty cycle of the main switch is 0.6 and the load inductance is sufficient enough to maintain a constant, ripple-free load current. The average steady-state diode current would be

(A) 1.6 A (B) 2 A (C) 2.6 A (D) 8 A

Solution: (B)

During chopper operation switch (s) conducts during T_{ON} and diode D conducts during T_{OFF}. Average diode current

$$I_D = I_D \times \frac{T_{OFF}}{T} = I_0 \times \frac{(1-\alpha)T}{T}$$

$$I_D = I_D(1-\alpha)$$

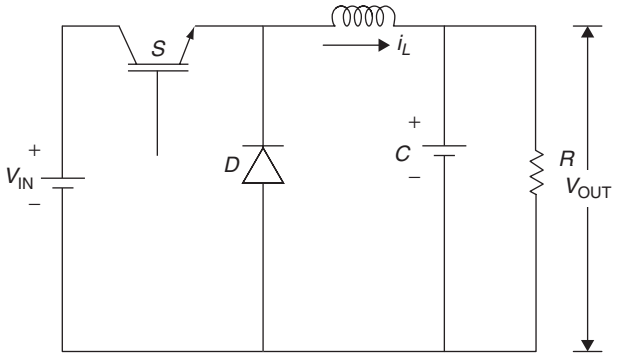
$$I_0 = \frac{V_0}{R}$$

$$V_0 = \alpha V_S = 0.6 \times 200 = 120 \text{ V}$$

$$I_0 = \frac{120}{24} = 5 \text{ A}$$

$$I_D = 5 \times (1 - 0.6) = 2 \text{ A}$$

Example 9: The chopper shown below is operated with a constant input source voltage V_{IN} and ripple free output voltage V_{OUT}. The value of duty ratio D at the boundary of continuous and discontinuous conduction of the inductor current I_L would be (assume that the switch S is operated with a switching time period T)



(A) D = $2L/RT$

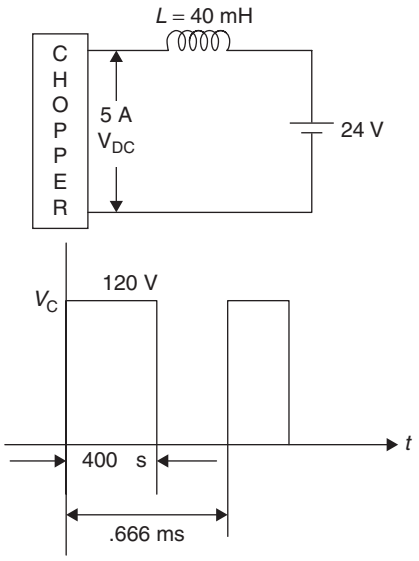
(B) D = RT/L

(C) D = $1 - \frac{V_s}{V_o}$

(D) D = $1 - 2\,L/RT$

Solution: (D)

Example 10: A battery charger employs a DC–DC which is operating at a duty ratio of 0.6 and supplying a charging current of 7.5 A is shown in Fig I. The chopper output voltage is as seen in Fig II. The peak-to-peak ripple current in the charging current is



(A) 0.48 A (B) 1.2 A (C) 1 A (D) 2.4 A

Solution: (C)

$$T_{ON} = 400\ \mu s \qquad T = 0.666 \text{ ms}$$

Charging current peak to peak

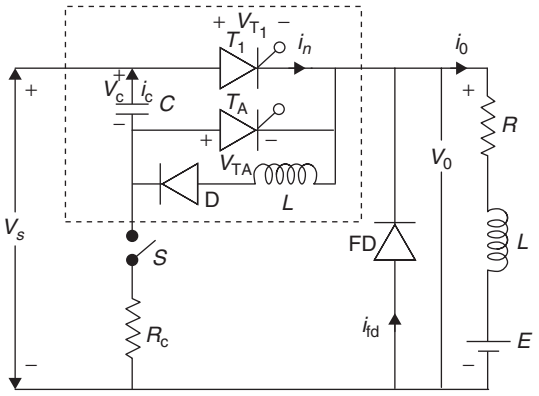
$$I_P = \left[\frac{V-E}{L}\right] T_{ON} = \left[\frac{120-24}{40\times10^{-3}}\right] \times 400 \times 10^{-6}$$

$$= 0.96 \text{ A} \cong 1 \text{ A}$$

Thyristor Chopper Circuits

The process of turning off a conducting thyristor is called commutation. In a DC chopper, it is essential to provide a separate commutation circuitry to commutate the main power SCR.

Voltage Commutated Chopper



Operating Modes

Chopper operation is divided into 4 modes.

Mode 1: For $0 < t < t_1$

T_1 and D_1 are ON T_2 and FD are kept OFF. The output voltage $V_0 = V_s$ and simultaneously the capacitor gets charged to a voltage $V_c = V_s$.

The direction of output current remains the same.

Mode 2: For $t_1 \le t \le t_2$

Only T_1 is kept ON and all other devices are turned OFF.

- The source voltage V_s appears across the output and i_0 flows through T_1 and then to the load.

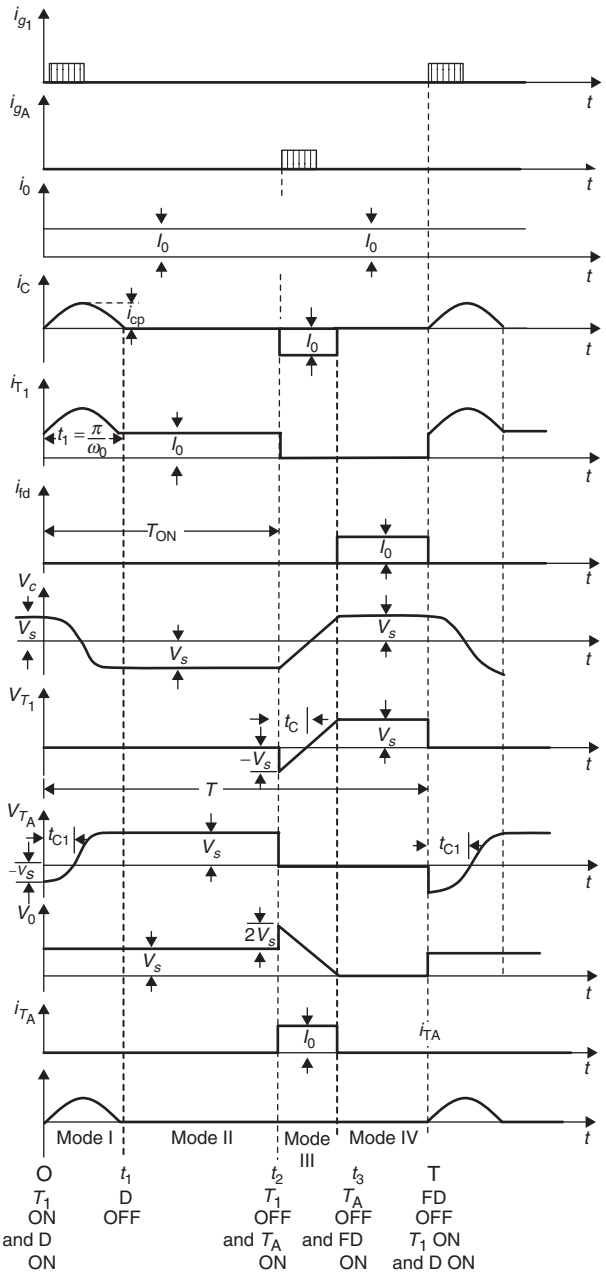
Mode 3: For $t_2 \le t \le t_3$

- Now T_A is operated and all other devices are turned off.
- Now V_s is in series with V_c; hence the total input voltage is $(V_s + V_c)$, which appears across load. $V_0 = V_s + V_c$.
- Load current i_0 flows through T_2 and then flows through load.

Mode 4: For $t_3 \le t \le T$

The capacitor is now fully discharged and all other devices except FD are switched OFF.

- Though there is no current flow from source to load since $V_0 = i_0$ in the load continues flowing in the same direction through free-wheeling diode.



For a constant load current I_0 commutating capacitor $C = \dfrac{t_c I_0}{V_s}$.

T_c = Commutation circuit turn off time t_c must be greater than t_q.

T_q = S.C.R. turn off time

$$T_c = t_q + \Delta t$$

$$C = \frac{(t_q + \Delta t) I_0}{V_s}$$

Commutating inductor $L \ge \left(\dfrac{V_s}{I_0}\right)^2 C$

Turn-off time $t_{c_1} = \dfrac{t_1}{2\omega_0} = \dfrac{\pi\sqrt{LC}}{2}$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$L = \left(\frac{2t_{c_1}}{\pi}\right)^2 \cdot \frac{1}{C}$$

Peak current through T_1 is

$$I_{T1P} = I_0 + V_s\sqrt{\frac{C}{L}}$$

Peak diode current $I_{DP} = V_s\sqrt{\dfrac{C}{L}}$ = Peak capacitor current.

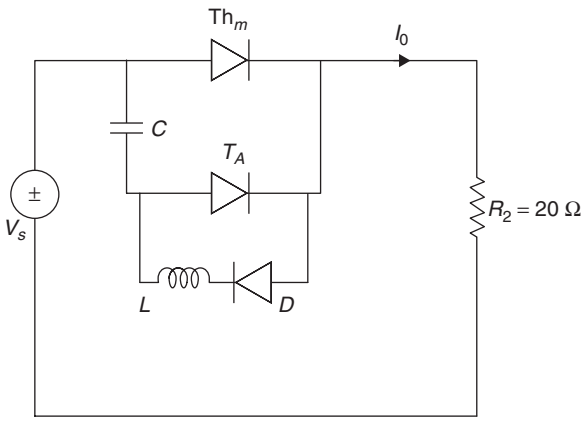
Effective on period $T'_{ON} = T_{ON} + \dfrac{2V_s}{I_0}C$

Commutating capacitor

$$I_0 = C\,\frac{(V_s - (-V_s))}{2t_c}$$

$$C = \frac{I_0 t_c}{V_s}$$

Example 11: For the current shown below if the circuit turn off time and peak value of capacitor current are 55 αs and 15 A, respectively. Then the values of '*L*' and '*C*', respectively, are



$V_s = 200$ V

(A) 1.2 mH, 24 μF (B) 2.4 mH, 44 μF
(C) 1.1 mH, 24 μF (D) 1.1 mH, 2.75 μF

Solution: (D)

$$i_0 = \frac{V_s}{R} = \frac{200}{20} = 10 \text{ A}$$

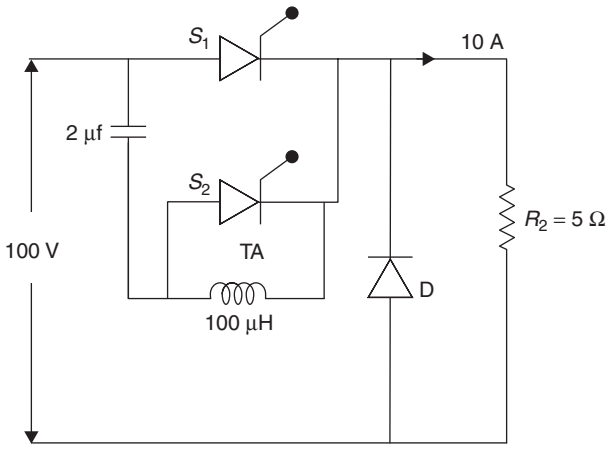
$$(t_c)_{OFF} = \frac{CV_s}{I_0} \quad C = \frac{(t_c)_{OFF} \times I_0}{V_s}$$

$$C = \frac{55\times10^{-6}\times10}{200} = 2.75\ \mu\text{F}$$

$$L = \left[\frac{V_s}{I_0}\right]^2 \times C = \left[\frac{200}{10}\right]^2 \times 2.75\times10^{-6}$$

$$L = 1.1 \text{ mH}$$

Example 12: The figure shown below is that of a chopper with S_M being the main switch rated for 200 V, 30 A and S_S being the auxiliary commutation device rated for 200 V, 20 A. The main device operates at a duty ratio of 75% and the load current is 10 A. The peak current through S_M is



(A) 40 A (B) 20 A (C) 30 A (D) 10 A

Solution: (B)

When S_M is triggered L, C and S_M forms a tank circuit. The current through circuit is given by

$$I_C = \frac{V_s}{\omega_0 L}\sin\omega_0 t \left[\text{where } \omega_0 = \frac{11}{\sqrt{LC}}\right]$$

Peak capacitor current

$$(I_C)_P = \frac{V_s}{\omega_0 L} = V_s \times \sqrt{\frac{C}{L}}$$

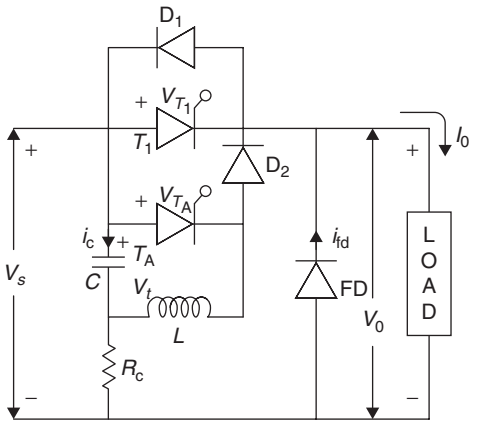
Peak current though S_1 is

$$I = (I_C)_p + I_0$$

$$= V_s\sqrt{\frac{C}{L}} + I_0 = 100 \times \sqrt{\frac{1\times10^{-6}}{100\times10^{-6}}} + 10$$

$$= 100 \times (0.1) + 10 = 20 \text{ A}$$

Current Commutated Chopper



Operating Modes

The chopper operates in five modes

Mode 1: For $t_1 < t < t_2$

- Initially devices T_1, T_A are switched ON and D_1, D_2, F.D. are turned OFF.
- This charge the capacitor $V_c = V_s$ (through T_A) and simultaneously source voltage appears across the load (through T_1) and current I_0 flows. When capacitor is fully charged T_A is turned off.

Mode 2: For $t_2 < t < t_3$

- When capacitor is fully charged D_2 is forward biased and is conducting. This circulates a current in the loop formed by C-L-D_1-T_1 and the current through T_1 opposes the flow of load current.

$$I_{T1} = i_0 - i_c$$

Mode 3: For $t_3 < t < t_4$

D_1 and D_2 are forward-biased and starts conducting. T_1 is turned OFF and T_A remains OFF as before.

- The source current splits up into diode D_1 and capacitor where

$$I_{D_1} = I_0 - I_c$$

- After this mode capacitor voltage is fully discharged and D_1 is turned off.

Mode 4: For $t_4 < t < t_5$

- During this mode, D_1, T_1, T_A and FD are turned OFF.
- The supply current flows through capacitor, charges the capacitor, and then reaches the load through D_2.
- Load current $I_0 = I$.

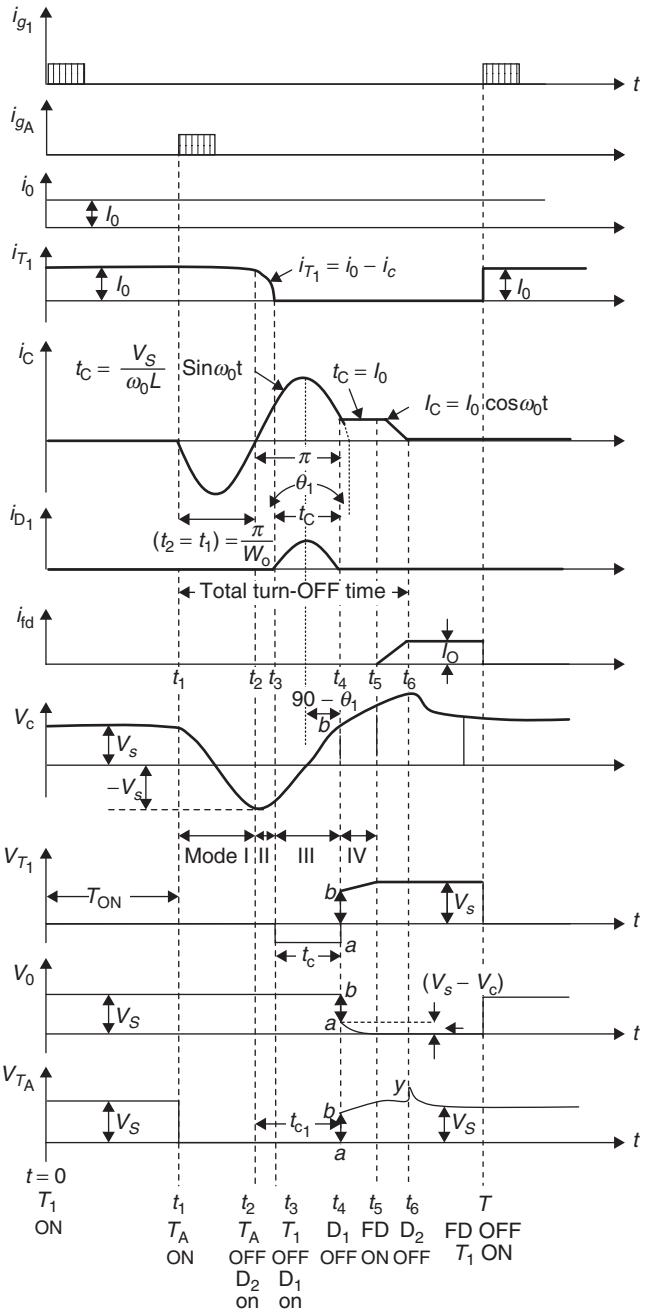
Mode 5: For $t_4 < t < t_6$

Once the capacitor is fully charged while mode 4 FD starts conducting and the load current keeps circulating in the same direction.

- D_2 and FD are the only conducting devices.
- The output current flowing through the load $I_0 = i_c + i_{fd}$.
- At t_6, D_2 is turned off as well and the load current circulates in the loop FD-LOAD.

Note:

1. In all modes of operations, the output voltage always remains positive.
2. The direction of current will always be from source to load and it is never fed back.



Peak Commutating Current (I_{cp})

$$I_{cp} = V_s\sqrt{\frac{C}{L}} > I_0$$

$$\boxed{I_{cp} = V_s\sqrt{\frac{C}{L}} = XI_0}$$

where $x = \frac{I_{cp}}{I_0}$ is greater than 1.

Circuit Turn-off time

$$t_c = \frac{1}{\omega_0}\left[\pi - 2\sin^{-1}\left[\frac{I_0}{I_{cp}}\right]\right]$$

Commutating Inductor

$$L = \frac{V_s t_c}{XI_0\left[\pi - 2\sin^{-1}(1/x)\right]}$$

Commutating Capacitor

$$C = \frac{X\ I_0\ t_c}{V_s\left[\pi - 2\sin^{-1}(1/x)\right]}$$

Turn-off Times

For main SCR, $t_4 - t_3 = t_c = \left[\pi - 2\sin^{-1}\left(\frac{1}{x}\right)\right]\cdot\sqrt{LC}$

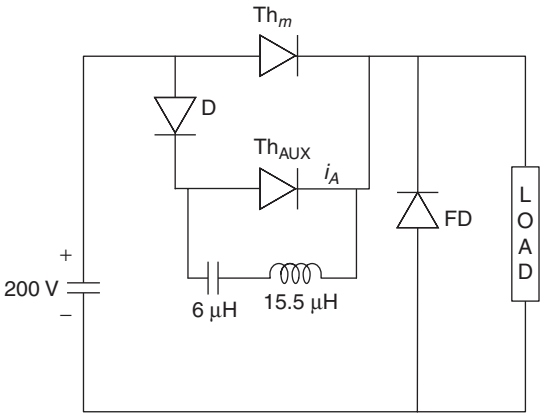
For auxiliary SCR

$$t_4 - t_2 = t_{c1} = \left[\pi - 2\sin^{-1}\left(\frac{1}{x}\right)\right]\cdot\sqrt{LC}$$

Example 13: Among the following which would yield best performance in a thyristor DC chopper?
(A) Source commutation
(B) Voltage commutation
(C) Load commutation
(D) Current commutation

Solution: (D)

Example 14: Given below is a current commutated DC–DC chopper where $(Th)_M$ and $(Th)_A$ are main and auxiliary thyristors, respectively.



If the auxiliary S.C.R. is triggered at $t = 0$, when Th_M is conducting and a constant load current of 12 A is flowing through the load. The main thyristor is turned OFF between

(A) $30\ \mu s < t \le 40\ \mu s$ (B) $20\ \mu s < t \le 30\ \mu s$
(C) $10\ \mu s < t \le 20\ \mu s$ (D) $0\ \mu s \le t \le 10\ \mu s$

Solution: (A)

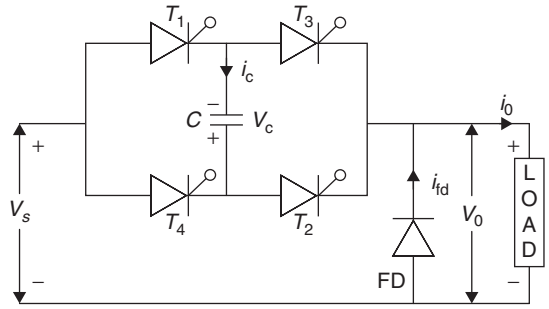
When $(Th)_A$ is triggered, a tank circuit is formed and conducts for $\pi\sqrt{LC}$ seconds

$$\pi\sqrt{LC} = \pi\sqrt{15.5\times10^{-6}\times6\times10^{-6}}$$
$$= 3.03\times10^{-5} = 30\ \mu s$$

After 30 μs an opposite current pulse will be injected and Th_M gets off. It is possible when capacitor current equals load current

$(Th)_M$ gets OFF when $30 < t \le 40$ μs

Load-commutated Chopper



Operating Modes

Mode 1: For $0 < t < t_1$

- T_1 and T_2 are turned ON and the current flows through
$$T_1 - C - T_2 - \text{LOAD}$$
- T_3 and T_4 are OFF
- The output voltage across the load $V_o = V_s$ and simultaneously the capacitor gets charged.

Mode 2: For $t_1 < t < t_2$

- All T_1, T_2, T_3, T_4 are turned off.
- The capacitor is fully charged and the polarity of the plates reversed when compared to initial condition.
- Free-wheeling diode D_1 starts conducting and the current through the load circulates in the loop.

Mode 3: For $t_2 < t < t_3$

- T_4 and T_3 conduct output voltage that appears across the load
$$V_o = V_s + V_c$$
- T_1 and T_2 are turned off.
- Since the load voltage is more than source voltage this chopper also is known as step-up chopper.

Multiphase Chopper

A multiphase chopper is one that consists of two or more choppers connected in parallel at the same frequency but with a proper phase shift. This type of operation enables the load and power supply to be subjected to an effective frequency which is a multiple of the chopping frequency.

- As a result, the supply harmonic current is reduced and the ripple amplitude decreases.

A multiphase chopper may be operated in two modes:

1. In phase operation mode
2. Phase-shifted operation mode

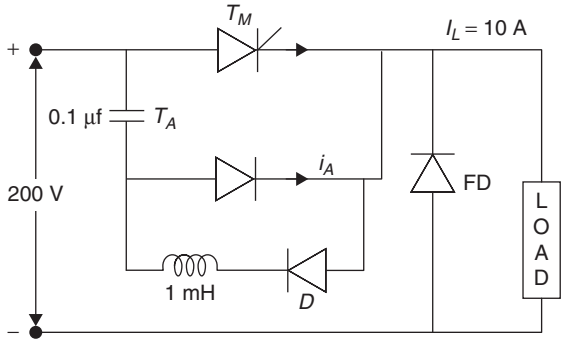
- In the in phase operation mode, all parallel connected choppers are on and off at the same instant.
- In the phase shifted operation mode, different choppers are on and off at different instant of time.
- The inductance L in series with each chopper is sufficiently large so that each chopper operates independent of each other.

Exercises

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

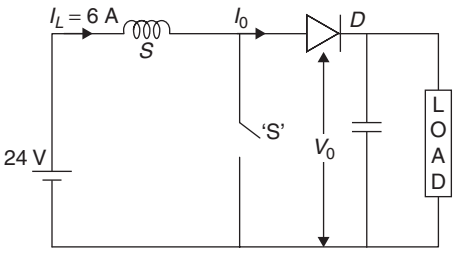
1. A voltage-commuted chopper circuit operated at 50 Hz as shown below.



For a load current of 5 A, the ratio of magnitudes of I_M to I_A (as indicated in the figure) would be

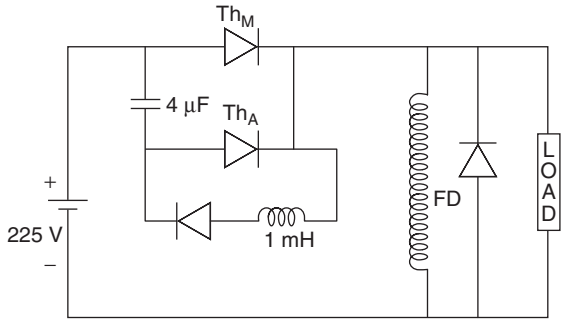
(A) 0.65 (B) 0.833 (C) 1 (D) 1.2

2. A step-up chopper shown below is operated at a duty cycle of 0.6. If a large capacitor is connected across the load and for a constant value of inductor current, then the average values of diode current and load voltage, respectively, are



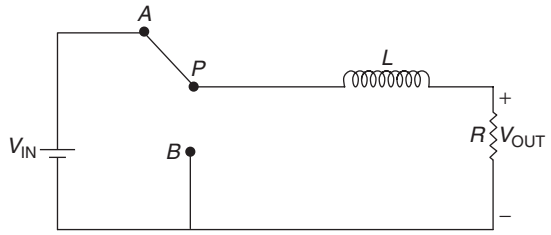
(A) 2.4 A, 60 V (B) 2 A, 40 V
(C) 8 A, 40 V (D) 2 A, 10 V

3. The figure shown below is a voltage commutated chopper operating with a duty ratio of 0.4 and at a frequency of 0.5 kHz. The load current is almost ripple free and remains constant at 15 A. The turn off time available to the thyristor Th_M is (assume the devices to be ideal)



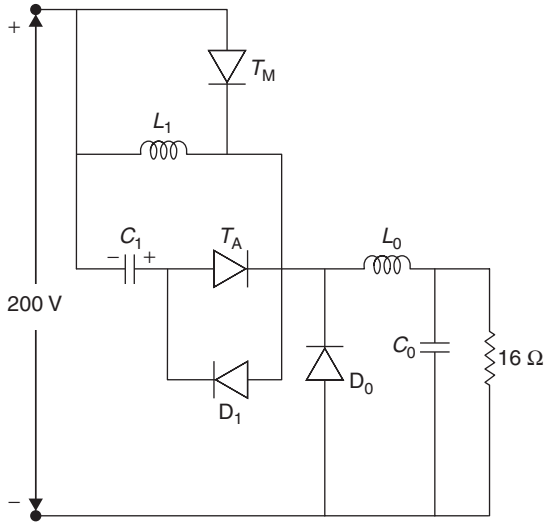
(A) 30 μs (B) 60 μs
(C) 20 μs (D) 40 μs

4. If the pole P of a single pole double throw switch is connected alternately to throws A and B as shown in the figure below. Which among the following converter does the figure given, represent?



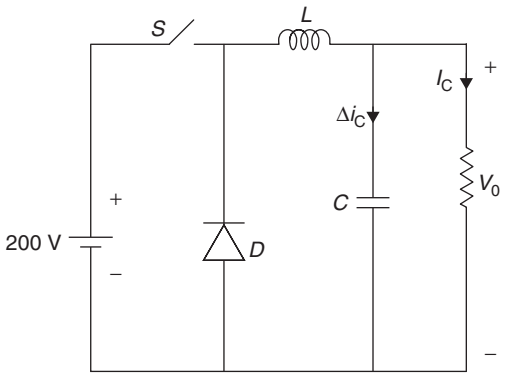
(A) Half-wave rectifier
(B) Step-up chopper [Boost converter]
(C) Step-down converter [Buck converter]
(D) None of the above

5. A chopper circuit shown in the figure below T_M is operated at 50% duty ratio. The maximum allowable reapplied $\frac{dV}{dt}$ on T_M is 10 V/μs. Assuming the ripple current through L_o to be negligible, the theoretical minimum values of C_1 should be



(A) 0.0625 μF
(B) 0.625 F
(C) 0.625 μF
(D) 0.625 mF

6. The ideal S in the circuit shown below is operated at 120 kHz at a duty ratio of 0.8. If I_0 is 10 A and $(\Delta i_c)_{p-p}$ is 1.6 A, the peak current in the switch S is



(A) 11.6 A
(B) 12.4 A
(C) 10.3 A
(D) 9.3 A

7. A 30 V battery charger feeds an ideal chopper which is operating at 400 Hz. It is supplying power to a resistive load of 1.5 , assuming the load is shunted by a perfect commutating diode and a loss less battery. The mean load current at an ON/OFF ratio of 1/1 would be
(A) 15 A
(B) 10 A
(C) 20 A
(D) 25 A

8. A voltage-commutated chopper has following parameters:

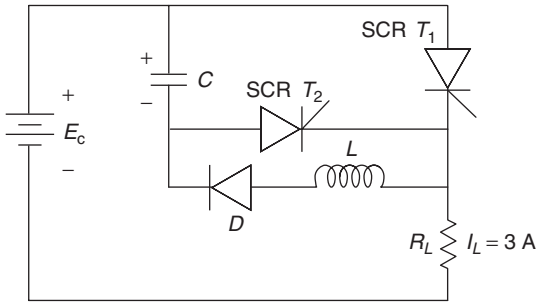
$$V_s = 230 \text{ V}$$

Load circuit parameter 5 Ω, 10 mH, 25 V
Commutation circuit parameters $L = 5$ μH, $C = 10$ μF
For a constant load current at 115 A, the effective ON period and peak current through the main thyristor are, respectively,
(A) 100 μS, 200 A
(B) 22.2 μS, 440 A
(C) 700 μS, 282.2 A
(D) 1000 μS, 440 A

9. In a type-A-chopper, source voltage is 110 V DC, ON period = 200 μS, OFF period = 150 μS and load RLE consists of $R = 4$ Ω, $L = 10$ mH, $E = 20$ V.

For continuous conduction, average output voltage and average output current for this chopper are, respectively,
(A) 40 V, 5 A
(B) 66.6 V, 28.3 A
(C) 62.8 V, 10.7 A
(D) 40 V, 20 A

Common Data for Questions 10 and 11:



For the above class D commutation circuit E_{DC} = 50 V, $I_{cc(max)}$ = 50 A, T_{OFF} of SCR_1 = 30 μS chopping frequency f = 500 Hz load variation required is 10% to 100% t_{DC} = 2 S.

10. The value of capacitor C is
(A) 27 μF (B) 45 μF
(C) 52 μF (D) 57 μF

11. The commutating inductor for allowing large voltage variation at the load is
(A) 90 μH (B) 67.5 μH
(C) 45 μH (D) 72 μH

Common Data for Questions 12 and 13:
A voltage commutated chopper has following parameters: V_s = 200 V, load parameters: 1 Ω, 2 mH, 5 V, commutation circuit parameters L = 25 μH and C = 50 μF

12. For a constant load current of 200 A, the effective ON period is
(A) 350 μS (B) 100 μS
(C) 700 μS (D) 1400 μS

13. The peak current through main thyristor is
(A) 200 A (B) 482.8 A
(C) 382.8 A (D) 282.8 A

Common Data for Questions 14 and 15:
A simple DC chopper is operating at a frequency of 2 kHz from 96 V DC source to supply a load of 8 Ω. The load time constant is 6ms. The average load voltage is 57.6 V.

14. The ON period of the chopper is
(A) 0.3 ms (B) 0.4 ms
(C) 0.6 ms (D) 0.5 ms

15. Average load current is
(A) 7.2 A (B) 14.4 A
(C) 28.8 A (D) 3.6 A

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. The features of chopper drives are
(A) Smooth control but slow response
(B) Smooth control, fast response but less efficient
(C) Smooth control, and fast response
(D) None of the above

2. Choppers can be used in future electric automobiles
(A) For speed control only
(B) For braking only
(C) For both speed control and braking
(D) None of these

3. Given a step-down chopper of supply voltage 80 V and average output voltage 40 V. Assuming continuous conduction mode and an ON period of 20 μs, the chopping frequency will be
(A) 10 kHz
(B) 20 kHz
(C) 50 kHz
(D) 25 kHz

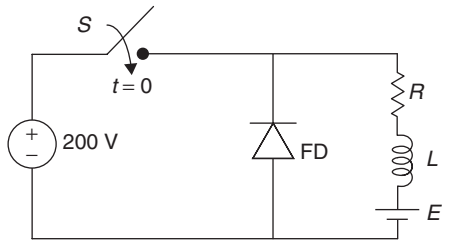
4. The per unit ripple in a step-down chopper would have its maximum value at a duty cycle of
(A) 20% (B) 40%
(C) 50% (D) 75%

5. A type-A chopper is connected to a 200 V DC source and is feeding an RLE load consisting of R = 4 Ω; L = 10 mH; E = 20 V. If the on and off periods are 200 μs and 300 μs, respectively, then the average output voltage and average output current for continuous conduction are, respectively,
(A) 40 V, 15 A (B) 60 V, 25 A
(C) 80 V, 15 A (D) 66.66 V, 28.33 A

6. If a step-down DC chopper is connected to a 230 V supply and a resistive load of R = 20 Ω; if the voltage drop across the converter switch is V_{ch} = 2 V. Then the chopper efficiency at α = 0.5 would be
(A) 93% (B) 95%
(C) 97% (D) 99%

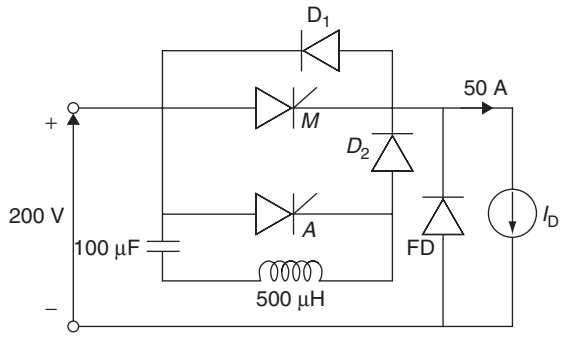
7. Which among the following is a correct match for DC chopper?
(A) Input Voltage—Continuous
Output Voltage—Continuous
(B) Input Voltage—Continuous
Output Voltage—Discontinuous
(C) Input voltage—Discontinuous
Output voltage—Continuous
(D) None of the above

8. For the chopper shown in the figure below if minimum instantaneous load current is I_1 = 32.75 A and peak instantaneous load current is I_2 = 26.05 A. The value of RMS value of load current at α = 0.5 would be
(A) 36.17 A (B) 22.13 A
(C) 63.71 A (D) 43.65 A



9. For the above problem, effective value of input resistance would be
(A) 20 Ω (B) 13.6 Ω
(C) 10 Ω (D) 18.36 Ω

10. The figure shows a current commutated chopper feeding a load such that a constant current of 50 A is maintained at the load. The peak commutating current in the circuit is



(A) 40 A (B) 60 A
(C) 20 A (D) 89 A

11. In a current-commutated chopper,
(A) Peak commutating current should be less than maximum load current
(B) Peak commutating current should be equal to maximum load current
(C) Peak commutating current should be greater than maximum load current
(D) There is no relation between peak commutating current and maximum load current

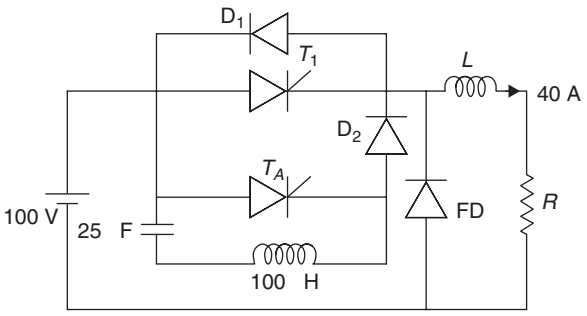
12. A 3-ϕ wound rotor induction motor is controlled by a chopper controlled resistance in its rotor circuit. A resistance of 2 Ω connected in the rotor circuit and a resistance of 4 Ω is additionally connected during OFF periods of the chopper. The OFF period of the chopper is 4 ms. The average resistance in the rotor circuit for the chopper frequency of 200 Hz is

(A) $\frac{26}{5}$ Ω (B) $\frac{24}{5}$ Ω
(C) $\frac{18}{5}$ Ω (D) $\frac{16}{5}$ Ω

13. A voltage commutated chopper has the following parameters:
V_s = 200 V, load circuit parameter 1 Ω, 2 mH, 5 V.
Commutation circuit parameter: L = 25 μH, C = 50 μF.
For constant load current at 100 A, the effective on period and peak current through the main thyristor are, respectively,
(A) 1000 μs, 200 A
(B) 700 μs, 382.8 A
(C) 500 μs, 282.8 A
(D) 1000 μs, 382.8 A

14. A DC–DC transistor chopper is supplied from a fixed voltage DC source feeds an RLE load and a free-wheeling diode. The chopper operates at 1 kHz and 50% duty cycle. Keeping the average DC value of current and duty cycle unchanged and by increasing chopper frequency; the ripple content of load current would
(A) Increase
(B) Reduce
(C) Remain the same
(D) Cannot be determined

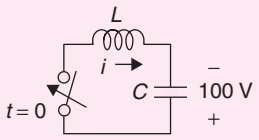
15. The figure shows a DC–DC chopper feeding on RL load such that the load current is 40 A. If the thyristor TA is fired at t = 0



(A) 64 μs (B) 128 μs
(C) 100 μs (D) 200 μs

Previous Years' Questions

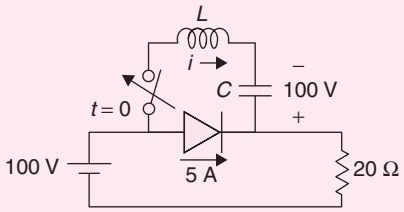
Common Data for Questions 1 and 2:
The *L–C* circuit shown in the figure has an inductance $L = 1$ mH and a capacitance $C = 10$ μF.



1. The initial current through the inductor is zero, while the initial capacitor voltage is 100 V. The switch is closed at $t = 0$. The current i through the circuit is:
[2010]

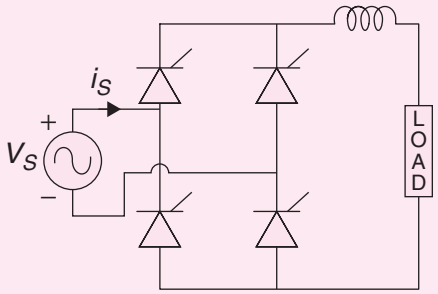
(A) $5\cos(5 \times 10^3 t)$ A (B) $5\sin(10^4 t)$ A
(C) $10\cos(5 \times 10^3 t)$ A (D) $10\sin(10^4 t)$ A

2. The *L-C* circuit of above question is used to commutate a thyristor, which is initially carrying a current of 5 A as shown in the figure below. The values and initial conditions of *L* and *C* are the same as in above question. The switch is closed at $t = 0$. If the forward drop is negligible, the time taken for the device to turn off is
[2010]

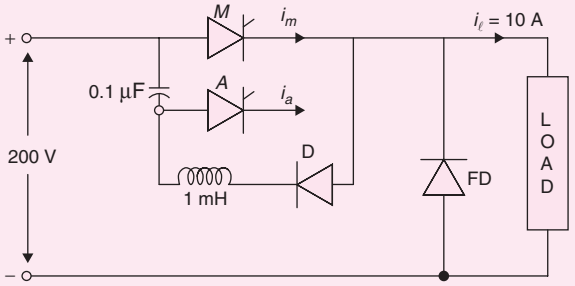


(A) 52 μs (B) 156 μs
(C) 312 μs (D) 26 μs

3. A fully controlled converter bridge feeds a highly inductive load with ripple-free load current. The input supply (v_s) to the bridge is a sinusoidal source. Triggering angle of the bridge converter is $\alpha = 30°$. The input power factor of the bridge is _____
[2010]



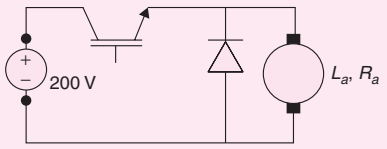
4. A voltage-commutated chopper circuit, operated at 500 Hz, is shown below.
[2011]



If the maximum value of load current is 10 A, then the maximum current through the main (*M*) and auxiliary (*A*) thyristors will be

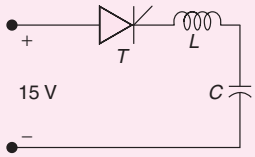
(A) $i_{M_{max}} = 12$ A and $i_{A_{max}} = 10$ A
(B) $i_{M_{max}} = 12$ A and $i_{A_{max}} = 2$ A
(C) $i_{M_{max}} = 10$ A and $i_{A_{max}} = 12$ A
(D) $i_{M_{max}} = 10$ A and $i_{A_{max}} = 8$ A

5. The separately excited DC motor in the figure below has a rated armature current of 20 A and a rated armature voltage of 150 V. An ideal chopper switching at 5 kHz is used to control the armature voltage. If $L_a = 0.1$ mH, $R_a = 1$ Ω, neglecting armature reaction, the duty ratio of the chopper to obtain 50% of the rated torque at the rated speed and the rated field current is
[2013]



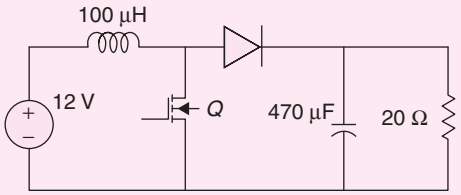
(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7

6. Thyristor *T* in the figure below is initially off and is triggered with a single pulse of width 10 μs. It is given that $L = \left(\frac{100}{\pi}\right)$ μH and $C = \left(\frac{100}{\pi}\right)$ μF. Assuming latching and holding current of the thyristor are both zero and the initial charge on *C* is zero, *T* conducts for
[2013]



(A) 10 μs (B) 50 μs
(C) 100 μs (D) 200 μs

Common Data for Questions 7 and 8:
In the figure shown below, the chopper feeds a resistive load from a battery source. MOSFET *Q* is switched at 250 kHz, with a duty ratio of 0.4. All elements of the circuit are assumed to be ideal.



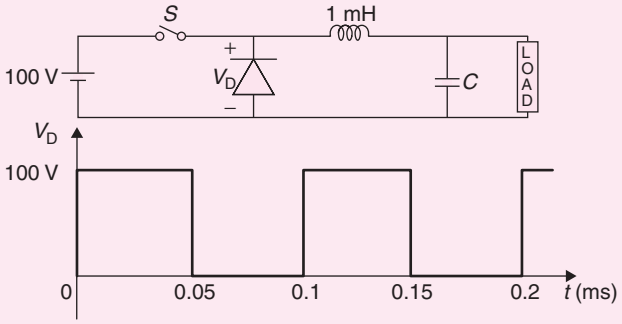
7. The average source current in ampere in steady state is **[2013]**

(A) 3/2 (B) 5/3 (C) 5/2 (D)

8. The PEAK-TO-PEAK source current ripple in ampere is **[2013]**

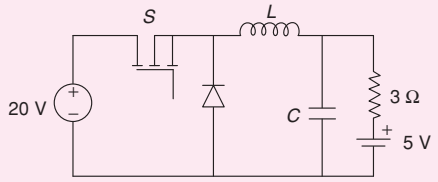
(A) 0.96 (B) 0.144
(C) 0.192 (D) 0.288

9. Figure (i) shows the circuit diagram of a chopper. The switch S in the circuit in figure (i) is switched such that the voltage v_D across the diode has the wave shape as shown in figure (ii). The capacitance C is large so that the voltage across it is constant. If switch S and the diode are ideal, the peak to peak ripple (in A) in the inductor current is _____ **[2014]**

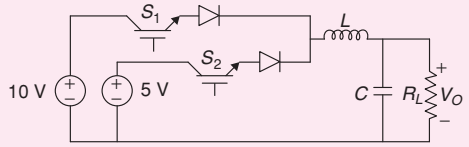


10. A step-up chopper is used to feed a load at 400 V DC from a 250 V DC source. The inductor current is continuous. If the 'off' time of the switch is 20 µs, the switching frequency of the chopper in kHz is ____. **[2014]**

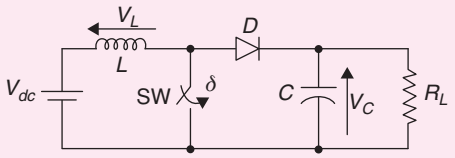
11. In the following chopper, the duty ratio of switch is 0.4. If the inductor and capacitor are sufficiently large to ensure continuous inductor current and ripple free capacitor voltage, the charging current (in Ampere) of the 5 V battery, under steady-state is _____. **[2015]**



12. The circuit shown is meant to supply a resistive load R_L from two separate DC voltage sources. The switches S_1 and S_2 are controlled so that only one of them is ON at any instant. S_1 is turned on for 0.2 ms and S_2 is turned on for 0.3 ms in a 0.5 ms switching cycle time period. Assuming continuous conduction of the inductor current and negligible ripple on the capacitor voltage, the output voltage V_o (in Volt) across R_L is _____. **[2015]**



13. A self commutating switch SW, operated at duty cycle δ is used to control the load voltage as shown in the figure



Under steady state operating conditions, the average voltage across the inductor and the capacitor respectively, are **[2015]**

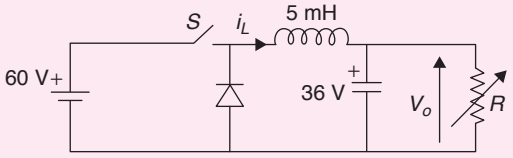
(A) $V_L = 0$ and $V_C = \dfrac{1}{1-\delta} V_{dc}$

(B) $V_L = \dfrac{\delta}{2} V_{dc}$ and $V_C = \dfrac{1}{1-\delta} V_{dc}$

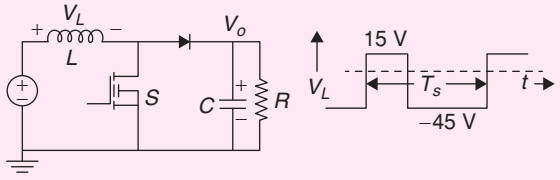
(C) $V_L = 0$ and $V_C = \dfrac{\delta}{1-\delta} V_{dc}$

(D) $V_L = \dfrac{\delta}{2} V_{dc}$ and $V_C = \dfrac{\delta}{1-\delta} V_{dc}$

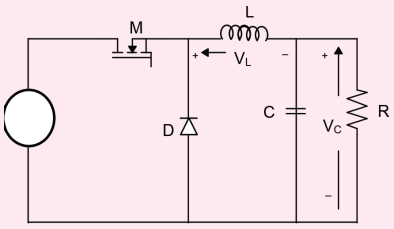
14. A buck converter feeding a variable resistive load is shown in the figure. The switching frequency of the switch S is 100 kHz and the duty ratio is 0.6. The output voltage V_O is 36 V. Assume that all the components are ideal, and that the output voltage is ripple-free. The value of R(in Ohm) that will make the inductor current (i_L) just continuous is _____. **[2015]**

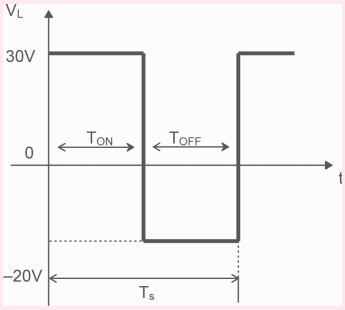


15. For the switching converter shown in the following figure, assume steady-state operation. Also assume that the components are ideal, the inductor current is always positive and continuous and switching period is T_s. If the voltage V_L is as shown, the duty cycle of the switch S is _____. **[2015]**

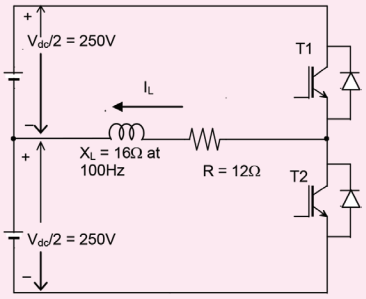


16. A buck converter, as shown in Figure (a) below, is working in steady state. The output voltage and the inductor current can be assumed to be ripple free Figure (b) shows the indicator voltage V_L during a complete switching interval. Assuming all devices are ideal, the duty cycle of the buck converter is _____. **[2016]**

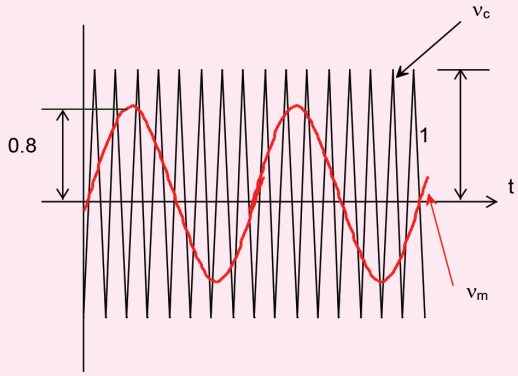




17. The switches T1 and T2 in Figure (a) are switched in a complementary fashion with sinusoidal pulse width modulation technique. The modulating voltage $\nu_m(t) = 0.8 \sin(200_t)$ *V* and the triangular carrier voltage (ν_c) are as shown in Figure (b). The carrier frequency is 5 kHz. The peak value of the 100 Hz component of the load current (i_L), in ampere, is **[2016]**

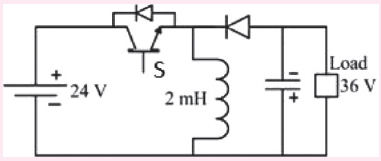


(a)



(b)

18. A buck-boost DC – DC converter, shown in the figure below, is used to convert 24V battery voltage to 36V DC voltage to feed a load of 72W. it is operated at 20KHz with an inductor of 2mH and output capacitor of 1000μF. all devices are considered to be ideal. The peak voltage across the Solid-state switch (S), in volt is _____. **[2016]**



19. A DC - DC boost converter, as shown in the figure below, is used to boost 360V to 400V, at a power of 4kW. All devices are ideal. Considering continuous inductor current, the rms current in the Solid state switch (*S*), in ampere is _____ **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** A	**3.** B	**4.** C	**5.** C	**6.** A	**7.** B	**8.** B	**9.** C	**10.** B
11. C	**12.** C	**13.** B	**14.** A	**15.** A					

Practice Problems 2

1. C	**2.** C	**3.** D	**4.** C	**5.** C	**6.** D	**7.** B	**8.** A	**9.** B	**10.** D
11. C	**12.** D	**13.** B	**14.** B	**15.** A					

Previous Years' Questions

1. D	**2.** A	**3.** 0.78	**4.** A	**5.** D	**6.** C	**7.** B	**8.** C	**9.** 2.5	**10.** 31.25
11. 1	**12.** 7	**13.** A	**14.** 2480 to 2520		**15.** 0.75	**16.** 0.4	**17.** 10	**18.** 60.03	**19.** 3.51

Chapter 4

DC–AC Converters (OR) Inverters

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Basic inverter principles and classifications
- Single-phase half-bridge inverter
- Gating sequence
- Single phase full bridge inverter
- Fourier analysis of single phase inverter output voltage
- I-Q half-bridge inverter
- I-Q full-bridge inverter
- Modified Mc Murray half-bridge inverter
- Three-phase bridge inverter
- Three-phase 120° mode V.S.I.
- Series inverter
- Parallel inverter
- PWM techniques

- Inversion is the conversion of DC power to AC power at a desired output voltage or current and frequency, where thyristors are supplied with DC and turned off by forced commutation.
- In most of the inverter circuits, voltage and frequency are controlled.
- The output of the inverter will have more AC power at lower frequencies and capacitors and inductors are used as energy storing elements.
- Phase-controlled converters when operated in inverter mode are called line commutated inverters but they require an existing AC supply at the terminals which will be used for its commutation. The voltage level, frequency and waveform on the AC side of line commutated inverters cannot be changed.
- In forced commutation method the SCR current is decreased to zero either by transferring the load current to a different path or by decreasing load current to zero.



Basic Inverter Principles and Classifications

Based on the output from inverter circuits they are broadly classified as

(a) Voltage-source inverter (VSI)
(b) Current-source inverter (CSI)

A VSI is one in which the DC input voltage is essentially constant and independent of the load current drawn. The inverter specifies the load voltage while the drawn current shape is dictated by the load.

A CSI is one in which the source. Hence the load current is predetermined and the load impedance determines the output voltage.

- VSI has a stiff DC voltage source at its input, it requires forced commutation. Load commutation is possible only, if the load is under damped.
- CSI is fed with stiff DC current source and its output current wave shape is not affected by the load.
- The inverter and its output can be single-phase, three-phase or multi-phase.
- The inverter and its output can be single-phase three-phase or multi-phase.
- Inverter output waveforms either voltage or current are usually rectilinear in nature and as such contain harmonics which may lead to reduced load efficiency and performance load harmonics reduction can be achieved by either filtering selected harmonic reduction chopping (or) pulse width modulation (PWM).
- The quality of inverter output is normally evaluated in terms of its harmonic factor 'ρ' distortion factor μ and total harmonic distortion THD for VSI. The in terms of the output voltage harmonics as

$$\Gamma_n = \left|\frac{V_n}{V_1}\right| = n\mu_n ; n > 1$$

The distortion factor for an individual harmonic is

$$\mu_n = \left|\frac{V_n}{nv_1}\right| = \frac{\Gamma_n}{n}$$

$$\text{THD} = \sqrt{\left[\sum_{n\geq z}^{\infty}\left(\frac{\gamma_n}{n}\right)^2\right]} V_1$$

$$= \sqrt{\sum_{n\geq z}^{\infty}\mu_n^{\,2}} = \sqrt{\sum_{n\geq z}^{\infty}\left(\frac{\rho_n}{n}\right)^2}$$

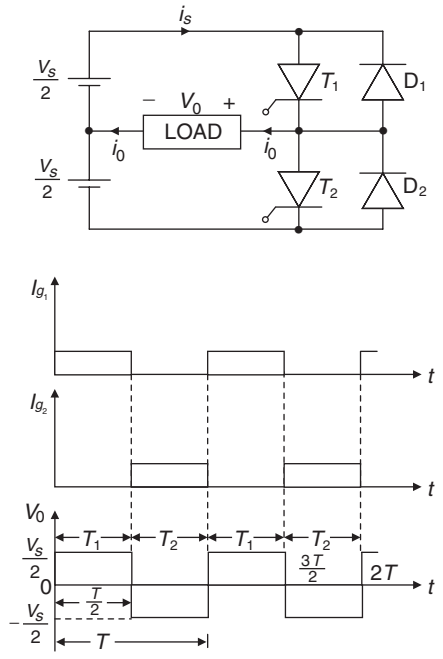
- The factor $\left(\frac{\gamma_n}{n}\right)$ is used since the harmonic currents produced in an inductive load attenuate with frequency.
- The harmonic currents produce unwanted heating and torque oscillations in AC motors, although such harmonic currents are not a drawback to the power delivered to a resistive heating load or incandescent load.

Depending on connection they are.
- Bridge Inverter
- Series Inverter
- Parallel Inverter

Bridge inverters are of two types:

1. Single-phase half-bridge inverters
2. Single-phase full-bridge inverters

Single-phase Half-bridge Inverter



- The inverter circuit consists of two choppers.
- When only T_1 is turned on for a time $\frac{T}{2}$, the instantaneous voltage across the load V_0 is $\frac{V_s}{2}$.
- If transistor T_2 is only turned on for a time $\frac{T}{2}$, $-\frac{V_s}{2}$ appears across the load.
- The logic circuit should be designed such that T_1 and T_2 are not turned on at the same time.
- This inverter requires a three wire DC source and when a transistor is off, its reverse voltage is V_s instead of $\frac{V_s}{2}$. Such an inverter is known as half-bridge inverter.
- For an inductive load current cannot change immediately with the output voltage. If T_1 is turned off at $t = \frac{T}{2}$, the load current would continue to flow through D_2, load and lower half of DC source, until the current flows to zero.
- When T_2 is turned off at $t = \frac{T}{2}$ the load current flows through D_1, load and upper half of the DC source.
- When diodes D_1 and D_2 conducts energy is fed back to the DC source and these diodes are known as feedback diodes.

Gating Sequence

1. Generate a square-wave gating signal V_{g_1} at an output frequency f_0 and a 50% duty cycle. The gating signal V_{g_2} should be a logic invert of V_{g_1}.
2. Signal V_{g_1} will drive switch T_1 through a gate isolating circuit, and V_{g_2} can drive T_1 without any isolating circuit.

Note:
- An AC output voltage can be obtained by alternatively connecting the positive and negative terminals of the DC source, across the load by turning on and off the switching devices accordingly.
- Feedback diodes are required to transfer the energy stored in the load inductance back to the DC source.

The output voltage does not depend on the nature of load.

By Fourier series

$$V_0 = \sum_{n=1,3,5}^{n} \frac{2Vs}{n\pi} \sin n\omega t$$

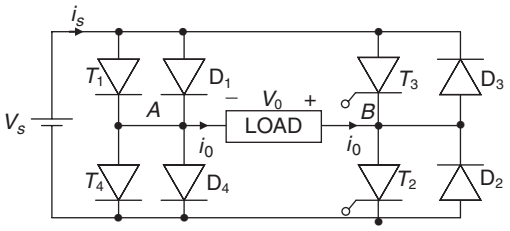
Frequency of output

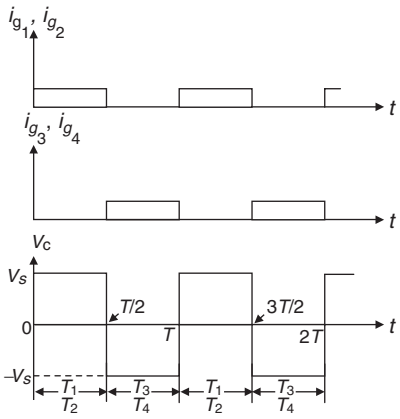
$$f = \frac{1}{T} \text{ voltage}$$

Disadvantages
- Mid-point supply is required
- Output voltage $= \pm \frac{V_s}{2}$

(i.e.) no proper utilization of thyristor rating

Single-phase Full-bridge Inverter





- It consists of four choppers.
- When transistors T_1 and T_2 are turned on simultaneously, the input voltage V_s appears across the load.
- If transistors T_3 and T_4 are turned ON at the same time, voltage across the load is reversed and is $-V_s$.

Switch states for a single-phase full-bridge VSI

State	State no.	Switch State*	v_{ao}	v_{bo}	v_o	Components Conducting
S_1, S_2 are on and S_4, S_3 are off	1	10	$V_s/2$	$-V_s/2$	V_s	S_1 and S_2 if $i_0 > 0$ D_1 and D_2 if $i_0 < 0$
S_4, S_3 are on and S_1, S_2 are off	2	01	$-V_s/2$	$V_s/2$	$-V_s$	D_4 and D_3 if $i_0 > 0$ S_4 and S_3 if $i_0 < 0$
S_1, S_3 are on and S_4, S_2 are off	3	11	$V_s/2$	$V_s/2$	0	S_1 and D_3 if $i_0 > 0$ D_1 and S_3 if $i_0 < 0$
S_4, S_2 are on and S_1, S_3 are off	4	00	$-V_s/2$	$-V_s/2$	0	D_4 and S_2 if $i_0 > 0$ S_4 and D_2 if $i_0 < 0$
S_1, S_2, S_3, and S_4 are all off	5	Off	$-V_s/2$ $V_s/2$	$V_s/2$ $-V_s/2$	$-V_s$ V_s	D_4 and D_3 if $i_0 > 0$ D_4 and D_2 if $i_0 < 0$

* If an upper switch is on and 0 if a lower switch is on.

- The above table shows five switch states.
- T_1 & T_4 acts as the switching devices S_1, S_4, respectively. If two switches one upper and one lower conduct at the same time such that output voltage is $\pm V_s$ the switch state is 1, whereas if these switches are off at the same time, the switch state is 0.

Gating Sequence

1. Generate two square-wave gating signals V_{g_1} and V_{g_2} at an output frequency f_0 and a 50% duty cycle. The gating V_{g_1} and V_{g_3} should be the logic invert of V_{g_2} and V_{g_4}, respectively.
2. Signals V_{g_1} and V_{g_3} drive Q_1 and Q_3, respectively, through gate isolating circuits. V_{g_2} and V_{g_4} can drive T_2 and T_4, respectively, without any isolation circuits.

Note:

- Peak reverse blocking voltage of each transistor and quality of output voltage for half- and full-bridge inverters are the same. However, for full-bridge inverters, the output power is four times higher and the fundamental component is twice that of half-bridge inverters.
- The full-bridge inverter requires four switching devices and four diodes. The output voltage switches between = V_s and $-V_s$. The RMS fundamental component V_1 of the output voltage is 0.9 V_s.
- The design of an inverter requires the determination of the average, RMS, and peak currents of the switching devices.

  The output voltage waveforms do not depend on the nature of the load. These waveforms can be represented as follows
- For single-phase, half-bridge inverter

$$V_0 = \sum_{n=1,3,5\ldots}^{\alpha} \frac{2V_s}{n\pi} \sin \omega t \text{ Volt}$$

- For single-phase, full-bridge inverter

$$V_0 = \sum_{n=1,3,5\ldots}^{\alpha} \frac{4V_s}{n\pi} \sin n\omega t \text{ Volt}$$

Here, n is the order of harmonic
$\omega \rightarrow$ Frequency of the output voltage in rad/sec
Load current

$$i_0 = \sum_{n=1,3,5....}^{\alpha} \frac{4V_S}{n\pi |Z_n|} \sin(n\omega t - \phi_n)$$

$$i_0 = \left[R^2 + \left(n\omega L - \frac{1}{n\omega C} \right)^2 \right]^{1/2}$$

where Z_n = Load impedance at frequency n.f.

Phase angle, $\phi_n = \dfrac{\tan^{-1}\left(n\omega L - \dfrac{1}{n\omega C} \right)}{R}$

If I_{01} is RMS value of fundamental component of load current, the fundamental load power

$$P_{01} = I_{01}^2 R = V_{01} I_{01} \cos \phi_1$$

$V_{01} \rightarrow$ RMS value of fundamental output voltage.

P_{01} does the useful work in most of the applications (e.g.: electric motor drives). The output power associated with harmonic current does no useful work and is dissipated as heat leading to rise in load temperature.

Steady State Analysis of Single-phase Inverter

If $(T/2 - t_1) > t_q$, load commutation takes place No force commutation is necessary.

In the under damped case, if duration $(T_2 - t_1) > t_q$, T_1, T_2 gets commutated naturally and therefore no commutation circuitry will be needed.

Fourier Analysis of Single-phase Inverter Output Voltage

1-Q Half-bridge Inverter

$$V_0 = \sum_{n=1,3}^{\infty} \frac{2V_s}{n\pi} \sin n\omega t$$

$$i_0 = \sum_{n=1,3}^{\infty} \frac{2V_s}{n\pi Z_n} \sin (n\omega t - \phi_n)$$

1-Q Full-bridge Inverter

$$V_0 = \sum_{n=1,3}^{\infty} \frac{4V_s}{n\pi} \sin n\omega t$$

$$i_0 = \sum_{n=1,3}^{\infty} \frac{4V_s}{n\pi Z_n} \sin (n\omega t - \phi_n)$$

Z_n is the impedance offered to n^{th} harmonic

$$z_n = \sqrt{R^2 + \left(n\omega L - \frac{1}{n\omega c} \right)^2}$$

$$Q_n = \tan^{-1}\left[\frac{n\omega L - \dfrac{1}{n\omega c}}{R} \right]$$

$V_{or} = V_s$ – full-bridge inverter

$$V_{or} = \frac{V_s}{2} \text{ – Half-bridge inverter}$$

Harmonic factor of n^{th} harmonic

$$\text{H.F.n} = \frac{V_n}{V_1}$$

where V_n = RMS value of nth harmonic component

V_1 = fundamental component RMS value

Total harmonic distortion (T.H.D.) – It is a measure of closeness in shape between a waveform and its fundamental component.

$$\text{THD} = \frac{1}{V_1}\left[\sum_{n=2,3}^{\alpha} V_n^2 \right]^{\frac{1}{2}} = \sqrt{\frac{V_{or}^2 - V_1^2}{V_1}}$$

Distortion of factor of n^{th} harmonic is defined as

$$\text{DFn} = \frac{V_{1\,\text{RMS}}}{V_{or}}$$

Solved Examples

Example 1: When a line commutated converter delivers real power from AC supply.

(A) It works as a rectifier at $\alpha > 90°$.

(B) It operates in the inverter mode at $\alpha < 90°$.

(C) Till $\alpha \leq 90°$ it operates as rectifier and beyond 90° it operates as inverter.

(D) Cannot be determined

Solution: (C)

Example 2: The false statement among the following choices regarding the operations below is "6 MOSFETs are connected in a bridge configuration must be operated as a 'voltage source inverter'.

(A) It saturation region MOSFETs can be operated as excellent current sources.

(B) The setup can be operated both as VSI and CSI.

(C) MOSFET's have inherent anti parallel diodes.

(D) MOSFET's are voltage-driven devices.

Solution: (A)

Example 3: A 200 V DC voltage is supplying power to an RCL load through a single-phase full bridge through a $R = 10\ \Omega$, $L = 60$ mH, C = 100 μF.

If the output frequency is 50 Hz, The maximum thyristor current is

(A) 5 A (B) 10 A (C) 15 A (D) 7.32 A

Solution: (B)

Output voltage $V_0 = \dfrac{4V_s}{\pi} \sin \omega t + \dfrac{4V_s}{3\pi} \sin 3\ \omega t + \dfrac{4V_s}{5\pi} \sin 5\ \omega t$

$= 254.65 \sin 314t + 84.88 \sin 942t + 50.93 \sin 1570t$

It is of the general form

$$V_0 = V_1 \sin \omega_1 t + V_2 \sin \omega_2 t + V_3 \sin \omega_3 t$$

$$I_{TH} = \sqrt{I_1^2 + I_2^2 + I_3^2}$$

$$= \sqrt{\left(\frac{V_1}{Z_1}\right)^2 + \left(\frac{V_2}{Z_2}\right)^2 + \left(\frac{V_3}{Z_3}\right)^2}$$

$$Z_1 = \sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 c}\right)^2}$$

$$Z_1 = \sqrt{20^2 + \left(314\times60\times10^{-3} - \frac{1}{314\times100\times10^{-6}}\right)^2}$$

$$= 23.86\ \Omega$$

Similarly

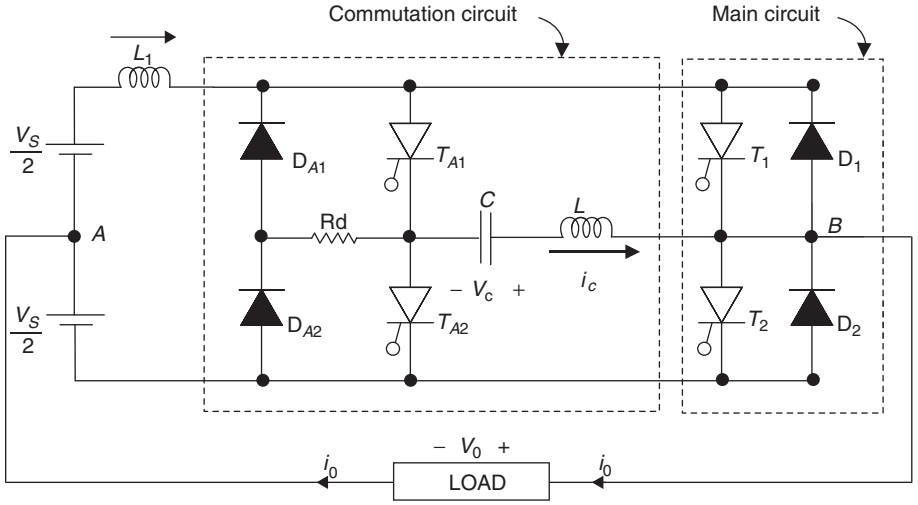
$$Z_2 = 50\ \Omega \qquad Z_3 = 90\ \Omega$$

$$I_{TH} = \sqrt{\left(\frac{254.66}{23.86}\right)^2 + \left(\frac{84.88}{50}\right)^2 + \left(\frac{50.93}{90}\right)^2}$$

$$I_{TH} = 10\ \text{A}$$

Modified McMurray Half-bridge Inverter

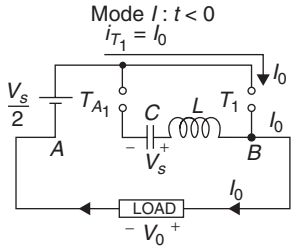
- The power circuit diagram for a single-phase modified Mc Murray half-bridge converter is as given below.



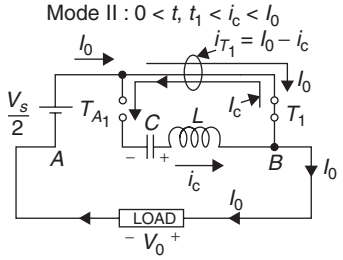
- It consists of a commutation circuit and a main circuit with two Thyristor and two diodes for each circuits.
- It operated in seven different modes which are as follows.

Operating Modes

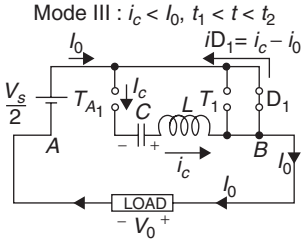
Mode *I* : $t < 0$



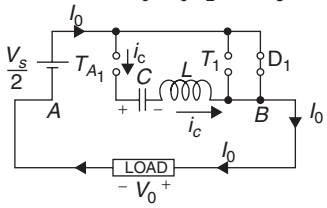
Mode II : $0 < t$, $t_1 < i_c < I_0$



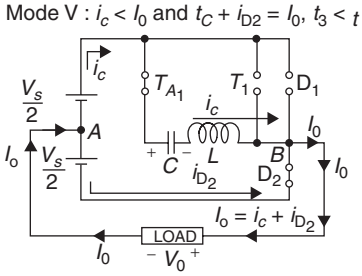
Mode III : $i_c < I_0$, $t_1 < t < t_2$

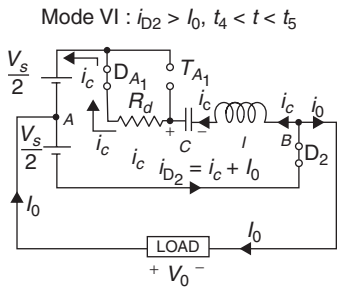


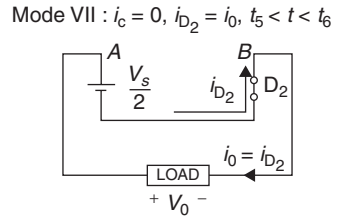
Mode IV : $i_c < I_0$, $t_2 < t < t_3$



Mode V : $i_c < I_0$ and $t_C + i_{D2} = I_0$, $t_3 < t < t_4$

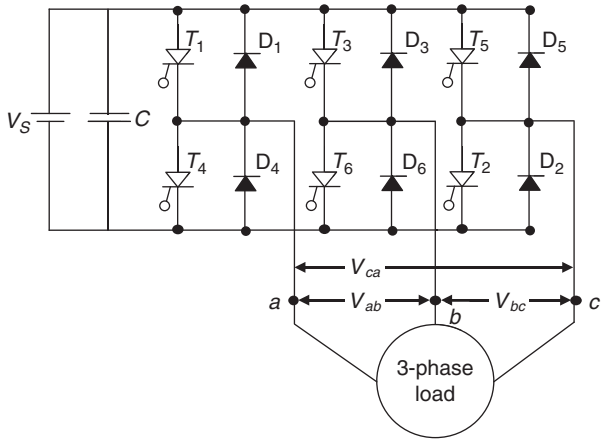






Three-phase, Bridge Inverter

- These are normally used for high power applications.
- Three, single-phase half (or full) -bridge inverters can be connected in parallel to form the configuration of a three-phase inverter.
- The gating signals of singe-phase inverters should be advanced or delayed by 120° with respect to each other to obtain three-phase balanced voltages.
- The transformer primary windings must be isolated from each other whereas secondary windings may be connected in Y or Δ. The transformer secondary is normally connected in Δ to eliminate triplen harmonics appearing in the output voltages and the circuit arrangement is shown as below.



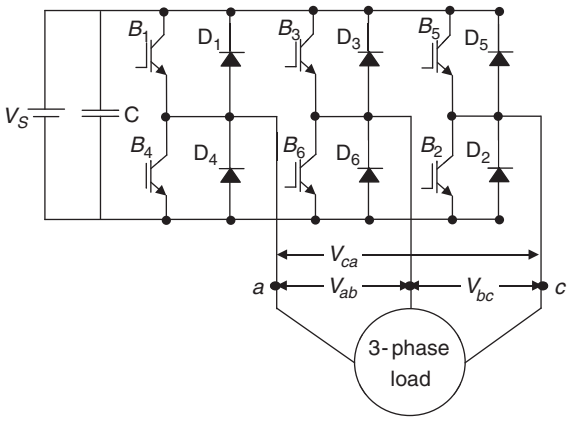
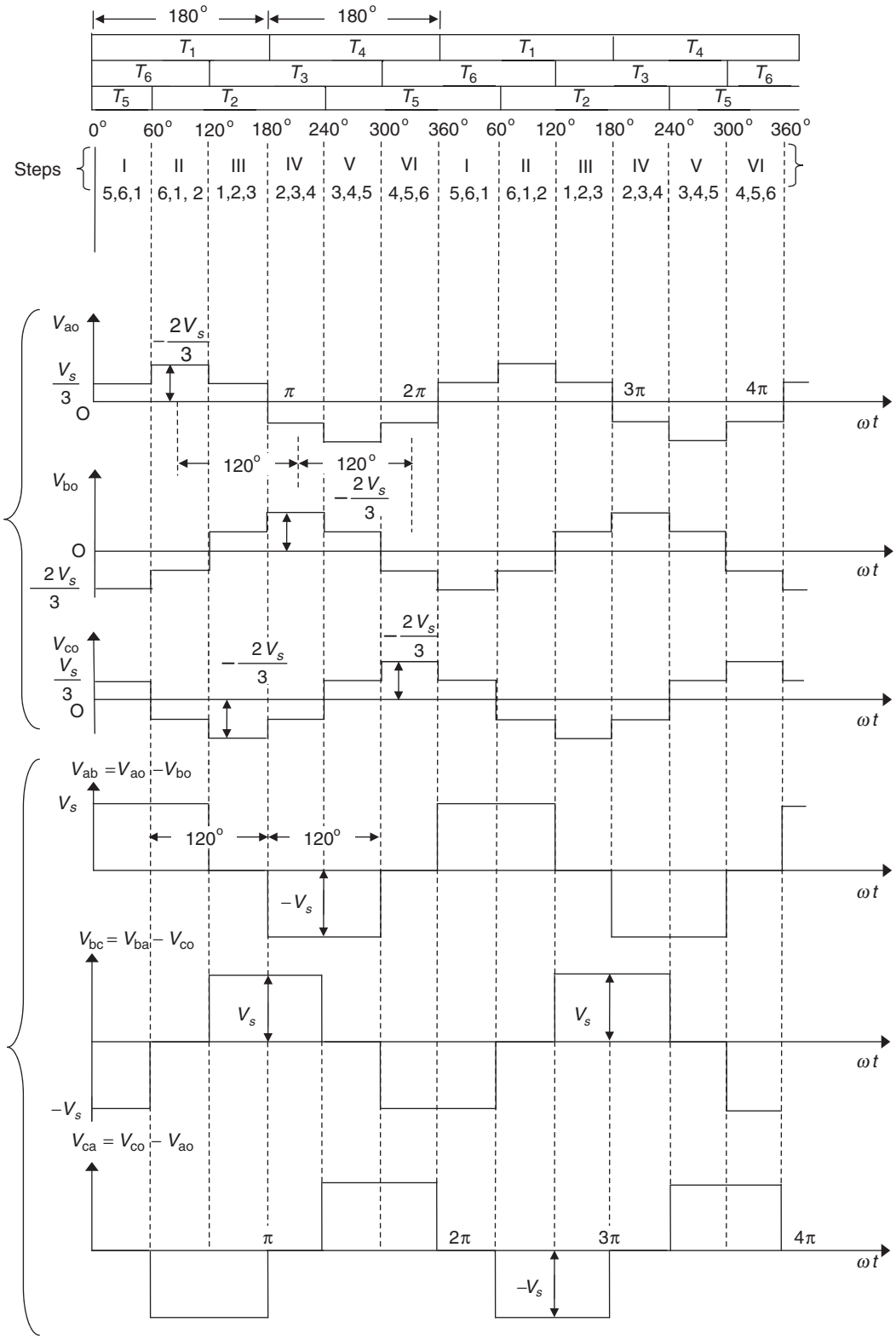


Figure 1 (a) Using thyristors; (b) Using IGBTs

- This arrangement requires three, single-phase transformers, 12 transistors and 12 diodes.
- If the output voltages of single-phase inverters are not perfectly balanced in magnitude and phases, the three-phase output voltages are unbalanced.
- Two types of control signals can be applied to the transistors to obtain 180° conduction or 120° conduction.
- The 180° conduction has better utilization of the switches and is the preferred method.

Three-phase, 180° Mode VSI

- Each transistor conducts for 180°. Three transistors remain on at any instant of time.
- When T_1 is switched ON, a is connected to the positive terminal of a DC input voltage. When T_4 is switched on, terminal a is brought to the negative terminal of DC source. There are six modes of operation in a cycle and the duration of each mode is 60°.
- The transistors are numbered in the sequence of gating as (123, 234, 345, 456, and 612).
- The gating signals are shifted from each other by 60° to obtain three-phase balance voltages.
- The switches of any leg of the inverter (S_1 and S_4, S_3 and S_6, S_5 and S_2) cannot be switched ON simultaneously. This would result in a short circuit across the DC link voltage supply.
- To avoid undefined states and thus undefined AC output line voltages the switches of any legs of the inverter cannot be switched of simultaneously. This can result in voltages that depend on the respective line current polarity.


180°
180°
T1
T4
T1
T4
T6
T3
T6
T3
T6
T5
T2
T5
T2
T5
0° 60° 120° 180° 240° 300° 360° 60° 120° 180° 240° 300° 360°
Steps
I II III IV V VI I II III IV V VI
5,6,1 6,1, 2 1,2,3 2,3,4 3,4,5 4,5,6 5,6,1 6,1,2 1,2,3 2,3,4 3,4,5 4,5,6
Vao
2Vs/3
Vs/3
O
π
2π
3π
4π
ωt
120°
120°
Vbo
2Vs/3
O
2Vs/3
ωt
Vco
2Vs/3
2Vs/3
Vs/3
O
ωt
Vab = Vao – Vbo
Vs
120°
120°
–Vs
ωt
Vbc = Vba – Vco
Vs
Vs
ωt
–Vs
Vca = Vco – Vao
π
2π
3π
4π
–Vs
ωt

The line output voltages

$$V_{ab} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{4V_s}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t + \frac{\pi}{6}\right)$$

$$V_{bc} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{4V_s}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t - \frac{\pi}{2}\right)$$

$$V_{ca} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{4V_s}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t + \frac{5\pi}{6}\right)$$

RMS value of n^{th} component of the line voltage,

$$V_{IN} = \frac{4V_s}{\sqrt{2}\cdot n\pi} \cos\frac{n\pi}{6}.$$

RMS value of fundamental line voltage

$$V_{L1} = \frac{4V_s}{\sqrt{2}\pi} \frac{\cos n\pi}{6} = 0.7797 V_s$$

From line voltage waveform V_{ab}

Line voltage is V_s from 0° to 120°

$$V_L = \left[\frac{1}{\pi}\int_0^{2\pi/3} V_s^2 d(\omega t)\right]^{1/2} = \sqrt{\frac{2}{3}} V_s = 0.8165\ V_s$$

RMS value of phase voltage

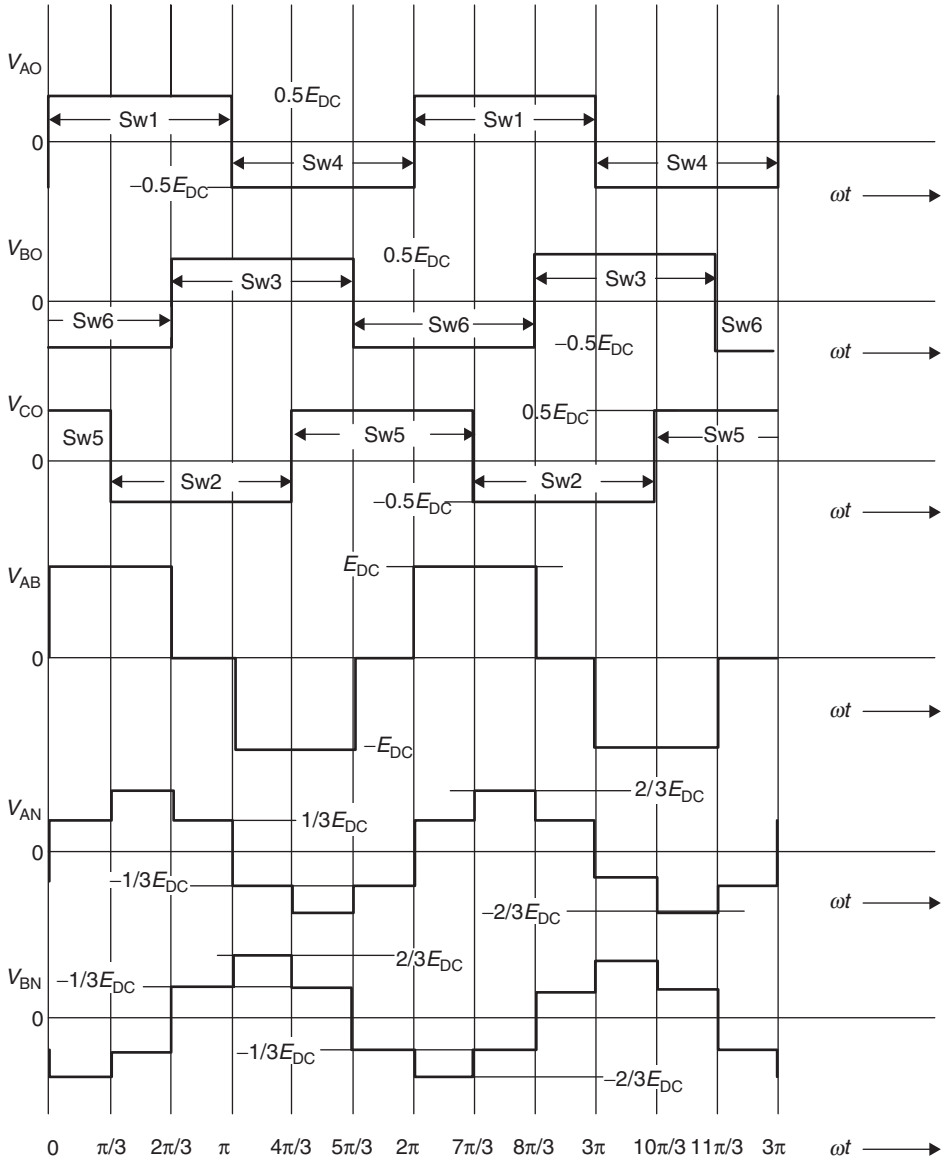
$$V_p = \frac{V_L}{\sqrt{3}} = \frac{\sqrt{2}V_s}{3} = 0.4714 V_s$$

RMS value of fundamental phase voltage

$$V_{p1} = \frac{\sqrt{2}V_s}{\sqrt{2}\pi} = 0.4502\ V_s = \frac{V_{L1}}{\sqrt{3}}$$

Three-phase 120° Mode V.S.I.

- For this mode, each thyristor conducts for 120° of a cycle. Like 180° conduction mode inverter also requires 6 steps, each of 60° duration for completing one cycle of the AC output voltage.
- The sequence of firing of six thyristors and voltage waveforms is as shown below.



Phase voltage waveforms are expressed as

$$V_{a0} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{2V_s}{n\pi}\cos\frac{n\pi}{6}\sin n\left(\omega t + \frac{\pi}{6}\right)$$

$$V_{b0} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{2V_s}{n\pi}\cos\frac{n\pi}{6}\sin n\left(\omega t - \frac{\pi}{2}\right)$$

$$V_{c0} = \sum_{n=1,3,5\ldots}^{\alpha} \frac{2V_s}{n\pi}\cos\frac{n\pi}{6}\sin n\left(\omega t + \frac{5\pi}{6}\right)$$

Line voltage

$$V_{ab} = \sum_{n=6K+1}^{\alpha} \frac{3V_s}{n\pi}\sin n\left(\omega t + \frac{\pi}{3}\right)$$

where $K = 0, 1, 2, \ldots\ldots\ldots$

V_{bc}, V_{ca} can be obtained in a similar manner.

RMS value of fundamental phase voltages

$$V_{p1} = \frac{2V_s}{\sqrt{2}\pi}\cos\pi/6 = 0.3898V_s$$

RMS value of phase voltage

$$V_{\text{ph}} = \left[\frac{1}{\pi}\int_0^{2\pi/3}\left(\frac{V_s}{2}\right)^2 d\omega t\right]^{1/2}$$

$$= \sqrt{\frac{2}{3}}\frac{V_s}{2} = \frac{V_s}{\sqrt{6}} = 0.4082V_s$$

RMS value of fundamental line voltage

$$V_{L1} = \frac{3V_s}{\sqrt{2}\pi} = 0.6752V_s + = \sqrt{3}\ V_{p1}$$

RMS value of line voltage

$$V_L = \sqrt{3}V_{ph} = \frac{V_s}{\sqrt{2}} = 0.7071V_s$$

Example 4: A thyristor bridge converter (three-phase, fully controlled) is uses as a line commuted inverter to feed 30 kW power at 220 V to a 3-∅, 215v (l – l), 50 Hz AC mains. The RMS value of thyristor current is (assume DC link current as constant)

(A) 39.68 A (B) 119.05 A
(C) 68.7 A (D) 78.73 A

Solution: (D)

$$P = V_0 I_0$$

$$I_0 = \frac{P}{V_0} = \frac{30\times10^3}{220} = 136.3$$

RMS value of thyristor current $(I_{\text{TH}})_{\text{RMS}} = I_0/\sqrt{3}$

$$(I_{\text{TH}})_{\text{RMS}} = \frac{136.36}{\sqrt{3}} = 78.73 \text{ A}$$

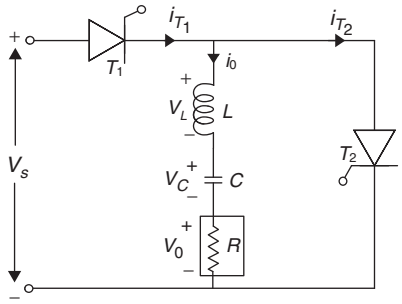
Example 5: Which among the following statement is TRUE regarding a three-phase VSI operated in 180° conduction mode?

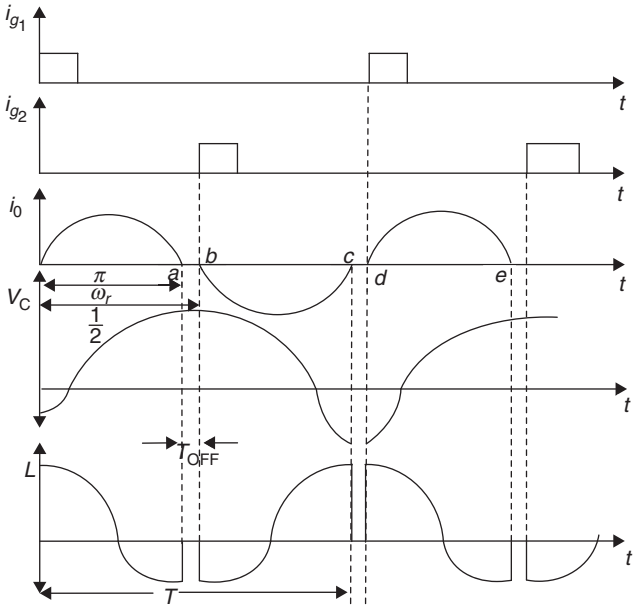
(A) Third harmonic component will not be present in line voltage but absent in phase voltage.
(B) Third harmonic component will not be present in phase voltage but absent in line voltage.
(C) Third harmonic component will be absent both in line and phase voltages.
(D) Both phase and line voltage will contain third harmonic component.

Solution: (C)

Series Inverter

- The time interval between the instant T_1 is turned off and instant T_2 is turned on is indicated by $T_{\text{OFF}} = ab$, where $T_{\text{OFF}} > t_{q,\text{ min}}$.





- After thyristor T_1 has commutated, upper plate of the capacitor attains positive polarity. Now when T_2 is turned on at instant b, capacitor begins to discharge and load current in the reverse direction builds up to some peak negative value and then decays to zero at instant C.
- After this time $T_{\text{OFF}} = cd$ must elapsed for T_2 to recover.
- At d, T_1 is again turned on and the process repeats. In this manner DC is converted to AC with the help of series inverter. $T_{\text{OFF}} = ab$ (or) cd is called the circuit turn-off time or dead zone time.
- The capacitor stores charge during one half cycle and releases the same amount of charge during the next half

cycle. Hence, the positive half cycle of current is identical with the negative half cycle of load current.

- In a practical series inverter, positive and negative half cycles may not be sine waves.

$$i(t) = \frac{V_S + V_\alpha}{\omega_r \cdot L} - e^{-\xi t} \sin \omega_r t$$

$\xi = \dfrac{R}{2L}$ is called damping factor

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ is resonant frequency is rad/sec

ω_r = circuit-ringing frequency in rad/sec

ω = operating (or) output frequency in rad/sec

Time period of oscillation $T_{osc} = \dfrac{\pi}{\omega_r}$

$$T_{osc} = \frac{\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$$

Output frequency

$$f = \frac{\frac{1}{2\pi}}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} + 2T_{OFF}}$$

Circuit-ringing frequency

$$f_r = \frac{1}{2\pi\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \text{ Hz i.e. } f < f_r$$

Voltage across inductor L

$$V_L = (V_s + V_{co}).\frac{\omega_0}{\omega_r}.e^{\theta t} \cos(\omega_r t + \psi)$$

where ω_0 = resonant frequency

$$\omega_0 = \sqrt{\omega_r^2 + \varepsilon^2}$$

$$\psi = \tan^{-1}\left(\frac{\xi}{\omega_r}\right)$$

- Series inverter produces an approximately sinusoidal waveform at high output frequency which is in the range of 200 Hz–100 kHz.

Parallel Inverter

- The basic inverter circuit for a single-phase parallel inverter, utilizing capacitor for its commutation is shown in Figure 1.
- It consists of two thyristors T_1 and T_2 and inductor L, an output transformer and a commutating capacitor C.
- Transformer turns ratio from each primary half to secondary winding is assumed unity. The output voltage and current are V_0 and i_0, respectively, L makes the source current constant at I_0.
- During the working of this inverter C, comes in parallel with the load via the transformer and hence it is called parallel inverter.

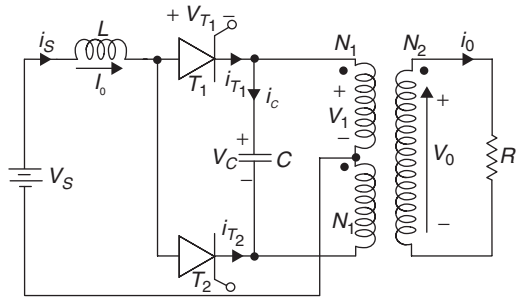


Figure 2

Operating Modes

Mode 1

In this mode thyristor T_1 is conducting and current flows in the upper half of primary winding thyristor T_2 is off.

- This current establishes magnetic flux that links both the halves of primary winding. As a result an emf, V_s is induced across upper as well as lower half of primary winding. In other words total voltage across primary winding is $2V_s$.
- This voltage charges the commutating capacitor C to a voltage of $2V_s$ with upper plate positive.
- Thyristor T_2 is forward biased through T_1 by the capacitor voltage $2V_s$. Eventually a steady state current I_o flows through V_s, L, T_1 and upper half of primary winding.

Mode 2

- At time $t = 0$, thyristor T_2 is turned on by applying a triggering pulse to its gate. At this time $t = 0$ capacitor voltage $2V_s$, appears as reverse bias across T_1, it is therefore turned off. A current I_o begins to flow through T_2, lower half of primary windings, V_s and L as shown.

  At the same time capacitor voltage $2V_s$ is applied across the total transformer primary and a capacitor current $-i_C$ is established. Negative sign before i_c means that current i_C flows opposite to its positive direction assumed initially.
- Before T_2 is on, i.e. at $t = 0^-$ mmf in the upper primary winding is I_0N_1 and zero in the lower primary winding. Soon after T_2 is on, i.e. at $t = 0^+$, mmfs linking both upper and lower halves cannot change suddenly. Therefore at $t = 0^+$, $-i_C = I_o$ such that mmf in the lower half remains zero and mmf in the upper half is equal to mmf at $t = 0^-$. After $t = 0^+$, capacitor C discharges and current I_C is such that it supplies load current i_0 and balances the primary and secondary ampere turns of the transformer.

  Capacitor current continues flowing till capacitor has charged from $+2V_s$ to $-2V_s$ at time $t = t_1$.
- Load voltage changes from V_s at $t = 0$ to $-V_s$ at $t = t_1$.

Mode 3

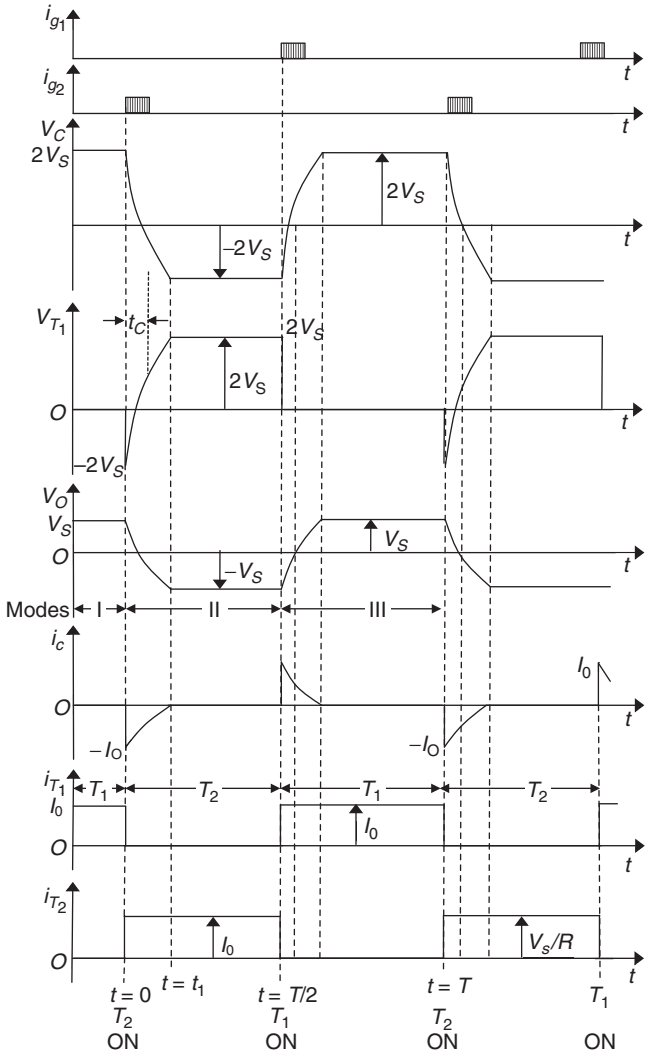
When capacitor has charged to $-2V_s$ with upper plate negative and lower plate positive, SCR T_1 may be turned on at any time. T_1 is triggered at $t = T/2$.

- Capacitor voltage $2V_s$ applies a reverse bias across T_2, it is therefore turned OFF. After T_2 is OFF, capacitor starts discharging current i_c is now positive. mmfs in the upper and lower halves remain unchanged from their valves just before $T/2$.

Current and Voltage Waveforms

When i_C decays to zero,

$$V_c = +2V_s,\ V_o = V_s,$$
$$i_{T1} = I_o = V_s/R.$$



At $t = 0+$ when T_2 is turned ON; $V_{T1} = -2V_s$, $i_C = -I_0$, $i_{T1} = 0$ and $i_{T2} = I_0$. As the turns ratio from whole primary to secondary winding is 2, the load voltage has half the amplitude of capacitor voltage. However, load voltage has the same waveform as the capacitor voltage.

$$V_c = 2V_s\left[2\exp\left(\frac{-n^2 t}{4RC}\right) - 1\right]$$

Circuit turn-off time, $t_c = \dfrac{4RC\ I_n 2}{n^2}$

Commutating capacitance, $C = \dfrac{n^2\, t_C}{4R.\ln 2}$

- Parallel inverter produces square and rectangular waveforms.

Inverter Control Methods and Techniques

Voltage Control in Single-phase Inverters

Inverter output voltage can be controlled by controlling the gain of the inverter.

- Inverter gain is defined as the ratio of output voltage to input DC voltage.
- The various methods for control of output voltage of inverters are listed as follows.
  1. External control of the AC output voltage
  2. External control of the DC input voltage
  3. Internal control of the inverter output voltage
- In the first two methods, extra circuits for the control of either DC input or AC output become necessary.

External Control of the AC Output Voltage

There are two possible methods to externally control the AC output voltage obtained from inverter output voltage terminal.

- The AC voltage control and
- The series inverter control

AC Voltage Control

In this method, an AC voltage controller is inserted between the output terminals of inverter and the input terminals of the load.

The load voltage is regulated through this voltage controller.

- This method gives rise to higher harmonic content in the output voltage, particularly when the output voltage from the AC voltage controller is a low level.
- This method is rarely employed except in the low power applications.

Series Inverter Control

This method of voltage control involves the use of two or more inverters in series.

- The series connection of inverters called multiple converter control does not augment the harmonic content even at low output voltage levels.

External Control of the DC Input Voltage through Variable DC Link

- In this method, the input DC voltage may be altered to vary the RMS voltage of the AC output.
- This can be achieved easily by using a phase-controlled thyristor bridge converter or a DC chopper.
- An LC filter is essential in between the variable DC source and the inverter.
- In this case, output voltage waveform and its harmonic content are not affected appreciably as the inverter output voltage is controlled through the adjustment of DC input voltage to the inverter.

Internal Control of the Inverter Voltage

The output voltage from an inverter can also be adjusted by incorporating a control mechanism within the inverter itself.

The most efficient method is by PWM Control.

PWM control

In this method, a fixed DC input voltage is given to the inverter and a controlled AC output voltage is obtained by adjusting the on and off periods of the inverter components.

The advantages are:

- Output voltage control can be obtained without any additional components.
- Lower-order harmonic can be eliminated or minimized along with their output voltage control.

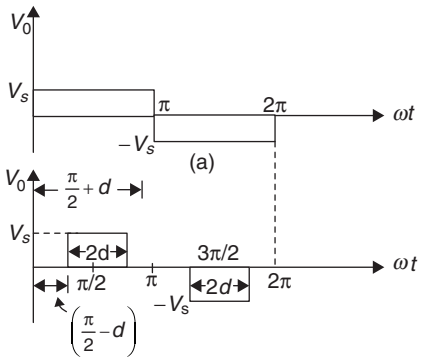
  As the higher-order harmonics can be filtered easily, the filtering requirements are minimized.
- These types of inverters are gradually taking over other types of inverters in industrial applications.
- PWM techniques are characterized by constant amplitude pulses. The width of these pulses is however modulated to obtain inverter output voltage control and to reduce its harmonic content.

PWM Techniques

- Forced commutation is essential.
- PWM differs from each other in the harmonic content in their respective output voltages.
- Thus, choice of a particular PWM technique depends upon the permissible harmonic content in the inverter output voltage.

Single-pulse Modulation

- The output voltage from single-phase full-bridge inverter is shown below.



- When this waveform is modulated, the output voltage is of the form shown in fig II.
- It consists of a pulse of width $2d$ located symmetrically located about $\pi/2$. The range of pulse width $2d$, varies from $0 < 2d < \pi$.
- The output voltage is controlled by varying the pulse width $2d$.
- The shape of output voltage wave shown is called quasi-square wave.

By Fourier analysis of waveform, we have

$$b_n = \frac{2}{\pi}\int_{(\pi/2-d)}^{(\pi/2+d)} V_s \sin\ n\omega t.\, d\omega t$$

$$b_n = \frac{4V_s}{\pi}\left[\sin\frac{n\pi}{2}\sin nd\right], a_n = 0$$

$$V_0 = \sum_{n=1,35\ldots}^{\alpha} \frac{4V_s}{\pi}\sin\frac{n\pi}{2}\sin nd\sin\left(n\omega t\right)$$

$$V_0 = \frac{4V_s}{\pi}\left[\sin d\sin\omega t - \frac{1}{3}\sin 3d\sin\omega t + \frac{1}{5}\sin 5d\sin 5\omega t\right]$$

Maximum value of fundamental component of output voltage

$$V_{o1m} = \frac{4V_s}{\pi}\Big|_{d=\pi/2}$$

For pulse width other than $2d = \pi$ rad, peak value of fundamental component $= \frac{4V_s}{\pi}\sin d$

To eliminate n^{th} harmonic from the inverter output voltage pulse width $2d = \frac{2\pi}{n}$

Peak value of n^{th} harmonic

$$V_{onm} = \frac{4V_s}{\pi}\sin nd$$

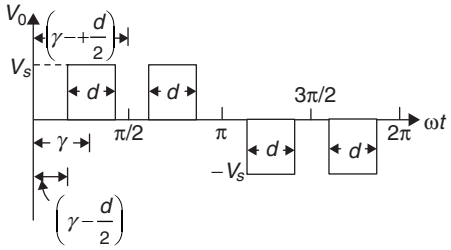
$$\frac{V_{onm}}{V_{o1m}} = \frac{\sin nd}{n}$$

RMS value of output voltage

$$V_{or} = \left[\frac{V_s^2 2d}{\pi}\right]^{1/2} = V_s\left[\frac{2d}{\pi}\right]^{1/2}$$

Multiple-pulse Modulation

- Multiple-pulse modulation (MPM) is an extension of single-pulse modulation.
- In MPM, several equidistant pulses per half cycle are used. For simplicity the effect of using two symmetrically spaced pulses per half cycle is investigated as shown in the waveform below.



By Fourier analysis of waveform,

$$b_n = \frac{2}{\pi}\int_0^{\pi} V_o \sin n\omega t \; d(\omega t)$$

$$= \frac{2}{\pi}\int_{(\gamma-d/2)}^{(\gamma+d/2)} V_s \sin n\omega t \; d(\omega t) \cdot 2$$

$$= \frac{4V_s}{\pi}\left[\cos n\omega t\right]_{\gamma-d/2}^{\gamma+d/2} = \frac{8V_s}{n\pi}\sin n\gamma \sin nd/2$$

$$V_0 = \sum_{n=1,3,4}^{\alpha} \frac{8V_s}{n\pi}\sin n\gamma \sin nd/2 \sin n\omega t$$

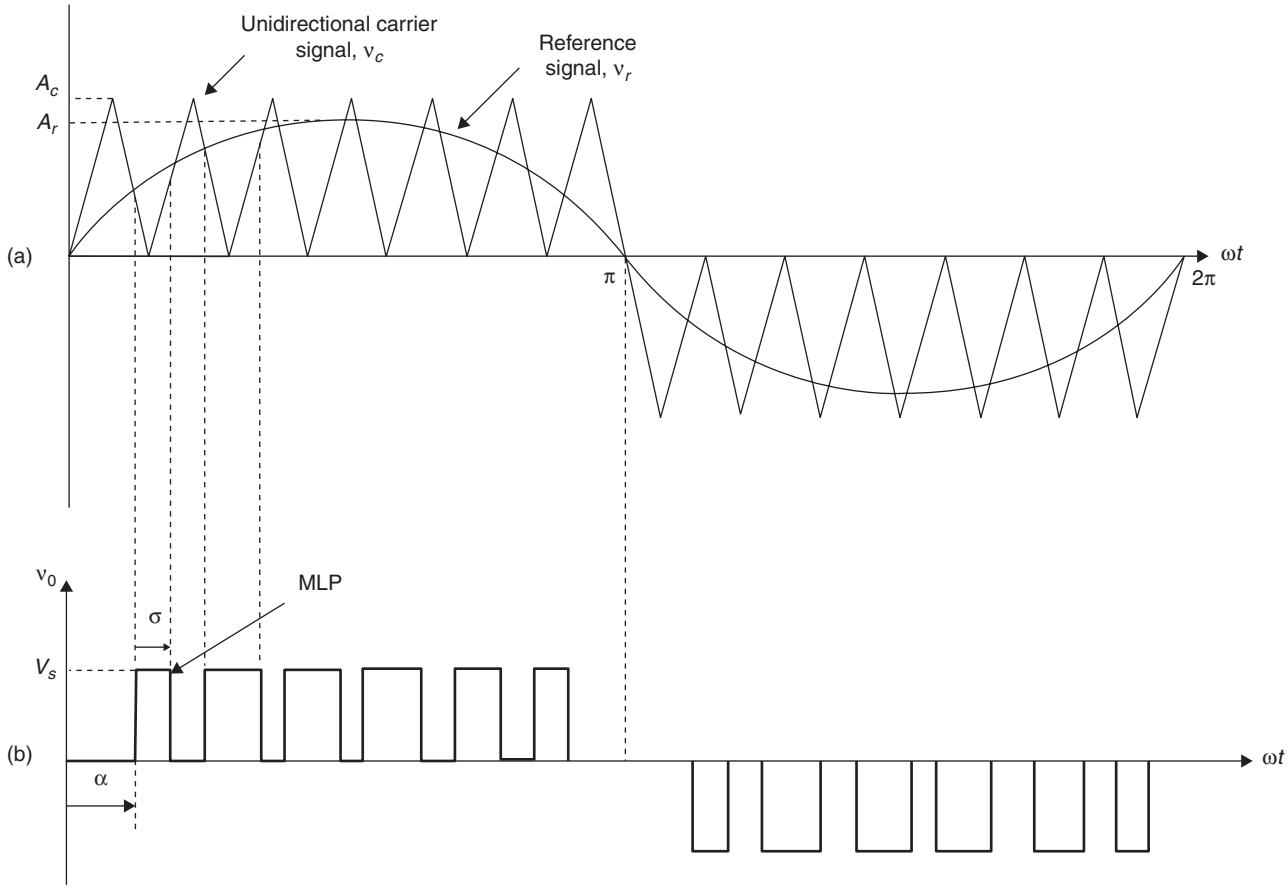
For n^{th} harmonic

$$V_n = \frac{8V_s}{n\pi}\sin n\,\gamma \sin\frac{nd}{2}$$

This equation shows that magnitude of V_n depends upon γ and d. This equation shows that when $\gamma = \pi/n$ (or) $d = 2\pi/n$, n^{th} harmonic can be eliminated from the output voltage. But this has the effect of reducing the fundamental component of output voltage.

Sinusoidal PWM

In this method of modulation, several pulses per half cycle are used as in case of MPM. In MPM, the pulse width is equal for all the pulses. But in SPWM, the pulse width is a sinusoidal function of the angular position of the pulse in a cycle as shown below.



- For realizing SPWM, a high-frequency triangular carrier wave V_c is compared with a sinusoidal reference. Wave V_r of the desired frequency. The intersection of V_c and V_r wave determines the switching instants and commutation of the modulated pulse. In the waveform figure, V_c is the peak value of triangular carrier wave and V_r that of the reference or modulating signal.

- When triangular carrier wave has its peak coincident with zero of the reference sinusoid, there are $N = \frac{f_c}{2f}$ pulses per half cycle.
- In case, zero of the triangular wave coincides with zero of the reference sinusoid there are $(N-1)$ pulses per half cycle.
- The ratio V_r/V_c is called modulation index (MI) and it controls the harmonic content of the output voltage waveform.
- The magnitude of fundamental component of output voltages is proportional to MI but MI can never be more than unity.
- Thus output voltage is controlled by varying MI.

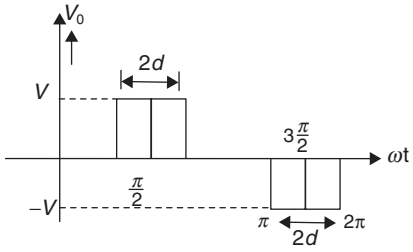
Example 6: A three-phase inverter employs a PWM switching scheme to
(A) Increase the high-order harmonics and minimize the low-order harmonics.
(B) Reduce the DC side load.
(C) Minimize the total harmonic distortion with modes filtering.
(D) Increase the battery life.

Solution: (A)

Example 7: When a triangular PWM control is applied to a three-phase BJJ base VSI
(A) DC side would have low-order harmonic voltages
(B) DC side would contain high-order harmonic voltages
(C) AC side would contain high-order harmonic voltages
(D) AC side would contain low-order harmonic voltages

Solution: (B)

Example 8: The figure shown below represents a single pulse of width $2d$, generated by a single-phase inverter generated in PWM mode. For a pulse width of 130° it was observed that the output voltage is free from 5th harmonic. The percentage of third harmonic that is present in the output voltage would be



(A) 24% (B) 19%
(C) 34.5% (D) 12.33%

Solution: (C)

Given $2d = 130° \Rightarrow d = 60°$

When $2d = \pi$

$$(V_0)_{max} = \frac{4V_s}{\pi}$$

when $2d \neq \pi$

Peak value of fundamental waveform

$$(V_0)_1 = \frac{4V_s}{\pi}\sin d \sin(\pi/2)$$

Peak value of nth harmonic

$$(V_0)_n = \frac{4V_s}{n\pi}\sin(nd)\sin(n\pi/2)$$

Peak value of third harmonic

$$(V_0)_3 = \frac{4V_s}{3\pi}\sin(3d)\sin(3\pi/2) \text{ at } 2d = \pi$$

$$\left(\frac{V_{03}}{V_{01}}\right) = \frac{\frac{4V_s}{3\pi}\sin(3\times 65)^\circ \sin 3\pi/2}{\frac{4V_s}{\pi}}$$

$$= \frac{4\sin(195)\sin 270}{3} = 0.345$$

$$= 34.5\%$$

Example 9: A single-phase VSI is operated in a single PWM mode with a width of 140° in each half cycle. The total harmonic distortion of the AC voltage waveform is
(A) 31.83% (B) 48.42%
(C) 65.65% (D) 28.63%

Solution: (D)

$2d = 140°$

$d = 70°$

Peak value of fundamental component

$$V_1 = \frac{4V_s}{\pi}\sin d \sin \pi/2$$

RMS value of fundamental component

$$(V_1)_{RMS} = \frac{V_1}{\sqrt{2}} = \frac{\frac{4V_s}{\pi}\sin 70 \sin \pi/2}{\sqrt{2}}$$

$$(V_1)_{RMS} = 0.846\ V_s$$

RMS value of output voltage

$$(V_0)_{RMS} = V_s\frac{2d}{\pi}$$

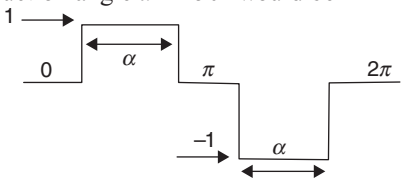
$$= V_s\sqrt{\left[\frac{140}{\pi}\times\frac{\pi}{180}\right]} = 0.88V_s$$

$$\text{T.H.D.} = \sqrt{\frac{(V_0)^2_{RMS} - (V_1)^2_{RMS}}{(V_1)^2_{RMS}}}$$

$$= \sqrt{\frac{(0.88V_s)^2 - (0.846V_s)^2}{(0.846V_s)^2}} = 0.2863$$

$$= 28.63\%$$

Example 10: The output waveform of a periodic output voltage of an inverter is as shown in the figure. The RMS value of fundamental component of the output voltage, when conduction angle $d = 150°$ would be



(A) 0.87 V (B) 0.78 V (C) 1.10 V (D) 0.90 V

Solution: (A)

$$V_{ON} = \frac{4Vs}{n\pi} \sin nd \sin n\pi/2$$

RMS value of fundamental component

$$(V_0)_{RMS} = \frac{V_{01}}{\sqrt{2}} = \frac{\frac{4\times1}{\pi}\sin 75 \sin 90}{\sqrt{2}}$$

$$= 0.87 \text{ V}$$

EXERCISES

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. A single-phase half-bridge inverter with supply voltage of 80 V feeds a resistive load of 305 Ω. The fundamental component of output voltage will have an RMS of

(A) $\frac{160}{\sqrt{2}\pi}V$ (B) $\frac{80}{\sqrt{2}\pi}V$

(C) $\frac{320}{\sqrt{2}\pi}V$ (D) 40 V

2. A single-phase half-bridge inverter feeds a resistive load. The percentage of 7th harmonic component with respect to fundamental component is

(A) 20% (B) $28\frac{2}{7}\%$

(C) $14\frac{2}{7}\%$ (D) 10%

3. If the RMS of output voltage at fundamental frequency of a single-phase full-bridge inverter is $\frac{72}{\sqrt{2}\pi}V$, the supply voltage at input is

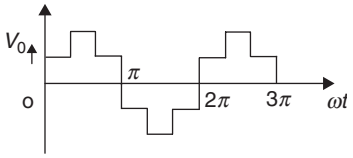
(A) 36 V (B) 72 V
(C) 9 V (D) 18 V

4. The ratio of output powers of a single-phase full-bridge inverter to that of a half-bridge inverter for the same supply voltage and load current when purely resistive load is employed is

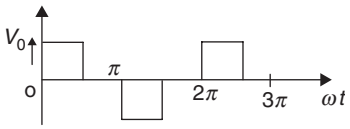
(A) 2:1 (B) 4:1
(C) 8:1 (D) 1:4

5. The phase voltage of a three-phase VSI operating in 180° conduction mode is of the nature.

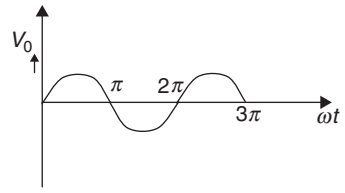
(A)



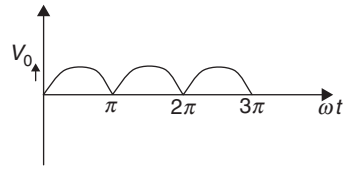
(B)



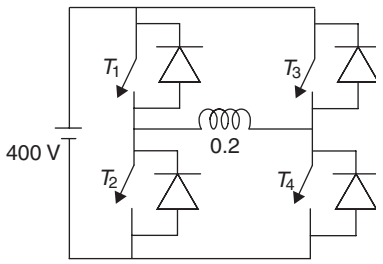
(C)



(D)



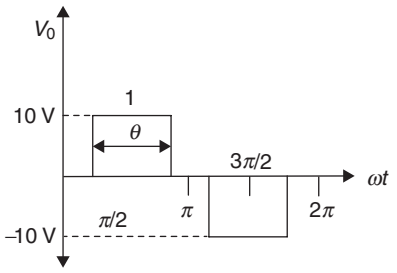
6. The figure shows a single-phase full-bridge inverter operating in square-wave mode at 100 Hz feeding a purely inductive load. The peak value of inductor current amounts to:



(A) 20 A
(B) 2.5 A
(C) 5 A
(D) 10 A

7. A single-phase inverter with supply voltage 10 V operating on single-pulse modulation has the output voltage waveform shown in figure.

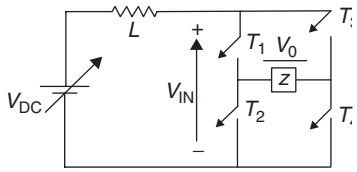
The RMS of fundamental component of output voltage when $\theta = 140°$ is



(A) 8.46 V (B) 5.16 V
(C) 5.78 V (D) 3.07 V

8. The seventh harmonic component can be eliminated from the output of a single-phase inverter operated with single-pulse modulation if pulse width amounts to:
(A) 103° (B) 72°
(C) 120° (D) 90°

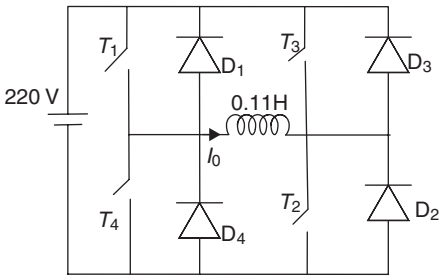
9. For a CSI feeding a capacitive load, the frequency of output voltage V_0 is F. The frequency of voltage V_{IN} will be



(A) F (B) 2 F
(C) $\frac{F}{2}$ (D) 4 F

10. If a single-phase CSI feeds a capacitive load, the waveforms of output voltage and output current, respectively, are:
(A) Square; square (B) Triangular; square
(C) Square; triangular (D) Triangular; sine

11. The figure below shows a single-phase VSI inverter operated at 60 Hz in 180° square-wave mode and is feeding a purely inductive load. Assuming that the load current does not have any DC components, the peak value of inductor current will be



(A) 32 A (B) 16 A
(C) 8 A (D) 6.23 A

12. A three-phase VSI feeds an induction motor at the base speed and rated voltage. In order to reduce the motor speed by regenerative braking, which of the following actions should be taken?
(A) Reduce the inverter output frequency.
(B) Increase the output voltage.
(C) Reduce the inverter output voltage.
(D) Increase the inverter output frequency.

13. Given a three-phase, 4 pole 600 V, 60 Hz synchronous motor fed by a three-phase VSI. In order to operate the motor at 1200 rpm under constant torque region, values of stator voltage and frequency, respectively, are:
(A) 600 V; 40 Hz (B) 400 V; 40 Hz
(C) 600 V; 60 Hz (D) 400 V; 60 Hz

14. A three-phase VSI feeds a 500 V, 50 Hz, 4-pole synchronous motor drive. The motor drives a fan load of 400 Nm at synchronous speed. The load torque at 1200 rpm will be:
(A) 320 Nm (B) 200 Nm
(C) 256 Nm (D) 400 Nm

15. Which of the following inverter control schemes allows simultaneous control of output voltage and output frequency in three-phase VSI?
(A) Sinusoidal PWM scheme
(B) 180° conduction mode
(C) 120°conduction mode
(D) All of the above

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. A single-phase bridge inverter using thyristor switches feeds a resistive load. For this operation,
(A) Forced commutation is employed.
(B) Natural commutation of thyristors occurs.
(C) The feedback diodes conduct.
(D) Commutation method will depend on magnitude of load current.

2. The output line voltage of a three-phase VSI operating under 180° conduction mode is free from:
(A) Even harmonics
(B) Odd harmonics
(C) Even and triplen harmonics
(D) Third-order harmonics

3. For a three-phase VSI.
(A) Shoot through problem is inherent in the 120° conduction mode.
(B) No shoot through problem is likely to occur in 120° conduction mode.
(C) Shoot through problem never occurs in 180° conduction mode.
(D) Sufficient dead band is inherently present in 180° conduction mode to prevent shoot through fault.

4. The ratio of RMS of third harmonic component to the fundamental component in the line voltage of a three-phase VSI in 180° conduction mode is:
(A) 1:0.33 (B) 0.33:1
(C) 0.9:1 (D) Zero

5. The advantages of using PWM switching scheme in VSI is
(A) Regulation of output voltage and frequency easily.
(B) Reduction in size of filter
(C) Reduction in harmonics
(D) All of the above

6. Which among the following is true?
(A) Feedback diodes are essential for VSI operation.
(B) CSI requires feedback diodes for proper operation.
(C) MOSFETs cannot be used in a VSI.
(D) None of the above

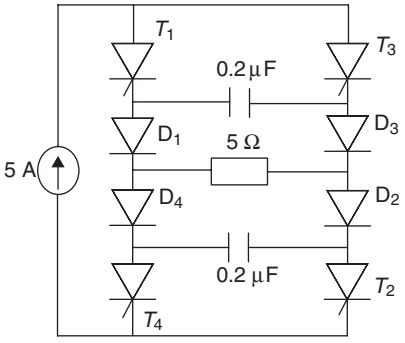
7. On operating a single-phase inverter under single-pulse modulation with a pulse width of 144°.
(A) 7^{th} harmonic will be eliminated.
(B) third harmonic will be eliminated.
(C) 5^{th} harmonic will be eliminated.
(D) 11^{th} harmonic will be eliminated.

8. A single-phase inverter operates with single pulse PWM with supply voltage 10 V and pulse width 120°. What will be the ratio of fifth harmonic component to the fundamental component?
(A) 0.2 (B) 0.4
(C) 0.33 (D) 0.9

9. Which of the following statements is true?
(A) CSI has inherent protection against short circuit current.
(B) CSI needs feedback diodes.
(C) CSI is not suitable for regenerative braking of a drive.
(D) None of the above.

10. Load commutation can be employed in a CSI if the load power factor is
(A) Leading
(B) Lagging
(C) Unity
(D) Does not depend on load power factor

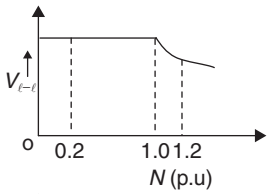
11. The figure shows a CSI feeding a resistive load. The circuit turn off line will be



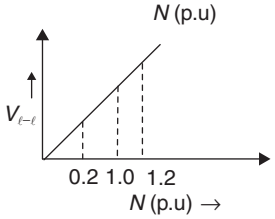
(A) 1.386 μs
(B) 0.693 μs
(C) 2.8 μs
(D) 0.347 μs

12. To control the speed of a three-phase induction motor fed by a three-phase VSI in constant torque region from 0.2 to 1.2 lines base speed, the variation of line to line voltage of VSI with speed will be of the nature.

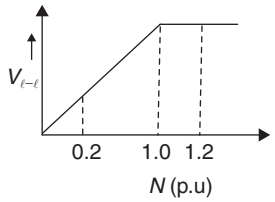
(A)



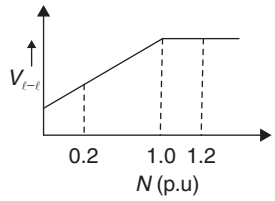
(B)



(C)



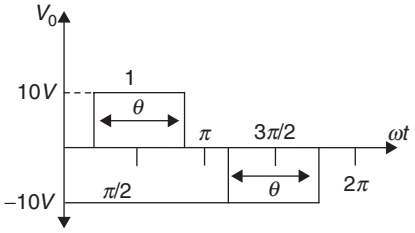
(D)



13. In order to operate a three-phase induction motor fed by a three-phase VSI in constant torque mode up to base speed,
(A) The air gap flux in the machine should be kept constant.
(B) The air gap flux should be increased with increase in speed.
(C) The air gap flux should be reduced with reduction in speed.
(D) Air gap flux has no bearing on this method of speed control.

14. If a three-phase VSI is operating in 180° conduction mode the output voltage will be controlled by:
(A) Varying switching frequency
(B) Varying input voltage and switching frequency together
(C) Varying input DC voltage alone
(D) Varying both output voltage and frequency

15. The figure shows the output voltage of a single-phase inverter. If $\theta = 150°$, the RMS of the output voltage is



(A) 2.362 V
(B) 5.625 V
(C) $\frac{10}{\sqrt{2V}}$
(D) 9.128 V

Previous Years' Questions

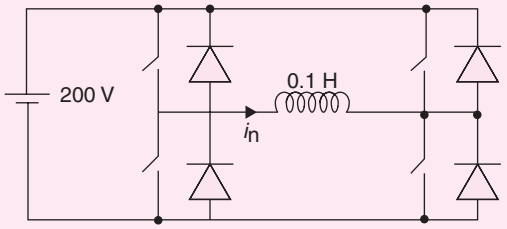
1. A VSI is used to control the speed of a three-phase, 50 Hz, squirrel cage induction motor. Its slip for rated torque is 4%. The flux is maintained at rated value. If the stator resistance and rotational losses are neglected, then the frequency of the impressed voltage to obtain twice the rated torque at starting should be **[2007]**

(A) 10 Hz (B) 5 Hz
(C) 4 Hz (D) 2 Hz

2. A 3-phase VSI is operated in 180° conduction mode. Which one of the following statements is true? **[2008]**

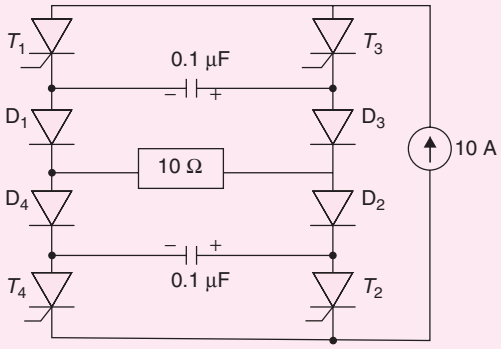
(A) Both pole voltage and line voltage will have third harmonic components.
(B) Pole voltage will have third harmonic component but line voltage will be free from third harmonic.
(C) Line voltage will have third harmonic component but pole voltage will be free from third harmonic.
(D) Both pole voltage and line voltage will be free from third harmonic components.

3. A single-phase VSI is feeding a purely inductive load as shown in the figure. The inverter is operated at 50 Hz in 180° square-wave mode. Assume that the load current does not have any DC component. The peak value of the inductor current i_0 will be **[2008]**



(A) 6.37 A (B) 10 A
(C) 20 A (D) 40 A

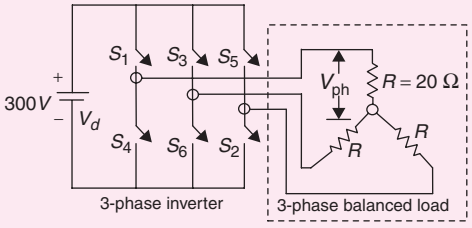
4. The CSI, shown in figure, is operated by alternately turning on thyristor pairs (T_1, T_2) and (T_3, T_4). If the load is purely resistive, the theoretical maximum output frequency obtainable will be **[2009]**



(A) 125 kHz (B) 250 kHz
(C) 500 kHz (D) 50 kHz

Common Data for Questions 5 and 6

In the 3-phase inverter circuit shown, the load is balanced and the gating scheme is 180°-conduction mode. All the switching devices are ideal.



5. The RMS value of load phase voltage is **[2012]**

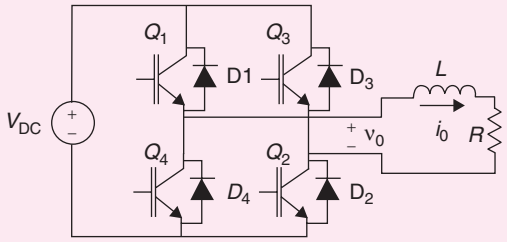
(A) 106.1 V (B) 141.4 V
(C) 212.2 V (D) 282.8 V

6. If the DC bus voltage $V_d = 300$, the power consumed by 3-phase load is **[2012]**

(A) 1.5 kW (B) 2.0 kW
(C) 2.5 kW (D) 3.0 kW

Common Data for Questions 7 and 8

The VSI shown in the figure below is switched to provide a 50 Hz, square-wave AC output voltage (V_0) across and R–L load. Reference polarity of V_0 and reference direction of the output current i_0 are indicated in the figure. It is given that $R = 3\ \Omega$, $L = 9.55$ mH.



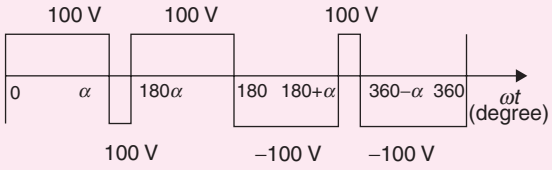
7. In the interval when $v_0 < 0$ and $i_0 > 0$, the pair of devices which conducts the load current is **[2013]**

(A) Q_1, Q_2 (B) Q_3, Q_4
(C) D_1, D_2 (D) D_3, D_4

8. Appropriate transition, i.e., zero voltage switching (ZVS)/zero current switching (ZCS) of the IGBTs during turn-on/turn-OFF is **[2013]**

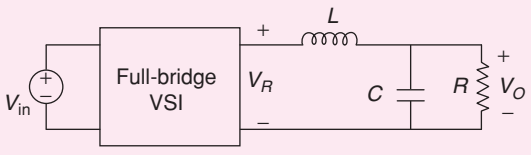
(A) ZVS during turn-OFF
(B) ZVS during turn-ON
(C) ZCS during turn-OFF
(D) ZCS during turn-ON

9. The figure shows one period of the output voltage of an inverter. α should be chosen such that $60° < \alpha < 90°$. If RMS value of the fundamental component is 50 V, then α in degree is _____ **[2014]**



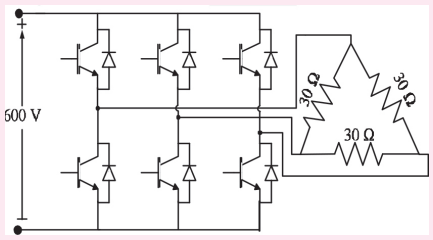
10. The single-phase full-bridge voltage source inverter (VSI), shown in figure, has an output frequency of 50 Hz. It uses unipolar pulse width modulation with switching frequency of 50 kHz and modulation index of 0.7. For $V_{in} = 100$ V DC, $L = 9.55$ mH, $C = 63.66$ μF, and $R = 5$ |, the amplitude of the fundamental component in the output voltage V_o (in Volts) under steady − state is _____. **[2015]**

11. A 3-phase 50 Hz square wave (6-step) VSI

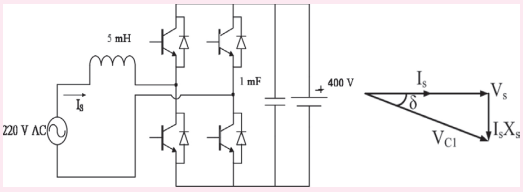


feeds a 3-phase, 4 pole induction motor. The VSI line voltage has a dominant 5th harmonic component. If the operating slip of the motor with respect to fundamental component voltage is 0.04, the slip of the motor with respect to 5th harmonic component of voltage is _____. **[2015]**

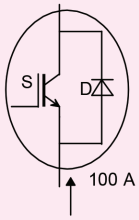
12. A three-phase Voltage Source Inverter (VSI) as shown in the figure is feeding a delta connected resistive load of 30Ω/phase. If it is fed from a 600V battery, with 180° conduction of solid-state devices, the power consumed by the load, in kW, is _____. **[2016]**



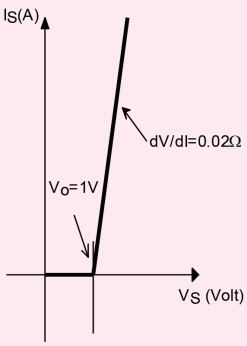
13. A single-phase bi-directional voltage source converter (VSC) is shown in the figure below. All devices are ideal. It is used to charge a battery at 400V with power of 5kW from a source $V_s = 220$ V(rms), 50HZ sinusoidal AC mains at unity p.f. If its AC side interfacing inductor is 5mH and the switches are operated at 20KHz, then the phase shift (δ) between AC mains voltage (V_s) and fundamental AC rms VSC voltage (V_{c1}), in degree, is _____. **[2016]**



14. A steady dc current of 100 A is flowing through a power module (S, D) as shown in Figure (a). The V-I characteristics of the IGBT (S) and the diode (D) are shown in Figures (b) and (c), respectively y. The conduction power loss in the power module (S, D), in watts, is _____. **[2016]**

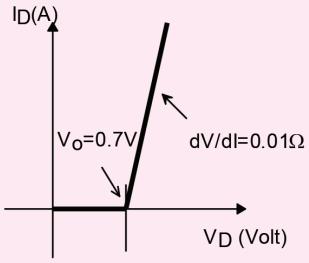


(a)



V-I characteristic of IGBT

(b)



V-I characteristic of diode

(C)

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** C	**3.** D	**4.** A	**5.** A	**6.** C	**7.** A	**8.** A	**9.** B	**10.** B
11. C	**12.** A	**13.** B	**14.** C	**15.** A					

Practice Problems 2

1. A	**2.** C	**3.** B	**4.** D	**5.** D	**6.** A	**7.** C	**8.** A	**9.** A	**10.** A
11. B	**12.** D	**13.** A	**14.** C	**15.** D					

Previous Years' Questions

1. C	**2.** D	**3.** B	**4.** B	**5.** B	**6.** D	**7.** D	**8.** D	**9.** 77.15
10. 63.05	**11.** 1.2	**12.** 24	**13.** 9.21	**14.** 170 W				

Chapter 5

AC Voltage Converters (OR) Controllers and Electric Drives

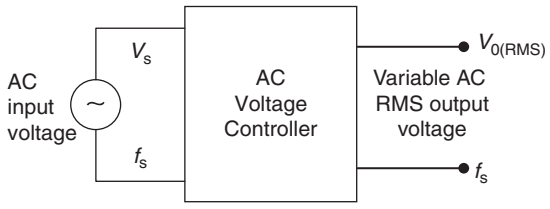
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- AC voltage controllers
- Single-phase half-wave AC voltage controller
- Single-phase full-wave AC voltage controller (AC regulator) or rms voltage controller with resistive load
- Single-phase full-wave AC voltage controller with RL load
- Two-stage sequence control
- Cyclo converters
- Electric drives
- Basic concept of speed control
- Controlling effective rsesistance in the rotor circuit
- Single-phase dual converter drives

AC Voltage Controllers

AC voltage controllers (AC line voltage controller) are employed to vary the RMS value of the alternating voltage applied to a load circuit by introducing thyristors between the load and a constant voltage AC source. The RMS value of alternating voltage applied to a load circuit is controlling the thyristors in the AC voltage controller circuits.



- An AC voltage controller is a type of thyristor power converter which is used to convert a fixed voltage, fixed frequency AC input supply to obtain a variable voltage AC output.

The RMS value of the AC output voltage and the AC power flow to the load is controlled by varying (adjusting) the trigger angle α.

Types of AC Voltage Controllers

They are classified into two types based on the type of input AC supply.

- Single-phase AC voltage controllers.
- Three-phase AC voltage controllers.
- Single-phase AC voltage Controllers operate with single-phase AC supply voltage of 230 V RMS at 50 Hz.
- Three-phase AC voltage controllers operate with three-phase AC supply of 400 V RMS at 50 Hz. Supply frequency. Each type of controller may be subdivided into
  - Unidirectional or half wave AC controller
  - Bidirectional or full wave AC controller

They are commonly used for

- Lightning/illumination control in AC power circuits.
- Induction heating.
- Industrial heating and domestic heating.
- Transformers tap changing (on load transformer tap changing).
- Speed control of induction motor (single-phase and poly-phase conduction).
- AC magnet controls.

AC Voltage Control Techniques

There are two different types of thyristor control used in practice to control the AC power flow.

- Phase angle control
- ON–OFF control

In ON–OFF control technique thyristors are used as switches to connect the load circuit to the AC supply (source) for a few cycles of the input AC supply and then to disconnect if for few input cycles. The thyristors thus act as high speed contactor (or switch).

Phase Control Technique

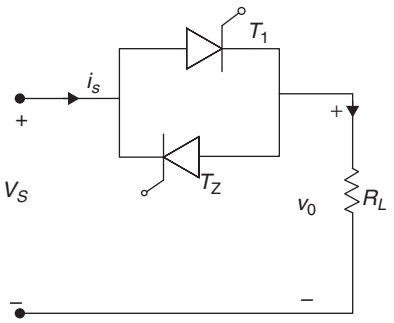
In phase control, the thyristors are used as switches to connect the load circuit to the input AC supply, for a part of every input cycle.

- By controlling the phase angle or the trigger angle α (delay angle), the output RMS voltage across the load can be controlled.
- They use AC line commutation or AC phase commutation.
- Since the input is AC, the commutation is line commutation.
- For applications up to 400 Hz, Triacs are more commonly used.
- Since no commutation circuit, they are more commonly used.
- Since no commutation circuit, they are very simple.

Principle of ON–OFF Control Technique (Integral Cycle Control)

The thyristors switches T_1 and T_2 are turned on by applying appropriate gate trigger pulses to connect the input AC supply to load for n number of input cycle during the time interval t_{ON}.

The thyristor switches T_1 and T_2 are turned off by blocking the gate trigger pulses for 'm' number of input cycles during the time interval t_{OFF}.



n = two input cycles – t_{ON}

m = one input cyclo – t_{OFF}

$$V_s = V_m \sin\omega t$$

$$= \sqrt{2}V_s \sin\omega t$$

V_s = RMS value of input AC supply

$$= \frac{V_m}{\sqrt{2}} = \text{RMS phase supply voltage}$$

$$t_{ON} = n \times T$$

$$t_{OFF} = m \times T$$

$$T = \frac{1}{F}$$

Output RMS voltage $V_{0(RMS)} = V_{i(RMS)}\sqrt{\frac{t_{ON}}{T_0}}$

$V_{i(RMS)}$ = input RMS supply voltage (V_s)

$$\frac{t_{ON}}{T_0} = \frac{t_{ON}}{T_{ON}+T_{OFF}} = \frac{nT}{nT+mT}$$

$$= \frac{n}{n+m} = k = \text{duty cycle}$$

$$V_{0(RMS)} = V_s\sqrt{\frac{n}{(n+m)}} = V_s\sqrt{k}$$

Duty Cycle: $K = \frac{n}{m+n}$

RMS load current

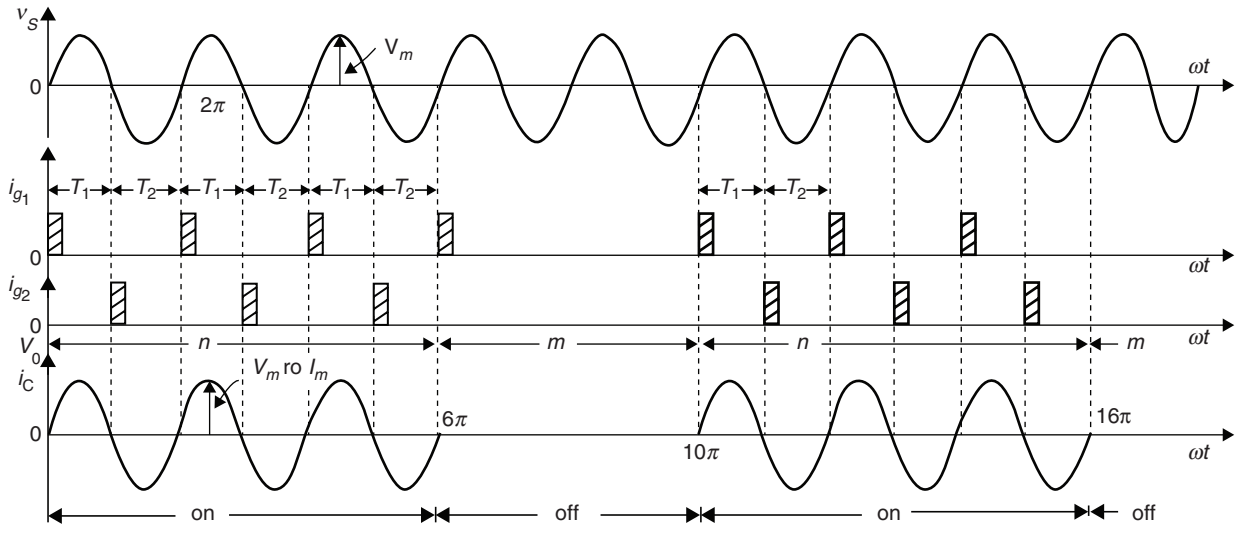
$$I_{0(RMS)} = \frac{V_{0(RMS)}}{Z} = \frac{V_{0(RMS)}}{R_L}$$

Output AC (Load) Power:

$$P_0 = I^2_{0(RMS)} \times R_L$$

Input Power Factor:

$$PF = \frac{P_0}{VA} = \frac{P_0}{V_S I_S}$$



$$\text{P.F.} = \sqrt{K} = \sqrt{\frac{n}{m+n}}$$

The average current of thyristor I_T (avg)

$$I_T = \frac{I_m n}{\pi(m+n)} = \frac{KI_m}{\pi}$$

RMS current of thyristor $I_{T_{RMS}} = \frac{I_m}{Q}\sqrt{k}$

Solved Examples

Example 1: A 1-ϕ AC voltage controller working on ON–OFF control technique has supply voltage of 230 V, RMS 50 Hz load = 50 Ω, The controller is ON for 30 cycles and OFF for 40 cycles. Calculate, ON and OFF time intervals RMS output voltage Input p.f., average and RMS thyristor currents.

Solution: $V_{(RMS)} = 230$ V

$$V_m = \sqrt{2}\times 230$$
$$= 325.269 \text{ V}$$
$$T = \frac{1}{f} = \frac{1}{50 \text{ Hz}} = 0.02 \text{ s}$$
$$T = 20 \text{ ms}$$
$$n = 30 \text{ ON}$$
$$M = 40 \text{ OFF}$$
$$t_{ON} = n \times T = 30 \times 20 \text{ ms} = 600 \text{ ms} = 0.6 \text{ s}$$
$$T_{OFF} = m \times T = 40 \times 20 \text{ ms} = 800 \text{ ms} = 0.8 \text{ s}$$

Duty cycle $k = \frac{n}{m+n} = \frac{30}{40+30} = 0.4285$

RMS output voltage

$$V_{0(RMS)} = V_{i\,(RMS)}\sqrt{\frac{n}{n+m}}$$

$$V_{0(RMS)} = 230 \text{ V}\sqrt{3/7} = 150.570 \text{ V}$$

$$I_{0(RMS)} = \frac{V_{o_{RMS}}}{z} = \frac{V_{o_{RMS}}}{R_L} = \frac{150.570}{50\ \Omega} = 3.01 \text{ A}$$

$$P_0 = I^2_{0\,(RMS)} \times R_L = (3.0114)^2 \times 50 = 453.426 \text{ W}$$

Input p.f. $= \sqrt{K} = \sqrt{\frac{n}{m+n}} = 0.6546$

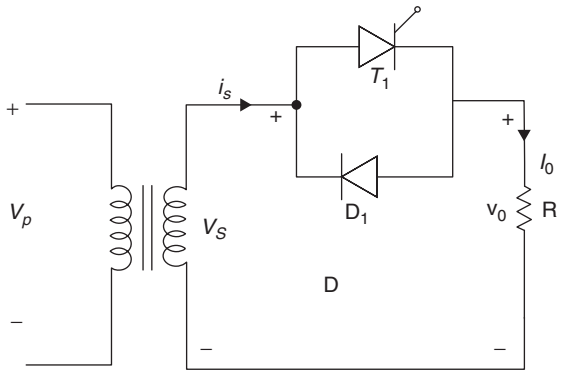
Average thyristor current rating

$$I_{T(Avg)} = \frac{k \times I_m}{\pi} = 0.88745 \text{ A}$$

RMS current rating of Thyristor

$$I_T \text{ (RMS)} = \frac{I_m}{2} \times \sqrt{k} = 2.12986 \text{ A}$$

Single-phase Half-wave AC Voltage Controller

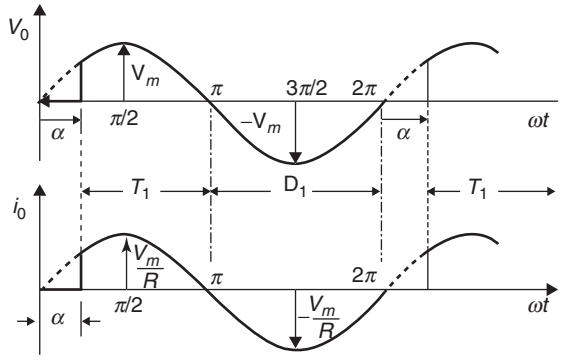


The half wave AC controller uses one thyristor and one diode connected in parallel across each other in opposite direction that is anode of thyristor T_1 is connected to the cathode of diode D_1 and the cathode of T_1 to the anode of D_1.

- The output voltage across the load V resistor 'R' and hence the AC power flow to the load is controlled by varying the trigger angle α.
- The thyristor T_1 is forward biased during the positive half cycle of input AC supply. It can be triggered and made to conduct by applying a suitable gate trigger pulse only during the positive half cycle of input supply. When T_1 is triggered it conducts and the load current flows through the thyristor T_1, the load and through the transformer secondary winding.

When the supply voltage reverses and becomes negative the diode D_1 becomes forward biased and hence turns ON and conducts.

The load current flows in the opposite direction during $\omega t = \pi$ to 2π radians when D_1 is ON and the output voltage follows the negative half cycle of input supply π to 2π.



Equations

Input AC supply voltage across the transformer secondary winding:

$$V_s = V_m \sin\omega t$$
$$V_s = V_{m\,(RMS)} = \frac{V_m}{\sqrt{2}}$$
$$= \text{RMS Value of secondary supply voltage}$$

Output load voltage

$$V_0 = V_L = 0 \text{ for } \omega t = 0 \text{ to } \infty$$
$$V_0 = V_L = V_m \sin\omega t \text{ for } \omega t = \alpha \text{ to } 2\pi$$

Output load current

$$I_0 = i_L = \frac{V_o}{R_L} = \frac{V_m \sin \omega t}{R_L} \text{ for } \omega t = \infty \text{ to } 2\pi$$
$$I_0 = i_L = 0 \text{ for } \omega t = 0 \text{ to } \infty$$

the RMS values of the output voltage $V_{0(RMS)}$

$$V_{0(RMS)} = \sqrt{\frac{1}{2\pi}\left[\int_{\alpha}^{2\pi} V_m^2 \sin^2 \omega t d(\omega t)\right]}$$

$$V_{0(RMS)}\ V_s = \sqrt{\frac{1}{2\pi}(2\lambda - \alpha) + \frac{\sin 2\alpha}{2}}$$

where $V_{1(RMS)} = V_s = \frac{V_m}{\sqrt{2}} =$ RMS value of input supply voltage (across the transformer secondary winding)

- Half wave AC controller has the drawback of limited range RMS output voltage control.

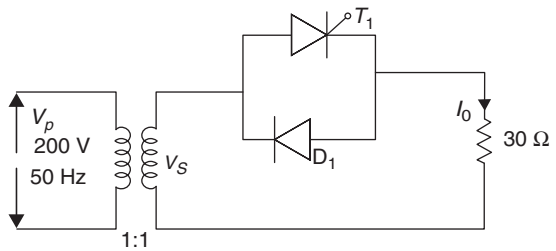
The average value (DC value) of output voltage

$$V_{0(DC)} = \frac{1}{2\pi}\int_{\alpha}^{2\pi} V_m \sin\omega t d(\omega t)$$
$$V_{0(DC)} = \frac{V_m}{2\pi}\int_{\alpha}^{2\pi} \sin\omega t d(\omega t)$$
$$V_{DC} = \frac{V_m}{2\pi}[\cos\alpha - 1]$$
$$V_m = \sqrt{2}V_s$$
$$\therefore \quad V_{DC} = \frac{\sqrt{2}V_s}{2\pi}(\cos\alpha - 1)$$

When α is varied from o to π, V_{DC} varies from o to $\frac{-V_m}{\pi}$

- The half wave AC voltage controller using a single thyristor and a single diode provides control on the thyristor and a single diode provides control on the thyristor only in one half cycle of the input supply. Hence, AC power flow to the load can be controlled only in one half cycle.

Common Data for Examples 2 and 3:



A single-phase voltage controller has the specifications as shown in the above figure. If T_1 is triggered at $\alpha = 30°$. Then

Example 2: RMS value of output voltage would be
(A) 218.47 V (B) 198 V
(C) 200 V (D) 226.67 V

Solution: (B)

$$\frac{V_P}{V_S} = \frac{N_P}{N_S} = \frac{1}{1}$$
$$V_s = V_P = 200 \text{ V}$$
$$(V_0)_{RMS} = \sqrt{\frac{1}{2\pi}\int_{\alpha}^{\pi} V_m^2 \sin^2 \omega t\, d\omega t}$$
$$V_0 = V_m\sqrt{\frac{1}{2\pi}\left[(2\pi - \alpha) + \frac{\sin 2\alpha}{2}\right]}$$
$$= V_m\sqrt{\frac{1}{2\pi}[5.76 + 0.433]}$$
$$= 200 \times 0.99 = 198 \text{ V}$$

Example 3: The input power factor would be
(A) 0.949 (B) 0.75
(C) 0.99 (D) 0.86

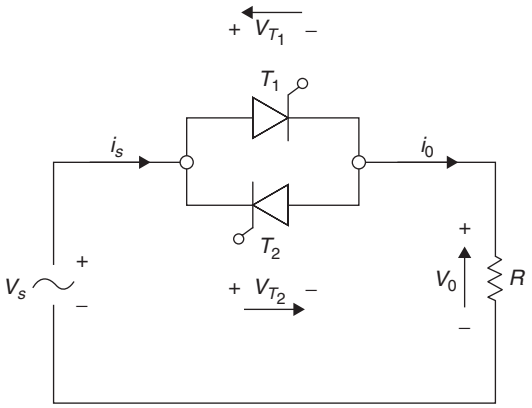
Solution: (C)

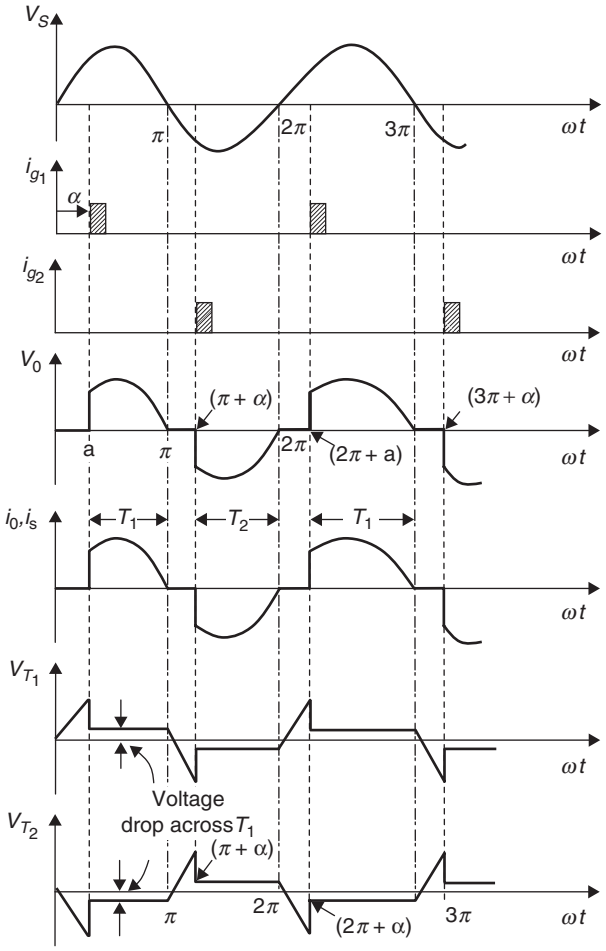
$$(I_0)_{RMS} = \frac{(V_0)_{RMS}}{R_L} = \frac{198}{30} = 6.6 \text{ A} = I_s$$
$$P_0 = \frac{(V_0)^2_{RMS}}{R_L}\ \frac{198^2}{30} = 1.3068 \text{ kW}$$
$$\text{Input power factor} = \frac{P_0}{V_S I_S} = \frac{1.3068 \times 10^3}{200 \times 6.6}$$
$$= 0.99$$

Single-phase Full-wave AC Voltage Controller (AC Regulator) or RMS Voltage Controller with Resistive Load

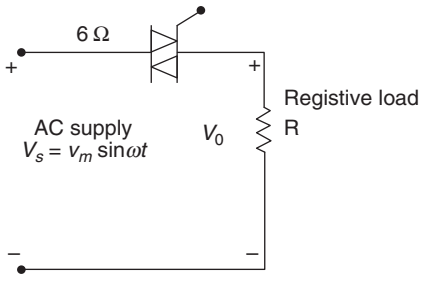
Single-phase full wave AC voltage controller circuit using two SCRs or a single triac is generally used in most of the AC control applications. The AC power flow to the load can be controlled in both the half cycles by varying the triggering angle α The RMS value of load voltage can be varied by varying the trigger angle α.

- There is no DC component of input supply current, because input supply current is alternating in the case of a full wave AC voltage controller and symmetrical in nature. It is also referred to as to a bi directional controller.





Instead of using two SCRs in parallel, a triac can be used for full-wave AC voltage TRIAC.



Circuit turn-off time, $t_c = \dfrac{\pi}{\omega}$ s

RMS output voltage

$$V_{or} = \left[\frac{1}{\pi}\int_{\alpha}^{\pi} V_m^2 \sin^2 \omega t \, d\omega t\right]^{1/2}$$

$$V_{or} = \frac{V_m}{\sqrt{2}}\left[\frac{1}{\pi}\left\{(\pi - \alpha) + \frac{1}{2}\sin 2\alpha\right\}\right]^{1/2}$$

$$I_{or} = \frac{V_{or}}{R} = \text{RMS value of load current}$$

Power delivered to the load

$$P = I_{or}^2 . R = \frac{V_m^2}{2\pi R}\left[(\pi - \alpha) + \frac{1}{2}\sin 2\alpha\right]$$

Power factor P.F. $= \dfrac{V_s I_1 \cos\phi_1}{V_s I_{RMS}} = \dfrac{\text{Real power}}{\text{Apparent power}}$

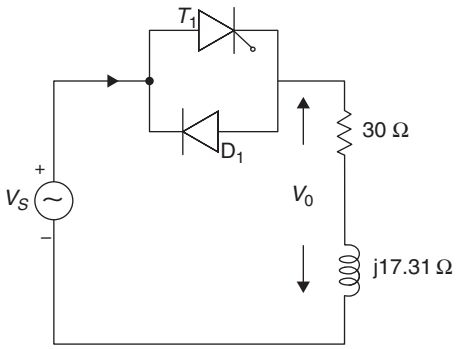
$$\text{P.F.} = \frac{I_1 \cos\phi_1}{I_{RMS}}$$

where ϕ_1 = Phase angle between V_s and I_1

I_1 = RMS value of fundamental component of source current

$$\text{P.F.} = \frac{V_{or}}{V_s} = \left\{\frac{1}{\pi}\left[(\pi - \alpha) + \frac{1}{2}\sin 2\alpha\right]\right\}^{1/2}$$

Example 4: The below figure shows a single-phase voltage controller circuit.



The output voltage V_0 is uncontrollable for firing angle ranging between

(A) $0 < \alpha < 45°$ (B) $0 < \alpha < 30°$

(C) $45° < \alpha < 135°$ (D) $30 < \alpha < 120°$

Solution: (B)

For controlling the load, the minimum value of firing angle α = load phase angle

$$\phi = \tan^{-1}\left[\frac{\omega L}{R}\right] = \tan^{-1}\left[\frac{17.31}{30}\right]$$

$$\phi = 30°$$

Range of firing angle for which output voltage is uncontrollable $0 < \alpha \le \phi$

$$0 < \alpha \le 30°$$

Example 5: A single-phase AC voltage controller feeding a pure resistive load has a load voltage of 150 V (RMS) when fed from a source of 200 V. The input power factor of the controller is

(A) 0.65 (B) 0.7 (C) 0.75 (D) 0.8

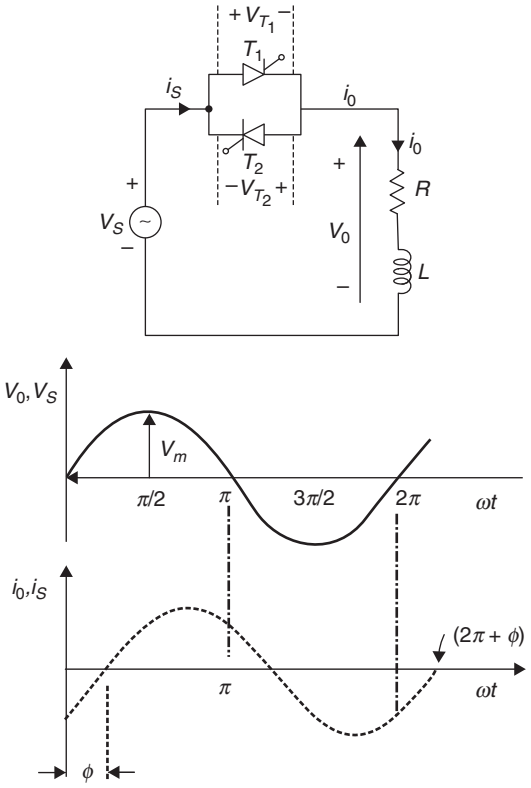
Solution: (C)

$$\text{P.F.} = \frac{V_0}{V_s} = \frac{150}{200} = 0.75$$

Single-phase Full-wave AC Voltage Controller with RL Load

The RMS Value of the output voltage and the load current may be varied by varying the trigger angle α. The circuit AC RMS voltage across the terminals of an AC motor (induction). It can be used to control the temperature of a furnace by varying the RMS output voltage.

- For very large load inductance 'L' the SCR may fail to commutate, after it is triggered and the load voltage will be a full sine wave as long as the gating signals are applied to the thyristors T_1 and T_2.



Circuit turn-off time, $t_c = \dfrac{\pi}{\omega}$ s

Load current

$$I_0 = \frac{V_m}{Z}\left[\sin(\omega t - \phi) - \sin(\alpha - \phi).\exp\left\{\frac{R}{L}\left(\frac{\alpha}{\omega} - t\right)\right\}\right]$$

At $\omega t = \beta$ (or) $t = \dfrac{\beta}{\omega}$

$$0 = \sin(\beta - \phi) - \sin(\alpha - \phi)\exp\left[\frac{R}{L}\left(\frac{\alpha - \beta}{\omega}\right)\right]$$

$$\sin(\beta - \phi) = \sin(\alpha - \phi)\exp\left[\frac{R}{L}\left(\frac{\alpha - \beta}{\omega}\right)\right]$$

Conduction angle $\gamma = (\beta - \alpha)$

Maximum thyristor

Conduction angle

$$\delta = (\beta - \alpha) = \pi \text{ radians} = 180° \text{ for } \alpha < \phi$$

RMS output voltage

$$V_{0(\text{RMS})}\ \frac{V_m}{\sqrt{2}} = \sqrt{\frac{1}{\pi}(\beta - \alpha) + \frac{\sin 2\alpha}{2} - \frac{\sin\frac{2\beta}{2}\beta}{2}}$$

The average thyristor current

$$I_{T(\text{Avg})} = \frac{1}{2\pi}\left[\int_{\infty}^{\beta} i_{T1}\, d(\omega t)\right]$$

Maximum value of $I_{T(\text{Avg})}$

Occur at $\alpha = 0$

$$\Rightarrow \qquad I_{T(\text{Avg})} = \frac{V_m}{z}$$

RMS Thyristor current $I_{T(\text{RMS})}$

$$I_{T(\text{RMS})} = \left[\frac{1}{2\pi}\int_{\infty}^{\beta} i_{T_1^2}(\omega t)\right]$$

Maximum value of $I_{T(\text{RMS})}$

Occurs at $\alpha = 0$

Thyristors should be rated for maximum.

$$I_{T(\text{RMS})} = \frac{I_m}{\sqrt{2}}$$

When a TRIAC is used in a single-phase full wave AC Voltage controller with RL load, then I_T (Avg) = 0 and maximum.

$$T_T\ (\text{RMS}) = \frac{I_m}{\sqrt{2}}$$

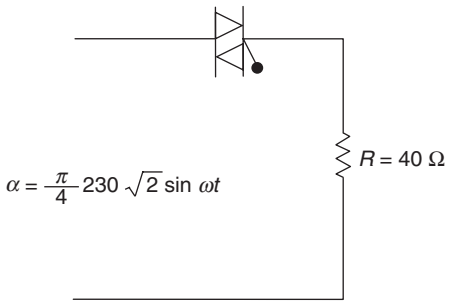
Example 6: A single-phase AC voltage controller is controlling current in a purely inductive load. If the firing angle of the S.C.R. is α, then the conduction angle will be

(A) 2π (B) $2\pi - \alpha$

(C) π (D) $\pi - \alpha$

Solution: (D)

Example 7: The triac circuit shown in figure controls the AC output power to the resistive load. The peak power dissipation in the load is



(A) 3968 W (B) 5290 W
(C) 7935 W (D) 10,580 W

Solution: (D)

For resistive load, peak power = $\frac{V_m^2}{R}$

$$V_m = 230 \times \sqrt{2} = \frac{\left(230 \times \sqrt{2}\right)}{10}$$

$$= 10{,}580 \text{ W}$$

Two-stage Sequence Control

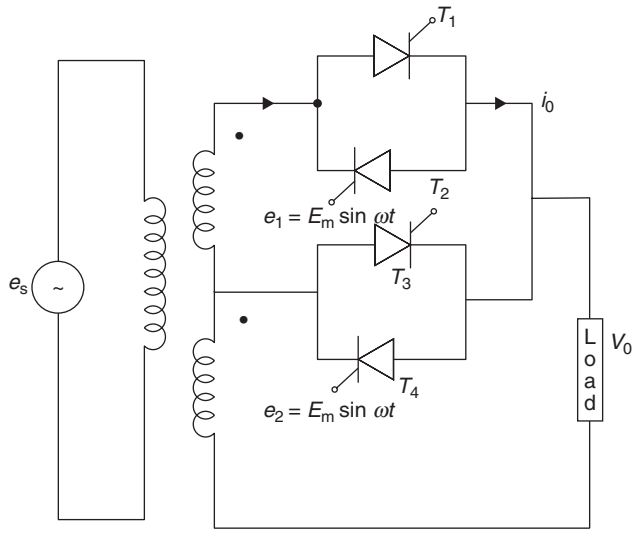
Two-stage sequence control of AC regulator employs two stages in parallel. The turn ratio from primary to each secondary is taken as unity.

For source voltage

$$e_s = E_m \sin \omega t.$$

$e_1 = e_2 = E_m \sin \omega t$ and the sum of two secondary voltages is $2E_m \sin \omega t$ for both R and RL loads, for obtaining output voltage control from zero to RMS value E, only thyristor pairs T_3, T_4 are used.

For zero output voltage, α is 18° for T_3 and T_4 and for E, α is zero for output voltage control from E to $2E$, α for thyristor pair T_3 and T_4 is always zero and for thyristor pair T_1, T_2 it is varied from zero to 180°.



Multistage Sequence Control of AC Regulators

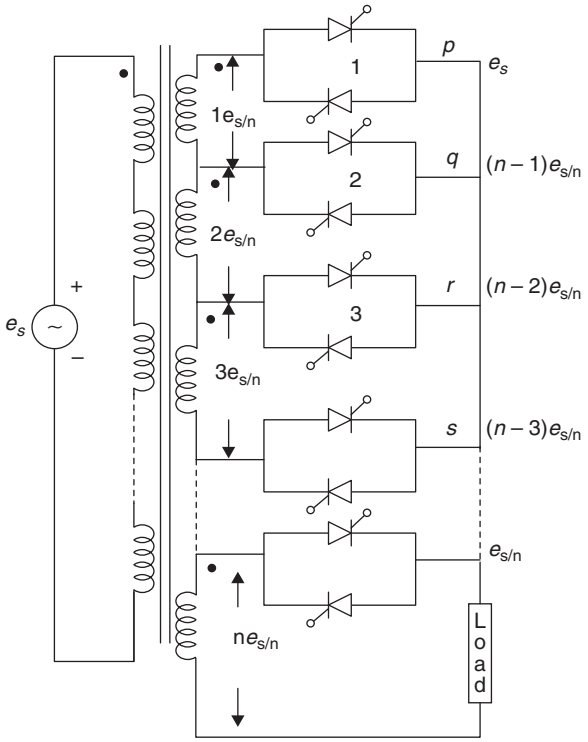
By using more than two stages of sequence control, is possible to have further improvement in power factor and reduction in harmonics than that in a two stage sequence control.

The transformer has n secondary windings. Each secondary is rated for $e_{s/n}$, where e_s is the source voltage.

The voltage node P with respect to K is e_s.

Voltage of terminal q is $(n-1)\, e_{s/n}$ and so on. If voltage control from $e_{sk} = (n-3)\, e_{s/n}$ to $e_{rk} = (n-2)\, e_{s/n}$ is required then thyristor pair 4 is triggered at $\alpha = 0°$ and the firing angle of thyristor pair 3 is controlled from $\alpha = 0°$to 180°, whereas e_s all other thyristor pairs are kept OFF.

The load voltage can be controlled from $e_{s/n}$ to e_s by an appropriate control of triggering the adjacent thyristor pairs.



The presence of harmonics in the output voltage depends upon the magnitude of voltage variation. If this voltage variation is a small fraction of the total output voltage the harmonic content in the output voltage is small. Multistage sequence control of AC regulators.

Example 8: AC voltage regulators are widely used in
(A) Traction drives
(B) Fan drives
(C) Synchronous motor drives
(D) Slip power recovery scheme of slip ring induction motor.

Solution: (B)

Cyclo Converters

A cyclo converter is an AC to AC converter that directly converts AC power at input frequency to an output AC power at a different frequency without employing an intermediate DC stage. It is therefore, a single stage frequency converter. They are of two types.

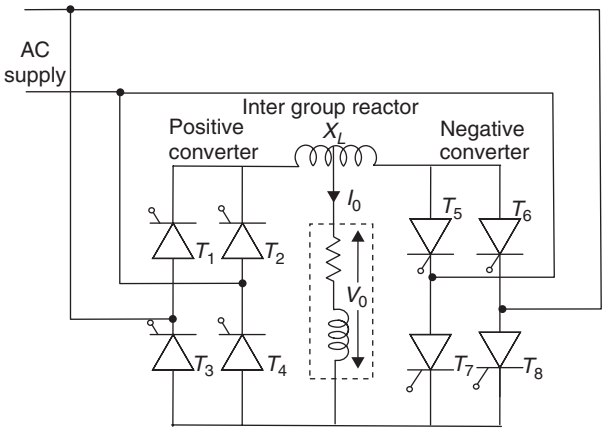
1. Step-down cyclo converter in which like a step-down transformer gives reduced voltage as output), provides an AC output whose frequency is less than that of the input, that is $f_0 < f_i$.
2. Step-up cyclo converter in which the output frequency is greater than the input frequency as in a step-up transformer whose output voltage is greater than the input voltage, $f_0 > f_i$.
   - A step-down cyclo converter uses thyristors as power switches. The maximum output frequency is limited to a fraction of supply frequency below 20 Hz in the case of 50 Hz input converter.
   - The thyristor are line commutated, there is no need for separate commutation circuits.
   - But a step-up cyclo converter needs forced commutation of thyristors, which makes the cyclo converter of this type relatively complex.
   - A frequency changer cyclo converter finds application in the following areas:
     - — Speed control of high power AC drives
     - — Induction heating
     - — For converting variable frequency voltage from a variable speed alternator to a constant frequency voltage
   - The majority of cyclo converters are step-down type.

The step-down cyclo converter circuits, namely:

1. Single-phase to single-phase cyclo converter
2. Three-phase to single-phase cyclo converter and
3. Three-phase to three-phase cyclo converter

- A step-down cyclo converter is essentially a wave synthesizer.
- In addition to frequency cyclo converter, output voltage can be varied by the application of phase control principle.
- Cyclo converter can be used to provide either variable frequency output from fixed frequency input or vice versa.
- A cyclo converter can also handle loads of lagging as well as leading power factor in addition to unity power factor loads.
- This is because cyclo converter permits power flow in either direction.
- The output voltage wave is a distorted sine wave the associated harmonic components can be filtered out if necessary the distortion is low for low frequency outputs but increases with output frequencies.

(a) Single-phase to Single-phase Cyclo Converter Feeding RL Load



Two single-phase fully controlled converters are connected in anti-parallel, as in dual converter. Converter 1 produces positive load current and when converter 2 conducts, the load current reverses.

- Two converters should not conduct together as their will produce a short circuit on the input.

The firing angles of SCRs in the two converters are kept the same so that the output load voltage is symmetrical for resistive loads,

- With R-load, they undergo natural commutation and produce discontinuous current operation.
- With inductive loads, line commutation taken place and the load current is continuous.
- The instantaneous value of output voltage will become negative because for inductive load, SCR conduct for duration more than $(\pi - \alpha)$ in each half cycle of the input.
- If only one converter conducts at a time, then it is known as non-circulating current scheme.
- In non-circulating current scheme, there is a dead time during the changeover of current.

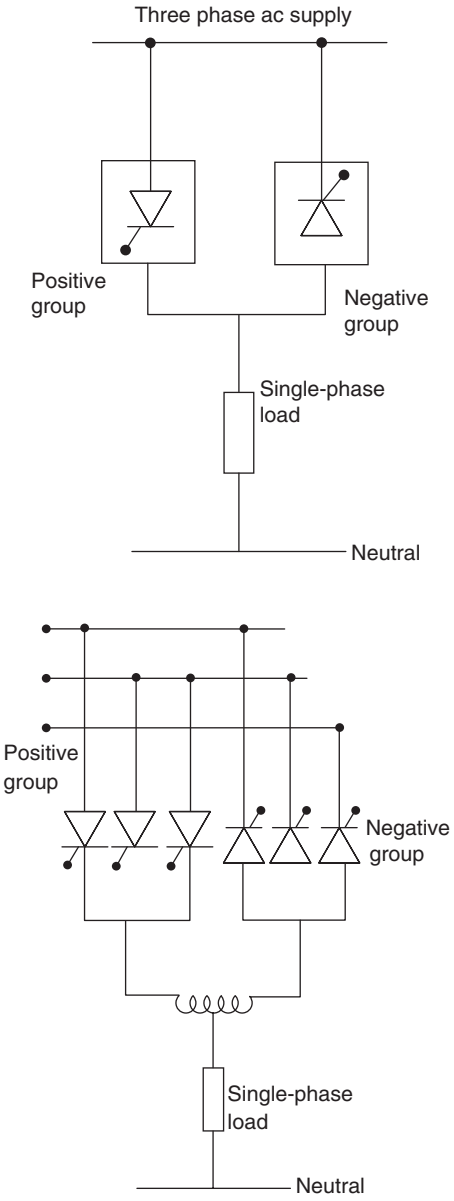
Example 9: The quality of output AC voltage of a cyclo converter is improved with
(A) Increase in output voltage with increased frequency.
(B) Decrease in output voltage with increased frequency.
(C) Increase in output voltage at reduced frequency.
(D) Decrease in output voltage at reduced frequency.

Solution: (A)

(b) Three-phase to Single-phase Cyclo Converter

- Many industrial AC voltages as input single phase to single, phase cyclo converter are seldom used in single-phase induction motor control because it can supply only non-sinusoidal output voltage.

- A very nearly sinusoidal output voltage can be fabricated from three-phase input voltage waves by means of three-phase to single-phase cyclo converters.



The positive group converter P is a three-phase half-wave-controlled rectifier that can conduct load current that flows downward towards the neutral.

- Negative group converter N is also a three-phase half-wave controlled rectifier which conducts the load current in the reverse.
- The inter group reactor is sometimes necessarily used to limit the circulating current common to P and N converters.
- The low frequency load voltage is fabricated by P and N converters making use of three-phase input supply by varying the firing angle of three thyristors of a three-phase half-wave circuit.

The average direct voltage output is given by

$$V_d = V_{do} \cos \alpha$$

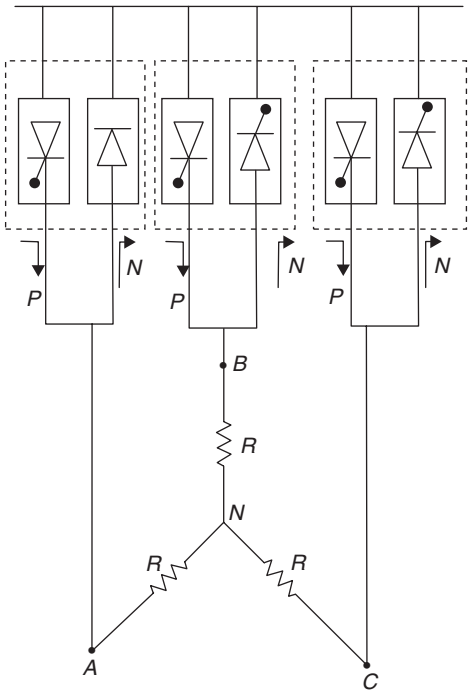
α = firing angle

V_{do} = maximum output voltage with zero firing delay.

(c) Three-phase to Three-phase Cycloconverter

A three-phase output can be obtained by a phase displacement of 120° from three similar three-phase to single-phase cyclo converters.

- It requires six thyristors per phase and 18 thyristors in all.
- If each phase consists of a three-phase bridge then 36 thyristors are required.
- The average value of the output voltage can be varied by varying the delay angle of the thyristors, whereas the frequency of the output voltage can be varied by changing the sequence of thyristor triggering.
- With a balanced load, the neutral connection is no longer necessary and can be omitted.



- Three-phase cyclo converters can handle large amounts of current and produce a smoother output waveform.
- It is a highly efficient, variable speed of drive because the large number of thyristor increases the pulse number, which causes a small amplitude ripple content in the load voltage waveform.

Example 10: In the construction of a three-phase cyclo converter, the number of switches required is

(A) 12 (B) 18 (C) 3 (D) 6

Solution: (B)

Example 11: Among the following, a cyclo converter fed induction motor drive is most suitable for
(A) Machine tool drive
(B) Compressor drive
(C) Paper mill drive
(D) Cement mill drive

Solution: (A)

Electric Drives

An electric drive is an electromechanical system that employs an electric motor as the prime mover instead of a diesel engine, steam or gas turbines, hydraulics, etc., to control the motion and processes of different machines and mechanisms. Typical applications of electric drives include fans, ventilators compressor pumps, hoists, cranes, conveyors, excavators, escalators, electric locomotives and cars.

- Drives that use electric motors as the prime movers are known as electric drives.
- It consists of a motor, micro controller, feedback circuit and a power electronic circuit Motor is a subset of drive.

(Output of converter = Input of motor); output voltage of converter acts as terminal voltage of motor.

Application of Choppers and Inverters

(a) Choppers

1. Speed control of DC series motor
2. Switch mode power supplies

(b) Inverters

1. Speed control of AC series motors
2. Induction heating
3. Air craft power supplies
4. Uninterruptable power supplies

Speed Control of Motors

Speed Control of DC Motors

Phase Control

When supply is AC the current is controlled to flow from supply to the motor for a fraction of each positive half cycle, this is repeated cyclically.

- It can be applied for all motor ratings.

Integral Cycle Control

- This method is useful for small fractional kW size motors.
- From AC supply, the current is made to flow for a number of complete cycles and quenched for a further number of cycles and the sequence of operation is repeated.
- 'ON' and 'OFF' duration ratio is adjusted by control.

Chopper Control

A DC voltage can be switched ON and OFF rapidly and repeatedly by means of a thyristor to a DC motor.

- The average DC voltage is controlled by varying ON/OFF time ratio.
- This type of control is used for traction motors.

Basic Concept of Speed Control

Speed of a DC motor, $\omega = \dfrac{E}{K_c \phi}$

So, speed of a DC machine can be controlled either by varying flux ϕ (or) by controlling supply voltage V, which is done by the following two methods.

1. Field flux control (Constant power drive)
2. Armature voltage control (Constant torque drive)

Speed Control of AC Motors

Speed of induction motor can be varied by varying its terminal voltage and motor resistance.

Speed Control Schemes

Speed of induction motor can be controlled by using suitable power electronic converter.

1. Stator voltage control
2. Stator frequency control
3. Stator voltage and frequency control
4. Stator current control
5. Static rotor resistance control
6. Slip energy recovery control

Stator Voltage Control

By controlling firing angle of the thyristors, the RMS value of the stator voltage can be regulated or controlled. Hence motor torque and speed can be controlled.

Stator Voltage and Frequency Control

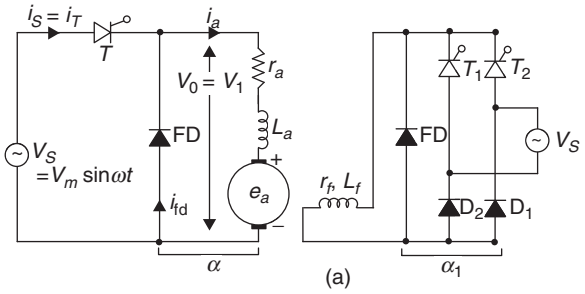
- In this V/f ratio is kept constant, so that air-gap flux, ϕ will remain constant.
- Control of stator voltage and frequency can be implemented by using three-phase inverters or cyclo converters.
- Generally voltage source inverter and PWM inverter are used for voltage and frequency control.

Stator Current Control ($T \propto I^2_1$)

- By controlling stator current I_1, we can control torque, hence we can control speed.
- Stator current control method is employed using current source inverters.

Controlling Effective Resistance in the Rotor Circuit

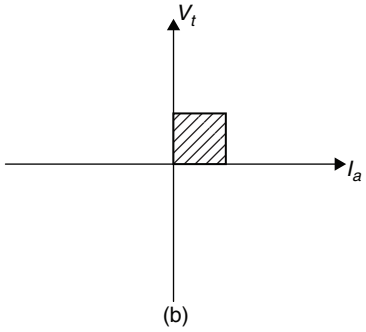
1. **Phase-Controlled Rectifier:** Scheme: In this scheme firing circuit is synchronized with rotor frequency.
2. **Chopper-Controlled Resistance Scheme:** It does not require synchronization with rotor frequency

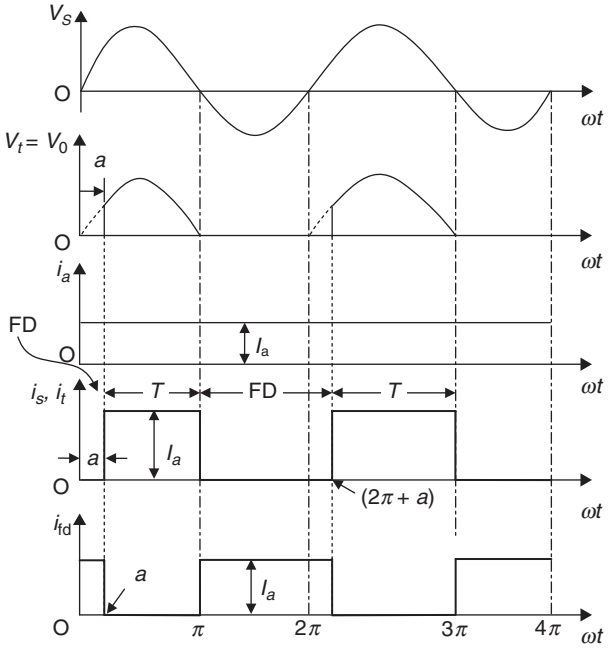


Single-phase Converter Drives

Single-phase Half-wave Converter Drives

- The diagram below is a separately excited DC motor, fed through single-phase, half-wave converter.





Average output voltage of converter (Single-phase, half-wave)

$$V_0 = V_t = \frac{V_m}{2\pi}(1 + \cos\alpha), \text{ where } 0 < \alpha < \pi.$$

For single-phase semi-converter in the field circuit, the average output voltage

$$V_t = \frac{V_m}{2\pi}(1 + \cos\alpha), \text{ where } 0 < \alpha < \pi.$$

RMS value of armature current $I_{ar} = I_a$.

RMS value of thyristor current

$$I_{sr} = I_a \left(\frac{\pi - \alpha}{2\pi}\right)^{1/2}$$

RMS value of free-wheeling diode current

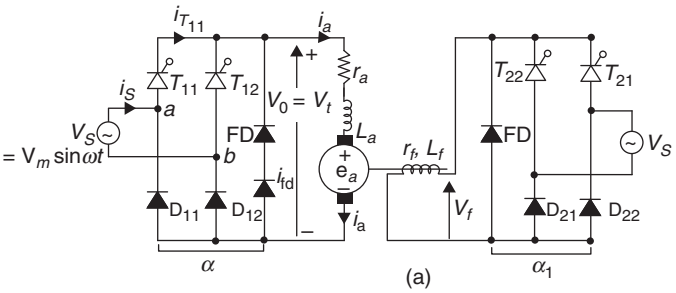
$$I_{fdr} = \sqrt{I_a^2 \left(\frac{\pi + d}{2\pi}\right)} = I_a \left(\frac{\pi + d}{2\pi}\right)^{1/2}$$

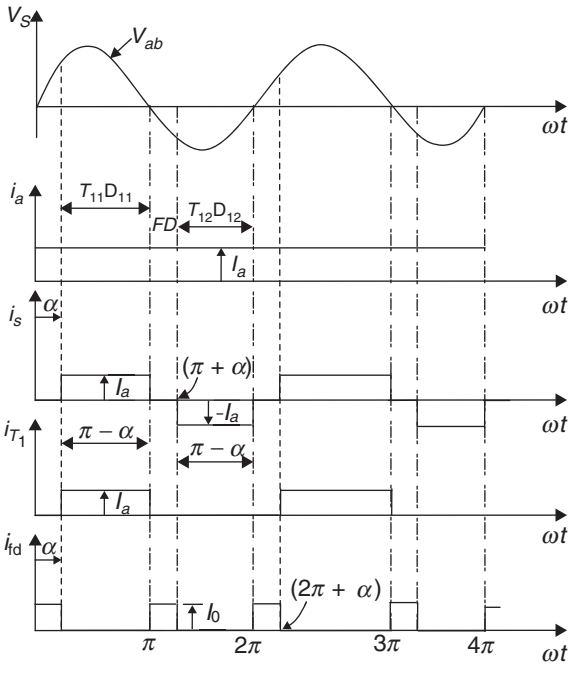
Apparent input power = (RMS source voltage) (RMS source current) = $V_s I_{sr}$

Power delivered to the motor

$$= E_a I_a + I_a^2 r_a = V_t I_a = (E_a + I_a r_a) \cdot I_a$$

Input supply of P.F. = $\dfrac{V_t I_a}{V_s I_{sr}}$





For single-phase semi-converter, average output voltage

$$V_0 = V_t = \frac{V_m}{\pi}(1 + \cos\alpha)$$

For field circuit $V_f = \frac{V_m}{\pi}(1 + \cos\alpha_1)$

RMS value of source current, $I_{sr} = I_a \left[\frac{\pi - \alpha}{\pi}\right]^{1/2}$

RMS value of free-wheeling diode current, $I_{fdr} = I_a \left[\frac{\alpha}{\pi}\right]^{1/2}$

RMS value of thyristor current, $I_{tr} = I_a \left[\frac{\pi - \alpha}{2\pi}\right]^{1/2}$

Input PF = $\frac{V_t . I_a}{V_s . I_{sr}}$

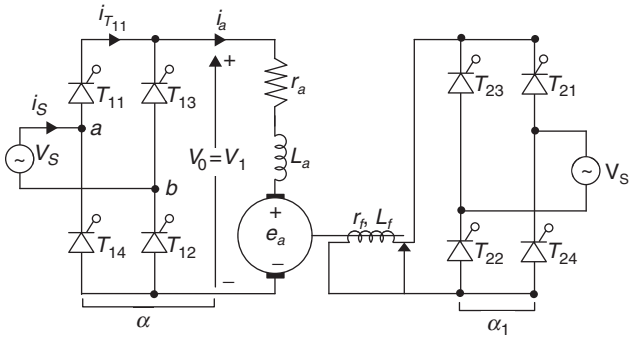
Example 12: A separately excited DC motor is required to be controlled from a three-ϕ source for operation in the first quadrant only the most preferred converter would be

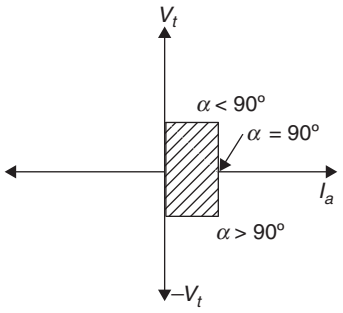
(A) Fully controlled converter
(B) Fully controlled converter with free-wheeling diode
(C) Half controller converter
(D) Sequence control of two series connected fully controlled converters

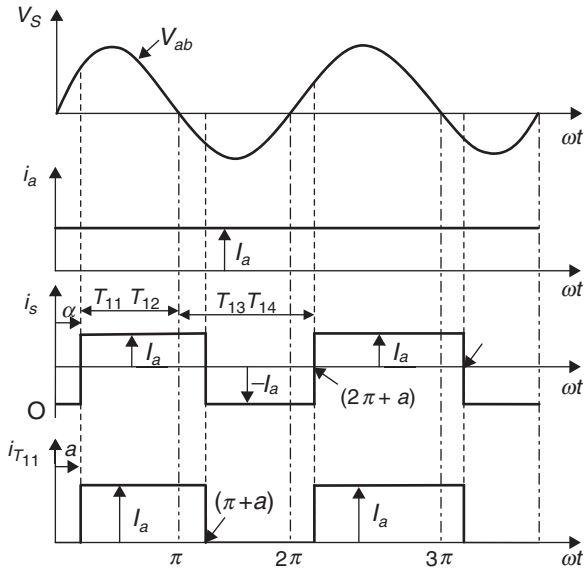
Solution: (C)

Single-phase Full Converter Drives

- A single-phase, full converter drive feeding a separately excited DC motor is in the figure below.







For armature converter 1

$$V_0 = V_t = \frac{2V_m}{\pi}\cos\alpha \text{ for } 0 < \alpha < \pi$$

For field converter 2

$$V_f = \frac{2V_m}{\pi}\cos\alpha_1 \text{ for } 0 < \alpha_1 < \pi$$

From the waveforms drawn,
RMS value for thyristor current

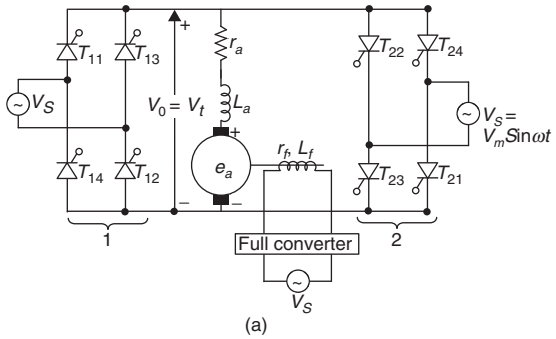
$$I_{tr} = \left(I_a^2 \frac{\pi}{2\pi}\right)^{1/2} = \frac{I_a}{\sqrt{2}}$$

Input supply p.f. = $\frac{V_t . I_a}{V_s . I_{sr}} = \frac{2V_m}{\pi}\cos\alpha \frac{I_a\sqrt{2}}{V_m . I_a}$

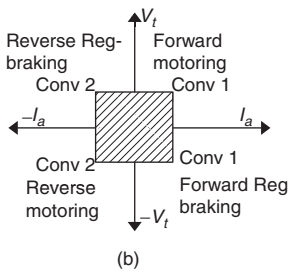
$$= \frac{2\sqrt{2}}{\pi}\cos\alpha$$

Single-phase Dual Converter Drives

- The figure shown below is a single-phase dual converter obtained by connecting two, full converters in anti-parallel.



(a)



(b)

Figure 1

- It is a four-quadrant converter.
- For working in first and fourth quadrants, converter 1 is in operation.
- For operation in second and third quadrants, converter 2 is energized.
- Four-quadrant operation demands the field winding of the motor to be energized from a single-phase or a three-phase full converter.
- When $V_t = \frac{2V_m}{\pi}\cos\alpha_1$ for $0 < \alpha_1 < \pi$ converter 1 operates.
- When $V_t = \frac{2V_m}{\pi}\cos\alpha_2$ for $0 < \alpha_2 < \pi$ converter 2 operates.
- Converter 1 with $\alpha_1 < 90°$ operates the motor in forward motoring mode in quadrant 1.
- Converter 1 with $\alpha_1 > 90°$ and with field excitation reversed operates the motor in forward regenerative braking mode in quadrant 4.
- Converter 2 with $\alpha_2 < 90°$ operates the motor in reverse motoring mode in quadrant 3.
- Converter 2 with $\alpha_2 > 90°$ and with field excitation reversed operates the motor in reverse regenerative braking mode in quadrant 2.

Example 13: The speed–torque characteristics of a DC motor and control methods suitable for the same are given in List I and II, respectively. The correct match would be

List I	List II
(P) Above base speed	(1) Field control
(Q) Above rated torque	(2) Armature control
(R) Below base speed	
(S) Below rated torque	

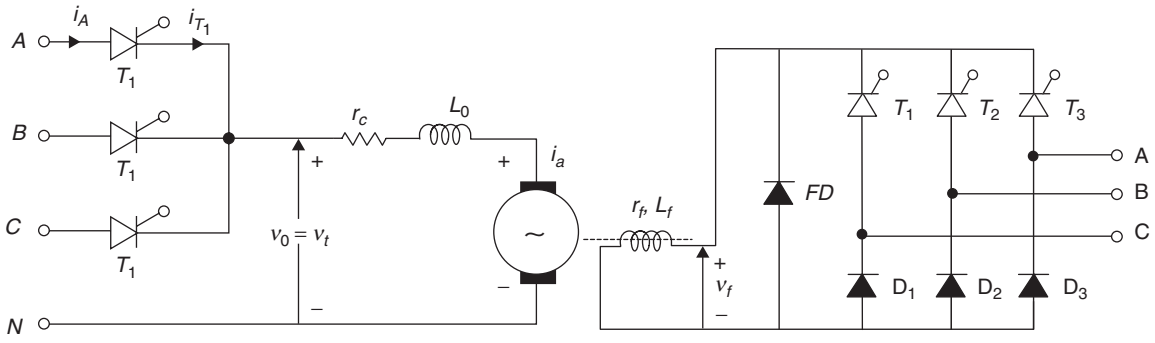
(A) P-1 Q-2 (B) P-2 Q-1
(C) P-1 R-2 (D) Q-2 S-1

Solution: (A)

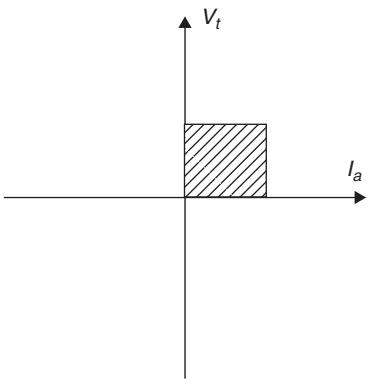
Three-phase Converter Drives

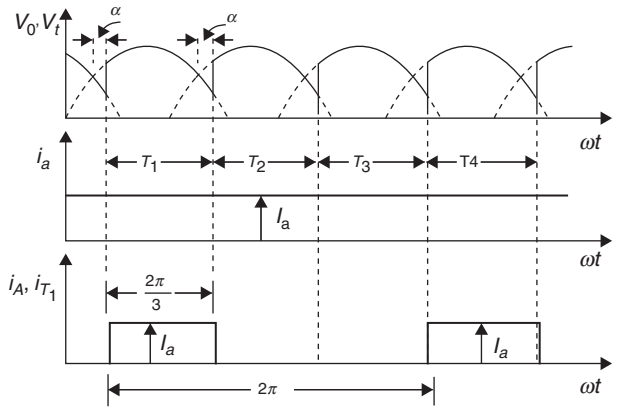
Three-phase, Half-wave Converter Drives

- Figure below shows a three-phase, half wave converter drive consisting of two converters and a separately excited DC motor.



(a) Circuit diagram





- The armature circuit of the DC motor is fed through a three-phase half-wave converter whereas the field is energized through a three-phase semi-converter.
- This converter offers one quadrant operation.
- Two-quadrant operation can be obtained from a three-phase, half-wave converter drive, in case motor field windings are energized from single-phase or three-phase full-converter.

Average value of output voltage

$$V_0 = V_t = \frac{3V_{ml}}{\pi}\cos\alpha \quad \text{for } 0 \le \alpha < \pi$$

(for continuous armature current)

V_{ml} = maximum line voltage

For three-phase semi-converter, average value of field voltage

$$V_f = \frac{3V_{ml}}{2\pi}(1 + \cos\alpha_1) \text{ for } 0 \le \alpha_1 < \pi.$$

RMS value of armature current

$$I_{ar} = I_a$$

RMS value of phase or line current

$$I_{sr} = \sqrt{I_a^2 \frac{2\pi}{3} \cdot \frac{1}{2\pi}} = \frac{I_a}{\sqrt{3}}$$

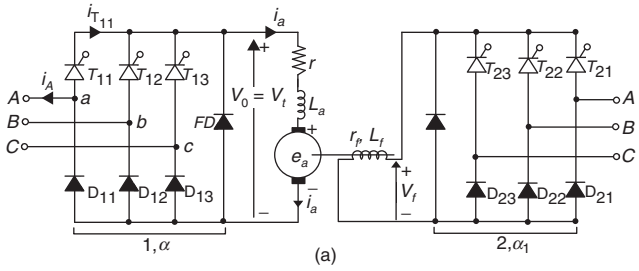
Average thyristor current

$$I_{TA} = I_a \cdot \frac{2\pi}{3} \cdot \frac{1}{2\pi} = \left(\frac{I_a}{3}\right)$$

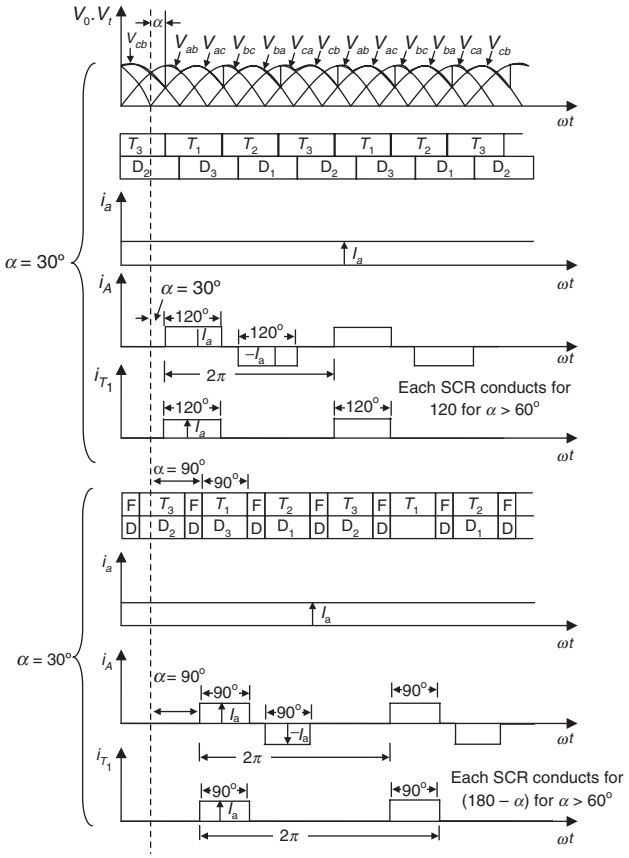
RMS value of thyristor current

$$I_{tr} = I_{sr} = \left(\frac{I_a}{\sqrt{3}}\right)$$

Three-phase Semi-converter Drives



(a)



For $\alpha < 60°$ RMS value of supply line current I_a is given by

$$I_{sr} = I_a\sqrt{\frac{2}{3}}$$

$$I_{tr} = \frac{I_a}{\sqrt{3}}$$

For $60 < \alpha < 180$ $I_{sr} = I_a\sqrt{\frac{180 - \alpha}{180}}$

$$I_{tr} = I_a\sqrt{\frac{180 - \alpha}{360}}$$

Average thyristor current = $\frac{I_a}{3}$ for $\alpha < 60°$.

For $60 < \alpha < 180$, $\frac{(180 - \alpha)}{360}$

For $60 < \alpha < 180°$

$$I_{f.da} = I_a\left(\frac{\alpha - 60}{120}\right)$$

$$I_{f.dr} = I_a\sqrt{\frac{\alpha - 60}{120}}$$

For converter 1, $V_0 = V_t = \frac{3V_{ml}}{\pi}(1 + \cos\alpha)$ for $0 < \alpha < \pi$

For converter 2, $V_f = \frac{3V_{ml}}{\pi}(1 + \cos\alpha_1)$ for $0 < \alpha_1 < \pi$

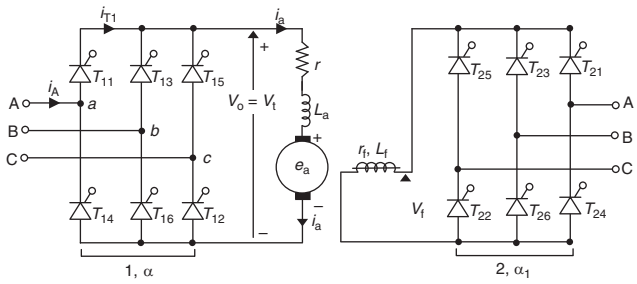
Example 14: A separately excited DC motor supplying a non-zero torque for steady-state operation, if fed from a three-phase semi-converter. The motor armature current is found to drop to zero at certain instances.

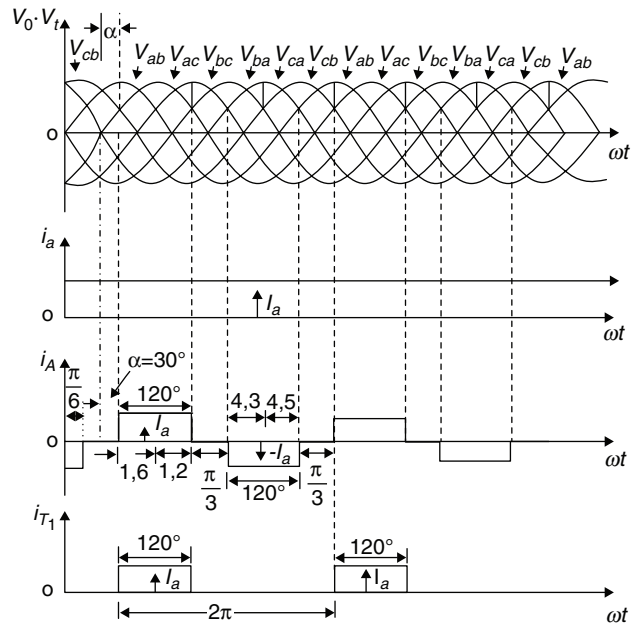
The assumed value of voltage is

(A) Zero
(B) Arbitrary
(C) Equal to instantaneous value of the AC phase voltage
(D) Equal to instantaneous value of the motor back emf

Solution: (D)

Three-phase Full Converter Drives





Average output voltage in armature circuit for converter 1

$$V_0 = V_t = \frac{3V_{ml}}{\pi}\cos\alpha$$

For field circuit $V_f = \frac{3V_{ml}}{\pi}\cos\alpha_1$

RMS value of source current

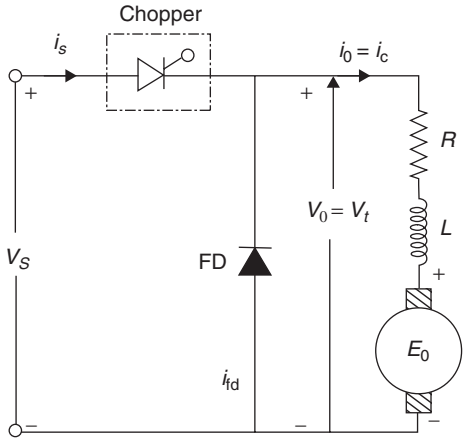
$$I_{sr} = I_a\sqrt{\frac{2}{3}}$$

$$I_{tr} = \frac{I_a}{\sqrt{3}}$$

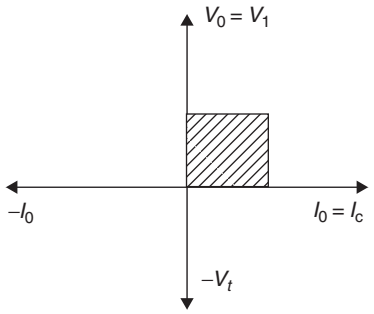
$$I_{TA} = \frac{I_a}{3}$$

Chopper-controlled Drives

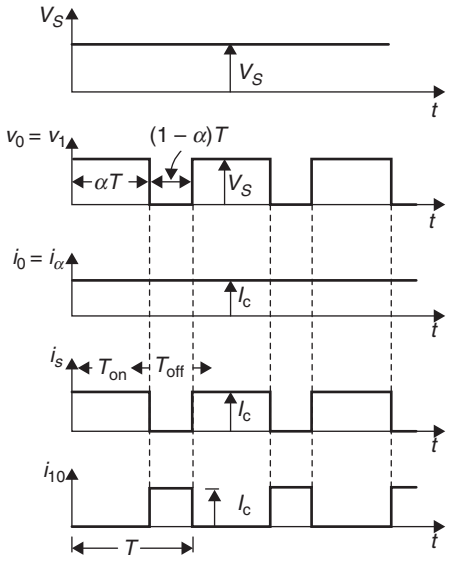
Power Control (or) Motoring Control



(a) Circuit diagram



(b) Quadrant diagram



(c) Waveforms

Figure 2

Average motor voltage

$$V_0 = V_t = \left(\frac{T_{ON}}{T}\right).V_s = \alpha V_s$$

$$V_0 = f T_{ON} V_s$$

where $\alpha = \frac{T_{ON}}{T}$ (duty cycle)

f = chopping frequency = $\frac{1}{T}$

Power delivered to motor

= (Average motor voltage) × (Average motor current)

$= V_t . I_a$

$= \alpha V_s \cdot I_a$

Input power to chopper

= (Average input voltage) (Average source current)

$= V_s \cdot \alpha I_a$

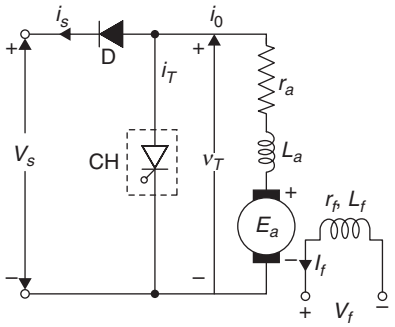
For the motor armature circuit

$$V_t = \alpha V_s = E_a + I_a (r_a + r_s)$$

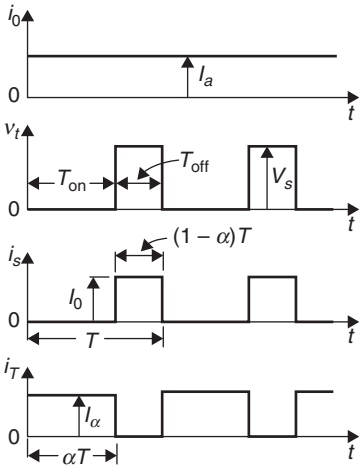
$$= K_m \cdot \omega_m + I_a (r_a + r_s)$$

$$\omega_m = \frac{\alpha V_s - I_a (r_a + r_s)}{K_m}$$

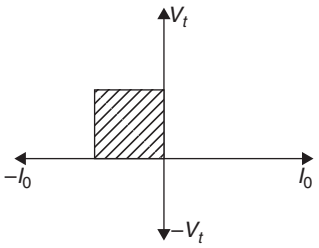
Regenerative Braking Control



(a) Circuit diagram



(b) Waveforms



(c) Quadrant diagram

Figure 3

$$V_{t\,avg} = (1 - \alpha) V_s$$

Power generated by motor

$$= V_t . I_a = (1 - \alpha) V_s . I_a$$

Motor emf generated

$$E_a = K_m \omega_m = V_t + I_a r_a$$

$$E_a = (1 - \alpha) V_s + I_a r_a$$

Motor speed during regenerative braking

$$\omega_m = \frac{(1-\alpha) V_s + I_a r_a}{K_m}$$

Conditions for controlling power during regenerative braking

$$0 \le (E_a - I_a r_a) \ge V_s$$

Min braking speed $\omega_{mn} = \frac{I_a r_a}{K_m}$

Max braking speed $\omega_{mx} = \frac{V_s + I_a r_a}{K_m}$

Therefore regenerative braking control is effective only when motor speed is less than ω_{mx} and more than ω_{mn}.

$$\omega_{mn} < \omega_m < \omega_{mx}$$

- Speed range for regenerative braking

$$= \frac{V_s + I_a r_a}{K_m} : \frac{I_a r_a}{K_m} = (V_s + I_a r_a) : I_a r_a$$

AC Drives

AC motors exhibit highly coupled, nonlinear and multi-variable structures as opposed to much simpler decoupled structures of separately excited DC motors.

They require control of frequency, voltage and current for variable speed applications. The power converters, inverters and AC voltage controllers can control the frequency, voltage or current to meet the drive requirements.

They are two types of AC drives:

- Induction motor drives
- Synchronous motor drives

AC drives are replacing DC drives and are used in many industrial and domestic applications.

The speed and torque of induction motors can be controlled by

- Stator voltage control
- Rotor voltage control
- Frequency control
- Stator voltage and frequency control
- Stator current control

To meet the torque–speed duty cycle of a drive, the voltage, current and frequency control are normally used stator voltage control.

Stator Voltage Control

The stator voltage can be varied by three-phase

- AC voltage controllers
- Voltage = fed variable DC link inverters, or
- Pulse width modulation (PWM) inverters.

However, due to limited speed range requirements the AC voltage controllers are normally used to provide the voltage control.

- The AC voltage controllers are very simple.
- However, the harmonic contents are high and the input PF of the controllers is low.
- They are used mainly in low-power applications such as fans, flowers, and centrifugal pumps, where the starting torque is low.
- They are also used for starting high power induction motors to limit the in – rush current.

Rotor Voltage Control

In a wound rotor motor an external three-phase resistors may be connected to its slip rings.

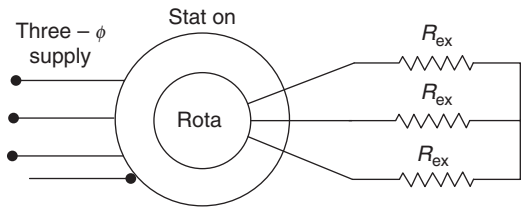
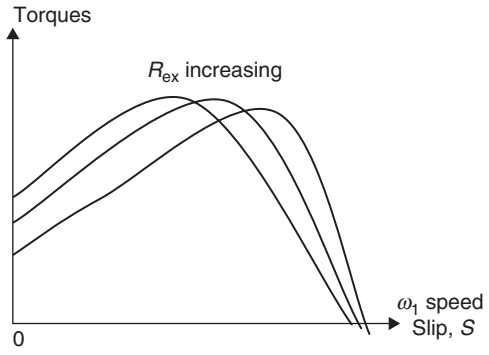


Figure 4 Rotor resistance

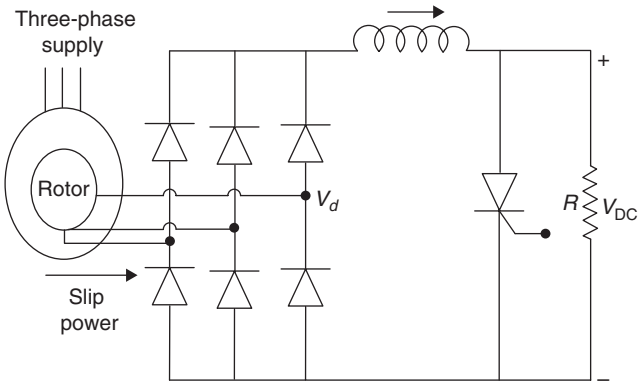


- The developed torque may be varied by varying the resistance R_{ex}.

Rotor Voltage Control

- This method increases the starting torque while limiting the starting current.
- However, this is an inefficient method and there would be imbalances in voltages and currents if the resistances in the rotor circuit are not equal.
- A wound rotor induction motor is designed to have a low-rotor resistance so that the running efficiency is high- and the full-load slip is low.
- The increase in the rotor resistance does not affect the value of maximum torque but increases the slip at maximum torque.
- The wound rotor motors are widely used in applications requiring frequent starting and braking with large motor torques (crane hoists).
- Because of the availability of rotor windings for changing the rotor resistance, the wound rotor offers greater flexibility for control.
- But cost and maintenance increases due to slip rings and brushes.
- Therefore wound rotor motor is less widely used as compared with the squirrel – cage motor.

The three-phase resistor may be replaced by a three-phase diode rectifier and a DC converter. Where the GTO or an IGBT operates as a DC converter switch.



The inductor L_d acts as a current source I_d and the DC converter varies the effective resistance which is given as $R_{ex} = R(1 - K)$.

where K = duty cycle of the DC converter motor speed can be controlled can be controlled by varying the duty cycle.

The portion of the air-gap power, which is not converted into mechanical power is called slip power which is dissipated in R.

Frequency Control

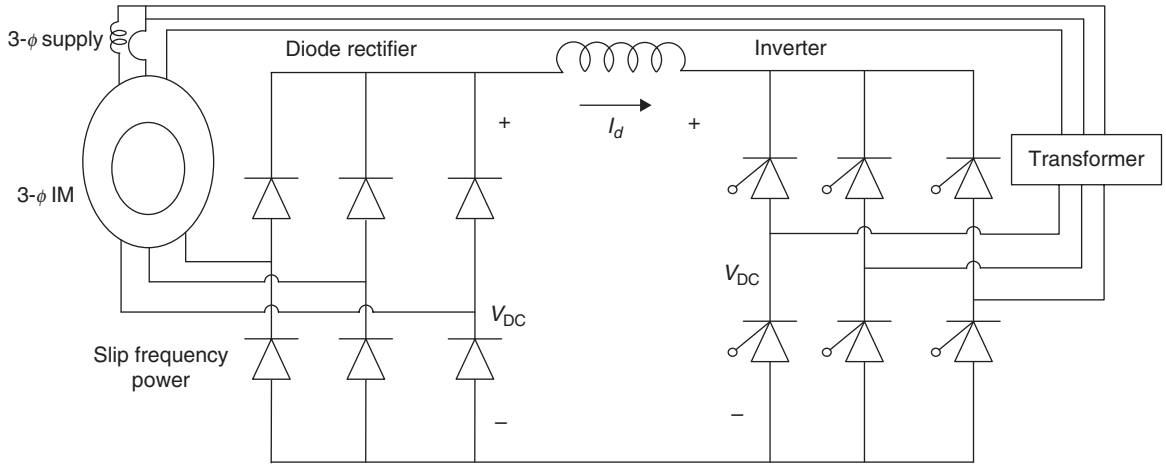
- The torque and speed of induction motors can be controlled by changing the supply frequency.
- If the voltage is maintained fixed at its rated value when the frequency is reduced below its rated value, the flux increases.

- This would cause saturation of the air-gap flux, and the motor parameters would not be valid in determining the torque–speed characteristics.
- At low frequency, the reactances decrease and the motor current may be too high. Therefore this type of frequency control is not normally used.

Slip Power Recovery Scheme

In chopper method of speed control, for slip ring induction motor, the slip power is wasted in the external resistance and it leads to poor efficiency of the drive. Two important slip power recovery schemes are listed as follows.

1. Static Kramers drive (sub synchronous speed)
2. Static Scherbius drive (super synchronous speed)



- Speed is sub-synchronous.
- The slip power can flow in only one direction, due to which

Static Kramers drive can offer speed below synchronous speed.

- The slip power from the rotor circuit is converted to DC voltages which is then converter to line frequency and pumped to the AC source.
- It will work as a constant torque drive.

Applications

Large power pumps compressor type loads where speed control is within narrow range and below synchronous speed.

Static Scherbius Drive

- Speed control below and above synchronous speeds can be obtained.

(a) DC link scherbius drive

- It consists of two-phase-controlled bridges smoothing inductor and transformer.
- For sub synchronous speed control, bridge works as a rectifier and bridge 2 acts as a line commutated inverter.
- Power flows from rotor circuit to supply.
- For super synchronous speed control, bridge 1 acts as a line commutated inverter and bridge 2 works as a rectifier.

(b) Cyclo converter scherbius drive

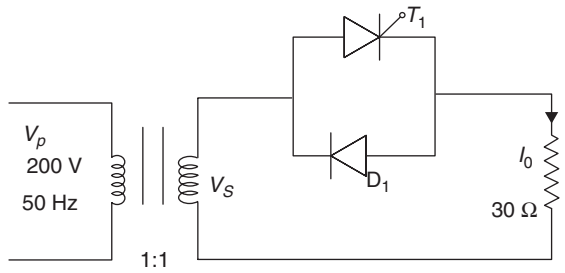
It allows the slip power to flow in either directions and machine can be controlled in both sub synchronous and super synchronous ranges motoring and regeneration features.

Exercises

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

Common Data for Questions 1 and 2:



A single-phase voltage controller has the specifications as shown in the above figure. If T_1 is triggered at $\alpha = 30°$. Then

1. The average value of thyristor current would be
(A) 1.55 A
(B) 2 A
(C) 2.23 A
(D) 2.8 A

2. The RMS value of thyristor current would be
(A) 2.65 A
(B) 3.56 A
(C) 4.65 A
(D) 5.56 A

3. The power factor for single-phase AC voltage controller feeding a resistive load is
(A) Unity for all values of firing angle
(B) $\left[\frac{1}{\pi}\left[(\pi-\alpha)-\frac{\sin 2\alpha}{2}\right]\right]^{1/2}$
(C) $\left[\frac{1}{\pi}\left[(\pi+\alpha)+\frac{\sin 2\alpha}{2}\right]\right]^{1/2}$
(D) $\left[\frac{1}{\pi}\left[(\pi-\alpha)+\frac{\sin 2\alpha}{2}\right]\right]^{1/2}$

4. The power factor of a single-phase AC regulator feeding a resistive load is
(A) (p.u. power)^2
(B) $\sqrt{\text{p.u. power}}$
(C) $\frac{(\text{p.u. power})^2}{\sqrt{2}}$
(D) $\frac{\sqrt{\text{p.u. power}}}{2}$

5. An AC voltage regulator using back to back connected SCRs is feeding an RL load. The SCR firing angle $\alpha < \theta$. The output load voltage waveform, if the SCRs are fired using short duration gate pulses will be
(A) Half-wave rectified
(B) Sinusoidal
(C) Full-wave rectified
(D) Triangular

6 A thyristor-controlled reactor is used to get
(A) Variable resistance
(B) Variable capacitance
(C) Variable inductance
(D) Improved reactor power factor

7. In a single-phase cyclo-converter circuit an intergroup reactor is used to
(A) Limit circuiting current
(B) Reduce voltage ripples
(C) Limit $\frac{di}{dt}$ in semiconductor switch
(D) Reduce current ripples.

8. It is required to drive a DC shunt motor at different speeds in both directions (forward and reverse) and also to brake it in both the directions. Which one of the following would you use?
(A) A full-controlled thyristor bridge
(B) A full-controlled thyristor bridge
(C) A dual converter
(D) A diode bridge

9. In AC voltage regulator, TRIACS cannot be used for a
(A) Back emf load
(B) Resistive load
(C) R–L Load
(D) Inductive load

10 A separately excited DC motor with a back emf constant 0.25 V/rpm, armature resistance of 4 and drawing an armature current of 10 A is driven by a single-phase half-controlled rectifier. The converter is working from a 230 V single-phase AC source with a firing angle of 45°. Under these operating conditions, the speed of the motor will be
(A) 273.5 rpm (B) 339 rpm
(C) 547 rpm (D) 457 rpm

11. In a self-controlled synchronous motor fed from a variable frequency inverter
(A) Stability problems exists
(B) The rotor speed is decided by the stator frequency.
(C) The stator frequency is decided by the rotor speed.
(D) Rotor poles invariably have damper windings.

12. A 3-φ, 380 V, 50 Hz, AC mains-fed thyristor bridge is feeding a 380 V DC, 12 kW, 1500 rpm, separately excited DC motor with a ripple free continuous current in the DC link. Under all operating conditions, neglecting the losses, the power factor of the AC mains at $(1/4)^{\text{th}}$ the rated speed is
(A) 0.2 (B) 0.3
(C) 0.6 (D) 0.8

13. Which among the following is best employed for the PWM inverter control for an AC motor?
(A) Cycle converter
(B) Controlled rectifier
(C) AC regulator
(D) Uncontrolled rectifier

14. A DC link is used in converter method of speed control of induction motor. The DC link is used to
(A) Vary the slip
(B) Trigger the thyristor of the inverter
(C) Provide a constant peak flux density to the gap and core saturation.
(D) Provide necessary DC voltage to the inverter

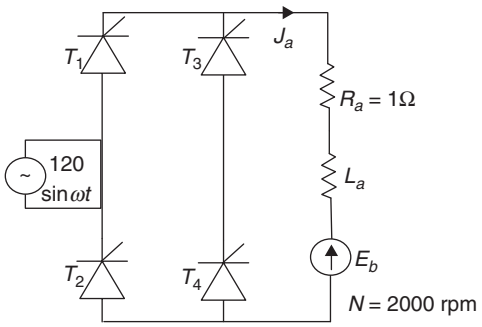
15. For the speed control of a 3-φ squirrel cage motor, the most suitable solid state converter used is
(A) Load-commutated inverter
(B) Cyclo converter
(C) Current source inverter
(D) Voltage source inverter

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. For a single-phase to single-phase cycloconverter,
(A) Natural commutation is employed in step-up converter.
(B) Forced commutation is employed in step-down converter.
(C) Natural commutation is employed in step-down converter.
(D) Forced commutation is employed in both step-up and step-down cyclo converter.

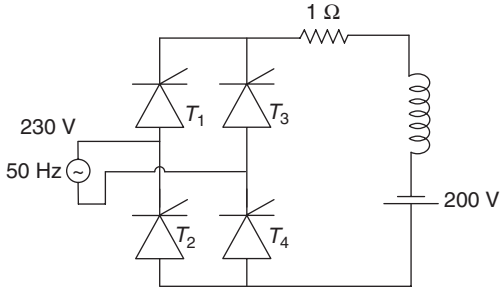
2. A single-phase fully controlled rectifier feeds a separately excited DC motor as shown in figure. If the back emf constant of the motor is 0.03 V/rpm and armature resistance is 1 Ω, calculate the armature current for a firing angle of 30°, assuming continuous conduction.



(A) 6.2 A (B) 12.4 A
(C) 5 A (D) 0.62 A

3. A single-phase half-controlled bridge feeds a separately excited DC motor. The average output voltage of the converter is 100 V at a firing angle of 0°. The firing angle at an average voltage of 75 V will be (assuming continuous conduction)
(A) 30° (B) 45°
(C) 60° (D) 90°

4. Regenerative braking of a separately excited DC motor fed by a single-phase fully controlled rectifier is shown in figure. If the firing angle is set at 160°, then the armature current (assuming a constant DC current in the armature) will be:



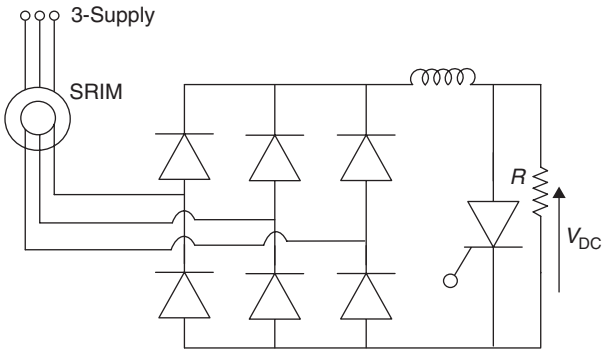
(A) 10 A (B) 15.6 A
(C) 1.2 A (D) 5.62 A

5. Which among the following converters is suitable for operation of a DC drive in forward motoring and forward regenerating mode?
(A) Single-phase half-controlled rectifier
(B) Three-phase half-controlled rectifier
(C) Single-phase fully-controlled rectifier
(D) None of the above

6. A single-phase half-controlled converter feeds a separately excited DC motor. At no load, the armature current becomes discontinuous. The instantaneous output voltage of the converter when current drops to zero amounts to:
(A) Instantaneous supply voltage
(B) Instantaneous back emf of motor
(C) Sudden spike in armature voltage
(D) Cannot be predicted

7. In a self-controlled permanent magnet synchronous motor drive,
(A) The stator supply frequency tracks the rotor speed.
(B) The stator has permanent magnets.
(C) The stator frequency is independently controlled.
(D) The damper windings in rotor are used for damping rotor oscillations.

8. The speed of a three-phase 600 V, 6 pole, 50 Hz slip-ring induction motor drive is controlled by employing a GTO chopper in the rotor circuit. The effective stator to rotor turns ratio is 1:0.6. The speed of the motor is maintained at 960 rpm. The rectified DC voltage in the rotor circuit is



(A) 19.5 V (B) 38 V
(C) 50 V (D) 28.6 V

9. Slip power recovery scheme can be used to control the speed of an induction motor drive
(A) Above synchronous speed only.
(B) Below synchronous speed only.
(C) Both above and below synchronous speed
(D) Above twice synchronous speed

10. Integral cycle control in a single-phase AC voltage controller
(A) Is suitable for application with small mechanical and thermal time constants.

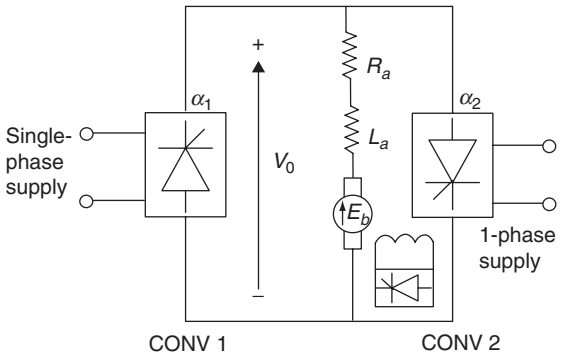
(B) Is suitable for applications with large mechanical and thermal time constants.
(C) Provides a sinusoidal output voltage.
(D) Only (a) and (c) are true.

11. A single-phase voltage controller feeds a resistive load of 20 Ω with an input voltage of 110 V, 50 Hz. The load is connected to supply for 8 cycles and disconnected for 4 cycles. The RMS of the output voltage will be:

(A) $\frac{110}{\sqrt{2}}$ V (B) 90 V

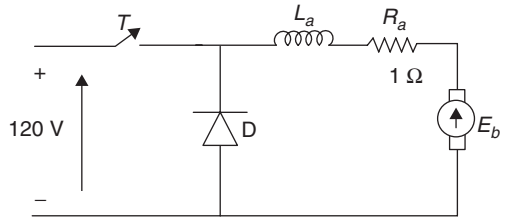
(C) $90\sqrt{2}$ V (D) 73 V

12. The figure shows a single-phase dual converter feeding a separately excited DC motor. For the motor to operate in forward motoring mode in circulating current mode,



(A) Converter 1 firing angle $\alpha_1 > 90°$; converter 2 firing angle $\alpha_2 < 90$ field reversed.
(B) Converter 1 firing angle $\alpha_1 < 90°$; converter 2 $\alpha_2 > 90°$; field correct polarity.
(C) Converter 1 firing angle $\alpha_1 < 90°$; field reversed.
(D) None of the above.

13. The figure shows a chopper circuit feeding a separately excited de motor with a back emf constant of k_b = 0.21 V/rpm at a speed of 276 rpm. The duty ratio of chopper is 0.5. Assuming continuous conduction, the armature current will be



(A) 1 A
(B) 3 A
(C) 5 A
(D) 2 A

14. In a salient-pole synchronous motor, if the field winding is disconnected from exciting voltage, the motor continues to rotate at synchronous speed. This is due to

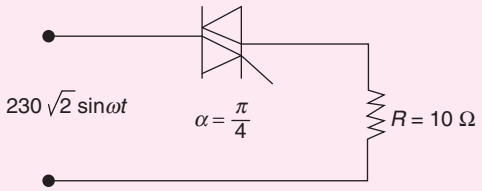
(A) Rotor inertia
(B) Reluctance torque
(C) Electromagnetic torque
(D) Permanent magnet in rotor

15. Rotor resistance control in a three-phase induction motor drive using a chopper in rotor circuit is used to control the drive.

(A) Above synchronous speed alone
(B) Below synchronous speed alone
(C) Both above and below synchronous speed
(D) Up to 0.2 times the base speed from standstill

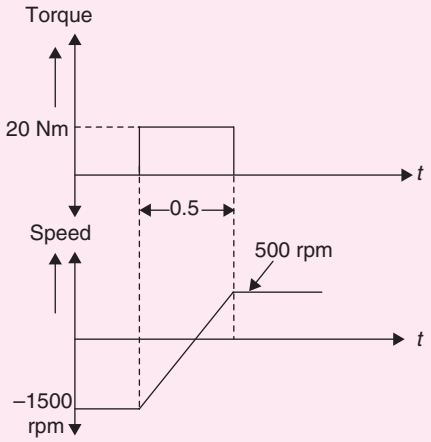
Previous Years' Questions

1. The triac circuit shown in figure controls the AC output power to the resistive load. The peak power dissipation in the load is **[2004]**



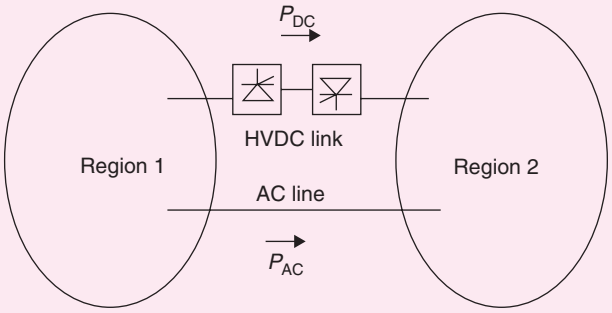
(A) 3968 W
(B) 5290 W
(C) 7935 W
(D) 10,580 W

2. A variable speed drive rated for 1500 rpm, 40 Nm is reversing under no load. Figure shows the reversing torque and the speed during the transient. The moment of inertia of the drive is **[2004]**



(A) 0.048 kgm²
(B) 0.064 kgm²
(C) 0.096 kgm²
(D) 0.128 kgm²

3. Two regional systems, each having several synchronous generators and loads are interconnected by an AC line and a HVDC link as shown in the figure. Which of the following statements is true in the steady state? **[2007]**

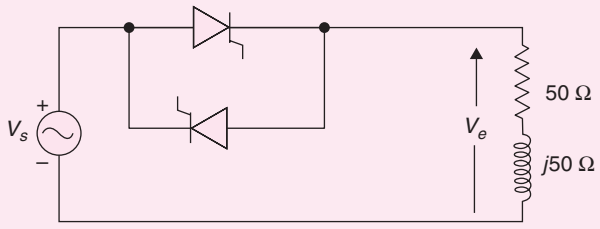


(A) Both regions need not have the same frequency.
(B) The total power flow between the regions ($P_{AC} + P_{DC}$) can be changed by controlling the HVDC converters alone.
(C) The power sharing between the AC line and the HVDC link can be changed by controlling the HVDC converters alone.
(D) The direction of power flow in the HVDC link (P_{DC}) cannot be reversed.

4. A three-phase, 440 V, 50 Hz AC mains-fed thyristor bridge is feeding a 440 V DC, 15 kW. 1500 rpm separately excited DC motor with a ripple-free continuous current in the DC link under all operating conditions. Neglecting the losses, the power factor of the AC mains at half the rated speed is **[2007]**

(A) 0.354 (B) 0.372
(C) 0.90 (D) 0.955

5. In the single-phase voltage controller circuit shown in the figure, for what range of triggering angle (α), the output voltage (V_0) is not controllable? **[2008]**



(A) $0° < \alpha < 45°$ (B) $45° < \alpha < 135°$
(C) $90° < \alpha < 180°$ (D) $135° < \alpha < 180°$

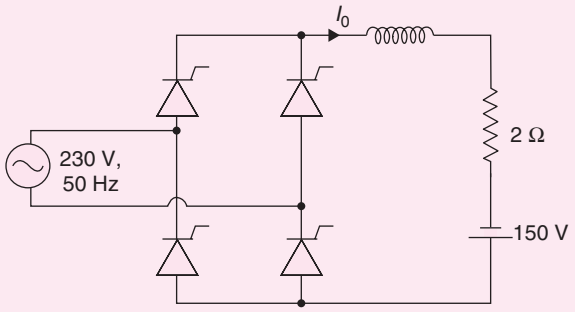
6. A 220 V, 20 A, 1000 rpm, separately excited DC motor has an armature resistance of 2.5 Ω. The motor is controlled by a step-down chopper with a frequency of 1 kHz. The input DC voltage to the chopper is 250 V. The duty cycle of the chopper for the motor to operate at a speed of 600 rpm delivering the rated torque will be **[2008]**

(A) 0.518 (B) 0.608
(C) 0.852 (D) 0.902

7. A 220 V. 1400 rpm, 40 A separately excited DC motor has an armature resistance of 0.4 Ω. The motor is fed from a single-phase circulating current dual converter with an input AC line voltage of 220 V (RMS). The approximate firing angles of the dual converter for motoring operation at 50% of rated torque and 1000 rpm will be **[2008]**

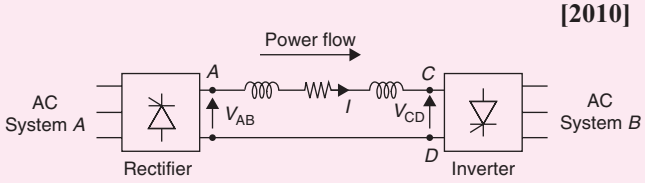
(A) 43°, 137° (B) 43°, 47°
(C) 39°, 141° (D) 39°, 51°

8. A single-phase fully controlled converter bridge is issued for electrical braking of a separately excited DC motor. The DC motor load is represented by an equivalent circuit as shown in the following figure. Assume that the load inductance is sufficient to ensure continuous and ripple free load current. The firing angle of the bridge for a load current of I_0 = 10 A will be **[2008]**



(A) 44° (B) 51°
(C) 129° (D) 136°

9. Power is transferred from system A to system B by an HVDC link as shown in the figure. If the voltages V_{AB} and V_{CD} are as indicated in the figure, and $I > 0$, then **[2010]**



(A) $V_{AB} < 0$, $V_{CD} < 0$, $V_{AB} > V_{CD}$
(B) $V_{AB} > 0$, $V_{CD} > 0$, $V_{AB} < V_{CD}$
(C) $V_{AB} > 0$, $V_{CD} > 0$, $V_{AB} > V_{CD}$
(D) $V_{AB} > 0$, $V_{CD} < 0$

10. In a constant V/f control of induction motor, the ratio V/f if maintained constant from 0 to base frequency, where V is the voltage applied to the motor at fundamental frequency f. Which of the following statements relating to low-frequency operation of the motor is TRUE? **[2014]**

(A) At low frequency, the stator flux increases from its rated value.
(B) At low frequency, the stator flux decreases from its rated value.
(C) At low frequency, the motor saturates.
(D) At low frequency, the stator flux remains unchanged at its rated value.

11. A single-phase SCR-based AC regulator is feeding power to a load consisting of 5 Ω resistance and 16 mH inductance. The input supply is 230 V, 50 Hz AC. The maximum firing angle, at which the voltage across the device becomes zero all throughout, and the RMS value of current through SCR, under this operating condition, respectively, are **[2014]**

(A) 30° and 46 A (B) 30° and 23 A
(C) 45° and 23 A (D) 45° and 32 A

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** C	**3.** D	**4.** B	**5.** B	**6.** D	**7.** A	**8.** C	**9.** D	**10.** C
11. B	**12.** A	**13.** D	**14.** C	**15.** B					

Practice Problems 2

1. C	**2.** A	**3.** C	**4.** D	**5.** C	**6.** B	**7.** A	**8.** A	**9.** C	**10.** B
11. B	**12.** B	**13.** D	**14.** B	**15.** B					

Previous Years' Questions

1. D	**2.** A	**3.** C	**4.** A	**5.** A	**6.** B	**7.** C	**8.** C	**9.** C	**10.** B
11. C									

TEST
Power Electronics and Drives

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. Commutation overlap in the phase-controlled AC to DC converter is due to
(A) Load inductance
(B) Harmonic content of load current
(C) Switching operation in the inverter
(D) Non linearity in the device characteristics.

2. Figure 1 shows a composite switch consisting of a power transistor in series with a diode. Assuming that the transistor switch and diode are ideal, *I–V* characteristic of the composite switch is

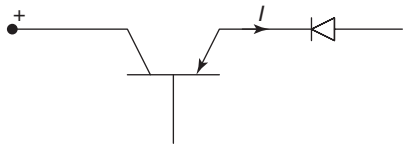
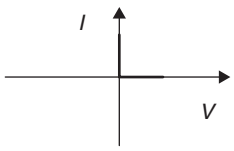
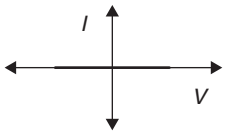
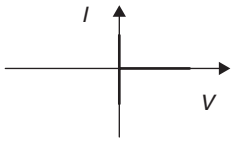
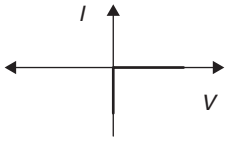


Figure 1

(A) 

(B) 

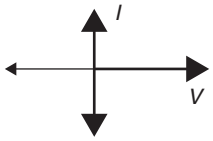
(C) 

(D) 

3. Analysis of voltage wave form of a single phase bridge converter shows that it contains *x*% of 6th harmonic. The 6th harmonic content of the voltage waveform of a 3-phase bridge converter would be _____
(A) Less than *x*% due to an increase in the number of pulses
(B) Equal to *x*%, the same as that of the single-phase converter
(C) Greater than *x*% due to changes in the input and output voltages of the converter
(D) Difficult to predict as the analysis of converters is not governed by any generalized theory

4. In a three phase controlled bridge rectifier, with an increase of overlap angle, the output dc voltage
(A) Decreases
(B) Increases
(C) Does not change
(D) Depends on load inductance

5. The wave form of the power electronic device shown is related to



(A) SCR (B) IGCT
(C) IGBT (D) FET

6. If an AC chopper is supplying power to an inductive load, then firing pulse to the SCR, it
(A) May have a width equal to turn-on time of the SCR.
(B) Should be a single pulse of long duration.
(C) Should be a series of pulses of short duration.
(D) Should be train of pulses duration equal to the conduction period of the SCR.

7. In a single phase full wave SCR circuit with R–L load
(A) Power is delivered to the source for delay angles less than 90°
(B) The SCR changes from converter to inverter at $\alpha = 90°$
(C) The negative dc voltage maximum is at $\alpha = 180°$
(D) To turn off the thyristor, maximum delay angle must be less than 360°

8. Consider a DC to dc converter (Chopper) having a load resistance of 10 Ω and an input voltage of 232 V. if the voltage drop across the converter during ON time is 2 V and the frequency, of operation is 1 kHZ at a duty cycle of 0.4, then the average output voltage of the converter will be
(A) 94.0 V
(B) 92.8 V
(C) 92.0 V
(D) 90.8 V

9. The correct sequence of the following devices in the increasing order of turn-off times is
(A) MOSFET, IGBT, BJT, Thyristor
(B) IGBT, MOSFET, BJT, Thyristor
(C) Thyristor, BJT, MOSFET, IGBT
(D) MOSFET, BJT, IGBT, Thyristor

10. Figure 2 shows a 3-ϕ inverter fed by constant voltage source V_{DC} and connected to a balanced resistive load at the output. Each switching device may conduct for 120° or for 180°. The waveform shown in Figure 3 is the

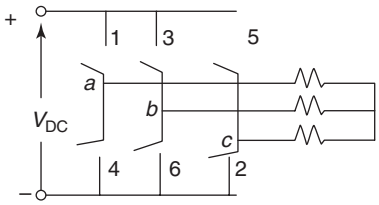


Figure 2

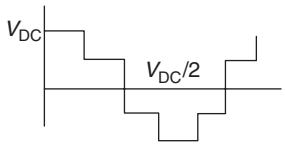
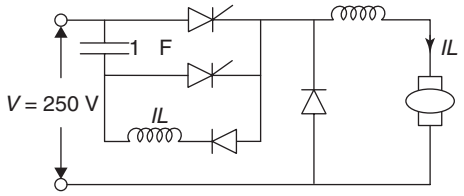


Figure 3

(A) Line voltage with 120° firing
(B) Load phase voltage with 120° firing
(C) Line voltage with 180° firing
(D) Load phase voltage with 180° firing

11. A voltage commutated chopper operating at 1 KHz is used to control the speed of the DC motor. The load current is assumed to be constant at 12 A



The minimum time for which SCR M should be ON.
(A) 140 μs (B) 70 μs (C) 210 μs (D) 280 μs

12. In the Figure 4, the ideal switch S is switched ON and OFF with a switching frequency $f = 20$ kHz. The circuit is operated in steady state at the boundary of continuous and discontinuous conduction so that the inductor current i is as shown in fig. The ON time of the switch and the peak value of current I_p is

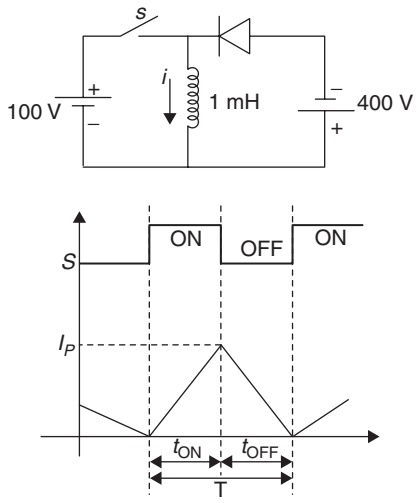


Figure 4

(A) 40 μs, 4 A
(B) 30 μs, 3 A
(C) 33.33 μs, 8 A
(D) 16.67 μs, 4 A

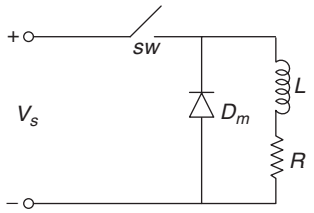
13. Match List–I with List–II and select the correct answer using the codes given below the lists

	List–I		List–II
a.	Semi converters	1.	Additional reactive power loading
b.	Source inductance of the converter	2.	Two quadrant operation with improved displacement factor on the AC side
c.	Sequence control of converters	3.	Unity displacement factor
d.	Current limit control of converters	4.	One quadrant operation with better displacement factor

Codes:

	a	b	c	d
(A)	2	1	4	3
(B)	2	3	4	1
(C)	4	1	2	3
(D)	4	3	2	1

14. The chopper in the figure has a load resistance $R = 0.25\ \Omega$ input voltage $V_s = 550$ V. The average load current $I_a = 200$ A and chopping frequency $f = 250$ Hz. The inductance L which would limit the maximum load ripple to 10% of I_a is



(A) 12.5 mH (B) 20.5 mH
(C) 27.5 mH (D) 38.5 mH

Common Data for Questions 15 and 16:
The single phase half controlled AC to DC bridge converter of Figure 5 supplies a 10 Ω resistor in series with a 100 V back emf load. The firing angle of the thyristors is set to 60°

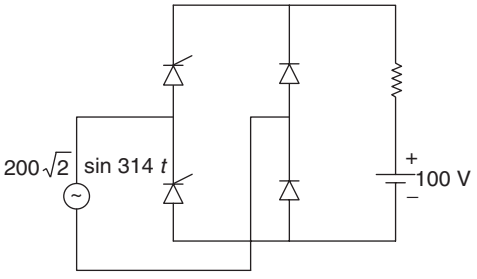


Figure 5

15. Average current through the resistor is
(A) 10.5 A (B) 5.4 A
(C) 7.2 A (D) 10 A

16. What will be the new average current through the resistor if a very large inductor is connected in series with the load.
(A) 10.5 A (B) 13.5 A
(C) 15 A (D) 8 A

17. In a 3 phase half converter, if the average output voltage obtained is 230 V. If it is required to obtain a maximum possible output voltage of 200% of average output voltages, then the per phase input voltage applied shall be
(A) 393.31 V (B) 268.31 V
(C) 681.23 V (D) 556.23 V

18. For a single phase fully controlled bridge converter fed from a 230 V, 50 Hz supply and connected to an *RL* load, the power factors at firing angles of 60° and 45° respectively are
(A) 0.64, 0.45 (B) 0.45, 0.64
(C) 0.78, 0.45 (D) 0.64, 0.78

19. A 230 V, 50 Hz supply feeds a full bridge converter connected to an *RL* load across the output, (The inductance is large enough to provide a constant and ripple free current). If $\alpha = 45°$, for an average out current of 5A to flow through the load, the value of R_L must be
(A) 29.3 Ω (B) 19.3 Ω
(C) 9.3 Ω (D) 22.76 Ω

Common Data for Questions 20 and 21:
The three-phase full-wave controller shown in figure 6 supplies a *Y*-connected resistive load of $R = 10\ \Omega$ and the line-to-line input voltage is 208 V, 60 Hz. The delay angle is $\alpha = \frac{\pi}{3}$

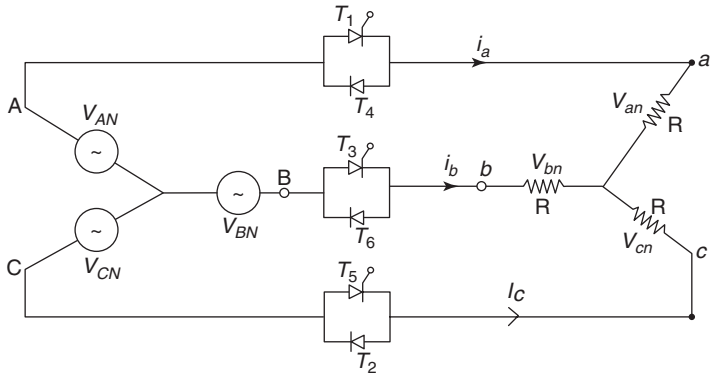


Figure 6

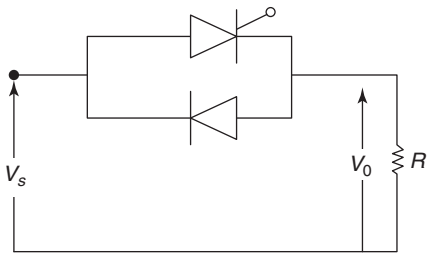
20. The rms output voltage per phase is
(A) 99.9 V
(B) 100.9 V
(C) 149.9 V
(D) 190 V

21. The input power factor is
(A) 0.64 lag
(B) 0.78 lag
(C) 0.84 lag
(D) 0.92 lag

22. A single phase half controlled bridge converter with input of 380 $\sin\omega t$, 50 Hz; supplies power to an *RL* load. The thyristors are operated in the range of 45° to 60° for the converter to supply a constant average output current of 10 A, the value of R should be
(A) 10 Ω
(B) 30 Ω
(C) 40 Ω
(D) 20 Ω

23. A Chopper circuit is operating on TRC principle at a frequency of 2 kHz on a 220 V DC supply. If the load voltage is 170 V, the conduction and blocking period of the thyristor in each cycle are, respectively
(A) 3.86 μs and 1.14 μs
(B) 38.6 μs and 11.4 μs
(C) 3.86 ms and 1.14 ms
(D) 0.386 ms and 0.114 ms

Common Data for Questions 24 and 25:
A 1-ϕ unidirectional AC voltage controller shown in the following figure has a resistive load of $R = 10\ \Omega$ and the input voltage is $V_s = 120$ V, 60 Hz. The delay angle of thyristor is $\alpha = \frac{\pi}{2}$.



24. The rms value of output voltage V_0 is
(A) 103.92 V
(B) 130.92 V
(C) 169.7 V
(D) 196.7 V

25. The input power factor is
(A) 0.75 lag
(B) 0.8 lag
(C) 0.866 lag
(D) None of these

Answer Keys

1. A	**2.** B	**3.** A	**4.** A	**5.** D	**6.** D	**7.** C	**8.** C	**9.** B	**10.** A
11. A	**12.** A	**13.** C	**14.** C	**15.** C	**16.** B	**17.** A	**18.** B	**19.** A	**20.** B
21. C	**22.** D	**23.** D	**24.** A	**25.** C					

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Electrical and Electronic Measurements

UNIT VI

Chapter 1: Error Analysis and R, L, C Measurements 3.763

Chapter 2: Electrical Measuring Instruments 3.784

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Chapter 4: Cathode Ray Oscilloscope and Electronic Voltmeters 3.815

Chapter 5: Magnetic Measurements and Transducers 3.828

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Chapter 1

Error Analysis and R, L, C Measurements

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Electrical measuring instruments
- Indicating instruments
- Error analysis
- Sources of error
- Power of a factor
- Resistance measurement
- Kelvin's double bridge method
- Potentiometer method
- Measurement of medium resistance
- Substitution method
- Wheatstone bridge method

Electrical Measuring Instruments

Classification

There are many ways in which instruments are classified. Broadly instruments are classified into two.

1. Absolute Instruments
2. Secondary instruments

Absolute Instruments

These instruments give the magnitude of the quantity under measurement in terms of physical constants of the instrument and their deflection.

- No previous calibration or comparison is required.

Secondary Instruments

They are those in which the electrical quantity to be measured can be determined from the deflection of instrument only when they have been pre calibrated against an absolute instrument.

Classification of Secondary Instruments

Indicating Instruments

They indicate the instantaneous value of electrical quantity at the time at which it is being measured. The indications can be by pointers moving over calibrated dial.

Example: Ordinary ammeters, Voltmeters, Wattmeters, etc.

Recording Instruments

They give a continuous record of variations of a quantity over a selected period of time.

- The system carries an ink pen which rests lightly on a chart or graph, which is moved at a uniform—low speed, perpendicular to the deflection of the pen.
- The path traced out by the pen presents a continuous record of the variations in the deflection of the instrument.

Integrating Instruments

These instruments measure and register either the total quantity of electricity (Amp hr) or the total amount of electrical energy in (W-h (or) kWh) supplied to a circuit in a given time, by a set of dials and pointers.

Electrical Principles of Operations

Measuring Instruments are generally classified according to the physical effects of an electric current or potential, utilized in their operation.

The effects generally utilized are

1. Magnetic Effect (Ammeters and Voltmeters)
2. Electrostatic Effect (Voltmeters only)
3. Electrodynamic Effect (Ammeters, Voltmeters and Wattmeter)
4. Electromagnetic Effect (Ammeters, Voltmeters and watt hour meters)

5. Thermal Effect (Ammeters and Voltmeters)
6. Chemical Effect (Amp-hour meter)

Error Analysis

Error is defined as the deviation of the true value from the desired value. It is expressed either as limiting error or as a percentage of error.

1. **Limiting Error (Guarantee Error)**
   Limiting error can be defined as the difference between the true value of the variable and measured value of the variable.
   $$\pm\delta A = A_n - A_a$$
   $A_a \rightarrow$ Actual value (true value)
   $A_n \rightarrow$ Nominal value (measured value)
   $\delta A \rightarrow$ Limiting error (absolute error)
2. **Relative (%) Limiting Error**
   $$\% \text{ Error} = \frac{\text{Limiting error value}}{\text{Nominal value}} \times 100$$
   $$\% \text{ Error} = \frac{\delta A}{A_n} \times 100$$
   $$\% \text{ Error} = \left[\frac{A_n - A_a}{A_n}\right] \times 100$$

Types of Errors

Static Error: It is the numerical difference between the actual value of a quantity and its value as obtained by measurement. They are subdivided into the following types.

1. Gross error
2. Systematic error
3. Random error

(a) **Gross Errors:** It is due to human mistakes in reading or in using instruments or errors in recording observations.

(b) **Systematic Errors:** These errors occur due to short comings of the instrument such as a defective or worn parts or ageing or effects of the environment on the instrument.

1. **Instrumental Errors:** These are inherent in measuring instruments because of their mechanical structure. These can be avoided by
   - Selecting a suitable instrument for the particular measurement application.
   - By applying correction factors after determining the amount of instrumental error of calibrating the instrument against a standard.
2. **Environment Errors:** These are due to conditions external to the measuring device including conditions in the area surrounding the instrument such as effects of changes in temperature, humidity, barometric pressure or effect of magnetic or electrostatic fields.
3. **Observational Errors:** These are errors introduced by the observer. The most common is the parallax error introduced in reading a meter scale and the error of estimation.
4. **Random Errors:** These are due to unknown causes and are normally small and follow the law of probability and are mathematically treated.

Sources of Error

1. Insufficient knowledge of process parameters and design conditions.
2. Poor maintenance.
3. Change in process parameters irregularities, upsets, etc.
4. Design limitations and poor design.
5. Due to person operating the instrument.

Rules

When two or more quantities, each of which are subjected to error are combined.

Sum

Suppose y is the final result of sum of measured quantities p, q, and r each of which are subjected to possible systematic errors $\pm d_p$, $\pm d_q$, $\pm d_r$

$$y = p + q + r$$

Relative limiting error

$$\frac{dy}{y} = \pm\left[\frac{p}{y}\frac{dp}{p} + \frac{q}{y}\frac{dq}{v} + \frac{r}{y}\frac{dr}{r}\right]$$

Resultant systematic error is equal to the sum of the products formed by multiplying the individual systematic errors by the ratio of each terms to the function.

Difference

$$y = p - q$$

Taking log and differentiating with respect to y and simplify, we get

Relative limiting error

$$\frac{\delta y}{y} = \pm\left(\frac{p}{y}\frac{\delta u}{U} - \frac{q}{y}\frac{\delta q}{q}\right)$$

Product

$$y = p.q.r$$

Taking log and differentiating with respect to y and simplify, we get

Relative limiting error

$$\frac{\delta y}{y} = \pm\left(\frac{\delta p}{P} + \frac{\delta q}{q} + \frac{\delta r}{r}\right)$$

Quotient

$$y = p/q$$

Taking logarithm and differentiating with respect to y, we get

Relative limiting error

$$\frac{\delta y}{y} = \pm\left(\frac{\delta p}{p} - \frac{\delta q}{q}\right)$$

Maximum possible error occurs when $\frac{\delta p}{P}$ is positive and $\frac{\delta q}{q}$ is negative or vice versa.

Power of a Factor

$$y = U^n$$

n may be + or _, integral or fraction. Taking logarithm and differentiating w.r.t. y.

Relative Limiting Error is

$$\frac{\delta y}{y} = \pm n\,\frac{\delta u}{U}$$

if $y = U^n V^m$

$$\frac{\delta y}{y} = \pm\left(\frac{n\delta u}{U} + \frac{m\delta v}{V}\right)$$

Statistical Analysis

1. **Arithmetic mean ($\bar{X}$):** The most probable value of the measured variable

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \frac{\Sigma X}{n}$$

$\bar{X}$ = arithmetic mean

$X_1, X_2, X_3, \ldots, X_n$ = readings

n = number of readings

2. **Deviation (d):** Departure of the observed reading from arithmetic mean of the group of readings

Deviation $X_1 = d_1 = X_1 - \bar{X}$

Deviation of $X_2 = d_2 = X_2 - \bar{X}$

Algebraic sum of all deviations for a given arithmetic mean = 0.

3. **Average Deviation (D):** Sum of absolute values of deviation divided by number of readings.

$$\bar{D} = \frac{|-d_1| + |-d_2| + 1 \ldots\ldots |-d_n|}{n} = \frac{\Sigma|d|}{n}$$

4. **Standard Deviation (S.D.):** The square root of sum of the individual deviation squared divided by number of readings.

$$\text{S.D.} = \sigma = \sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_n^2}{n}} = \sqrt{\frac{\Sigma d^2}{n}}$$

(for $n > 20$)

$$\text{S.D.} = S = \sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_n^2}{n-1}} = \sqrt{\frac{\Sigma d^2}{n-1}}$$

(for $n < 20$)

5. **Variance (V)**

V = (Standard deviation)^2

$$V = \text{S.D.}^2 = \sigma^2 = \frac{d_1^2 + d_2^2 + \ldots + d_n^2}{n}$$

$$V = a^2 = \frac{\Sigma d^2}{n} (\text{for } n > 20)$$

6. **Probable error of one reading**

$$r_1 = 0.6745\ \frac{\sqrt{\Sigma|d|^2}}{n-1}$$

7. **Probable error of mean**

$$r_m = \frac{r_1}{\sqrt{n-1}}$$

Solved Examples

Example 1: The limiting errors of measurement of power consumed and the current passing through a resistance are ± 2.5% and ± 0.5%, respectively, the limiting error of measurement of resistance will then be

(A) ±3.5% (B) ±5%

(C) ±0.5% (D) ±1.0%

Solution: (A)

(Limiting error in the measurement of resistance R) = (Limiting error in measurement of power + 2 × Limiting error in measurement of current)

$= \pm (2.5 + 2 \times 0.5)$

$= \pm 3.5\%$.

Example 2: A 0–160 V voltmeter has an accuracy of 1% full scale reading. The voltage is 80 V. The limiting error is

(A) 1% (B) 2%

(C) 2.5% (D) 3%

Solution: (B)

Magnitude of error $\partial A = \varepsilon_r A_s$

$= 0.01 \times 160 = 1.6$ V

Relative error = $\varepsilon_r = \frac{1.6}{80} = 0.02$ V

Voltage between limits

$$80(1 \pm 0.02) = 80 \pm 1.6 \text{ V}$$

$$\%\text{ limiting error} = \frac{1.6}{80} \times 100 = 2\%.$$

Example 3: A 0–200 V voltmeter has an error of ±2% of full-scale deflection. The range of readings if true voltage is 20 V would be

(A) 16–24 V (B) 19.6–20.4 V

(C) 10–30 V (D) None of the above

Solution: (A)

$$2\% \text{ of F.S.d} = 200 \times \frac{2}{100} = 4$$

Range would be $20 \pm 4 = 16–24$ V

Example 4: If a wattmeter reads 350 W on its 700 W full scale, the range of reading, if it has an error of ± 2% of true value, is

(A) 343–357 (B) 336–364
(C) 300–400 (D) None of the above

Solution: (A)

Basic Terminologies

1. **Accuracy:** It is the closeness with which an instrument reading approaches the true value of the quantity being measured.
   - It is specified in terms of limits of errors and is expressed as either percentage of scale or range of percentage of true value.
2. **Precision:** It is a measure of the reproducibility of the measurements, i.e. for a fixed value of a quantity precision is a measure of the degree of agreement within a group of measurements.
   - An indication of the measurement is obtained from the number of significant figures in which it is expressed.
   - The more the significant figure the greater the precision of measurement.
3. **Linearity:** When the output quantity of a measurement system is directly proportional to the input

   or

   Whenever the deflection θ (output) of the instrument is proportional to the current I (input). The system is said to be linear.
4. **Sensitivity:** It is the ratio of magnitude of the output signal or response to the magnitude of input signal or the quantity being measured.

$$\text{Sensitivity} = \frac{\text{Magnitude output response}}{\text{Magnitude of input}}$$

5. **Deflection Factor (or) Scale Factor (or) Inverse Sensitivity**

$$\text{S.F.} = \frac{\text{Magnitude of input}}{\text{Magnitude of output response}}$$

6. **Dead Time:** Dead time is defined as the time required by a measurement system to begin to respond after the measured quantity has been changed.
7. **Dead Zone:** It is defined as the largest change of input quantity for which there is no output of the instrument. It is also called THRESHOLD. It defines the smallest measurable input.
8. **Resolution:** The smallest increment in input (the quantity being measured).Which can be detected with certainly by an instrument is called resolution.

Resistance Measurement

Classification of Resistance

1. **Low Resistances:** Resistance of the order 1 Ω and below.
2. **Medium Resistances:** Resistance of the range above 1 Ω to 0.1 MΩ.
3. **High Resistances:** Resistance of the order 0.1 MΩ and upwards.

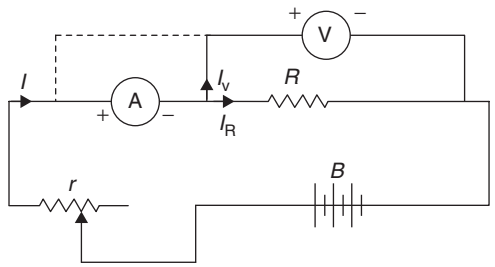
Measurement of Low Resistance

Ammeter–Voltmeter Method

- Voltmeter can be connected in 2 ways.
- Current through ammeter = current through unknown resistance + current through voltmeter.

$$I = I_R + I_V$$

[Steps of operation as explained in section. (III) (1)-measurement of medium resistance]



True value of unknown resistance $R = \frac{V}{I_R}$

$$= \frac{V}{I - I_v} = \frac{V}{I - \frac{V}{R_v}}$$

$$= \frac{V/I}{\left(1 - \frac{V}{IR_v}\right)} = \frac{R'}{\left(1 - \frac{R'}{R_v}\right)}$$

$$R = \frac{R'}{\left(1 - \frac{R'}{R_v}\right)}$$

where V = Voltage reading voltmeter
I = Current reading in ammeter
R_v = Resistance of voltmeter
R' = Circuit resistance

Accuracy of this method of measurement of low resistance is limited, since it depends on accuracy of voltmeter and ammeter.

Example 5: Given below are the test results for A–V method of measurement of resistance

$V = 200$ V (% error ± 2%)
$I = 16.67$ A (% error ± 2%)

The most approximate value of resistance measured (in Ω) would range between

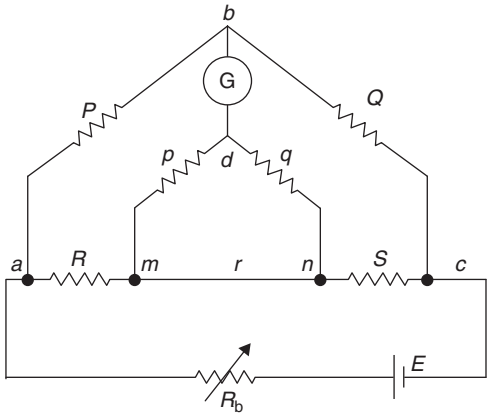
(A) $12 \pm 4\%$ (B) $12 \pm 2\%$
(C) 12 exactly (D) Cannot be determined

Solution: (A)
In the A–V method of measurement of resistance. The % errors in ammeter and voltmeter readings would get added up

i.e. % Error = $(\pm 2\%) + (\pm 2\%) = \pm 4\%$

$$R = \frac{V}{I} = \left(\frac{200}{16.67}\right) = 12 \pm 4\ \%$$

Kelvin's Double Bridge Method



- Kelvin's bridge is a modification of the wheat stone's bridge and provides increased accuracy.
- It incorporates the idea of second set of ratio arms, hence the name double bridge and the use of 4 terminal resistances for low arms.
- The first of ratio arms is P and Q-Second set of ratio arms p and q is used to connect galvanometer to a point d at the approximate potential between points m and n to eliminate the effect of connecting r between R and S.

Step 1: The ratio P/Q is made equal to p/q under balanced condition $I_G = 0$, which means $E_{AB} = E_{amd}$

$$E_{AB} = \frac{P}{P+Q} E_{AC}$$

$$E_{AB} = \left[\frac{P}{p+q}\right] I \left[R + S + \frac{(p+q)r}{p+q+r}\right] \quad (1)$$

$$E_{amd} = E_{am} + E_{md}$$

$$= IR + V_{mn}\left[\frac{p}{p+q}\right]$$

$$= IR + I(R_{eq})_{mn}\left[\frac{p}{p+q}\right]$$

$$= IR + I\left[\frac{(p+q)r}{p+q+r}\right]\left[\frac{p}{p+q}\right]$$

$$E_{amd} = I\left[R + \frac{p}{p+q}\left[\frac{(p+q)r}{p+q+r}\right]\right]$$

$$= I\left[R + \frac{Pr}{p+q+r}\right] \quad (2)$$

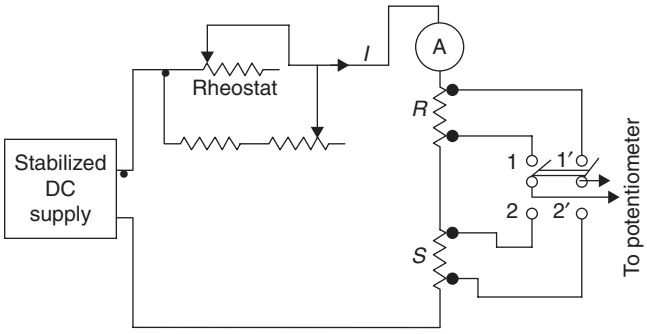
$$\left[\frac{P}{P+Q}\right] I \left[R + S + \frac{(p+q)r}{p+q+r}\right] = I\left[R + \frac{pr}{p+q+r}\right]$$

$$R = \frac{P}{Q} S + \frac{qr}{p+q+r}\left[\frac{P}{Q} - \frac{p}{q}\right]$$

If $P/Q = p/q$ then

$$R = \frac{P}{Q}.\ S$$

Potentiometer Method



- The unknown resistor R is connected in series with standard resistors.
- The current through the circuit is controlled with the help of a rheostat.
- A two-pole double throw switch is used, when switch is in position 1–1′ it connects unknown resistance to potentiometer and let the reading of potentiometer is V_R.

$$V_R = IR \quad (3)$$

- When switch is in position 2–2′ this connects the standard resistor to potentiometer suppose the reading of potentiometer is V_s

$$V_s = IS \quad (4)$$

From (3) and (4) $\frac{R}{S} = \frac{V_R}{V_s}$ $R = \frac{V_R}{V_s}.S$

Factors For Accuracy

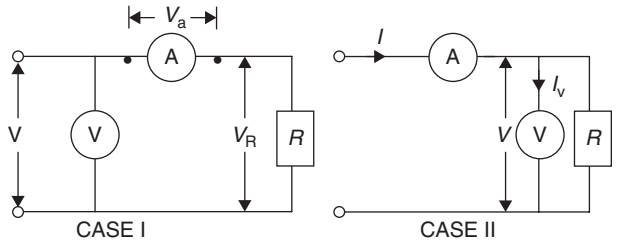
1. It mainly depends upon the assumption that there is no change in the value of current when two different measurements are taken. A stable DC supply is absolutely necessary.
2. S and R should be of same order or range.
3. The ammeter is within the capacity of resistors.

4. It is desirable that the current flowing through circuit be so adjusted that the value of voltage drop across each resistor is of the order of 1 V.

Measurement of Medium Resistance

Ammeter–Voltmeter Method

Two types of connections employed for ammeter voltmeter method are shown.



Case I:

- Ammeter measures true value of the current through resistance.
- Voltmeter does not measure true value across resistance but it indicates the sum of voltage across ammeter and measured resistance.
- Voltage across ammeter $V_a = IR_a$ (where R_a is resistance of ammeter)

Measured value of resistance $R_{m1} = \frac{V}{I}$

$$R_{m1} = \frac{V_R + V_a}{I} = \frac{IR + IR_a}{I} = R + R_a$$

True value of resistance $R = R_{m1} - R_a$

$$R = R_{m1}\left(1 - \frac{R_a}{R_{m1}}\right)$$

It is clear that $R = Rm$ only when $R_a = 0$.

$$\text{Relative Error } E_r = \frac{R_{m1} - R}{R} = \frac{R_a}{R}$$

- The error in measurement would be small when $R >>> R_a$.

Note: Case 1 to be used when measuring High resistance values.

Case II:

- Voltmeter measures the true value of voltage but ammeter measures the sum of currents through the resistance and voltmeter.
- Current through voltmeter $I_V = \frac{V}{R_V}$

(R_v is the resistance of voltmeter)

Measured value of resistance $R_{m2} = \frac{V}{I}$

$$= \frac{V}{I_R + I_V} = \frac{V}{\left(\frac{V}{R}\right) + \left(\frac{V}{R_v}\right)} = \frac{R}{1 + \left(\frac{R}{R_v}\right)}$$

True value of resistance $R = \frac{R_{m2} R_v}{R_v + R_{m2}}$

$$R = R_{m2}\left(\frac{1}{1 + \frac{R_{m2}}{R_v}}\right)$$

- It is clear that $R = R_{m2}$ only when $R_v = \mu$.

Since $R_v >>> R_{m2}$ and $\frac{R_{m2}}{R_v}$ is very small

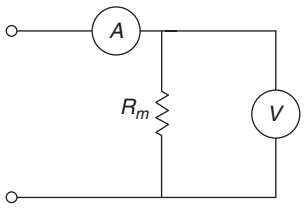
We can write $R = R_{m2}\,(1 + R_{m2}/R_v)$

$$\text{Relative error } E_r = \frac{R_{m2} - R}{R} = \frac{R_{m2}^2}{R R_v}$$

$E_r = \frac{-R}{R_v}$ Error measurement would be small when $R <<< R_v$.

Note: Case 2 to be used to measure Low Resistance Value.

Example 6: The connection diagram for an ammeter voltmeter method of measuring resistance is as given bellow:



The readings of the voltmeter and ammeter are 80 V and 4 mA, respectively. If the voltmeter has a sensitivity of 1000 Ω/V and a f.s.d of 120 V then the percentage error due to relative loading will be.

(A) 16.66% (B) 13.33 %
(C) −16.66% (D) −13.33%

Solution: (C)

Total circuit resistance = $R_t = \frac{80}{4 \times 10^{-3}} = 20$ kΩ

Resistance of milli ammeter is very small and hence negligible.

Voltmeter resistance $R_v = 1000 \times 120 = 120$ kΩ

Total circuit resistance = R_v/R_m

$$R_t = \frac{R_v R_m}{R_v + R_m}$$

$$R_m = \frac{R_v R_t}{R_v - R_t} = \frac{20 \times 120}{120 - 20}$$

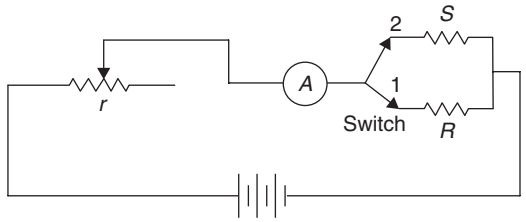
$$= 24\ \%\ \text{error} = \frac{\text{Measured value} - \text{True value}}{\text{True value}} \times 100$$

$$= \frac{20 - 24}{24} \times 100$$

$$= -16.66\%$$

Substitution Method

- In the connection diagram shown below, R is the unknown resistance while S is a standard variable resistance.
  A is an ammeter and r is regulating resistance. There is a switch for putting R and S into circuit alternately.

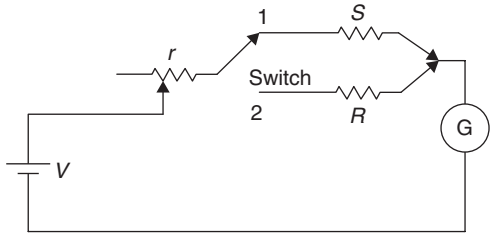


Step 1: Switch is put at position 1 and R is connected to the circuit. r is adjusted till the ammeter pointer is at a chosen scale mark.

Step 2: Switch is thrown to position 2 putting S in the circuit. The value of S is varied till the same deflection as was obtained with R in the circuit is obtained. The setting of the dials is read.

- Since substitution of one resistance for another has left the current unaltered, and provided that the emf of battery and position of r are unaltered the two resistances must be equal.
- Thus $R = S$.
- More accurate than ammeter–voltmeter Method.
- Accuracy is greatly affected if there is any change in battery emf during the time of readings on the two settings taken.
- Accuracy of measurement depends on the constancy of the battery emf and of the resistance of the circuit excluding R and S, upon sensitivity of the instrument, and upon the accuracy with which standard resistance S is known.

Example 7: The experiments set-up given below measures the value of unknown resistance in two steps.



- When $V = 24$ V, r is set at 400 Ω and $S = 80$ Ω, galvanometer shows a deflection of 30°.
- When switch is thrown to position 2 the supply voltage drops down to 22 V and the galvanometer still shows a deflection of 30°.

The value of unknown resistance R is

(A) 80 Ω (B) 60 Ω
(C) 40 Ω (D) None of the above

Solution: (C)

The galvanometer deflects in proportion to the current flowing through the circuit

$$\theta \propto I$$

Step 1: $I = \dfrac{24}{400 + 80} = \dfrac{24}{480} = 0.05$

Step 2: $I = \dfrac{22}{400 + R}$

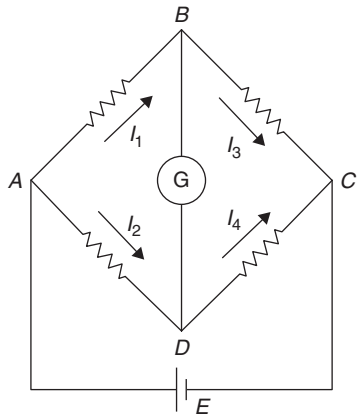
$$= \frac{22}{400 + R} = 0.05$$

$$400 + R = \frac{22}{0.05} = 440$$

$$R = 40\ \Omega.$$

Wheatstone Bridge Method

- Most widely used methods for measurement of medium resistance.
- Wheatstone bridge consists of known variable resistance P, Q and S. The unknown resistance is R.
- The resistance are arranged to form two parallel circuits and a sensitive galvanometer is bridged across two circuits between B and D.
- While balancing the bridge the values of resistance P, Q and S are so adjusted that



$$I_1 P = I_2 R \quad (5)$$

$$I_3 Q = I_4 S \quad (6)$$

But $$I_1 = I_3 = \frac{E}{P+Q}$$

$$I_2 = I_4 = \frac{E}{R+S}$$

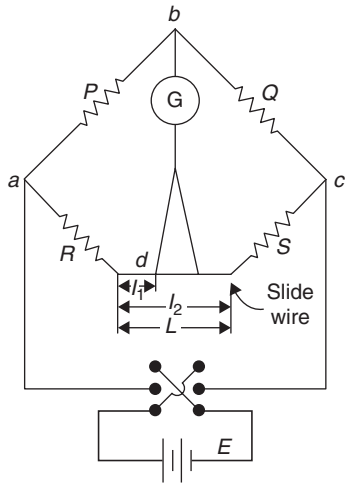
(5) ÷ (6) $$\frac{P}{Q} = \frac{R}{S}$$

$$R = S\,(P/Q)$$

Carey–Foster Bridge Method

- This bridge is especially suited for the comparison of two nearly equal resistances.
- The circuit consists of length L is included between R and S. Resistance P and Q are adjusted so that $\frac{P}{Q}$ is approximately equal to R/S.
- Exact balance is obtained by adjustment of the sliding contact on slide wire. Let l_1 be the distance of the sliding contact from the left hand end of the slide wire.
- R and S are interchanged and balance is again obtained. Let the distance now be l_2.

Let r be resistance/unit length of slide wire.



For first balance $$\frac{P}{Q}=\frac{R+l_1\ r}{S+(L-l_1)r} \quad (7)$$

For second balance $$\frac{P}{Q}=\frac{S+l_2\ r}{R+(L-l_2)r} \quad (8)$$

Adding (7) and (8) and add 1 on the both sides

$$\frac{R+l_1\ r}{S+(L-l_1)r}+1=\frac{S+l_2\ r}{R+(L-l_2)r}+1$$

$$\frac{R+S+L_r}{S+(L-l_1)r}=\frac{S+R+L_r}{R+(L-l_2)r}$$

$$(S-R)=(l_1-l_2)\ r$$

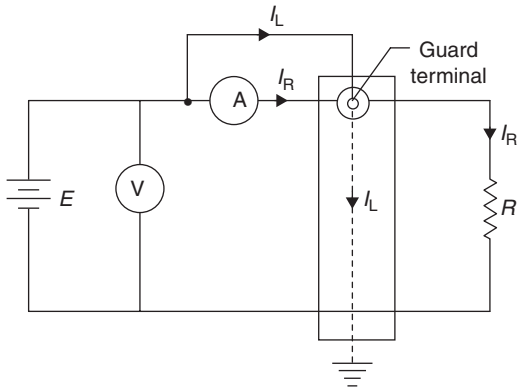
- Thus, S–R is obtained from resistance per unit length of the slide wire together with $(l_1 - l_2)$.
- The slide wire is then calibrated, i.e. r is obtained and again determining the difference in length $(l_1'-l_2')$.
- Suppose S is known as S' is its value when shunted by a known resistance.

$$\frac{(S-R)}{l_1-l_2}=\frac{S'-R}{l_1'-l_2'}$$

$$\Rightarrow \quad R=\frac{S(l_1'-l_2')-S'(l_1-l_2)}{(l_1'-l_2'-l_1+l_2)}$$

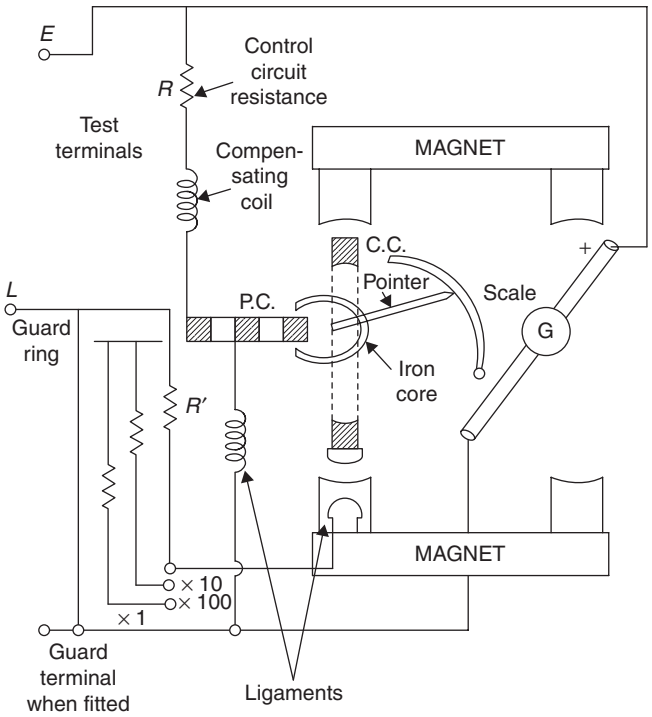
Measurement of High Resistance

Direct Deflection Method



- A guard terminal surrounding resistance terminal is connected to the battery side and micro ammeter.
- The guard terminal and resistance terminal are almost at the same potential and hence there is no flow of current between them.
- The leakage current I_L which would otherwise flow through the μ ammeter bypasses the μ ammeter.
- The μ ammeter indicates the current I_R only.
- The resistance values is determined by the readings of voltmeter and ammeter as $R=\frac{V}{I_R}\ \Omega$.

Megger Method



Construction

- It consists of a hand-driven DC generator and a direct reading ohmmeter.

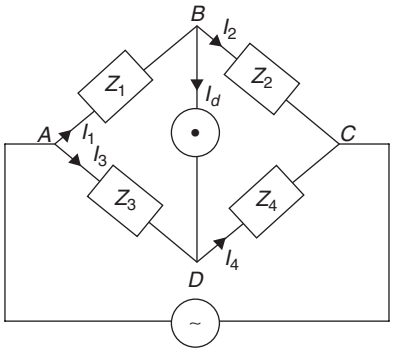
- Permanent element consist of three coils, 1 current coil, potential or pressure coil and compensating coil.
- The coils are connected to the circuit by flexible leads or ligaments.
- E and L are test terminals.
- Current coil is connected in series with terminal L.
- R' protects the current coil in case the test terminals are short circuited and controls the range of the instrument.
- Pressure coil (PC) in series with a compensating coil and protection resistance R is connected across generator terminals.
- Compensating coil is provided for better scale proportions and to make instrument astatic.

Working

- When, current from the generator flows through the pressure coil, the coil tends to align itself at right angles to the permanent magnet field.
- When test terminals are open, i.e. infinite resistance no current flows through current coil and pressure coil governs the motion of moving elements causing it to move to its extreme counter clockwise position, i.e. infinite resistance. Current coil produces clockwise torque on moving element.
- When terminals L and E are short circuited, the current flowing through the current coil (if external resistance = 0) is large enough to move the pointer to its extreme clockwise position zero.
- For any resistance connected between L and E, the opposing torque of the coils balances each other so that pointer comes to rest at some intermediate point on the scale.

Bridges

- Measurement of inductance and capacitance may be made conveniently and accurately by employing AC bridge networks.
- The AC bridge is a modified form of the Wheatstone bridge. It consists of four arms, a source of excitation and a balance detector which is sensible to small alternating potential differences.



- If B and D are at same potential $V_{AB} = V_{AD}$

$$I_1Z_1 = I_3Z_3 \qquad (9)$$

- In this condition $V_{BC} = V_{DC}$

$$I_2Z_2 = I_4Z_4 \qquad (10)$$

Divide (9) by (10)

$$\frac{I_1Z_1}{I_2Z_2} = \frac{I_3Z_3}{I_4Z_4}$$

But in balance condition $I_D = 0$.

$$I_1 = I_2 \text{ and } I_3 = I_4.$$

Also

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

$$Z_1Z_4 = Z_2Z_3$$

In polar form, $Z_1Z_4(\angle\theta_1 + \theta_4) = Z_2Z_3(\angle\theta_3 + \theta_2)$

Two conditions for balancing bridge

1. $Z_1Z_4 = Z_2Z_3$
2. $\angle\theta_1 + \theta_4 = \angle\theta_2 + \theta_3$

Measurement of Inductance

Measurement of Self-inductance

Maxwell's Inductance Bridge

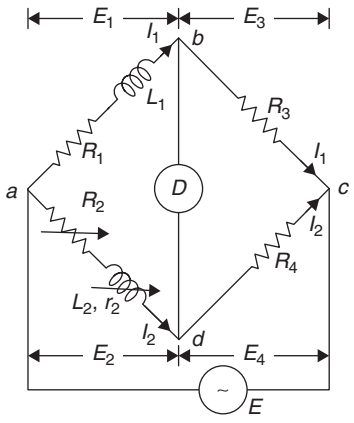
- This bridge circuit measures an inductance by comparing with a variable standard self-inductance.

Let L_1 = Unknown inductance of resistance R_1

L_2 = Unknown inductance of fixed resistance r_2

R_2 = Variable resistance connected in series with inductor L_2.

R_3, R_4 = Known non-inductive resistances



From concept of bridges (1.3)

$$Z_1Z_4 = Z_2Z_3$$

$$(R_1 + j\omega L_1)\,R_4 = (R_2 + j\omega L_2)\,R_3$$

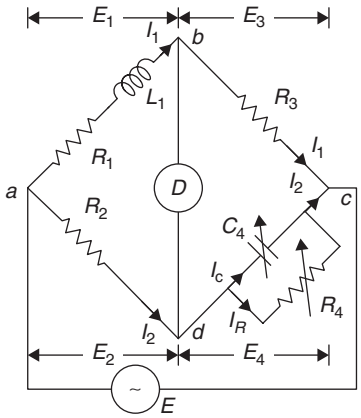
Equating real and imaginary parts separately

$$R_1R_4 = R_2R_3 \text{ (or) } R_1 = \left(\frac{R_3}{R_4}\right).(R_2 + r_2)$$

$$j\omega L_1\,R_4 = j\omega L_2\,R_3 \text{ (or) } L_1 = \left(\frac{R_3}{R_4}\right)L_2$$

Maxwell's Wien Bridge

- In this bridge, an inductance is measured by comparison with a standard variable capacitance.
- The connection diagram is as given below.



$$Z_1 = R_1 + j\omega L_1;\ Z_3 = R_3$$

$$Z_2 = R_2\ Z_4 = \frac{1}{\frac{1}{R_4} + \frac{1}{jxc}}$$

$$Z_4 = \frac{1}{\frac{1}{R_4} + j\omega C_4} = \frac{R_4}{1 + j\omega C_4 R_4}$$

At balance

$$Z_1 Z_4 = Z_2 Z_3$$

$$(R_1 + j\omega L_1)\left(\frac{R_4}{1 + j\omega C_4 R_4}\right) = R_2\ R_3$$

$$R_1 R_4 + j\omega L_1 R_4 = R_2 R_3 + j\omega R_2 R_3\ C_4 R_4.$$

Equating real and imaginary parts

$$R_1 = \frac{R_2\ R_3}{R_4}$$

$$L_1 = C_4 R_2 R_3$$

Advantages

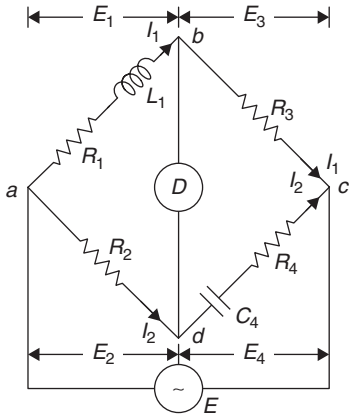
1. If we choose R_4 and C_4 as variable elements, the two balance equations are independent.
2. Frequency does not appear in any of the equations.
3. This bridge yields simple expressions for L_1 and R_1 in terms of known bridge elements.
4. This bridge is useful for measurement of a wide range of inductance at power and audio frequencies.

Disadvantages

1. Bridge requires a variable standard capacitor which may be very expensive if calibrated to a high degree of accuracy.

Hay's Bridge

- It is modification of Maxwell's Bridge.
- It uses a resistance in series with a standard capacitor.



L_1 = Unknown inductance having a resistance

R_2, R_3, R_4 = Known non-inductive resistance

C_4 = Standard capacitor

$$Z_1 = R_1 + j\omega L_1,\ Z_2 = R_2,\ Z_3 = R_3$$

$$Z_4 = (R_4 - j/\omega C_4)$$

At balance

$$(R_1 + j\omega L_1)\ (R_4 - j/\omega C_4) = R_2 R_3$$

$$(R_1 R_4) + \frac{L_1}{C_4} + j\omega L_1 R_4 - jR_1/\omega C_4 = R_2\ R_3$$

$$R_1 R_4 + \frac{L_1}{C_4} = R_2 R_3 \qquad (11)$$

$$\omega L_1 R_4 - \frac{R_1}{\omega C_4} = 0 \qquad (12)$$

Solving (11) and (12), we get

$$L_1 = \frac{R_2\ R_3\ C_4}{1 + \omega^2\ R_4^2\ C_4^2}$$

$$R_1 = \omega^2\ R_4\ C_4 \times \frac{R_2\ R_3\ C_4}{1 + \omega^2\ R_4^2\ C_4^2}$$

$$R_1 = \frac{\omega^2 R_2\ R_3\ R_4 C_4^2}{1 + \omega^2\ R_4^2\ C_4^2}$$

Q-factor of the coil $Q = \omega L_1/R_1 = \dfrac{1}{\omega C_4 R_4}$

$$\therefore \qquad L_1 = \frac{R_2\ R_3\ C_4}{1 + (1/Q)^2}$$

Advantages

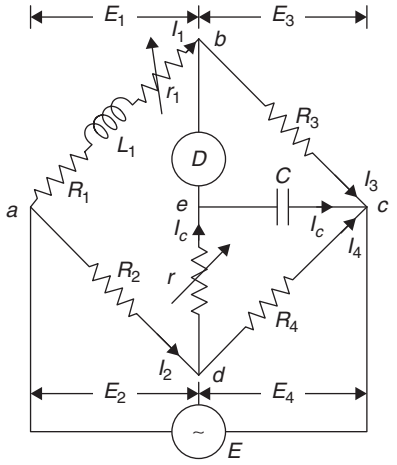
1. This bridge gives very simple expression for unknown inductance for high Q coils and is suitable for coils with $Q > 10$.
2. The expression for Q-factor is simple small when compared to Maxwell's bridge which requires a parallel R_4 with a very high value.

Disadvantages

1. This bridge is suitable for measurement of high Q inductors ($Q > 10$).
2. For inductor with $Q < 10$ this bridge is not suited and Maxwell's bridge is preferred.

Anderson's Bridge

- It is a modification of Maxwell's Wien bridge.
- In this bridge self-inductance is measured in terms of a standard capacitor.



Let L_1 = Inductance to be measured

R_1 = Resistance of L_1

r_1 = Resistance connected in series with L_1

r, R_2, R_3, R_4 = Known non-inductive resistor

C = Fixed standard capacitor

$I_1 = I_3$ and $I_2 = I_c + I_4$

At balance,

$$I_1 R_3 = I_c \times \frac{1}{j\omega c}$$

$$\therefore \quad I_c = I_1 j\omega C R_3$$

Writing the other balance equations

$$I_1 (r_1 + R_1 + j\omega L_1) = I_2 R_2 + I_c r \qquad (13)$$

and

$$I_c \left(r + \frac{1}{j\omega c} \right) = (I_2 - I_c) R_4 \qquad (14)$$

Sub value of I_c in (14) and (13),we get

$$I_1 (r_1 + R_1 + j\omega L_1) = I_2 R_2 + I_1 j\omega C R_3 r$$

$$I_1 (r_1 + R_1 + j\omega L_1 - j\omega C R_3 r) = I_2 R_2 \qquad (15)$$

$$J\omega C R_3 I_1 \left(r + \frac{1}{j\omega c} \right) = (I_2 - I_1 j\omega C R_3) R_4$$

$$I_1 (j\omega C R_3 r + j\omega C R_3 R_4 + R_3) = I_2 R_4 \qquad (16)$$

From (15) and (16), we can write

$$I_1 (r_1 + R_1 + j\omega L_1 - j\omega C R_3 r)$$

$$= I_1 \left(\frac{R_2 R_3}{R_4} + \frac{j\omega C R_2 R_3 r}{R_4} + j\omega C R_2 R_3 \right)$$

Equating real and imaginary parts,

$$R_1 = \frac{R_2 R_3}{R_4} - r_1$$

$$L_1 = C \frac{R_3}{R_4} [\, r(R_4 + R_2) + R_2 R_4 \,]$$

Advantages

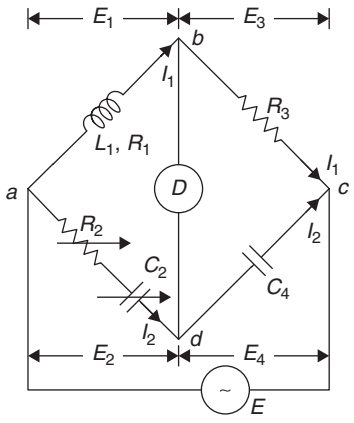
1. In case adjustments are made by manipulating control over r_1 and r_2, they become independent of each other. This is a marked superiority over sliding conditions met with low Q coils when measuring with Maxwell's bridge.
2. For low Q coils, it is much easier to attain balance in case of Anderson's Bridge than in Maxwell's bridge.
3. A fixed capacitor can be used instead of a variable capacitor.
4. Bridge may also be used to determine capacitance in terms of inductance.

Disadvantages

1. It is more complex than its prototype Maxwell's bridge. Anderson's bridge has more parts and is more complicated to set up and manipulate.
2. Balance equations are not simple and in fact are much more tedious.
3. An additional junction point increases the difficulty of shielding the bridge.

Owen's Bridge

- This bridge may be used for measuring inductance in terms of capacitance



L_1 = Unknown self-inductance of resistance R_1
R_2 = Variable non-inductive resistor
R_3 = Fixed non-inductive resistor
C_2 = Variable standard capacitor
C_4 = Fixed standard capacitor

At balance

$$(R_1 + j\omega L_1)\left(\frac{1}{j\omega C_4}\right) = \left(R_2 + \frac{1}{j\omega C_2}\right)R_3$$

Equating real and imaginary terms,

$$L_1 = R_2 R_3 C_4$$

$$R_1 = R_3 \frac{C_4}{C_2}$$

Advantages

1. Examining the equations for balance we obtain two independent equations in case C_2 and R_2 are made variable. Since R_2 and C_2 are in the same arm, convergence to balance condition is much easier.
2. The balance equations are quite simple and do not contain any frequency component.
3. A wide range of inductance can be measured.

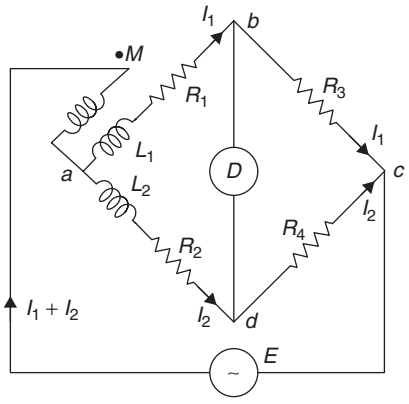
Disadvantages

1. The variable capacitor is expensive and its accuracy is about 1%.
2. For high Q-coils, the value of C_2 tends to become target.

Measurement of Mutual Inductance

Heaviside's Bridge

- This bridge measures mutual inductance in terms of known self-inductance.



M = Unknown mutual inductance
L_1 = Self-inductance of secondary of M
L_2 = Known self-inductance
R_1, R_2, R_3, R_4 = Non-inductive resistors

At balance $V_{BC} = V_{DC}$

$$I_1 R_3 = I_2 R_4$$

$$(I_1 + I_2)\, j\omega M + I_1(R_1 + R_3 + j\omega L_1) = I_2(R_2 + R_4 + j\omega L_2)$$

$$I_2\left(\frac{R_4}{R_3} + 1\right)j\omega M + I_2\left(\frac{R_4}{R_3}\right)(R_1 + R_3 + j\omega L_1) = I_2(R_2 + R_4 + j\omega L_2)$$

$$j\omega M\left(\frac{R_4}{R_3} + 1\right) + \frac{R_4}{R_3}.R_1 + R_4 + j\omega L_1 \frac{R_4}{R_3} = R_2 + R_4 + j\omega.L_2$$

$$R_1 = \frac{R_2 R_3}{R_4}$$

$$M = \frac{L_2 - L_1\frac{R_4}{R_3}}{\frac{R_4}{R_3} + 1} = \frac{R_3 L_2 - R_4 L_1}{R_3 + R_4}$$

For the value of M to be measured, the value of L must be known.

If $R_3 = R_4$, we get $M = \frac{L_2 - L_1}{2}$ and $R_1 = R_2$

This method can be used for measuring self-inductance.

$$L_2 = \frac{M(R_3 + R_4) + R_4 L_1}{R_3}$$

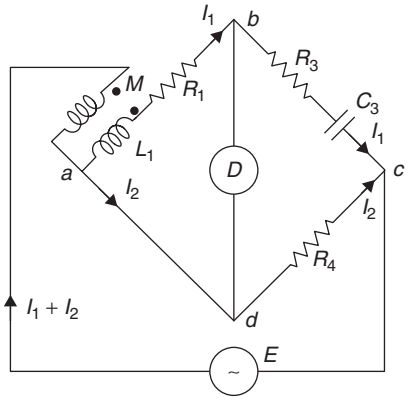
$$= M\left(1 + \frac{R_4}{R_3}\right) + \left(\frac{R_4}{R_3}\right)L_1$$

$$R_2 = R_1 R_4 / R_3$$

If $R_4 = R_3 L_2 = 2M + L_1$ and $R_2 = R_1$

Carey Foster Bridge

- The bridge is used for the measurement of standard capacitance in terms of mutual inductance and mutual inductance in terms of standard capacitance.



At balance $V_{AB} = V_{AD} = 0$

$$I_1 [R_1 + j\omega L_1] - j\omega M [I_1 + I_2] = 0$$

$$I_1 [R_1 + j\omega L_1] = j\omega M (I_1 + I_2) \quad (17)$$

Also $$V_{BC} = V_{CD}$$

$$I_1\left(R_3 - \frac{j}{\omega C_3}\right) = I_2 R_4$$

$$I_2 = \frac{I_1\left(R_3 - \frac{j}{\omega C_3}\right)}{R_4} \quad (18)$$

Sub (18) from (17)

$$I_1 [R_1 + j\omega L_1] = j\omega M \left(I_1 + \frac{I_1\left(R_3 - \frac{j}{\omega C_3}\right)}{R_4}\right)$$

$$R_1 + j\omega L_1 = j\omega M \left(\frac{R_3 - \frac{j}{\omega C_3}}{R_4} + 1\right)$$

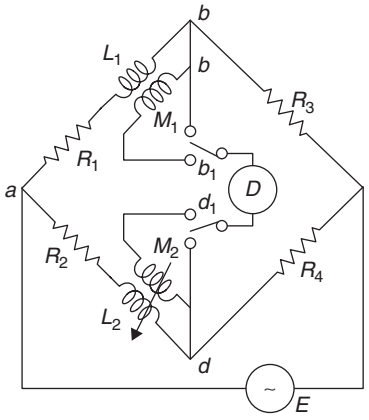
Equating real and imaginary parts, we get

$$R_1 = \frac{M}{C_3 R_4} \text{ or } C_3 = \frac{M}{R_1 R_4}$$

$$L_1 = M\left(\frac{R_3}{R_4} + 1\right) \text{ or } M = \frac{L_1}{(1 + R_3/R_4)}$$

Campbell's Bridge

- This bridge measures an unknown mutual inductance in terms of a standard mutual inductance.



$M_1 \rightarrow$ Unknown mutual inductance
$L_1 \rightarrow$ Self-inductance of secondary of M_1
$M_2 \rightarrow$ Variable standard mutual inductance
$L_2 \rightarrow$ Self-inductance of secondary M_2
$R_1, R_2, R_3, R_4 \rightarrow$ Non-inductive resistances.

There are two steps in balancing process.

1. Detector is connected between b and d. The circuit now becomes a simple self-inductance comparison bridge. The requirement for balance is

$$\frac{L_1}{L_2} = \frac{R_1}{R_2} = \frac{R_3}{R_4}$$

Bridge may be balanced by R_3 (or R_4) and R_1 (or R_2).

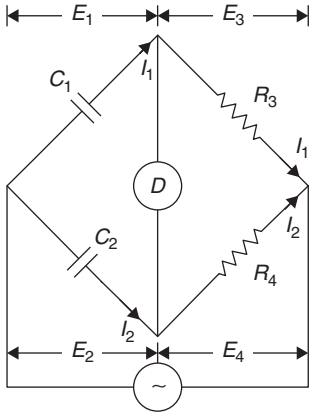
2. Detector is connected between b' and d' keeping adjustments as in step 1 above, the variable mutual inductance M_2 is varied to get balance $\frac{M_1}{M_2} = \frac{R_3}{R_4}$

$$M_1 = M_2\left(\frac{R_3}{R_4}\right)$$

Measurement of Capacitance

De Sauty's Bridge

- This bridge is the simplest method of comparing two capacitances.



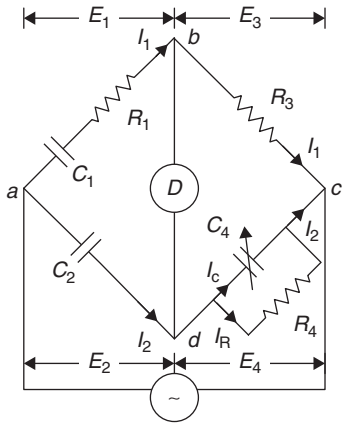
C_1 = Capacitor whose capacitance is to be measured
C_2 = A standard capacitor
R_3, R_4 = Non-inductive resistors

At balance

$$\left(\frac{1}{j\omega C_1}\right) R_4 = \left(\frac{1}{j\omega C_2}\right) R_3$$

$$C_1 = C_2\left(\frac{R_4}{R_3}\right)$$

Schering Bridge



C_1 = Capacitor whose capacitance is to be determined
r_1 = Series resistance representing loss in C_1
C_2 = Standard capacitor,
R_2 = Non-inductive resistance

C_4 = A variable capacitor
R_4 = Variable non-inductive resistance in parallel with variable capacitor C_4.

$$Z_1 = \left(r_1 + \frac{1}{j\omega C_1}\right) \quad Z_2 = \frac{1}{j\omega C_2}$$

$$Z_3 = (R_3) \quad Z_4 = \left(\frac{R_4}{1 + j\omega C_4 R_4}\right)$$

At balance

$$\left(r_1 + \frac{1}{j\omega C_1}\right)\left(\frac{R_4}{1 + j\omega C_4 R_4}\right) = \frac{1}{j\omega C_2} \cdot R_3$$

$$\left(r_1 + \frac{1}{j\omega C_1}\right) R_4 = \frac{R_3}{j\omega C_2}(1 + j\omega C_4 R_4)$$

$$r_1 R_4 - \frac{j R_4}{\omega C_1} = -j\frac{R_3}{\omega C_2} = \frac{R_3 R_4 C_4}{C_2}$$

Equating real and imaginary terms

We obtain $r_1 = R_3 C_4 / C_2$

$$C_1 = C_2\left(\frac{R_4}{R_3}\right)$$

Two independent balance equations are obtained if C_4 and R_4 are chosen as variable elements.

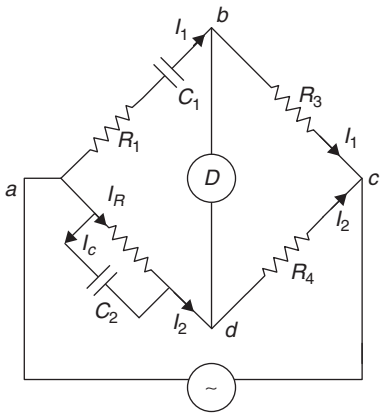
Dissipation factor $D_1 = \tan\delta = \omega C_1 r_1 = \omega C_2 \left(\frac{R_4}{R_3}\right)$

$D_1 = \omega C_4 R_4$

δ = Loss angle of the capacitor

$\sin\delta$ = Power factor of the capacitor

Wien's Bridge



- Wien's bridge is primarily used for the measurement of frequency.
- Its balance equation involves frequency even the performance of individual bridge elements is independent of frequency.
- It may be employed in harmonic distortion analyser, audio and HF oscillators.

$$Z_1 = R_1 - \frac{j}{\omega C_1} \quad Z_2 = \frac{R_2}{1 + j\omega C_2 R_2} S$$

$$Z_3 = R_3 \; Z_4 = R_4$$

At balance

$$\left(R_1 - \frac{j}{\omega C_1}\right) R_4 = R_3 \; \frac{R_2}{(1 + j\omega C_2 R_2)}$$

$$R_4\left(R_1 - \frac{j}{\omega C_1}\right)(1 + j\omega C_2 R_2) = R_2 R_3$$

Equating real and imaginary terms,

$$R_1 R_4 + R_2 R_4 \frac{C_2}{C_1} = R_2 R_3$$

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} - \frac{R_1}{R_2}$$

$$\frac{-R_4}{\omega C_1} + \omega C_2 R_1 R_2 R_4 = 0$$

$$\omega^2 = \frac{1}{R_1 R_2 C_1 C_2}$$

$$C_1 = \frac{1}{\omega^2 R_1 R_2 C_2}$$

For measurement of frequency

$$\omega = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \quad \text{and} \quad f = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} \text{Hz}$$

Example 8: In Wien's bridge method of measurement of frequency, among the following; the combination of bridge elements which could not result in frequency measurement of 2.5 kHz would be.

1. $C_1 = 0.2\ \mu F$, $C_2 = 0.8\ \mu F$
   $R_1 = 100\ \Omega$, $R_2 = 253.3\ \Omega$
2. $C_1 = 0.1\ \mu F$, $C_2 = 0.8\ \mu F$
   $R_1 = 200\ \Omega$, $R_2 = 253.3\ \Omega$
3. $C_1 = 0.8\ \mu F$, $C_2 = 3.16 \times 10^{-4}$ F
   $R_1 = 200\ \Omega$, $R_2 = 253.3\ \Omega$

(A) 1 only (B) 2 only
(C) 3 only (D) 1, 2, and 3

Solution: (C)

Wien's Bridge method to calculate frequency

$$f = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}}$$

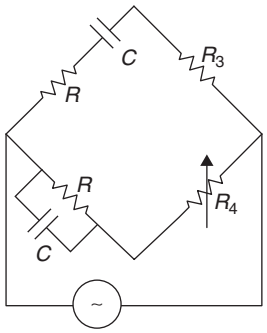
1. $$= \frac{1}{2\pi\sqrt{0.2\,\mu \times 0.8\,\mu \times 100 \times 253.3}} = 2.5 \text{ kHz}$$

2. $$= \frac{1}{2\pi\sqrt{0.8\,\mu \times 0.1\,\mu \times 200 \times 253.3}} = 2.5 \text{ kHz}$$

3. $= \dfrac{1}{2\pi\sqrt{0.8\mu \times 3.16 \times 10.9 \times 200 \times 258.3}} = 44.47$ kHz

1 and 2 would measure 2.5 kHz
(3) alone measures 44.47 Hz

Example 9: If R and C are variables connected as shown in the Wien bridge given which of the following is true, when bridge is balanced and $f = \dfrac{1}{2\pi CR}$



(A) $R_4 = R_3$ (B) $R_4 = 2R_3$
(C) $R_4 = R_3/2$ (D) $R_4 = 3R_3$

Solution: (C)
At balance

$$R_4\left(R + \frac{1}{j\omega C}\right) = R_3\left(\frac{R \cdot \dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}}\right)$$

$$\frac{R_4\left(1 + j\omega RC\right)}{j\omega C} = \frac{R_3 R}{\left(1 + j\omega RC\right)}$$

$$R_4\left(-C^2\omega^2R^2 + 1 + 2j\omega\, RC\right) = jR_3RC_\omega$$

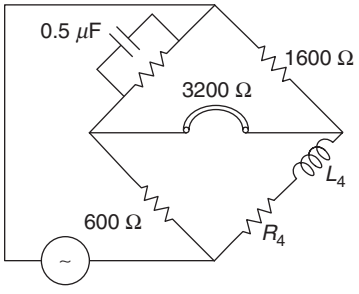
Equating real and imaginary terms

$$2\omega\, CR\, R_4 = \omega\, CR\, R_3$$

$$R_4 = \frac{R_3}{2}$$

Example 10: In the Maxwell's bridge, the component values shown are when the bridge is balanced determines R_4 and L_4

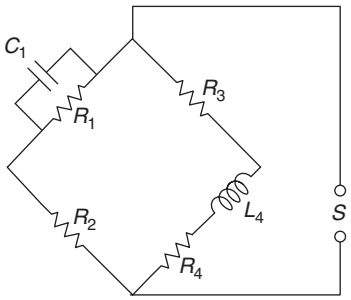
(A) 300 Ω, 0.48 H (B) 30 Ω, 4.8H
(C) 37.5 Ω, 75 mH (D) 75Ω, 7.5 mH



Solution: (A)
At balance

$$\left(\frac{R_1 \cdot \dfrac{1}{j\omega C_1}}{R_1 + \dfrac{1}{j\omega C_1}}\right)\left(R_4 + j\omega L_4\right)$$

$$= R_2\, R_3$$



$$R_1R_4 + j\omega L_4R_1 = R_2R_3 + j\;\; R_2R_3R_1C_1\omega$$

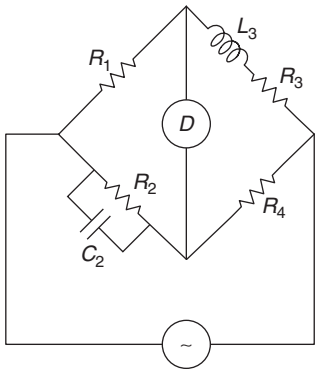
$$R_4 = \frac{R_2R_3}{R_1} \quad L_4 = R_2R_3C_1$$

for the given problem

$$R_1 = 1600 \times \frac{600}{3200} = 300\ \Omega$$

$$L_1 = 1600 \times 600 \times 0.5 \times 10^{-6}$$
$$= 0.48\ \text{H}$$

Example 11: For a balanced bridge shown below what are the expressions for R_3, L_3, Q_3, respectively.



(A) $\dfrac{R_4}{R_1}R_2, \dfrac{R_4\,R_1}{C_2}, \omega C_2R_2$

(B) $\dfrac{R_2}{R_4}.R_1, \dfrac{R_1\,R_4}{C_2}, \omega C_2R_2$

(C) $\dfrac{R_1\;R_4}{R_2}, R_1\,R_4\,C_2, \omega\,R_2C_2$

(D) $\dfrac{R_2\;R_1}{R_4}, R_1\,R_4\,, C_2, \omega\,R_2C_2$

Solution: (C)

At balance $R_1R_4 = (R_3 + j\omega L_3)\left(\frac{R_2}{1 + j\omega R_2 C_2}\right)$

$$(1 + j\omega R_2C_2)(R_1 R_4) = R_2 (R_3 + j\omega L_3)$$

Equating real and imaginary parts

$$R_1R_4 = R_2R_3$$

$$R_2 = \left(\frac{R_1 R_4}{R_3}\right)$$

$$\omega C_2R_2R_1R_4 = \omega R_2L_2$$

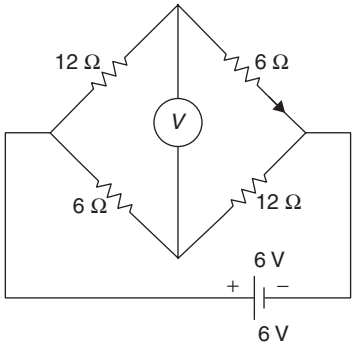
$$L_2 = R_1 R_4 C_2$$

$$Q_2 = \omega C_2R_2$$

Example 12: The reading of high impedance voltmeter in the given bridge is

(A) Zero (B) 4 V

(C) 6.66 V (D) 2 V



Solution: (D)

Current in either branch is $I = \frac{6}{18} = \left(\frac{1}{3}\right)$ A

$$V_{BC} = 12 \times \frac{1}{3} - 6 \times \frac{1}{3} = 2 \text{ V}.$$

Example 13: A slide potentiometer has 5 wires 2 m each. With the help of a standard voltage source of 1.234 V, it is standardized by keeping the jockey at 123.4 cm. If resistance of potentiometer wire is 1000 Ω, then the value of working current will be

(A) 1 mA (B) 0.5 mA

(C) 0.1 mA (D) 10 mA

Solution: (D)

Total length of slide wire = 5 × 2

= 10 m = 1000 cm

Total resistance of slide wire = 1000 Ω

Resistance per cm = 1 Ω

Resistance of 123.4 cm wire = 123.4 Ω

Which corresponds to a voltage of 1.234 V

$$\text{Current} = \frac{1.234}{123.4} = 10 \text{ mA}.$$

EXERCISES

Practice Problems I

1. The Maxwell's inductance capacitance bridge is used to measure inductance of

(A) Low Q coils

(B) High Q coils

(C) Both (A) and (B)

(D) None

2. Match list I (terms) with List II definitions and select the correct answer using the codes given below

List I	List II
A. Resolution	1. Closeness with which the instrument reading approaches the true value
B. Sensitivity	2. Reproducibility of measurements
C. Accuracy	3. Smallest change in measured value to which the instrument can respond
D. Precision	4. Ratio of response of the instrument to the input variable.

Codes

(A) A B C D
4 3 1 2

(B) A B C D
3 4 2 1

(C) A B C D
4 3 2 1

(D) A B C D
3 4 1 2

3. Which of the following statements are correct? The application of instrument in wrong manner in the procedure of measuring result in an AC

(A) Systematic error

(B) Instrument error

(C) Random error

(D) Gross error

4. Commonly used detectors for AC bridges are

(A) Headphones

(B) Vibration galvanometer

(C) Tunable amplifier detector

(D) All of the above

5. Slide wire in a potentiometer is made of

(A) Platinum–silver alloy

(B) Silver alloy

(C) Chromium alloy

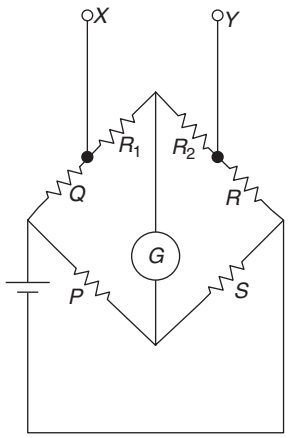
(D) Lead alloy

6. Accuracy of an instrument is
(A) The measure of consistency and reproducibility of measurements
(B) The smallest change in measurableinput
(C) The ratio of change in output signal to the change in input signal
(D) The closeness with which an instrument reading approaches the true value of the quantity being measured

7. A symmetrical square-wave voltage is read on an average response electronic voltmeter whose scale is calibrated in terms of RMS value of a sinusoidal wave. The error in the reading is
(A) −3.1% (B) +6.4%
(C) −11% (D) +11%

8. The current coil of a wattmeter has a resistance of 0.05 Ω and its pressure coil has a resistance of 10,000 Ω. The wattmeter is used to measure the power of a load taking 40 A at 230 V and 0.7 p.f. The percentage error in the reading if the current coil is on the load side is
(A) 1.24% (B) 2.48%
(C) 0.62% (D) 1.96%

9. A Wheatstone bridge is modified by taking leads as shown in the figure



If an external resistance R_3 is connected across the leads such that $R_1 = 600\ \Omega$, $R_1 = 800R_2$, $P = 400R_2$, $P = 1.5Q$, $S = 0.6667R_2$

The value of unknown resistance R would be
(A) 0.25 Ω (B) 0.5 Ω
(C) 1.33 Ω (D) 1.5 Ω

10. In a Carey Fosters bridge, a resistance of 2.0250 Ω is compared with a known standard resistance of 2 Ω. The slide wire is 2 meters long and has a resistance of 0.1 Ω in 100 divisions. The ratio arms are actually 20.01 Ω and 19.9 Ω, respectively, the balance positions on the slide wire are obtained at
(A) 25 and 175 cm
(B) 15 and 185 cm
(C) 54 and 146 cm
(D) None of the above

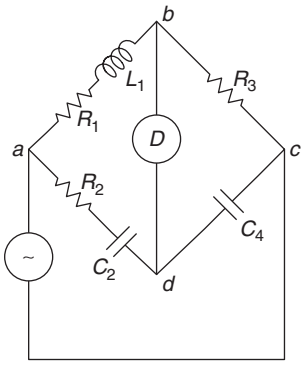
11. Kelvins double bridge was employed to measure an unknown resistance. The bridge had the following components Outer ratio arms 50.12 Ω and 100 Ω Inner ratio arms 50.155 Ω and 100 Ω known standard resistance of 50.015 μΩ. Resistance link connecting the standard resistance and unknown resistance is 350 μΩ

The value of unknown resistance will be
(A) 15 μΩ (B) 25 μΩ
(C) 30 Ω (D) 25 Ω

12. The drawback of Maxwell's inductance capacitance bridge over Hay's bridge is because
(A) Its balance equation do not contain frequency term
(B) It cannot be used for low-quality factor coil
(C) It cannot be used to measure coils with high Q-factor
(D) None of the above

13. Frequency measurement can be performed using a
(A) Wien's bridge
(B) Heaviside's Bridge
(C) Schering Bridge
(D) Maxwell's Bridge

14.



The properties of a sample sheet metal are measured using an Owen's bridge at 1 kHz. At balance, arm ab is test specimen $R_3 = 50\ \Omega$, $C_4 = 0.05\ \mu F$, $R_2 = 417\ \Omega$ in series with $C_2 = 0.062\ \mu F$. The effective impedance of the specimen under test condition would be
(A) 40.85 mΩ
(B) 40.85 μΩ
(C) 40.85 Ω
(D) 4.085 Ω

15. For a measuring instrument when a set of measurements were noted down it was observed that the readings had a wide range, the instrument has
(A) High accuracy
(B) Low accuracy
(C) High precision
(D) Low precision

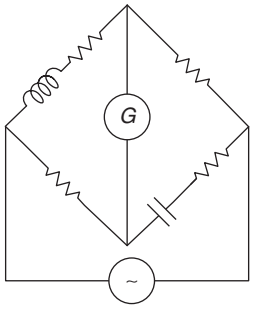
Practice Problems 2

1. Kelvin's double bridge is used to
(A) Measure capacitance
(B) Measure frequency
(C) Measure high resistance
(D) Measure low resistance

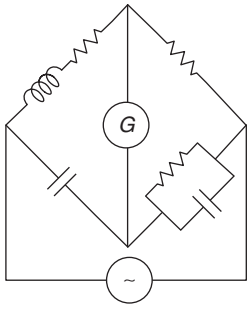
2. Carey Foster slide wire bridge is used to
(A) Measure low resistance
(B) Measure high resistance
(C) Measure the deviation of a unknown resistance from a standard one
(D) Measure the deviation of an unknown capacitance from a standard one

3. Hay's bridge from the following is

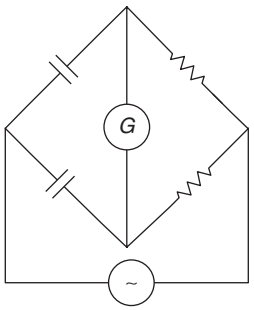
(A)



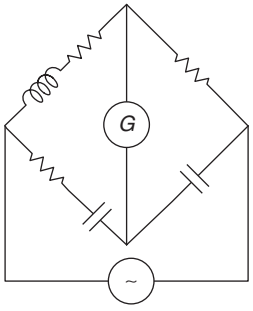
(B)



(C)



(D)



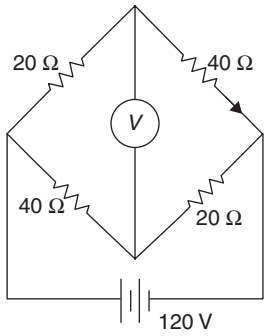
4. Group I represents the bridges available for measurement of certain parameters with reasonable accuracy. Group II represents the parameters that are to be measured. Select the correct choice of item in group I for the corresponding item in group II.

Group I	Group II
P. Wheatstone Bridge	1. Resistance in milli-ohm range
Q. Kelvin double	2. Low values of bridge capacitance
R. Schering bridge	3. Comparison of inductance with standard capacitor
S. Hay's bridge	4. Mutual inductance of a coil
T. Carey Foster bridge	
U. Wien's bridge	

(A) 1 = *Q*; 2 = *R*; 3 = *T*; 4 = *S*
(B) 1 = *Q*; 2 = *T*; 3 = *V*; 4 = *S*
(C) 1 = *Q*; 2 = *R*; 3 = *S*; 4 = *T*
(D) 1 = *P*; 2 = *R*; 3 = *S*; 4 = *U*

5. A slide wire potentiometer has 20 wires at 50 cm each. The bridge is standardized with a voltage of 1.232 V and keeping the jockey at 123.2 cm. The total resistance of the potentiometer is 2 kΩ, value of working current is
(A) 2 mA
(B) 1 mA
(C) 5 mA
(D) 20 mA

6. The reading shown by high impedance voltmeter is

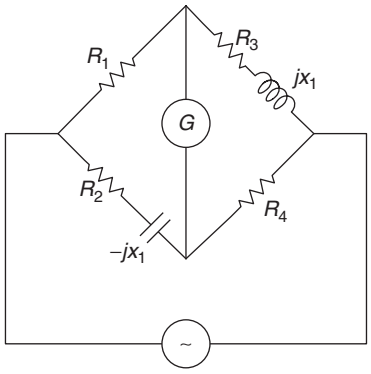


(A) 10 V
(B) 120 V
(C) 30 V
(D) 40 V

7. A DC potentiometer is used to measure up to 3 V with a slide wire of 900mm. A standard cell of 1.53 V attains balance at 660 mm. A test cell obtains balance at 720 mm. The emf of test cell is
(A) 1.83 V
(B) 2 V
(C) 1.67 V
(D) 1.52 V

8. The potentiometer wire should have
(A) Low resistivity and high temperature coefficient
(B) High resistivity and high temperature coefficient
(C) High resistivity and low temperature coefficient
(D) Low resistivity and low temperature coefficient

9. In the bridge circuit shown below, the sequence that is most suitable for balancing the bridge is



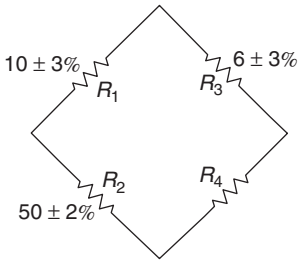
(A) First adjust R_1 and then adjust R_4
(B) First adjust R_3 and then adjust R_2
(C) First adjust R_1 and then adjust R_2
(D) First adjust R_3 and then adjust R_4

10. The current through a resistance R is measured as

$$I = 6\text{ A} \pm 0.25\%.$$

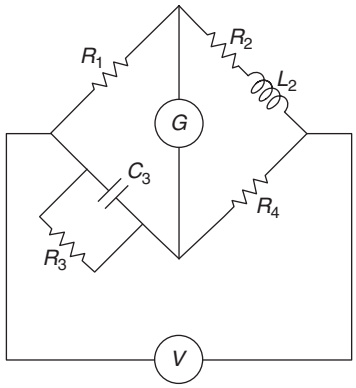
If the resistance $R = 100 \pm 0.5\%$, the uncertainty in the measurement of power is
(A) 3600 W ± 0.45%
(B) 3600 W ± 1%
(C) 3600 W ± 0.05%
(D) 3600 W ± 1.5%

11.



At balanced condition, the value of R_4 is
(A) 30 ± 6%
(B) 30 ± 4%
(C) 30 ± 8%
(D) 30 ± 3%

Common Data for Questions 12 and 13:



The above bridge is used to measure the inductance L_2. Resistance R_3 is adjusted and the bridge is balanced $R_1 = 500\ \Omega$,

$$R_3 = 1000\ \Omega \text{ and } R_4 = 800\ \Omega,\ C_3 = 0.6\ \mu\text{F}$$

12. The value of L_2 and R_2 is
(A) 0.12 H, 200 Ω
(B) 0.24 H, 200 Ω
(C) 0.12 H, 400 Ω
(D) 0.24 H, 400 Ω

13. The Q-factor of the coil is
(A) 3.77
(B) 3.14
(C) 1.87
(D) 2.56

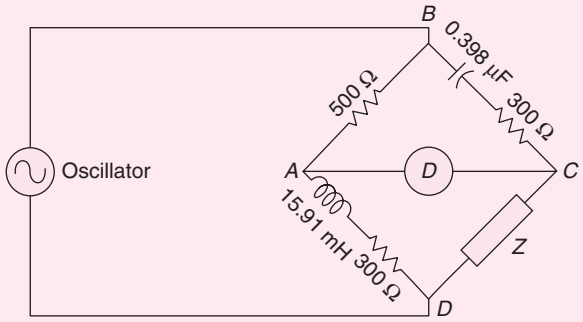
Common Data for Questions 14 and 15:
A voltmeter having a full-scale deflection of 100 V has a specified accuracy of ±1%

14. At full-scale deflection of 100 V, the error will be
(A) ± 0.01%of 100 V
(B) ± 0.1% of 100 V
(C) ±1 % of 100 V
(D) ± 99%of 100 V

15. When the voltmeter is reading 50 V, the error will be
(A) ±0.5% of measured voltage
(B) ±1% of measured voltage
(C) ±1.5% of measured voltage
(D) ±2% of measured voltage

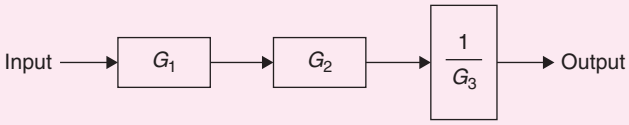
Previous Years' Questions

1. The AC bridge shown in the figure is used to measure the impedance Z. If the bridge is balanced for oscillator frequency $f = 2$ kHz, then the impedance Z will be **[2008]**



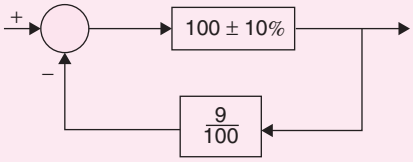
(A) $(260 + j0)\ \Omega$ (B) $(0 + j200)\ \Omega$
(C) $(260 - j200)\ \Omega$ (D) $(260 + j200)\ \Omega$

2. The measurement system shown in the figure uses three sub-systems in cascade whose gains are specified as G_1, G_2 and $\frac{1}{G_3}$. The relative small errors associated with each respective subsystems G_1, G_2 and G_3 are ε_1, ε_2 and ε_3. The error associated with the output is: **[2009]**



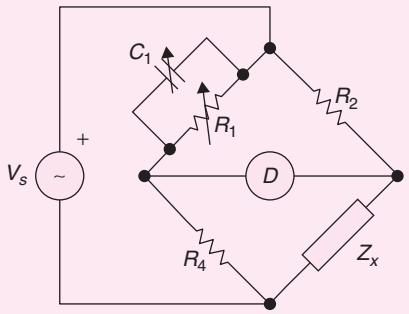
(A) $\varepsilon_1 + \varepsilon_2 + \frac{1}{\varepsilon_3}$ (B) $\frac{\varepsilon_1 \cdot \varepsilon_2}{\varepsilon_3}$
(C) $\varepsilon_1 + \varepsilon_2 - \varepsilon_3$ (D) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

3. As shown in the figure, a negative feedback system has an amplifier of gain 100 with ±10% tolerance in the forward path, and an attenuator of value 9/100 in the feedback path. The overall system gain is approximately: **[2010]**



(A) $10 \pm 1\%$ (B) $10 \pm 2\%$
(C) $10 \pm 5\%$ (D) $10 \pm 10\%$

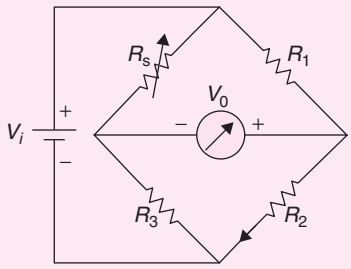
4. The bridge circuit shown in the figure below is used for the measurement of an unknown element Z_x. The bridge circuit is best suited when Z_x is a. **[2011]**



(A) Low resistance (B) High resistance
(C) Low Q inductor (D) Lossy capacitor

5. The bridge method commonly used for finding mutual inductance is **[2012]**
(A) Heaviside Campbell bridge
(B) Schering bridge
(C) De Sauty's bridge
(D) Wien bridge

6. A strain gauge forms one arm of the bridge shown in the figure below and has a nominal resistance without any load as $R_s = 300\ \Omega$. Other bridge resistances are $R_1 = R_2 = R_3 = 300\ \Omega$. The maximum permissible current through the strain gauge is 20 mA. During certain measurement when the bridge is excited by maximum permissible voltage and the strain gauge resistance is increased by 1% over the nominal value, the output voltage V_0 in mV is **[2013]**

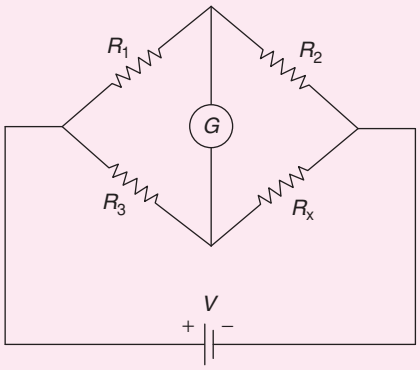


(A) 56.02 (B) 40.83
(C) 29.85 (D) 10.02

7. Suppose that resistors R_1 and R_2 and connected in parallel to given an equivalent resistor R. If resistors R_1 and R_2 have tolerance of 1% each, the equivalent resistor R for resistors $R_1 = 300\ \Omega$ and $R_2 = 200\ \Omega$ will have tolerance of **[2014]**
(A) 0.5% (B) 1%
(C) 1.2% (D) 2%

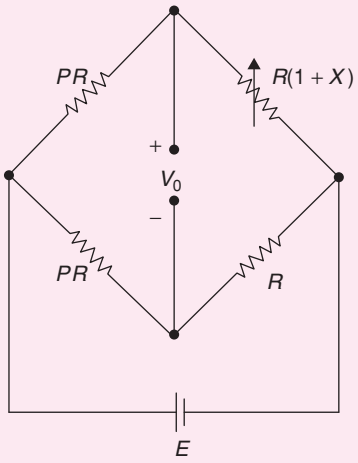
8. When the Wheatstone bridge shown in the figure is used to find the value of resistor R_x, the galvanometer G indicates zero current when $R_1 = 50\ \Omega$, $R_2 = 65\ \Omega$ and $R_3 = 100\ \Omega$. If R_3 is known with ±5% tolerance on

its nominal value of 100 Ω, what is the range of R_x in Ohms? **[2015]**



(A) [123.50, 136.50] (B) [125.89, 134.12]
(C) [117.00, 143.00] (D) [120.25, 139.75]

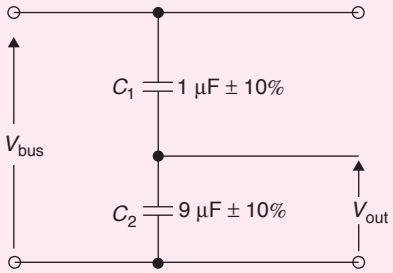
9.



An unbalanced DC Wheatstone bridge is shown in the figure. At what value of p will the magnitude of V_0 be maximum? **[2015]**

(A) $\sqrt{(1+x)}$ (B) $(1+x)$
(C) $\sqrt{(1+x)}$ (D) $\sqrt{(1-x)}$

10. A capacitive voltage divider is used to measure the bus voltage V_{bus} in a high-voltage 50 Hz AC system as shown in the figure. The measurement capacitors C_1 and C_2 have tolerances of ±10% on their nominal capacitance values. If the bus voltage V_{bus} is 100 kV rms, the maximum rms output voltage V_{out} (in kV), considering the capacitor tolerance, is _____. **[2015]**



Answer Keys

Exercises

Practice Problems 1

1. D	**2.** D	**3.** B	**4.** D	**5.** A	**6.** D	**7.** D	**8.** A	**9.** A	**10.** C
11. B	**12.** C	**13.** A	**14.** C	**15.** D					

Practice Problems 2

1. D	**2.** C	**3.** A	**4.** C	**5.** C	**6.** D	**7.** C	**8.** C	**9.** D	**10.** B
11. C	**12.** D	**13.** A	**14.** C	**15.** D					

Previous Years' Questions

1. A	**2.** C	**3.** A	**4.** C	**5.** A	**6.** C	**7.** B	**8.** A	**9.** A
10. 11.75 to 12.25								

Chapter 2

Electrical Measuring Instruments

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Indicating instruments
- Types of torques
- Moving coil instruments
- Errors in PMMC instruments
- Electrodynamometer type
- Instruments
- Deflecting torque
- Moving iron instruments
- Deflecting torque
- Classification of instruments

Indicating Instruments

- They consist of a pointer moving over a calibrated scale attached to the moving system pivoted in jewel bearings.
- The following torques are needed for proper operation.

Types of Torques

Deflecting Torque (T_d)

- When the instrument is connected in the electrical circuit to measure given electrical quantity, there must be an arrangement for providing operating or deflecting torque.
- Deflecting torque causes the moving system to move from zero position. When the instrument is connected in the circuit.

Controlling Torque (T_c)

Under the action of deflecting torque, the pointer will continue to move indefinitely and shall be independent of the value of electrical quantity to be measured.

- Here, a controlling torque is introduced which should oppose T_d and should increase with deflection of moving system so that the pointer is brought to rest at a position when two opposing torques are equal.
- Controlling torque limits the movement of the pointer so that the magnitude of deflection is always the same for given value of electrical quantity to be measured.
- It brings the pointer back to zero position when T_d is removed. If it was not provided the pointer once deflected would not return to zero position on removing the deflecting torque.
- T_c (or) restoring (or) balancing torque is provided either by a spring or by gravity. A hair spring is attached to the moving system the spring is twisted in opposite direction, which produces a restoring torque. Gravity control is obtained by attaching a small adjustable weight to the moving system.

Damping Torque

- If the moving system is acted upon by a T_D and T_c, then the pointer, due to its inertia will oscillate about final position before coming to rest.
- In order to avoid these oscillations of the pointer and bring it to its deflected position, damping torque is provided which opposes the movement of the pointer and operates 'only when the system is moving'.
- If the instrument is underdamped the pointer will oscillated about the final position before coming to rest.
- For overdamped instrument the pointer is slow and lethargic.
- For dead beat system, the degree of damping is adjusted such that pointer moves and quickly comes to its final position.
- Damping force is produced by air friction, eddy currents and fluid friction.
- In air friction damping, an aluminium piston attached to the moving system is arranged to travel in a fixed air chamber closed

at one end. The necessary damping effect is produced by movement of piston in the chamber.
- Fluid friction is similar in action to the air friction. Damping is produced by movement of piston in oil.
- The most efficient type of damping is obtained by eddy current damping. A non-magnetic metallic disc is attached to the moving system, cuts the magnetic flux. Hence eddy currents are produced in the disc, which produces a damping force in such a direction to oppose the cause producing them.

Types of Indicating Instruments

1. Moving coil instruments
2. Moving iron instruments
3. Dynamometer type instruments
4. Electrostatic instruments
5. Hot wire type instruments
6. Induction type instruments

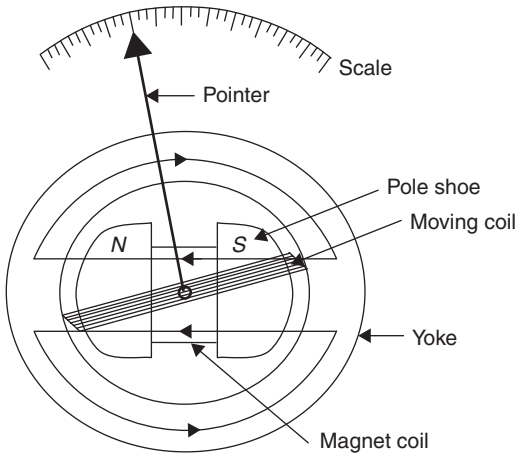
Moving Coil Instruments

They are again classified into the following two types:

1. Permanent magnet moving coil instruments
2. Dynamometer type instruments

Permanent Magnet Moving Coil Instruments

- It is a DC-indicating instrument.
- It is employed either as ammeter or voltmeter and can be used for DC work only.



Principle

This type of instrument is based on the principle of Permanent Magnet Moving Coil (PMMC), i.e. when a current-carrying conductor is placed in a magnetic field, mechanical force acts on the conductor. The coil placed in the magnetic field and carrying operating current is attached to the moving system. With the movement of the coil the pointer moves over the scale.

Construction

It consists of a powerful permanent magnet with soft iron pieces and light rectangular coil of many turns of fine wire wound on aluminium former inside which is an iron core.

- The purpose of the coil is to make field uniform. The coil is mounted on the spindle and acts as the moving element.
- The current is led into and out of the coil by means of the two control hair springs, one above and the others below the coil.
- The springs also provide controlling torques. Eddy current damping is provided by aluminium former.

Working

- When the instrument is connected in the circuit, operating current flows through the coil. This current carrying coil is placed in the magnetic field produced by the permanent magnet and therefore mechanical force acts on the coil. As the coil is attached to the moving system the pointer moves over the scale.
- These instruments work for DC circuits only, since negative current leads to opposite side deflection of the pointer (below zero).

Deflecting Torque

When current is passed through the coil, a deflection torque due to the reaction between permanent magnet field and magnetic field of the coil.

$$F = BIN\,L$$

where B = Flux density in Wb/m^2
L = Active length of the coil side in meters
N = Number of turns of coil
I = Current flowing through the coil
F = Force acting on each coil side

Deflecting torque $T_d\,\omega$ = Force × perpendicular distance

$$T_d\,\omega = F \times 2r \text{ Nm}$$

r = distance of the coil side from the axis of rotation in meters

$$T_d = NBI\,l \times 2r \text{ Nm}$$

If $A = l \cdot 2r$ is the surface area of the coil

$$T_d = NBIA \text{ Nm}$$

N, B and A are constants

$$\Rightarrow \quad T_d \propto I \qquad (1)$$

Since control is by springs

$$T_c \propto \theta \qquad (2)$$

The pointer will come to rest at $T_d = T_c$

From (1) and (2) $\theta \propto I$

Therefore, such instruments have uniform scale.

Note: PMMC reads average values.

Errors in PMMC Instruments

The main sources of error in moving coil instruments are due to

1. Weakening of permanent magnets due to ageing and temperature effects.
2. Change of resistance of moving coil with temperature.

Advantages

1. Uniform scale.
2. Very effective eddy current damping.
3. Power consumption is low.
4. No hysteresis loss.
5. As working field is very strong, such instruments are not affected by stray fields.
6. Such instruments require small operating current.
7. Very accurate and reliable.

Disadvantages

1. Such instruments cannot be used for AC.
2. Costlier as compared to moving iron instruments.
3. Errors are caused due to the ageing of control springs and the permanent magnet.

Solved Examples

Example 1: The cross-sectional area of the coil of a P.M.M.C instrument with a spring constant of 0.28×10^{-6} Nm/rad is 7.2×10^{-4} m^2. The air gap flux density is 3.6 m Wb/m^2. The number of turns required to produce a deflection of 60° when a current of 10 mA flows through the coil would be

(A) 136 (B) 44 (C) 22 (D) 11

Solution: (D)

For P.M.M.C. $T_d = NBAI$ $T_c = K\theta$

At the instance when reading is taken

$$T_d = T_c$$

$$NBIA = K\theta$$

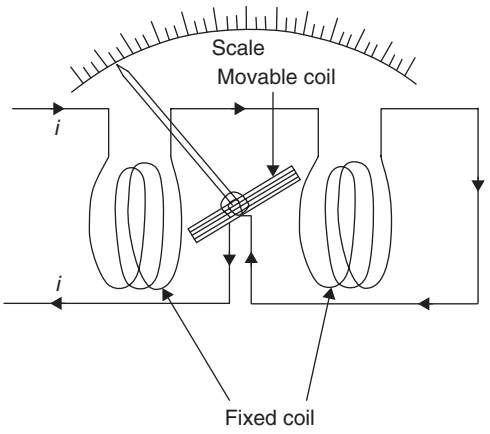
$$N = \frac{K\theta}{BIA}$$

$$= \frac{0.28\times10^{-6}\times\frac{\pi}{3}}{3.6\times10^{-3}\times7.2\times10^{-4}\times10\times10^{-3}}$$

$$= 11 \text{ turns.}$$

Electrodynamometer Type Instruments

- It is a modified form of PMMC.
- Here, operating field is produced not by a permanent magnet but by another fixed coil. The moving system and control system are similar to those of permanent magnet type.
- They can be used as ammeters and voltmeters but are generally used as wattmeters.



Principle

These instruments are based on the principle that the mechanical force exists between the current-carrying conductors.

Construction

- It consists of a fixed coil and moving coil.
- Fixed coil is split into two equal parts which are placed close together and parallel to each other. The moving coil is pivoted between two fixed coils.
- The fixed and moving coils may be exited separately or they may be connected in series depending upon the use to which the measurement is put. The moving coil is attached to the moving system such that under the action of deflecting torque the pointer moves over the scale.
- The controlling torque is provided by two springs which also serves the additional purpose of leading the current into and out of the moving coil. Air friction damping is provided in such instruments.

Working

- When instrument is connected in the circuit, operating current flows through the coils, due to which mechanical force exist between the coils.
- The result is that the moving coil moves the pointer over the scale. The pointer comes to rest at a position where deflecting torque is equal to controlling torque.
- By reversing the current, the field due to fixed coils as well as the current in the moving coil is reversed, so that the direction of deflection remains unchanged.
- These instruments can be used for both AC and DC.

Deflecting Torque

$$T_d \propto I_f I_m$$

I_f = Current through fixed coil

I_m = Current through moving coil

$I_f = I_m$ [since it is a series connection]

$$T_d \propto I^2$$

Since the control is by springs $T_c \propto \theta$

The pointer will come to rest at

$$T_d = T_c$$

$$\theta \propto I^2$$

Hence the scale of these instruments is non-uniform being crowded in their lower parts and spread out at the top.

Electrodynamometer type Ammeter $\theta = \frac{I^2}{K}\left(\frac{dM}{d\theta}\right)$

Electrodynamometer type Voltmeter $\theta = \frac{V^2}{KZ^2}\left(\frac{dM}{d\theta}\right)$

where K is the spring constant in Nm/rad
M is mutual inductance of the coil

Errors

1. **Torque/Weight ratio:** A low torque/weight ratio indicates a heavy moving system and therefore the friction losses in electrodynamometer type of instrument are larger than in other types. Frictional errors are high.
2. **Frequency:** Frequency error is large as a result of variation of self-reactance of the coils, with frequency.
3. **Eddy currents:** The effect of eddy current is to produce a torque as a result of coupling between moving coil and adjacent metal parts. The metal present in coil supports, in the shields, etc. produces a frequency error in the instrument.
4. **External magnetic fields:** Since the operating field in electrodynamometer instruments is weak, therefore these instruments must be protected from external magnetic fields.
5. **Temperature changes:** High currents are carried by coils which produce heat. Self-heating of coils produce errors.

Advantages

- These instruments can be used for both AC and DC measurements.
- They are free from hysteresis and eddy current errors.

Disadvantages

- Since torque/weight ratio is small therefore such instruments have frictional errors which reduce sensitivity.
- Scale is not uniform.
- A good amount of screening of the instruments is required to avoid the effect of stray fields.
- These instruments are costlier and so they are rarely used.

Example 2: The variation of mutual inductance M with deflection θ for a certain dynamometer type ammeter is given as $M = [-3\cos(\theta + 60°)]$ mH. If the pointer deflects 30° for a direct current of 25 mA.The magnitude of deflecting torque produced would be.

(A) 1.875 μN
(B) 3.6 μN
(C) 15 μN
(D) 18.75 μN

Solution: (A)

$$M = -3\cos(\theta + 60°)$$

$$\left(\frac{dM}{d\theta}\right) = 3\sin(\theta + 60°)$$

$$\left(\frac{dM}{d\theta}\right)_{\theta = 30°} = 3\sin(30 + 60) = 3 \text{ mH/degrees}$$

$$T_d = I^2\left(\frac{dM}{d\theta}\right) = (25 \times 10^{-3})^2 \times 3 \times 10^{-3}$$

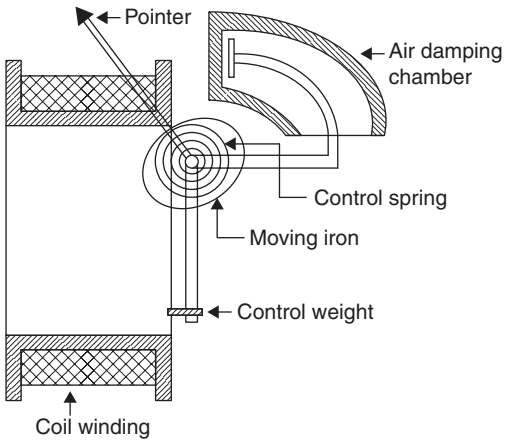
$$T_d = 1.875 \text{ μN}.$$

Moving Iron Instruments

Classification of moving iron Instruments

1. Attraction type
2. Repulsion type

Attraction Type Moving Iron Instrument



Principle

- When an unmagnetized soft iron piece is placed in the magnetic field of a coil, then piece is attracted towards coil.
- The moving system is attached to a soft iron piece and operating current is passed through a coil placed near it.
- The operating current sets up the magnetic field which attracts the iron piece and moves the pointer over the scale.

Construction

- It consists of a hollow cylindrical coil or solenoid which is kept fixed.
- An oval-shaped iron piece is attached to the spindle in such a way that it can move in or out of the coil.

- The pointer is attached to the spindle so that it is deflected with the motion of soft iron piece.
- The controlling torque on the moving system is provided by spring control method while damping is provided by air friction.

Working

- When instrument is connected in the circuit, the operating current flows through the coil which sets up magnetic field in the coil.
- In other words coil behaves like a magnet and therefore it attracts the soft iron piece towards it, which makes the pointer attached to move from 0 position.
- If the current in the coil is reversed the direction of magnetic field also reverses and so does the magnetism produced in the soft iron piece.
- Therefore the direction of the deflecting torque unchanged.

Deflecting Torque

- The force pulling the soft iron piece towards the coil depends upon

1. Field strength H produced by the coil
2. Pole strength m developed by the piece

$$F = mH$$

$$T_d \propto F \propto H^2$$

If permeability of iron is assumed constant, then

$$H \propto I$$

$$T_d \propto I^2$$

$$T_d = \frac{1}{2} I^2 \left(\frac{dL}{d\theta}\right)$$

If the controlling torque provided by springs

$$T_c \propto \theta$$

$$T_c = K\theta$$

In steady position, $T_d = T_c$

$$\theta = I^2 \text{ for DC}$$

$$\theta = I_{RMS} \text{ for AC}$$

$$\theta = \frac{I^2}{2k}\left(\frac{dL}{d\theta}\right)$$

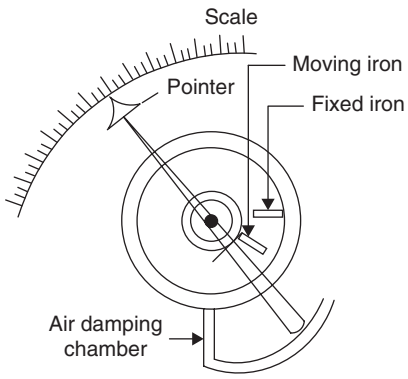
Since $\theta \propto I^2$, scale of such instrument is non-uniform.

In order to make the scale, uniform, tongue-shaped iron piece is used.

Repulsion Type Moving Iron Instruments

Principle

It is based on the principle of repulsion between the two iron pieces similarly magnetized.



Construction

It consists of a fixed cylindrical hollow coil which carries operating current. Inside the coil there are two soft iron pieces or vanes, one of which is fixed and other is movable.

- The fixed iron piece is attached to the coil whereas the movable iron piece is attached to the pointer shaft.
- Under the action of deflecting torque, the pointer attached to the moving system moves over the scale.
- The controlling torque is produced by spring control and damping torque is provided by air friction damping.

Working

- When the instrument is connected in the circuit, current flows through the coil, which sets up magnetic field in the coil.
- The magnetic field magnetizes both iron pieces in the same direction, i.e. both become similar magnets and hence they repel each other.
- Due to this movable iron piece moves as the other piece is fixed and cannot move, hence the pointer moves from zero position.

Deflecting Torque

- The deflecting torque results due to the repulsion between the similarly magnetized iron pieces.

If two pieces develop pole strengths m_1, and m_2, respectively, then

Instantaneous deflecting torque $\propto$ repulsive force $\propto m_1 m_2$

Assuming constant permeability $H \propto$ current through the coil,

Deflecting torque $T_d \propto I^2$

Controlling torque $T_c \propto \theta$

In the steady position of deflection

$$T_d = T_c$$

$$\theta \propto I^2$$

$$\theta \propto I^2 \text{ (for DC)}$$

$$\theta \propto I_{RMS} \text{ (for AC)}$$

Since $\theta \propto I^2$, the scale of such instruments are non-uniform being crowded in the beginning.

Comparison between Attraction and Repulsion Types of Instruments

- An attraction type instrument will have a lower inductance than corresponding repulsion type instrument.
- Repulsion instruments are suitable for more economical production in manufacture and a nearly uniform scale is more easily obtained they are therefore much more common than attraction type.

Errors

1. **Errors with both DC and AC**
   (a) **Hysteresis error:** This error occurs as the value of flux density is different for the same current for ascending and descending values.
   The value of flux density is higher for descending values of current and therefore instrument tends to read higher for descending values of current, than for ascending values.
   (b) **Temperature error:** this occurs due to the temperature coefficient of spring. As temperature increases, (due to self-heating and series resistance), resistance increases causing decrease in current for a given voltage. This produces a decreased deflection.
   (c) **Stray magnetic fields:** Such errors may be appreciable as the operating magnetic field is weak and hence can be easily distorted.
2. **Errors with AC only**
   (a) **Frequency error:** Changes in frequency may cause errors due to changes of reactance of the working coil and also due to changes of magnitude of eddy currents set up in metal parts of the instrument.
   (b) **Eddy current:** At low frequencies the eddy current error increases with square of frequency while at high frequencies the error is practically constant.

For these reasons moving iron instruments are unsuitable for frequencies above 125 Hz.

Advantages

1. These are cheap, robust and simple in construction.
2. They can be used for both AC and DC.
3. They have high operating torque.
4. They are reasonably accurate.

Disadvantages

1. Such instruments have non-uniform scale.
2. They are not very sensitive.
3. Errors are introduced due to changes in frequency in case of AC measurements.
4. Higher power consumption.

Note: Moving Iron Instruments indicate RMS values.

Example 3: In the measurement of voltage or current using moving iron instruments it indicates
(A) High values of the measuring quantity for ascending values.
(B) High values of the measuring quantity for descending values.
(C) Same values of measurement for both ascending and descending values.
(D) None of the above.

Solution: (B).

Induction Type Instruments

These instruments are based on the principle of inductor motor.

Principle

When a drum or disc of a non-magnetic conducting material is placed in a rotating magnetic field, eddy currents are induced in it.

- The reaction between the rotating flux and eddy current produced by it creates a torque which rotates the disc or drum.
- The rotating flux is produced by the current or voltage to be measured.
  The eddy current is proportional to flux

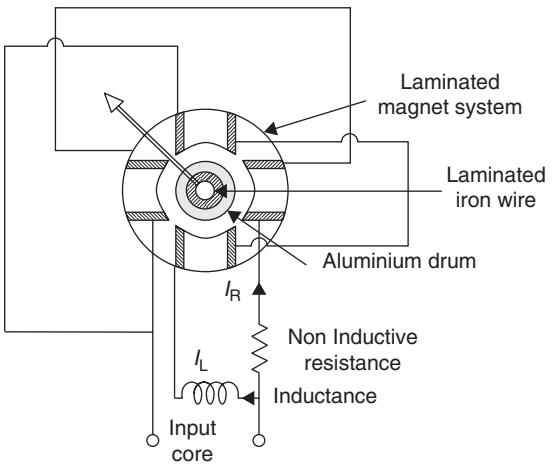
$$R \propto \phi_1 \propto I^2.$$

- A single-phase supply is converted into two phases in the instrument; this is done by split phase or shaded pole arrangement.

Classification of Instruments

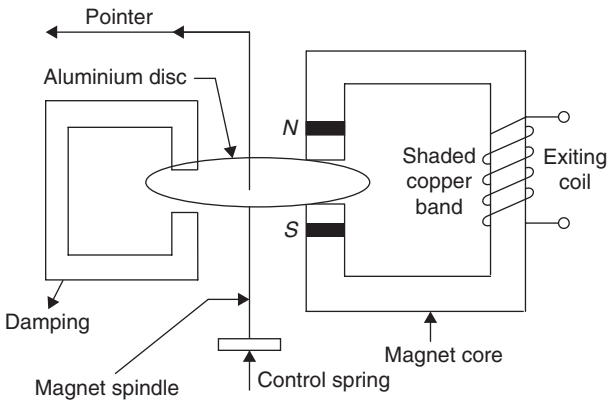
1. Split-phase type
2. Shaded-pole type

Split-phase Type Instruments



Construction

- It is also called Ferraris type instrument.
- It consists of a laminated magnet with the pairs of poles at right angles to each other.
- Coils are wound on the poles, the opposite poles being connected in series.
- The coils on two pairs of poles are connected in parallel. One set of coil is connected through an inductance and another with a high resistance to create a phase difference of 90°.
- The input to both the coils is the current to be measured. In the centre of the yoke and coil is an aluminium drum. Inside the drum there is a cylindrical laminated iron core to strengthen the magnetic field.



Working

- When the instrument is connected in the circuit and current flows through the coil, a rotating magnetic field is produced.
- This field induces eddy current in the drum and a torque is produced by the reaction of magnetic field and current. This torque deflects the pointer attached to the drum. Controlling torque is produced by the spring.

Deflecting Torque

Deflecting torque produced is given by

$$T \propto \frac{I_R\ I_L\ f}{z} \cos\theta \sin\alpha$$

For an ammeter $T \propto \frac{I_R}{z} \cos\theta \sin\alpha$

For a voltmeter $T \propto \frac{V^2}{Z} \cos\theta \sin\alpha$

where I_R = Current through resistor
I_L = Current through inductor
F = Supply frequency
Z = Impedance of eddy current path
θ = Phase angle between voltage and current in resistor
α = Phase angle between currents in resistor and inductor

θ = Almost zero and R is very much larger than reactive part of eddy paths

$$T \propto \frac{I^2}{R} \sin\alpha$$

Obviously α should be as high as possible for high torque.

Shaded-pole Type Instruments

Construction

- A band of copper is placed in pole faces, this makes the two fluxes of the shaded and unshaded portions differ in phase by 90°. A metallic disc rotates between the pole faces.
- Damping is provided by another magnet.

Working

- The current flowing through the exiting coil sets up flux.
- Eddy currents are induced in the copper band.
- Flux of the eddy current opposes the flux in the magnetic core and a two-way flux, same as Ferraris type instruments is induced.

Measurement of Electric Voltage and Current

Moving iron instruments are used as ammeter and voltmeter is only. They work on both AC and DC systems.

Ammeter

- An instrument which is used to measure electric current in a circuit is called an ammeter.
- It is always connected in series with the circuit and carries current to be measured. This current flows through the operating coil to produce the desired deflecting torque.
- Since it is to be connected in series it should have low resistance. Hence when a moving iron instrument is used as ammeter, the operating coil is provided with a few turns of thick wire so that it has low resistance.

Shunts

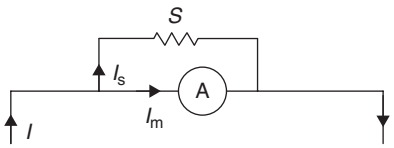
- A resistance placed in parallel with an instrument to control the current passing through it, when placed in a circuit carrying a fairly large current.
- A length of wire with resistance of constant temperature within the box of the instrument can be used as shunt. Or there may be an external shunt having very low resistance.

General Requirements of a Shunt

1. The temperature coefficient of shunt and instrument should be low and nearly identical.
2. The resistance of shunt should not vary with time.
3. It should carry the current without excessive temperature rise.
4. It should have a low thermal emf.

- Shunts for low current are enclosed in meter casing but for currents above 200 A they are mounted separately.
- Constantan is very useful material for AC circuit since its comparatively high thermal emf being unidirectional is ineffective on these circuits.
- Manganin is usually used as a shunt for DC instruments since it gives a low value of thermal emf with copper.

Shunt with Ammeter



Low resistance of shunt is connected across the coil of ammeter.

Let I be the circuit current to be measured.

R_m = Ammeter resistance

S = Shunt resistance

I_m = Full-scale deflection of ammeter

I_s = Shunt current

$I = I_s + I_m$

$I_s = I - I_m$.

Since voltage across shunt and ammeter is same

$$I_m R_m = (I - I_m)R_s$$

$$I_m (R_m + R_s) = IR_s$$

$$I = \frac{I_m (R_m + R_s)}{R_s}$$

$$\text{Circuit current} = \text{Full-scale deflection} \times \left(\frac{R_m + R_s}{R_s} \right)$$

Instrument Constant

It is the ratio of current to be measured to the full-scale deflection.

$$M = \frac{I}{I_m} = \frac{R_m + R_s}{R_s}$$

- With different shunts the same instrument will have different instrument constants.

Shunts for AC Instruments

In order to maintain the current division between shunt and instrument, constant for all frequencies, the ratio of impedances of the instrument and leads to the shunt must remain constant.

L_m = Inductance of the instrument

L_s = Inductance of the shunt

Then $\dfrac{\sqrt{R_m^2 + \omega^2 L_m^2}}{\sqrt{R_s^2 + \omega^2 L_s^2}}$ should be constant.

This is possible only if time constant of the shunt and instruments are the same.

$$\text{i.e., } \frac{L_m}{R_m} = \frac{L_s}{R_s} = K$$

$$\text{Multiplying factor } N = \frac{I}{I_m} = \frac{I_m + I_s}{I_m}$$

$$N = \frac{R_m}{R_m} + 1$$

where

$$I_m = \frac{V}{\sqrt{R_m^2 + \omega^2 L_m^2}}$$

$$I_s = \frac{V}{\sqrt{R_s^2 + \omega^2 L_s^2}}$$

Voltmeter

An instrument which is used to measure voltage between two points in a circuit is called voltmeter.

- It is always connected in parallel with the portion whose voltage is to be measured.
- The current flowing through the operating coil of the meter produces a deflecting torque.
- Since voltmeter is connected in parallel and in order that its connection does not disturb the circuit condition therefore the resistance of the voltmeter should be very high.
- To do so a high resistance is connected in series with the coil of the instrument.
  Let R_0 be the total resistance (resistance of the operating coil + high series resistance) of the instrument current flowing through instrument

$$I_v = \frac{V}{R_v}$$

$$\theta \propto I_v \propto \frac{V}{R_v}$$

$$\theta \propto V$$

Deflection is proportional to voltage and hence measures voltage.

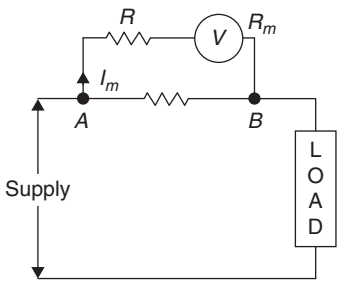
Multipliers

- A high resistance in series with a galvanometer is connected to limit the current flowing through the meter, so that it does not exceed the value for full-scale deflection and thus prevent the instrument from being damaged. Such a resistance is called multiplier.

Multiplier with Voltmeter

High resistance is connected with a voltmeter coil

$$V = I_m (R + R_m) \qquad (3)$$



From given figure

V_{AB} = Voltage across voltmeter
V = Voltage across voltmeter
R = High resistance in series with voltmeter coil
R_m = Voltmeter resistance
I_m = Full-scale deflection current for voltmeter

Equation (3) can also be written as $R + R_m = \frac{V}{I_m}$

$$R = \frac{V}{I_m} - R_m$$

Multiplier constant $m = \frac{I_m(R_m + R_s)}{I_m R_m}$

$$m = 1 + \frac{R}{R_m}$$

Required high resistance

$$R = \frac{\text{Maximum voltage to be measured}}{f.s.d.\ \text{Voltmeter current}} - \text{Voltmeter resistance}$$

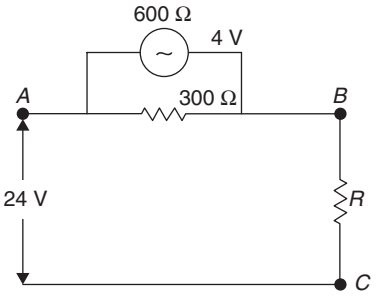
Example 4: Determine the current through the voltmeter if it reads half of full-scale deflection of 200 V and sensitivity of voltmeter is 500 Ω/V.

Solution: Half of f.s.d 200 V = 100 V
Resistance of the voltmeter

$$R_v = S_v \times V$$
$$= 500 \times 100$$
$$= 50\ \text{k}\Omega$$
$$I = \frac{V}{R} = \frac{100\ \text{V}}{50\ \text{k}\Omega} = 2\ \text{mA}.$$

Example 5: A voltmeter having internal resistance 600 Ω is connected across 300 Ω resistor reads 4 V, when the above combination is connected in series with a resistance and is given a supply voltage of 24 V the resistance R will be
(A) 150 Ω
(B) 500 Ω
(C) 1.0 kΩ
(D) 2.0 kΩ

Solution: (C)



R_{eq} of parallel combination

$$\frac{600 \times 300}{600 + 300} = 200\ \Omega$$
$$V_{AC} = V_{AB} + V_{BC}$$
$$V_{BC} = V_{AC} - V_{AB}$$
$$= 24 - 4 = 20\ \text{V}$$
$$\frac{V_{BC}}{R_{BC}} = \frac{V_{AB}}{R_{AB}}$$
$$\frac{20}{R} = \frac{4}{200}$$
$$R = \frac{200 \times 20}{4} = 1000\ \Omega$$
$$R = 1\ \text{k}\Omega$$

Example 6: A meter has a full-scale deflection of 120° at a current of 2 A. The response of the meter is square law. Assuming spring control, the current for deflection at 60° will be
(A) 1.414 A (B) 0.707 A
(C) 2 A (D) 1 A

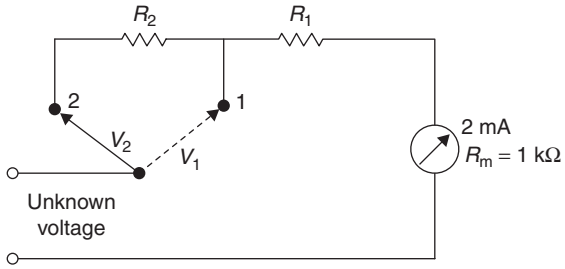
Solution: (A)
We know that deflecting torque

$$T_d = K I^2$$
$$T_c = C\,\theta$$
$$I^2 = \left(\frac{C}{K}\right)\theta$$

Suppose (C/K) is a constant

$$\frac{I_1^2}{I_2^2} = \frac{\theta_1}{\theta_2} \quad I_2^2 = \frac{I_1^2 \times \theta_2}{\theta_1}$$
$$= 2^2 \times \left(\frac{60}{120}\right)$$
$$I_2 = \sqrt{2} = 1.414\ \text{A}.$$

Example 7: For the voltmeter circuit shown in the given figure, it has a f.s.d. of 2 mA and an internal resistance of 1 kΩ. The values of series resistance required for a range of $V_1 = 12$ V and $V_2 = 60$ V will be, respectively.



(A) 9.9 kΩ and 40 kΩ (B) 8 kΩ and 30 kΩ
(C) 20 kΩ and 30 kΩ (D) 5 kΩ and 29 kΩ

Solution: (D)
For 12 V, the f.s.d can be calculated as

$$2 \times 10^{-3} = \frac{12}{(1000 + R_1)}$$

$$R_1 \times 2 \times 10^{-3} = 12 - 2$$

$$R_1 = 5\ \text{k}\Omega$$

For 60 V, the f.s.d can be calculated as

$$2 \times 10^{-3} = \frac{60}{(1000 + R_1 + R_2)}$$

$$(R_1 + R_2) \times 2 \times 10^{-3} = 60 - 2$$

$$R_2 = 29\ \text{k}\Omega$$

Example 8: What will happen if a voltmeter is connected like an ammeter in series to the load

Solution: There will be almost no current in the circuit.

Instrument Transformers

- Instrument transformers are used in AC systems for measurement of current, voltage, power and energy.
- They are also used in connection with measurement of power factor, frequency and indication of synchronism.
- They also have a wide application in protection circuit of power systems for operation of various relays.

Advantages

1. These enable single instruments to cover a large current or voltage range.
2. Minimal power loss.
3. Measuring instruments can be located far from the high voltage circuit by using long loads connecting the instrument transformer to the instrument. Hence the instrument is isolated from high voltages and stray fields providing safety in use to both instrument and the observer.
4. By using a C.T. with a suitably split and hinged core upon which the secondary winding is wound, the current in a heavy bus bar is measured without breaking the circuit.
5. Due to the inductance of the instrument winding being in parallel with the non-inductive shunt, shunt on AC does not give a linear relation between instrument current and bypassed current. On the other hand, in case of C.T, the whole current passes through primary a proportional voltage is induced in the secondary which sends a current in direction proportion around the secondary circuit.

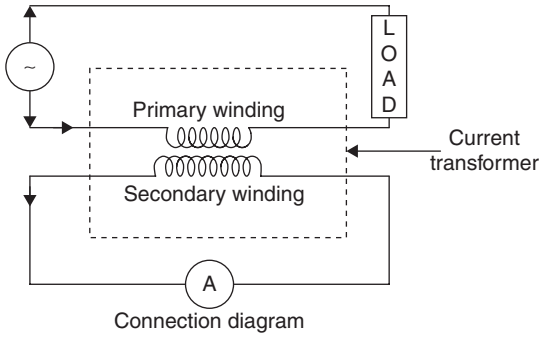
Disadvantages

They cannot be used for DC.

Types of Instrument Transformers

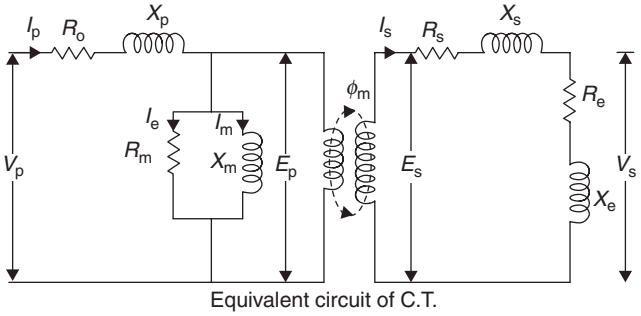
1. Current transformer (Series)
2. Potential transformer (Parallel)

Current Transformer (C.T.)



Connection diagram

- The primary winding of the C.T. is connected in series with the load, and carries the load current to be measured. It consist of a very few turns.
- Secondary winding is directly connected across the load and has more turns.
- The amount of power C.T. can handle is small. The product of voltage and current on secondary side of C.T. when it is supplying the instrument with its maximum rated value of current is called rated burden and is measured in V.A. Its secondary is kept short circuited always.



Equivalent circuit of C.T.

Ratio of Current Transformers

Transformation ratio

$$R = \frac{\text{Primary winding current}}{\text{Secondary winding current}}$$

Nominal ratio $K_n = \dfrac{\text{Rated primary current}}{\text{Rated secondary current}}$

$$\text{Turns ratio} = \frac{\text{Number of turns in secondary}}{\text{Number of turns in primary}}$$

For the phasor diagram

E_p = Primary induced voltage
E_s = Secondary induced voltage
n = Turns ratio
I_p = Primary current
I_s = Secondary current
I_o = Exciting current
I_m = Magnetizing component of exciting current
I_c = Loss component of exciting current
ϕ = Working flux of C.T.
α = Angle between I_o and ϕ

$\delta = \tan^{-1}\left(\dfrac{X_s + X_e}{R_s + R_e}\right)$ = angle between E_s and I_s

$\Delta = \tan^{-1}\left(\dfrac{X_e}{R_e}\right)$ = phase angle of secondary burden

θ = Phase angle of transformers

From the phasor diagram, transformation ratio can

$$R \approx \frac{nI_s + I_o \sin(\delta + \alpha)}{I_s} \qquad \left[\because I_m = I_o \cos\alpha,\ I_e = I_o \sin\alpha\right]$$

$$R \approx \frac{nI_s + I_m \sin\delta + I_e \cos\delta}{I_s}$$

$$\text{Phase angle } \theta \approx \frac{180}{\pi}\left(\frac{I_m \cos\delta - I_e \sin\delta}{nI_s}\right)$$

Errors

1. **Ratio angle error:** It is mainly on account of magnetizing component of excitation current

$$\text{Ratio error} = \frac{\text{Nominal ratio} - \text{Actual ratio}}{\text{Actual ratio}}$$

$$\text{Ratio error} = \frac{K_n - R}{R}$$

2. **Phase angle error:** It is mainly on account of magnetizing component of excitation current

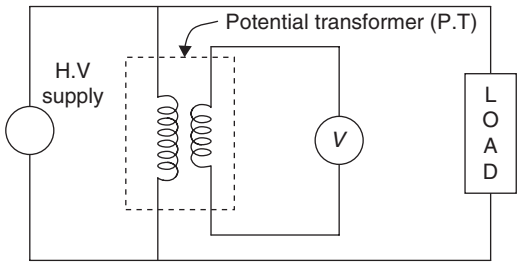
$$\theta = \frac{180}{\pi}\left(\frac{I_m \cos\delta - I_e \sin\delta}{nI_s}\right)\text{degree}$$

$$\theta = \frac{180}{\pi}\left(\frac{I_m}{I_p}\right)\text{degree}$$

Design Features

- The coils are wound close together to reduce the secondary leakage reactance. They are separately wound and insulated by tape and varnish for lower voltages.
- The number of exciting ampere turns are kept very small when compared to the total primary ampere turns, to reduce the current ratio and phase angle errors.
- I_m and I_e are kept small by using core of high permeability and low magnetic materials.
- The core reluctance is reduced by reducing the length of magnetic path in the core and increasing the area of the magnetic path in the core. The transformer has a rectangular or ring shaped core.

Potential Transformers



Connection diagram

- They are used for measurement of high voltages by means of low range voltmeters or for energizing the potential coils of wattmeters, energy meters.
- The primary winding is connected across the line carrying the voltage to be measured that the voltage circuit is connected across the secondary winding.
- In the phasor diagram,
  θ = Angle between V_p and V_s reversed
  θ = Phase angle of the transformer
  β = Phase angle between I_p and V_s reversed
  Δ = Phase angle of secondary load circuit

Design Features

- The ratio error produced is compensated by increasing the turn on the secondary winding by an appropriate small amount. The increased secondary turns compensate the voltage drop in potential transformers.
- By minimizing the no load current errors of potential transformers are controlled to a great extent.
- The reluctance of the core and flux density in the core is kept low.
- High-quality core material is used to reduce core loss.
- The windings are wound one over the other to reduce leakage to minimum.

Ratios

1. Transformation ratio $R = \dfrac{\text{Primary winding voltage}}{\text{Secondary winding voltage}}$

2. Nominal ratio $K_n = \dfrac{\text{Rated primary voltage}}{\text{Rated secondary voltage}}$
3. Turns ratio

$$n = \frac{\text{Number of turns of primary winding}}{\text{Number of turns of secondary winding}}$$

Transformation ratio from phasor diagrams

$$R = \frac{V_p}{V_s}$$

$$= n + \frac{nI_s(R_s \cos\Delta + X_s \sin\Delta) + I_s R_p + I_m X_p}{V_s}$$

$$\theta = \frac{I_s}{V_s}\left(X_s \cos\Delta - R_s \sin\Delta\right) + \frac{I_e X_p - I_m R_p}{nV_s}$$

Errors

1. Ratio error $= \dfrac{K_n - R}{R} \times 100$
2. Phase angle error

$$\theta = \frac{I_s}{V_s}\left(X_s \cos\Delta - R_s \sin\Delta\right) + \frac{I_e X_p - I_m R_p}{n\,V_s}$$

Example 9: When the primary winding of a current transformer is energized with the secondary winding open circuited.

(A) A large value of flux is produced in the core due to the whole of the primary winding current producing a large voltage on the secondary side.
(B) When large magnetization force is taken off, it leaves behind a huge residual magnetism.
(C) Large secondary voltage may damage the insulation.
(D) All of the above.

Solution: (D)

Example 10: The size of the potential transformers

(A) Is much greater than that of power transformers of same VA rating.
(B) Is the same as that of power transformers of the same VA rating.
(C) Is smaller than that of power transformer of same VA rating.
(D) Cannot be determined.

Solution: (A)
The size of potential transformers would be much greater than that of power transformers of the same VA rating since they are designed for low ratio and phase angle errors which requires large sized cores and winding conductors.

Example 11: Calculate the transformation ratio of a current transformer rate 60/6 A. Its secondary side being purely resistive has magnetizing and loss component of exciting current as 20 A and 1.2 A, respectively.

(A) 10.02 (B) 10.20
(C) 10.002 (D) 100.2

Solution: (B)
Transformation ratio

$$R = n + \frac{I_e \cos\delta + I_m \sin\delta}{I_s}$$

$$n = 60/6 = 10$$

$$I_e = 1.2$$

$$I_m = 20 \text{ A}$$

$$\cos\delta = \frac{R_s}{Z_{eq}} \quad \sin\delta = \frac{X_s}{Z_{eq}}$$

$$[Z_{eq} = R \text{ since } X_s = 0]$$

$$\cos\delta = 1 \text{ sin } \delta = 0$$

$$R = 10 + \frac{(1.2\times1) + (20\times0)}{6} = 10.2.$$

Example 12: The magnetizing and loss component of exciting current of a current transformer rated 1000/5 A, are 15 A and 9 A, respectively. The phase angle between secondary winding induced voltage and current is 40°. The phase angle error of the transformer is

(A) 0.657 (B) 0.125
(C) 0.457 (D) 0.327

Solution: (D)

$$n = \frac{1000}{5} = 200$$

$$I_m = 15 \text{ A}$$

$$I_p = 1000 \text{ A}$$

$$\delta = 40°$$

$$I_e = 9 \text{ A}$$

$$I_s = 5 \text{ A}$$

$$\theta = \frac{180}{\pi}\left(\frac{I_m \cos\delta - I_e \sin\delta}{n\,I_s}\right)$$

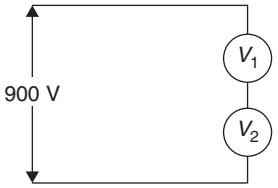
$$= \frac{180}{\pi}\left(\frac{15\cos40 - 9 \text{ sin } 40}{200\times5}\right)$$

$$= \frac{180}{\pi}\left(\frac{11.5 - 5.78}{1000}\right) = 0.327°$$

EXERCISES

Practice Problems I

1. Two 500 V full-scale voltmeters V_1 and V_2 having resistances 200 kΩ/V and 250 kΩ/V, respectively, are connected in series to measure 900 V. Then



(A) V_1 and V_2 reads 400 V each
(B) V_1 reads 400 V and V_2 reads 500 V
(C) V_1 reads 500 V and V_2 reads 400 V
(D) V_1 and V_2 reads 0 V

2. Which of the following is an indicating type instrument?
(A) Tachometer (B) C.R.O.
(C) Energymeter (D) E.C.G.

3. A moving coil milli ammeter with a resistance of 1.6 Ω is connected with a shunt of 0.52 Ω. What will be the current flowing through the instrument if it is connected in a circuit in which a current of 400 mA is flowing?
(A) 97.98 mA (B) 97.02 mA
(C) 98.28 mA (D) 99 mA

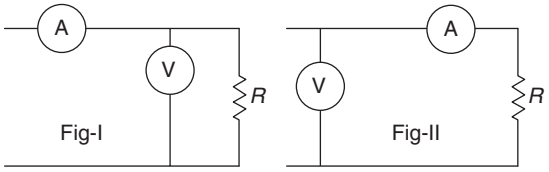
4. In ammeter, the shunt and the moving coil are made of the same material because
(A) Temperature error can be eliminated
(B) Temperature error cannot be eliminated
(C) Both A and B
(D) None of these

5. The error occurring in current transformer is due to
(1) The magnetic leakage in secondary winding
(2) Power consumption in the metering circuit
(3) The exiting mmf required by the primary winding to produce flux
(4) The non-linear relation between flux density in the core and magnetizing force Of these statements which are correct?
(A) 1 and 4 (B) 1, 2 and 3
(B) 1, 3, 4 (D) 2 and 3

6. The error produced due to the inductance in voltage coil of an electrodynamometer type wattmeter.
(A) Increases as P.F. decreases
(B) Decreases as P.F. decreases
(C) Constant irrespective of P.F. of the circuit
(D) No relation between P.F. and inductance of the coil

7. What is the correct sequence of the following types of ammeter and voltmeter with increasing accuracy?
(1) Moving iron
(2) P.M.M.C.
(3) Induction
(A) 1, 3, 2 (B) 1, 2, 3
(C) 3, 1, 2 (D) 2, 1, 3

8. Ammeter voltmeter method is employed to measure the resistance of resistor R. From among the circuits shown below the least erroneous results are obtained by adopting



(A) Circuit I for low resistance and circuit II for high resistance
(B) Circuit I for high resistance and circuit II for low resistance
(C) Circuit I should not be employed for resistance measurement
(D) Circuit II should not be employed for resistance measurement

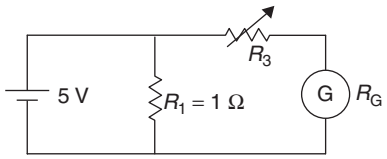
9. The coil of a PMMC instrument has 20 turns on a rectangular former of 3.5 cm × 1.5 cm swings in a uniform field of 0.05 Wb/m^2. If a steady current of 40 mA is flowing through the coil, the deflecting torque is
(A) 210×10^{7} Nm (B) 210×10^{-6} Nm
(C) 210×10^{-7} Nm (D) 210×10^{-5} Nm

10. A sinusoidal voltage of 230 V RMS 50 Hz is connected to a rectifier type ammeter which do not allow the flow of current in one direction and offers a resistance of 30 Ω to flow of current in other direction. The current recorded by ammeter is
(A) 2.95 A (B) 10.35 A
(C) 7.67 A (D) 3.45 A

11. A 10 mA ammeter has a resistance of 50 Ω. It has to be converted to a 1 A ammeter. The value of shunt resistance is
(A) 5 Ω (B) 0.05 Ω
(C) 0.5 Ω (D) 50 Ω

12. A current transformer has a rating of 200/5 A. The magnetizing and loss component of exciting current are 2 A and 1 A, respectively, and secondary has a burden of pure resistance. The transformation ratio of rated current is
(A) 40 (B) 40.1
(C) 40.2 (D) 41

13. A galvanometer is tested in the circuit where $E = 5$ V, $R_1 = 1.0$ Ω and $R_2 = 4500$ Ω with R_3 set as 600 Ω, the deflection in galvanometer is 160 mm. When R_3 is changed to 1280 Ω the deflection is changed to 80 mm. The resistance of the galvanometer is



(A) 80 Ω (B) 79 Ω (C) 100 Ω (D) 99 Ω

14. A 1000 A/5 A, 50 Hz current transformer has a bar primary. The secondary burden is a pure resistance of 0.5 Ω and it is drawing a current of 5 A. If the core of the C.T requires 400 AT for magnetization, then the percentage ratio error is

(A) –10.8% (B) +7.1%
(C) +10.8% (D) –7.1%

15. A simple slide wire is used for measurement of current in a circuit. The voltage drop across a standard resistor of 0.1 Ω is balanced at 60 cm. Find the magnitude of current if the standard cell emf of 1.45 V is balanced at 50 cm
(A) 14.5 A
(B) 14.7 A
(C) 17.4 A
(D) 24.5 A

Practice Problems 2

1. Rectifier moving coil instruments respond to
(A) Average value irrespective of nature of waveform
(B) RMS value irrespective of nature of waveform
(C) Peak value irrespective of nature of waveform
(D) RMS value for symmetrical square waveform

2. A certain measuring instrument has a resistivity of 20,000 Ω/V. The current required to deflect the meter to full scale is
(A) 20 μA (B) 50 μA (C) 20 mA (D) 50 mA

3. A (0–5) A PMMC ammeter reads 3 A when connected in a certain circuit. If the bottom control spring snaps, the reading will be
(A) 3 mA (B) 6 mA (C) 1.5 mA (D) Zero

4. A (0–5) A PMMC ammeter does not have any controlling mechanism. Now, if a current of 2 A (DC) is passed through the coil, the reading shown by the Meter would be (frictional opposition is overcome by torque produced
(A) 2 A
(B) 5 A
(C) Pointer rotates continuously.
(D) Pointer does not rotate.

5. In moving coil meters, damping is provided by
(A) Damping vane in the airtight chamber
(B) Coil spring attached to the moving coil
(C) Aluminium frame of the coil
(D) Eddy current disk

6. A PMMC instrument has a 50 turn coil and carries a current of 5 mA. The flux density in the air gap is 0.5 Wb/m^2. The coil dimensions are: length = 2 cm, and Diameter = 1 cm. The torque will be
(A) 25 μNm (B) 25 Nm
(C) 25 mNm (D) None of the above

7. The inductance of a certain MI ammeter is $L = 30 + 2\theta - \frac{\theta^2}{18}$ μH. The control spring torque is 35 μNm/radian. The deflection of the pointer in radian when it carries a current of 6 A is
(A) π^c (B) 1^c (C) $\frac{\pi^c}{3}$ (D) 2^c

8. A PMMC voltmeter is connected across a series combination of a DC source of 4 V and AC source of $4\sin\omega t$. The meter reads
(A) $\sqrt{6}$ V (B) 2 V
(C) 4 V (D) $2\sqrt{2}$ V

9. A moving iron voltmeter reads correctly on 230 V DC. If 230 V, 50 Hz AC is applied to it, the reading of the voltmeter is
(Coil has a resistance of 800 Ω and an inductance of 1 H and a series non inductive resistance of 1500 Ω)
(A) 225.78 V (B) 227.88 V
(C) 230 V (D) 232.12 V

10. An unshielded moving iron voltmeter is used to measure the voltage in an AC circuit. If a stray DC magnetic field having a component along the axis of the meter coil appears, the meter reading would be
(A) Unaffected
(B) Decreased
(C) Increased
(D) Either increased or decreased depending on the direction of the field

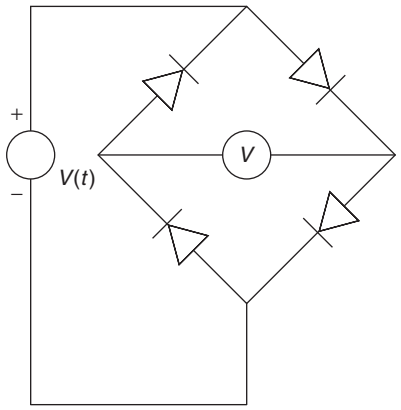
11. A moving iron type ammeter and a PMMC type ammeter are connected in series in a resistive circuit supplied from a half-wave rectified voltage source. If the moving iron type meter reads 8 A, the reading of PMMC instrument is likely to be
(A) 5.09 A (B) Zero
(C) 8 A (D) 6.18 A

12. Two voltmeter of range (0–500) V are connected in series. The internal resistances of the voltmeter are 5000 Ω and 15,000 Ω respectively. A voltage of 400 V is applied across them. The voltage shown by two meter are
(A) 150 V, 250 V (B) 200 V, 200V
(C) 100 V, 300 V (D) None of these

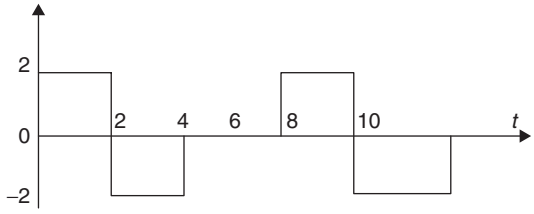
13. The mutual inductance of a dynamometer ammeter varies with deflection θ as
$M = 10 \sin(\theta + 30)$ mH
The deflecting torque produced by a DC current of 0.5 A corresponding to a deflection of 30° is

(A) 1.25 m Nm (B) 1.5 m Nm
(C) 1 m Nm (D) 0.75 m Nm

Common Data for Questions 14 and 15:



$V(t)$ is as shown below.



14. The voltmeter is PMMC. The reading shown is
(A) 2 V (B) 1 V
(C) $\sqrt{2}$ V (D) 0 V

15. If the voltmeter is M.I, then reading shown is
(A) 2 V (B) 1 V
(C) $\sqrt{2}$ V (D) 0 V

Previous Years' Questions

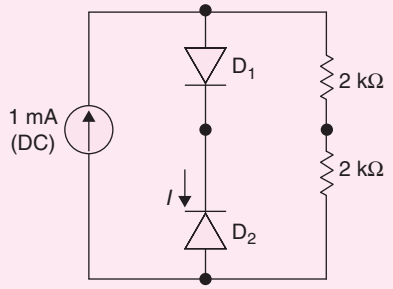
1. A moving iron ammeter produces a full-scale torque of 240 μNm with a deflection of 120° at a current of 10 A. The rate of change of self-inductance (μH/radian) of the instrument at full scale is **[2004]**
(A) 2.0 μH/radian (B) 4.8 μH/radian
(C) 12.0 μH/radian (D) 114.6 μH/radian

2. A 50 Hz, bar primary CT has a secondary with 500 turns. The secondary supplies 5 A current into a purely resistive burden of 1 Ω. The magnetizing ampere-turns is 200. The phase angle between the primary and secondary current is **[2004]**
(A) 4.6° (B) 85.4°
(C) 94.6° (D) 175.4°

3. The core flux in the CT of problem 83, under the given operating condition is **[2004]**
(A) 0 (B) 45.0 μWb
(C) 22.5 mWb (D) 100.0 mWb

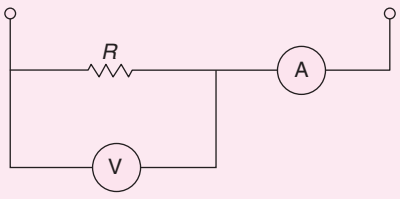
4. A PMMC voltmeter is connected across a series combination of a DC voltage source $V_1 = 2$ V and an AC voltage source $V_2(t) = 3\sin(4t)$ V. The meter reads **[2005]**
(A) 2 V (B) 5 V
(C) $\left(2+\frac{\sqrt{3}}{2}\right)$ V (D) $\left(\frac{\sqrt{17}}{2}\right)$ V

5. Assume that D_1 and D_2 in Fig. Q23 are ideal diodes. The value of current I is: **[2005]**



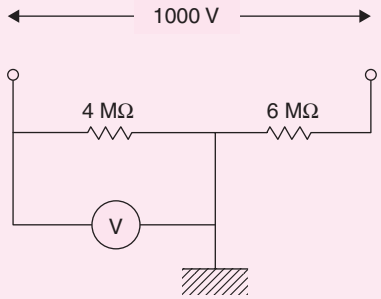
(A) 0 mA (B) 0.5 mA
(C) 1 mA (D) 2 mA

6. The set-up in Fig. Q.66 is used to measure resistance R. The ammeter and voltmeter resistances are 0.01 Ω and 2000 Ω, respectively. Their readings are 2 A and 180 V, respectively, giving a measured resistance of 90 Ω. The percentage error in the measurement is: **[2005]**



(A) 2.25% (B) 2.35%
(C) 4.5% (D) 4.71%

7. A 1000 V DC supply has two 1-core cables as its positive and negative leads; their insulation resistances to earth are 4 MΩ and 6 MΩ, respectively, as shown in Fig. Q67. A voltmeter with resistance 50 kΩ is used to measure the insulation of the cable. When connected between the positive core and earth, the voltmeter reads **[2005]**



(A) 8 V (B) 16 V (C) 24 V (D) 40 V

8. A current of $-8 + 6\sqrt{2}$ (sin $\omega t + 30°$) A is passed through three meters. They are a centre zero PMMC meter, a true RMS meter and a moving iron instrument. The respective readings (in A) will be **[2006]**

(A) 8, 6, 10 (B) 8, 6, 8
(C) –8, 10, 10 (D) –8, 2, 2

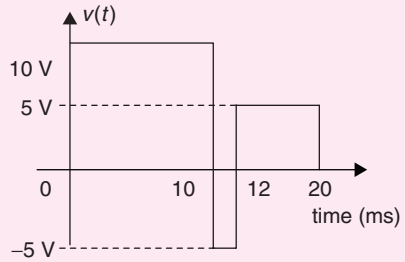
9. A 200/1 current transformer (CT) is wound with 200 turns on the secondary on a toroidal core. When it carries a current of 160 A on the primary, the ratio and phase errors of the CT are found to be –0.5% and 30 minutes, respectively. If the number of secondary turns is reduced by 1 the new ratio error (%) and phase error (min) will be, respectively, **[2006]**

(A) 0.0, 30 (B) –0.5, 35
(C) –1.0, 30 (D) –1.0, 25

10. An ammeter has a current range of 0–5 A, and its internal resistance is 0.2 Ω. In order to change the range to 0–25 A, we need to add a resistance of **[2010]**

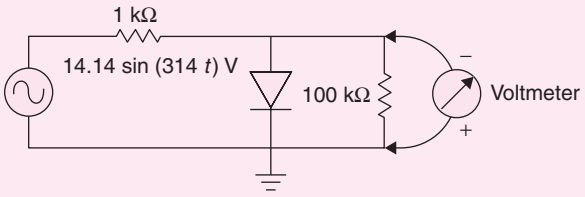
(A) 0.8 Ω in series with the meter
(B) 1.0 Ω in series with the meter
(C) 0.04 Ω in parallel with the meter
(D) 0.05 Ω in parallel with the meter

11. A periodic voltage waveform observed on an oscilloscope across a load is shown. A permanent magnet moving coil (PMMC) meter connected across the same load reads. **[2012]**



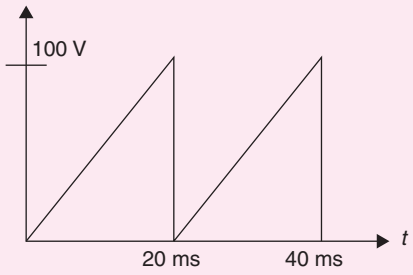
(A) 4 V (B) 5 V (C) 8 V (D) 10 V

12. The input impedance of the permanent magnet moving coil (PMMC) voltmeter is infinite. Assuming that the diode shown in the figure below is ideal, the reading of the voltmeter in volts is **[2013]**



(A) 4.46 (B) 3.15 (C) 2.23 (D) 0

13. The saw-tooth voltage waveform shown in the figure is fed to a moving iron voltmeter. Its reading would be close to _____ **[2014]**



14. Two ammeters X and Y have resistances of 1.2 Ω and 1.5 Ω, respectively, and they give full-scale deflection with 150 mA and 250 mA, respectively. The ranges have been extended by connecting shunts so as to given full-scale deflection with 15 A. The ammeters along with shunts are connected in parallel and then placed in a circuit in which the total current flowing is 15 A. The current in amperes indicated in ammeter X is _____. **[2014]**

15. A dc voltage with ripple is given by $v(t) = [100 + 10 \sin \omega (t) - 5 \sin (3 \omega (t))]$ volts.

Measurements of this voltage $v(t)$, made by moving-coil and moving-iron voltmeters, show readings of V_1 and V_2 respectively. The value of $V_1 - V_2$, in volts, is _____. **[2015]**

16. A(0 – 50 A) moving coil ammeter has a voltage drop of 0.1 V across its terminals at full scale deflection. The external shunt resistance (in milliohms) needed to extend its range to (0 – 500 A) is _____.

17. Match the following. **[2015]**

Instrument Type	Used for
P. Permanent magnet moving coil	1. DC only
Q. Moving iron connected through current transformer	2. AC only
R. Rectifier	3. AC and DC
S. Electrodynamometer	

(A) P–1 Q–2 R–1 S–3
(B) P–1 Q–3 R–1 S–2
(C) P–1 Q–2 R–3 S–3
(D) P–3 Q–1 R–2 S–1

Answer Keys

Exercises

Practice Problems 1

1. B **2.** C **3.** C **4.** A **5.** B **6.** D **7.** C **8.** A **9.** C **10.** D
11. C **12.** C **13.** B **14.** D **15.** C

Practice Problems 2

1. A **2.** B **3.** D **4.** C **5.** C **6.** A **7.** B **8.** C **9.** B **10.** D
11. A **12.** C **13.** A **14.** B **15.** C

Previous Years' Questions

1. B **2.** A **3.** B **4.** A **5.** A **6.** C **7.** A **8.** C **9.** A **10.** D
11. A **12.** A **13.** 57.73 **14.** 10.13 **15.** 0.312 **16.** 0.22 to 0.23 **17.** C

Chapter 3

Measurement of Power and Energy

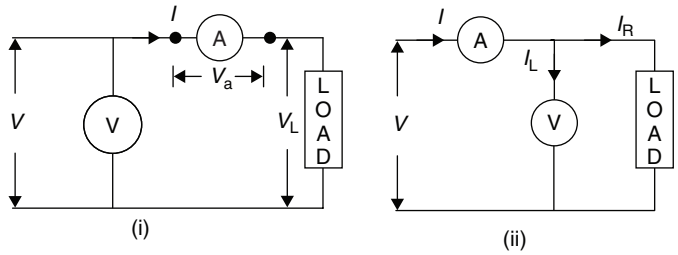
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Measurement of power
- Measurement of power in AC circuits
- Deflecting torque
- Dynamometer type wattmeter
- Deflecting torque
- Measurement of power in three-phase circuits
- Braking torque
- Ficticious or phantom loading
- Shunt magnet
- Braking magnet

Measurement of Power

Measurement of Power in DC Circuits



- The power taken by a load from a DC supply is given by the product of readings of an ammeter and voltmeter when connected in a circuit.

$$\text{Power} = V \times I \text{ W}$$

Figure (i): When ammeter is connected between voltmeter and load.

- Voltmeter indicates the voltage drop across the load and also V_a (or) voltage drop across the ammeter.
- If R_a is the resistance of the ammeter

$$V_a = I_a R_a.$$

- Power consumed by the load = $V_L I$

$$= (V - V_a)I = VI - V_a I$$

$$= VI - I^2 R_a$$

= (Power indicated by the instruments) – (Power loss in ammeter)

Power indicated by instruments = (Power consumed by load) + (Power loss in ammeter)

Figure (ii): Power Consumed by load = VI_L

$$= V(I - I_V)$$

$$= V\left(I - \frac{V}{R_v}\right)$$

$$= VI - \frac{V^2}{R_v}$$

Power indicated by instruments = (Power consumed in load) + (Power loss in voltmeter)

In case I and II power indicated by the instruments = power consumed by the load plus power consumed by instrument nearer to load terminal.

- In order to obtain true power, corrections must be applied for power loss in instruments.

Measurement of Power in AC Circuits

The power in AC circuit is given by

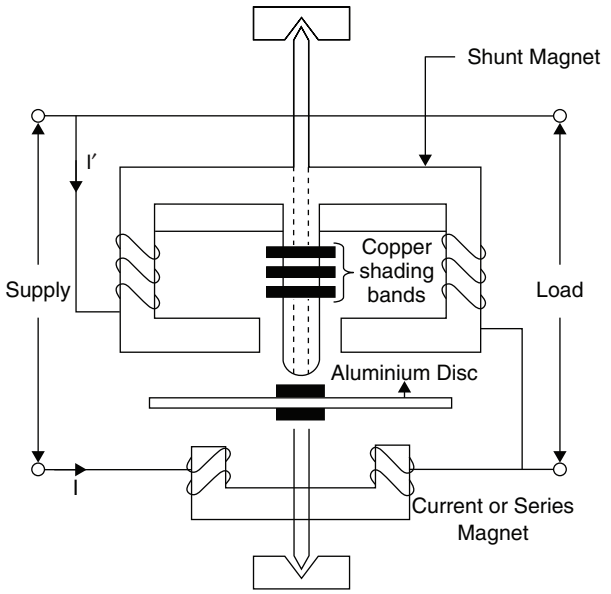
$$P = VI \cos\phi$$

V = Voltage across the load
I = Current through the load
$\cos\phi$ = Power factor of the load

- Voltmeter–ammeter method is not applicable in AC circuits because of involvement of power factor.
- There are two types of wattmeters.

Induction Type Wattmeter

- This type of wattmeter is used to measure AC power only.



Principle

- The principle of operation is same as that of induction ammeter and voltmeter.
- The only area where it differs is that two separate coils in place of one coil with phase-split arrangement.

Construction

- It consist of two laminated electromagnets.
- One electromagnet is called the shunt magnet is connected across the supply and carries current proportional to applied voltage. The coil of this magnet is highly inductive so that the current in it lags behind supply voltage lags behind by 90°.
- The other electromagnet is called series magnet and is connected in series with supply and carries load current. The coil of this magnet is highly non inductive so that the angle of lag or lead is determined wholly by the load.
- A thin aluminium disc mounted on the spindle is placed in between two magnets so that it cuts the fluxes in both the magnets.
- The controlling torque is provided by spiral springs.
- Damping is usually electromagnetic and is usually done by a permanent magnet embracing the aluminium disc.
- Two or more copper rings, called the shading rings are provided on the central limb of the shunt magnet. By adjusting their position the shunt magnet flux can be made to lag behind supply voltage exact by 90°.

Working

- When wattmeter is connected in the circuit, the shunt magnet carries current proportional to supply voltage and the series magnet carries the load current.
- The two fluxes produced by the magnets induce eddy currents in the aluminium disc. The interaction between fluxes and eddy currents produce the deflecting torque on the disc, causing the pointer connected to the moving system to move over the scale.

Deflecting Torque

Let V = Applied voltage

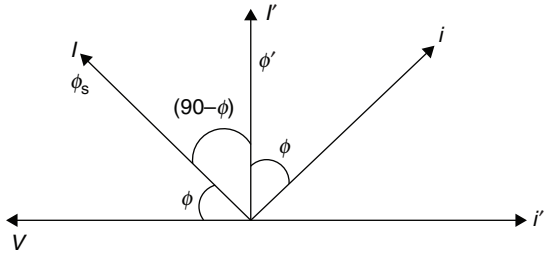
I = Load current carried by series magnet

I' = Current carried by shunt magnet

$\cos\phi$ = Lagging p.f. of the load

- Current I' in the shunt magnet lags applied voltage V by 90° and so does the flux ϕ' produced by it.
- The current I in the series magnet is the load current and hence lags behind the applied voltage by ϕ. The flux ϕ_s produced by this current is in phase with it.
- The two currents I' and I and the corresponding fluxes ϕ' and ϕ_s are $(90 - \phi°)$ apart.
- The flux ϕ' induces eddy currents i' in the aluminium disc which lags behind ϕ' by 90°.

Similarly ϕ_s induces eddy current i, which lags ϕ_s by 90°.



Mean deflecting torque

$$T_d \propto \phi' \phi_s \sin(90 - \phi)$$

$$[\phi' \propto V \text{ and } \phi_s \propto I]$$

$$T_d \propto VI\cos\phi$$

$$T_d \propto \text{AC power}$$

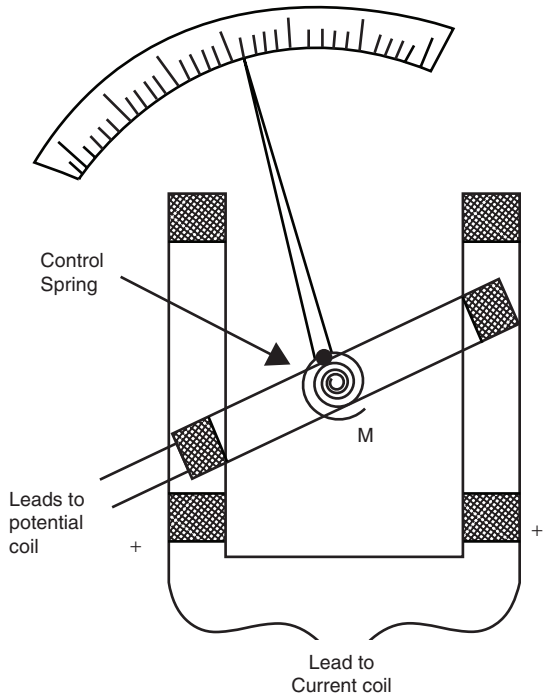
Since control is by springs $T_c \propto T_c$

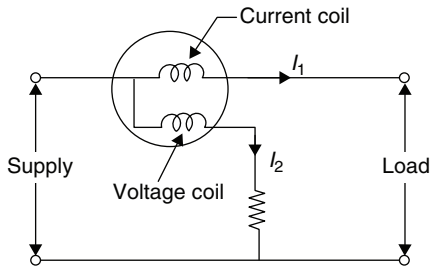
$$\theta \propto \text{Power}$$

Such instruments have uniform scale.

Dynamometer Type Wattmeter

- It is the most commonly employed for measurement of power in AC and DC.





Principle

It is based on the principle that mechanical force exists between two current carrying conductors.

Construction

- It essentially consists of two coils namely fixed coil and moving coil.
- Fixed coils are split into two equal parts which are placed close together and parallel to each other.
- The moving coil is pivoted between the two fixed coils and is placed on the spindle to which the pointer is attached.
- Fixed coils carry circuit current and are connected in series with the load. It is therefore called the current coil.
- The moving coil is connected across the load and carries current proportional to voltage. It is therefore called potential coil.
- To limit the current through the potential coil a high resistance is connected in series to it.
- Controlling torque is provided by the springs which also serve the additional purpose of leading current into and out of the moving coil.
- Damping is achieved by air friction.

Working

- Due to current in fixed and moving coils, mechanical force exist between them hence the moving coil moves the pointer over the scale. The pointer comes to rest at a position where deflecting torque is equal to controlling torque.
- Reversing the current reverses the field due to fixed coil as well as the current in the moving coil so that the direction of the deflection torque remains unchanged. Therefore such instruments can be used for the measurement of AC as well as DC power.

Deflecting Torque

Operation on DC

Let V be the voltage across the load and; be the current through the load. Current through fixed coil $I_f \propto I$; current through moving coil $I_m \propto V$. Deflecting Torque T_d is due to I_m and I_f

$$T_d \propto I_m I_f$$

$$\propto VI$$

$$T_d \propto \text{Power}$$

Operation on AC

Let e be the instantaneous voltage across load, i be the instantaneous current through load in an AC circuit

If the load has a power factor of $\cos\phi$ (lagging)

$$e = E_m \sin\omega t$$

$$i = I_m \sin(\omega t - \phi)$$

Current through fixed coil $I_f \propto$ i

Current through moving coil $I_m \propto e$

Due to large inertia of the moving system, the deflection will be proportional to the average torque.

Mean deflecting torque $T_d \propto$ Average of $I_m I_f \propto$ Average of $e\ i$

$$T_d \propto \text{Average of } E_m \sin\omega t$$

$$I_m \sin(\omega t - \phi) \propto EI \cos\phi$$

$$T_d \propto \text{Power}$$

Scale

$$T_d \propto \text{power and } T_c \propto \theta \text{ (spring controlled)}$$

$$\theta \propto \text{power}$$

Hence such instruments have uniform scale.

Errors

Potential Coil Inductance

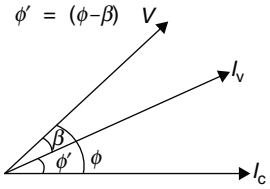
In an idealized wattmeter, potential coil is assumed to be purely resistive. In practice the potential coil has inductance also, due to which the current through the coil lags the voltage by an angle β and is not in phase with it.

$$\beta = \tan^{-1}\left(\frac{\omega L}{R_p}\right) = \tan^{-1}\left(\frac{\omega L}{r_p + R}\right)$$

where r_p = Resistance of potential coil
R = Resistance in series with potential coil

- For a lagging power factor, the angle between I_c and I_v is less than ϕ.

$$\phi' = (\phi - \beta)$$



Actual wattmeter reading $\propto VI\cos(\phi - \beta)\cos\beta$
But true power = $VI\cos\phi$

$$\text{Correction factor} = \frac{\text{True power}}{\text{Actual wattmeter reading}}$$

$$= \frac{\cos\phi}{\cos\beta\ \cos(\phi - \beta)}$$

$$\text{For leading p.f.} = \frac{\cos\phi}{\cos\beta\ \cos(\phi + \beta)}$$

$$= \frac{1}{1 + \tan\phi\ \tan\beta}$$

Compensation for error can be provided by connecting a capacitor across a portion of the external resistance R.

Potential Coil Capacitance

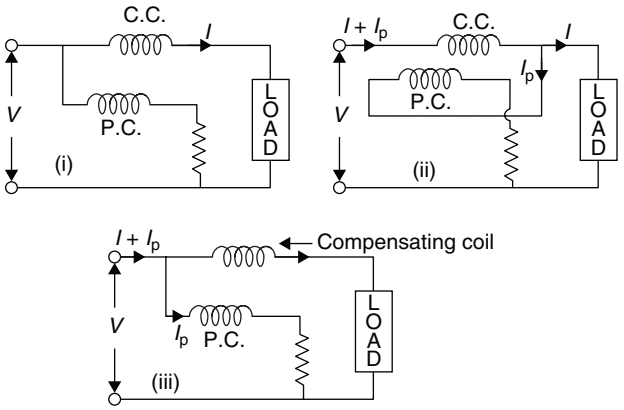
- The potential coil may have capacitance in addition to inductance due to in turn capacitance of the series resistance.
- The effect of capacitance opposite to that may be produced by inductance the wattmeter reads high on lagging power factor.

Mutual Inductance of Coils

Errors are caused due to mutual inductance of current and potential coils of the wattmeter.

- These errors are low at lower frequencies but they increase and become important as frequency is increased.
- The effect of mutual inductance is to increase the phase angle for the connection at which potential coil is connected on the load side and to decrease the phase angle when the current coil is connected on load side. Change in phase angle is $\tan^{-1}\frac{\omega M}{R_p}$.

Errors Because of Connections



Case (i)

- The potential coil is connected on the supply side and the voltage across it is equal to the sum of voltages across current coil and the load.
- The wattmeter measures power consumed by the load and power loss in current coil.

Case (ii)

- The current coil is on supply side and it carries the load current plus the potential coil current.
- The wattmeter reads power loss in potential coil plus the power consumed by the load.
- The above connection holds good only if power loss in the instrument is low and can be used if load currents are large.
- The power taken by potential coil is constant if the voltage is constant and becomes a smaller percentage of total power as a larger amount of power is measured.
- In cases where load current is large and power factor is small.

Case (ii) results in larger error since total power measured is small.

- In such cases, case (iii) can be used where a compensating coil is used to compensate the error caused by power loss in P.C.

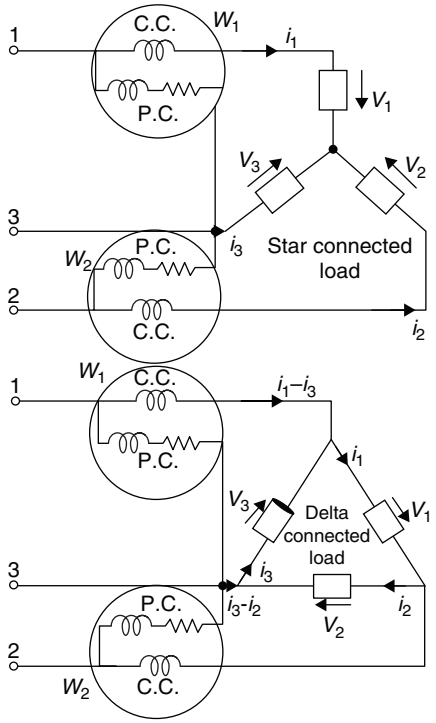
Case (iii)

- Compensating coil connected in series with the potential coil and is identical and coincident with current coil.
- It opposes the field of C.C., the C.C. carries a current of $I + I_p$ and P.C carries a current of I_p. The resultant field is due to I only. Hence, the error is neutralized.

Eddy Current Errors

- Eddy currents are induced within the solid metal parts and within the thickness of the conductors by alternating magnetic field of the C.C.
- These currents produce a field of their own and alter the magnitude and phase of the C.C. field.

Measurement of Power in Three-phase Circuits



- Two-wattmeter method is most commonly used method to measure power in 3ϕ circuits.
- The C.C. of two wattmeters are inserted in any two lines and potential coil is connected from its own current coil to the line without a C.C.
- The power supplied is sum of the two wattmeter readings.

For both connections,

Power consumed by the load

$$P = W_1 + W_2$$

$$W_1 = VI\cos(30° - \phi)$$

$$W_2 = VI\cos(30 + \phi)$$

Load power factor ϕ

$$= \tan^{-1}\left[\frac{\sqrt{3}(W_1 - W_2)}{W_1 + W_2}\right]$$

Case 1: For Purely Resistive Load

- Both wattmeter readings are equal

$$W_1 = W_2 \tan\phi = 0° \cos\phi = 1$$

Case 2: For Purely Reactive Load

- Both wattmeter readings are equal and opposite

$$W_1 = -W_2 \tan\phi = \propto \cos\phi = 0$$

Case 3: For Lagging Power Factor

- Only one wattmeter reads

$$W_1 = 0 \tan\phi = \sqrt{3} \cos\phi = 0.5$$

$$\phi = 60° \text{ lagging}$$

Case 4: For Leading Power Factor

- When p.f. leads by ϕ

$$W_1 = VI\cos(30 + \phi)$$

$$W_1 = VI\cos(30 - \phi)$$

i.e. readings of two wattmeter are interchanged.

Case 5: When $60 < \phi < 90°$ and $0 < \cos\phi < 0.5$

- W_1 is +ve and W_2 is –ve. As one wattmeter cannot indicate –ve reading pressure coil has to be reversed.

$$W_T = W_2 - W_1$$

Solved Examples

Example 1: Calculate the power factor of a balanced 3-ϕ inductive load, if the readings of two wattmeters (W_1 and W_2) to measure power were 6 and 4, respectively,
(A) 0.94 (B) 0.85 (C) 0.90 (D) 0.5

Solution: (A)

$$\tan\phi = \sqrt{3}\left[\frac{W_1 - W_2}{W_1 + W_2}\right]$$

$$= \sqrt{3}\left[\frac{6-4}{6+4}\right] = \frac{\sqrt{3}}{5}$$

$$\cos\phi = \frac{5}{\sqrt{28}} = 0.94 \text{ lag.}$$

Example 2: One single-phase wattmeter having a meter constant of 300 operates on 230 V, 6 A. if it makes 1800 revolutions in 6 hours the p.f. of the load will be
(A) 0.87 (B) 0.67 (C) 0.47 (D) 0.27

Solution: (D)

Energy consumed = $230 \times 6 \times 6 = 8280$ Wh

$= 8.28$ kWh

Meter constant = 800 rev/kWh

$$\text{kWh} = \frac{1800}{800} = 2.25$$

$$\text{Power factor} = \frac{2.25}{8.28} = 0.27.$$

Example 3: The resistance of two coils of a wattmeter are 10 mΩ and 1 kΩ, respectively, and both being non inductive. The voltage across load is 20 V and load current is 10 A. The error in the reading when voltage coil is connected in one of the two ways would be

(A) 0.1% too high (B) 0.2% too high
(C) 0.15% too high (D) zero

Solution: (B)

$r_{cc} = 10\ \text{m}\Omega = 0.01\ \Omega$, $r_{vc} = 1000\ \Omega$

Two possible power losses are

1. $I_c^2 r_{cc} = 10^2 \times 0.01 = 1\ \text{W}$
2. $\dfrac{V^2}{r_{vc}} = \dfrac{20^2}{1000} = 0.4\ \text{W}$

Two percentage errors are

1. $\dfrac{1}{200} \times 100 = 0.5\%$
2. $\dfrac{0.4}{200} \times 100 = 0.2\%.$

Example 4: In electrodynamometer type wattmeters, laminated conductors (or) stranded wires are used as current coils (which are designed to carry heavy currents) because it minimizes
(A) Hysteresis losses
(B) Eddy current losses
(C) Iron losses
(D) All of the above

Solution: (B).

Example 5: If an electrodynamometer type wattmeter was used for power measurement with very large value of load current them
(A) The pressure coil should be connected to the supply side.
(B) The current coil should be connected to the supply side.
(C) The pressure coil should be connected to the load side.
(D) The current coil should be connected to the load side.

Solution: (C)

Example 6: A two-wattmeter method employed in the measurement of power flowing in a three-phase three-wire balanced load system. Wattmeters W_1 reads 3500 W and W_2 reads −700 W. If the circuit is operating at 440 V, 50 Hz and if the entire power measured was to be appeared on W_1, the value of capacitance which must be introduced into each phase should be?
(A) 0.41 mF (B) 4.1 mF
(C) 40 μF (D) 0.41 μF

Solution: (A)

$W_1 = 3500\ \text{W}$

$W_2 = -700\ \text{W}$

Total power

$P = W_1 + W_2$

$= 3500 + (-700)$

$= 2800\ \text{W}$

$$\tan\phi = \sqrt{3}\left(\frac{W_1 - W_2}{W_1 + W_2}\right) = \sqrt{3} \times (1.5)$$

$$\phi = \tan^{-1}(2.6) \cong 69°$$

$$\cos\phi = 0.358$$

Power consumed by each phase $= \dfrac{2800}{3} = 933.33\ \text{W}$

$$V_{ph} = \frac{440}{\sqrt{3}} = 254\ \text{V}$$

$$I_{ph} = \frac{933.33}{254 \times 0.358} = 10.26\ \text{A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = 24.76\ \Omega$$

$$R_{ph} = Z_{ph}\cos\phi = 8.86\ \Omega$$

$$X_{ph} = \sqrt{Z_{ph}^2 - R_{ph}^2} = 23.12\ \Omega$$

In order that one wattmeter reads zero (i.e. $W_2 = 0$) p.f. $= \cos\phi = 0.5$

$$\tan\phi = 1.73$$

$$\tan\phi = \frac{X}{R}$$

$$\Rightarrow \quad X = R\tan\phi$$

$$X_{ph} = 8.86 \times 1.73$$

$$= 15.33\ \Omega$$

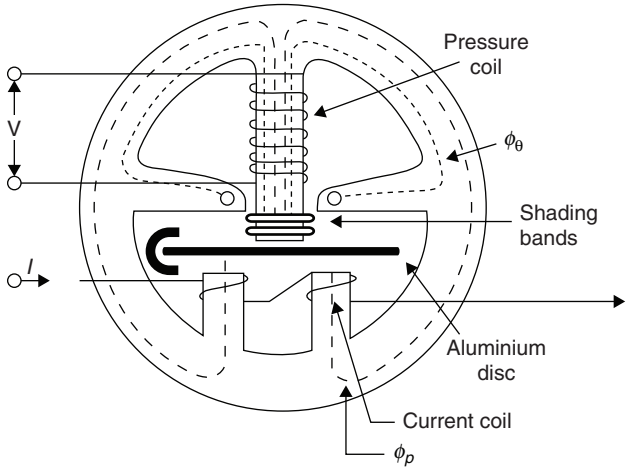
Capacitive reactance required

$$= 23.12 - 15.33 = 7.79\ \Omega$$

$$C = \frac{1}{2\pi \times 50 \times 7.79} = 4.086 \times 10^{-4}$$

$$= 0.41\ \text{mF}.$$

Measurement of Energy



Induction type single-phase energy meter is extensively used to measure energy supplied to a single-phase circuit.

Energymeter is an integrating instrument which works on the principle of induction. It consists of

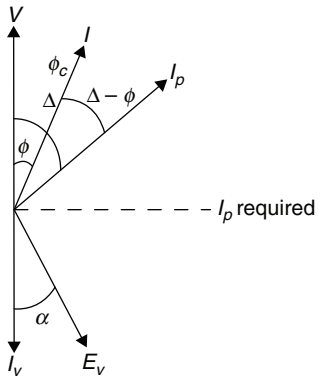
1. Driving mechanism
2. Rotating mechanism
3. Braking mechanism
4. Recording or Registering mechanism

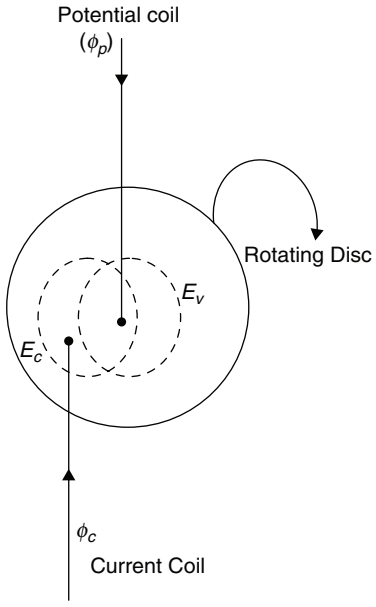
Driving Torque

Torque, $T_d \propto I_1 I_2 \cos\Phi \frac{dM}{d\theta}$

Torque due to pressure coil current and current coil flux

$$T_d \propto I \cos[(10 + \alpha) + (\Delta - \Phi)].$$



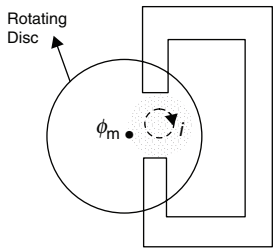


Approximate equation, $T_d \propto VI \sin(\Delta - \Phi)$

If $\Delta = 90°$, $T_d \propto VI \cos\Phi \propto$ power.

$$\text{Error} = VI [\sin(\Delta - \Phi) - \cos\Phi]$$

Braking Torque



Braking torque, $T_B \propto (N\phi_M) Id$

where N = Speed of the rotating disc

Φ_m = Magnetic flux of paramount magnet

d = Distance between centre of rotating disc and permanent magnet

$$T_B \propto N$$

At equilibrium $T_d = T_B$

$$\int VI \cos\phi dt = \int N dt$$

$$\text{Energy} = \int N dt$$

Braking magnet is used to produce the braking torque, which keeps the disc rotation constant speed depending on the load.

Speed adjustment: To decrease the speed, permanent magnet is kept far away from the disc.

$$T_B \propto N \phi_m d$$

$$A \propto \frac{T_B}{d\phi_m}$$

$$N \propto \frac{1}{d}$$

Formula in Energy Meter

Energy = Power × Time = $VI \cos\Phi \times t$.

$$\text{Energy} = \frac{VI \cos\phi}{1000} \times \frac{t}{3600} \quad \text{kWhr}$$

where t in terms of seconds and 1 kWhr = 1 unit

$$\text{Energymeter constant, } K = \frac{\text{No. of revoultions}}{\text{kWhr}}$$

% Creeping error

$$= \frac{\text{No. of revoultions due to creeping}}{\text{Revolutions due to load}} \times 100.$$

% Energy meter error

$$= \frac{\text{Energy recorded} - \text{True energy consumed}}{\text{True energy consumed}} \times 100.$$

Errors and Compensation in Energy Meter

1. Phase lag/Quadrature/Inductive compensation/Lag coil. For measurement of true energy by the energy meter the potential coil should be highly inductive. But in practice the presence coil current lagging its voltage by 'Δ', which is less than 90°. To improve this, a lag coil is connected below the potential coil. Otherwise, shaded bands are similar to the lag coil. which are made of copper. By adjusting the position of shading bands its flux is adjusted so that resultant flux is kept at 90° with respect to voltage.
2. Light load or friction compensation: A shading loop is kept between central limb of the shunt magnet and the rotating disc which compensates light load and friction produced in the bearings.
3. Creeping: If the friction is over compensated by keeping the shading loop very nearer to the potential coil then the energy meter records the energy without any load, due to potential coil is energized, is called creeping.

   To avoid or reducing the effect of creeping two holes are kept in the opposite sides of the rotating discs.
4. Overvoltage compensation: The saturable shunt magnets are kept between the side limb and central limb of the shunt magnet. For normal voltages an air gap between the saturable shunt magnet and the side limbs produces maximum reluctance but under high voltages the reluctance breaks so the flux being diverted and hence meter will be protected.
5. Overcurrent Compensation: Under heavy load currents, the magnetic flux diverts through a saturable series magnet kept between the side limbs of series magnet hence meter will be protected.
6. Temperature Compensation: A permanent magnet made up of MUTEMP will be less affected by the surrounding temperature that is the reason braking magnet is made up of MU TEMP.

Fictitious or Phantom Loading

It is used to reduce the energy loss due to actual loading so that the energy meter can be tested at the desired load conditions at nominal supply voltage. A variable resistance is connected in series with the low voltage supply of current coil. By varying this, current through the current coil can be adjusted which is equivalent to the load current. This called fictitious loading.

Operating Principle

- The operation of the instrument depends on the passage of alternating current through suitably located coils. Producing a rotating magnetic field which interacts with a metallic disc suspended near to the coils and cause the disc to rotate.
- Current coil carries line current and produces field in phase with the line current.
- Pressure coil is made highly inductive so that the current through it lags behind supply voltage by 90°.
- A phase difference of 90° exists between the fluxes produced by the two coils.
- This sets up rotating field which interacts with the disc to cause it to rotate.

Construction

Series Magnet

- It consists of a number of U-shaped laminations assembled together to form a core. A thick wire of few turns is wound on both legs of U-shaped core.
- The wound coil is known as current coil and is connected in series with the load so that it carries load current.
- The series magnet is placed underside the aluminium disc and produces magnetic field proportional to and in phase with the current.

Shunt Magnet

- It consists of number of M-shaped laminations assembled together to form a core.
- A fine wire of large turns is wound on the control limb of this magnet. The wound coil is known as pressure coil and is connected across the load so that it carries current proportional to supply voltage. Shunt magnet is placed above aluminium disc.
- In order to obtain deflecting torque current in the pressure coil must lag behind supply voltage by 90°. This is obtained by placing a copper ring (compensating loop) over the control limb of the shunt magnet.
- This copper acts as short circuited transformer secondary. As its resistance the current circulating in the ring will lag nearly by 90° behind the supply voltage.

Braking Magnet

- The speed of aluminium is controlled to required value by the C-shaped permanent braking magnet. The magnet is mounted so that the disc revolves in the air gap between the polar extremities.
- As the disc rotates, currents are induced in the disc because it cuts the flux, produced by the braking magnet.
- The direction of current in disc is such that it opposes the rotation of the disc. Since the induced currents in the disc are proportional to the speed of the disc, braking torque is proportional to disc speed $T_B \propto N$.

Recording Mechanism

The number of revolutions of the disc is a measure of electrical energy passing through the meter and is recorded on dials which are geared to shaft.

Working

When energy meter is connected in the circuit to measure electrical energy the current coil carries the load current and pressure coil carries current proportional to supply voltage.

- The current coil produces eddy currents in the disc which react with the field due to the pressure coil. Thus a driving force is created which causes the disc to rotate.
- The braking magnet provides the braking torque on the disc. By altering the position of this magnet desired speed can be obtained.
- The spindle is geared to the recording mechanism so that electrical energy consumed in the circuit is directly registered in kWh.

Example 7: A single-phase energy meter is operating on 380, 50 Hz supply with a load of 15 A for two hours at u.p.f. The meter makes 1260 revolutions in that period. The meter constant is

(A) 695 rev/kWh (B) 1/111 rev/kWh
(C) 0.15 rev/kWh (D) 111 rev/kWh

Solution: (D)

$$\text{Energy consumed} = \frac{380\times15\times1\times2}{1000}$$

$$= 11.4 \text{ kWh}$$

Revolutions = 1260

$$\text{Meter constant} = \frac{1260}{11.4}$$

$$= 110.53$$

$$= 111 \text{ rev/kWh.}$$

Example 8: A 220 V, 6 A single-phase energy meter at u.p.f., half load, rated voltage makes 120 revolutions in 4 min. Calculate the error at half load (meter constant = 2400 revs/kWh)

(A) 13.04% slow (B) 12% fast
(C) 13.04% fast (D) 12% slow

Solution: (D)

$$\text{Energy consumed} = \frac{120}{2400} = 0.05 \text{ kWh}$$

Actual energy consumed

$$= \frac{220\times\left(\frac{6}{2}\right)\times1\times\left(\frac{4}{60}\right)}{1000}$$

$$= 0.044$$

Error = 0.05 – 0.044 = 0.006 kWh

$$\text{Hence meter is fast by } \frac{0.006}{0.05}\times100 = 12\%$$

Example 9: One single-phase wattmeter operating on 230 V and 10 A for 5 h makes 1380 revolutions. Meter constant is 150 power factor of the load will be

(A) 0.8 (B) 1
(C) 0.7 (D) 0.6

Solution: (A)

$$\text{Energy consumed} = \frac{1380}{150} = 9.2 \text{ kWh}$$

$$\text{Actual energy consumed} = 230 \times 10 \times 5$$

$$= 11.5 \text{ kW}$$

$$\cos\phi = \frac{9.2}{11.5} = 0.8 .$$

Example 10: The direction of rotation of a household single-phase induction type wattmeter can be reversed by

(A) Reversing the supply terminals without opening the meter connection.
(B) Reversing the load terminals without opening the meter connection.
(C) Reversing either the supply or load terminals after opening the meter connections.
(D) Reversing both supply and load terminals after opening the meter connections.

Solution: (C).

Example 11: If a household induction energy meter is running slower it can be made to run faster by

(A) Light load adjustment.
(B) Making the position of the braking magnet come closer to the centre of the rotating disc.
(C) Making the position of the braking magnet to move away from the centre of the rotating disc.
(D) Lag adjustment.

Solution: (B).

Example 12: Which among the following options is true? Creeping in a single-phase induction type energymeter may be due to

(1) Overcompensation of friction
(2) Vibration
(3) Overvoltage
(A) 1 and 2 only
(B) 2 and 3 only
(C) 1 and 3 only
(D) 1, 2 and 3

Solution: (D).

Important Definitions

1. **Creep:** Continuous rotation of the meter even when there is no current flowing through the current coil and only pressure coil is energized due to over compensation of friction over voltage and vibration. It can be prevented by drilling two holes on diametrically opposite side of the disc.
2. Phantom loading for testing of energy meter is used to test meters having a large current rating for which loads may not be available.

3. Braking torque provided by a permanent magnet $T_B \propto$
   - Square of flux of the permanent magnet
   - Speed of the meter
   - Distance of permanent magnet from the centre of the revolving disc
4. In order to obtain true value of energy shunt magnet flux should lag behind applied voltage by 90°.
5. Maximum torque is produced when phase angle between two fluxes is 90°.
6. Shading bands are actuated to provide a constant torque irrespective of load and also compensates static friction.
7. Induction type energy meter can be slowed down by light load adjustment and is usually done at 1% of full-load current.

EXERCISES

Practice Problems I

Q No 1–4 consists of two statements one is assertion (A) and the other is Reason (R). Examine these two statements and select the answer using code given below

(A) Both A and R are individually correct and R is correct explanation of A
(B) Both A and R are individually correct and R is not the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

1. A: General purpose dynamometer type wattmeter cannot indicate the correct value of power at low p.f.

R: The pressure of self-inductance in the pressure coil circuit introduces an error in the indicated valve which increases with decrease in p.f. of the load.

2. A: Power factor meters commonly used are of dynamometer type.

R: Dynamometer type p.f. meter is more accurate than moving iron one.

3. A: A good energy meter should be provided with a low load fraction compensation loop arrangement.

R: The frictional force at bearings causes small errors at light loads.

4. A: For an energymeter, careful design and treatment of braking magnet during its manufacture are essential in order to ensure constancy of brake magnet strength during the use of meter.

R: Steady rotational speed of energymeter disc is directly proportional to flux of the brake magnet.

5. Creep error is associated with which one of the following meters:
(A) Moving iron meter
(B) Energymeter
(C) Electrodynamometer
(D) Wattmeter

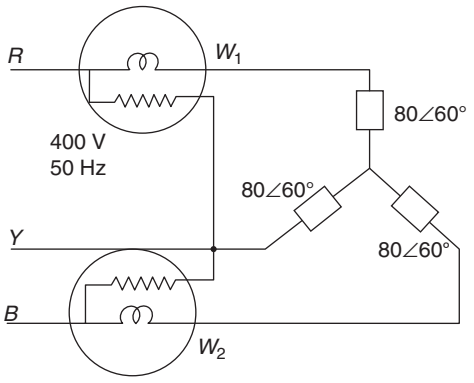
6. Holes drilled on diametrically opposite sides of the disc of an induction type energy meter to
(A) Avoid creep on no load
(B) Balance the disc
(C) Dissipate the energy due to eddy currents
(D) Increase the deflecting torque

7. In a 3-phase power measurements the reading of one of the wattmeters is zero when the power factor of the load is
(A) 0.866 (B) 0.707
(C) 0.5 (D) 0

8. Two wattmeters connected in a 3-ϕ circuit to measure power shows reading of 600 W and 300 W. The p.f. of the circuit is
(A) 0.5 (B) 0.707
(C) 0.866 (D) 1

9. A 230 V, 10 A single-phase energy meter makes 90 revs in 3 min at half full load, rated voltage and p.f. If the meter constant is 1800 rev/kWh then its error will be
(A) 13.04% slow
(B) 13.04% fast
(C) 15% slow
(D) 15% fast

10. Two wattmeters W_1 and W_2 are used to measure the power in a 3-ϕ star-connected circuit as shown below. If the phase sequence is RYB then wattmeter would read.



(A) 1732 W, 1268 W (B) 1500 W, 1500 W
(C) 0 W, 3000 W (D) 1250 W, 1750 W

Common Data for Questions 11 and 12:
A wattmeter is rated at 5 A and 12 V. The current coil has a resistance of 0.02 Ω and reactance of 0.04 Ω. The potential coil circuit may be assumed to be purely resistive having a resistance of 6230 Ω.

The load is 5 A at a p.f. of 0.15 lagging. The voltage across the load is 12 V.

11. The power lost in pressure coil is
(A) 0.01 W (B) 0.1 W
(C) 0.2 W (D) 0.023 W

12. The power indicated by wattmeter and error, respectively, are
(A) 9.023 W, 0.25 (B) 9.023 W, 2.5
(C) 9 W, 0.25 (D) 8.5 W, 0.25 W

13. A single-phase energymeter is operating on 200 V, 50 Hz supply with a load of 15 A at u.p.f. for 2 h. The meter makes 960 revolutions during this period. The meter constant is
(A) 0.16 rev/kWh
(B) 1/160 rev/kWh
(C) 256 rev/kWh
(D) 160 rev/kWh

14. The pressure coil of a wattmeter is connected across *R* phase and neutral of a 3-ϕ balanced system having phase sequence RYB. The current coil is connected in the *R* phase. The wattmeter measures a power of 6000 W. If the power of the circuit is 0.707 lag, and the wattmeter pressure coil is reconnected across *Y* and *B*, the power measured by the meter would be
(A) −273 W (B) 1273 W
(C) 1039 W (D) 1800 W

15. Two-wattmeter method is used to measure power in a 3-ϕ circuit operated at 400 V and p.f. of 0.35. The power measured by them s 10 kW. The individual readings of meters are
(A) 12.73 kW, −2.73 kW (B) 3 kW, 2 kW
(C) 13.26 kW, −3.26 kW (D) 11.28 kW, −1.28 kW

Practice Problems 2

1. Two wattmeters, W_1 and W_2, are used to measure power in a 3-ϕ circuit having a phase sequence of RYB. The current coil of W_1 is connected in *R* and that of W_2 is connected in *B*. If reading of W_2 is zero, the power factor of the circuit is
(A) 0.5 lead (B) 0.5 lag
(C) Zero lead (D) Zero lag

2. Holes are drilled on the opposite sides of the disc of an induction type energy meter to
(A) Avoid creep on no load
(B) Balance the disk
(C) Dissipate the energy due to eddy currents
(D) Increase the deflecting torque

3. A dynamometer type power factor meter has
(A) One current coil and one voltage coil
(B) One current coil and two voltage coils
(C) One voltage coil and two current coils
(D) Two current coils and two voltage coils

4. A 10 mA ammeter has a resistance of 50 Ω. It has to be converted to a 1 A ammeter. The value of shunt resistance is
(A) 5 Ω (B) 0.05 Ω
(C) 0.5 Ω (D) 50 Ω

5. High AC voltages are measured with
(A) Potential transformer with voltmeters
(B) Current transformers with voltmeters
(C) Inductive voltmeter
(D) Magnetic voltmeter

6. To measure a voltage above 100 kV, the type of measuring arrangement used is
(A) Potential transformer with voltmeter
(B) Capacitive potential transformer
(C) Current transformer with voltmeter
(D) All of these

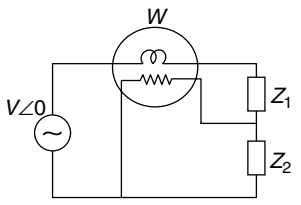
7. An electrodynamometer type instrument finds its major use as
(A) Indicator type of instrument only
(B) Standard instrument only
(C) Transfer instrument only
(D) Both as standard and transfer instrument

8. A current transformer has a rating of 200/5 A. The magnetizing and loss component of exciting current are 2 A and 1 A, respectively, and secondary has a burden of pure resistance. The transformation ratio at rated current is
(A) 40 (B) 40.1
(C) 40.2 (D) 41

9. An electrodynamometer type wattmeter is used to measure power in a circuit where the load current is small. Then,
(A) Voltage coil is connected on load side
(B) Current coil is connected on load side
(C) Does not matter whether current coil or voltage coil is connected on load side.
(D) None of these

10. Highly erroneous results are obtained if power factor of a non-sinusoidal wave form is measured by electrodynamometer type power factor meters. This is true of
(A) Single-phase meter only
(B) Three-phase meter only
(C) Both single- and three-phase meters
(D) None of these

11. A wattmeter is connected as shown in the figure



The wattmeter reads
(A) Zero
(B) Power consumed by Z_1
(C) Power consumed by Z_2
(D) Total power connected by Z_1 and Z_2

12. The error produced due to the inductance in the voltage coil of an electrodynamometer type wattmeter
(A) Increases as power factor decreases
(B) Decreases as power factor decreases
(C) Constant irrespective of the power factor of the circuit
(D) No relation between p.f. and inductance of voltage coil

13. The burden of a CT is expressed in
(A) Power rating of the resistor
(B) VA rating
(C) Current rating of secondary winding
(D) Current and voltage rating of secondary

14. The core flux in the current transformer under this operating condition is
(A) 56 μWb (B) 112 mWb
(C) 112 μWb (D) 56 mWb

Question no. 15 consists of two statements: one is Assertion (A) and the other is Reason (R). You have to examine these two statements and select the answer using the code given below.
(A) Both A and R are individually correct and R is the correct explanation of A.
(B) Both A and R are individually correct, but R is not the correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

15. A: General purpose dynamometer type wattmeter cannot indicate the correct value of power at low power factors.
R: The pressure of self-inductance in the pressure coil circuit introduces an error in the indicated value which increases with decrease in power factor of the load.

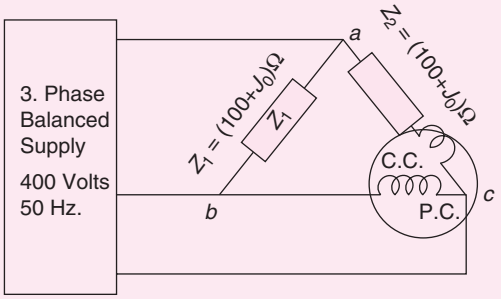
Previous Years' Questions

1. Two wattmeters, which are connected to measure the total power on a three-phase system supplying a balanced load, read 10.5 kW and –2.5 kW, respectively. The total power and the power factor, respectively, are **[2005]**
(A) 13.0 kW, 0.334
(B) 13.0 kW, 0.684
(C) 8.0 kW, 0.52
(D) 8.0 kW, 0.334

2. An energymeter connected to an immersion heater (resistive) operating on an AC 230 V, 50 Hz, AC single-phase source reads 2.3 units (kWh) in 1 hour. The heater is removed from the supply and now connected to a 400 V peak-to-peak square-wave source of 150 Hz. The power in kW dissipated by the heater will be **[2006]**
(A) 3.478 (B) 1.739
(C) 1.540 (D) 0.870

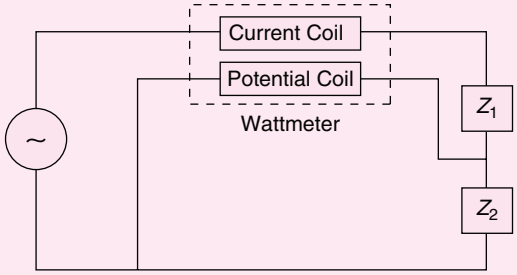
3. The pressure coil of a dynamometer type wattmeter is **[2009]**
(A) Highly inductive (B) Highly resistive
(C) Purely resistive (D) Purely inductive

4. The figure shows a three-phase delta-connected load supplied from a 400 V, 50 Hz, 3-phase balanced source. The pressure coil (PC) and current coil (CC) of a wattmeter are connected to the load as shown, with the coil polarities suitably selected to ensure a positive deflection. The wattmeter reading will be **[2009]**



(A) 0 (B) 1600 W
(C) 800 W (D) 400 W

5. A wattmeter is connected as shown in the figure. The wattmeter reads **[2010]**



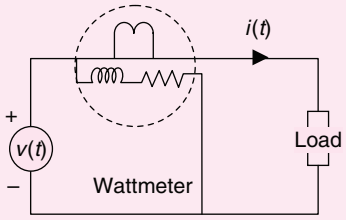
(A) Zero always
(B) Total power consumed by Z_1 and Z_2
(C) Power consumed by Z_1
(D) Power consumed by Z_2

6. The active power drawn by converter is **[2011]**
(A) 181 W (B) 500 W
(C) 707 W (D) 887 W

7. The input power factor of the converter is **[2011]**
(A) 0.31 (B) 0.44
(C) 0.5 (D) 0.71

8. A two-phase load draws the following phase currents: $i_1(t) = I_m \sin(\omega t - \phi_1)$, $i_2(t) = I_m \cos(\omega t - \phi_2)$. These currents are balanced if ϕ_1 is equal to **[2012]**
(A) $-\phi_2$ (B) ϕ_2
(C) $(\pi/2 - \phi_2)$ (D) $(\pi/2 + \phi_2)$

9. For the circuit shown in the figure, the voltage and current expressions are $v(t) = E_1 \sin(\omega t) + E_3 \sin(3\omega t)$ and $i(t) = I_1 \sin(\omega t - \varphi_1) + I_3 \sin(3\omega t - \phi_3) + I_5 \sin(5\omega t)$. The average power measured by the wattmeter is **[2012]**



(A) $\frac{1}{2} E_1 I_1 \cos\phi_1$

(B) $\frac{1}{2}\left[E_1 I_1 \cos\phi_1 + E_1 I_3 \cos\phi_3 + E_1 I_5\right]$

(C) $\frac{1}{2}\left[E_1 I_1 \cos\phi_1 + E_3 I_3 \cos\phi_3\right]$

(D) $\frac{1}{2}\left[E_1 I_1 \cos\phi_1 + E_1 I_1 \cos\phi_1\right]$

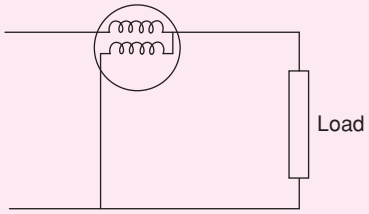
10. Power consumed by a balanced three-phase, 3-wire load is measured by the two-wattmeter method. The first wattmeter reads twice that of the second. Then the load impedance angle in radians is **[2014]**
(A) $\pi/12$ (B) $\pi/8$
(C) $\pi/6$ (D) $\pi/3$

11. An incandescent lamp is marked 40 W, 240 V. If resistance at room temperature (26°C) is 120 Ω, and temperature coefficient of resistance is 4.5×10^{-3}/°C, then its 'ON' state filament temperature in °C is approximately _____. **[2014]**

12. While measuring power of a three-phase balanced load by the two-wattmeter method, the readings are 100 W and 250 W. The power factor of the load is _____. **[2014]**

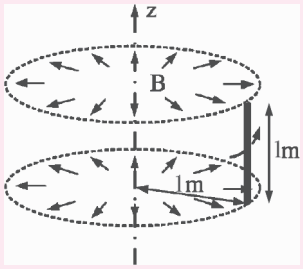
13. A 3-phase balanced load which has a power factor of 0.707 is connected to a balanced supply. The power consumed by the load is 5 kW. The power is measured by the two-wattmeter method. The readings of the two wattmeters are **[2015]**
(A) 3.94 kW and 1.06 kW
(B) 2.50 kW and 2.50 kW
(C) 5.00 kW and 0.00 kW
(D) 2.96 kW and 2.04 kW

14. The coils of a wattmeter have resistances 0.01 Ω and 1000 Ω; their inductances may be neglected. The wattmeter is connected as shown in the figure, to measure the power consumed by a load, which draws 25 A at power factor 0.8. The voltage across the load terminals is 30 V. The percentage error on the wattmeter reading is _____. **[2015]**



15. An energy meter, having meter constant of 1200 revolutions/kWh, makes 20 revolutions in 30 seconds for a constant load. The load, in kW, is _____. **[2016]**

16. A rotating conductor of 1m length is placed in a radially outward (about the z - axis) magnetic flux density (B) of 1 Tesla as shown in figure below. Conductor is parallel to and at 1m distance from the z - axis. The speed of the conductor in r.p.m. required to induce a voltage of 1V across it, should be _____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. A	**2.** D	**3.** C	**4.** C	**5.** B	**6.** A	**7.** C	**8.** C	**9.** C	**10.** C
11. D	**12.** A	**13.** D	**14.** C	**15.** A					

Practice Problems 2

1. A	**2.** A	**3.** B	**4.** C	**5.** A	**6.** B	**7.** D	**8.** C	**9.** B	**10.** A
11. C	**12.** D	**13.** B	**14.** C	**15.** A					

Previous Years' Questions

1. D	**2.** B	**3.** B	**4.** C	**5.** D	**6.** B	**7.** B	**8.** D	**9.** C	**10.** C
11. 2470	**12.** 0.8029		**13.** A	**14.** 0.14 to 0.16		**15.** 2	**16.** 9.54		

Chapter 4

Cathode Ray Oscilloscope and Electronic Voltmeters

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Horizontal deflection system
- Deflection of moving electrons in CRO tube
- Measurement of voltage and currents
- Measurement of frequency
- Lissajous method
- Analog and digital voltmeters
- Digital voltmeters
- Continuous balance type DVM
- Ramp type DVM
- Successive approximation type DVM

Cathode Ray Oscilloscope

Cathode ray oscilloscope (CRO) is a very useful and versatile laboratory instrument used for display, measurement analysis of waveforms and other phenomena in electrical and electronic circuits.

- It displays an input signal V_s another signal or time.

Construction, Operation and Working

- Cathode ray tube (CRT)
- Vertical amplifier
- Horizontal amplifier
- Time base
- Trigger base
- Trigger circuits
- Power supplies

In a CRO, CRT generates the electron beam, accelerates it to a high velocity, deflects the beam to create the image and the electron beam becomes finally visible on the phosphor screen.

- The time base of the oscilloscope generates correct voltage to supply the CRT to deflect the spot at a constant time dependent rate.
- Vertical amplifier increases the potential of input signal which is to be viewed, to a level that will provide a usable deflection of electron beam.
- To synchronize the horizontal deflection with the vertical input such that horizontal deflection starts at the same point of input vertical signal each time it sweeps, a synchronizing or triggering circuit is used.

Horizontal Deflection System

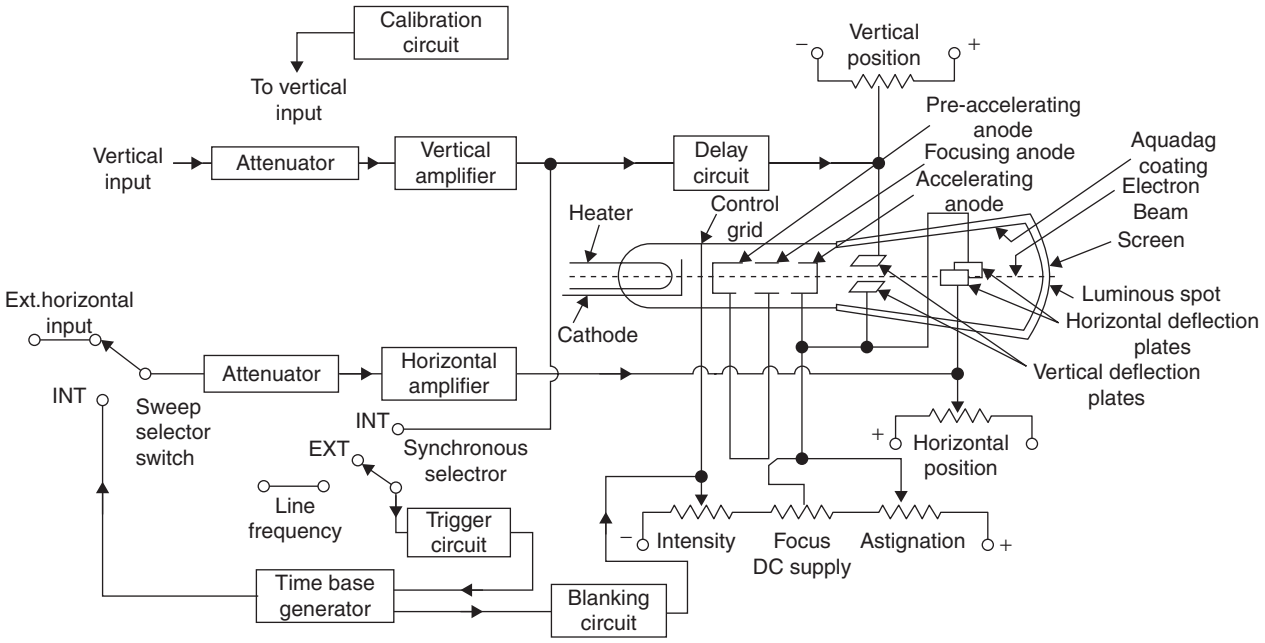
- It consists of a time base generator a trigger circuit and a horizontal amplifier.
- Time base generator controls the rate at which the beam is scanned across the face of the CRT and is adjusted from the front panel.
- The trigger circuit ensures that the horizontal sweep starts at the same point of vertical input signal.
- Horizontal amplifier to increase the amplitude to the signal generated in the sweep generator to the level required by the horizontal deflection plates of the CRT.

Vertical Deflection System

- It provides an amplified signal of proper level to drive the vertical deflection plates without introducing any appreciable deflection to the system.
- Attenuator sets the sensitivity of the oscilloscope.

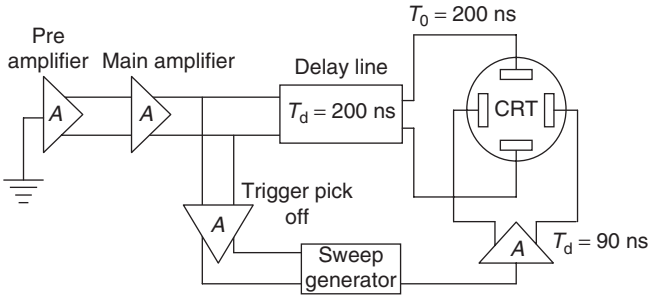
Vertical Delay Line

- All electronic circuitry in the oscilloscope causes a certain amount of time delay in transmission of signal voltages to the deflection plates.
- The horizontal signal is initiated by the portion of output signal applied to vertical CRT plates.
- Signal processing in horizontal channel consists of generating and shaping a trigger pulse that starts the sweep generator whose



output is fed to horizontal amplifier and then to horizontal deflection plates.

- The whole process takes time of the order of 80 ns or so. To allow the operator to observe the leading edge of the signal waveform, the signal drive for the vertical CRT plates must be delayed by at least the same amount of time.
- The signal voltage to the CRT plates is delayed by some-time and horizontal sweep is started prior to the vertical deflection.



CRO Control Panel

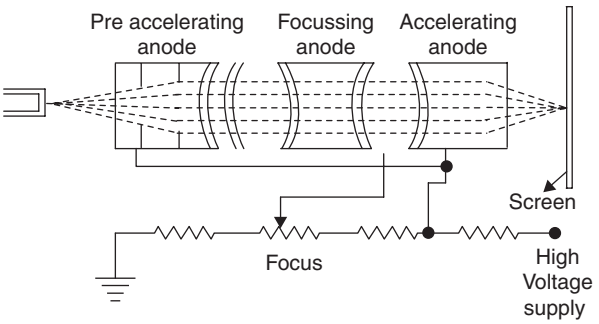
Intensity Control

It regulates the bias on the control grid and affects the electron beam intensity.

- If the negative bias on the grid is increased, the intensity of electrons beam is decreased by reducing the brightness of the spot.

Focus Control

- It regulates the positive potential on the focusing anode, which makes electron beam quite narrow and the spot on the screen is a pin point.



Horizontal Position Control

- It regulates the amplitude of DC potential which is applied to the horizontal plates, in addition to the usual saw-tooth wave.

By adjusting this control the spot can be moved to right or left as required.

Vertical Position Control

It regulates the amplitude of DC potential which is applied to vertical deflection plates in addition to the signal.

- By adjusting this control the image can be moved up or down as required.

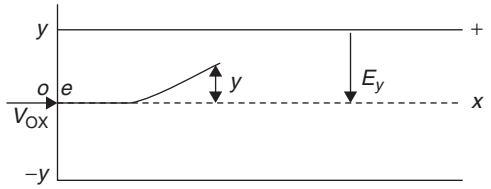
Deflection of Moving Electrons in CRO Tube

Force on the electron in a uniform electric field of intensity E

$$F = QE = -eE = ma$$

$$a = \frac{F}{m} = \frac{-eE}{m} \text{ m/s}^2$$

where a = Acceleration of electrons in m/s^2
F = Force of electron in N
m = Mass of electron in kg



- An electron entering the field in the positive X-direction with an initial velocity V_{ox} will experience a force.
- As the field acts along Y-axis only, there will be no force acting along Z- or X-axis.
- The acceleration means constant velocity and since the electron enters the field in positive X direction with an initial velocity V_{ox}, it will continue to travel along X-axis at this velocity.

In the direction of Y-axis

$$F = ma_y$$

$$a_y = \frac{F}{m} = \frac{-eE_y}{m}$$

Since $V = V_o + at$

$$x = x_o + V_o t + 1/2\ at^2$$

Putting initial conditions, we have

$$V_y = a_y t = \frac{-eE_y t}{m}$$

$$y = 1/2\ a_y t = \frac{-eE_y t^2}{2m}$$

Distance travelled in time t

$$x = x_o + V_{ox} t + 1/2\ a_x t^2$$

At $x_0 = 0$ and $a_x = 0$

$$X = V_{ox} t$$

$$\Rightarrow \quad t = \frac{x}{V_{ox}}$$

Vertical deflection as a function of horizontal distance travelled

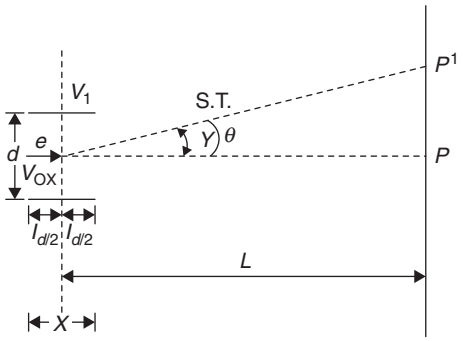
$$y = \left[\frac{-eEy}{2V_{ox}^2 m}\right] x^2 \text{ m}$$

Hence, path of an electron travelling through an electric field of constant intensity and entering the field at right angles to the lines of force is a parabola.

Consider two parallel deflection plates at a distance d and connected to a source of potential difference V_d.

Hence,

$$E = \frac{V_d}{d} \text{ V/m}$$



Slope of the parabola at a distance $X = l_d$ is defined as

$$\tan\theta = \frac{d_y}{d_x} = \frac{-e\ E_y\ l_d}{mV_{ox}^2}$$

$$X = \frac{y}{\tan\theta} = \frac{\left(\frac{e\,E_y\,l_d^2}{2m\,V_{ox}^2}\right)}{\left(\frac{e\,E_y\,l_d}{m\,V_{ox}^2}\right)} = \frac{l_d}{2}$$

Deflect on the screen $D = L\tan\theta$

$$D = L\ \frac{e\,E_y\,l_D}{mV_{ox}^2}$$

Kinetic energy of the electrons entering the deflection plate is

$$1/2\ m\ V_{ox}^2 = e\ V_a$$

$$V_{ox}^2 = \frac{2e\ V_a}{m}$$

$$D = \frac{L_e\,E_y\,L_D}{mV_{ox}^2} = \frac{L\ l_D}{2V_a} \cdot \frac{V_D}{d}$$

where V_d = Deflection voltage in volts
V_a = Accelerating voltage in volts

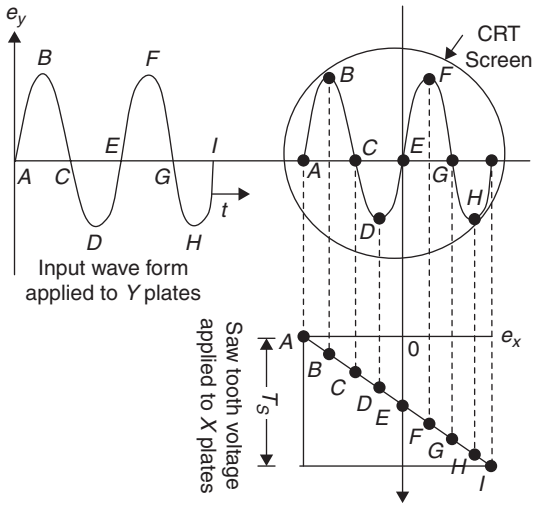
Deflection sensitivity S is defined as the deflection on the screen (in metre) per volts of deflection voltage

$$S = \frac{D}{V_D} = \frac{L\ l_D}{2d\ V_a} \text{(m/V)}$$

Deflection factor $G = \frac{1}{s} = \frac{2d\ V_a}{L\ l_D}$ V/m

Applications of CRO

Observation of Waveform on CRO



- When horizontal saw-tooth voltage and an input voltage is applied simultaneously to y plates, the beam is under influence of the following two forces.
  1. One in horizontal direction moving the beam at a line or rate from right to left.
  2. Second in vertical direction moving the beam up and down.
- Since deflection is proportional to applied voltage, the horizontal movement is proportional to voltage applied to X plates at any instant and since ramp voltage is linear it traces a straight line on CRT screen.
- The vertical deflection is proportional to input voltage applied to y plates and hence up and down according to the magnitude and polarity of the input voltage.
- At the end of one sweep cycle, the sweep voltage abruptly drops down and the spot is immediately transferred to its original position. The process is repeated again with a result that a stationary image is seen on the screen.

Measurement of Voltage and Currents

- From the expression of electrostatic deflection, deflection is proportional to the deflection plate voltage. Thus CRT measures voltage.
- Direct voltage may be obtained from static deflection of spot. Alternating voltages may be obtained from the length of the line produced when voltage is applied to Y plates when no voltage is applied to X plates. The length of the line corresponds to peak-to-peak voltage.

$$V_{RMS} = \frac{V_{pp}}{2\sqrt{2}}$$

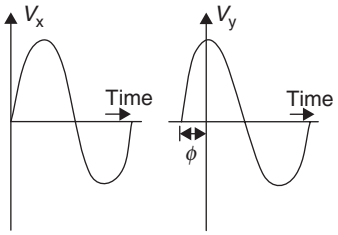
- The y-shift control is adjusted so that positive peak of test voltage coincides with some datum line on the screen, and then the shift control is operated so that negative peak coincides with the datum.
- The movement of control is arranged to read directly V_{pp}.
- The value of current can be obtained by measuring voltage drop across a known resistor connected in the circuit.

Measurement of Phase Difference

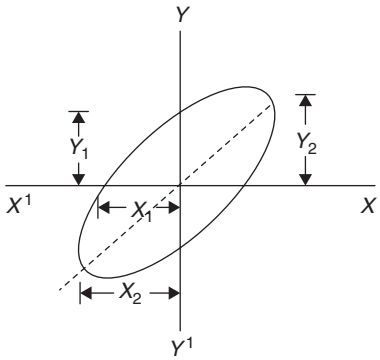
When two sinusoidal voltages are applied to deflection plates of CRO, the phase difference ϕ is determined as follows:

$$V_x = V_x \sin\omega t$$

$$V_y = V_y \sin(\omega t + \phi)$$



Input waveforms



O/P on CRO

$$\sin\phi = \frac{Y_1}{Y_2} = \frac{X_1}{X_2}$$

Measurement of Frequency

Direct Method

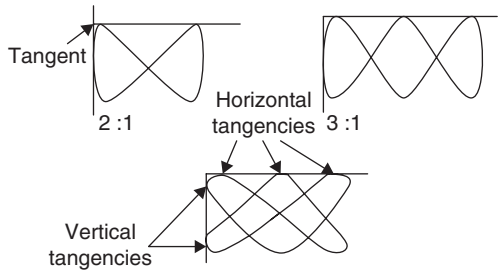
A signal frequency is determined by measuring its time period T and calculating the frequency by $f = \frac{1}{T}$.

- The input waveform is fed into the vertical input, with vertical sensitivity sweep speed and triggering controls are adjusted for at least one complete stable cycle of waveform.
- The number of horizontal divisions for a complete cycle is determined and time period is found by

T = (Number of horizontal divisions in one cycle) × (Sweep speed in s/division)

Lissajous Method

Frequency of a signal can be accurately measured by Lissajous patterns. The signal of unknown frequency is applied to *Y* plates. The signal of known frequency is applied to *X* plates. Depending on the frequency, ratio patterns are obtained.



For all the cases of patterns obtained the ratio of the two frequencies is

$$\frac{f_y}{f_x} = \frac{\text{Number of horizontal tangencies}}{\text{Number of vertical tangencies}}$$

f_y = Frequency of signal applied to Y plates
f_x = Frequency of signal applied to X plates

For open Lissajous patterns, free end is treated as 1/2 tangency.

Solved Examples

Example 1: In a CRO, post acceleration is required if the frequency of the signal is
(A) Greater than 10 Hz
(B) Greater than 10 MHz
(C) Greater than 1 MHz
(D) Less than 1 MHz

Solution: (B).

Example 2: The non-viewing side of the CRO phosphorus screen has a thin aluminium film deposited because
(A) It protects the screen from getting negatively charged.
(B) It prevents phosphorus burn by acting as heat sink.
(C) It minimizes light scatters from phosphorus.
(D) All of the above.

Solution: (D).

Example 3: In a CRO, the design of horizontal amplifier should be for signals with
(A) High frequency and fast rise time
(B) High amplitude and fast rise time
(C) High amplitude and slow rise time
(D) Low amplitude and fast rise time

Solution: (C).

Example 4: In a CRO when an unknown signal is applied to the vertical plate is being synchronized with the sweep signal applied to horizontal plates. If the frequency of the signal is lower than that of the unknown signal, then
(A) The pattern on CRO screen moves towards the right.
(B) The pattern on CRO screen moves towards the left.
(C) The pattern on CRO screen moves toward the top.
(D) The pattern non-CRO screen moves towards the bottom.

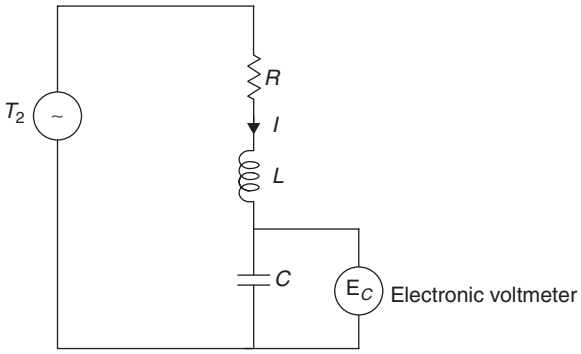
Solution: (A).

Example 5: Focusing anode in a CRO is located
(A) Between accelerating and accelerating anode
(B) Before pre-accelerating anode
(C) After accelerating anode
(D) None of the above

Solution: (C).

Q-meter

It is also called voltage magnifier.



Q-meter works based on the principle of RLC series resonance.

$$\text{Quality factor, } Q = \frac{X_L}{R} = \frac{X_C}{R}$$

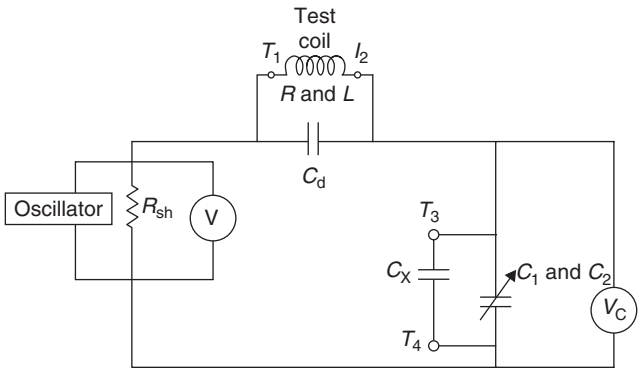
Q-meter is an instrument is designed is very useful in measuring the characteristics of inductive coil and capacitors.

Under resonant condition $I = \frac{V}{R}$

$$V_c = IX_C = \frac{VX_C}{R}$$

$$V_c = VQ$$

Practical Q-meter



Applications of Q-meter

1. Measurement of quality factor of the coil:

$$Q_T = Q_m\left[1+\frac{R_{sh}}{R}\right], \text{ since } R_{sh} <<< R$$

$$Q_T = Q_m$$

$$\% \text{ Error} = \frac{-R_{sh}}{R_{sh}+R}$$

2. Measurement of unknown capacitance
   If a test capacitance is connected between terminals T_3 and T_4 by varying capacitance C to C_1, the circuit is resonated to a frequency f_1.

$$f_1 = \frac{1}{2\pi\sqrt{L(C_1+C_x)}}$$

   after removing C_x between T_3 and T_4, by adjusting capacitance C to the new value C_2

$$f_1 = \frac{1}{2\pi\sqrt{L(C_1+C_x)}} = \frac{1}{2\pi\sqrt{LC_2}}$$

$$C_X = C_2 - C_1$$

3. Measurement of resistance:

   At resonant frequency, $Q = \dfrac{\omega_0 L}{R}$

$$R = \frac{\omega_0 L}{Q} = \frac{2\pi f_0 I}{Q}$$

4. Measurement of distributed (or) self-capacitance of an inductive coil:
   The internal capacitance of the inductive coil can be measured at resonance by adjusting C to a new value C_1.

   Resonant frequency $f_1 = \dfrac{1}{2\pi\sqrt{(C_1+C_d)L}}$

   Again the capacitor C is adjusted to C_2 to get the resonant frequency $f_2 = 2f_1$.

$$\frac{1}{2\pi\sqrt{L(C_2+C_d)}} = \frac{2}{2\pi\sqrt{L(C_1+C_d)}}$$

$$C_d = \frac{C_1-4C_2}{3}$$

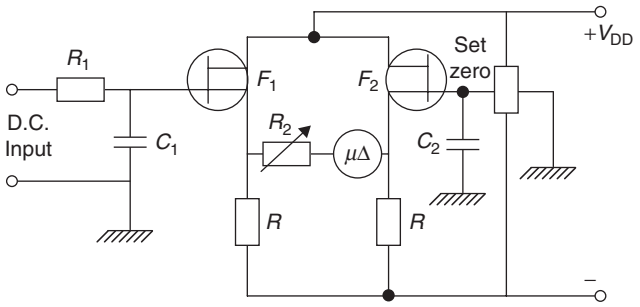
   For $f_2 = nf_1$

$$C_d = \frac{C_1-n^2C_2}{n^2-1}$$

Analog and Digital Voltmeters

Analog Electronic Voltmeter

- These voltmeters use a moving coil meter in which the torque acting on the movement is proportional to the average current flowing through it.



- A circuit of DC electronic voltmeter is a bridge circuit with the two FETs forming one half of the bridge and source resistor R forming the other half.
- The moving coil meter is coupled between two source resistor R. The circuit R_1C_1 acts as an attenuator to AC signal super imposed on DC voltage.
- The bridge is balanced by set zero control which changes the bias applied to F_2. the capacitor C_2 decouples the gate F_2 to earth.
- If the gate F_1 is raised to a positive potential, the potential of the source is raised by a similar amount and current flow through the meter from F_1 to F_2. A variable resistor R_2 in series with meter is used for calibration

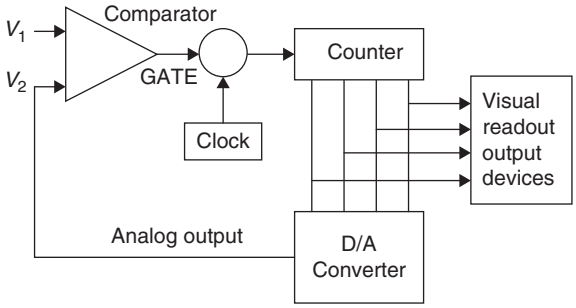
  Sensitivity of this instrument = $\dfrac{g_m}{2}$ where g_m is the mutual conductance of FET.
  For measuring AC quantities, the instruments are listed as follows.

1. **Rectifier Amplifier:** In which a DC amplifier is used and both AC or DC quantities can be measured.
2. **Amplifier Rectifier Type:** In which an AC amplifier is used and rectification takes place at output. Only AC quantities are measured.

Digital Voltmeters

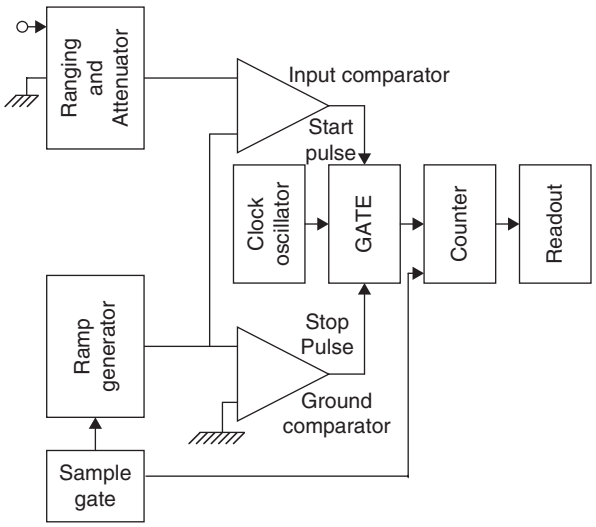
- These display the value of AC or DC voltages being measured directly as discrete numerals in the decimal number system. Analog to digital conversion is used.
- Voltage is changed to a proportional time interval and it starts and stops a clock oscillator. The oscillator output is applied to an electronic counter.

Continuous Balance Type DVM



- Initially the counter is set to zero at the beginning of the measuring process afterwards the counter holds a succession of numbers.
- These are converted into analog signal V_2 by the digital to analog converter. The output is compared with input signal in an electronic comparator.
- If $V_1 > V_2$, output from the comparator is highest. The electronic gate is open and clock pulses are allowed to be fed into the counter.
- The number stored in the counter increases as it counts clock pulses.
- If $V_1 = V_2$, output comparator suddenly falls to zero closing the circuit.

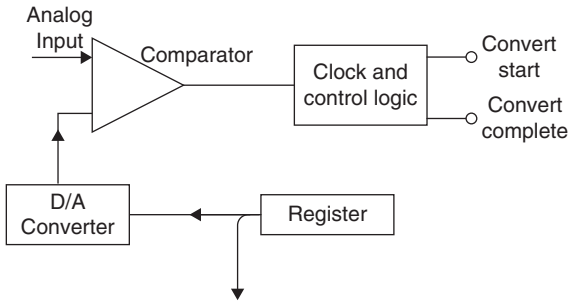
Ramp Type DVM



- The principle is to measure the time that a linear ramp voltage takes to change from the level of input voltage to zero voltage (or vice versa).
- This interval is measured with an electronic time interval counter and the count is displayed as a number of digits on electronic indicating tubes of the output read out.
- At start of measurement, ramp voltage is initiated. The ramp voltage value is continuously compared with the unknown voltage. At the instant the value of ramp voltage is equal to that of unknown voltage, a coincidence circuit called and input comparator generates a pulse which opens a gate.
- The ramp voltage continues to decrease till it reaches ground level. At this instant another comparator called the ground comparator generates a pulse and closes the gate. During the time interval the pulses passing through the gate are counted and displayed.

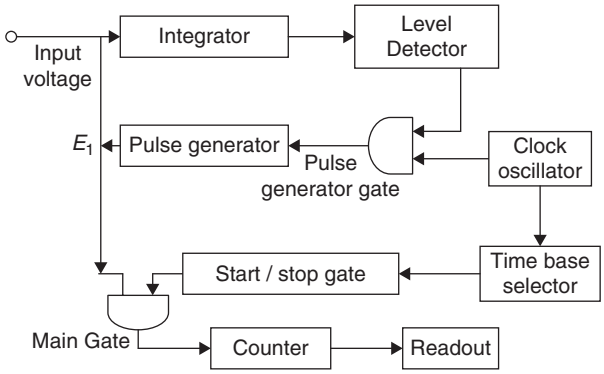
Successive Approximation Type DVM

- The essential difference between the successive approximation type DVM and continuous balance DVM lies in the sequence of test voltages used to balance the voltmeter.



- Let us try the sequence 8, 4, 2, 1. The first clock pulse in the successive approximation type sets the counter to 8. this is compared with V_1 and since $V_1 > V_2$, the comparator allows the count to continue in the up direction.
- The next clock pulse adds another 4 to the count which now becomes 12; this is compared with V_1 and the count continues in the up directions. The next clock pulse adds 2 to the count stored in the counter making it 14. The comparator must then instruct the counter to count down by previous count reducing the number in counter to 12 again.
- The comparator now recognized that $V_2 > V_1$ and the counter then counts up by next sample number, i.e. 1 making the stored number equal to 13.
- At the point after 5 steps DVM balances.
- Clearly this type of DVM is more rapid in operation than the more simple continuous balance DVM, but only at an increase in complexity and cost.

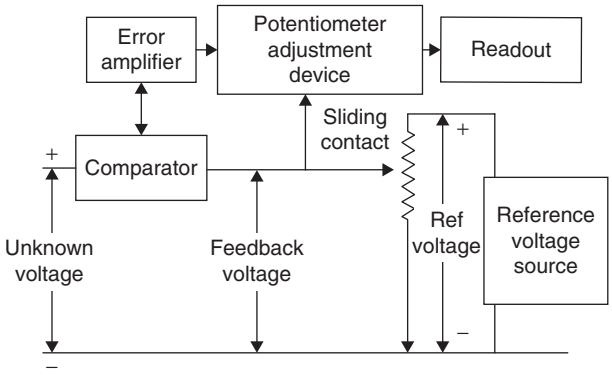
Integrating Type DVM



- It measures the true average value of the input voltage over a fixed measuring period. Here, voltage to frequency conversion is used.
- The voltage frequency conversion function has a feedback control system which controls the rate of pulse generation in proportion to magnitude of the input pulse.
- A train of pulse whose frequency depends on the unknown voltage is generated. The number of pulses appearing in a definite time interval is counted. Since the frequency of these pulses is a function of the unknown voltage, then the number of pulses counted in that period of time is an indication of the unknown voltage.

- The unknown voltage E_i is applied to the input of the integrator and output voltage E_o starts to rise. The slope of E_o is determined by the value of E_i. This voltage is fed to a level detector and when E_o reaches that level, the detector produces an output pulse.
- Greater the value of E_i, sharper will be the slope of E_o and quicker the E_o will reach its reference level. The output pulse of the level detector opens the pulse level gate, permitting pulses from a clock oscillator to pass through pulse generator.
- This generator produces an output pulse of fixed amplitude and width for every pulse it receives. This output pulse, whose polarity is opposite to that of E_i and has greater amplitude, is fed back to the integrator. Hence, the net input to the integrator is now a reversed polarity.
- As a result of this reversed input E_o drops back to its original level and there is no output from the detector to the pulse generator and the gate gets closed.
- When the output voltage pulse from the pulse generator has passed E_i is restored to its original value and E_o starts to rise again. Once again when it reaches the level of reference voltage, the gate is opened, the cycle is repeated.
- Thus the waveform E_o is a saw-tooth wave, whose rise time depends upon E_i and fall time upon width of the output pulse from the pulse generator.
- The frequency of the saw-tooth is measured by coupling the number of pulses in a given interval of time. Pulses from clock oscillator are applied to a time base selector. The first pulse passes through start/stop gate producing an output which is applied to the main gate.
- The same output from the pulse generator also passes through the main gate.
- The next pulse from time base closes the start/stop and main gate. The counter thus counts the number of pulses that have passed during a known interval of time.

Potentiometric Type DVM



- In this, DVM unknown voltage is compared with a reference voltage whose value is fixed by the setting of calibrated potentiometer.
- Under balanced conditions, the value of unknown voltage is indicated by dial setting of the potentiometer. The balance is achieved automatically.
- The unknown voltage is filtered and attenuated to a suitable level. This is applied to a comparator reference voltage is obtained from a fixed voltage source is applied to potentiometer R. The feedback voltage depends upon the position of sliding contact.
- This feedback voltage and unknown voltage are compared in the comparator. The error signal so obtained is amplified and is fed to potentiometer adjustment device which moves sliding contact of the potentiometer proportional to the error signal.
- The sliding contact moves to such a place where feedback signal equals unknown voltage. In such a case, there is no error signal and the movement of sliding contact is stopped.
- The position of potentiometer adjustment device at this point is indicated in numerical form on the digital read out and it is the unknown voltage.

Example 6: Which among the following is a false statement with regard to a potentiometric type DVM?
(A) It is a self-balancing type DVM.
(B) The error detector employed can be a chopper.
(C) The sliding contact moves proportional to the error signal received at the potentiometer adjustment device.
(D) None of the above.

Solution: (D)
Choices a, b, c are true regarding a potentiometric type DVM and hence.

Example 7: Identify the correct matching list I to that in List II.

List I
(P) Ramp type DVM
(Q) Integrating type DVM
(R) Successive approximation type DVM
(S) Potentiometric type DVM

List II
(1) Ground comparator
(2) Level detection
(3) Rapid and simple than continuous balance type DVM

(A)	P – 1	Q – 2	R – 3
(B)	P – 1	Q – 2	S – 3
(C)	P – 2	Q – 1	S – 3
(D)	P – 3	Q – 2	S – 1

Solution: (A).

Example 8: In a dual-slope integrating type DVM an input voltage of 1.6 V and reference voltage of 3.2 V are used for a supply frequency of 50 Hz. Calculate the total conversion time if the first integration is carried out for 20 periods.
(A) 0.2 s (B) 0.02 s
(C) 0.05 s (D) 2 s

Solution: (A)

In a dual-slope integrating type

$$\text{DVM}\left(\frac{t_1}{t_2}\right)V_{IN} = V_{ref}$$

First integration time $t_1 = 20 \times \frac{1}{50}$

$= 0.4$ s

$V_{IN} = 1.6$ V; $V_{ref} = 3.2$ V

Total conversion time

$$t_2 = \frac{V_{IN}}{V_{ref}}\ t_1 = \frac{1.6 \times 0.4}{3.2}$$

$t_2 = 0.2$ s

Example 9: A digital voltmeter has a voltage controlled generator which provides a width of 5 μs/V of unit signal, uses a clock of 20 MHz. The pulse count corresponding to an input signal of 12 V would be

(A) 600 (B) 1200
(C) 1800 (D) 2400

Solution: (B)

$$\text{Pulse count} = (20\times10^6) \times (5\times10^{-6}) \times 12$$

$$= 1200.$$

Example 10: If a 24 V DC voltage is applied to an average response rectifier type electronic AC voltmeter, the meter would read

(A) 24 V (B) 23.34 V
(C) 25.5 V (D) 26.64 V

Solution: (D)

A rectifier type instrument is calibrated to read RMS values for sinusoidal waveforms

The meter would display 1.11 times the average value

$$\text{Readout} = 24 \times 1.11 = 26.64 \text{ V.}$$

Example 11: In a DVM, the number of pulses counted by the counter, when the ramp voltage falls down from 12 V to 0 V in 24 ms and the oscillator frequency of 325 kHz is

(A) 7800 (B) 8000
(C) 288 (D) 3900

Solution: (A)

Assuming the gate opens when ramp is 12 V and closes when ramp reaches 0

$$\text{Number of pulses} = (325\times10^3) \times (24\times10^{-3}) = 7800$$

Example 12: An integrating DVM measures

(A) Peak-to-peak value (B) True average value
(C) RMS value (D) Peak value

Solution: (B)

EXERCISES

Practice Problems I

1. The type of converter used in digital meter is
(A) R2R ladder type A–D converter.
(B) Dual-slope integrating type A/D converter.
(C) Any kind of A–D converter.
(D) Flash converter.

2. The electrons in a CRT having a cathode anode voltage of 1000 V leave the cathode with zero velocity. The maximum velocity of the beam of electron in km/s is

(A) 45.6×10^3 (B) 30.6×10^3
(C) 10.5×10^3 (D) 18.7×10^3

3. When two voltages having the same frequency and equal magnitude but with a phase difference of 90° is applied to a CRO, the trace on the screen is a

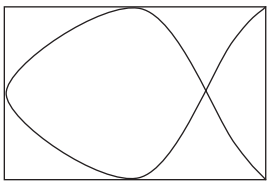
(A) Circle (B) Ellipse
(C) Oval (D) Straight line

4. A voltage of $10 \sin(314t + 30°)$ is examined on a CRO having 10 divisions on the horizontal scale.

The line base is set to 10 ms/div. The number of cycles of signal displayed on the screen will be

(A) 1 (B) 2.5
(C) 5 (D) 10

5. Two frequencies of value f_h and f_v are applied to the horizontal and vertical plates of a CRO. The Lissajous pattern observed is as shown below. If f_h = 50 Hz, the value of f_v is



(A) 50 Hz (B) 100 Hz
(C) 150 Hz (D) 200 Hz

6. The meter that is suitable for measuring 10 mV at 50 Hz is
(A) VTVM
(B) CRO
(C) Moving iron voltmeter
(D) Electrostatic voltmeter

7. The vertical amplifier of an oscilloscope has a bandwidth of 20 MHz. The fastest rise time that an input may have to be displayed without distortion is

(A) 17.5 μs (B) 17.5 ns
(C) 1.75 μs (D) 1.75 ns

8. A CRT has an anode voltage of 3000 V and parallel deflecting plates 2.5 cm long and 5 mm apart. The screen is 25 cm from the centre of the plates. The input voltage is applied to the deflecting plates through amplifiers having an overall gain of 100. The input voltage required to deflect the beam through 5 cm is.
(A) 100 V (B) 240 V
(C) 2.4 V (D) 2 V

9. By digital voltmeters we can measure
(A) AC voltage
(B) DC voltage
(C) Both A and B
(D) Integrating type is used to measure DC only and the rest are used to measure both AC and DC

10. An average response rectifier type electronic AC voltmeter has DC voltage of 20 V applied to it. The meter reading will be.
(A) 22.2 V
(B) 11.1 V
(C) 10 V
(D) 14.4 V

11. The number of comparators needed in a parallel conversion type 8 bit A to D converter is
(A) 8 (B) 16
(C) 255 (C) 256

12. In a compensated shunt type peak reading VTVM, a bucking battery with a series adjustable resistance is used for
(A) Providing power for the circuit
(B) Zero setting
(C) Purpose of amplifying low level signal
(D) All of these

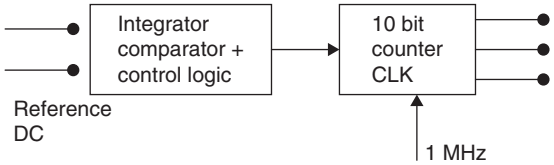
13. In comparison with a conventional VTVM, a balanced bridge vacuum tube voltmeter has
1. Higher input impedance.
2. Effects due to changes in value characteristics are minimized.
3. Meter zero has less tendency to shift.
4. Power supply fluctuation have a smaller effect on measuring circuit.

Select the correct answer using codes below.
(A) 2 and 4 (B) 1 and 3
(C) 1, 2 and 3 (D) 1, 2, 3 and 4

14. Rectifier type electronic voltmeter uses negative feedback. This is done to
(A) Increase the overall gain
(B) Improve the stability
(C) Overcome non-linearity of diodes
(D) None of these

15. The simplified block diagram of a 10 bit A–D converter of dual-slope integrator type is as shown.

The 10-bit counter at the output is clocked by a 1 MHz clock. Assuming negligible timing overhead for the control logic, the maximum frequency of the analog signal that can be converted using this A/D converter is approximately

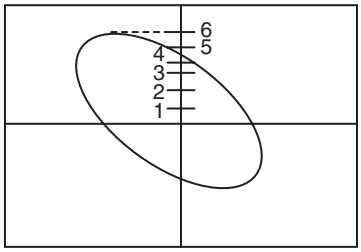


(A) 2 kHz (B) 1 kHz
(C) 500 Hz (D) 250

Practice Problems 2

1. The Lissajous pattern is used to measure
(A) Phase angle
(B) Frequency
(C) Both phase angle and frequency
(D) Does not measure phase angle or frequency

2. Two voltages having the same frequency is applied to a CRO in the *X*–*Y* mode. The Lissajous pattern obtained is as shown. The phase difference between the voltages would be.



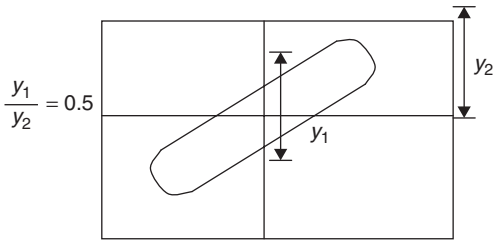
(A) 41.81° (B) 45° (C) 180° (D) 138.2°

3. When the horizontal deflecting plates of a CRO are kept at ground potential and 30 V DC is applied to the vertical defecting plate, the picture observed on the screen would be
(A) A spot approximately 3 cm away from the centre.
(B) A vertical line 2 cm long.
(C) A vertical line approximately 3 cm long.
(D) Two spots 2 cm vertical above each other.

4. A sinusoidal waveform, when observed on an oscilloscope, has a peak-to-peak amplitude of 14 cm. If the vertical resistivity setting is 5 V/cm, the RMS value of voltage will be
(A) 24.8 V (B) 49.6 V
(C) 9.9 V (D) 3.54 V

5. To the *y* input of a CRO, a signal defined by 10sin 100*t* is applied. To the *x*-input the signal 10cos100 *t* is applied. The gain for both *x*-channel and *y*-channel are the same. The screen shows
(A) Sinusoidal (B) A straight line
(C) An ellipse (D) A circle

6. Two sine waves of the same frequency are impressed on the X and Y plates of a CRO and the Lissajous figure seen is shown in the given figure. The phase difference between the signals is



(A) 30° or 330° or 150° or 210°
(B) 30° or 330° or 150°
(C) 30° or 330°
(D) 30°

7. Two inputs of a CRO are fed with two stationary periodic signals. In the X-Y mode, the screen shows a figure which changes from ellipse to circle and back to ellipse with its major axis changing orientation slowly and repeatedly. The following inference can be made from this
(A) The signals are not sinusoidal.
(B) The amplitude of signals are very close but not equal.
(C) The signals are sinusoidal with their frequencies very close but not equal.
(D) There is a constant but small phase difference between the signals.

8. Level detector is used in integrated type DVM circuit. It works much like a
(A) Voltage comparator (B) Pulse generator
(C) Counter (D) None of these

9. The advantage of VTVM over a non-electronic voltmeter is
(A) Lower power consumption
(B) Lower input impedance
(C) Ability to measure wider ranges of voltage and resistance
(D) Greater portability

10. A 10-bit A–D converter is used to digitize an analog signal in the 0–5 V range. The maximum peak to ripple voltage that can be allowed in DC supply voltage is
(A) Nearly 100 mV (B) Nearly 50 mV
(C) Nearly 25 mV (D) Nearly 5 mV

11. In a CRO, a Lissajous pattern of eight keeps on changing the shape when ratio of frequencies is
(A) Not exactly 1:1 (B) Not exactly 2:1
(B) Exactly 3:1 (D) Exactly 4:1

12. The number of base circuits that a dual-trace CRO can have is
(A) 1 (B) 2 (C) 3 (D) 4

13. Group II represents the figures obtained on a CRO screen when the voltage signals $V_x = V_{xm} \sin\omega t$ and $V_y = V_{ym} \sin(\omega t + \phi)$ are given to X and Y plates and ϕ is changed. Choose the correct value of ϕ from group I to match with corresponding figure of group II

Group I

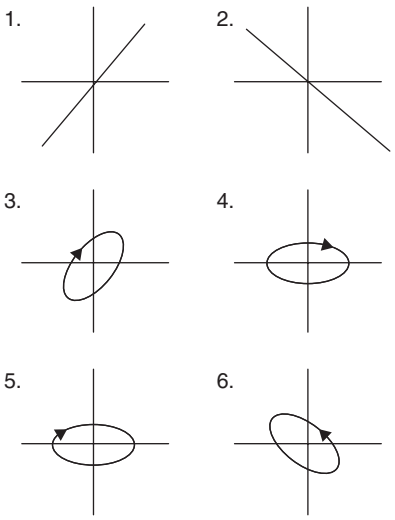
P $\phi = 0$

Q $\phi = \frac{\pi}{2}$

R $\pi < \phi < \frac{3\pi}{\alpha}$

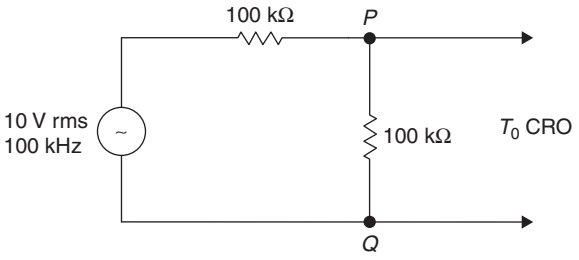
S $\phi = \frac{3\pi}{2}$

Group II



(A) P =1, Q = 3, R = 6, S = 5
(B) P = 2, Q = 6, R = 4, S = 5
(C) P = 2, Q = 3, R = 5, S = 4
(D) P = 1, Q = 5, R = 6, S = 6

14. A CRO probe has an impedance of 500 kΩ in parallel with a capacitance of 10 pF. The probe is used to measure the voltage between P and Q. The measured voltage will be

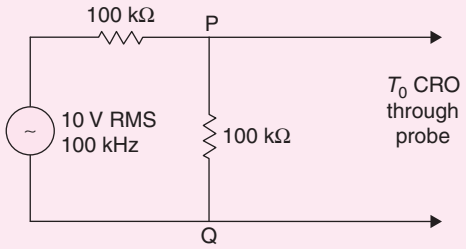


(A) 3.53 V (B) 4.37 V
(C) 4.54 V (D) 5.00 V

15. The simultaneous application of signals $x(t)$ and $y(t)$ to the horizontal and vertical plates, respectively, of an oscilloscope produces a vertical figure of '8' on display. If X and Y are constants and $x(t) = X \sin(4t + 30)$, then $y(t)$ is
(A) $Y\sin(4t - 30)$ (B) $Y\sin(2t + 15)$
(C) $Y\sin(8t + 60)$ (D) $Y\sin(4t + 30)$

Previous Years' Questions

1. A CRO probe has an impedance of 500 kΩ in parallel with a capacitance of 10 pF. The probe is used to measure the voltage between P and Q as shown in figure. The measured voltage will be **[2004]**

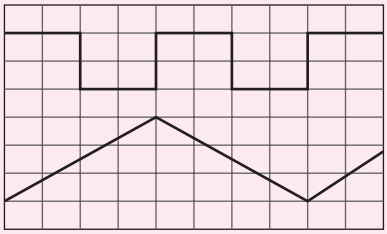


(A) 3.53 V (B) 4.37 V
(C) 4.54 V (D) 5.00 V

2. The Q-meter works on the principle of **[2005]**
(A) Mutual inductance (B) Self-inductance
(C) Series resonance (D) Parallel resonance

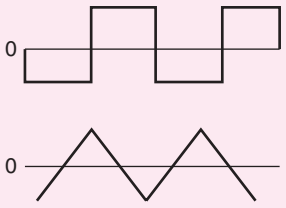
3. The simultaneous application of signals $x(t)$ and $y(t)$ to the horizontal and vertical plates, respectively, of an oscilloscope, produces a vertical figure-of-8 display. If P and Q are constants, and $x(t) = P\sin(4t + 30)$, then $y(t)$ is equal to **[2005]**
(A) $Q\sin(4t - 30)$ (B) $Q\sin(2t + 15)$
(C) $Q\sin(8t + 60)$ (D) $Q\sin(4t + 30)$

4. The time/div and voltage/div axes of an oscilloscope have been erased. A student connects a 1 kHz, 5 V p–p square wave calibration pulse to channel of the scope and observes the screen to be as shown in the upper trace of the figure. An unknown signal is connected to channel 2 (lower trace) of the scope. If the time/div and V/div on both channels are the same, the amplitude (p-p) and period of the unknown signal are, respectively. **[2006]**



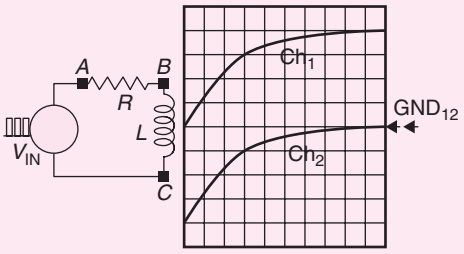
(A) 5 V, 1 ms (B) 5 V, 2 ms
(C) 7.5 V, 2 ms (D) 10 V, 1 ms

5. A sampling wattmeter (that computes power from simultaneously sampled values of voltage and current) is used to measure the average power of a load. The peak-to-peak voltage of the square wave is 10 V and the current is a triangular wave of 5 A p–p as shown in the figure. The period is 20 ms. The reading in W will be **[2006]**



(A) 0 W
(B) 25 W
(C) 50 W
(D) 100 W

6. The probes of a non-isolated, two-channel oscilloscope are clipped to points A, B and C in the circuit of the adjacent figure. V_{IN} is a square wave of a suitable low frequency. The display on Ch_1 and Ch_2 are as shown on the right. Then the 'Signal' and 'Ground' probes S_1, G_1 and S_2, G_2 of Ch_1 and Ch_2, respectively, are connected to points. **[2007]**

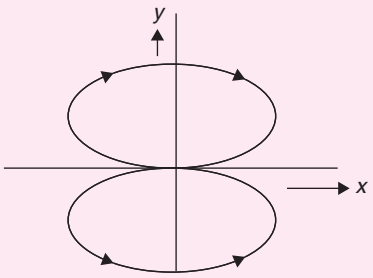


(A) A, B, C, A
(B) A, B, C, B
(C) C, B, A, B
(D) B, A, B, C

7. For a step-input e_i, the overshoot in the output e_0 will be **[2007]**
(A) 0, since the system is not under-clamped
(B) 5%
(C) 16%
(D) 48 %

8. If the above step response is to be observed on a non-storage CRO, then it would be best to have the e_i as a **[2007]**
(A) Step function
(B) Square wave of frequency 50 Hz
(C) Square wave of frequency 300 Hz
(D) Square wave of frequency 2.0 kHz

9. Two sinusoidal signals $p(\omega_1 t) = \text{A}\sin\omega_1 t$ and $q(\omega_2 t)$ are applied to X and Y inputs of a dual-channel CRO. The Lissajous figure displayed on the screen is shown below. The signal $q(\omega_2 t)$ will be represented as **[2008]**



(A) $q(\omega_2 t) = A\sin\omega_2 t,\ \omega_2 = 2\omega_1$

(B) $q(\omega_2 t) = A\sin\omega_2 t,\ \omega_2 = \dfrac{\omega_1}{2}$

(C) $q(\omega_2 t) = A\cos\omega_2 t,\ \omega_2 = 2\omega_1$

(D) $q(\omega_2 t) = A\cos\omega_2 t,\ \omega_2 = \dfrac{\omega_1}{2}$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** D	**3.** A	**4.** C	**5.** B	**6.** B	**7.** B	**8.** C	**9.** C	**10.** A
11. C	**12.** C	**13.** D	**14.** C	**15.** B					

Practice Problems 2

1. C	**2.** D	**3.** C	**4.** A	**5.** D	**6.** C	**7.** D	**8.** A	**9.** A	**10.** D
11. B	**12.** C	**13.** D	**14.** A	**15.** B					

Previous Years' Questions

1. B	**2.** C	**3.** B	**4.** C	**5.** A	**6.** B	**7.** C	**8.** C	**9.** D

Chapter 5

Magnetic Measurements and Transducers

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Magnetic measurements
- Ballistic test
- Measurement of magnetizing force
- Determination of B–H curve
- Measurement of iron loss
- Measurement of displacement
- Measurement of speed of rotation
- Measurement of strain
- Measurement of force or pressure
- Measurement of temperature
- Thermocouple
- Pyrometers

Magnetic Measurements

Magnetic measurements are performed to get knowledge of characteristics of various magnetic materials. There are two types of tests performed on magnetic materials.

Ballistic Test

- To measure magnetic flux, permeability field strength, plotting of B–H curve and hysteresis loop.

AC Test

- To know the power loss in magnetic material under the influence of AC.

Note: Main difficulties in these measurements.

- Magnetic flux cannot be measured directly.
- Flux does not confine itself to a definite path.
- Magnetic materials are not homogenous.
- Non-uniformity in magnetic properties between different batches of the material.
- When ferromagnetic materials are subjected to an alternating magnetic field, power loss due to hysteresis effects and eddy currents is present.
- Hysteresis loss depends upon frequency maximum flux density of the magnetic field to which the specimen is subjected

$$P_h = K_h f B_{max}^{1.6}.$$

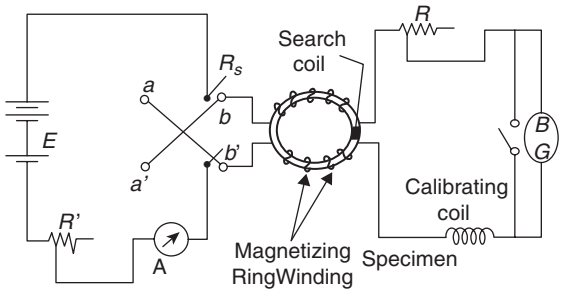
P_h = Hysteresis loss in watts/cubic meter volume of material
f = Frequency of alternating flux in Hz
B_{max} = Maximum flux density (Wb/m^2)
K_h = Hysteresis loss constant of the given material

Eddy current loss depends on waveshape of variation of flux with time, frequency and maximum flux density of the magnetic field to which the specimen is subjected.

$$\text{Eddy current loss } P_e = K_e K_f^2 f^2 B_{max}^2$$

P_e = Eddy current loss in Watts/m^3 volume of material
K_f = Form factor of the alternating flux
K_e = Eddy current loss constant for the given material

Measurement of Magnetic Flux



- A current I is flowing through the magnetizing winding uniformly wound on the ring specimen.
- Let ϕ be flux in 'Wb' produced by magnetizing current of 'I' A.
- Let t be time of reversal of flux rate of change of flux = $\frac{2\phi}{t}$ Wb/s.
- Average emf induced in search coil

$$e = \frac{2N\phi}{Rt} \text{ V}$$

(N = Number of turns in search coil)

- Current through ballistic galvanometer circuit $\frac{2N\phi}{Rt}$ A

R = Total resistance of the galvanometer

- Quantity of electricity discharged through galvanometer $Q = \frac{2N\phi}{Rt} \times t = \frac{2N\phi}{R}$ coulombs.
- Quantity of electricity discharged through galvanometer can be written as

$$Q = K\theta_1.$$

K = Ballistic galvanometer constant
θ_1 = First deflection of galvanometer

When primary current is reversed then

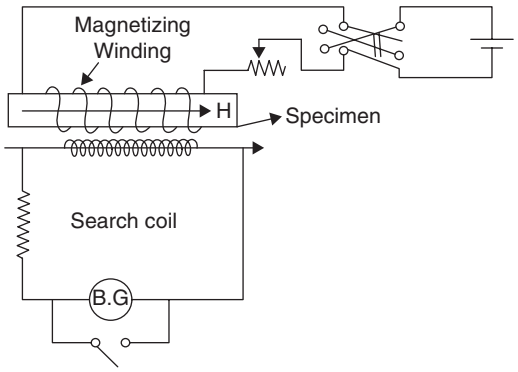
$$\frac{2N\phi}{R} = K\theta_1$$

$$\phi = \frac{RK\theta_1}{2N} \text{ Wb}$$

$$B = \frac{\phi}{A_s} = \frac{RK\theta_1}{2NA_s} \text{ Wb/m}^2$$

where A_s = Cross-sectional area of the specimen.

Measurement of Magnetizing Force (H)



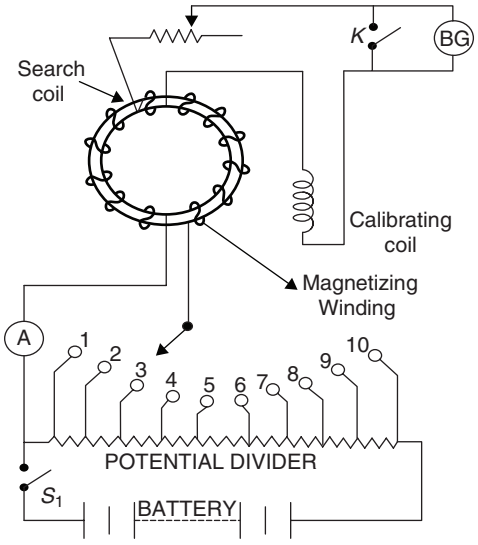
The value of flux density B_0 in search coil is measured in a similar manner described above for determination of B in the specimen.

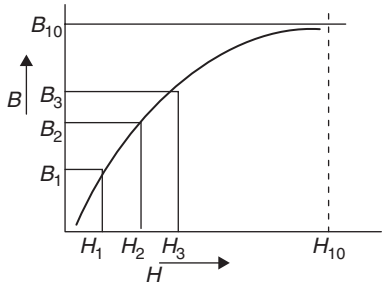
Now $$H = \frac{B_0}{\mu_0} \text{ AT/m}$$

Determination of B–H Curve

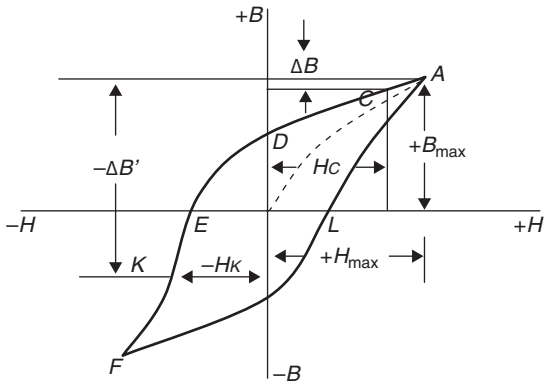
B–H curve can be determined by the following two methods.

1. **Method of Reversal**
   - Same circuit which was used for determination of ϕ and B can be utilized for this purpose.
   - A layer of thin tape is put on the ring and a search coil insulated by paraffined wax is wound over the search coil and the magnetizing winding is uniformly wound over this tape.
   - After demagnetizing the test is started by setting the magnetizing current to its lowest test value.
   - With galvanometer key K closed, the iron specimen is brought into a reproducible cyclic magnetic state by throwing the reversing switch forward and backward about 20 times.
   - K is now opened and the value of B corresponding to H is measured by the reversing switch S and noting the throw of galvanometer.
   - The value of B corresponding to H can be calculated by ϕ/A.
   - The above procedure is repeated for various values of H up to maximum testing point and B–H curve may be plotted.
2. **Step-by-Step Method**

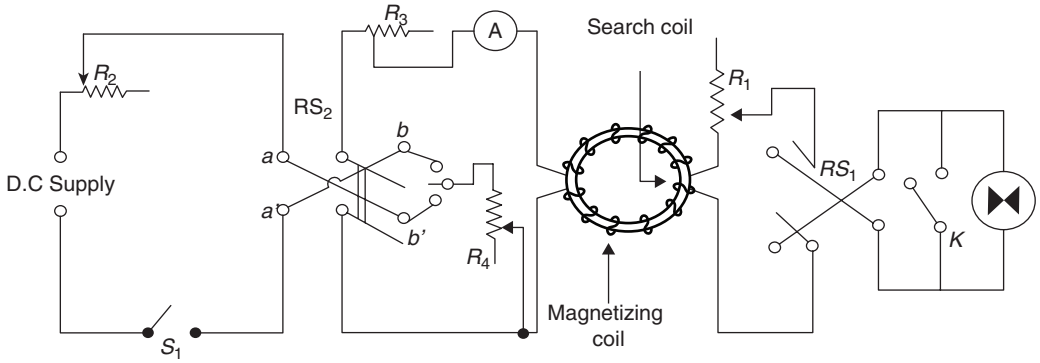




- The magnetizing winding is supplied through a potential divider having a large number of tappings. The tappings are arranged so that the magnetizing force H may be increased in a number of suitable steps up to the desired maximum value. The specimen before being tested is demagnetized.
- Switch S_2 is set on tapping 1 and S_1 is closed. The throw of the galvanometer corresponding to this increase in flux density in the specimen from 0 to some value above B_1 is observed.
- B_1 can be calculated from the throw of galvanometer. H_1 may be calculated from the value of current flowing in magnetizing winding at tapping 1.
- The magnetizing force is then increased to H_2 by switching S_2 suddenly to tapping 2 and corresponding increase in ΔB. is determined from the throw of galvanometer.
- B_2 corresponding to H_2 is given by $B_1 + \Delta B$.
- This process is repeated for other values of H up to maximum point and complete B–H curve is thus obtained.



Determination of Hysteresis Loop

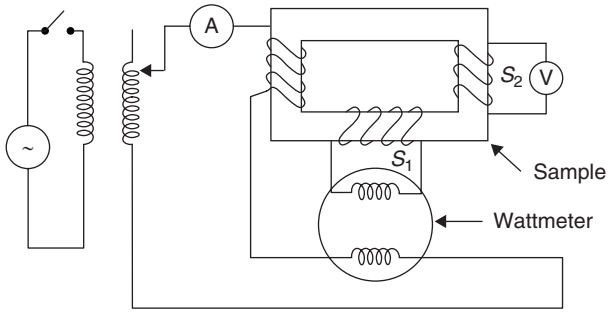


- The galvanometer is connected to the circuit by the Key K and reversing switch RS_2 is placed on contacts aa'.
- Value of B_{max} is found corresponding to magnetizing force H_{max} from the deflection of the galvanometer, observed on reversing switch RS_2 and point A on hysteresis loop is found.
- The switch S_2 is put to contact 2 in order to connect resistance R_4 across the magnetizing winding and to reduce the magnetizing force to H_k.
- The corresponding reduction in flux density is obtained from the galvanometer throw and the point K is found.
- The galvanometer is then short circuited by closing K and reversing RS_2 is put to contacts bb'. Switch S_2 is moved to off position and RS_2 is moved back to aa'. The specimen is brought back to point A again.
- Section AC of the loop is obtained by adjusting R_4 to give different reduced value of H and finding corresponding reduction in B.
- For CDE of the loop galvanometer is short circuited, S_2 is put in off position and RS_2 to contacts aa'. Again S_2 is put on to contacts', key is opened and RS_2 to bb'.

Corresponding galvanometer throw is noted. Specimen magnetization is brought back to K opening switch S_2 and RS_2 to aa'. Rest of the portion of the loop is obtained, as two halves are identical.

Measurement of Iron Loss

Wattmeter Method



- The primary winding is connected to a source of supply through an auto-transformer in series with the current coil of the wattmeter.
- Two single-layer secondary coils are connected in series, in a group of four, one on each side of the core to give exactly similar and separate windings RMS value of the emf induced in secondary winding S_2.

$$E = 4K_f B_{max} A f N_2 \text{ V}$$

K_f = Maximum flux density in Wb/m^2
A = Area of cross section
f = Supply frequency
N_2 = Number of turns on secondary winding

Hence $B_{max} = \dfrac{E}{4K_f A_f N_2}$ Wb/m^2

From the circuit diagram,

Voltage applied to pressure coil circuit $V = I_p R_p$

Induced emf in secondary winding

S_2 = Reading of voltmeter

$$V = I_p(R_p + R_s)$$

Total power loss in the iron, i.e. iron loss and copper loss in the winding of the secondary as well as in the pressure coil

$$= \frac{WE}{V} = \frac{W I_p (R_p + R_s)}{I_p R_p} = \frac{W(R_p + R_s)}{R_p}$$

$$= W\left(1 + \frac{R_p}{R_s}\right)$$

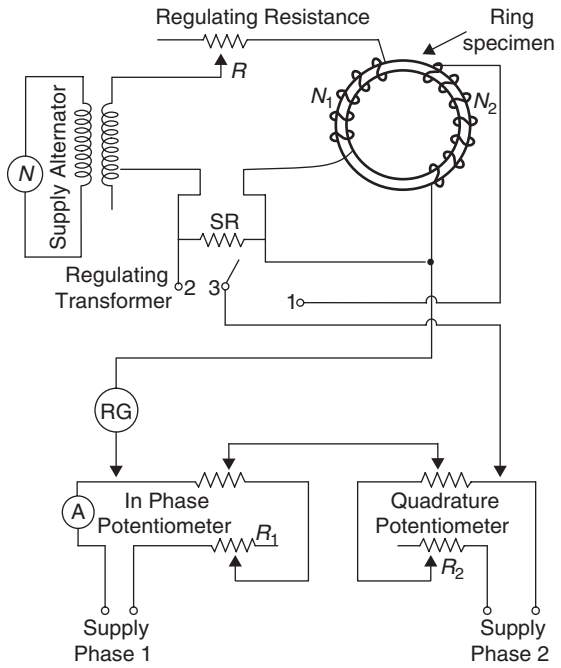
Total copper loss in both R_p and R_s

$$= I_p^2 (R_p + R_s) = \frac{E^2}{R_p + R_s}$$

Iron loss $P_i = W\left(1 + \frac{R_s}{R_p}\right) - \frac{E^2}{R_p + R_s}$

This method is not sufficient sensitive for testing laminated core material at low flux densities.

AC Potentiometer Method



- The primary winding (N_1 turns) on the specimen is supplied from sinusoidal voltages alternator. A standard resistance SR is connected in series with primary winding

$$B_{max} = \frac{E}{4K_f A_f N_2} = \frac{E}{4.44 A_f N_2}$$

E = RMS value of emf induced in secondary
A = Area of cross section of specimen
F = Supply frequency
K_f = Form factor

- The switch S is kept to position 1. The quadrature potentiometer is set at 0, in phase potentiometer is adjusted for 0 deflection in vibration galvanometer.
- The setting of in phase potentiometer gives E directly.
- Switch is now put to position 2 and both potentiometers are adjusted for 0 deflection.
- Value of $I_e R_s$ is given by in phase potentiometer and $I_m R_s$ by quadrature potentiometer.

Solved Examples

Example 1: Ballistic tests are conducted for measurement of which magnetic quantity
(A) Determination of hysteresis loop of the specimen
(B) Determination of flux density of the specimen
(C) Determination of magnetic field intensity of the specimen
(D) All of the above

Solution: (D).

Example 2: Ring type specimen are used for ballistic test due to the reason that
(A) They could be used for large quantity material with very small permeability.
(B) In comparison to the bar type specimen, they require less labour to prepare.
(C) They are free from end effects.
(D) All of the above

Solution: (D).

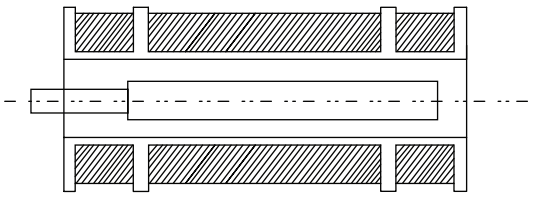
Transducers

- The input quantity for most instrumentation system is non-electric.
- In order to use non-electrical methods and techniques for measurement the non-electric quantity are converted into an electrical signal by a device called transducers.

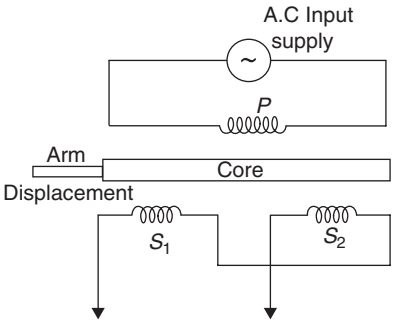
Measurement of Displacement

- An inductive transducer for translating linear motion into electrical signal is used. A plate is placed at right angles to the axis of the coil.
- The displacement of the plate causes a change in inductance of the coil.

Linear Variable Differential Transformer (LVDT)



- It consists of single primary winding P and two secondary winding S_1 and S_2 wound on a cylindrical former.
- Primary winding is connected to an AC source.
- A movable soft iron core is placed inside the former. The displacement to be measured is applied to an arm attached to the soft iron core.
- When the core is in null position equal voltages are there in two secondary winding. When core is moved to right or left difference of two voltages is produced.



- In null position $E_{s_1} = E_{s_2}$

$$E_0 = 0$$

When core is moved to left

$$E_{s_1} > E_{s_2} E_0 = E_{s_1} - E_{s_2}$$

When core is moved towards right

$$E_0 = E_{s_2} - E_{s_1}$$

Hence, the difference of output voltages of secondary winding gives the amount of displacement.

Advantages

1. Almost linear characteristics
2. Low-power consumption
3. Output impedance is constant
4. Infinite resolution

Disadvantages

1. Affected by vibrations
2. Larger displacements necessary for appreciable differential output
3. Performance affected by temperature
4. Sensitive to stray magnetic fields

Example 3: Two secondary windings in an LVDT are connected in a subtractive polarity to obtain
(A) High value of output voltage
(B) Phase sensitive output voltage
(C) A null or reference point for displacement of the core
(D) Both (B) and (C)

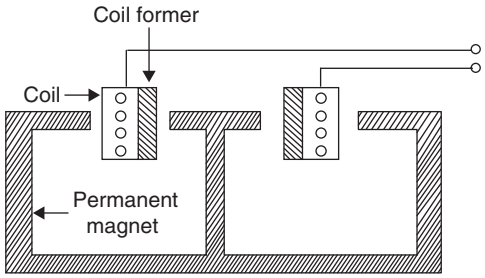
Solution: (D).

Example 4: A high permeability nickel iron hydrogen annealed material is used as core in LVDT
(A) To produce low harmonics
(B) To produce low null voltage
(C) To have high sensitivity
(D) All of the above

Solution: (D).

Measurement of Velocity

Moving Coil Velocity Pick Up



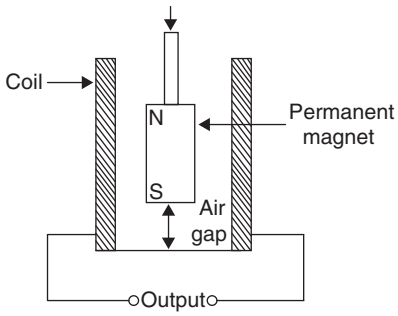
EMF induced in a coil is given by

$$E = Blv \sin\theta \text{ V}$$

B = Flux density of magnetic field in Tesla
l = Length of moving conductor in 'm'
V = Velocity of conductor in magnetic field 'm/s'
θ = Angle of conductor with the direction of field

Since B, l and θ are constants, induced emf is a function of its velocity.

Moving Magnet Velocity Pickup

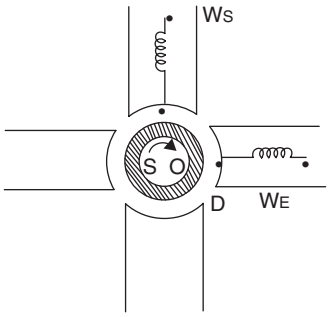


- Here, the coil is kept fixed and a permanent magnet moves inside the coil which induces an emf in the coil proportional to the velocity of the magnet.
- It allows more number of turns on the coil which ensures a sufficient high output voltage.

Measurement of Speed of Rotation

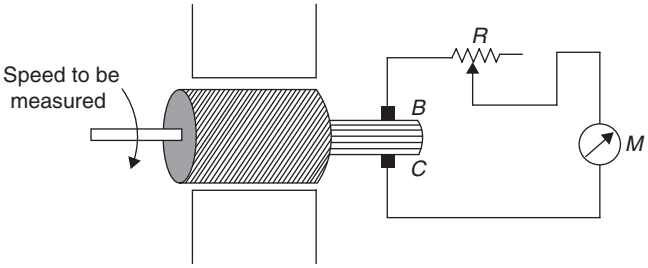
Inductive Tachometer

It works on the principle of variable coupling. The object whose angular velocity is to be measured is coupled to a shaft *S* of the rotor as shown.

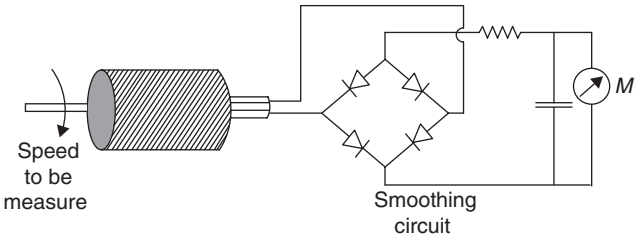


DC Tachometer Generator

- It consists of a small armature which is coupled to a machine whose angular velocity is to be measured.
- The emf is measured using a moving coil voltmeter.



AC Tachometer Generator



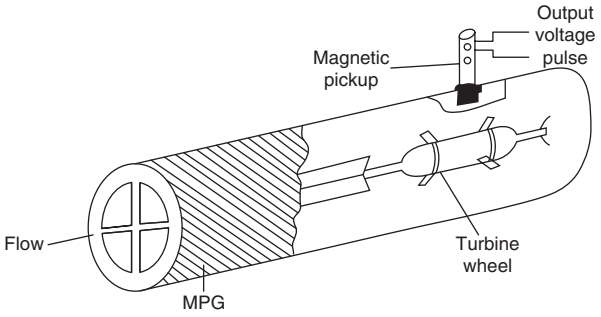
- Rotation of magnet causes an emf to be induced whose amplitude and frequency are proportional to speed of rotation.

Measurement of Flow

Devices used for sensing the rate of fluid flow operate on the principle of placing an obstruction in the path of fluid causing a change in fluid pressure which is dependent upon the rate of flow.

- By measuring the difference in pressure before and after the obstruction by means of differential pressure sensor, the rate of flow may be determined.

Turbine Flowmeter



It is used in the measurement of liquid gases and gases of very low flow rate. It consists of a multi bladed rotor called the turbine wheel which is mounted 90° to the axis of flowing liquid.

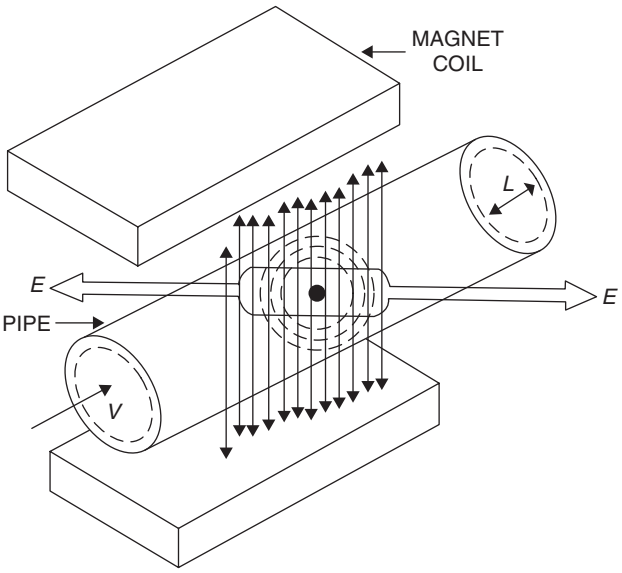
- The rotor is supported by the ball or sleeve bearing on the shaft. It is free to rotate about its axis.
- The flowing liquid strikes the turbine blades (rotor) imparting a force to the blade surface which causes the rotation of the rotor.
- At a steady rotational speed the speed of the rotor is proportional to the fluid velocity and hence to the volumetric flow rate.
- The speed of rotation is monitored by magnetic pickup which is fitted to the outside of the meter housing.
- A voltage pulse is generated as each blade passes the magnetic pickup coil. The total number of pulses give a measure of the flow
  The *K*-factor is given as number of pulses generated per gallon flow.

$$K = \frac{T_K f}{Q}$$

K = Pulses per unit volume
T_K = Time constant in minutes
f = Frequency in Hz
Q = Volumetric flow rate in gmp

Magnetic Flowmeter

- It works on the principle of Faraday's law of electromagnetic induction (which is used in generators).
- In case of flowmeter, electrically conductive flowing liquids work as the conductor.



Induced voltage $E = C.B.L.V$

where C = Dimensional constants
B = Length of the conductor in 'm'
V = Velocity of conductor in 'm/s'

The equation to convert a velocity measurement to volumetric flow rate is

$$Q = V.A$$

where Q = Volumetric flow rate
V = Fluid velocity
A = Cross-sectional area of flowmeter

$$V = \frac{E.A}{C.B.L}$$

For a given flowmeter A, C, B, and L are constants.

$$Q = K.E$$

where $K = \frac{A}{C.B.L}$

Therefore, induced voltage is linear and is directly proportional to volumetric flow rate.

The length L is the distance between the electrodes and equals the pipe diameter. As liquid passes through the pipe section it also passes through the magnetic field setup by the magnetic coils thus inducing a voltage in the liquid, which is detected by a pair of electrodes mounted on the pipe wall.

The amplitude of the induced voltage is proportional to the velocity of the flowing liquid.

Advantages

- It can measure slurry and greasy materials.
- It can handle corrosive fluids.
- It has a very low pressure drop.
- It is totally obstruction less.
- It is available in different pipe dimensions and several construction materials.
- It is capable of handling low volume flows (with minimum size less than 3.175 mm) and very high volume flow rate (with size as large as 3.04 m).
- It can be used as bidirectional meter.

Disadvantages

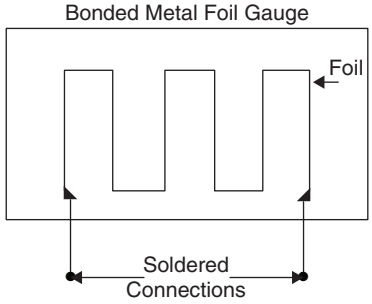
- It is relatively expensive.
- It is relatively heavy especially in large sizes.
- It must be full at all times.
- It must be explosion proof when installed in hazardous area.

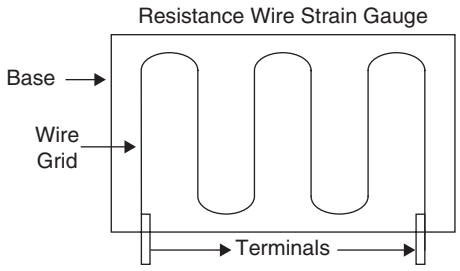
Measurement of Strain

Strain Gauge

These are used for measurement of strain and associated stress.

These are based on the principle that if a metal conductor is stretched or compressed, its resistance changes on account of change on length and diameter of the conductor also there is a change in resistivity of conductor due to strain.

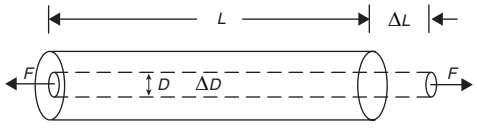




Consider a strain gauge made of circuit wire.

Let the length = L, area = A, diameter = D be the dimensions of the wire before straining.

Let ρ be the resistivity of the material of the wire



Resistance of the unstrained conductor

$$R = \rho\frac{L}{A}$$

On application of stress, resistance changes by ΔR due to change in length ΔL and change in cross-sectional area by ΔR.

$$\frac{dR}{ds} = \frac{d}{ds}\left(\frac{\rho L}{A}\right)$$

$$= \frac{\rho}{A}\frac{dL}{ds} - \frac{\rho L}{A^2}\frac{dA}{ds} + \frac{L}{R}\frac{d\rho}{ds}$$

Dividing the equation by $R = \frac{\rho L}{A}$

$$\frac{1}{R}\frac{dR}{dS} = \frac{\Delta L}{L} - \frac{\Delta A}{A} + \frac{\Delta \rho}{\rho} \quad (1)$$

Under strained conditions

$$\text{Area } A = \pi/4\ D^2$$

$$\frac{\partial A}{\partial S} = 2\ \pi/4\ D\ \frac{\partial D}{\partial S}$$

$$\frac{\partial A}{A} = \frac{2\partial D}{D}$$

Change in parameters $\frac{\Delta D}{D} = V\left(\frac{-\Delta L}{L}\right)$

$$V = \frac{\text{Lateral stress}}{\text{Longitudinal strain}} = \frac{-\partial D/D}{\partial\ L/L}$$

$$= \text{Poisson's ratio}$$

$$\frac{\Delta A}{A} = -2V\frac{\Delta L}{L} \quad (2)$$

Sub (2) in (1) we get

$$\frac{\Delta R}{R} = \frac{\Delta L}{L} + 2V\frac{\Delta L}{L} + \frac{\Delta \rho}{\rho}$$

$$\frac{(\Delta R/R)}{(\Delta L/L)} = 1 + 2v + \frac{(\Delta \rho/\rho)}{(\Delta L/L)} = G$$

G (gauge factor indicates the strain sensitivity of the gauge in resistance per unit strain. Neglecting change in resistivity of the material under strained condition the gauge factor will be

$$G = 1 + 2v$$

Characteristics of Strain Gauges

1. Element should be of low temperature coefficient of resistance.
2. Should have high value of gauge factor.
3. Should have linear characteristics.
4. Should have very high resistance.
5. Should have very good frequency response.

Types of Strain Gauge

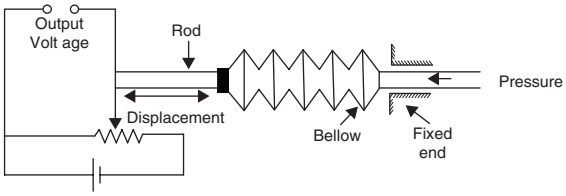
1. Wire strain gauge
2. Foil strain gauge
3. Semiconductor gauge

Measurement of Force or Pressure

Measurement of force or pressure is done by changing the applied force or pressure into displacement.

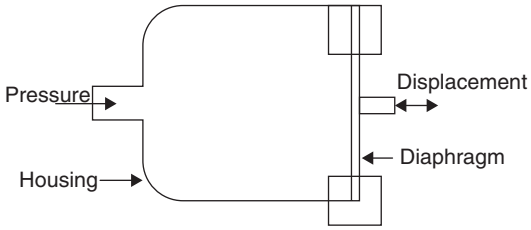
Pressure Sensors

Bellows



- These are corrugated shaped having thin walled tube. These provide high force and wide pressure range.

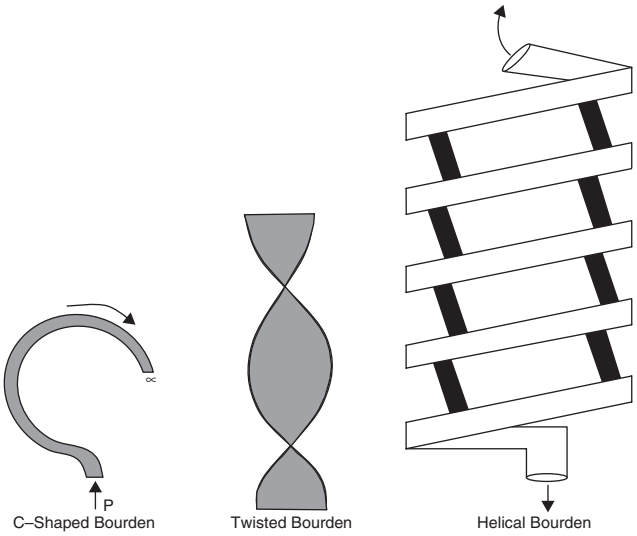
Diaphragm



- These are made of metal alloys such as bronze phosphor bronze, beryllium, copper, stainless steel, ferrous nickel alloy, etc.
- These have a high accuracy and good response.

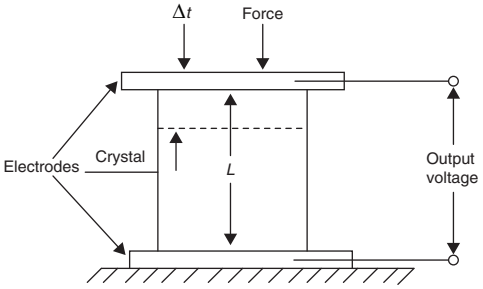
Bourdon Tube

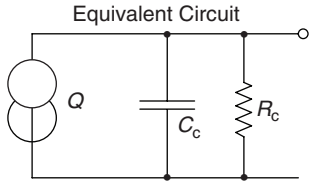
These are curved on twisted metallic tube having elliptical cross section and are made of a long thin-walled cylinder.

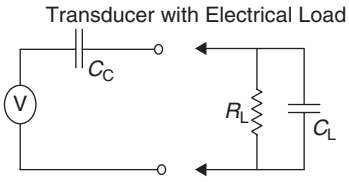


Piezo Electric Transducer

- These transducers are used for measurement of force and pressure.
- Materials used are quartz, crystal Rochelle salt, etc.
- When mechanical force is applied on to it, then due to dimensional changes, a potential difference appears across opposite faces of the materials.







- Charge Q is generated by the crystal across leakage resistance R_C and capacitance C_C.
- This is equivalent to a voltage source of $V = \frac{Q}{C_C}$ since R_C is very high it is omitted.
- Total impedance offered to voltage source

$$Z_s = \frac{1}{j\omega C_C} + \frac{R_L}{1 + j\omega C_L R_L}$$

$$= \frac{1 + j\omega R_L (C_C + C_L)}{j\omega C_C (1 + j\omega C_L R_L)}$$

$$VZ_L = V_{OUT} Z_s$$

$$V_{OUT} = \frac{Z_L}{Z_S} V$$

$$= \left[\frac{R_L}{1 + j\omega C_L R_L} \cdot \frac{j\omega C_C (1 + j\omega C_L R_L)}{1 + j\omega R_L (C_C + C_L)}\right]$$

$$= \frac{j\omega C_C R_L}{1 + j\omega R_L (C_C + R_L)} . V \quad (3)$$

$$V = \frac{Q}{C_C} = \frac{dF}{C_C} \quad (4)$$

Sub (3) in (4)

$$V_{OUT} = \left[\frac{1 + j\omega R_L C_C)}{1 + j\omega R_L (C_C + C_L)}\right] \frac{dF}{C_C}$$

$$V_{OUT} = \left[\frac{1 + j\omega R_L}{1 + j\omega R_L (C_L + C_C)}\right] dF$$

This relates output voltage V_{OUT} of transducer with applied force F.

In general for a piezoelectric transducer, if P is the pressure applied (experienced) by the crystal, g is the voltage sensitivity, t is the thickness $V_{OUT} = Pg\,t$ then (or) $P = \frac{V_{OUT}}{gt}$

$$\text{Force} = P \times A$$

where A is the cross-sectional area.

Example 5: Which among the following is an inverse transducer?
(A) Piezo electric crystals
(B) Capacitive transducer
(C) LVDT
(D) Electrical resistance potentiometer

Solution: (A).

Measurement of Temperature

Resistance Thermometers

For a change in temperature, change in electrical resistance is constant for a given conductor.

- Since electrical resistance can be determined accurately, hence the temperature. Platinum is mostly used for the measurement of temperature.

$$R_t = R_0(1 + \alpha t - \beta t^2)$$

where R_t = Resistance of resistor at t°C
R_0 = Resistance of resistor at 0°C; α and β are constants

For platinum $\alpha = 3.7 \times 10^{-3}$

$$\beta = 5.7 \times 10^{-7}$$

For small changes of temperature

$$R_t = R_0 (1 + Kt)$$

where K = Fundamental constant

$$K = \frac{R_{100} - R_0}{100\ R_0}$$

$$R_t = R_0 + \frac{R_{100} - R_0}{100} t$$

Advantages

1. Quick response
2. Good stability accuracy
3. Wide temperature range

Disadvantages

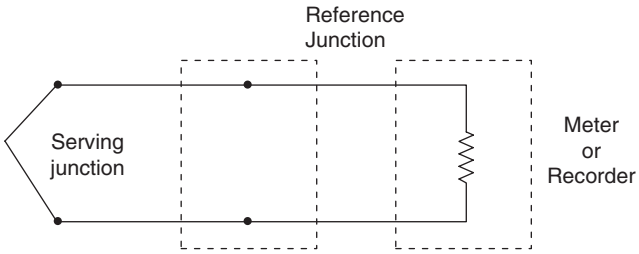
1. Expensive
2. Large in size
3. Required power supply and bridge circuit sets

Example 6: Which among the following should be the requirement of a conductor material to be used in a resistance temperature detector?

(A) The change in resistance of the material per unit change in temperature should be as large as possible.
(B) The material should have low resistivity.
(C) The resistance of the material should have a non-continuous and non-stable relationship with temperature.
(D) All of the above.

Solution: (A).

Thermocouple



- These consist of a pair of dissimilar metal wires joined together at one end and terminated at the other end which is maintained at a known constant temperature.
- When a temperature difference exists between the sensing junction and the reference junction, an emf is produced that causes a current in the circuit.

Let

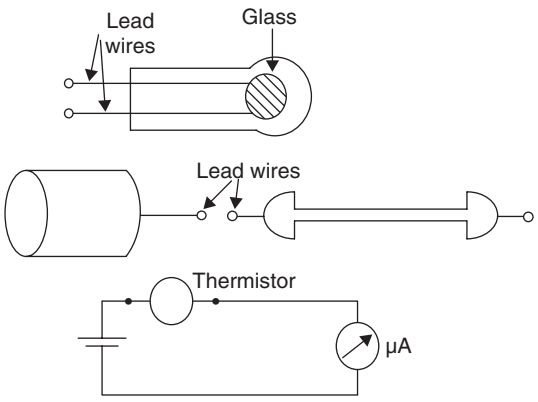
e = Thermoelectric emf in volts
T_0 = Absolute temperature of cold junction
T = Absolute temperature of hot junction
(α and β are constants)

Then

$$e = \alpha (T - T_0) + \beta (T^2 - T_0^2)$$

Thermistors

These are semiconductor with a negative temperature coefficient of resistance. They are widely used in low-temperature range applications.



The non-linear resistance temperature relation is given by

$$R_1 = R_0 \left[\beta \left(\frac{1}{T} - \frac{1}{10} \right) \right]$$

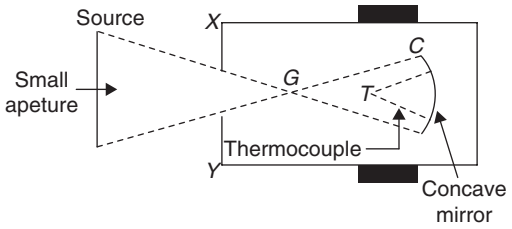
R_1 = Resistance at temperature T_1
R_0 = Resistance at temperature T_0
β = Characteristic constant of material

- Temperature is measured by connecting a thermistor in series with a battery and a micro ammeter.
- The micro ammeter is calibrated to read temperature directly in case of using constant emf battery and limited current.

Pyrometers

- These are used in the measurement of high temperature in the order of thousands.

Radiation Pyrometer



- These operate on the basis of Stefan–Boltzmann law of radiation, which states that the energy radiated by a noted black body is proportional to the fourth power of its absolute temperature

$$W = K \left(T_2^4 - T_1^4 \right)$$

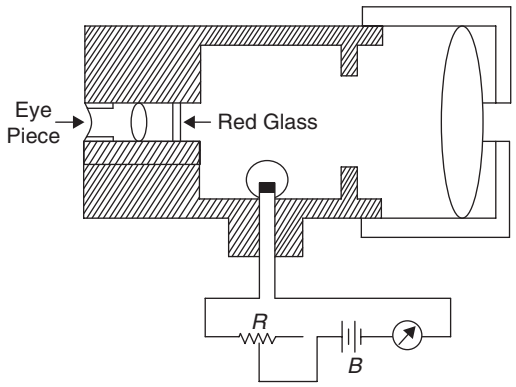
W = Energy per m^2/S needed by the cool body

K_1 = Constant = 1.279×10^{-8} gm Calories/sec/m^2

T_1 = Absolute temperature of a colder body near it
T_2 = Absolute temperature of a hot body near it

Note: Black Body: (i.e. a body for which the radiated energy for a given temperature is maximum).

Optical Pyrometers



- It consists of a standard lamp fed from source of variable voltage, galvanometer calibrated to read temperature directly by measurement of current flowing through the lamp and the lens which is used to focus light radiated from the hot body upon the plane containing lamp filament.
- The light from the hot body and that from the lamp are viewed through and eyepiece. A red glass is used to make the light monochromatic, which is very helpful in comparing brightness.
- The current flowing through the standard lamp is adjusted till the brightness of the lamp filament is equal to that of the hot body viewed through red glass by means of the eye piece.
- When brightness of both are equal, the filament disappears into the surrounding field of light from the hot body.
- The current producing the equality of brightness indicates the actual temperature of the hot body, which is given directly by the indicating instrument.

Example 7: The error observed in a transducer when the measured value deviates from the actual value by a constant factor called
(A) Zero error
(B) Hysteresis error
(C) Sensitivity error
(D) Non-conformity error

Solution: (C).

Example 8: Which among the following is an active transducer?
(A) Photo emissive cell
(B) Selsyn
(C) Strain gauge
(D) Photovoltaic cell

Solution: (D).

Example 9: For any transducer the lower limit of useful working range is determined by
(A) Dynamic response
(B) Minimum useful input level
(C) Cross sensitivity
(D) Transducer error and noise

Solution: (D).

Example 10: The choice of a transducer for a particular application should be done based on its
(A) Input characteristics only
(B) Output characteristics only
(C) Transfer characteristics only
(D) All of the above

Solution: (D).

EXERCISES

Practice Problems I

1. Which among the following statements given below is true regarding an LVDT?
(1) Exhibits linear characteristics up to a displacement of ±0.005 m
(2) Has a linearity of 0.05%
(3) Has zero resolution
(4) Has a low sensitivity
(A) 1 and 2 only (B) 1, 2 and 3 only
(C) 1 and 3 only (D) 1, 2, 3, and 4

2. A thermistor with increase in temperature exhibits
(A) Only negative change of resistance
(B) Only positive change of resistance
(C) Based on the material used, can exhibit either positive or negative change of resistance
(D) None of the above

3. In wire wound strain gauges, the change in résistance is due to
(A) Change in length and diameter
(B) Change in length only
(C) Change in diameter only
(D) Change in resistivity

4. Materials when applied with mechanical force them if generates electrostatic charge or voltage then they are called
(A) Photo resistive (B) Piezo electric
(C) Photo voltaic (D) Thermo electric

5. The principle of operation of LVDT is based on the variation of
(A) Permeance
(B) Self-inductance
(C) Reluctance
(D) Mutual Inductance

6. Two secondary voltages in an LVDT
(A) Are always in phase
(B) Vary equally based on the position of the core
(C) Vary unequally based on the position of the core
(D) Do not depend on core position

7. LVDT finds its application in
(A) Movement of human body joints
(B) Movement of fingers
(C) Movement of limbs
(D) Motion of heart walls

8. The capacitance microphone is used for the detection of
(A) The rate of heartbeat
(B) The sound of heartbeat
(C) Flow of blood
(D) Foot pressure

9. To sense which among the following quantities is fibre optic sensor used
(A) Resistance
(B) Current
(C) Power
(D) Displacement

10. A iron ring with a mean diameter of 0.2 m and a cross section of 60 mm^2 is wound with a magnetizing winding of 640 turns and a secondary winding of 440 turns. On a current reversal of 20 A in the magnetizing coil if the flux linking the search coil is 30 μWb then the relative permeability of the specimen would be
(A) 15.23 (B) 17.44
(C) 19.6 (D) 23.33

11. A specimen of iron stamping subjected to magnetic testing has the following specifications
Weight of the specimen = 5 kg
Cross-sectional area = 840 mm^2
No. of primary winding turns = 260
No. of secondary winding turns = 260
Primary winding resistance = 20 Ω
Secondary winding resistance = 40 kΩ
Ammeter reading connected on primary side = 0.175 A
Wattmeter reading connected on primary side = 40 W
Voltmeter reading connected on secondary side = 125 V
For the above test conditions, iron loss per kg of the specimen would be
(A) 79.26 W/ kg (B) 15.84 W/kg
(C) 23.79 W/kg (D) 7.926 W/kg

12. In a wattmeter method to determine the flux density of a specimen the following results were observed at 50 Hz operating frequency

Current	0.1	0.2	0.3	0.4
Voltage	49.875	99.75	149.625	199.5
Power	29.27	58.54	87.81	117.08

It was calculated that iron loss per kg is 5.14 the steel stampings are made using 102 plates of 60 mm wide × 0.978 mm thick and weighs 22 kg number of magnetizing turns in the coil are 1200, allowing a copper loss of 4. The maximum flux density in the specimen would be
(A) 0.125 Wb/m^2 (B) 1 Wb/m^2
(C) 1.25 Wb/m^2 (D) 0.125 Wb/m^2

13. In the measurement of displacement using an LVDT, for a displacement of 1 mm the LVDT reads 1 mV. The output of the LVDT is connected to a (0–10) V voltmeter through an amplifier of gain 200. The sensitivity of the instrument would be
(A) 1 mV/m (B) 1 V/m
(C) 1 V/mm (D) 0.1 V/mm

14. The parameters defined below are that of a piezoelectric transducer
(1) Change sensitivity = 8×10^{-6} C/cm
(2) Crystal capacitance = 2 nF
(3) Cable capacitance = 0.4 nF
An oscilloscope of input resistance 2 MΩ in parallel with C = 0.8 nF is used for readout. Then the voltage sensitivity constant will be
(A) 2500 V/m
(B) 2500 kV/cm
(C) 2.5 kV/cm
(D) 2.5 V/cm

15. The major disadvantages of strain gauges are
(1) Low fatigue life
(2) Poor linearity
(3) They are brittle, expensive and temperature-sensitive
(A) 1 and 2 (B) 2 and 3
(C) 1 and 3 (D) 3 only

Practice Problems 2

1. One of the major applications of bonded wire strain gauge is in
(A) Construction of transducers
(B) Pressure measurement
(C) Stress Analysis
(D) Both stress analysis and construction of transducers

2. Piezoelectric crystal (substance) among the following is/are
(1) Cadmium sulphate
(2) Barium titanate
(3) Lead titanate
(4) Lead zirconate
(A) 1, 2 and 4 (B) 2, 3 and 4
(C) 1, 2 and 3 (D) 1, 2, 3 and 4

3. In the construction of a LVDT, the windings are preferably wound on
(A) Ferrite (B) Copper core
(C) Aluminium core (D) Steel core

4. LVDT is n/a _____ transducer
(A) Resistive (B) Magnetostriction
(C) Eddy current (D) Inductive

5. Variable resistance transducer works on the principal of
(A) Resistance variation of the material by movement of magnetic field
(B) Resistance variation due to deformation
(C) Coupling of two coils changes with displacement
(D) Displacement of contact slider on a resistance

6. While using strain gauge for measurement, dummy strain gauges are used to
(A) Calibrate the strain gauge
(B) Compensate for different expansion
(C) Increase the sensitivity of the instrument
(D) Compensate temperature error

7. Frequent calibration of a DC tacho generator has to be done while they are used for measurement of shaft speed because
(A) Of heating effect due to armature currents
(B) Of the wearing off of the contacts
(C) Of weakening of magnetic field strength of the permanent magnets
(D) All of the above

8. The time constant of a thermo couple would be low
(A) When it is uncovered
(B) When it is enclosed in a case
(C) Time constant remains the same whether uncovered or enclosed
(D) None of the above

9. An average flow rate of an effluent in a pipe of 10 cm diameter is measured using an electromagnetic flow meter. The flow rate is symmetrical and uniform. The peak value of flux density in the liquid is 0.2 Wb/m^2 and the output from the flow meter electrodes is through an amplifier of gain 2000. The impedance between electrodes is 500 kΩ and the meter has an output impedance of 5 MΩ. The average flow rate of the effluent when the peak to peak voltage at the amplifier output is 0.4 V is
(A) 5.5 mm/s (B) 5.5 cm/s
(C) 5.5 m/s (D) 5.5×10^{-3} mm/s

10. Strain gauge factor of a metal of nominal resistance of 240 Ω is 4. The value of change in resistance in response to strain if it undergoes a strain of 2×10^{-5} is
(A) 0.01 Ω (B) 4×10^{-5} Ω
(C) 0.02 Ω (D) 480 Ω

11. Over a temperature range of 50°C to 100°C a thermistor has a temperature coefficient of resistance of –6%. If the resistance of the thermistor at 50°C is 200 Ω then the resistance at 65°C would be
(A) 10 Ω (B) 20 Ω
(C) 30 Ω (D) 40 Ω

12. A piezo electric crystal used to measure force has its dimensions as 10 mm × 10 mm × 3 mm and a voltage sensitivity of 0.11 V-m/N. If the voltage developed is 200 V, the force across the crystal would be
(A) 60 MN (B) 60 μN
(C) 60 KN (D) 60 N

13. In the method of measurement of linear velocity using an electromagnetic transducer, the measured value of velocity is directly proportional to
(A) The rate of change of flux
(B) The rate of change of reluctance
(C) The rate of change in the length of the air gap
(D) All of the above

14. The range of measuring temperature for which radiation pyrometers are used would be
(A) 0–500°C
(B) 1200–2500°C
(C) 500°C–1000°C
(D) –250°C–500°C

15. In Resistance Temperature Detection (RTD), platinum is the commonly used metal because
(A) Pure form of platinum is available for commercial applications, which is stable over high temperature ranges
(B) At higher temperature, the resistivity of platinum increases less rapidly
(C) For temperature ranging from 0 to 100°C, it has a constant temperature coefficient of resistance
(D) All of the above

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** C	**3.** A	**4.** B	**5.** B	**6.** C	**7.** C	**8.** B	**9.** D	**10.** C
11. D	**12.** A	**13.** B	**14.** C	**15.** B					

Practice Problems 2

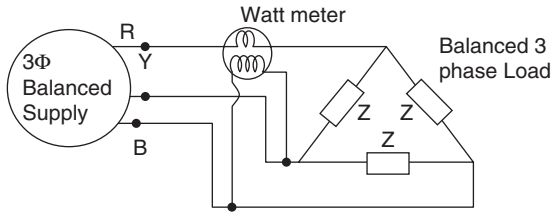
1. D	**2.** B	**3.** A	**4.** C	**5.** D	**6.** D	**7.** C	**8.** A	**9.** A	**10.** C
11. B	**12.** D	**13.** D	**14.** B	**15.** D					

TEST
Electrical and Electronic Measurements

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. A wattmeter is connected as shown in the figure. The reading of the wattmeter is equal to (ϕ is p.f. angle)



(A) $V_L I_L \cos\phi$ (B) $\sqrt{3}\, V_L I_L \cos\phi$
(C) $V_L I_L \sin\phi$ (D) $\sqrt{3}\, V_L I_L \sin\phi$

2. It is desired to measure parameters of 500/250 V, 10 KVA single phase transformer. The following wattmeters are available in a laboratory.

W_1: 500 V, 30 A, LPF

W_2: 500 V, 15 A, LPF

W_3: 250 V, 30 A, HPF

W_4: 250 V, 15 A, HPF

The wattmeters used in O.C test and S.C test of the transformer will respectively be

(A) W_1 and W_2 (B) W_2 and W_4
(C) W_1 and W_3 (D) W_2 and W_3

3. The limiting errors of measurement of power consumed by and the current passing through a resistance are ± 2% and ± 1.5% respectively. The limiting error of measurement of resistance will then be

(A) ±3.5% (B) ± 5%
(C) ± 1.5% (D) ± 1.75%

4. A galvanometer with a full scale current of 5 mA has a resistance of 200 Ω. The multiplying power (the ratio of measured current to galvanometer current) of a 20 Ω shunt with the galvanometer is

(A) 10 (B) 100
(C) 20 (D) 11

5. A PMMC voltmeter is connected across a series combination of a DC voltage source $V_1 = 3$ V and an AC voltage source $V_2(t) = 2 \sin 4t$. The meter reads

(A) 3 V (B) 2 V
(C) $3 + \frac{2}{\sqrt{2}}\, V$ (D) $\frac{2}{\pi}\, V$

6. The X- and Y-inputs of a CRO are respectively $V \sin \omega t$ and $-V \sin \omega t$. The resulting lissajous pattern will be _____.

(A) Straight line (B) A circle
(C) An ellipse (D) A figure of eight

7. A DC voltmeter has a resistivity of 1000 W/Volt. When it measures half full scale in 100 V range, the current through the voltmeter is

(A) 100 mA (B) 50 mA
(C) 25 mA (D) 10 mA

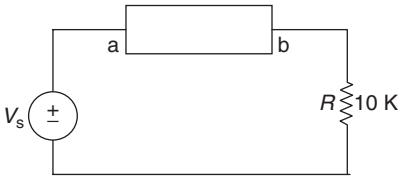
8. In a wheatstone bridge, the known resistance are correct to within ±0.1%
The accuracy to which an unknown resistance can be measured is

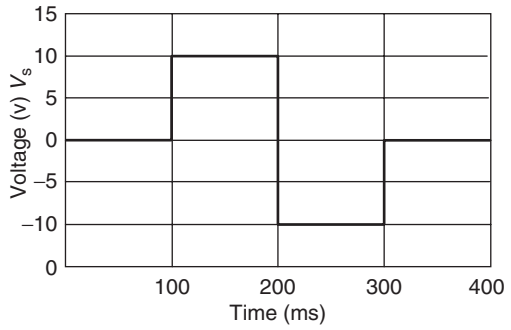
(A) ±0.4% (B) ±0.3%
(C) ±0.2% (D) ±0.1%

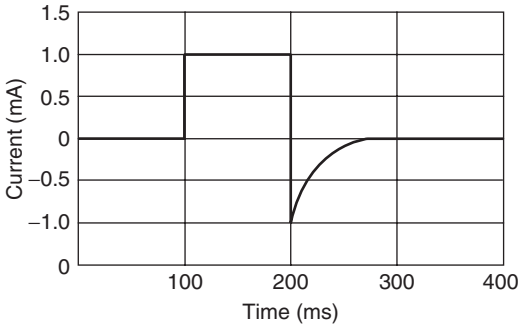
9. In a semiconductor strain gauge, the change in resistance on application of strain is mainly due to the change in its

(A) Length (B) Diameter
(C) Resistivity (D) Length and diameter

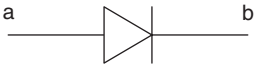
10. The following circuit has a voltage source V_s as shown in the graph. The current through the circuit is also shown.

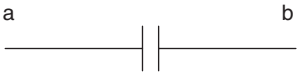


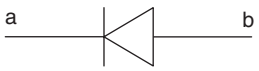


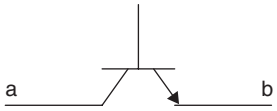


The element connected between a and b could be

(A) 

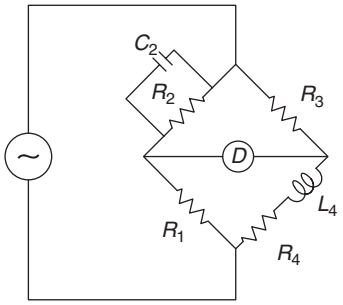
(B) 

(C) 

(D) 

11. In eddy current damping systems, the disc employed should be of
(A) Conducting and magnetic material
(B) Conducting but non-magnetic material
(C) Magnetic but non-conducting material
(D) Non-conducting and non-magnetic material

12. For the AC bridge circuit shown in the figure at balance, the value of R_4, L_4 and the quality factor of inductive coil will be respectively.



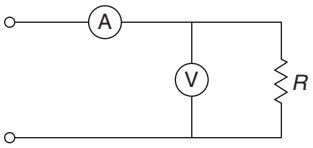
(A) $\frac{R_1}{R_3}R_2, \frac{R_1R_3}{C_2}, \omega C_2R_2$

(B) $\frac{R_2}{R_1}R_3, \frac{R_1R_3}{C_2}, \omega C_2R_2$

(C) $\frac{R_2}{R_1}R_3, R_1R_3C_2, \omega C_2R_3$

(D) $\frac{R_1}{R_2}R_3, R_1R_3C_2, \omega C_2R_2$

13. The circuit below is used to measure resistance R. The ammeter and voltmeter resistance are 0.02 Ω and 3000 Ω respectively. Their readings are 3 A and 180 V, respectively, giving a measured resistance of 60 Ω. The percentage error on the measurement is



(A) 2.04%
(B) 3.5%
(C) 4.71%
(D) 1.68%

14. Two wattmeters which are connected to measure the total power on a three phase system supplying a balanced load read 10.5 kW and –2.5 kW respectively. The total power and power factor respectively are
(A) 13.0 kW, 0.334
(B) 13.0 kW, 0.684
(C) 8.0 kW, 0.52
(D) 8.0 kW, 0.334

15. The ratio error of a given $\frac{1000}{5}A$ current transformer is zero when 10 VA, upf burden at rated current. If the secondary has 196 turns and a winding resistance of 0.04 W, the iron loss of the CT at this operating condition is
(A) 224.5 mW
(B) 324.5 mW
(C) 424.5 mW
(D) 524.5 mW

16. A resistance of 110 Ω is specified using significant figures as indicated below
(1) 110 Ω
(2) 110.0 Ω
(3) 0.000110 MΩ
for the above three indications, which one of the following statements is correct?
(A) 1 represents greater precision than 2 and 3
(B) 2 represents greater precision but 1 and 3 represent same precision
(C) 2 and 3 represent same precision
(D) 3 is more precise than 1 and 2

17. The inductance of a certain moving-iron ammeter is expressed as $L = 10 + 3\theta - \frac{\theta^2}{4}\mu H$, where θ is the deflection in radians from the zero position. The control spring torque is 25×10^{-6} Nm/radian. The deflection of the pointer in radian when the meter carries a current of 5 A, is _____.
(A) 2.4
(B) 2.0
(C) 1.2
(D) 1.0

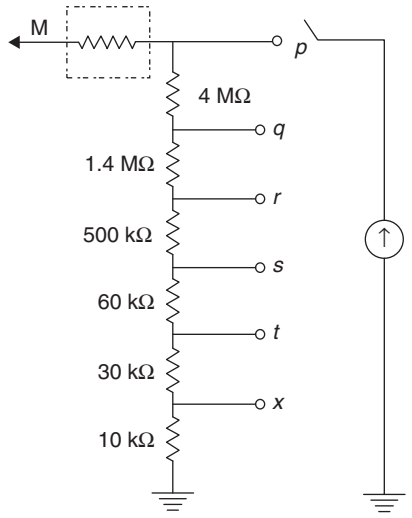
18. A bridge rectifier type instrument used for measurement of voltage, has its scale calibrated in terms of rms value of sine wave. It indicates a voltage of 3.16 V while measuring voltage of triangular wave slope. The percentage error in measurement is
(A) 4.01
(B) 3.87
(C) –3.87
(D) –4.01

19. Four ammeters M_1, M_2, M_3 and M_4 with the following specifications are available

Instrument	Type	Full scale value	Accuracy % of full scale
M_1	$3^1/_2$ digital dual slope	20	±0.10
M_2	PMMC	10	±0.20
M_3	Electro dynamic	5	±0.50
M_4	Moving Iron	1	±1

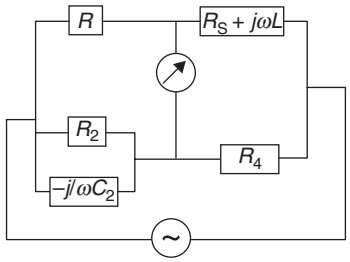
A current of 1 A is to be measured. To obtain minimum error in the reading, one should select meter
(A) M_1
(B) M_2
(C) M_3
(D) M_4

20. Given figure shows input attenuation of a multimeter. If the meter reads full scale with 10 V at '*M*' with the range switch at position '*P*', then the required voltage at '*M*' to obtain full-scale deflection with the range switch at position '*r*' will be



(A) 10 V
(B) 100 V
(C) 200 V
(D) Undeterminable because resistance of the dc probe is not known.

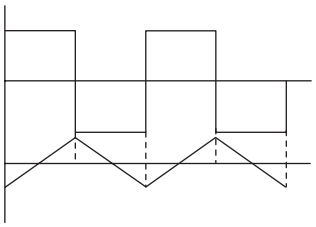
21. A balanced Maxwell's bridge is shown in the figure. The inductive coil parameters are



(A) $R = R_2R_3/R_4,\ L = R_2R_3C_4$
(B) $R = R_2R_3/R_4,\ L = RR_4C_2$
(C) $R = R_4/R_2R_3,\ L = 1/R_2R_3C_4$
(D) $R = R_4/R_2R_3,\ L = 1/RR_4C_2$

22. In a 220 V Induction watt hour metre, the phase-angle between applied voltage and the flux due to it has been changed to 92° instead of 90°. If the current flowing is 8.5 A, the errors introduced in the reading of this meter at unity and 0.5 lagging power factor are respectively.
(A) –1.1 W, 55.9 W
(B) 1.1 W, –55.9 W
(C) –1.1 W, –55.9 W
(D) 1.1 W, 55.9 W

23. For measuring average power of a load a sampling wattmeter that computes power from simultaneously sampled values of voltage and current is used. The peak to peak voltage of the square wave and the current of a triangular wave is 20 V and 10 A p–p as shown in the figure. If the period is 40ms, the reading in wattmeter W, will be



(A) 0 W
(B) 10 W
(C) 20 W
(D) 15 W

Common Data for Questions 24 and 25:
A 50 Hz bus bar primary CT has a secondary with 500 turns. The secondary supplies 5A current into a purely resistive burden of 1W. The magnetizing ampere turns is 200.

24. The phase angle between primary and secondary current is
(A) 4.6°
(B) 85.4°
(C) 94.6°
(D) 175.4°

25. The core flux in the CT under the given operating condition is
(A) 0
(B) 45.0 μWb
(C) 22.5 μWb
(D) 100 μWb

Answer Keys

1. C	**2.** D	**3.** B	**4.** D	**5.** A	**6.** A	**7.** B	**8.** B	**9.** D	**10.** A
11. B	**12.** D	**13.** A	**14.** D	**15.** A	**16.** B	**17.** C	**18.** C	**19.** D	**20.** B
21. B	**22.** A	**23.** A	**24.** A	**25.** B					

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Power Systems

UNIT VII

Chapter 1: Power System Generation, Transmission and Distribution 3.847

Chapter 2: Fault Analysis 3.905

Chapter 3: Load Flow Analysis and Protection 3.927

Chapter 4: Power Systems Stability 3.956

Chapter 5: HVDC and Facts 3.971

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Chapter 1

Power System Generation, Transmission and Distribution

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Basics of power system
- Structure of power systems
- Basic power generation concepts
- Thermal station
- Coal and ash handling plant
- Types of boilers
- Types of super heaters
- Types of condensers
- Types of hydro power stations
- Mass curve

Basics of Power System

Complex Power

The total power in a single-phase AC system at any instant is the product of the instantaneous voltage and instantaneous current. For a single-phase AC circuit, the voltage and current phasors are conventionally
written as

$$u_{an} = V_{max} \cos\omega t$$

$$i_{an} = I_{max} \cos(\omega t - \theta)$$

The instantaneous power is

$$S = u_{an}\, i_{an} = V_{max}\, I_{max} \cos\omega t \cos(\omega t - \theta)$$

The angle 'θ' is positive for current lagging the voltage and negative for leading current.

$$\text{'}S\text{'} = \frac{V_{max}\, I_{max}}{2} \cos\theta \left(1 + \cos 2\omega t\right) + \frac{V_{max}\, I_{max}}{2} \sin\theta \sin^2 \omega t$$

The above equation shows that the term containing '$\cos\theta$' is always positive and has an average value of $P = \frac{V_{max}\, I_{max}}{2} \cos\theta$

RMS value of current and voltage will give the power expression $P = |V|\,|I| \cos\theta$;

This average power 'P' is also called the real or active power. Unit for this power is watt.

The term containing $\sin\theta$ component of the instantaneous power 'S' is called the 'instantaneous reactive power'. The maximum value of the pulsating power, designated Q is called reactive power or reactive volt ampere. The units for Q is VARS (Volt Ampere reactive), the reactive power is

$$Q = \frac{V_{max}\, I_{max}}{2} \sin\theta$$

$$Q = |V||I| \sin\theta$$

The square root of the sum of the squares of P and Q is equal to the product of $|V|$ and $|I|$.

$$\sqrt{P^2 + Q^2} = \sqrt{\left(|V||I|\cos\theta\right)^2 + \left(|V||I|\sin\theta\right)^2} = |V||I|$$

In a simple series circuit, when 'Z' is equal to $R + jX$

$$P = |I|^2\, |Z|\, \cos\theta = |I|^2 R$$

$$Q = |I|^2\, |Z|\, \sin\theta = |I|^2 X$$

$\because$ $\quad |Z|\cos\theta = R$

$\because$ $\quad |Z|\sin\theta = X$

The cosine of the phase angle 'θ' between the voltage and the current is called the power factor. An inductive circuit is said to have a lagging power factor and a capacitive circuit is said to have a leading power factor. Power factor indicated whether the current

is lagging or leading the applied voltage, the power factor is therefore

$$\cos\theta = \cos\left\{\tan^{-1}\left(\frac{Q}{P}\right)\right\} \text{ or}$$

$$\cos\theta = \frac{P}{\sqrt{P^2+Q^2}}$$

Phasor representation of the complex power is given as

$$|S| = \sqrt{P^2+Q^2}$$

$$\theta = \tan^{-1}\left(\frac{Q}{P}\right)$$

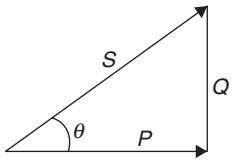


Figure 1 Power triangle of inductive load

Solved Examples

Example 1: A 15 HP, 440 V, three-phase motor is operating at full load, 85% efficiency and 90% power factor lagging, the real and reactive power drawn by the motor is

(A) 13,169 W, 14,662 VAR
(B) 13,169 W, 6365 VAR
(C) 6378 W, 13169 VAR
(D) 14,662 W, 13,169 VAR

Solution: (B)

$$\text{Current } (I) = \frac{15 \text{ HP} \times 746}{\sqrt{3}\times 440 \times 0.85 \times 0.9} = \mathbf{19.2\ A}$$

Real power drawn by the motor

$$= \sqrt{3}\times V_L \cdot I_L \cos\theta = \sqrt{3}\times 440 \times 19.2 \times 0.9$$

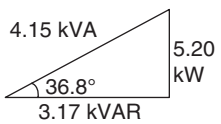
$$= 13{,}169.12 \text{ W}$$

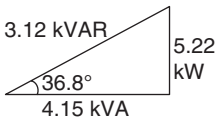
Reactive power drawn by the motor $= \sqrt{3}\, V_L\, I_L \sin\theta$

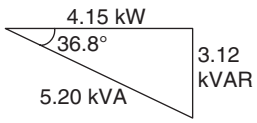
$$P = \sqrt{3}\times 440 \times 19.2 \times 0.435$$

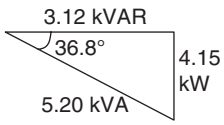
$$= 6365 \text{ VAR}$$

Example 2: A 25 kVA, 200 V, 50 Hz, 3-ϕ alternator supplies a line current of 15 A per phase at 0.8 lagging power factor and at rated voltage under balance steady-state condition. For this operating condition, the power triangle is

(A)


(B)


(C)


(D)


Solution: (C)

Real power

$$\sqrt{3}\, V_L\, I_L \cos\theta = \sqrt{3}\times 200 \times 15 \times \cos\theta = 4.15 \text{ kW}$$

Reactive power

$$\sqrt{3}\, V_L\, I_L \sin\theta = \sqrt{3}\times 200 \times 15 \times \sin\theta = 3.12 \text{ kVAR}$$

$$\text{Complex power} = \sqrt{P^2+Q^2} = 5.20 \text{ kVA}$$

Example 3: A single-phase load is supplied by a single-phase voltage source. If the current flowing from the load to the source is $10\ \angle -160^\circ$A and if the voltage at the load terminals is $100\angle 60^\circ$ then the

(A) Load absorbs both real and reactive power
(B) Load delivers both real and reactive power
(C) Load absorbs real power and delivers reactive power
(D) Load delivers real power and absorbs reactive power

Solution: (A)

$$\text{Voltage across load } (V) = 100\ \angle 60^\circ V$$

$$\text{Current entering load } (I) = -10\ \angle -160^\circ V$$

$$= 10\ \angle 20^\circ$$

Complex power $= VI$

$$= 100\angle 60^\circ \times 10\angle 20^\circ = 173.6 + j984.8$$

P and Q are positive, so load absorbs both P and Q.

Structure of Power Systems

The flow of the electric power from the generating station to the consumer is called an electric power system or electric supply systems. It consists of the following important components.

1. Generating station
2. Transmission network
3. Distribution network

The figure shows the line diagram of a typical transmission, distribution scheme and the complete flow of an electric power from generation to the consumer premises.

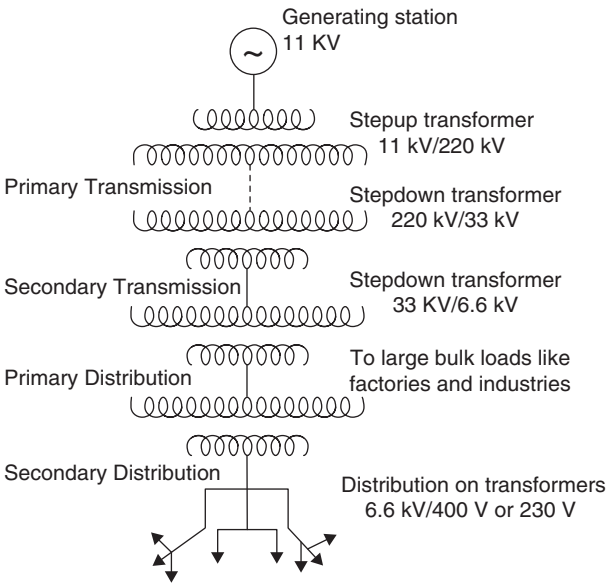


Figure 2 Line diagram of typical transmission and distribution on system

At the generating stations, an electric power is generated with the help of three-phase alternators running in parallel. In the scheme shown in figure, the voltage level is 11 kV but the voltage level can be 6.6 kV, 22 kV or 33 kV depending upon the capacity of the generating station. After generating station, actual transmission and distribution starts, the overall scheme can be divided into four sections which are,

Primary Transmission

It is basically with the help of overhead transmission lines. For the economic aspects, the voltage level is increased to 132 kV, 22 kV or more, with the help of step-up transformers. Hence, this is also called high voltage transmission.

Secondary Transmission

The primary transmission line continuous transmission towers till the receiving stations. At the receiving station, the voltage level is reduced to 22 kV or 33 kV using the step-down transformers. This is secondary transmission; the conductors used for the secondary transmission are called feeders.

Primary Distribution

At substations the voltage level is reduced to 6.6 kV, 3.3 kV or 11 kV with the help of step-down transformers. It uses three-phase three wire underground systems. And the power is further transmitted to the local distribution. For the large consumers like factories and industries, the power is directly transmitted to such loads from a substation.

Secondary Distribution

At the local centres, there are step down distribution transformers. The voltage levels of 6.6 kV, 11 kV is further reduced to 400 V using distribution transformers and service mains to the consumers. This is secondary distribution also called low voltage distribution.

Basic Power Generation Concepts

Electricity is not often generated at a power station by electromechanical generators, primarily driven by heat engines fuelled by chemical combustion or nuclear fission but also may by other means such as kinetic energy of flowing water and wind. The types of electric generation concepts are as follows:

1. Thermal power plants
2. Hydroelectric power plants
3. Nuclear power plants
4. Diesel power plants
5. Wind plants
6. Solar power generation, etc.

Thermal Station

Definition

A generating station which converts heat energy of coal combustion into electrical energy is known as steam station (thermal station).

Advantages

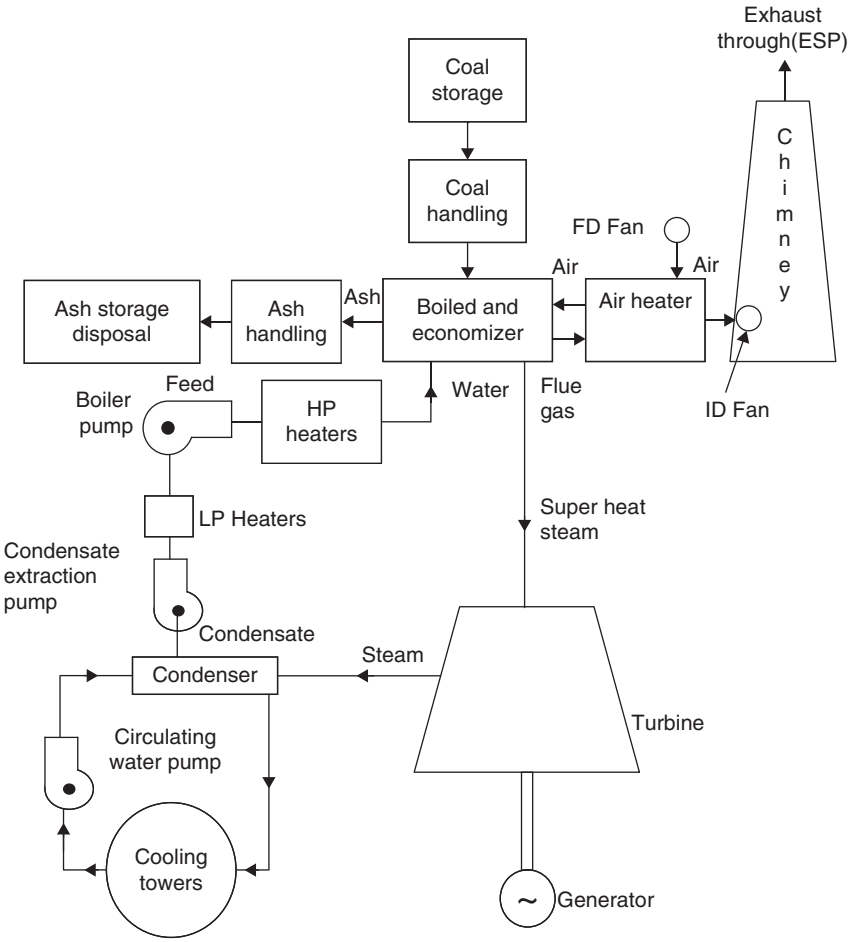
1. Less initial cost.
2. It requires less space as compared to hydroelectric power station.
3. Cost of generation is less than that of diesel power station.

Disadvantages

1. It pollutes the atmosphere due to production of large amount of smoke and fumes.
2. Its running cost is more compared to hydro electric plant.

Points to decide the selection of site for thermal power station

1. Large quantity of cooling water should be available for the condensers, etc.
2. The fuel to be used in the power station should be cheap and available nearby or ample facilities for its transportation to the site should be available.
3. The soil should be such that, it should provide good and firm foundation to the building and the plants.
4. The site should not be surrounded by residential buildings.
5. The future expansion of power station should be possible.
6. Ample accommodation for the operational and maintenance staff should be available nearby at reasonable rates.
7. Facility for the disposal of ash, etc. should be available.
8. The cost of land should be reasonable.



Schematic Arrangement of Steam Power Station

1. Coal and ash handling arrangement
2. Steam-generating plant
3. Steam turbine
4. Alternator
5. Feed water
6. Cooling arrangement

Coal and Ash Handling Plant

The coal is transported to the power station by road or rail and is stored in a coal storage plant. Storage of coal is primarily a matter of protection against coal strikes, failure of transportation system and general coal shortages.

From coal storage plant, coal is delivered to the coal handling plant where it is pulverized. The pulverized coal is fed to the boiler by conveyor belt. The coal is burnt in the boiler and the ash produced after complete combustion of coal is moved to the ash handling plant and then delivered to the ash storage plant for disposal.

Steam-generating Plant

1. **Boiler:** The heat of combustion of coal in the boiler is utilized to convert water into steam at high temperature and pressure.
2. **Superheater:** The steam produced in the boiler is wet and is passed through a superheater where it is superheated (i.e. steam temperature increased above that of boiling point of water).
   - Advantage is that efficiency is increased
3. **Economizer:** An economizer is a feed water heater, i.e. increases temperature of feed water from extracting heat of flue gases.
4. **Air Preheater:** It increases temperature of the air supplied for coal burning by delivering heat from flue gases.

Advantages

- Thermal efficiency increases.
- Steam capacity per square metre of boiler surface is increased.

Steam Turbine

The dry and superheated steam from the superheater is fed to the steam turbine and it converts into mechanical energy. After giving heat energy to the turbine, the steam is exhausted to the condenser, which condenses exhausted steam.

Alternator

The alternator converts mechanical energy into electrical energy. The electrical energy is delivered to busbars.

Feed Water

The condensate from the condenser is used as feed water to the boiler. The feed water on its way to the boiler is heated by water heaters and economizer. This helps in raising the overall efficiency of the plant.

Cooling Arrangement

Water is circulated through the condenser, which takes up heat of the exhausted steam. In case availability of water throughout the year is not assured, cooling tower is used.

Types of Boilers

1. Water tube boiler
2. Fire tube boiler

- In water tube boiler, water flows through the tubes and the hot gases of combustion flow over these tubes. Advantages are less space requirement, high pressure and less liable to explosion, etc.
- In fire tube boiler, the hot products of combustion pass through the tubes surrounded by water. Disadvantage is low pressure.

Types of Superheaters

1. Radiant superheater
2. Convection superheater

- In Radiant superheater, it is placed in the furnace between water walls and receives heat from the burning fuel through radiation process. Disadvantage is that it may get over heated and temperature of superheater falls with increase in steam output.
- In convection superheater, it is placed in the boiler tube bank and receives heat from flue gases entirely through the convection process.

Advantage is temperature of superheater increases with increase in steam output.

Types of Air-pre Heaters

1. Recuperative type
2. Regenerative type

- In recuperative type, air-heater consists of a group of steel tubes. The flue gases are passed through the tubes while air flows externally to the tubes.
- In regenerative type, air-heater consists of slowly moving drum. The flue gases flow continuously on one side of the drum and air on the other side.

Types of Condensers

1. Jet Condenser
2. Surface condenser

- In a jet condenser, cooling water and exhausted steam are mixed together. Therefore, the temperature of cooling water and condensate is the same when leaving the condenser.

Advantages

Low initial cost, less floor area required, less cooling water required and low maintenance charges.

Disadvantages

Condensate is wasted and high power is required for pumping water.

- In a surface condenser, there is no direct contact between cooling water and exhausted steam. It consists of a bank of horizontal tubes enclosed in a cast iron shell. The cooling water flows through the tubes and exhausted steam flows over the surface of the tubes.

Advantages

Condensate can be used as feed water, less pumping power required and creation of better vacuum at turbine exhaust.

Disadvantages

High initial cost, large floor area requirement and high maintenance charges.

Types of Steam Turbines

1. Impulse turbine
2. Reaction turbine

- In an impulse turbine, the steam expands completely in the stationary nozzles (or flexi blades), the pressure over the moving blades remaining constant. In doing so, the steam attains a high velocity and impinges against the moving blades.
- In reaction turbine, the steam is partially expanded in the stationary nozzles; the remaining expansion takes place during its flow over the moving blades.

Efficiency of Steam Power Station

Thermal Efficiency

The ratio of heat equivalent to the mechanical energy transmitted to the turbine shaft, to the heat of combustion of coal is known as thermal efficiency.

$$\eta_{\text{Thermal}} = \frac{\text{Heat equivalent to the mechanical energy transmited to the turbine shaft}}{\text{Heat of coal combustion}}$$

- The thermal efficiency of a modern steam power station is about 30%

Overall Efficiency

The ratio of heat equivalent of the electrical output to the heat of combustion of coal is known as overall efficiency.

$$\eta_{\text{overall}} = \frac{\text{Heat equivalent of the electrical output}}{\text{Heat of combustion of coal}}$$

- The overall efficiency of steam power station is about 29%
- Overall efficiency = Thermal efficiency × Electrical efficiency

Hydroelectric Power Station

Definition

A generating station which utilizes the potential energy of water at a high level for the generation of electrical energy is known as a hydroelectric power station.

Advantages

1. It is quiet and clean.
2. It requires less maintenance.
3. No fuel requirement.
4. It can be put into service instantly.
5. It has longer life.
6. In addition to the generation of electrical energy, they also help in irrigation and controlling floods.

Disadvantages

1. It involves high capital cost.
2. Availability of huge amount of water.
3. Skilled and experienced hands are required to build the plant.
4. Increased cost of transmission lines because of remote location of plants.
5. Long dry season may affect the delivery of power.

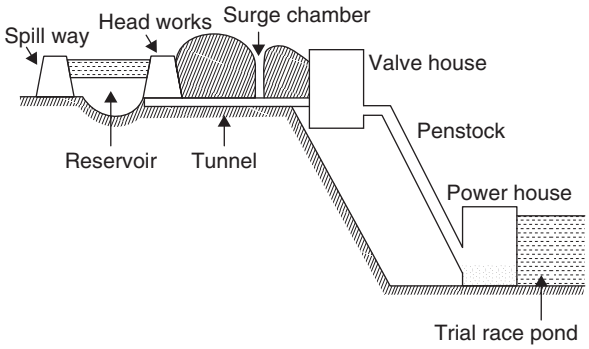


Figure 3 A typical layout of a Hydro plant

Selection of Site for Hydroelectric Station

1. Sufficient quantity of water at reasonable level should be available.
2. The site should allow for strong foundation with minimum cost.
3. No possibility of future sources of leakage of water.
4. Reservoir should have large catchment area.
5. Local supplies of sand, gravel, etc., should be available.
6. The land for the construction of the plant should be available at reasonable price.
7. The site selected for hydroelectric plant should be accessible by rail and road so that necessary equipment and machinery could be easily transported.

Components of Hydroelectric Plant

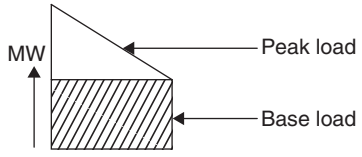
1. **Dam:** A dam is a barrier which stores water and creates water head.
2. **Spillways:** It discharges the surplus water to downstream side of the dam.
3. **Surge tank:** A surge tank is located near the beginning of the conduit. It overcomes the abnormal pressure in the conduit when load on the turbine falls. It acts as a reservoir during increase of load on the turbine.
4. **Penstocks:** Penstocks are open or closed conduits which carry water to the turbines.
   - Low-pressure penstocks
     E.g.: Cement canals
   - High pressure penstocks
     E.g.: Steel pipes
5. **Trash rock:** It is used to avoid the entry of debris into the pen stock.
6. **Forebay:** It is a small pondage where extra water is stored in it and regulates the load variations.
7. **Reservoir:** The area to store water.
8. **Turbines:** It converts energy of falling water into mechanical energy.
9. **Alternator:** It converts mechanical energy into electrical energy.

Types of Turbines

1. Impulse turbine
   It has high power output and low efficiency.
   E.g.: Pelton wheel
2. Reaction turbines
   It has low power output and high efficiency turbines.
   E.g.: Kaplan, Francis and Propeller turbines.

Types of Hydro Power Stations

- Based on the water availability
  1. Run off river plants
     (a) Without pondage
     (b) With pondage
- Based on the head available
  1. Low head (<30 m)
  2. Medium head (30–50 m)
  3. High head (>50 m)
- Based on the nature of load
  (1) Base load plant
  (2) Peak load plant



Type of Alternator used for Hydroelectric Power Station

Air cooled salient pole construction, vertical configuration, large diameter and small axial length alternator is used.

Types of turbines used for hydroelectric power station based on head

1. **High head:** Pelton wheel turbines
2. **Medium head:** Francis, propeller turbines
3. **Low head:** Kaplan turbines.

Power

$$P = \frac{0.736\, QWh\eta}{75}\ \text{kW}$$

where

Q = Discharge in m^3/s
W = Water density in 1000 kg/m^3
h = Height of water in metre
η = Efficiency

Specific Speed

With available head of 1 m to develop 1 HP power output, the speed at which the turbine will rotate is called specific speed.

$$N_s = \frac{N\sqrt{P}}{H^{\frac{5}{4}}}$$

where

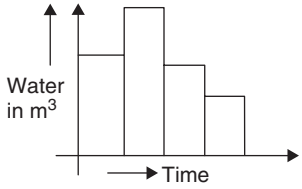
P = Power in HP
N = Actual speed in rpm
H = Head of water

$$N_s = \frac{1.145N\sqrt{P}}{H^{\frac{5}{4}}}$$

if P is in kW.

Hydrograph

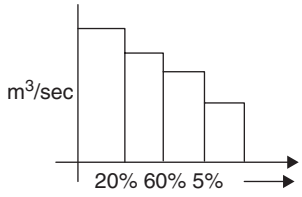
It is a graph between discharge of water in m^3 to time.



- It gives the variation of water discharge from time to time.
- The area under this graph gives the total quantity of water available at a particular period.

Flow Duration Curve

It is drawn from hydrograph.



It is drawn in descending order of water discharge.

Mass Curve

A mass curve is a graph drawn between cumulative volume of water versus time.

- A mass curve gives the storage requirement.

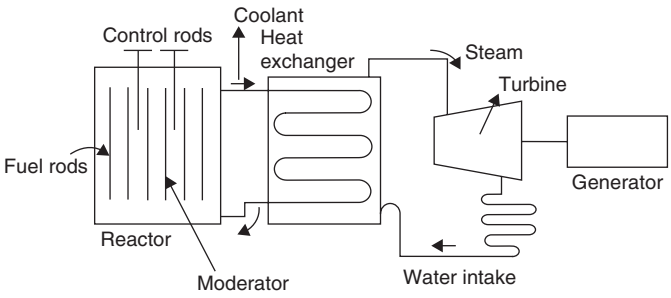
Nuclear Power Station

Definition

A generating station in which nuclear energy is converted into electrical energy is known as nuclear power station.

Advantages

1. Fuel requirement is quite small.
2. It requires less space.
3. Running charges are low.
4. It can be located near the load centres because it does not require large quantities of water and it need not be near coal mines.
5. Such plants can ensure continued supply because of large deposits of fuel available.
6. Reliability operation.



Disadvantages

1. Fuel is expensive and difficult to recover.
2. Capital cost is very high.
3. Radioactive pollution problems.
4. It requires greater technical staff.
5. Not suited for varying loads.

Selection of site for nuclear power station

1. Plant should be located where ample quantity of water is available.
2. Plant should have adequate arrangement for disposal of radioactive waste.
3. It should be quite distant from populated areas as there is a danger of presence of radioactivity in the atmosphere near the plant.
4. Transportation facilities should be available.

Schematic Arrangement of Nuclear Power Station

1. Nuclear reactor
2. Heat exchanger
3. Steam turbine
4. Alternator

Components of Nuclear Reactor

1. Reactor core: It consists of fuel rods.
2. Moderator
   - The purpose of moderator is to slow down the fast neutrons.
   - It should have low molecular weight.
   - Materials that can be used as moderator are Beryllium, Graphite, H_2O and Heavy water (D_2O).
3. Control rods: The purpose of control rods is to absorb the fast neutrons and they control the value of K (multiplication factor).
   - Materials that can be used for control rods are cadmium, boron and Haffinium.
4. Reflector: The purpose of this reflector is to bounce back escaping neutrons into reactor core.
   Materials: Reactor grade graphite
5. Shielding: The purpose of shielding is to avoid a, b, g radiations outside the reactor core.
   Materials: Stainless steel, etc.
6. Coolant: The purpose of coolant is to absorb the heat from the reactor core and exchange with water in heat exchanger. Materials are
   Liquid coolants $\rightarrow H_2O, D_2O$
   Liquid metals $\rightarrow$ Na and K
   Gases $\rightarrow CO_2, H_2$

Heat Exchanger

The coolant gives up heat to the heat exchanger which is utilized in rising the steam. After giving up heat, the coolant is again fed to the reactor.

Steam Turbine

The steam produced in the heat exchanger is led to the steam turbine through a valve. After doing a useful work in the turbine, the steam is exhausted to the condenser. The condenser condenses the steam which is fed to the heat exchanger through feed water pump.

Alternator

Alternator is a device which converts mechanical energy into electrical energy.
Multiplication factor (K)

$$K = \frac{\text{No. of neutrons released in fission}}{\text{No. of neutrons released in preceding fission}}$$

- The value of K can be controlled by control rods.
- $K > 1$, for increasing generation
  $K = 1$, for constant generation
  $K < 1$, for decreasing generation
  $K = 0$, for shut-down generation

Fission Material

It is the material used as fuel in nuclear power station.

1. Natural Uranium
   - 99.3% U^{238}
   - 0.7% U^{235}
2. Enriched Uranium
3. Thorium
4. Plutonium

- 'Enriched Uranium' is used in Indian nuclear power stations
- Fission materials can be divided into two types.
  1. Fissile materials
  2. Fertile materials
- Fissile materials are those with which chain reactions start directly.
  Ex: U_{235}, U_{233} and Pu_{239}
- Fertile materials are those with which chain reactions start indirectly.
  Ex: U^{238}, Th^{232}

Critical Mass

It is the mass to be added in the chain reaction to maintain the 'K' value at '1' only.

Comparison of Various Power Plants

S. No.	Item	Steam Power station	Hydroelectric power plant	Diesel power plant	Nuclear power plant
1.	Site	Located at water and coal available places	Located at where reservoirs can be obtained by constructing dam Example: hill areas	Located at any place	Located away from thickly populated areas
2.	Initial cost	Lower than hydro and nuclear plants	Very high	Less compared to other plants	Highest
3.	Running cost	Higher than hydro and nuclear plants	Very less	Highest	Except hydro, it has minimum running cost
4.	Efficiency	Least about 25%	About 85%	About 35%	More efficient than steam power station
5.	Maintenance cost	Quite high	Quite low	Less	Very high
6.	Transmission and distribution	Quite low	Quite high	Least	Quite low
7.	Starting	Requires lot of time	Started instantly	Started quickly	Started easily

Economic Aspects

Commonly used terms in system operation are discussed in the following sections.

Connected Load

It is the sum of the continuous ratings of all load-consuming apparatus connected to the system.

Maximum Demand

It is the highest load demand on the power system during a given period (day, month, and year).

Firm Power

It is the power intended to be always available.

Cold Reserve

It is the reserve generating capacity available for service but is not in operation.

Hot Reserve

It is the reserve generating capacity in operation but not in service.

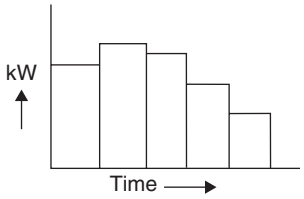
Spinning Reserve

It is the generating capacity which is connected to the bus and is ready to take load.

Factors Affecting the Cost of Generation

Load Curve

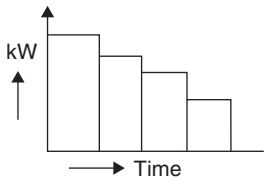
The curve showing the variation of load on power station with reference to time is known as a load curve.



- The variation of quantity w.r.t time, it is known as 'chronological curve'
- Load curve can have –ve or +ve slopes.
- The area under the curve gives the total number of units generated.
- Average demand = $\frac{\text{Total no. of units generated in a period}}{\text{No. of hours in that period}}$

Load Duration Curve

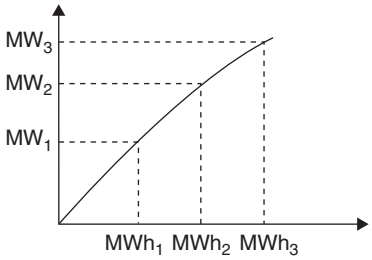
It is a graph drawn from load curve by rearranging load elements in descending order w.r.t their duration of occurrence.



- The curve will have only negative slope.
- The area under this curve gives the no. of units generated.

Integrated Load Duration Curve

It is a graph drawn between MW versus MWh.



- It gives the cumulative value of energy when load is changing.
- By using integrated load curve, we can estimate the number of units generated when the load is changing from MW_1 to MW_2.

Load Factor (LF)

It is the ratio of average demand to maximum demand

$$\text{Load factor} = \frac{\text{Average demand}}{\text{Maximum demand}}$$

- Load factor is less than 1.
- Higher is the load factor, lesser will be the cost per unit.

Demand Factor (DF)

It is the ratio of maximum demand to connected load

$$\text{Demand factor} = \frac{\text{Maximum demand}}{\text{Connected load}}$$

- It is usually less than 1.
- Low demand factor results in less capital cost.

Diversity Factor (Div.F)

$$\text{Diversity factor} = \frac{\text{Sum of individual maximum demands}}{\text{Maximum demand on power station}}$$

- It is usually more than 1 only.
- Div.F↑, Max. Demand ↓, Installed capacity ↓ cost ↓

Utilization Factor (UF)

$$\text{U.F} = \frac{\text{Maximum Demand}}{\text{Installed capacity}}$$

- It is always ≤ 1.
- If UF is high, i.e. maximum demand is high, then the power station is having proper utilization.
- Reserve capacity = Installed capacity – Maximum demand.
- If reserve capacity is zero, utilization factor is one.

Capacity Factor (CF)

$$\text{Capacity factor} = \frac{\text{Average demand}}{\text{Installed capacity}}$$

$$= \frac{\text{Actual energy produced}}{\text{Max energy that could have been produced}}$$

- $CF = LF \times UF$

Plant use Factor

$$\text{Plant use factor} = \frac{\text{Station output in kWh}}{\text{Plant capacity} \times \text{Hours of use}}$$

- Plant use factor is more than or equal to plant capacity factor.

Example 4: A generating station has a connected load of 550 MW. The maximum demand is 380 MW. The number of units generated per annum is 20×10^8. The demand factor and load factor are

(A) 0.69 and 0.60 (B) 0.60 and 0.69
(C) 1.45 and 0.60 (D) 0.69 and 1.66

Solution: (A)

$$\text{Demand factor} = \frac{\text{Maximum demand}}{\text{Connected load}}$$

$$= \frac{380}{550} = 0.69$$

Average demand

$$= \frac{\text{No. of units generated per annum}}{8760}\text{ kW}$$

$$= \frac{20 \times 10^8}{8760} = 228.31 \text{ MW}$$

$$\text{Load factor} = \frac{\text{Average demand}}{\text{Maximum demand}}$$

$$= \frac{228.31}{380} = 0.60$$

Example 5: A generating station has a maximum demand of 1000 MW. The annual load factor is 70% and plant capacity factor is 60%. The reserve capacity is

(A) 700 MW (B) 1.166 MW
(C) 167 MW (D) 1000 MW

Solution: (C)

Average demand = $0.7 \times 1000 = 700$ MW

$$\text{Plant capacity} = \frac{\text{Average demand}}{\text{Capacity factor}}$$

$$= \frac{700}{0.5} = 1166.6 \text{ MW}$$

Reserve capacity = Plant capacity – Maximum demand
= 1166.6 – 1000 = 166.6 MW ≅ 167 MW

Example 6: Which of the following stations has minimum running cost?

(A) Hydroelectric (B) Nuclear
(C) Diesel (D) Coal

Solution: (A)

No fuel cost

Example 7: For a hydroelectric power plant with an available head of 60 m, which of the following turbines is most likely to be used?

(A) Pelton wheel (B) Francis
(C) Kaplan (D) None of the above

Solution: (A)

Pelton wheels are typically used for high heads.

Power System Representation

Single Line Diagram

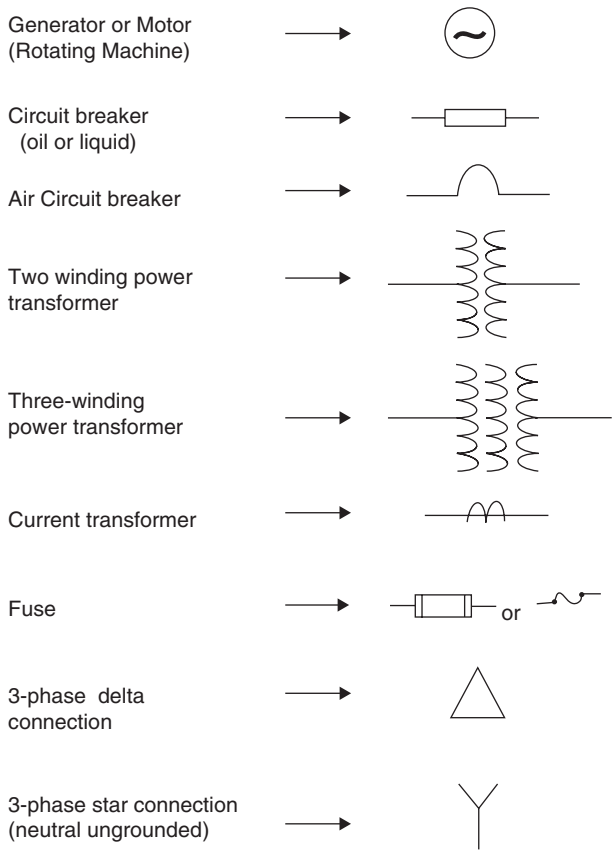


Figure 4 (*Continued*)

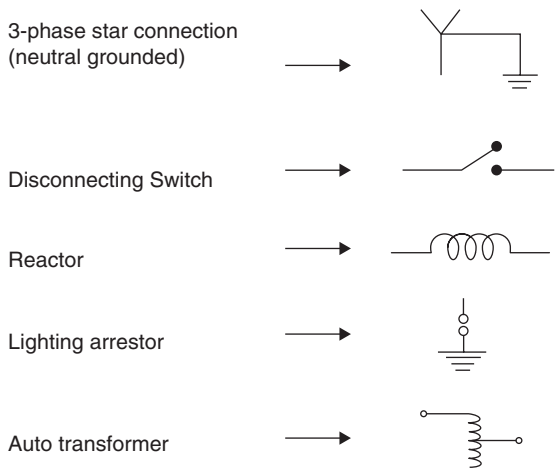


Figure 4 Symbols for representing single line diagram

Almost all modern power systems are three-phase systems with the phases of equal magnitude and equal phase difference (i.e. 120°). These three-phase balanced systems are always solved as a single-phase circuit composed of one of the three lines and neutral return and this is sufficient to give a complete analysis. Such a simplified diagram of an electric system is called a 'one-line diagram' or 'single-line diagram'. Combined with a standard set of symbols for electric components, such one-line diagrams provide a compact way to represent information. The purpose of the one-line diagram is to supply in concise form, the significant information about the system, the importance of different features of a system varies with the problem under consideration, and the amount of information included on the diagram depends on the purpose for which the diagram is intended. Figure 3 below shows the symbols for representing the components of a three-phase power system.

Above Figure 4 shows the single line diagram of an electrical power system. Two generators grounded through reactors are connected to a bus and through a step-up transformer to a transmission line. Another generator, grounded through reactor, is connected to a bus and through a transformer to the opposite end of the transmission line. A load is connected at each bus.

Impedance and Reactance Diagrams

The impedance diagram on single-phase basis for use under balanced operating conditions can be easily drawn from the online diagram. For the system of Figure 4, the impedance diagram is shown in Figure 5.

No currents flow in the ground under balanced conditions and the neutral of the generators are at the potential of the neutral of the system, so the impedance diagram does not include the current limiting impedances shown in the one line diagram between the neutral of the generator and ground.

Since the shunt current of a transformer is usually insignificant compared with the full load current, the shunt admittance is usually omitted in the equivalent circuit of the transformer.

The inductive reactance of a system is much larger than its resistance. So, the resistance is neglected in fault calculations.

Synchronous motor loads are always included in making fault calculations, since their generated emfs contribute to the short-circuit current. Induction motors are represented by a generated emf in series with an inductive reactance if the diagram is to be used to determine the current immediately after the occurrence of a fault. Induction motors are ignored in computing the current a few cycles after the fault occurrence because the current contributed by an induction

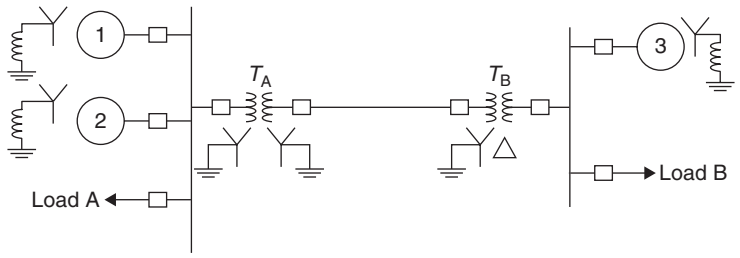


Figure 5 Single-line diagram of an electrical power system

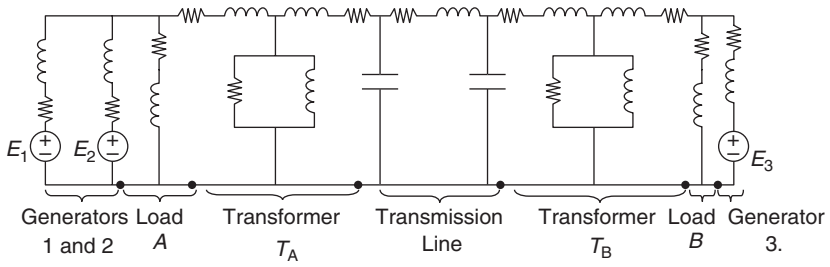


Figure 6 The per-phase impedance diagram

motor dies out very quickly after the induction motor is short circuited. Per-phase reactance diagram after neglecting all static loads, resistances, shunt admittances of each transformer and the capacitance of the transmission line is shown in Figure 6. The per-phase impedance and reactance diagrams are sometimes called the per-phase positive sequence diagram.

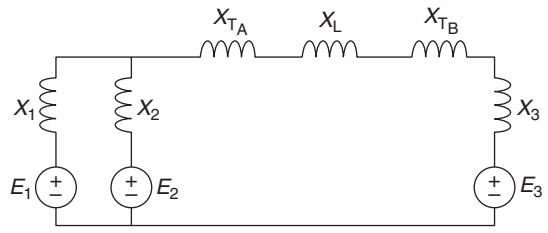


Figure 7 Simplified reactance diagram

Per Unit System

The quantities in a power system (i.e., voltage, current, volt amperes and impedance) are often expressed as a per cent or per unit of a base or reference value specified for each. The per-unit value of any quantity is defined as the ratio of the actual quantity to its base quantity.

$$\text{Per Unit Value} = \frac{\text{The actual value in any units}}{\text{The base or reference value in the same units}}$$

The ratio in per cent is 100 times the value in per unit. Both the per cent and per-unit methods of calculation are similar and more informative than the use of actual quantities. The per-unit method has advantages over per cent calculation method because the product of two quantities expressed in per unit is expressed in per unit itself, but the product of two quantities expressed in per cent must be divided by 100 to obtain the result in per cent.

There are several reasons for using a per-unit system.

1. Similar apparatus (generators, transformers, lines) will have similar per-unit impedances and losses expressed on their own ratings, regardless of their absolute size.
2. Use of the constant $\sqrt{3}$ is reduced in three-phase calculations.
3. Per-unit quantities are the same on either side of a transformer, independent of voltage level.
4. By normalizing quantities to a common base, the calculations are simplified.

A per unit system provides units for power, voltage, current and impedance. Only two of these are independent, usually power and voltage. Generally base values of power and voltage are chosen. Once the base power and the base voltage are chosen, the base current and the impedances are determined by the natural laws of electrical circuits.

The relationship between quantities in a per-unit system depends on whether the system is single phase or three phase.

Single Phase

Assuming that the independent base values are power and voltage,

Base volt amperes = S_{base} = 1 p.u.

Base voltage = V_{base} = 1 p.u.

Base active power = $P_{base} = S_{base}.\cos\phi$

Base reactive power = $Q_{base} = S_{base}.\sin\phi$

$$\text{Base current} = I_{base} = \frac{(VA)_{base}}{V_{base}} = \frac{S_{base}}{V_{base}} = 1\,\text{p.u.}$$

$$\text{Base impedance} = Z_{base} = \frac{V_{base}}{I_{base}} = \frac{V^2_{base}}{S_{base}}$$

$$\text{Base admittance} = Y_{base} = \frac{1}{Z_{base}}$$

If the actual impedance is Z (ohms), its per-unit value is given by

$$Z_{(p.u.)} = \frac{Z}{Z_{base}} = \frac{Z\,(\text{ohms}) \times (VA)_{base}}{V^2_{base}}$$

$$= \frac{Z\,(\text{ohms}) \times S_{base}}{V^2_{base}}$$

Three Phase

Power and voltage are specified in the same way as single-phase system. But the difference is that the power is specified as total power (not per phase), and voltage is line-to-line voltage.

In three-phase systems the relations

$P_{base} = S_{base} \cdot \cos\phi$ and $Q_{base} = S_{base} \sin\phi$ holds good.

The apparent power S equals

$$S_{base} = \sqrt{3}\, V_{base}\, I_{base}$$

$$I_{base} = \frac{S_{base}}{\sqrt{3}\, V_{base}} = 1\text{ p.u.}$$

$$Z_{base} = \frac{\frac{V_{base}}{\text{phase}}}{I_{base}} = \frac{V^2_{base}}{S_{base}} = 1\,\text{p.u.}$$

$$Y_{base} = \frac{1}{Z_{base}} = \frac{S_{base}}{V^2_{base}} = 1\text{ p.u.}$$

If the actual impedance is Z (ohms), its per-unit value is given by

$$Z_{p.u.} = \frac{Z}{Z_{base}} = \frac{Z\,(\text{ohms}) \times S_{base}}{V^2_{base}}$$

If the impedance has to be represented in a new base value denoted as $Z_{\text{base new}}$ (Referred to $S_{\text{base new}}$ and $V_{\text{base new}}$)

$$Z_{p.u._{new}} = Z_{p.u._{old}} \times \left(\frac{V_{\text{base old}}}{V_{\text{base-new}}}\right)^2 \times \frac{S_{\text{base-new}}}{S_{\text{base-old}}}$$

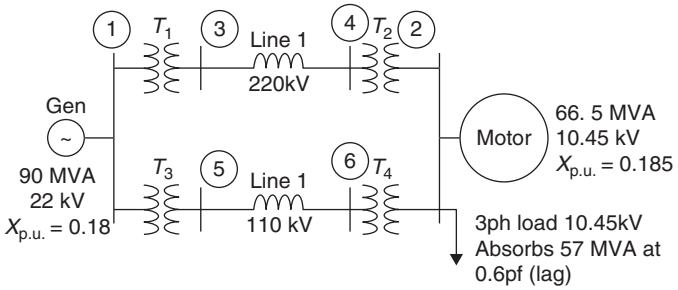
Note:

$$Z_{\text{p.u.}} \propto S_{\text{base}}$$

$$Z_{\text{p.u.}} \propto \frac{1}{V^2_{\text{base}}}$$

Example:

Single line diagram of a power system is shown in the figure. The system contains one generator, four transformers and two transmission lines. The system ratings and reactances are indicated in the figure. Draw the equivalent impedance diagram for the given system and per unit equivalent diagram with $S_{\text{base}} = 100$ MVA and $V_{\text{base}} = 22$ kV.



T_1: 50 MVA, 22/220 kV. $X_{\text{p.u.}} = 0.1$,

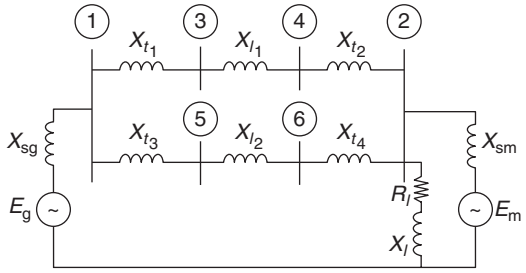
T_2: 40 MVA, 220/11 kV. $X_{\text{p.u.}} = 0.06$,

T_3: 40 MVA, 22/110 kV. $X_{\text{p.u.}} = 0.064$,

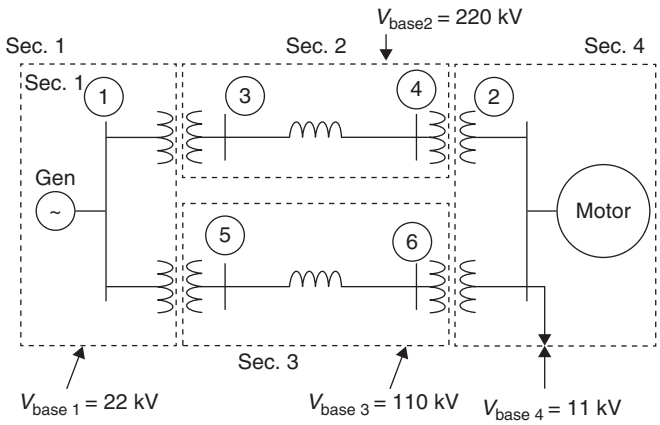
T_4: 40 MVA, 110/11 kV. $X_{\text{p.u.}} = 0.08$,

Line 1 : 48.4 Ω, Line 2 : 65.43 Ω

Reactance Diagram



Per Unit Representation



Base impedance with given base values

$$Z_{\text{base}} = \frac{V^2_{\text{base}}}{S_{\text{base}}}$$

For Section I $Z_{\text{base}_1} = \dfrac{22^2 \times 10^6}{100 \times 10^6} = 4.84\ \Omega$

For Section II $Z_{\text{base}_2} = \dfrac{220^2 \times 10^6}{100 \times 10^6} = 484\ \Omega$

For Section III $Z_{\text{base}_3} = \dfrac{110^2 \times 10^6}{100 \times 10^6} = 121\ \Omega$

For Section IV $Z_{\text{base}_4} = \dfrac{11^2 \times 10^6}{100 \times 10^6} = 1.21\ \Omega$

Per unit impedance calculations $Z_{\text{p.u.}} = \dfrac{Z_{\text{actual}}}{Z_{\text{base}}}$

$$Z_{\text{p.u. new}} = Z_{\text{p.u. old}} \times \frac{S_{\text{base new}}}{S_{\text{base old}}} \times \left(\frac{V_{\text{base old}}}{V_{\text{base new}}} \right)^2$$

For generator 1, new per unit reactance

$$X_{\text{sg}} = X_{\text{sg, old}} \times \frac{100}{90} \times \left(\frac{22}{22} \right)^2 = 0.2 \text{ p.u.}$$

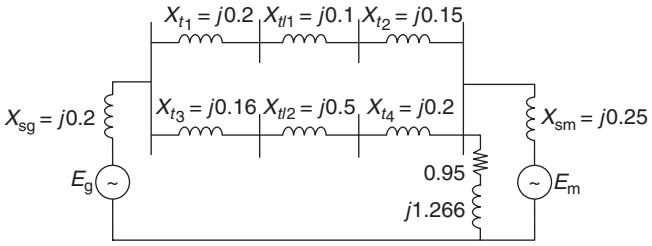


Figure 8 Per-unit impedance diagram

For Transmission line 1, $X_{l_{\text{p.u.}}} = \dfrac{48.4}{484} = 0.1$ p.u.

For Transmission line 2, $X_{l_{\text{p.u.}}} = \dfrac{65.43}{121} = 0.5$ p.u.

For Transformer 1, $X_{t_1} = 0.1 \times \dfrac{100}{50} \times \left(\dfrac{22}{22} \right)^2 = 0.2$ p.u.

For Transformer 2,

$$X_{t_2} = 0.06 \left(\frac{100}{40} \right) \left(\frac{22}{22} \right)^2 = 0.15 \text{ p.u.}$$

For transformer 3,

$$X_{t_3} = 0.064 \left(\frac{100}{40} \right) \left(\frac{220}{220} \right)^2 = 0.16 \text{ p.u.}$$

For transformer 4,

$$X_{t_4} = 0.08 \left(\frac{100}{40} \right) \left(\frac{220}{220} \right)^2 = 0.2 \text{ p.u.}$$

For motor, $X_{Sm} = 0.185\left(\frac{100}{66.5}\right)\left(\frac{10.45}{11}\right)^2 = 0.25$ p.u.

For three-phase load:
Power factor: $\cos^{-1}(0.6) = 53.13^\circ$
Load $S_{3\phi} = 57\angle 53.13$

$$Z_{act} = \frac{V^2_{rated}}{S^*} = \frac{10.45^2}{57\angle -53.13} = 1.1495 + j1.5326\ \Omega$$

$$\text{Per unit impedance of 3-}\phi\text{ load} = \frac{1.1495 + j1.532}{1.21}$$
$$= 0.95 + j1.2667 \text{ p.u.}$$

Example 8: The base impedance and base voltage of a 345 kV system are chosen to be 3000 A and 300 kV, respectively, the base impedance of the system is
(A) 115 Ω (B) 100 Ω
(C) 10 Ω (D) 0.01 Ω

Solution: (B)

$$\text{Base impedance} = \frac{300\times10^3}{3000} = \frac{V_b}{I_b}$$
$$= 100\ \Omega$$

Example 9: Let a 10 kVA, 400/200-V transformer be approximately represented by a 4 Ω reactance referred to the low-voltage side, considering the rated values as base quantities, what is the transformer reactance as a per unit quantity.
(A) 1 p.u. (B) 0.5 p.u.
(C) 0.25 p.u. (D) 1.5 p.u.

Solution: (A)

$$\text{Base impedance } (Z_B) = \frac{V^2}{S_B} = \frac{200^2}{10,000} = 4\ \Omega$$

The per unit reactance referred to the low-voltage side is

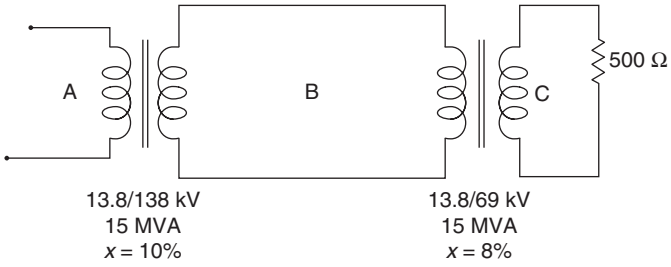
$$\text{Per unit reactance} = \frac{4}{4} = 1 \text{ p.u.}$$

Example 10: The per-unit impedance of an alternator corresponding to base values 13.2 kV and 25 MVA is 0.18 p.u. The per unit value of the impedance for base values are 13.8 kV and 60 MVA in p.u. will be
(A) 0.395 p.u.
(B) 0.472 p.u.
(C) 0.329 p.u.
(D) 0.082 p.u.

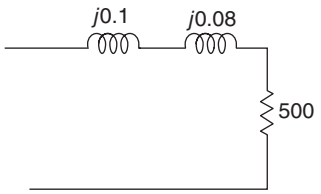
Solution: (A)

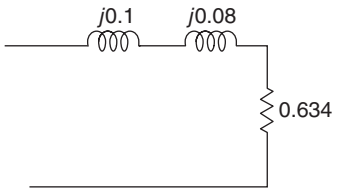
$$Z_{p.u.new} = Z_{p.u.old}\left(\frac{V_{base\ new}}{V_{base\ old}}\right)^2 \times \left(\frac{VA_{base\ old}}{VA_{base\ new}}\right)$$
$$= 0.18 \times \left(\frac{60}{25}\right)\left(\frac{13.2}{13.8}\right)^2 = 0.395 \text{ p.u.}$$

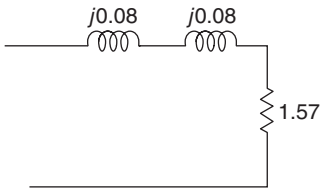
Example 11: A single-phase system is shown in figure below:

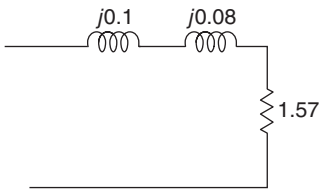


With the base in 'A' circuit chosen as 13.8 kV and 15 MVA, the impedance diagram is

(A)


(B)


(C)


(D)


Solution: (D)

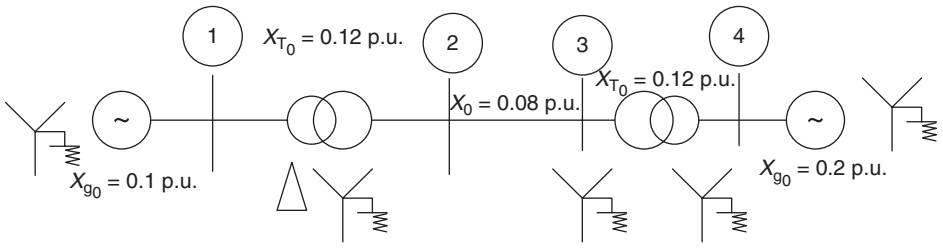
Chosen base values 13.8 kV and 15 MVA

Transformer 1 $X_{p.u.}$ = 0.1 p.u. [old and new base values are same]

Transformer 2 $X_{p.u.}$ = 0.08 p.u. [old and new values are same]

$$\text{Load } Z_{p.u.} = Z_{actual} \times \frac{S_{base}}{V^2_{base}} = 500 \times \frac{15\times10^6}{(69\times10^3)^2} = 1.57 \text{ p.u.}$$

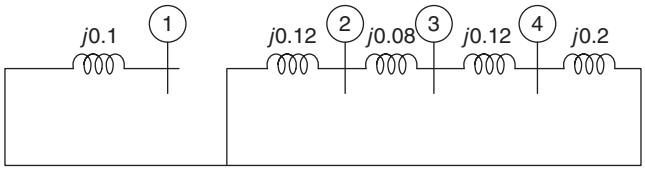
Example 12: The zero sequence reactance in p.u. are indicated in the network shown in the given below. The zero sequence driving point reactance of node-3 will be



(A) $j0.2$ p.u. (B) $j0.32$ p.u.
(C) $j0.123$ p.u. (D) $j0.52$ p.u.

Solution: (C)

Reactance diagram is given by



Driving point reactance at node 3 = $j(0.12 + 0.08) || j(0.12 + 0.2) = j\dfrac{0.2 \times 0.32}{0.52} = 0.123$ p.u.

Mechanical Design of Transmission Lines

Introduction

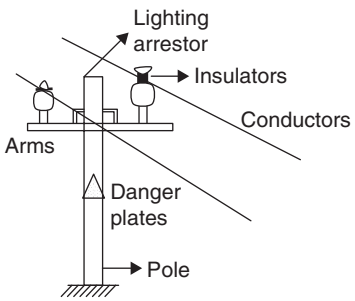
Electric power can be transmitted or distributed by means of overhead lines and underground cables, the underground cables are used rarely because of two main reasons. They are (i) high initial cost (ii) high Insulation cost. Hence, the power is transmitted by overhead lines by proper mechanical factors like strength weather conditions and other external interferences taken into consideration because of the continuity of operations in the line.

Components required for overhead lines

The successful operation of an overhead line depends to a great extent upon the mechanical design of the line.

The components of the OH lines are

1. Conductors
2. Poles
3. Insulators



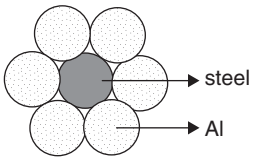
Conductor Materials

The conductor is one of the important items as most of the capital outlay is invested for it. The proper choice of material and size of the conductor is of considerable importance. The conductors should have the following properties

1. High electrical conductivity
2. High tensile strength
3. Low cost
4. Low specific gravity

All above properties are not found in single material. So, combination of different materials will give above properties.

The common materials used for overhead lines are copper, aluminium steel-cored, aluminium galvanized steel and cadmium copper. Conductors are preferably stranded in order to maintain flexibility. In stranded conductors one central wire around this some no of conductors are used based on the voltage designed according to the 6, 12, 18, 24…..wires.



If there are 'n' layers, then, $3n(n + 1) + 1$ is total no of individual layers. In the manufacture of stranded conductors, the layers of wires are twisted in opposite directions so that layers are bounded together.

Copper

Copper is an ideal material for OH lines due to its high conductivity and tensile strength.

- High electrical conductivity
- Greater tensile strength
- High current density
- High cost
- Non-availability

Specific gravity of copper is 8.9 gm/cc. Due to disadvantage of copper, they are rarely used.

Aluminium

- Aluminium is cheap and light in weight compared to copper.

- Smaller conductivity and low tensile strength.
- The conductivity of aluminium is 60% that of copper.
- The diameter of Al conductor is 1.26 times the diameter of copper for equal resistances.
- The increased cross sections of Al give greater surface to wind pressure, so supporting towards should be in greater in strength.
- Specific gravity of Al is 2.71 gm/cc.

Galvanized Steel

- Galvanized steel conductors can be used for extremely long spans.
- They are suitable in rural areas where cheapness is the main consideration.
- It has poor conductivity due to high resistance of steel.

Cadmium Copper

Conductor material in this case is copper alloyed with cadmium. Addition of 2% cadmium to copper increases 50% tensile strength and 15% reduction of conductivity. Therefore, these conductors can be useful for exceptionally long spans.

ACSR (Aluminium Conductor Steel Reinforced)

- The aluminium conductor is reinforced with a core of galvanized steel wires.
- The diameter of both steel and Aluminium is same.
- The cross section of two metals is generally in the ratio of 1:6.
- They are represented by x/y.
- Steel-cored aluminium spans can be used for towers of smaller heights.
- Steel-cored aluminium conductor will produce smaller Sag, and hence longer spans can be used with smaller tower heights.

Example 13: An ACSR conductor consists of 30 Al and 7 steel stands

Solution: $\frac{30}{7}$ Ans.

Example 14: ACSR conductors represented by 7/54

Solution: 7 steel and 54 Al

Diameter of Stranded Conductor

$D = (2n - 1)d$

$\because$ d = diameter of each strand

n = no of layers

Poles

The supporting structure for overhead lines are poles and towers.

The poles or towers should have following properties

1. High mechanical strength
2. Cheap in cost
3. Longer life
4. Light in weight

Different types of poles

1. Wooden poles
2. Steel poles and steel towers
3. RCC (Reinforced concrete) poles

Insulators

The overhead line conductors are supported on the poles in such a way that the current should not flow to earth by poles, so insulators are properly fixed to conductor to provide necessary insulation between line conductors and support.
The Insulators should have following properties

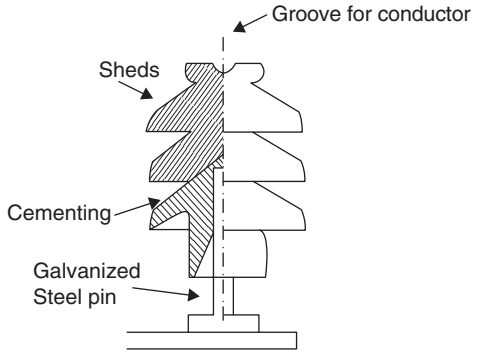
1. High mechanical strength
2. High relative permittivity
3. High electrical

Different types of Insulators

1. Pin type
2. Suspension type
3. Strain type
4. Shackle type

Pin-type Insulator

- The pin type is secured (fixed) to cross arm on the pole.
- Pin-type insulators are used for transmission and distribution of electrical power at voltage up to 33 kV.



Causes of Insulator Failure

- Insulators are required to withstand both mechanical and electrical stress.
- The electrical breakdown of the insulator can occur either by flash – over or puncture.

Flashover: an arc occurs between the live conductor and insulator pin.
Puncture: the discharge occurs from the conductor to pin through the body of the insulator.

$$\text{Safety factor of insulator} = \frac{\text{Puncture strength}}{\text{Flash} - \text{Overvoltage}}$$

Mechanical–Electrical

1. Flashover: Arc occurs between line conductor and insulator or pin (i.e. Earth) and the discharge jumps over air gap, following the shortest path. After flashover,

insulator continues to work properly, unless destroyed by the heat produced during flashover.

2. Puncture: Discharge focus from pin to the body of insulator. This destroys the insulator and it must be replaced.

Suspension-type Insulator

- Suspension-type insulator consists of a number of porcelain discs connected in series by metal links in the form of a string. The conductor is suspended at the bottom end of this string while the other end of the string is secured to the cross arm of the tower.
- Each disc is designed for low voltage 11 kV. For example, 6 discs → 66 kV.
- These insulators are cheaper than pin-type insulators for voltages beyond 33 kV.
- We can replace the damaged disc from the whole string and we can add the desired number of discs.
- These insulators are used with steel towers.

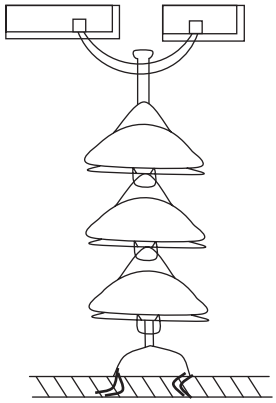
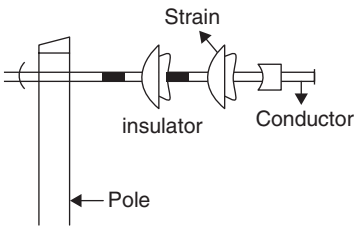


Figure 9 Suspension-type insulators

Strain Insulators

When there is a dead end of the line or there is corner or sharp curve, the line is subjected to greater tension.
Strain insulator is used for low voltages line (<11 KV).

- By assembling suspension insulators, we can use as the strain insulator.



Potential Distribution Over a String of Insulators

A string of suspension insulators consist of a number of porcelain discs connected in series through metallic links.

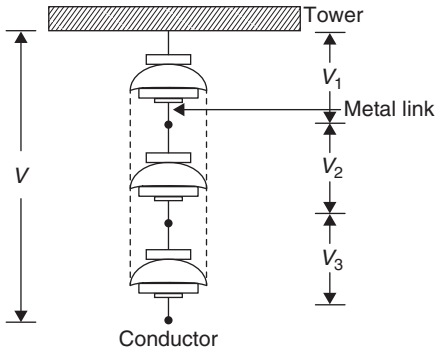


Figure 10 A typical suspension-type Insulator

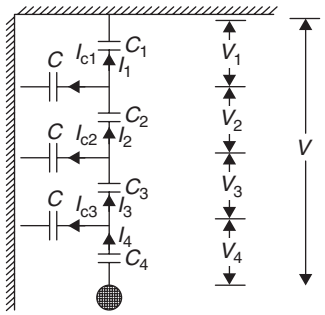


Figure 11 Potential distribution across a string of Insulators

Porcelain portion of each disc between two metallic links forms a capacitance. This capacitance is known as self-capacitance. If there were mutual capacitance alone, the potential distribution across each unit would have been the same. But in actual case, capacitance also exists between metal fitting and tower or earth. This capacitance is known as shunt capacitance. Due to the shunt capacitance, charging current is not the same through all the discs of the string. Therefore, voltage across each disc will be different.
Presence of self-capacitance and shunt capacitance results in the

1. Non-uniform potential distribution of voltage across the individual discs of suspension-type insulators.
2. The disc nearest to the conductor has maximum voltage across it.
3. The disc nearest to the conductor is under maximum electrical stress and is likely to be punctured.
4. Since the insulator capacitances are ineffective for DC, the voltage across each disc would be same for DC voltage across the string.

The unequal potential distribution is generally expressed in terms of string efficiency:

String Efficiency

The ratio of voltage across the whole string to the product of discs and the voltage across the disc nearest to the conductor is known as 'string efficiency'.

$$\text{String efficiency} = \frac{\text{Voltage across the string}}{n \times \text{Voltage across disc nearest to conductor}}$$

n = no. of discs in the string.

$$\text{Let } K = \frac{\text{Capacitance per insulator}}{\text{Capacitance to ground}} = \frac{KC}{C}$$

where $C_1 = C_2 = C_3 = C_4 = KC$

Let V be the operating voltage (Line to ground) and V_1, V_2, V_3 and V_4 the voltage drops across the units as shown in the figure.

$$V = V_1 + V_2 + V_3 + V_4.$$

From the figure, we can write

$$I_2 = I_1 + I_{c1}$$

$\omega kcV_2 = \omega kcV_1 + \omega cV_1$ ($\because \omega$ = supply frequency)

$$V_2 = V_1\left(\frac{1+K}{K}\right) = V_1\left(1+\frac{1}{K}\right).$$

$$V_3 = V_1\left(1+\frac{3}{K}+\frac{1}{K^2}\right)$$

$$V_4 = V_1\left(1+\frac{6}{K}+\frac{5}{K^2}+\frac{1}{K^3}\right).$$

V_1 can be expressed in terms of V and from this V_2, V_3, etc. can be obtained normally.

If $k = 5$, then $V_2 = 1.2\ V_1$, $V_3 = 1.64\ V_1$, $V_4 = 2.40\ V_1$

$$\therefore \quad V_1 < V_2 < V_3 < V_4.$$

As 'k' increases, the division of voltage becomes uniform. The 100% string efficiency can be achieved by increasing 'K' value till the potential distribution across discs become uniform.

Methods of Improving String Efficiency

1. Selection of K
2. Grading of units (discs)
3. Static shielding

Selection of K

Performance of string (string efficiency) depends on the value of 'K'. As K is increased the division of voltage becomes more equalized. To increase the 'K' value, longer cross arm for the tower to be used.

Grading of Units

Insulators of different dimensions are chosen so that each disc has different capacitance. The insulators are capacitance graded such a way that the top unit has the minimum capacitance and the capacitance increases progressively as the bottom unit is reached. This method has a disadvantage that a large number of different sized insulators are required.

Static Shielding

The string efficiency in this method is improved by providing grading in the form of a large metal ring surrounding the bottom unit and connected to the metal work at the bottom of this unit and therefore to the line. The guard ring introduces capacitance between the metal fittings and the line conductor. The guard ring is connected in such a way that shunt capacitance currents i_{c_1}, i_{c_2}, i_{c_3}, etc. are equal to metal fitting line Capacitance currents i_{m_1}, i_{m_2}, i_{m_3}, etc. The result is that same charging current 'I' flows through each unit of string. Consequently, there will be uniform potential distribution across the units.

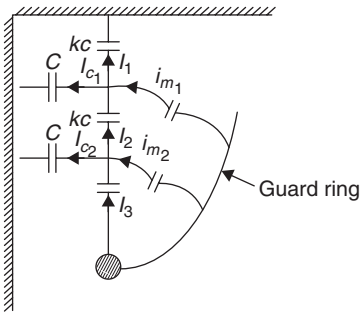


Figure 12 Static shielding

Common Data for Examples

A 3-ϕ overhead transmission line is being supported by three disc insulators. The potential drop across top unit and middle unit are 10 kV and 12 kV, respectively.

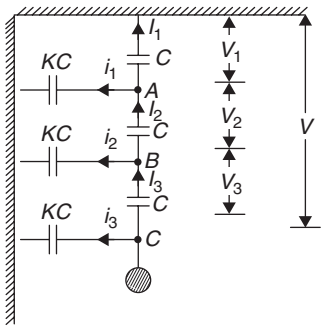
Example 15: The ratio of capacitance between pin and earth to self-capacitance of each unit is

(A) 0.2 (B) 5

(C) 0.833 (D) 1.2

Solution: (A)

The equivalent circuit of string insulators is shown below



Conductor location

Let 'K' be the ratio of capacitance between pin and earth to self-capacitance.

'C' is self-capacitance of each unit.

Capacitance between pin to earth = KC

Apply KCL at node A

$$I_2 = I_1 + i_1$$

$$V_2\ \omega c = V_1\ \omega c + V_1\ k\ \omega c$$

$$V_2 = V_1(1 + k)$$

$$k = \frac{V_2 - V_1}{V_1} = \frac{12-10}{10} = 0.2$$

Example 16: The string efficiency of the system is
(A) 10.6% (B) 12%
(C) 78% (D) 92%

Solution: (C)

From the given data,

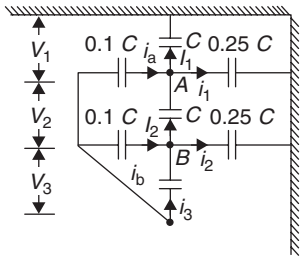
$$V_1 = 10 \text{ kV}, V_2 = 12 \text{ kV}$$
$$V_3 = V_2 + (V_1 + V_2)\, k$$
$$= 12 + (10 + 12) \times 0.2 = 16.4 \text{ kV}$$

Voltage between line and earth $= V_1 + V_2 + V_3$
$= 38.4$ kV

$$\text{String efficiency} = \frac{\text{Voltage across string}}{\text{No. of insulators} \times V_3} \times 100$$

$$= \frac{38.4}{3 \times 16.4} \times 100 = 78\%$$

Example 17: For the following figure, the voltage across middle insulator as a percentage of line voltage to earth



(A) 29.85% (B) 40.65%
(C) 70.15% (D) 59.35%

Solution: (A)

Applying Kirchhoff's law at node 'A'

$$I_2 + i_a = I_1 + i_1$$
$$V_2 \omega c + 0.1\ \omega c (V_2 + V_3) = V_1\ \omega c + 0.25 V_1\ \omega c$$
$$V_2 + 0.1 V_2 + 0.1 V_3 = 1.25 V_1$$
$$1.1 V_2 + 0.1 V_3 = 1.25 V_1$$

At node 'B' $I_3 + i_b = I_2 + i_2$

$$0.25 V_2 + 1.25 V_2 = 1.1 V_3$$

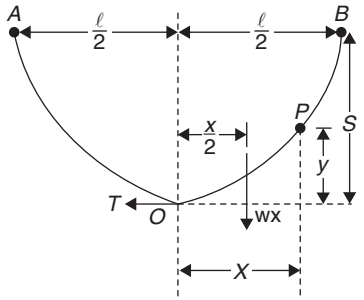
From the above two equations, we have

$$V_1 = 0.988\ V_2$$
$$V_3 = 1.362\ V_2$$
$$V = V_1 + V_2 + V_3$$
$$V = 0.988\ V_2 + V_2 + 1.362\ V_2 = 3.35\ V_2$$
$$V_2 = \frac{1}{3.35} V = 029.85\ V$$
$$V_2 = 29.85\% \text{ of } V$$

Sag Calculation in OH Lines

While erecting an overhead line, it is very important that conductors are under safe tension. In order to permit safe tension in the conductors, they are not fully stretched but are allowed to have a dip or sag.

The difference in level between points of supports and the lowest point on the conductor is called sag.

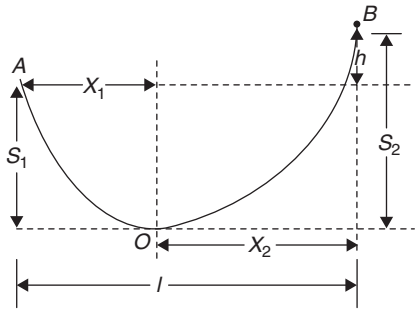


Consider a conductor between two equilateral supports A and B with 'O' as the lowest point as shown above.

$$\text{Sag } S = \frac{Wl^2}{8T}$$

where l = Length of span
W = Weight per unit length of conductor
T = Tension in the conductor

Let A and B be at different levels form the ground.



$$S_1 = \frac{Wx_1^2}{2T} \qquad \text{where } x_1 = \frac{l}{2} - \frac{Th}{Wl}$$

$$S_2 = \frac{Wx_2^2}{2T} \qquad \text{where } x_2 = \frac{l}{2} + \frac{Th}{Wl}$$

where h = Difference in levels between two supports.

$$S_2 - S_1 = \frac{Wl}{2T}\left[x_2 - x_1\right].$$

and $$l = x_2 + x_1$$

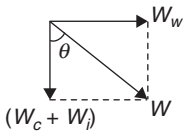
Effect of Wind and Ice

The effect of ice covering (w_i) is to increase the weight of the conductor (w_c) and thus increases vertical sag. To guard against this, the tension on the conductors at the time of erection is suitably decreased.

If the wind pressure is also considered, the wind pressure acting on the projected area of the conductor per metre converted to local should be added to the weight. If W_w is wind pressure.

Resultant weight (per conductor)

$$W = \sqrt{(W_c + W_i) + W_w^2}$$



Example 18: If the ultimate strength is 5750 kg, sag is 2 m and factor of safety is 2 and the overhead line has a span of 250 m. The weight of the conductor is

(A) 0.736 kg (B) 184 kg
(C) 92 kg (D) 23 kg

Solution: (B)

Allowable maximum tension

$$(T) = \frac{\text{Ultimate strength}}{\text{Safety factor}} = \frac{5750}{2} = 2875\,\text{kg}$$

Total sag(s) = 2 m

$$S = \frac{WL^2}{8T}$$

$$W = \frac{8ST}{L^2} = \frac{2\times 8\times 2875}{(250)^2}$$

$$W = 0.737\ \text{kg/m}$$

Half-span length = $l + \frac{w^2 l^2}{6T^2}$

$$= \frac{250}{2} + \frac{(0.737)^2\left(\frac{250}{2}\right)^2}{6\times(2875)^2} = 125\ \text{m}$$

Total length = 250 m

Weight of conductor = (0.737 × 250) kg
= 184 kg

Example 19: An overhead line conductor is subjected to a horizontal wind load of 1.78 kg/m and vertical ice loading of 1.08 kg/m. If the maximum possible sag is 6 m. Safety factor is 2. Weight of conductor is 0.844 kg/m^2. The tension is 7950 kg. The permissible span between two supports is

(A) 135 m (B) 275 m
(C) 13 m (D) 27 m

Solution: (B)

Total weight $w = \sqrt{(w_c + w_i)^2 + w_w^2}$

$$= \sqrt{(0.844 + 1.08)^2 + (1.78)^2}$$

$$= 2.621\ \text{kg/m}$$

Maximum possible sag $D = \frac{wl^2}{2T}$

$$6 = \frac{2.621\times l^2}{2\times(7950/2)}$$

$$l = \sqrt{\frac{12\times 3975}{2.621}} = 135\ \text{m}$$

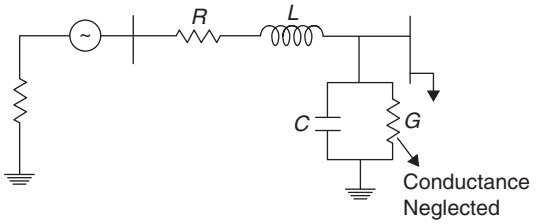
The span length = 2 × 135 = 270 m.

Electrical Design of Overhead Lines

An AC transmission has resistance inductance and capacitance uniformly distributed along its length. These four are called parameters. These parameters will determine the efficiency of transmission system.

Parameters of Transmission line:

1. Resistance (R)
2. Inductance (L)
3. Capacitance (C)
4. Conductance (G)



The transmission line consists of a series combination of resistance, inductance and a parallel combination of capacitance, conductance. Conductance is normally neglected due to leakage over line insulation is almost always in overhead transmission line.

The series line parameters, i.e., inductance resistance are uniformly distributed along the line and they together form series impedance. Inductance is the most dominant line parameter for a power system. It is the inductive reactance which limits the transmission capacity of line.

Resistance

It opposes the flow of current in a line conductor. It is the cause for power loss in the Transmission line



$$R = \frac{\rho l}{a}$$

R = Resistance
ρ = Resistivity
l = Length of conductor
a = Area of conductor

The variation of resistance of metallic conductor with temperature is practically linear.

$$R_2 = R_1\,[(1 + \alpha_1(t_2 - t_1)]$$

$$\alpha_1 = \frac{\alpha_0}{1 + \alpha_0 t_1}$$

α_0 = temperature coefficient

Skin Effect

An alternating current flowing through the conductor does not distribute uniformly. Rather it has the tendency to concentrate near the surface of the conductor. This effect is known as skin effect.

The skin effect depends on

1. Nature of material
2. Diameter of wire
3. Frequency
4. shape of wire

When frequency < 50 Hz and conductor diameters < 1 cm, skin effect can be negligible.

Flux Linkages

Inductance of a circuit is defined as the flux linkages per unit current.

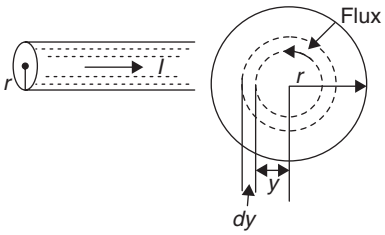
There are two important cases of flux linkages.

Flux linkages due to a single current carrying conductor

The magnetic lines of force will exist inside the conductor and outside of the conductor. Contribution of both fluxes will form the inductance

For example

A long straight cylindrical conductor of radius 'r' metre and carrying a current of 'I' Amperes



Internal Flux

Figure shows the cross-sectional view of a long cylindrical conductor carrying current I

The mmf around a concentric closed circular path of radius (internal) of the conductor is

[$\because$ H_y = magnetic field intensity] (AT/m)

$\oint H_y \, ds = I_y$ (i) (Amperes law), where

I_y = current enclosed (A)

By symmetry, H_y is constant and is in direction of ds all along the circular path. Therefore from equation (1)

$$2\pi y \, H_y = I_y \tag{2}$$

Assuming uniform current density

$$I_y = \left(\frac{\pi y^2}{\pi r^2}\right) I = \left(\frac{y^2}{r^2}\right) I \tag{3}$$

From Equations (2) and (3), we obtain

$$H_y = \frac{yI}{2\pi r^2} \text{ AT/m} \tag{4}$$

The flux density (B_y) at 'y' metre from the centre of conductor is

$$B_y = \mu H_y = \frac{\mu y I}{2\pi r^2} \text{ Wb/m}^2 \tag{5}$$

where μ is the permeability of the conductor.

Consider now an infinitesimal tabular element of thickness dy and length 1 metre. The flux in the tabular element ($d\phi = B_y dy$) weber links the fractional turn $\frac{\ell y}{\ell} = \frac{y_2}{r_2}$ resulting in flux linkages of

$$d\lambda = \left(\frac{y^2}{r^2}\right) d\phi = \left(\frac{y^2}{r^2}\right) \frac{\mu y I}{2\pi r^2} \, dy \text{ Weber-turns} \tag{6}$$

Integrating, we get the total internal flux linkages as

$$\lambda_{\text{int}} = \int_0^r \frac{\mu I}{2\pi r^4} y^3 dy = \frac{\mu I}{8\pi} \text{ Wb-T/m}$$

For relative permeability $\mu_r = 1$ (non-magnetic conductor)

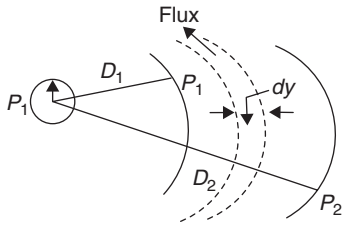
$$\mu = 4\pi \times 10^{-7} \text{ Wb-T/m}$$

and

$$L_{\text{int}} = \frac{1}{2} \times 10^{-7} \text{ H/m}$$

Flux Linkage Due to Flux Between Two Points External to Conductor

Figure shows two points P_1 and P_2, at distance D_1 and D_2 from a conductor which carries current of I amperes. The magnetic field external to the conductor is concentric circles around the conductor and therefore all the flux between P_1 and P_2 lies within the concentric cylindrical surfaces passing through P_1 and P_2.



Magnetic field intensity at distance y from the conductor is

$$H_y = \frac{1}{2\pi y} \text{ AT/m}$$

The flux $d\phi$ contained in the tabular element of thickness dy is

$$d\phi = \frac{\mu I}{2\pi y} dy \text{ Wb}$$

The flux $d\phi$ links all the current in the conductor once and only once.

Flux linkages,

$$d\lambda = 1 \times d\phi = \frac{\mu I}{2\pi y} dy$$

Total flux linkages of the conductor due to flux between points

$$\lambda_{12} = \int_{D_1}^{D_2} \frac{\mu I}{2\pi y} dy = \frac{\mu}{2\pi} \ln \frac{D_2}{D_1} \text{ Wb-turns/m}$$

where 'ln' stands for natural logarithm.

Since $\mu_r = 1$ $\mu = 4\pi \times 10^{-7}$

$$\lambda_{12} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \text{ Wb-turns/m}$$

The inductance of the conductor by the flux included between points P_1 and P_2 is then

$$L_{12} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \text{ H/m}$$

Or

$$L_{12} = 0.461 \log \frac{D_2}{D_1} \text{ mH/km}$$

Flux linkages due to External flux:

Let the external point be at distance D from the centre of the conductor, due to external flux, these linkages can be is obtained by substituting $D_1 = r$ and $D_2 = D$, i.e.

$$\lambda_{ext} = 2 \times 10^{-7} \ln \frac{D}{r}$$

Total flux linkages of the conductor due to internal and external flux are

$$\lambda = \lambda_{int} + \lambda_{ent}$$

$$= \frac{I}{2} \times 10^{-7} + 2 \times 10^{-7} \ln \frac{D}{r}$$

$$= 2 \times 10^{-7} \left(\frac{1}{4} + \ln \frac{D}{r} \right) = 2 \times 10^{-7} \ln \frac{D}{re^{-1/4}}$$

$$= 2 \times 10^{-7} \ln \frac{D}{r'}$$

where $r' = re^{-1/4} = 0.77888\, r$

$$\lambda = 2 \times 10^{-7} \ln \frac{D}{r'} \text{ Wb-T/m}$$

Inductance of the conductor due to flux up to an external point

$$L = 2 \times 10^{-7} \ln \left(\frac{D}{r'} \right) \text{H/m}$$

Inductance (*L*)

- It opposes sudden changes in currents. The voltage induced can be obtained by

$$e = \frac{d\phi}{dt} \text{ V}$$

ψ = flux linkages

$$e = \frac{d\phi}{di} \cdot \frac{di}{dt} V = L \frac{di}{dt} V$$

where $L = \frac{d\phi}{di}$ is defined as the inductance of circuit in henry

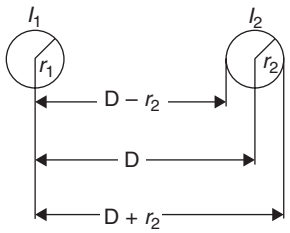
$$L = \frac{\phi}{i} \text{ Henry}$$

y = Li (Wb-T)

If the current is alternative

$$\lambda = LI$$

Inductance of a single-phase two wire line



The flux linkage caused by current in Conductor 1 is given by

$$\lambda_1 = 2 \times 10^{-7} \ln \frac{D}{r'}$$

Inductance of the conductor due to current in conductor 1 only is

$$L_1 = 2 \times 10^{-7} \ln \frac{D}{r_1'}$$

Due to current in conductor 2 only is

$$L_2 = 2 \times 10^{-7} \ln \frac{D}{r_2'}$$

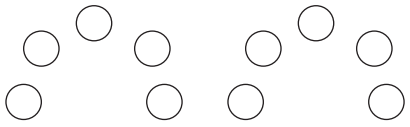
$$L = L_1 + L_2 = 4 \times 10^{-7} \ln \frac{D}{\sqrt{r_1' r_2'}} \text{ H/m}$$

If $r_1' = r_2' = r'$; then

$L = 4 \times 10^{-7} \ln D/r'$ H/m

$L = 0.921 \ln D/r'$mH/m.

Inductance of Composite Conductor Live



$$\lambda_i = 2 \times 10^{-7} \frac{I}{n} \left(\ln \frac{1}{D_{i_1}} + \ln \frac{1}{D_{i_2}} + \ldots\ldots + \ln \frac{1}{D_{ii}} \ldots . \ln \frac{1}{D_{in}} \right)$$

$$= 2 \times 10^{-7} \frac{I}{m_1} \left(\ln \frac{1}{D_{i_1}} + \ln \frac{1}{D_{i_2}} + \ldots . in \frac{1}{D_{im_1}} \right)$$

$$= 2 \times 10^{-7} \text{ I} \ln \frac{\left(D_{i_1} \ldots\ldots D_{i_2} \ldots\ldots D_{in'} \right)^{1/m'}}{\left(D_{i_1} \; D_{i_2} \; D_{il'} \; D_{in} \right)^{1/n}} \text{ Wb-T/m}$$

The inductance of filament is then

$$L_i = \frac{\lambda_i}{I/n} = 2n\times10^{-7} L_n \frac{\left(D_{il'}.....D_{ij'}.....D_{im'}\right)}{\left(D_{i_1}\ D_{i_2}....D_{ii}.....D_{in}\right)} \text{H/m}$$

$$L_{avg} = \frac{L_1 + l_2 + l_3 + ...L_n}{n}$$

∵ Conductor A is composed of n filaments electrically in parallel

$$L_A = \frac{L_{avg}}{n} = \frac{L_1 + l_2 + ...L_n}{n^2}$$

$$L_A = 2\times10^{-7} L_n \frac{\left[\left(D_{1i'}....D_{1j'}....D_{1m'}\right)...\left(D_{il'}...D_{1j'}..D_{im'}\right)...\left(D_{nl'}...D_{nj'}...D_{nm'}\right)\right]^{1/m'n}}{\left[\left(D_{1i}'....D_{1j}....Dn'\right)...\left(D_{il}'...D_{1j}'...Dn'\right)...\left(D_{nl}'...D_{nj}...Dn\right)\right]^{1/n^2}}$$

$$L_A = 2\times10^{-7} \ln\times\left(\frac{\text{Products of all possible mutual inductances}}{\text{Geometric mean radius}}\right)$$

$$L_A = 2\times10^{-7} L_n \frac{\text{DM}}{\text{DSA}} \text{H/m} = 0.461\, L_{ong} \frac{\text{DM}}{\text{DSA}} \text{mH/km}$$

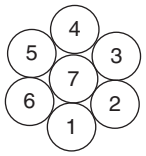
Total inductance of line

$$L = L_A + L_B$$

Problems on Self-GMD

Example 20: A conductor is composed of seven (7) identical copper strands each having a radius r, as shown in Figure 1. Find the self-GMD of the conductor.

$D_{14} = 4r$
$D_{12} = 2r$
$D_{26} = 2\sqrt{3}\, r$



Solution: The self-GMD of seven strand conductor is the 49th root of the 49 distances. Thus,

$$D_S = \left((r')^7 \left(D_{12}^2\, D_{26}^2\, D_{14}\, D_{17}\right)^6 \left(2r\right)^6\right)^{1/49}$$

Substituting the value of various distances,

$$D_S = \left(\left(0.7788r\right)^7 \left(2^2 r^2 \times 3 \times 2^2 r^2 \times 2^2 r^2 \times 2^2 r \times 2r \times 2r\right)^6\right)^{1/49}$$

$$D_S = \frac{2r\left(3\left(0.7788\right)\right)^{1/7}}{6^{1/49}}$$

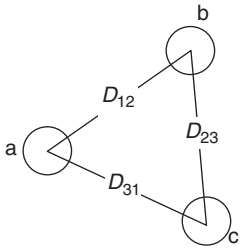
$$D_S = 2.177r$$

Inductance of Three-phase Lines

The basic equations developed can however be easily adapted to the calculation of the inductance of three-phase lines.

Figure shows conductor of a three-phase live

Unsymmetrical spacing:



Practically transposition is difficult but mathematical it can be assumed.

$$L/\text{phase} = 2\times10^{-7} \ln\left(\frac{\text{GMD}}{\text{GMR}}\right) \text{H/M}$$

$$L/\text{phase} = 2\times10^{-7} \ln\left(\frac{\text{DM}}{\text{DSA}}\right) \text{H/M}$$

$$\text{GMD or DM} = \sqrt[3]{XYZ} \text{ if } D_{12} = X,\ D_{23} = Y,\ D_{31} = Z$$

GMR or DSA $= r'$

Symmetrical lines

$$X = Y = Z = d \text{ (equilateral spacing)}$$

$$L/\text{phase} = 0.2 \text{ Ln}\left(\frac{\text{GMD}}{\text{GMR}}\right) \text{mH/km}$$

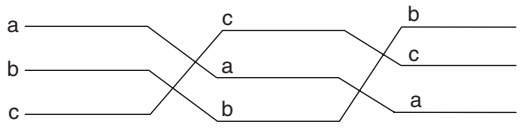
GMD $= d$
GMR $= r'$

Inductive reactance

$$X = 2\pi FL$$

Unsymmetrical Spacing

When three-phase line conductors are not equidistant from each other, the conductor spacing is said to be unsymmetrical. So, flux linkages are different. Interchanging the positions of the conductor at regular intervals along the line such an exchange of positions is known as transposition.



$I_a + I_b + I_c = 0$

$I_a = I\,(1 + j0)$
$I_b = I\,(-0.5 - j0.866)$
$I_c = I\,(-0.5 + j0.866)$

$$\lambda_{a1} = 2\times10^{-7}\left(I_a \ln\frac{1}{r_a'} + I_b \ln\frac{1}{D_{12}} + I_c \ln\frac{1}{D_{31}}\right) \text{Wb-T/m}$$

$$\lambda_{a2} = 2\times10^{-7}\left(I_a \ln\frac{1}{r_a'} + I_b \ln\frac{1}{D_{23}} + I_c \ln\frac{1}{D_{12}}\right) \text{Wb-T/m}$$

$$\lambda_{a3} = 2 \times 10^{-7} \text{b} \left(I_a \ln \frac{1}{r_a'} + I_b \ln \frac{1}{D_{31}} + I_c \ln \frac{1}{D_{23}} \right) \text{Wb-T/m}$$

Average flux linkages of conductor 'a' are

$$\lambda_a = \frac{\lambda_{a_1} + \lambda_{a_2} + \lambda_{a_3}}{3}$$

$$= 2 \times 10^{-7} \left(I_a \ln \frac{1}{r_a} + I_b \ln \frac{1}{(D_{12} D_{23} D_{31})^{1/3}} + I_c \ln \frac{1}{(D_{12} D_{23} D_{31})^{1/3}} \right)$$

$$I_b + I_c = -I_a$$

$$\lambda_0 = 2 \times 10^{-7} I_a \ln \frac{(D_{12} D_{23} D_{31})^{1/3}}{r_a'}$$

$D_{eq} = (D_{12} D_{23} D_{31})^{1/3}$ = equivalent equilateral spacing

$$L_a = 2 \times 10^{-7} \ln \frac{D_{eq}}{r_a'} = 2 \times 10^{-7} \ln \frac{D_{eq}}{D_S} \text{ H/m}$$

$D_m = D_{eq}$

$r_a = r_b = r_c$

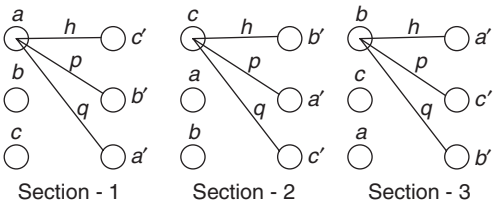
$L_a = L_b = L_c$

where D_s = self-GMD, D_m = Mutual GMD

Double-circuit Three-phase Line

The double-circuit three-phase line is built so as to increase transmission reliability, at enhanced cost.

The three sections of the transposition cycle of two parallel circuit three-phase line with vertical spacing are shown below.



$D_{eq} = (D_{ab} D_{bc} D_{ca})^{1/3}$

D_{ab} = mutual GMD between phases a and b in Section 1 of transposition cycle

$= (D_P \ D_P)^{1/4} = (D_P)^{1/2}$

D_{bc} = mutual GMD between phases b and c in Section 1 of transposition cycle

$= (D_p)^{1/2}$

D_{ca} = mutual GMD between phases a and c in Section 1 of transposition cycle

$= (2Dh)^{1/2}$

$D_{eq} = 2^{1/6} \ D^{1/2} \ p^{1/3} \ h^{1/6}$

$D_{sa} = (r'q\,r'q)^{1/4} = (r'q)^{1/2}$

$D_{sb} = (r'h\,r'h)^{1/4} = (r'h)^{1/2}$

$D_{sc} = (r'q\,r'q)^{1/4} = (r'q)^{1/2}$

Equivalent self-GMD $D_s = (D_{sa} D_{sb} D_{sc})^{1/3}$

$= (r')^{1/2} \ q^{1/3} \ h^{1/6}$

$$L = 2 \times 10^{-7} \ln \left[2^{1/6} \left(\frac{D}{r'} \right)^{1/2} \left(\frac{p}{q} \right)^{1/3} \right] \text{H/phrase/m.}$$

Mutual inductance between the two circuits

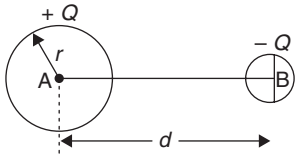
$$M = 2 \times 10^{-7} \ln \left(\frac{p}{q} \right)^{2/3}$$

$M \to 0$. $\therefore$ mutual impedance is zero.

$$L = 1 \times 10^{-7} \ln \frac{3\sqrt{2}D}{r'}$$

Capacitance

The capacitance and conductance together forms the shunt admittance of a transmission line. When an alternating voltage is applied to the line, the line capacitance draws a leading sinusoidal current called the charging current which is drawn even when the line is open circuited at the far end.



$$V_A = \int_r^\infty \frac{Q}{2\pi x \varepsilon_0} dx + \int_d^\infty \frac{-Q}{2\pi x \varepsilon_0} dx$$

$$= \frac{Q}{2\pi\varepsilon_0} \left(\log_e \frac{\infty}{r} - \log_e \frac{\infty}{d} \right) \text{V}$$

$$= \frac{Q}{2\pi\varepsilon_0} \log_e \frac{d}{r} \text{V}$$

Potential difference between conductor B and central 'infinite'

$$V_B = \int_r^\infty \frac{-Q}{2\pi x \varepsilon_0} dx + \int_d^\infty \frac{Q}{2\pi x \varepsilon_0} dx$$

$$= \frac{-Q}{2\pi\varepsilon_0} \left[\log_e \frac{d}{r} \right] \text{V}$$

$$= \frac{-Q}{2\pi\varepsilon_0}\log_e\frac{d}{r}\text{V}$$

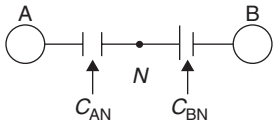
$$V_{AB} = 2V_A = \frac{2Q}{2\pi\varepsilon_0}\log_e\frac{d}{r}\text{V}$$

$$C_{AB} = Q/V_{AB} = \frac{Q}{\frac{2Q}{2\pi\varepsilon_0}\log_e\frac{d}{r}}\text{F/m}$$

$$C_{AB} = \frac{\pi\varepsilon_0}{\log_e\frac{d}{r}}\text{F/m}$$

Capacitance to Neutral

The aim is to know the capacitance between one of the conductors and a neutral point between them. Since potential of the midpoint between the conductors is zero, the potential difference between each conductor and ground or neutral is half the potential differences between the conductors.



Capacitance of neutral

$$C_N = C_{AN} = C_{BN} = 2C_{AB}$$

$$C_N = \frac{2\pi\varepsilon_0}{\log e\frac{d}{r}}\text{F/m.}$$

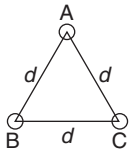
Capacitance of a Three-phase

The capacitance of each conductor is considered instead of capacitance from conductor to conductor
Two cases:

1. Symmetrical spacing
2. Unsymmetrical spacing

Symmetrical Spacing

The three conductors A, B and C are having charges of Q_A, Q_B and Q_C per metre (equidistant of d metre) from each other, the capacitance from line conductor to central in the symmetrical spaced line.



$$V_A = \int_r^\infty \frac{Q_A}{2\pi x\varepsilon_0}dx + \int_d^\infty \frac{Q_B}{2\pi X\varepsilon_0} + \int_d^\infty \frac{Q_C}{2\pi X\varepsilon_0}dx$$

$$= \frac{1}{2\pi\varepsilon_0}\left(Q_A\log_e\frac{1}{r} + Q_B\log_e\frac{1}{d} + Q_C\log_e\frac{1}{d}\right)$$

$$= \frac{1}{2\pi\varepsilon_0}\left[Q_A\log_e\frac{1}{r} + (Q_B + Q_C)\log_e\frac{1}{d}\right]$$

Assume balanced supply, then

$$Q_A + Q_B + Q_C = 0$$

$$Q_B + Q_C = -Q_A$$

$$V_A = \frac{1}{2\pi\varepsilon_0}\left(Q_A\log_e\frac{1}{r} - Q_A\log_e\frac{1}{d}\right) = \frac{Q_A}{2\pi\varepsilon_0}\log_e\frac{d}{r}\text{V}$$

Capacitance of conductor A with respect to neutral,

$$C_A = \frac{Q_A}{V_A} = \frac{Q_A}{\frac{Q_A}{2\pi\varepsilon_0}\log_e\frac{d}{r}}\text{F/M} = \frac{2\pi\varepsilon_0}{\log_e\frac{d}{r}}\text{F/m}$$

Unsymmetrical Spacing

Three-phase transposed line having unsymmetrical spacing for balanced condition $Q_A + Q_B + Q_C = 0$
Capacitance from conductor to neutral is

$$C_A = \frac{Q_A}{V_A} = \frac{2\pi\varepsilon_0}{\log_e\frac{\sqrt[3]{d_1d_2d_3}}{r}}\text{F/m}$$

Example 21: A single-phase transmission line has two parallel conductor **4 m** apart, radius of each conductor being 2 cm. calculate the capacitance of line per km .given that $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Solution: Conductor radius = 2 cm
Spacing of conductors = 4 m = 400 cm
Capacitance of the line

$$= \frac{\pi\varepsilon_0}{\log_e\frac{d}{r}}\text{F/m}$$

$$= \frac{\pi\times 8.85\times 10^{-12}}{\log_e\frac{400}{2}}\text{F/m}$$

$$= 5.247 \times 10^{-12}\text{ F/m}$$

$$= 5.247 \times 10^{-3}\,\mu\text{F/km}$$

Example 22: A three-phase overhead transmission line has its conductors arranged at the corners of an equilateral triangle of 8 m side. Calculate the capacitance of each line conductor per km. diameter of each conductor is 2.5 cm.

Solution: Conductor radius $r = 2.5/2 = 1.25$ cm
Spacing of conductors $d = 8$ m = 800 cm
Capacitance of each line conductor

$$= \frac{2\pi\varepsilon_0}{\ln\frac{d}{r}}\text{F/m} = \frac{2\times\pi\times 8.854\times 10^{-12}}{\ln\left(\frac{800}{1.25}\right)}\text{F/m}$$

$$= 8.609703 \times 10^{-12}\text{ F/m}$$

$$= 8.6097\times 10^{-3}\,\mu\text{F/km}$$

Example 23: Calculate the capacitance of a 400 km long three-phase 50 Hz overhead transmission line consisting of 3 conductors of each diameter 2 cm and spaced 4 m at the corner of an equilateral triangle.

Solution: Equilateral spacing $d = 4$ m $= 400$ cm

$R = 2/2 = 1$ cm

Capacitance of each conductor to neutral

$$\frac{2\pi\varepsilon_0}{\log_e \frac{d}{r}} = \frac{2\pi\varepsilon_0}{\log_e 400/1}\text{ F/m} = \frac{2\pi \times 8.854\times10^{-12}}{\log_e 400/1}\text{ F/m}$$

$$= 9.28 \times 10^{-9}\text{ F/km} = 9.28 \times 10^{-3}\ \mu\text{F/km}$$

Example 24: A long two-wire line composed of solid ground conductors is 0.5 cm and the distance between their centres is 2 m. If this distance is double, then the inductance per unit length.

(A) Halves
(B) Increases but does not double
(C) Doubles
(D) Decreases but does not halve

Solution: (B)

Inductance of conductors per phase $= 0.2 \ln\left(\frac{d}{r'}\right)$ mH/km

$$L_1/\text{phase} = 0.2 \ln\left(\frac{2}{0.778\times0.5\times10^{-2}}\right)\text{mH/km}$$

After distance is doubled

$$L_2/\text{phase} = 0.2 \ln\left(\frac{4}{0.778\times0.5\times10^{-2}}\right)\text{mH/km}$$

L_2/phase $> L_1$/phase but L_2/phase $\neq 2L_1$/phase

Inductance per unit length has increased but not doubled

Example 25: The conductor of 15 km long, single-phase, two-wire line is separated by a distance of 1.5 m. The diameter of each conductor is 0.75 cm. If the conductors are of copper, the inductance of the circuit is

(A) 2.22 mH (B) 22.8 mH
(C) 1.10 mH (D) 11.0 mH

Solution: (B)

Inductance of 1-ϕ circuit $= 2L$ per phase

$$= 2 \times 0.2 \ln\frac{d}{r'}$$

Distance (d) = 1.5 m,

Radius (r') $= 0.778 \times 0.75 \times 10^{-2}$m

$$L = 0.4 \ln\left(\frac{1.5}{0.778\times0.75\times10^{-2}}\right)\text{mH/km}$$

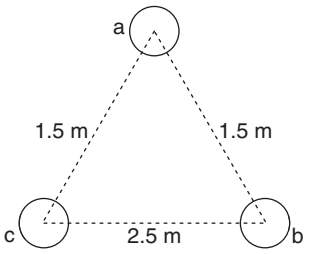
$$= 2.2\text{L mH/km}$$

Length of the line is 10 km

Total inductance $= L \times 10$ km

$L = 22.2$ mH

Example 26: A 100 km long three-phase transmission lines are shown in the figure. The diameter of each conductor is 0.5 cm. The value of the inductance per phase is



(A) 1.36 mH (B) 136 mH
(C) 6.817 mH (D) 681 mH

Solution: (B)

Inductance per phase $= 2 \times 10^{-7}$. $\ln\frac{\text{GMD}}{\text{GMR}}$

$$\text{GMD} = \sqrt[3]{D_{ab}D_{bc}D_{ca}} = \sqrt[3]{1.5\times1.5\times2.5} = 1.778\text{ m}$$

$$\text{GMR} = 0.778\, r = 0.778 \times 0.25\text{ cm}$$

$$= 0.1945 \times 10^{-2}\text{ m}$$

$$\text{Inductance per phase} = 0.2 \ln\frac{\text{GMD}}{\text{GMR}}\text{mH/km}$$

$$= 0.2\ln\left(\frac{1.778}{0.1945\times10^{-2}}\right)$$

$$= 1.36\text{ mH/km}$$

Total inductance per phase $= 1.36 \times 100$ km $= 136$ mH

Example 27: A 125 km long transmission line is loaded at 110 kV. If the loss of line is 20 MW and the load is 175 MVA the resistance of the line is

(A) 7.9 Ω/phase (B) 4.56 Ω/phase
(C) 13.68 Ω/phase (D) 45.6 Ω/phase

Solution: (A)

Given line loss expression is given by

$$P_{loss} = I_L^2 R$$

$$I_L = \frac{P_L}{V} = \frac{175\times10^6}{110\times10^3} = 1.6\text{ kA}$$

$$P_{loss} = 20 \times 10^6 = I_L^2 R = \left(1.6\times10^3\right)^2 R$$

$$R = 7.9\ \Omega\text{/phase}$$

Common Data Questions

Example 28: A 100 km long, three-phase, 110 kV 50 Hz overhead transmission line has three conductors each diameter 1.5 cm. The conductors are spaced 2 m at the corners of equilateral triangle. The capacitance of the line is

(A) $1 \times 10^{-3}\ \mu$F (B) 2.29 F
(C) 1.5 F (D) 1.5 μF

Solution: (A)

Capacitance of the line (C) $= \frac{2\pi \in_0}{\ln\frac{d}{r}}$

$$= \frac{2\pi\times8.854\times10^{-12}}{\ln\left(\frac{200}{0.75}\right)} = 10\times10^{-12}\text{ F/m}$$

Capacitance of 100 km line = $10 \times 100 \times 10^{-12}$

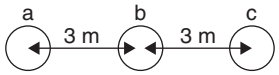
$$C = 1 \times 10^{-9}$$

Example 29: The capacitance to neutral is

(A) 8.4 μF/km (B) 0.0119 μF/km

(C) 8.4 F/km (D) 0.0084 F/km

Solution: (B)



$$C_n = \frac{0.242}{\ln \frac{D_{eq}}{r}}$$

$$D_{eq} = (3 \times 3 \times 6)^{1/3} = 3.78 \text{ m}$$

$$C_n = \frac{0.0242}{\ln \frac{3.78}{0.5}} = 0.0119 \ \mu\text{F/km}$$

Example 30: The changing current per kilometre line is

(A) 0.379 A/km (B) 0.167 A/m

(C) 0.379 A/m (D) 0.23 A/km

Solution: (D)

$$X_n = \frac{1}{\omega C_n} = \frac{1}{2 \times \pi \times 50 \times 0.0119 \times 10^{-6}}$$

$$= 267.48 \times 10^3 \ \Omega/\text{km}$$

$$\text{Charging current} = \frac{V_n}{X_n} = \frac{\left(\frac{100}{\sqrt{3}}\right) \times 1000}{267.4 \times 10^3}$$

$$= 0.23 \text{ A/km}$$

Performance of Lines

Performance of transmission lines is to find the efficiency and Regulation.

The efficiency of the transmission line is %

$$\text{Efficiency} = \frac{\text{Power delivered at the receiving end}}{\text{Power sent from sending end}} \times 100$$

$$\frac{\text{Power delivered at the receiving end}}{\text{Power sent from sending end}} \times 100$$

Receiving end in a transmission line is where the load is connected and the supply is from the sending end.

Regulation

The regulation of the line is defined as the change in the receiving end voltage, expressed as the percentage of full load voltage from no - load to full load, keeping the sending end voltage and frequency constant.

$$\% \text{ Regulation} = \frac{V_{R_1} - V_{R_2}}{V_{R_2}} \times 100$$

where V_{R1} is the receiving end voltage under No-load condition.

V_{R2} is the receiving end voltage under full load condition.

Note that V_{R1} and V_{R2} are magnitudes only

Regulation is calculated on

1. Speed
2. Voltage

Performance of rotating equipment is evaluated based on efficiency and speed regulation.

Performance of static equipment is evaluated based on Efficiency and voltage regulation.

Since transmission line comes under static equipment, performance is evaluated when the system reaches steady state rather than in transient state.

- The loss in a transmission line is due to real power but not due to reactive power. Hence, the efficiency is calculated in terms of real power.

Classification of Overhead Transmission Lines

- In steady-state condition, for a line whose lengths is less than 160 km, the voltage or current does not vary much and for the line lengths of about 160 km, the parameters can be assumed as lumped but not distributed. These lines are called as electrically short transmission lines. These electrically short transmission lines are again divided as short and medium transmission lines.

Short Transmission Lines

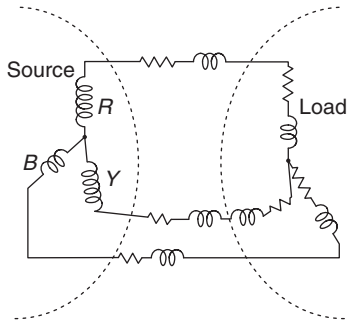
The lines of lengths up to 80 km are known as short transmission lines in which the effect of shunt capacitance is neglected.

Medium Transmission Lines

The lines of lengths above 80 km and below 160 km are known as medium transmission lines in which the shunt capacitance is assumed to be lumped at the centre (or) half of the capacitance is lumped at each end of the line

Long Transmission Lines

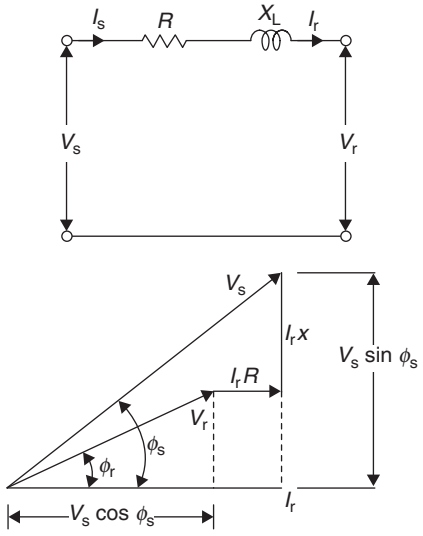
For line lengths more than 160 km, all the parameters are distributed and in some cases where the line length is less than 250 km, the analysis is done using nominal-π and T.



- The above figure shows a balanced 3-ϕ source which is connected to a balanced star load. As the sum of all the current in a balanced polyphase network is zero, hence the current through the wire connected between the star point of the load and neutral of the system is zero. Hence, the star point of the load and the neutral point of the source are at the source potential. Hence, for all purposes, a 3-ϕ balanced system can therefore be analysed on a single-phase basis.

Short Transmission Line

- The parameters considered here are the lumped parameters. Lumped parameters are those which can be placed at one point (or) two points to the maximum, and these parameters have physical and electrical separation. These parameters are used to study the steady-state behaviour of the long transmission lines.
- The inductance will be high and capacitance will be less in short transmission lines. Hence the capacitance is neglected. Therefore the parameters present are only resistance and inductance.
- The equivalent circuit and the vector diagram for a short transmission line are shown in the below figure.



The vector diagram is drawn taking I_r, the receiving end current, as the reference

From the vector diagram,

$$V_s \cos\phi_s = V_r \cos\phi_r + I_r R \quad (7)$$

$$V_s \sin\phi_s = V_r \sin\phi_r + I_r X \quad (8)$$

Squaring and adding equations (7) and (8),

$$V_s^2 = V_r^2 + 2 I_r R V_r \cos\phi_r + 2 I_r X V_r \sin\phi_r + I_r^2 (R^2 + X^2)$$

$$\Rightarrow V_s = V_r \sqrt{1 + \frac{2 I_r R \cos\phi_r}{V_r} + \frac{2 I_r X \sin\phi_r}{V_r} + \frac{I_r^2}{V_r^2}\left(R^2 + X^2\right)}$$

The last term in the square root can be neglected.

$$V_s = V_r \left\{1 + \left(\frac{2 I_r R}{V_r}\right)\cos\phi_r + \frac{2 I_r X}{V_r}\sin\phi_r\right\}^{1/2}$$

Using Binomial expansion, and limiting only to two terms,

$$V_s \approx V_r + I_r R \cos\phi_R + I_r X \sin\phi_R$$

Here, V_s is the sending end voltage corresponding to a particular load current and power factor condition. From the equivalent circuit of short line, we can see that the receiving end voltage under no load V_r' is same as the sending end voltage under full-load condition, i.e.

$$V_r' = V_s$$

$$\therefore \quad \% \text{ Regulation} = \frac{V_s - V_r}{V_r} \times 100$$

$$= \left(\frac{I_r R}{V_r}\cos\phi_r + \frac{I_r X}{V_r}\sin\phi_r\right) \times 100$$

$$\text{Per unit Regulation} = \frac{I_r R}{V_r}\cos\phi_r + \frac{I_r X}{V_r}\sin\phi_r$$

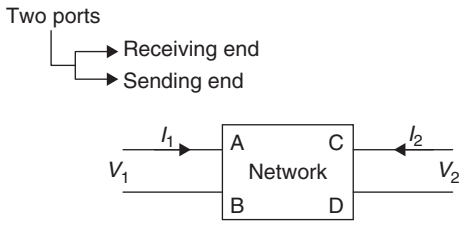
$$= u_r \cos\phi_r + u_x \sin\phi_r$$

where u_r and u_x are the per unit values of the resistance and reactance of the line, respectively.

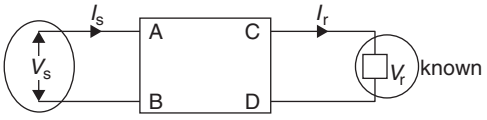
- The performance of transmission line can be analysed at a faster rate with the help of less no. of equations by representing the transmission line in a 2-port network model rather R–L–C model.

Two-port Network Model

Port: pair of terminals (phase and Neutral) is known as ports.



As we want V_1, I_1 from V_2, I_2, we take the transmission line as A B C D networks.



The unknown sending end electrical quantities are expressed in terms of known receiving end electrical quantities along with certain parameters.

Those parameters are called A B C D parameters.

$$V_s = A V_r + B I_r \text{ kV/phase}$$

$$I_s = C V_r + D I_r \text{ kA/phase}$$

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

If $A = D \rightarrow$ Symmetrical network

If $A D - B C = 1$ $\rightarrow$ Reciprocal network

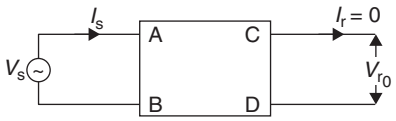
(or)

$-A D + B C = 1$

- Electrical network may be symmetrical (or) unsymmetrical but always reciprocal network

Units for A, B, C, D

Open Circuit at the Receiving End



$$V_s = A V_{r_0}$$

$$I_s = C V_{r_0}$$

$$A = \frac{V_s}{V_{r_0}} | I_{r=0} \text{ (unitless)}$$

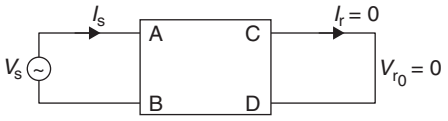
$$C = \frac{I_s}{V_{r_0}} \Big|_{I_r = 0}$$

'C' open circuit admittance in Mho's

$$\% \text{ Regulation} = \frac{|V_{r_0}| - |V_r|}{|V_r|} \times 100$$

$$\% \varepsilon = \frac{\left|\frac{V_s}{A}\right| - |V_r|}{|V_r|} \times 100$$

Short Circuit at Receiving End



When the receiving and is short circuited $V_r = 0$. Then ABCD parameters are given by

$$V_s = B I_r$$
$$I_s = D I_r$$

$$\Rightarrow \quad \frac{V_s}{I_s} = \frac{B}{D}$$

$$B = \frac{V_s}{I_r} \Big|_{V_r = 0}$$

(Short-circuit impedance in ohms)

$$D = \frac{I_s}{I_r} \Big|_{V_r = 0}$$

- For short transmission lines, regulation is equal to voltage drop but this is not possible in medium and long transmission lines, as shunt capacitance is present.
- In case of short transmission line, if load is open, then $|V_r| = |V_s|$ and $I_s = 0$. But this is not true in case of medium and long transmission lines as shunt capacitance is present and hence $I_s \neq 0$ but $I_r = 0$ and $V_s = V_c$.

Condition for Maximum Regulation

The expression is given by

$$\% \varepsilon = \% V_r \cos\phi_r + \% V_x \sin\phi_r$$

The condition for maximum regulation can be obtained from

$$\frac{d\%\varepsilon}{d\phi_r} = 0$$

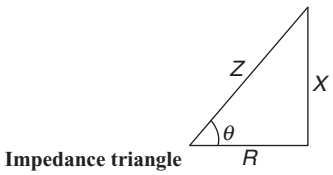
$$\Rightarrow \quad -\% V_r \sin\phi_r + \% V_x \cos\phi_r = 0$$

$$\Rightarrow \quad \tan\phi_r = \frac{\%V_x}{\%V_r} = \frac{\frac{I_r X}{V_r} \times 100}{\frac{I_r R}{V_r} \times 100} = \frac{X}{R}$$

$$\Rightarrow \quad \phi_r = \tan^{-1}\left(\frac{X}{R}\right)$$

X – Line reactance Impedance triangle

R – Line resistance



Impedance triangle

$$\tan\theta = \frac{X}{R}$$

When the phase angle of load is equal to the impedance angle, the regulation is maximum.

$$\Rightarrow \quad \tan\phi_R = \frac{X}{R} = \tan\theta$$

$$\Rightarrow \quad \phi_R = \theta$$

Condition for Zero Regulation

$$\% \varepsilon = \% V_r \cos\phi_r - \% V_x \sin\phi_r = 0$$

$$\Rightarrow \quad \tan\phi_r = \frac{\frac{I_r R}{V_r} \times 100}{\frac{I_r X}{V_r} \times 100} \frac{\%V_r}{\%V_x} = \frac{R}{X}$$

$$\Rightarrow \quad \tan\phi_r = \frac{R}{x} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \quad \phi_r = \frac{\pi}{2} - \theta$$

$$\Rightarrow \quad \phi_r + \theta = \frac{\pi}{2}$$

- Max regulation occurs for lagging load,

$$\% \varepsilon = \% V_r \cos\phi_r + \% V_x \sin\phi_r$$

- Min regulation occurs for leading load,

$$\% \varepsilon = \% V_r \cos\phi_r - \% V_x \sin\phi_r$$

(negative value)

- We get % regulation as '0' only in the leading loads provided

(i) $\frac{X}{R} = 1.0$

(ii) 0.707 lead p.f.

Example 31: A single-phase overhead transmission line delivers 1100 kW at 33 kV at 0.8 p.f. lagging. The total resistance and inductive reactance of the line are 15 Ω and 20 Ω, respectively. Find the (i) sending end voltage (ii) sending end power factor and (iii) transmission efficiency.

Solution: Load power factor, $\cos\phi_r = 0.8$ lag

Total line impedance $\rightarrow Z = R + jX_L$

$= 15 + j20$

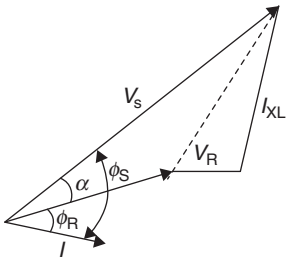
Receiving end voltage, $V_R = 33$ kV

$$\therefore \text{Line current, } I = \frac{\text{kW} \times 10^3}{V_R \times \cos\phi_R}$$

$$= \frac{1100 \times 10^3}{33,000 \times 0.8} = 41.66$$

$$\cos\phi_R = 0.8$$

$$\Rightarrow \quad \sin\phi_R = 0.6$$



$$\overrightarrow{V_R} = V_R + j0 = 33,000 \text{ V}$$

$$I = I(\cos\phi_R - j \sin\phi_R)$$

$$= 41.67\ (0.8 - j\ 0.6)$$

$$= 33.33 - j25 = 41.67 \angle -36.86°$$

(i) Sending end voltage, $\overrightarrow{V_s} = \overrightarrow{V_R} = +\vec{I} * \vec{Z}$

$$= 33,000 + (33.33 - j25.0)\ (15 + j20)$$

$$= 33,999.95 + 291.6\, j$$

$$\therefore \text{Magnitude of } V_s = \sqrt{(33,999.95)^2 + (291.6)^2}$$

$$= 34,001.20 \text{ V}$$

(ii) Angle between $\overrightarrow{V_s}$ and $\overrightarrow{V_r}$

$$\alpha = \tan^{-1}\left(\frac{291.6}{33,999.95}\right) = 0{:}49°$$

$\therefore$ Sending power factor angle,

$$\phi_s = \phi_r + \alpha = 36°86' + 0°49' = 37°35'$$

$\therefore$ Sending end power factor, $\cos\phi_s = \cos(37°\ 35') = 0.79$ lagging

(iii) Line losses $= I^2R = (41.67)^2 \times 15 = 26.045$ kW

Output = 1100 kW

Power sent = 1100 + 26.045 kW

= 1126.045 kW

$\therefore$ Transmission efficiency

$$= \frac{\text{Power delivered}}{\text{Power sent}} \times 100$$

$$= \frac{1100}{1126.045} \times 100 = 97.68\%$$

Example 32: What is the maximum length in km for a single-phase transmission line having copper conductor of 0.775 cm^2 cross-section over which 200 kW at unity power factor and at 3300 V are to be delivered? The efficiency of transmission is 90%. Take specific resistance as 1.725 μΩ cm.

Solution: Receiving end power = 200 kW

Transmission efficiency = 0.9

$$\therefore \text{Sending end power} = \frac{2,00,000}{0.9}$$

$$= 2,22,222 \text{ W}$$

$\therefore$ Line losses = 2,22,222 – 2,00,000

= 22,222 W

$$\text{Line current } I = \frac{200 \times 10^3}{3300 \times 1} = 60.6 \text{ A}$$

Let R Ω be the resistance of the 2 conductors, hence line losses = $2\ I^2R$

$$22,222 = 2\ I^2R$$

$$\Rightarrow \quad R = \frac{22,222}{2 \times (60.6)^2} = 3.025\ \Omega$$

$$\therefore \quad R = \frac{\rho I}{A} \Rightarrow l = \frac{R \times A}{\rho}$$

$$= \frac{3.025 \times 0.775}{1.725 \times 10^{-6}} = 1.36 \times 10^6 \text{ cm}$$

$$= 13.6 \text{ km}$$

Medium Transmission Lines

We know that the transmission lines with lengths between 80 km and 160 km are categorized as medium length lines, in which the parameters are assumed to be lumped.

The shunt capacitance is assumed to be either at the middle of the line or half of the total capacitance is concentrated at each end of the line. These two configurations are classified as—nominal-T and nominal-π, respectively. The circuits are shown below.

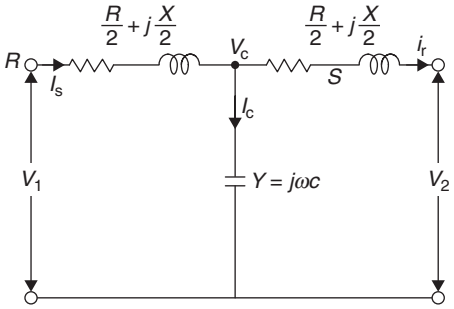


Figure 13 Nominal-T network

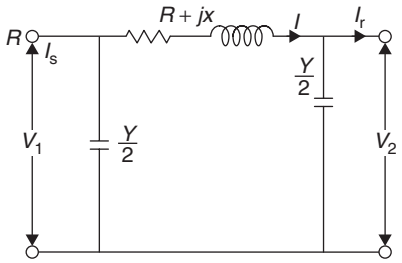


Figure 14 Nominal-π network

Nominal-T Network

To analyse the medium length lines by this network, it would be convenient to take receiving end current as the reference vector rather than taking the receiving end voltage (V_r) as the reference. The vector diagram for lagging power factor load is given below.

From the network diagram, in order to calculate the regulation, we have to first calculate the sending end voltage V_s.

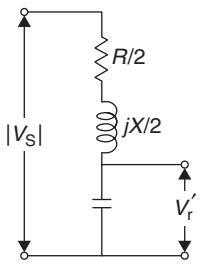
$$V_C = (|V_r|\cos\phi_r + j\cdot|V_r|\sin\phi_r) + I_r\left(\frac{R}{2}+j\frac{x}{2}\right)$$

$$I_c = j\omega c\, V_c$$

$$I_s = I_c + I_r = I_r + j\omega c\, V_c$$

$$V_s = V_c + I_s\left(\frac{R}{2}+j\frac{x}{2}\right)$$

$$= (|V_r|\cos\phi_r + j\,|V_r|\sin\phi_r) + I_r\left(\frac{R}{2}+j\frac{x}{2}\right) + I_s\left(\frac{R}{2}+j\frac{x}{2}\right)$$



To calculate the regulation, we have to find V_r', the receiving end no load voltage keeping the sending end voltage V_s fixed in magnitude. The above nominal-T network reduces to the network beside.

$$V_r' = \frac{|V_s|\left(\frac{-j}{\omega c}\right)}{\frac{R}{2}+j\frac{x}{2}-\frac{j}{\omega c}}$$

$\therefore$ The regulations for the nominal-T network is obtained as

$$\%\text{ regulations} = \frac{V_r' - V_r}{V_r}\times 100$$

The efficiency is obtained as

$$\%\ \eta = \frac{\text{Power at receiving end}}{\text{Power at receiving} + \text{Loss}}\times 100$$

In order to determine A, B, C, D constants for nominal-T,

$$V_c = V_r + I_r\left(\frac{Z}{2}\right)$$

$$I_c = V_c Y$$

$$I_s = I_r + I_c = I_r + V_c Y \Rightarrow I_r + \left(V_r + I_r\frac{Z}{2}\right)Y$$

$$V_s = V_c + I_s\frac{Z}{2} = V_r + I_r\frac{Z}{2} + \left\{I_r + \left(V_r + I_r\frac{Z}{2}\right)Y\right\}\frac{Z}{2}$$

$$= V_r\left(1+\frac{YZ}{2}\right) + I_r\left(\frac{Z}{2}+\frac{Z}{2}+\frac{YZ^2}{4}\right)$$

$$= V_r\left(1+\frac{YZ}{2}\right) + I_r\left(Z+\frac{YZ^2}{4}\right)$$

$$I_s = I_r\left(1+\frac{YZ}{2}\right) + V_r Y = YV_r + \left(1+\frac{YZ}{2}\right)I_r$$

By comparing the above equations with the standard ones,

$$V_s = AV_r + BI_r$$

$$I_s = CV_r + DI_r$$

$$\Rightarrow \quad A = 1+\frac{YZ}{2},\ B = Z\left(1+\frac{YZ}{4}\right),\ C = Y,\ D = \left(1+\frac{YZ}{2}\right)$$

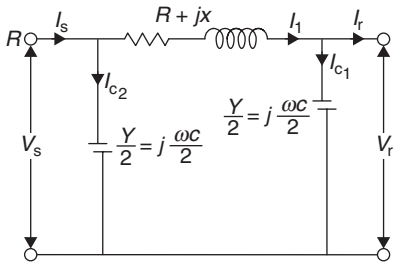
We can see that $A = D$,

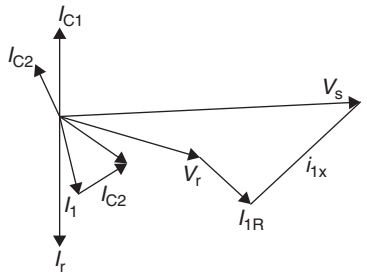
$$\text{and AD} - \text{BC} = \left(1+\frac{YZ}{2}\right)^2 - YZ\left(1+\frac{YZ}{4}\right)$$

$$= 1 + \frac{Y^2Z^2}{4} - YZ - YZ - \frac{Y^2Z^2}{4} = 1$$

Nominal-π Representation

Nominal-π representation of medium length transmission lines and their phasor diagram are shown below.





From the above representation,

$$I_{c_1} = jV_r\frac{\omega C}{2},\ I_1 = |I_r|(\cos\phi_r - j\sin\phi_r) + jV_r\frac{\omega C}{2}$$

$$V_s = V_r + I_1 Z$$

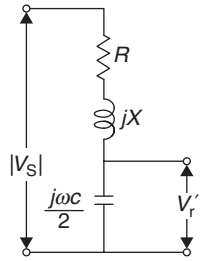
$$= V_r + \left\{|I_r|(\cos\phi_r - j\sin\phi_r) + j\frac{V_r\omega C}{2}\right\}(R + jX)$$

and $$I_s = I_1 + I_{c2} = I_1 + j\frac{V_s\omega C}{2}$$

$$I_s = |I_r|(\cos\phi_r - j\sin\phi_r) + \frac{jV_r\omega C}{2} + \frac{jV_s\omega C}{2}$$

$$V_s = V_r + \left\{|I_r|(\cos\phi_r - j\sin\phi_r) + \frac{jV_r\omega C}{2}\right\}(R + jX)]$$

To calculate the regulation, we have to find no load receiving end voltage by keeping the sending end voltage constant. The nominal-π circuit for no-load reduces as follows.



$$V_r' = \frac{|V_s|\left(\frac{-2j}{\omega C}\right)}{R + jX - \frac{2j}{\omega C}}$$

$$\therefore \quad \%\ \text{Regulation} = \frac{V_r' - V_r}{V_r} \times 100$$

and $$\%\ \eta = \frac{P}{P + 3I_1^2 R} \times 100$$

To find the A, B, C, D constants for nominal-π network,

$$I_c = V_r\frac{Y}{2}$$

$$I_1 = I_r + I_c = I_r + \frac{V_r Y}{2}$$

$$V_s = V_r + I_1 Z = V_r + \left(I_r + \frac{V_r Y}{2}\right)Z = \left(1 + \frac{YZ}{2}\right)V_r + ZI_r$$

$$I_s = I_1 + I_{c2} = I_1 + \frac{V_s Y}{2} = I_r + \frac{V_r Y}{2} + \left\{V_r\left(1 + \frac{YZ}{2}\right) + ZI_r\right\}\frac{Y}{2}$$

$$= V_r\left(Y + \frac{Y^2 Z}{4}\right) + \left(1 + \frac{YZ}{2}\right)I_r$$

Comparing the coefficients of equations with the standard ones,

$$A = 1 + \frac{YZ}{2},$$

$$B = Z$$

$$C = Y\left(1 + \frac{YZ}{4}\right)$$

$$D = \left(1 + \frac{YZ}{2}\right)$$

It is seen that $A = D$,

and $$AD - BC = \left(1 + \frac{YZ}{2}\right)^2 - ZY\left(1 + \frac{YZ}{4}\right) = 1$$

Example 33: A three-phase 50 Hz overhead transmission line 100 km long has the following constants:

Resistance/km/phase = 0.1 Ω

Inductive reactance/km/phase = 0.2 Ω

Capacitive susceptance/km/ phase = 0.04×10^{-4} mho

Determine (i) sending end current (ii) sending end voltage (iii) sending end power factor and (iv) transmission efficiency when supplying a balanced load of 10,000 kW at 66 kV, p.f. 0.8 lagging, use nominal-T method.

Solution: Total resistance/phase, $R = 0.1 \times 100$
$= 10\ \Omega$

Total resistance/phase, $X_L = 0.2 \times 100$
$= 20\ \Omega$

Capacitive susceptance,

$$Y = 0.04 \times 10^{-4} \times 100 = 4 \times 10^{-4}\ \Omega$$

Receiving end voltage/phase

$$V_R = \frac{66,000}{\sqrt{3}} = 38,105\ \text{V}$$

Load current, $I_R = \frac{10,000 \times 10^3}{\sqrt{3} \times 66 \times 0.8} = 109\ \text{A}$

$$\cos\phi_r = 0.8, \Rightarrow \sin\phi_r = 0.6$$

Impedance per phase, $\vec{Z} = R + jX_L = 10 + j20$

Taking receiving end voltage as the reference vector, we have,

Receiving voltage, $\vec{V_R} = V_R + j0 = 38{,}105$ V

Load current, $\vec{I_R} = I_R(\cos\phi_r - j\sin\phi_r)$

$= 109\,(0.8 - j0.6) = 87.2 - j65.4$

Voltage across C, $\vec{V_1} = \vec{V_R} + \vec{I_R}\frac{\vec{Z}}{2}$

$= 38{,}105 + (87.2 - j\,65.4)\,(5 + j\,10)$

$= 38{,}105 + 436 + j\,872 - j\,327 + 654$

$= 39.195 - j\,545$

Charging current, $\vec{I_c} = jY\vec{V_1}$

$= j\,4 \times 10^{-4}\,(39{,}195 - j\,545)$

$= 0.218 + j\,15.6$

Sending end current

$\vec{I_s} = \vec{I_R} + \vec{I_c}$

$= (87.2 - j\,65.4) + (0.218 + j\,15.6)$

$= 87.4 - j\,49.8 = 100\angle -29°40' = 100$ A

Sending end voltage,

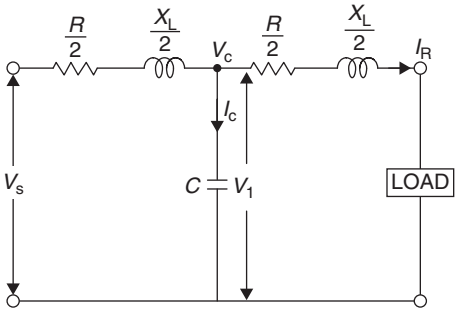
$\vec{V_s} = \vec{V_1} + \vec{I_s}\frac{\vec{Z}}{2}$

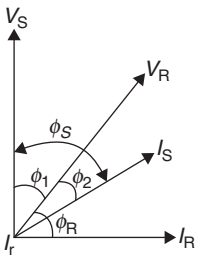
$= (39{,}195 - j\,545) + (87.4 - j\,49.8)\,(5 + j\,10)$

$= 40130 + j\,80$

$\therefore$ Line-to-line sending end voltage $= 40130 \times \sqrt{3}$

$= 69{,}507$ V $= 69.507$ kV





$\phi_S = \theta_1 + \theta_2 = 6' + 29°40' = 29°46'$

$\therefore$ Sending end power factor, $\cos\phi_s$

$= \cos 29°46 = 0.868$

$\therefore$ Sending end power $= 3\,V_s I_s \cos\phi_s$

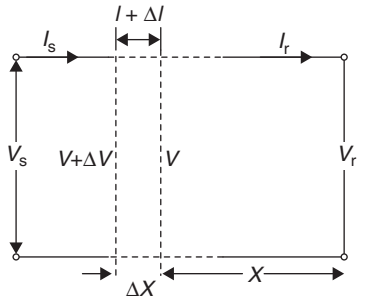
$= 3 \times 40{,}130 \times 100 \times 0.868 = 10{,}449.85$ kW

Power delivered $= 10{,}000$ kW

$\therefore$ Transmission efficiency $= \dfrac{10000}{10{,}449.855} \times 100 = 95.7\%$

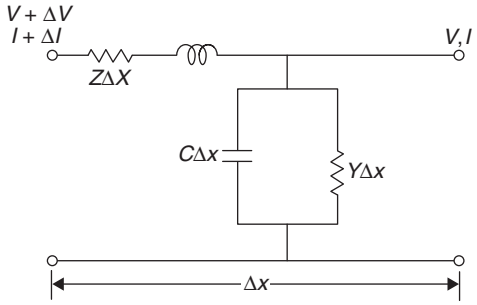
Long Transmission Lines

If the lines are more than 160 km in length, for the analysis to be accurate, the parameters must be taken as distributed rather than lumped for the voltage and currents.



Let Z = Series impedance per unit length
y = Shunt admittance per unit length
l = Length of the line
$Z = zl$ = Total series impedance
$Y = yl$ = Total shunt admittance

For analysis, the elemental length dx is redrawn



We take the receiving end as the reference for measuring distances. Take an elemental length dx of the line at a distance 'x' from the receiving end. Let the voltage and current at a distance 'x' are V and I and at a distance $x + dx$, $V + \Delta V$ and $I + \Delta I$, respectively.

$$\Delta V = Iz\,\Delta x \Rightarrow \Delta I = Vy\,\Delta x$$

$$\Rightarrow \quad \frac{\Delta V}{\Delta x} Iz \Rightarrow \frac{\Delta I}{\Delta x} = Vy$$

When $\Delta x \to 0$, the equations become,

$$\frac{dV}{dx} = Iz \tag{9}$$

$$\frac{dI}{dx} = Vy \tag{10}$$

Differentiating Equation (9),we get

$$\Rightarrow \frac{d^2V}{dx^2} = Z\frac{dI}{dx} = Z.\,y.\,V \Rightarrow \frac{d^2V}{dx^2} = Z\,yV = 0$$

The solution of the above equation is

$$V = A\ \exp\left(\sqrt{yz}\ .x\right) + B\ \exp\left(-\sqrt{yz}\ .x\right)$$

Let $Z_c = \sqrt{\frac{z}{y}}$ and $\gamma = \sqrt{yz} = \alpha + j\beta$

where Z_c is known as characteristic impedance and 'γ' is the propagation constant.

The above equations can be rewritten as

$$V = A\,e^{\gamma X} + B\,e^{-\gamma X}$$

$$I = \frac{1}{Z_c}\left(A\,e^{\gamma X} + B\,e^{-\gamma X}\right)$$

The constants can be determined by the boundary conditions, at $x = 0$,

$$V = V_r \text{ and } I = I_r \Rightarrow V_r = A + B$$

$$I_r = \frac{1}{Z_c}(A - B) \Rightarrow \text{A} = \frac{V_r + I_r Z_c}{2} \text{ and B} = \frac{V_r + I_r Z_c}{2}$$

Substituting 'A' and 'B' in above equations

$$V = \frac{V_r + I_r Z_c}{2}e^{\gamma X} + \frac{V_r - I_r Z_c}{2}e^{-\gamma X}$$

and $$I = \frac{1}{Z_c}\left[\frac{V_r + I_r Z_c}{2}e^{\gamma X} - \frac{V_r - I_r Z_c}{2}e^{-\gamma X}\right]$$

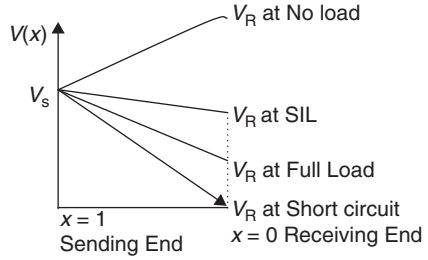
We can observe that V and I are the functions of the distance x, receiving end voltage V_r and current I_r and the parameters of the line, which means they will vary as we move from receiving end towards the sending end.

$$Z_c \text{ and } \gamma \text{ are complex} \Rightarrow Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{r + j\omega L}{g + j\omega c}}$$

For a loss – less line, $r = 0$, $g = 0$, $\Rightarrow Z_c = \sqrt{\frac{L}{c}}$

The Surge impedance (Z_c) is purely resistive and is known as surge impedance of the line.

Voltage Profiles



During high frequencies (or) surges, normally the losses are neglected and hence, the characteristic impedance becomes surge impedance. '**Surge impedance loading'** of a line is the power transmitted when the line is terminated through a resistance equal to surge impedance. A line terminated in its characteristic impedance is called a flat line or an infinite line.

The lower value of surge impedance in case of cables is due to relatively large capacitance and low inductance of the cables.

The propagations constant $\gamma = \alpha + j\beta$, the real part is known as attenuation constant (α) and the quadrature component β is known as the phase constant and is measured in radians per unit length.

In the above equation, the first part, $\frac{V_r + I_r Z_c}{2}e^{\alpha X}\ e^{j\beta X}$ is called the incident voltage wave as its value increases as 'X' is increased, this means that the magnitude of voltage wave decreases in magnitude as it travels towards the receiving end. Similarly, the second part of the equation $\frac{V_r - I_r Z_c}{2}e^{-\alpha X}e^{-j\beta X}$ is called reflected voltage. At any point on the line, the voltage wave is the sum of the incidence and reflected waves. The above result is true even for the current equation.

The above equations for voltage and current can be re-arranged as follows,

$$V = V_r\frac{e^{\gamma X} + e^{-\gamma X}}{2} + I_r Z_c\frac{e^{\gamma X} - e^{-\gamma X}}{2}$$

$$= V_r \cos \text{h}\ \gamma X + I_r Z_c \sin \text{h}\ \gamma X$$

and $$I = \frac{1}{Z_c}\left[V_r\frac{e^{\gamma X} + e^{-\gamma X}}{2} + I_r Z_c\frac{e^{\gamma X} + e^{-\gamma X}}{2}\right]$$

$$= \frac{1}{Z_c}\left[V_r \sin \gamma X + I_r Z_c \cos \text{h}\,\gamma X\right]$$

$$= \frac{V_r}{Z_c}\sin \text{h}\gamma X + I_r \cos \text{h}\gamma X$$

The above equations can be re-written for $x = l$, where $V = V_s$ and $I = I_s$

$$V_s = V_r \cos \text{h}\ \gamma l + I_r Z_c \sin \text{h}\ \gamma l$$

$$I_s = V_r\frac{\sin \text{h}\ \gamma I}{Z_c} + I_r \cos \text{h}\ \gamma I$$

Comparing the above equations with standard equations,

$$A = \cos \text{h}\ \gamma l$$

$$B = Z_c \sin \text{h}\ \gamma l$$

$$C = \frac{\sin \text{h}\ \gamma l}{Z_c},\ D = \cos \text{h}\ \gamma l.$$

We can see that $A = D$ and

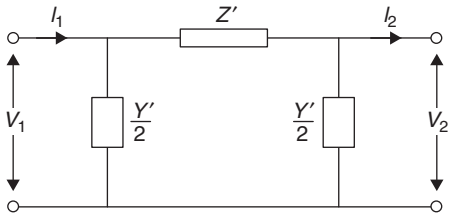
$$\text{AD} - \text{BC} = \cos \text{h}^2\ \gamma I - \sin \text{h}^2\ \gamma I = 1$$

Equivalent Representation of a Long Line

For Line lengths more than 160 km, the parameters are assumed to be distributed. The equivalent-T and Equivalent-π networks are represented. For long lines, nominal-T and nominal-π networks may not be used as these networks consider the lines with lumped parameters.

Equivalent-π Representation

The sending end and receiving end voltages and currents are found form this equivalent network.



V_1 – Sending end voltage
V_2 – Receiving end voltage
I_1 – Sending end current
I_2 – Receiving end current.

$$V_1 = \left(1+\frac{Y'Z'}{2}\right)V_r + Z'I_r \qquad (11)$$

$$I_1 = y'\left(1+\frac{Y'Z'}{4}\right)V_r + \left(1+\frac{Y'Z'}{2}\right)I_r \qquad (12)$$

Comparing the above two equations with the standard ones, i.e.,

$$V_1 = V_2 \cos h\,\gamma l + \frac{T}{2} Z_c \sin h\,\gamma l$$

$$I_1 = V_2 \frac{\sin h\,\gamma l}{Z_c} + \frac{T}{2}\cosh \gamma l$$

$$\Rightarrow \quad 1+\frac{Y'Z'}{2} = \cos h\,\gamma l,\ Z' = Z_c \sin h\,\gamma l,$$

$$Y'\left(1+\frac{Y'Z'}{4}\right) = \frac{\sin h\,\gamma l}{Z_c} \text{ and } 1+\frac{Y'Z'}{2} = \cos h\,\gamma l$$

$$\text{Consider} = Z' = Z_c \sin h\,\gamma l$$

$$= \sqrt{\frac{z}{y}} \times \sin h\,\gamma l \times \frac{l}{l\sqrt{yz}} \times \sqrt{yz}$$

$$= Zl\,\frac{\sin h\,\gamma l}{\gamma l} = Z\frac{\sin h\,\gamma l}{\gamma l}.$$

From the above expression, we can observe that the equivalent series impedance, the lumped impedance must be multiplied with $\frac{\sin h\,\gamma l}{\gamma l}$.

In order to get the shunt parameters of the equivalent-π network, substitute Z' from the above equation, i.e.,

$$1+\frac{Y'Z'}{2} = \cos h\,\gamma l \Rightarrow 1+\frac{Y'Z'}{2} \times Z_c \sin h\,\gamma l = \cos h\,\gamma l$$

(as $Z = zl$ and $\gamma = \sqrt{yz}$)

$$\Rightarrow \quad \frac{Y'}{2} Z_c \sin h\,\gamma l = \cos h\,\gamma l - 1$$

$$= \cos h^2\frac{\gamma l}{2} + \sin h^2\frac{\gamma l}{2} - \cos h^2\frac{\gamma l}{2} + \sin h^2\frac{\gamma l}{2}$$

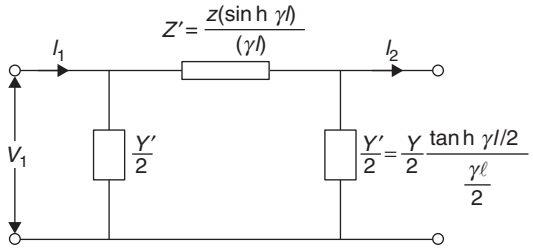
$$\Rightarrow \quad Y' Z_c \sin h\frac{\gamma l}{2}\ \cos h\frac{\gamma l}{2} = 2\sin h^2\frac{\gamma l}{2}$$

$$\Rightarrow \quad \frac{Y'}{2} = \frac{1}{Z_c}\tan h\frac{\gamma l}{2} = \sqrt{\frac{y}{z}} \times \frac{\sqrt{yzl}}{2} \times \frac{\tan h\,\gamma l/2}{\sqrt{yz}\,l/2}$$

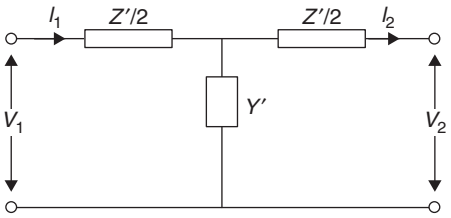
$$= \frac{Y}{2}\frac{\tan h\,\gamma l/2}{\gamma l/2}.$$

where 'Y' is the total shunt admittance of the nominal-π network. Hence, we can conclude that the total shunt admittance of the nominal-π network should be multiplied with $\frac{\tan h\,\gamma L/2}{\gamma L/2}$

Hence, the equivalent circuit can be redrawn as



Equivalent T-Representation of Long Line



V_1 and V_2 are the sending end and receiving end voltages, respectively, whereas I_1 and I_2 are the sending end and receiving end currents,respectively.

From the network of nominal-T,

We have,

$$V_S = V_r\left(1+\frac{YZ}{2}\right) + I_r Z\left(1+\frac{YZ}{4}\right)$$

$$I_S = Y V_r + \left(1+\frac{YZ}{2}\right)I_r$$

As the above network is similar to nominal-T network, we can use the above equations to analyse the equivalent-T network.

$$V_1 = \left(1+\frac{Y'Z'}{2}\right)V_2 + Z'\left(1+\frac{Y'Z'}{4}\right)I_2$$

$$I_1 = \left(1+\frac{Y'Z'}{2}\right)I_2 + Y'V_2$$

Comparing the above equations with the equations of long transmission lines, i.e.

$$V_1 = V_s \cosh \gamma l + I_2 Z_c \sinh \gamma l$$

$$I_1 = V_2 \frac{\sinh \gamma l}{Z_c} + I_s \cosh \gamma l$$

$$\left(1+\frac{YZ}{2}\right) = \cosh \gamma l,\ Z'\left(1+\frac{Y'Z'}{4}\right) = Z_c \sinh \gamma l$$

$$1+\frac{Y'Z'}{2} = \cosh \gamma l,\ Y' = \frac{\sinh \gamma l}{Z_c}.$$

To find the shunt branch parameters of the equivalent-T circuit,

$$Y' = \frac{1}{Z_c}\sinh \gamma l = \sqrt{\frac{y}{z}} \times \sinh \gamma l \times \frac{\sqrt{yz}\times l}{\sqrt{yz}\times l}$$

$$= Y\frac{\sinh \gamma l}{\gamma l} \text{ (where } \gamma = \sqrt{yz}\text{).}$$

Hence, to find the equivalent shunt admittance, the lumped shunt admittance should be multiplied with $\frac{\sinh \gamma l}{\gamma l}$.

To find the series impedance, consider the equation.

$$\left(1+\frac{Y'Z'}{2}\right) = \cosh \gamma l$$

Substitute $Y' = \frac{\sinh \gamma l}{Z_c}$ in the above equation,

$$\Rightarrow \quad 1+\frac{Z^1}{2}\times\frac{\sinh \gamma l}{Z_c} = \cosh \gamma l$$

$$\Rightarrow \quad \frac{Z'}{2}\times\frac{2\sinh\frac{\gamma l}{2}\cosh\frac{\gamma l}{2}}{Z_C} = \cosh \gamma l - 1 = 2\sinh^2\frac{\gamma l}{2}$$

$$\Rightarrow \quad \frac{Z'}{2} = Z_C \tanh\frac{\gamma l}{2} = \sqrt{\frac{\gamma}{y}}\tanh\frac{\gamma l}{2}$$

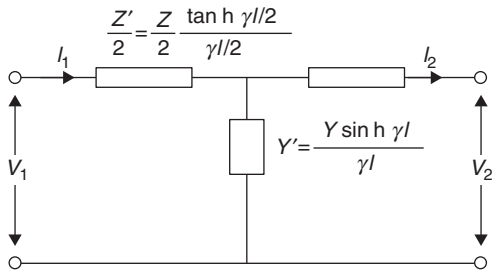
Multiply and divide by $\frac{\sqrt{yz}.l}{2}$

$$\Rightarrow \quad \frac{Z'}{2} = \sqrt{\frac{z}{y}}\times\frac{\sqrt{yz}.l}{2}\times\frac{\tanh(\gamma l/2)}{\frac{\sqrt{yz}.l}{2}}$$

$$= \frac{Z}{2}\frac{\tanh\frac{\gamma l}{2}}{\frac{\gamma l}{2}} \text{ (where } z = x.l \text{ and } \gamma = \sqrt{zy}\text{)}$$

Therefore to get the series equivalent impedance, the lumped series impedance of the nominal-T network should be multiplied with $\frac{\tanh\frac{\gamma l}{2}}{\frac{\gamma l}{2}}$.

The equivalent T-representation is as follows.



Calculation of Receiving End Voltages and Currents

We know that

$$V_S = AV_r + B\,I_r \tag{13}$$

$$I_S = CV_r + D\,I_r \tag{14}$$

where V_s, I_s are sending end voltage and current, respectively, while V_r, I_r are receiving end voltage and current, respectively.

Multiply (13) with C and (14) with A

$$\Rightarrow \quad CV_S = VA\,V_r + CB\,I_r \tag{15}$$

$$AI_S = AC\,V_r + AD\,I_r \tag{16}$$

Subtract (15) from (16),

$$AI_S - CV_S = AC\,V_r + AD\,I_r - CAV_r - EBI_r$$

$$\Rightarrow \quad AI_s - CV_s = I_r(AD - BC)$$

We know that $AD - BC = 1$ and $A = D$, receiving end current is expressed as

$$\Rightarrow \quad I_r = -CV_S + DI_S \tag{17}$$

Multiply (13) with 'D' and (14) with 'B'

$$DV_s = AD\,V_r + DB\,I_r \tag{18}$$

$$BI_s = BC\,V_r + BD\,I_r \tag{19}$$

Subtract (18) from (19),

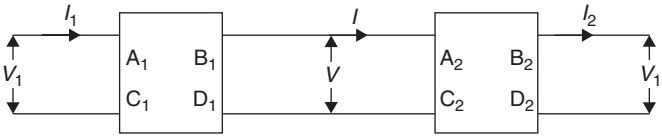
$$DV_s - BI_s = (AD - BC)\,V_r$$

We know that $AD - BC = 1$, receiving end voltage is expressed as

$$\Rightarrow \quad V_r = DV_s - BI_s \tag{20}$$

Constants of Two Networks in Series

Two networks are said to be connected in series when the output of one network is connected to the input of the other network. Let the constants of the, network be A_1, B_1, C_1, D_1 and the constants of the networks be A_2, B_2, C_2, D_2, provided there are two networks connected is series as shown in the below figure.



From the figure,

$$V = D_1V_1 - B_1I_1$$
$$I = -C_1V_1 + A_1I_1$$

And,
$$V = A_2V_2 + B_2I_2$$
$$I = C_2V_2 + D_2I_2$$

From the above equations, we can see that

$$D_1V_1 - B_1I_1 = A_2V_2 + B_2I_2 \quad (21)$$

$$-C_1V_1 + A_1I_1 = C_2V_2 + D_2I_2 \quad (22)$$

Multiply Equation (21) with A_1 and (22) with B_1 and adding the equations,

$$\begin{aligned} & A_1D_1V_1 - A_1B_1I_1 = A_1A_2V_2 + A_1B_2I_2 \\ + \quad & -B_1C_1V_1 + A_1B_1I_1 = B_1C_2V_2 + B_1D_2I_2 \end{aligned}$$

$$V_1(A_1D_1 - B_1C_1) = (A_1A_2 + B_1C_2)V_2 + (A_1B_2 + B_1D_2)I_2 \quad (23)$$

Multiply (21) with C_1 and (22) by D_1 and the equations,

$$\begin{aligned} & C_1D_1V_1 - B_1C_1I_1 = A_2C_1V_2 + B_2C_1I_2 \\ + \quad & -C_1D_1V_1 + A_1D_1I_1 = C_2D_1V_2 + D_1D_2I_2 \end{aligned}$$

$$I_1(A_1D_1 - B_1C_1) = V_2(A_2C_1 + C_2D_1) + I_2(B_2C_1 + D_1D_2) \quad (24)$$

We know that $A_1D_1 - B_1C_1 = 1$

Equations (23) and (24) represent the combination of two network and representing as a single network with a single constants.

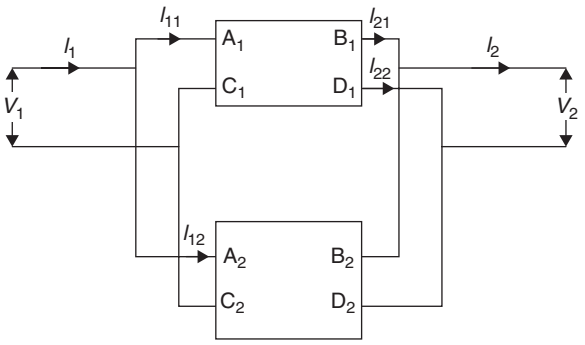
$$\therefore \quad A = A_1A_2 + B_1C_2$$
$$B = A_1B_2 + B_1D_2$$
$$C = A_2C_1 + C_2D_1$$
$$D = B_2C_1 + D_1D_2$$

Therefore the above relations represented in matrix form as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

Constants for Two Networks in Parallel

The two networks connected in parallel are as shown below and the constants for the combination of two networks are calculated as follows.



From the above network, we can write that

$$V_1 = A_1V_2 + B_1I_{21} \quad (25)$$

$$V_1 = A_2V_2 + B_2I_{22} \quad (26)$$

And the combination of the two networks as a single network can be written is the form of equations as

$$V_1 = AV_2 + BI_2$$

where
$$I_2 = I_{21} + I_{22}$$

Multiply Equation (25) with B_2 and (26) with B_1 and adding these results in

$$\begin{aligned} & V_1B_2 = A_1B_2V_2 + B_2B_1I_{21} \\ + \quad & V_1B_1 = A_2B_1V_2 + B_1B_2I_{22} \end{aligned}$$

$$V_1(B_1 + B_2) = (A_1B_2 + A_2B_1)V_2 + B_1B_2(I_{21} + I_{22})$$

$$\Rightarrow \quad V_1 = \frac{A_1B_2 + A_2B_1}{B_1 + B_2}V_2 + \frac{B_1B_2}{B_1 + B_2}I_2 \quad (27)$$

From Equation (27), we can infer that

$$A = \frac{A_1B_2 + A_2B_1}{B_1 + B_2} \text{ and } B = \frac{B_1B_2}{B_1 + B_2}$$

As the transmission line in a symmetric network,

$$A = D = \frac{A_1B_2 + A_2B_1}{B_1 + B_2} = \frac{D_1B_2 + D_2B_1}{B_1 + B_2}$$

And $AD - BC = 1$ and substituting A, B, D in this equation, we get

$$\Rightarrow \quad \left(\frac{A_1B_2 + A_2B_1}{B_1 + B_2}\right)\left(\frac{A_1B_2 + A_2B_1}{B_1 + B_2}\right) - \frac{B_1B_2}{B_1 + B_2} \times C$$

$$\Rightarrow \quad C = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}$$

Example 34: A 220 kV, 20 km long, 3-ϕ transmission line has the following A, B, C, D constants.

$A = D = 0.96\angle 3^\circ$
$B = 55\ \angle 65^\circ\ \Omega$/phase
$C = 0.5 \times 10^{-4}\ \angle 90^\circ$ S/phase

Its charging current per phase is

(A) 6.3 A (B) 11 A
(C) 127 A (D) 220 A

Solution: (A)

Admittance per phase $(Y) = C$

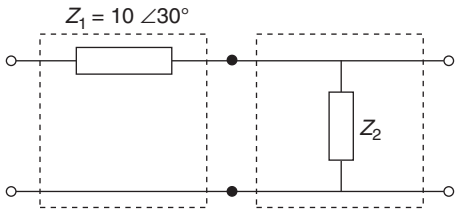
$$= 0.5 \times 10^{-4} \angle 90^{\circ}$$

Charging current per phase $= V_{pn} \times Y$

$$= \frac{220}{\sqrt{3}} \times 10^{3} \times 0.5 \times 10^{-4}$$

$$= \frac{11}{\sqrt{3}} \text{ A} = 6.35 \text{ A}$$

Example 35: Two networks are connected in cascade as shown in figure with the usual notation the equivalents A, B, C and D constants are obtained. Given that $C = 0.025 \angle 40^{\circ}$, the value of Z_2 is



(A) $40\angle 40^{\circ}$ (B) $20\angle 40^{\circ}$
(C) $10\angle 20^{\circ}$ (D) $20\angle 20^{\circ}$

Solution: (A)

For the given two part network

$$\text{The open-circuit admittance} = \frac{1}{Z_2}$$

$$\therefore C = \frac{1}{Z_2}$$

$$Z_2 = \frac{1}{C} = 40\angle -40^{\circ}\Omega$$

Example 36: The total reactance and total susceptance of a lossless overhead EHV line operating at 50 Hz are given by 0.04 p.u. and 1.1 p.u., respectively. If the velocity of wave propagation is 30×10^5 km/s, then the approximate length of line is

(A) 200 km (B) 2000 km
(C) 20 km (D) 2 km

Solution: (B)

Velocity of wave propagation $(\vartheta) = \frac{1}{\sqrt{(\text{L/km})/(\text{C/km})}}$

Total reactance of line $(X_L) = 0.04$ p.u.

$$\text{L/km} = \frac{X_L}{2\pi f l} = \frac{0.04}{2\pi \times 50 \times l}$$

Total susceptance of line = 1.1 p.u.

$$\frac{1}{X_c} = 2\pi f C = 1.1 \text{ p.u.}$$

$$C = \frac{1.1}{2\pi \times 50}$$

$$\text{C/km} = \frac{1.1}{2\pi \times 50 \times l}$$

$$\text{Velocity of propagation} = \frac{1}{\sqrt{LC}}$$

$$30 \times 10^5 = \frac{l}{\sqrt{\frac{0.04}{2\pi \times 50} \times \frac{1.1}{2\pi \times 50}}}$$

$$l = 2124.5 \text{ km}$$

Example 37: A single-phase 50 Hz generator supplies an inductive load of 5 MW at a power factor of 0.7 lagging by means of an overhead transmission line 20 km long. The line resistance and inductance are 0.019 Ω and 0.62 mH per km. The voltage at receiving end is required to be kept at 10 kV.

Calculate

1. The sending end voltage and voltage regulation of the line.
2. The value of capacitors to be placed in parallel with the load such that the regulation is reduced to 60% of that obtained in the previous problem.
3. Calculate the transmission efficiency in case (i) and (ii).

Solution: Given transmission line is 20 km, so it is considered as a short line.

$\therefore$ $I = I_R = I_S$ given by

$$I_R = \frac{5 \times 10^3}{10 \times 0.7} = \frac{P}{V \cos\phi} = 714 \text{ A}$$

Source voltage

$$|V_s| = |V_R| + |I|(R\cos\phi_R + X\sin\phi_R)$$

$$= 10{,}000 + 714(0.38 \times 0.7 + 3.89 \times 0.714)$$

Sending end voltage = 12.173 kV

$$\text{Voltage regulation} = \frac{12.173 - 10}{10} \times 100$$

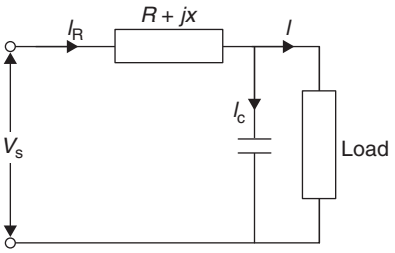
$$= 21.73\%$$

(ii) Desired voltage regulation = 60% of 21.73

$$= 0.6 \times 21.73 = 13.03\%$$

$$\text{Regulation } \frac{V_s - 10}{10} = 0.13$$

$$V_s = 11.3 \text{ kV}$$



Source voltage in the above circuit after placing capacitor to improve voltage regulation

$$V_s = V_r + I_R(R\cos\phi_r + X\sin\phi_r)$$

$$V_s - V_r = 0.13 \text{ kV} = I_R(R\cos\phi_r + X\sin\phi_r)$$

Since capacitance does not draw any real power

$$|I_R| = \frac{5000}{10\times\cos\phi_R}$$

From the above two equations, $\cos\phi_r = 0.8685$, $I_R = 575.7$ A

Now capacitor current $I_c = I_R - I$

$$= 575(0.8685 - j0.495) - 714(0.7 - j0.714)$$

$$= 0.195 + j225.17$$

Capacitor current $I_C = j225.17$ A

$$X_c = \frac{1}{\omega c} = \left|\frac{V_R}{I_C}\right| = \frac{10\times1000}{225.17}$$

$$C = \frac{225.17}{10^4\times\omega} = \frac{225.17}{10^4\times314} \Rightarrow C = 71.71\ \mu\text{F}$$

(iii) Efficiency of transmission

$$\eta = \frac{\text{Output}}{\text{Output} + \text{Losses}}$$

Case (i): $\eta = \dfrac{5000}{5000 + 714^2\times0.38\times10^{-3}} = 0.962\%$

Case (ii): $\eta = \dfrac{5000}{500 + 575^5\times0.38\times10^{-3}} = 97.5\%$

Example 38: A 200 km, 3-ϕ, 50 Hz transmission line delivering 25 MVA at 0.85 lagging power factor to a balanced load at 132 kV. The line conductors are spaced equilaterally 2.5 m apart. The conductor resistance is 0.1 Ω/km and its effective diameter is 1.5 cm. Calculate the sending end voltage and voltage regulation.

Solution: Transmission line parameters

$$L = 0.461\log\frac{D}{r^1} = 0.461.\log.\frac{250}{0.778\times0.75} = 1.21 \text{ mH/km}$$

$$C = \frac{0.0242}{\log\frac{D}{r}} = \frac{0.0242}{\log\frac{250}{0.75}} = 0.0096\ \mu\text{F/km}$$

$$R = 0.1\times200 = 20\ \Omega$$

$$X_L = 2\pi fL = 2\pi\times50\times1.21\times10^{-3}\times200 = 76.02\ \Omega$$

$$Z = R + jX_L = 20 + j76.02 = 78.6\angle75.2^\circ$$

$$Y = j\omega cl = j.314\times0.0096\times10^{-6}\times200$$

$$Y = 6.02\times10^{-4}\angle90^\circ$$

$$I_R = \frac{25\times1000}{\sqrt{3}\times132}\angle31.78 = 109.3\angle-31.78$$

$$V_R\ (\text{per phase}) = \frac{132}{\sqrt{2}}\angle0^\circ = 76.2\angle0^\circ \text{ kV}$$

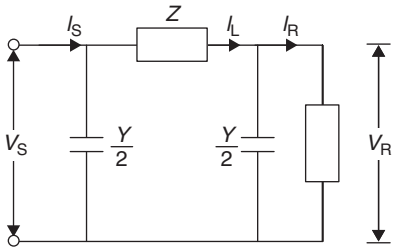


Figure 15 Medium line π–representation

Source voltage

$$V_s = \left(1 + \frac{1}{2}YZ\right)V_R + ZI_R$$

$$= \left[1 + \frac{1}{2}\times6.02\times10^{-4}\angle90\times78.6\angle75.2\right]\times76.2$$

$$+ 78.6\angle75.2\times109.3\angle-31.78\times10^{-3}$$

$$= 76.2 + 1.8\angle166.2 + 8.59\angle43.42$$

$$V_S = 80.69 + 6.3j = 80.93\angle4.48^\circ$$

$$|V_s|(\text{line}) = 80.93\times\sqrt{3} = 140.17 \text{ kV}$$

$$1 + \frac{1}{2}Yz = 1 + \frac{1}{2}\times6.02\times10^{-4}\angle90\times78.6\angle75.2$$

$$= 1 + 0.023\angle165.2 = 0.977 + j0.006$$

$$|V_{R_0}|(\text{line no load}) = \frac{140.17}{0.977} = 143.47 \text{ kV}$$

$$\text{Voltage regulation} = \frac{143.47 - 140.17}{140.17} = 2.4\%$$

Example 39: A 50 Hz, 250 km long transmission line has a total series impedance of 30 + j 100 Ω and a total shunt admittance of 10^{-3} Ω. The receiving end load is 40 MW at 220 kV with 0.8 lagging power factor. Calculate the sending end voltage, current, power and power factor using.

(i) Short line approximation
(ii) Nominal π-method

Solution: Impedance of the line $Z = 30 + j100\ \Omega$

$= 104.4\ \angle 73.3$

$Y = 10^{-3}\ \angle 90^{o}\ \because$

The receiving end load is 40 MW at 220 kV, 0.8 p.f. lagging

$$\therefore I_R = \frac{40}{\sqrt{3}\times 220\times 0.8}\angle -36.9^\circ$$

$$= 0.1312\angle -36.9^\circ \text{kA} \qquad V_R = \frac{220}{\sqrt{3}}\angle 0^\circ = 127\angle 0^\circ \text{ kV}$$

(i) Short line approximation

$V_s = V_R + I_R Z = 127 + (0.164\ \angle -36.9 \times 104.4\ \angle 73.3)$

$= 127 + 17.12\ \angle 36.4$

$= 141.14\ \angle 4.127^{o}$

$= 140.7 + 10.15j$

$|V_s|_{\text{line}} = 244.46^\circ$ kV

$I_s = I_R = 0.1312\ \angle -36.9$

Sending end power factor

$= \cos(4.127 + 36.9) = 0.754$ lagging

Sending end power $= \sqrt{3}\times 244.4\times 0.1312\times 0.754$

$= 41.87$ MW

(ii) Nominal-π method

$$A = D = 1+\frac{1}{2}YZ = 1+\frac{1}{2}\times 10^{-3}\angle 90\times 104.4\angle 73.3^\circ$$

$= 1 + 0.0522\ \angle 163.3 = 0.95\ \angle 0.90^{o}$

$B = Z = 104.4\ \angle 73.3^{o}$

$$C = Y\left(1+\frac{1}{4}YZ\right) = Y\left(1+\frac{1}{4}\times 10^{-3}\angle 90\times 104:4\angle 73.3\right)$$

$= 9.75 \times 10^{-4}\ \angle 90.4^{o}$

$V_s = AV_R + BI_R = 0.95\ \angle 0.90 \times 127 +$
$104.4\ \angle 73.3 \times 0.13\ \angle -36.9^{o}$

$= 120.6\ \angle 0.9 + 13.57\ \angle 36.4$

$V_s = 131.88\ \angle 4.32$

$|V_s|_{\text{line}} = 228.4$ kV

$I_s = CV_R + DI_R$

$= 9.75 \times 10^{-4}\ \angle 90.4 \times 127 + 0.95\ \angle 90^{o}$
$\times 0.13\ \angle -36.9^{o}$

$= 0.123\ \angle 90.4 + 0.1235\ \angle -36$

$I_s = 0.111\ \angle 26.96^{o}$

Sending power factor = cos(26.96 – 4.32) = cos 22.64 = 0.922 leading

Sending end power

$= \sqrt{3}\times 228.4\times 0.111\times 0.922 = 40.5$ MW

Underground Cables

Underground transmission has more advantages over overhead transmission like low maintenance, cost, good looking, free from damage due to lightning and thunder storms and has no interference with communication circuit. The underground transmission is done by underground cables.

The cable used in underground transmission generally has three parts.

1. The core (or) conductor.
2. The insulating material.
3. A metallic sheath of lead is provided around the insulation to provide external protection.

The characteristics of insulating material used in cables are

(i) High dielectric strength.
(ii) Good mechanical properties and the insulating material should not affected by chemicals.
(iii) High insulation resistance.

Depend upon the operating voltage and temperature, different types of insulating materials used in cables those are listed as follows.

(i) Polyvinyl chloride (PVC)
(ii) Impregnated paper
(iii) Vulcanized rubber
(iv) Polythene

Classification of Cables

The underground cables are classified depending upon the operating voltages are

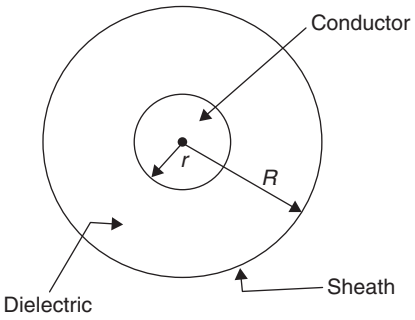
1. Low-tension cable (LV-cable): These cables are generally used for an operating voltage below 1 kV.
2. High-tension cable (HV-cable): These cables are used for an operating voltage below 11 kV.
3. Super tension cable (ST-cable): These cables are used for an operating voltage up to 33 kV.
4. Extra high-tension cable (EHT-cable): These cables are used for an operating voltage of 66 kV.
5. Extra super tension cable: These cables are used for an operating voltage of above 132 kV.

Dielectric Stress in Cable

In a cable the current carries in a core (or) conductor of the cable. So, the stress is more on the surface of the core and it decreases towards the surface of the cable (or) sheath of the cable.

Maximum gradient at the surface of the conductor is

$$g_{\max} = \frac{V}{r\ln\left(\frac{R}{r}\right)}$$



Minimum gradient at the sheath of the cable

$$g_{min} = \frac{V}{R \ln\left(\frac{R}{r}\right)}$$

Here, V is the voltage applied to the cable
r – Radius of the conductor (or) core
R – Radius of the cable

Grading of Cable

Grading of cable is done according to the stress on the dielectric material of the cable. There are two methods of grading.

1. Capacitance grading
2. Inter sheath grading

Capacitance Grading

In this method, different dielectric materials having different permittivities are used and these materials are used in cables by two ways.

1. The factor of safety for all the materials is same, i.e., the material which has highest product of dielectric strength and permittivity should be placed near to the core of the cable.
2. The same working stress for different materials but the material which has highest permittivity placed near to the conductor of the cable.

Intersheath Grading

In this method, different metallic sheaths are used in between the core and sheath of the cable for the same dielectric material by applying different voltages to different metallic sheath from operating voltage to earth potential. So, the stress on the same material also changed corresponding to the voltage.

Laying and Heating of Cables

Generally there are three methods of laying of cables

1. Draw-in-system
2. Direct laying
3. Solid system

In cable if the rate of heat generation is more than the rate of heat dissipation, then the temperature of the cable Increases. The sources of heat generation in cables are

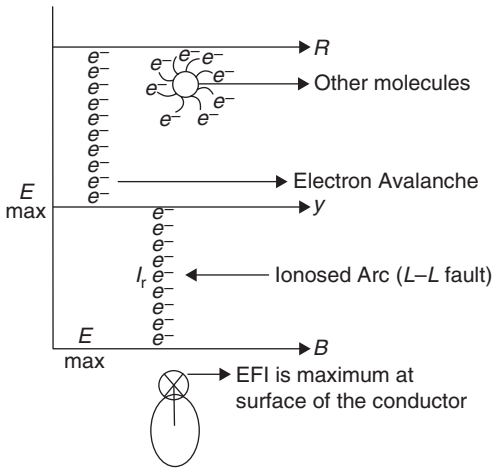
1. Dielectric loss
2. Copper loss in the core
3. Sheath losses

Corona

The ionization of air surrounding the power conductor is corona. There are free electrons normally present in the free space due to radioactivity and cosmic rays. The gradient around the surface of the conductor increases with increases in the potential between the conductors. The free electrons present in the air will move with certain velocity depending on the field strength. These moving electrons collide with the molecules of the air and some of the fast moving electrons may also dislodge electrons from the air molecules, thereby increasing the number of electrons. Hence the process of ionization is thus cumulative and may lead to electron avalanche. Hence the air surrounding the conductor may be ionized.

When a voltage higher than the critical voltage is applied between two parallel polished wires, the glow is quite obvious. After it is operated for some time, reddish heads or tufts form along the wire, bluish white glow is appeared around the surface of the wire.

It is noticed that the reddish tufts or beads are formed when the conductor is negative, and a smoother bluish white glow is formed when the conductor is positive. The corona due to AC is same as the corona due to DC when viewed through stroboscope. Due to corona, a hissing noise is observed and ozone gas is formed which is detected by its characteristic odour.

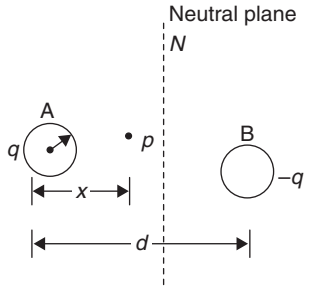


Overhead transmission lines are used for all economic reasons than underground cables. If line-to-line operating voltage ≥ 275 kV, then there will be abnormal electric field intensity (EFI) at the surface of the conductor.

EFI (increase or decrease a potential difference a mobility of electrons).

Critical Disruptive Voltage

Consider a single-phase transmission line. Let 'r' be the radius of each conductor and 'd' be the distance of separation such that 'd' is far greater than 'r'. Let 'q' be the charge per unit length of the conductor on one of the conductors and hence '$-q$' is on the other. If the operating voltage is 'V', the potential of conductor 'A' with respect to neutral plane 'N' will be $\frac{V}{2}$ and that of 'B' will be '$\frac{-V}{2}$'. Consider a point 'p' at a distance 'x' where we want to find the electric field intensity.



The field due to 'A' will be repulsive and due to 'B' would be attractive by bringing a unit positive charge from infinity to that point 'p'. Hence the EFI's due to A and B at 'p' would be additive

$$E = \frac{q}{2\pi\varepsilon_0 x} + \frac{q}{2\pi\varepsilon_0 (d-x)} = \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{x} + \frac{1}{d-x}\right]$$

The potential difference between the conductors,

$$V = -\int_{d-r}^{r} E\,dx = \int_{r}^{d-r} \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\ln x - \ln(d-x)\right]r^{d-r}$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\ln(d-r) - \ln\left(d-(d-r)\right) - \ln r + \ln(d-r)\right]$$

$$= \frac{q}{2\pi\varepsilon_0} \times 2\ln\left[\frac{d-r}{r}\right] = \frac{q}{\pi\varepsilon_0}\ln\left[\frac{d-r}{r}\right]$$

as $d >> r$, $d - r \cong d$,

$$\Rightarrow \quad V = \frac{q}{\pi\varepsilon_0}\ln\frac{d}{r}$$

Now the EFI at the point, which is 'x' distance from the centre of the conductor 'A' is given by,

$$E = \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{x} + \frac{1}{d-x}\right] = \frac{q}{2\pi\varepsilon_0} \cdot \frac{d}{x(d-x)}$$

From (1), substitute 'q' in above equation.

$$q = \frac{\pi\varepsilon_0 V}{\ln\frac{d}{r}}$$

$$\Rightarrow \quad E = \frac{\pi\varepsilon_0 V}{\ln\frac{d}{r}} \times \frac{1}{2\pi\varepsilon_0} \times \frac{d}{x(d-x)} = \frac{V}{2\ln\frac{d}{r}} \times \frac{d}{x(d-x)}$$

Let $V' = \frac{V}{2}$ which is line to neutral voltage. In case of three-phase system, $V' = \frac{V}{\sqrt{3}}$, where 'V' is line-to-line voltage

$$\Rightarrow \quad E = \frac{V'd}{x(d-x)\ln\frac{d}{r}}$$

From the above expression, it is clear the EFI is inversely proportional to 'x', i.e. as 'x' decreases, EFI increases and it is maximum when $x = r$, i.e., at the surface of the conductor and this value i given by

$$E_{max} = \frac{V'd}{(d-r)r\ln\frac{d}{r}} \approx \frac{V'}{r\ln\frac{d}{r}} \Rightarrow V' = E_{max}\, r \ln\frac{d}{r}$$

Critical disruptive voltage is hence defined as the voltage at which the complete disruption of dielectric occurs. This voltage gives the gradient at the surface of the conductor equal to the breakdown strength of the air. This dielectric strength is normally denoted by g_0 and is equal to 30 kV/cm peak at normal temperature and pressure, i.e. 25°C and 76 cm of Hg. At any other temperature and pressure,

$$g_0' = g_0 \times \delta$$

where 'δ' is the air density correction and is given by

$$\delta = \frac{3.92\,b}{273 + t}$$

where 'b' is the barometric pressure in cm of Hg and 't' the temperature in °c.

Hence, the critical disruptive voltage is given by

$$V' = rg_0\delta \ln\frac{d}{r} \text{ kV.}$$

where the gradient 'g' is nothing but electric field intensity (EFI).

The above expression is derived based on the consideration that the conductor is solid and the surface is smooth. ACSR conductors are used for higher voltages. The potential gradient for ACSR conductors will be greater than the potential gradient of an equivalent smooth conductor, and hence the breakdown voltage of a stranded conductor will be comparatively less than a smooth and solid conductor. The irregularities on the surface of such a conductor are increased further due to the deposition of dust and dirt on its surface and hence the breakdown voltage is further reduced.

Hence, the final expression for the critical disruptive voltage after considering atmospheric conditions and the surface of the conductor is

$$V' = rg_0\delta m_0 \ln\frac{d}{r} \text{ kV.}$$

When the voltage applied corresponds to the critical disruptive voltage, corona creates but may not be visible as the charged ions in the air must receive some finite energy to cause further ionization by collision. For a radial field, it must reach a gradient g_v at the surface of the conductor to cause a gradient g_0, a finite distance away from the surface of the conductor. 'g_v' is not constant as 'g_0' is, and is a function of the size of the conductor.

$$g_v = g_0\delta\left(1-\frac{0.3}{\sqrt{r\delta}}\right)\text{kV/cm}$$

If 'V_v' is the critical visual disruptive voltage, then

$$V_v = g_v r \ln\frac{d}{r}$$

$$\Rightarrow \quad g_v = \frac{V_v}{r\ln\frac{d}{r}} = g_0\delta\left(1+\frac{0.3}{\sqrt{r\delta}}\right)$$

$$\Rightarrow \quad V_v = rg_0\,\delta\left[1+\frac{0.3}{\sqrt{r\delta}}\right]\ln\frac{d}{r}\text{ kV}$$

In case the irregularity factor is taken into account,

$$V_v = g_0 m_v \delta r\left[1+\frac{0.3}{\sqrt{r\delta}}\right]\ln\frac{d}{r}$$

$$= 21.1\, m_v\,\delta r\left[1+\frac{0.3}{\sqrt{r\delta}}\right]\ln\frac{d}{r}\text{ kV RMS}$$

where 'r' is the radius in cm. The irregularity factor m_v has the following values:

m_v = 1.0 for polished wires

= 0.98 to 0.93 for rough conductor exposed to atmospheric conditions

= 0.72 for local corona on stranded conductors

Since the surface of the conductor is irregular, the corona does not start immediately on the whole surface but it takes place at difference points of the conductor which are pointed and this is known as local corona.

Example 40: A three-phase line has conductors 4 cm in diameter spaced equally 2 m apart. If the dielectric strength of the air is 30 kV (max) per cm, find the disruptive critical voltage for the line. Take air density factor, d = 0.951 and irregularity factor = 0.9.

Solution: Conductor radius = $\frac{4}{2}$ = 2 cm

Conductor spacing = 2 m = 200 cm

Dielectric strength of the air,

g_0 = 30 kV/cm (max)

= 21.2 kV (RMS) per cm

Disruptive critical voltage

$$V' = rg_0\delta m_0 \ln\left(\frac{d}{r}\right)\text{kV.ph}$$

$$= 2\times 21.2\times 0.951\times 0.9\times\ln\left(\frac{200}{2}\right)$$

= 167.28 kV/ph

$\therefore$ Line voltage (RMS) = $\sqrt{3}\times 167.28$

= 289.72 kV

Example 41: A 132 kV line with 2 cm diameter conductors is built so that corona takes place if the line voltage exceeds 220 kV (RMS). If the value of potential gradient at which ionization occurs can be taken as 30 kV per cm, find the spacing between the conductors.

Solution: If the line is 3-ϕ, Radius of the conductor, $r = \frac{2}{2} = 1$ cm. Dielectric strength of the air, $g_0 = \frac{30}{\sqrt{2}} = 21.2$ kV (RMS). Disruptive voltage/phase = $\frac{220\text{ kV}}{\sqrt{3}} = 127.020$ kV. If the conductors are assumed to be smooth conductors, then irregularity factor $m_0 = 1$. At standard temperature and pressure, air density factor d = 1

$$\therefore \qquad V' = rg_0\delta m_0 \ln\left(\frac{d}{r}\right)\text{kV}$$

where 'd' is spacing between the conductors.

$$127.020 = 1\times 21.2\times 1\times 1\times\ln\left(\frac{d}{1}\right)$$

$$\Rightarrow \qquad \ln\left(\frac{d}{1}\right) = 5.9915 \Rightarrow d = 400.014\text{ cm}$$

Corona Loss

The electric field produces ions in the space which further leads to the production of space charge which move around the conductor. If there is a continuous transmission of energy, there is a continuous motion for these charges. Hence, it can indirectly be said that the energy required for the charges to remain in motion is derived from the supply system. In order to maintain the flow of energy over the conductor in the field where in this additional energy would have been otherwise absent, it is necessary to supply this additional loss from the supply system. This additional power is referred as corona loss.

An empirical relation was derived by peek by conducting a number of experiments on corona loss.

$$P = 241\times\frac{(f+25)}{\delta}\sqrt{\frac{r}{d}}\left(V_P - V_C\right)^2\times 10^{-5}\text{ kW/km/phase}$$

where 'f' is the supply of frequency,
'δ' – Air density correction factor,
V_p – The operating voltage in kV
V_0 – Critical disruptive voltage

and $\delta = \frac{3.92\,b}{273+t}$,

where b = Atmospheric pressure
t = Temperature.

The above equation is for a fair weather condition. For a foul weather condition, the power loss can be obtained by taking V_0 as 0.8 times the fair weather value. The above empirical formula has certain limitations and is applicable only if the frequency is in between 25 and 120 Hz, the conductor radius is greater than 0.25 cm and the ratio of $\frac{V_p}{V_0} > 1.8$.

Example 42: A 3-ϕ 220 kV, 50 Hz transmission line consists of 3 cm diameter conductor spaced 4 m apart in an equilateral triangle form. If the temperature is 40°C, atmospheric pressure is 76 cm, calculate the corona loss per km of the line. Take $m_0 = 0.9$.

Solution: We know that

$$P = 241 \times 10^{-5} \frac{(f+25)}{\delta}\sqrt{\frac{r}{d}}\ (V_p - V_0)^2 \text{ kW/km/phase}$$

$$\delta = \frac{3.92 \times b}{273+t} = \frac{3.92 \times 76}{273+40} = 0.952$$

$g_0 = 21.2$ kV/cm (RMS)

Critical disruptive voltage per phase

$$V_0 = m_0 g_0\, \delta\, r \log_e \frac{d}{r} \text{ kV}$$

$$V_0 = 0.9 \times 21.2 \times 0.952 \times 1.5 \times \log_e \frac{200}{1.5}$$

$$= 27.24 \times 4.80 = 133.281 \text{ kV}$$

Supply voltage per phase,

$$V_p = \frac{220}{\sqrt{3}} = 127 \text{ kV}$$

$$\therefore P = 241 \times 10^{-5} \times \frac{(50+25)}{0.952} \times \sqrt{\frac{1.5}{200}} \times (127 - 133.281)^2 \text{ kW/km/phase}$$

$$= 0.01644 \times 39.450 = 0.648 \text{ kW/km/phase}$$

$\therefore$ Total corona loss per km for the three phases

$$= 3 \times 0.648 = 1.944 \text{ kW.}$$

Example 43: Which of the following will minimize the corona loss?
(A) Reduction in conductor size
(B) Reduction in smoothness of the conductor
(C) Sharp points are provided in the line hardware
(D) Current density is conductor is reduced.

Solution: (D)

Example 44: The corona loss on a particular system at 50 Hz is 1.2 kW/km/phase. The corona loss at 60 Hz would be
(A) 1.2 kW/km/ph (B) 1.36 kW/km/ph
(C) 1.14 kW/km/ph (D) 1.42 kW/km/ph

Solution: (B)

Corona loss $\alpha (f + 25)$

$$\frac{P_2}{P_1} = \frac{f_2 + 25}{= f_1 + 25}$$

$$P_2 = \frac{60+25}{50+25} \times P_1 = 1.36 \text{ kW/km/ph}$$

Factors Affecting Corona Loss

The factors that affect corona loss on overhead transmission lines:

1. Electrical factors
2. Atmospheric factors and
3. Factors with respect to conductors

Electrical Factors

According to the above power loss expressions corona loss is a function of frequency. Hence if the frequency increases, the corona loss also increases proportionately. Hence we can infer that DC corona loss is comparatively less than AC corona loss. Hence if in case of AC supply, there may be harmonic which may further increase the corona loss.

Field Around the Conductor

The field around the conductor is a function of the voltage. Along with .voltage, field also depends on the configuration of the conductors. The height of the conductors from the ground also effects the corona loss. The smaller the height, the greater will be the corona loss.

Atmospheric Factors

Pressure and Temperature Effect

From the expression for power loss, we can observe that it is a function of air density correction factor 'δ' and even critical disruptive voltage is a function of 'δ'. The lower the value of 'δ', the higher is the loss. For lower values of 'δ', the pressure should be low and the temperature should be high. Hence the corona loss is low in plain areas and high in hilly areas.

Dust, Rain, Snow and Storm Effect

The particles of dust deposit on the conductor, thereby reducing the critical disruptive voltage for local corona and also the rains, snow and hail storm reduce the critical disruptive voltage and the corona loss increases.

Factors with Respect to Conductor

Diameter of the Conductor

From the expression, we can see that the corona loss is directly proportional to the radius of the conductor and also we know that loss $\alpha\ (V_p - V_0)^2$, where V_0 is the critical disruptive voltage which is inversely proportional to the radius of the conductors. Hence, the larger the size of the conductor, the lower is the corona loss.

Number of Conductors/phase

If the operating voltage ≥ 380 kV, it is found that using one conductor per phase produces large coroner loss and also large radio interference in the communication lines running parallel to power lines. Hence, using '2' or more conductors per phase reduces the corona loss, i.e. using bundled conductors. By using bundled conductors, the self-GMD of the conductor reduces increases the critical disruptive voltage and thereby the corona loss reduces.

Heating of the Conductor by Load Current

The current passing through the conductor produces I^2R losses which would help to reduce the corona loss. If there is no such heat produced, then the conductor's temperature would be slightly less than the surrounding temperature. In the absence of heating, due to high humidity at times, additional corona is introduced.

Methods of Reducing Corona Loss

The losses can be reduced by

1. Large diameter conductors
2. Hollow conductors
3. Bundled conductors
4. By increasing the conductor's spacing

Advantages and Disadvantages of Corona

Advantages

1. Due to corona formation, the air surrounding the conductor also becomes conducting and therefore virtually the diameter of the conductor increases. This increased diameter reduces electrostatic stresses between the conductors.
2. Corona reduces the effects of transients produced by surges by dissipating them as corona loss.

Disadvantages

1. Corona reduces the energy to be transmitted thereby reducing the transmission efficiency of the transmission line.
2. The ozone produced during the corona may cause corrosion of the conductor.
3. The current drawn by the conductor during the time of corona is non-sinusoidal and produces non-sinusoidal voltage drop which may cause inductive interference with neighbouring communication lines.
4. Due to corona, the effective capacitance of the conductors is increased as the effective diameter of the conductor is increases. The charging current flowing also increases.

Example 45: A 3-ϕ line is operating at 220 kV and 50 Hz. The line has a conductor of 1.3 cm diameter arranged in a 3 m delta connection. Assume air density factor of 1.05 and the dielectric strength of air to be 21.1 kV/cm, the corona losses will be (m = 0.75)

(A) 5 kW/km (B) 10 kW/km
(C) 15 kW/km (D) 20 kW/km

Solution: (C)

$$\text{Phase voltage} = \frac{220}{\sqrt{3}}\text{kV} = 127\text{ kV}$$

Critical disruptive voltage

$$(E_0) = mg_0\delta\left(\ln\frac{d}{r}\right)\text{kV/phase}$$

$$= 0.75 \times 1.05 \times 21.1\ \text{In}\left(\frac{300}{0.65}\right)$$

$$= 101.93 \approx 102\text{ kV}$$

Critical disruptive voltage $E_0(L-L)$

$$= \sqrt{3} \times 102 = 176.6\text{ kV}$$

Corona loss in the system is given by

$$P = 241 \times 10^{-5}\left(\frac{f+25}{8}\right)\sqrt{\frac{r}{D}}\left(E - E_0\right)^2\text{ kW/phase/km}$$

$$= 241 \times 10^{-5} \times \frac{75}{\delta} \times 29.09$$

$$= 5.007\text{ kW/phase/km}$$

$$= 15.02\text{ kW/km}$$

Skin Effect

When a current-carrying conductor is carrying direct current (DC), the current is uniformly distributed over the whole cross section of the conductor. But, if an alternating current is flowing through the conductor, the current does not distribute uniformly, instead, it is concentrated more on the surface of the conductor compared to the current flowing through the centre of the conductor. This is known as skin effect.

The tendency of the alternating current to concentrate near the surface of a conductor is known as skin effect.

Due to skin effect, the effective area of cross section of the conductor through which the current flows is reduced. Hence, the resistance of the conductor increases while carrying an alternating current. The conductor consisting of a large number of strands each carrying part of the total current. The inner strands carrying currents give rise to flux which link only the inner strands whereas the strands outside produce flux which links both inner and outside strands. The flux linkages per ampere are more for the inner strands than for the outer strands. Thus the inductance as well as impedance of the inner strands is more than that of

the outer strands, resulting in more current to flow in outer strands. Skin effect depends on the following factors.

1. Nature of the material
2. Radius of the conductor
3. Frequency of supply
4. Shape of wire

Skin effect is more if the frequency of supply is more and is proportional to the size of the conductor.

Proximity Effect

The alternating magnetic flux produced in the conductor due to current flowing through the neighbouring conductor gives rise to circulating currents which apparently increases the resistance of the conductor. This phenomenon is called proximity effect. Consider a two-wire system in which the lines of flux link with inner and outer elements of the conductor but most of them link with the elements farther apart compared to the inner elements. Hence the inductance of the farthest elements are more compared to the nearer elements, thereby impedance of the farther elements are more compared to nearer elements. This gives raise to non-uniform distribution of current. The effective resistance is thereby increased due to non-uniform distribution of current. The proximity effect is more in case of underground cables, where the distance between overhead line transmission lines is more Proximity effect depends on conductor's size, frequency, distance between conductors and permeability of conductor material.

Ferranti Effect

A long-transmission line operated under no load or lightly loaded condition, the receiving end voltage is more than the sending end voltage. This phenomenon is known as Ferranti effect. This is explained as follows.

We know the equation of voltage

$$V = \frac{V_r + I_r Z_c}{2} e^{\alpha X} e^{j\beta X} + \frac{V_r + I_r Z_c}{2} e^{-\alpha X} e^{-j\beta X}$$

under no-load condition,

The above equation reduces to

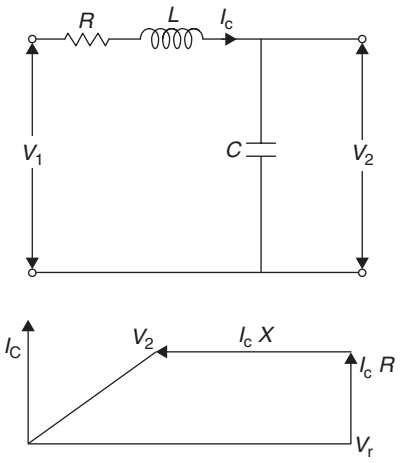
When $x = L, I_r = 0$

$$V_s = \frac{V_r}{2} e^{\alpha L} e^{j\beta L} + \frac{V_r}{2} e^{-\alpha L} e^{-j\beta L}$$

At $L = 0, V_r = \frac{V_r}{2} + \frac{V_r}{2}$

As 'L' increases, the incident component of sending end voltage increases exponentially and turns the vector anticlockwise through an angle βL whereas the reflected sending end voltage decreases by the same amount and is rotated clockwise through the same angle βL. The sum of these two components of sending end voltage gives a voltage which is smaller than the receiving end voltage V_r.

Ferranti effect can be explained by approximating the distributed parameters of the long transmission line by lumped impedance as shown below.



As the capacitive reactance of the line is more than the inductive reactance, under no load condition the line current is of leading power factor. The charging current produces drop in the reactance of the line which is in phase opposition to the receiving end voltage and therefore the sending end voltage becomes smaller than the receiving end voltage. This is shown in the above phasor diagram.

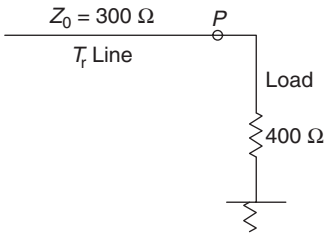
Example 46: The surge impedance of a 200 km long overhead line is 200 Ω. For a 100 km length of the same line, the surge impedance will be

(A) 200 Ω (B) 100 Ω
(C) 400 Ω (D) 150 Ω

Solution: (A)

Surge impedance is independent of length of the line

Example 47: The reflection co-efficient for the transmission line shown in figure at 'P' is



(A) 0.25 (B) 0.142
(C) 0.33 (D) 7

Solution: (B)

Reflection co-efficient $= \frac{Z_L - Z_S}{Z_L - Z_S}$

$$= \frac{400 - 300}{700} = 0.142$$

Example 48: An overhead line having a surge impedance of 400 Ω is connected in series with an underground cable having a surge impedance of 80 Ω. If surge of 40 kV travels from the line towards the cable junction, the value of the transmitted voltage wave at the junction is

(A) 160 kV (B) 40 kV
(C) 80 kV (D) 13.3 kV

Solution: (D)

The transmitted voltage $= \frac{2Z_L}{Z_L + Z_S} \times V$

$= \frac{2 \times 80}{80 + 400} \times 40 \times 10^3$

$= 13.33$ kV

Consider a power system network as shown in the figure.

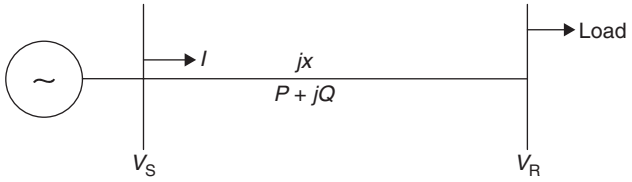


Figure 16 Power system single-line diagram

Generally the load is Inductive load. The complex power at the sending end is

$$S = V_s^* I = P - jQ$$

$$I = \frac{P - jQ}{V_s^*}$$

The receiving end voltage $V_R = V_S - j\left(\frac{P - jQ}{V_s}\right)x$

($\because V_s$ – reference voltage)

$$V_R = \left[V_s - \frac{Q}{V_s}X\right] - j\frac{P}{V_2}X$$

From the above equation V_R is not considerable affect by the variation of real power then by the variation reactive power of the load, Bus voltage is varied. So, by controlling the reactive power the voltage is maintained within the specified limits.

Methods of Voltage Control

The different voltage control methods are

1. Shunt reactors
2. Shunt capacitors
3. Tap-changing transformers
4. Synchronous condensers
5. Series capacitors
6. Static VAR system

Shunt Reactors

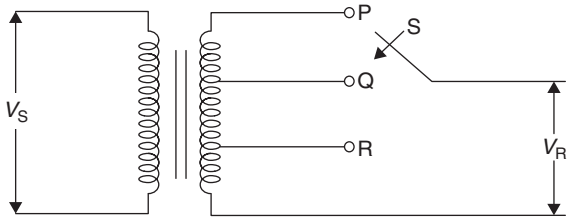
In this method, inductors are connected across the load. Shunt reactors generally used when the bus voltage more than the specified voltage, i.e., under light load condition ((or) Ferranti effect).

Shunt Capacitor

In this method, capacitors connected in parallel with the load. Shunt capacitors are generally used when the bus voltage less than the specified voltage, i.e., under voltage condition. By connecting shunt capacitors, the VARS power required by the load decreases so that the reactive power drawn by the load from the source decreases and the power factor, voltage of the bus Increases.

Tap-changing Transformer

In this method, the voltage of the line changes by changing the position of the winding connection as shown in the figure.

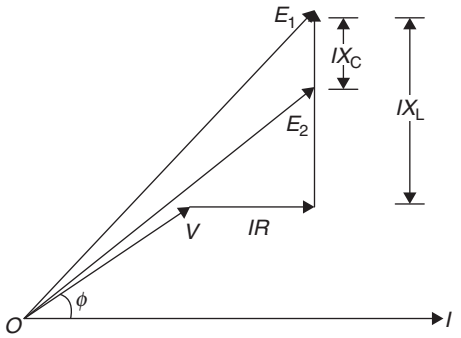


The above figure shows that the secondary voltage of the transformer changes by changing the corresponding switch position.

Series Capacitors

In this method, a capacitor is connected in series with the line. So, the resultant reactance of the line decreases then the corresponding voltage drop of the line also decreases.

The change in voltage by connecting series capacitor is $\Delta V = IX_c \sin\phi$.



In the above phasor diagram E_1 represent the sending end voltage before connecting the series capacitor and E_2 represent the sending end voltage after connecting the series capacitor.

Therefore by connecting series capacitor the voltage drop decreases by IX_c factor.

Power Factor Improvement

As the electrical energy is generated, transmitted and distributed in the form of alternating wave, hence power factor is considered. As most of the loads are inductive in nature, therefore they have low lagging power factor. Low power factor causes the increase active power loss. To ensure the most favourable conditions for a supply system from engineering and economical standpoint, it is important to have power factor close to unity.

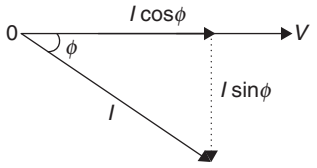
Power Factor

The cosine of angle between voltage and current in an AC circuit is known as power factor.

In an AC circuit, there is generally phase difference 'ϕ' between voltage and current. The term cosϕ is called the power factor of the circuit.

If the circuit is inductive, the current lags the voltage and power factor is said to be lagging. In case of capacitive circuit, current leads the voltage and the power factor is said to be leading.

Consider an inductive circuit taking a lagging current 'I' form the supply voltage 'V' the lagging angle being equal to 'ϕ'. The vector diagram for the above circuit is shown below.



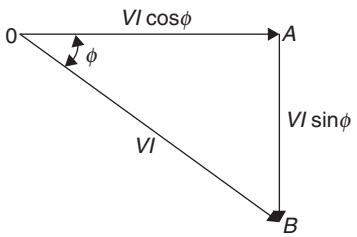
The current 'I' is resolved into two perpendicular components,

1. $I\cos\phi$ in phase with 'V'
2. $I\sin\phi$, 90° out of phase with V. (for L and C)

$I\cos\phi$ is known as active (or) wattful component while the component $I\sin\phi$ is called Reactive (or) wattless component. If the reactive component is small, then the phase angle 'ϕ' is small and hence the power factor cosϕ will be high. Hence the circuit having small reactive current ($I\sin\phi$) will have high power factor and vice versa. The value of power factor can never be more than unity.

Power Triangle

Power factor can also be analysed by the power drawn by the AC circuit. If each side of the current triangle is multiplied by voltage V, then we get the power triangle.



OA = $VI\cos\phi$ which represents the active power in watts (or) kW.

AB = $VI\sin\phi$ represents the reactive power in VAR or kVAR.

OB = VI represents the apparent power in VA or kVA

From the power triangle, we can infer that,

1. The apparent power in AC circuit has two components, i.e. active and reactive power at right angles to each other.

$$OB^2 = OA^2 + AB^2$$

$\Rightarrow$ (Apparent power)^2 = (Active power)^2 + (Reactive power)^2

2. Power factor $\cos\phi = \dfrac{OA}{OB} = \dfrac{\text{Active power}}{\text{Apparent power}}$

3. For leading currents, the power triangle should be reversed. To improve the power factor of the load, the device taking leading reactive power (e.g. capacitor) is connected in parallel with the load, neutralizing partly the lagging reactive power.

Disadvantages of Low Power Factor

We know that $P_L = V_L I_L \cos\phi$

$$\Rightarrow \quad I_L = \frac{P}{V_L \cos\phi} \text{ (for single-phase)}$$

$$P = \sqrt{3} V_L I_L \cos\phi$$

$$\Rightarrow \quad I_L = \frac{P}{\sqrt{3} V_L \cos\phi} \text{ (for three-phase)}$$

From the above expression, we can conclude that for the fixed power and voltage, the load current is inversely proportional to the power factor.

Due to the low power factor, the current drawn from the source is large resulting in the following disadvantages.

1. The electrical machinery is always rated in kVA. While designing the machine, the power factor of the load is not known.

$$\text{kVA} = \frac{\text{kW}}{\cos\phi}$$

From the above formula, kVA rating of the Equipment is inversely proportional to the power factor. Hence, lower the power factor, the kVA of the machinery should be increased.

2. The large current drawn at low power factor causes more copper losses (I^2R) in all the elements of the supply system. Hence the efficiency is reduced.

Economics of Power Factor Improvement

By using power factor improvement devices the power factor of the load Increases. Therefore net maximum demand power drawn from the supply decreases then the corresponding total current and transmission losses decreases

which leads to the net saving on the economic cost. But by considering the cost and maintain of power factor correction equipment the total saving is decreases. Generally P is the expenditure per kVA per annum of the power factor correction equipment and q is the rate per kVA of maximum demand per annum. Therefore the corresponding economic power factor improvement is $\cos\phi$, i.e.,

$$\cos\phi = \sqrt{1-\left(\frac{P}{q}\right)^2}$$

The power factor is improved by different methods.

1. Static VAR compensator
2. Synchronous condenser
3. STATCOM
4. Static capacitor

Static VAR Compensator

In this method, a capacitor bank fixed or switched (or) fixed, capacitor and switch reactor bank are used in parallel. These type of compensations are generally used when reactive power leading and lagging by corresponding switching ON and OFF of the capacitors and reactors into the circuit. By changing the reactive power, the power factor of the circuit also changes.

Static Capacitor

In the above method both capacitors and reactors are connected but In these method only capacitors are used in parallel with the load then there is net improvement of the power factor and also the reactive power drawn from the supply also reduces.

Synchronous Condenser

It is a synchronous motor operating under no-load and over excitation condition. When these motor is connected in parallel with the load then the motor supplying reactive power to the load. There for the corresponding power factor of the load Increases.

Example 49: A shunt reactor of 100 MVAR is operated at 98% of its rated voltage and 95% of its rated frequency the reactive power absorbed by the reactor is

(A) 98.92 MVAR (B) 101.09 MVAR
(C) 95.32 MVAR (D) 104.32 MVAR

Solution: (B)

Reactive power absorbed by reactor $= \dfrac{V^2}{X_L}$

$$Q_1 = \frac{V_1^2}{2\pi f_1 L} = 100 \text{ MVAR}$$

$$V_2 = 0.98\ V_1$$

$$f_2 = 0.95 f_1$$

New case reactive power absorbed $= \dfrac{V_2^2}{2\pi f_2 L}$

$$Q_2 = \frac{(0.98V_1)^2}{2\pi(0.95 f_1)L}$$

$$Q_2 = \frac{0.98^2}{0.95} Q_1 = 1.0109 Q_1$$

$$Q_2 = 101.09 \text{ MVAR}$$

Example 50: A balanced delta connected load of $(8 + j6)\ \Omega$ per phase is connected to a 400 V, 60 Hz, 3-ϕ supply lines. If the input power factor is to be improved to 0.9 by connecting a bank of star connected capacitor, the required kVAR of the bank is.

(A) 28.8 kVAR (B) 18.5 kVAR
(C) 10.7 kVAR (D) 20.4 kVAR

Solution: (C)

Per-phase load current $= \dfrac{400}{8+j6} = 40\angle -36.86°$

Load power factor = cos 36.86 = 0.8 lagging

Capacitor placement affects the reactive component of load but real power component of the load remains unchanged

$$\therefore\ I_1 \cos\phi_1 = I_2 \cos\phi_2$$

$$40 \times 0.8 = I_2 \times 0.9 \Rightarrow I_2 = 35.55 \text{ A}$$

$I_2 = I_1 + I_c$ from the system configuration

$I_c = I_2 - I_1$ [$\therefore \phi_2 = \cos^{-1} 0.9$]

$= 35.55\ \angle -25.84 - 40\ \angle -36.86$

$I_c = 8.5\ \angle 90°$

Reactive power of capacitor bank

= Q at 0.8 power factor – Q at 0.9 power factor

$$= \sqrt{3} V_{L_1} I_{L_1} \sin\phi_1 - \sqrt{3} V_L\ I_{L_2}\ \sin\phi_2$$

$$= \sqrt{3} \times 400 \times \sqrt{3} \times 40 \times 0.6 - \sqrt{3} \times 400 \times \sqrt{3} \times 35.5 \times \sin 25$$

= 10.7 kVAR

Example 51: At an industrial substation with a 4 MW load, a capacitor of 2.5 MVAR is installed to maintain the load factor at 0.97 lagging. If the capacitor goes out of service the load power factor is

(A) 0.9 (B) 0.75
(C) 0.8 (D) 0.842

Solution: (B)

Initial phase difference $\phi_1 = \cos^{-1} 0.97$

$= 14.07°$

kVAR supplied by capacitor bank = $P(\tan\phi_1 - \tan\phi_2)$

$4 \times 10^6\ (\tan\phi_1 - \tan 14.07) = 2.5 \times 10^6$

$\phi_1 = 41.2°$

Power factor before adding capacitor bank

$$\cos\phi_1 = 0.752$$

Example 52: A loss less transmission line having surge impedance loading of 2000 MW. A series capacitive compensation of 30% the surge impedance loading of the compensated transmission line will be

(A) 1673 MW (B) 2390 MW
(C) 600 MW (D) 2600 MW

Solution: (B)

Surge impedance loading (SIL) $= \frac{V^2}{Z_C}$

$$\text{Surge impedance loading } \alpha \frac{1}{Z_C}$$

When the capacitive compensation of 30% is provided, the series impedance of the line is reduced to 70% of its original value which in turn reduced the characteristic impedance.

Characteristic impedance $\alpha \sqrt{Z_{series}}$

$$\frac{Z_{C_1}}{Z_{C_2}} = \frac{\sqrt{Z_{series}}}{\sqrt{0.7 \times Z_{series}}}$$

Characteristic impedance after compensation

$$Z_{C_2} = 0.836\, Z_{C_1}$$

Since surge impedance loading $\alpha = \frac{1}{Z_C}$

$$\frac{SIL_1}{SIL_2} = \frac{Z_{C_2}}{Z_{C_1}}$$

$$SIL_2 = SIL_1 \times \frac{Z_{C_1}}{Z_{C_2}}$$

$$= 1.195 \times 2000 = 2390 \text{ MW}$$

EXERCISES

Practice Problems I

Directions for questions 1 to 27: Select the correct alternative from the given choices.

1. A generating station has a maximum demand of 60 MW. The plant capacity factor is 0.5 and the load factor is 0.6. The plant running as per schedule is fully loaded. The daily energy produced will be

(A) 720 MWh (B) 964 MWh
(C) 864 MWh (D) 820 MWh

2. The inductance in mH/km of a single-phase overhead transmission line of diameter 1 cm and distance of separation 60 cm is

(A) 0.05 + 0.2 ln 60

(B) $0.05 + 0.2 \ln\left(\frac{60}{0.5}\right)$

(C) $0.2 \ln\left(\frac{60}{0.5}\right)$

(D) 0.2 ln 60

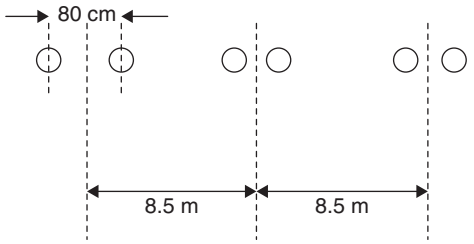
3. A 50 km transmission line when excited with a source of frequency 500 Hz, it will be modelled as

(A) Short line (B) Medium line
(C) Long line (D) Data insufficient

4. An existing AC system has a string efficiency of 80%. The same line, if used for DC transmission, then string efficiency will be

(A) 100% (B) >80%
(C) <80% (D) 0%

5. The figure shows a single circuit 460 kV line using two bundled conductors per phase. If diameter of each conductor is 5 cm, the inductance/km/phase is



(A) 0.24 mH/km/phase
(B) 0.76 mH/km/phase
(C) 0.89 mH/km/phase
(D) 0.90 mH/km/phase

6. A conductor is composed of seven identical strands of copper, each having a radius r. The self-GMD of the conductor will be

(A) 2.17 r (B) 2.645 r
(C) 2.141 r (D) 1.21 r

7. Determine the capacitance for a 2-conductor 1-ϕ line operating at 50 Hz, the conductors being of 2 cm in diameter and spaced 3 m apart.

(A) 9.74×10^{-9} F/km (B) 4.87×10^{-9} F/km
(C) 3.05×10^{-9} F/km (D) None of these

8. A 1-ϕ transmission line having $R = 1.4\ \Omega$ and $X = 0.8\ \Omega$ is delivering a load of 200 kVA at 2500 V. The voltage at sending end when p.f. of the load is 0.8 p.f. lag is

(A) 2.069 kV (B) 2.840 kV
(C) 2.627 kV (D) 3.54 kV

9. The parameters of a single circuit 3-ϕ, 50 Hz transmission line are

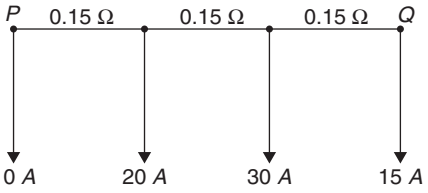
$R = 0.2\ \Omega$/km, $L = 1.3$ mH/km

$C = 0.01\ \mu$F/km, $L = 120$ km

The efficiency of the line if it delivers 40 MW at 132 kV and 0.8 p.f. lag is

(A) 85.4% (B) 90.0%
(C) 98% (D) 92.6%

10. A DC distribution system is shown with load currents as marked. The two ends of the feeder P and Q are fed such that $V_P - V_Q = 3$ V. The value of voltage V_P for a minimum voltage of 220 V at any point along the feeder is

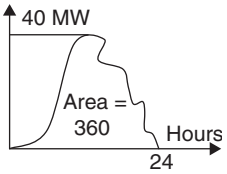


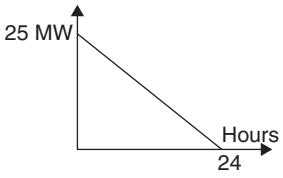
(A) 220 V (B) 223 V
(C) 226 V (D) 229 V

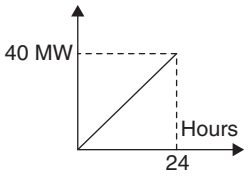
11. An underground cable with a surge impedance of 100 Ω is connected in series with an overhead line having a surge impedance of 400 Ω. If a source of 50 kV travels from the line towards the line cable junctions, the value of the transmitted voltage wave at the junction is

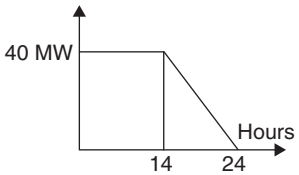
(A) 20 kV (B) 30 kV
(C) –30 kV (D) 80 kV

12. For the figures given below, which curve has the least load factor?

(A)


(B)


(C)


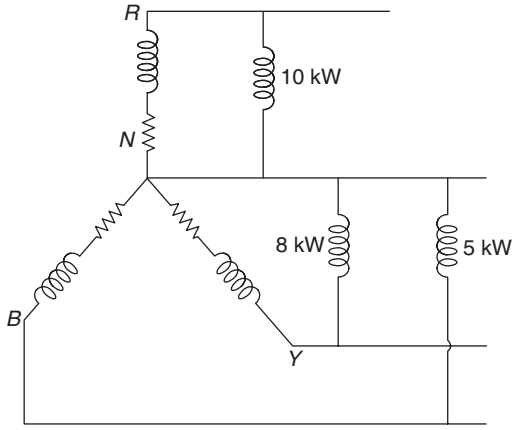
(D)


(A) Curve A (B) Curve B
(C) Curve C (D) Curve D

13. A 275 kV transmission line has the following line constants. $A = 0.85\ \angle 5°$, $B = 200\ \angle 75°$. The power that can be received at unity p.f. if voltage profile at each end is to be maintained at 275 kV is

(A) 118 MW (B) 218 MW
(C) 98 MW (D) None of these

14. In a 3-ϕ, 4-wire system, the line voltage is 400 V and now inductive loads of 10, 8 and 5 kW are connected between the three line conductors and the neutral as shown in figure. Calculate the current in the neutral conductor.



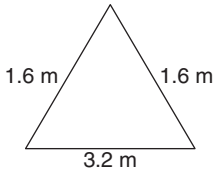
(A) 15.8 A (B) 16.8 A
(C) 17.8 A (D) 18.8 A

15. A 3-ϕ transmission line consists of 1 cm radius conductors spaced symmetrically 4 m apart. Dielectric strength of air is 30 kV/cm. The line voltage for corona to commence is

(A) 110 kW (B) 220 kV
(C) 330 kV (D) 400 kV

Common Data for Questions 16 and 17:

The conductors of a 3-ϕ transmission line are arranged as shown, radius of each line = 0.4 cm



The operating voltage is 132 kV.

16. The capacitance per phase is

(A) 8.9×10^{-9} F/km
(B) 9.2×10^{-6} F/km
(C) 11.5×10^{-9} F/km
(D) 7.5×10^{-7} F/km

17. The charging current when the transmission line is operated at 264 kV is

(A) 0.312 A/km
(B) 0.257 A/km
(C) 0.427 A/km
(D) 0.214 A/km

Common Data for Questions 18 and 19:

A 3-ϕ 50 Hz transmission line is 400 km long. The voltage at the sending end is 220 kV. The line parameters are $r = 0.125\ \Omega$/km, $x = 0.4\ \Omega$/km, $y = 2.8 \times 10^{-6}\ \because$/km

18. The sending end current and receiving end voltage are.
(A) 76 A, 121 kV (B) 152 A, 242 kV
(C) 152 A, 226 kV (D) 76 A, 242 kV

19. The maximum permissible line length of receiving end no load voltage is not to exceed 235 kV
(A) 138 km (B) 238 km
(C) 338 km (D) 438 km

20. A 100 MVA, 11 kV alternator has synchronous reactance 0.3 p.u. The p.u. synchronous reactance on a 50 MVA, 33 kV base is
(A) 1.67 p.u. (B) 0.45 p.u.
(C) 0.016 p.u. (D) 0.167 p.u.

21. A 3-ϕ induction motor rated at 400 V, 50 Hz is running at a p.f. of 0.707 lag. The input to the motor is 8 kVA. It is proposed to improve the p.f. of the system to 0.9 by a Δ connected bank of capacitors. The value of capacitance/phase required is
(A) 0.58 μF (B) 58 μF
(C) 0.19 μF (D) 19 μF

22. A shunt capacitor rated 100 MVAR is operated at 99% of rated voltage and 98% of rated frequency. The reactive power absorbed would be approximately
(A) 100 MVAR (B) 98 MVAR
(C) 96 MVAR (D) 95 MVAR

23. The per-unit impedance of a circuit element is 0.30. If the base kV and base MVA are halved, then the new value of per unit impedance is
(A) 0.9 (B) 0.6
(C) 0.3 (D) 0.15

24. Two buses 1 and 2 are connected by an impedance $(0 + j10)\ \Omega$. The bus 1 voltage is $400\angle 45$ and bus 2 voltage is $400\angle 0°$V. The real and reactive powers supplied by bus 1, respectively, are
(A) +11314 W, +4686 VAR
(B) +11314 W, –4686 VAR
(C) –11314 W, –2687 VAR
(D) –5656 W, +2687 VAR

25. A load is drawing 500 kW at 0.707 lag. A synchronous motor improves the p.f. of the load to 0.96. The motor carries a load of 300 kW. The leading reactive power supplied by the motor is
(A) 500 kVAR (B) 233.33 kVAR
(C) 266.67 kVAR (D) 250 kVAR

26. The voltage phasor of a circuit is $20\angle 15°$V and current phasor is $5\angle -45°$A. The active and reactive powers in the circuit are
(A) 50 W and –86.67 VAR
(B) 50 W and 86.67 VAR
(C) 86.67 W and 50 VAR
(D) 86.67 W and –50 VAR

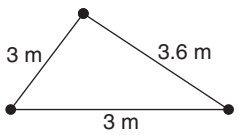
27. A 240 V single-phase AC source is connected to a load with an impedance of $20\angle 45$. A capacitor is connected in parallel with the load. If the capacitor supplies 2400 VAR, the real power supplied by source is
(A) 3394 W (B) 2400 W
(C) 994 W (D) None of these

Practice Problems 2

Directions for questions 1 to 40: Select the correct alternative from the given choices.

1. A 1-ϕ two conductor circuit has a spacing of 3 m between them and each conductor has a diameter of 10 mm. The inductance of each circuit per km is
(A) 1.24 mH/km (B) 1.19 mH/km
(C) 2.38 mH/km (D) 0.59 mH/km

2. The conductors of a 3-ϕ 50 Hz circuit are each of diameter 21 mm. The spacing is as shown. The inductance is



(A) 10.5×10^{-4} H/km (B) 8.2×10^{-4} H/km
(C) 5.6×10^{-3} H/km (D) None of these

3. A two conductor 1-ϕ line operating at 50 Hz has a diameter of 20 mm and spacing between the conductors is 3 m. Height of conductor above the ground is 6 m. The capacitance of line to neutral will be
(A) 9.7 pF/km (B) 10.2 pF/km
(C) 8.7 μF/km (D) 2.4 μF/km

4. The power losses in a 3-ϕ 20 km line delivering 10 MW at 11 kV, 50 Hz, 0.8 p.f. lagging is 10% of power delivered. The resistance of each conductor is
(A) 0.774 Ω (B) 2.14 Ω
(C) 1.54 Ω (D) 0.86 Ω

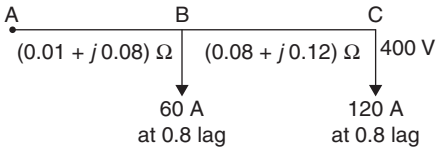
5. If bundled conductors are used instead of single conductors, then
(A) Inductance and capacitance increase
(B) Inductance and capacitance decrease
(C) Inductance increases and capacitance decreases
(D) Inductance decreases and capacitance increases

6. A 400 V, 3-ϕ 4-wire system supplies a Y-connected load. Resistance of each line is 0.2 Ω and of neutral is 0.4 Ω. The load impedances are
$Z_R = 6 + j10$, $Z_Y = 10\ \Omega$
$Z_B = 8 - j10\ \Omega$. Current is neutral is
(A) 16.56 A (B) 17.56 A
(C) 18.56 A (D) 19.56 A

7. The non-uniform distribution of voltage across the units in a string of suspension-type insulator is due to
(A) Non-uniform distance between the cross arm and the units
(B) Existence of the stray capacitance between the metallic junctions of the units and tower body
(C) Non-uniform distance of separation of units from tower body
(D) Unequal self-capacitance of the units

8. A long overhead transmission line is terminated by its characteristic impedance. The ratio of voltage to current at different points along the line
(A) Progressively increases from receiving end to the sending end
(B) Progressively decreases from receiving end to the sending end
(C) Remains the same at the two ends being maximum at the centre of the line
(D) Remains the same at all points

9. A 1-ϕ, AC distributor supplies two single-phase loads as shown in the figure.



The voltage drop from A to C is
(A) 21 V (B) 26.7 V
(C) 41 V (D) 51 V

10. The use of hollow conductors reduce corona loss as
(A) Provides better ventilation of the conductor
(B) Eddy currents in the conductor are eliminated
(C) Current density is reduced
(D) For a given cross section radius of conductor is increases

11. An insulator string has three units. The capacitance of the link between two units to earth is one eight of the capacitance of each insulator. Each unit has a safe working voltage of 18.5 kV. The system voltage is
(A) 55.5 kV (B) 46.7 kV
(C) 37 kV (D) 36.7 kV

12. The increase in resistance of a conductor carrying alternating current, due to non-uniform distribution of current is called
(A) Hall effect (B) Proximity effect
(C) Skin effect (D) Ferranti effect

13. The transmission capacity of a line is related to its inductance varies in
(A) Direct proportion
(B) Inverse proportion
(C) Square
(D) Independent of inductance

14. A lossless radial transmission line with surge impedance loading
(A) Has sending end voltage higher than receiving end voltage
(B) Has flat voltage profile and unity p.f. at all points along it
(C) Takes positive VAR at sending end and zero VAR at receiving end
(D) Takes negative VAR at sending end and zero VAR at receiving end

15. The surge impedance of a 400 km line is 500 Ω. For a 200 km length of same line, the impedance will be
(A) 250 Ω (B) 1000 Ω
(C) 500 Ω (D) 750 Ω

16. In a string of suspension insulators, the voltage distribution across the different units could be made uniform by using grading ring because it
(A) Decreases the capacitances of upper insulators units causing equal voltage drop
(B) Increases capacitances of lower insulator units to cause equal voltage drop
(C) Forms capacitances with link pins which helps to cancel the charging current from link pins
(D) Forms capacitances with link pins to carry the charging current from the link pins

17. The cause of Ferranti effect in transmission lines is
(A) Line inductance
(B) Line capacitance
(C) Line capacitance and inductance
(D) Line resistances an capacitance

18. Series capacitors are used in a transmission line to
(A) To increase the phase angles at sending end and receiving ends under heavy load conditions
(B) Reduce the phase angle between sending end and receiving ends under heavy load conditions
(C) To reduce the capacitive current
(D) None of these

19. The load factor and capacity factor of a plant are equal if
(A) Peak load is equal to capacity of plant
(B) Average load is half the capacity of the plant
(C) Average load is same as peak load
(D) Group diversity factor is equal to peak diversity factor

20. The voltage across the units in a two unit suspension insulator is 40% and 60%. The ratio of capacitance of insulator to that of earth will be

(A) 0.4 (B) 0.45
(C) 0.5 (D) 0.55

21. The receiving end voltage of a long transmission line at no load condition is
(A) Less than sending end voltage
(B) Greater than sending end voltage
(C) Equal to sending end voltage
(D) Zero voltage

22. The level of insulation of a 400 kV extra high voltage line is decided based on
(A) Lightning over voltage
(B) Switching over voltage
(C) Corona inception voltage
(D) Radio and TV interference

23. The transmission line feeding power on either side of main transmission line is
(A) Primary distribution
(B) Primary transmission
(C) Secondary transmission
(D) Secondary distribution

24. Suppose the conductors are dead ended and there is a change in the direction of transmission line, insulators used are
(A) Strain type (B) Pin type
(C) Shackle type (D) Suspension type

25. For a fixed inductive shunt compensation, series capacitive compensation
(A) Increases virtual surge impedance loading
(B) Decreases vertical surge impedance loading
(C) Increases effective length of line
(D) None of these

Common Data for Questions 26 and 27:
A 50 Hz, 3-ϕ, transmission line 30 km long has a total series impedance of $(40 + j125)\ \Omega$ and shunt admittance of 10^{-3} mho. The load is 50 MW at 220 kV with 0.8 lagging p.f.

26. Sending end voltage using nominal π is
(A) 216 kV (B) 204 kV
(C) 232 kV (D) 238 kV

27. The sending end power factor is
(A) 0.8 lag (B) 0.9 lag
(C) 0.98 lag (D) 1

28. A pure inductance absorbs reactive power from the AC line when
(A) Both applied voltage and current decreases
(B) Both applied voltage and current increases
(C) Applied voltage increases and current decreases
(D) Applied voltage decreases and current increases

29. The power factor of the load is corrected by connecting a static capacitor in parallel with the load. The power taken by the load
(A) Increases (B) Decreases
(C) Remains same (D) Is uncertain

30. Shunt reactors are connected in transmission lines to
(A) Limit the fault current
(B) Improve stability limit
(C) Absorb the excess reactive power
(D) Generate requisite reactive power

31. The excitation of a synchronous generator connected to an infinite bus bar delivering power at a lagging power factor is increased. Then
(A) Current delivered increases
(B) Terminal voltage increases
(C) Voltage angle d increases
(D) Both A and C

32. The voltage of an infinite bus bar is $1\angle 0°$ and that of an alternator connected to it is $0.95\angle 0°$. The alternator acts as a
(A) Shunt coil
(B) Shunt capacitor
(C) Shunt resistance
(D) None of these

33. The power dispatch through a transmission line can be increased by installing
(A) Shunt capacitance
(B) Series capacitance
(C) Shunt reactance
(D) Series reactance

34. The combined frequency regulation of machines in an area of capacity 3000 MW and operating at a normal frequency of 60 Hz is 0.15 p.u. The regulation in Hz/MW is
(A) $\frac{3}{1000}$ (B) $\frac{0.3}{1000}$
(C) $\frac{1}{200}$ (D) $\frac{1}{100}$

35. Which amongst the following is not true?
(A) Synchronous phase modifiers are installed at sending end
(B) Synchronous phase modifiers are installed at load end
(C) Synchronous phase modifiers are specially designed synchronous motors
(D) Synchronous phase modifiers do not carry load

36. In EHV lines, series capacitance compensation is used to
(A) Improve the stability of the system
(B) Reduce the loading on the line
(C) Improve the protection of the line
(D) Reduce the voltage profile

II. Question Nos.: 37 and 38
A line operating at 60 Hz is 81% series compensated.

37. The resonating frequency is
(A) 60 Hz (B) 66.6 Hz
(C) 50 Hz (D) 54 Hz

38. Now if the line is 100% series compensated. Resonant frequency is
(A) 60 Hz (B) 66.6 Hz
(C) 50 Hz (D) 54 Hz.

III. Question Nos.: 39 and 40
A20 kVA, 2000/200 V single-phase transformer has an equivalent impedance of $Z = 8 + j15$ when referred to the HV side.

39. The p.u. value of impedance when referred to the LV side is
(A) $2.13\angle 32°$ (B) $0.0085\angle 62°$
(C) $0.085\angle 62°$ (D) $0.213\angle 32°$

40. If the transformer has an exciting current of 10 A on HV side, p.u. value of current referred to LV side is
(A) 1 p.u. (B) 2 p.u.
(C) 0.5 p.u. (D) 0.25 p.u.

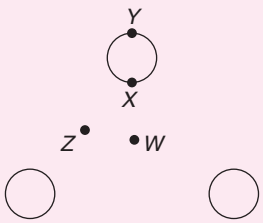
Previous Years' Questions

1. The concept of an electrically short, medium, and long line is primarily based on the **[2006]**
(A) nominal voltage of the line
(B) physical length of the line
(C) wavelength of the line
(D) power transmitted over the line

2. A 400 V, 50 Hz, three-phase balanced source supplies power to a star connected load whose rating is $12\sqrt{3}$ kVA, 0.8 p.f. (lag). The rating (in kVAR) of the delta connected (capacitive) reactive power bank necessary to bring the p.f. to unity is **[2006]**
(A) 28.78 (B) 21.60
(C) 16.60 (D) 12.47

3. The A, B, C, D constants of a 220 kV line are:
$A = D = 0.94\ \angle 1°$, $B = 130\angle 73°$, $C = 0.001\angle 90°$. If the sending end voltage of the line for a given load delivered at nominal voltage is 240 kV, then % voltage regulation of the line is **[2006]**
(A) 5 (B) 9
(C) 16 (D) 21

4. Consider a bundled conductor of an overhead line, consisting of three identical sub-conductors placed at the corners of an equilateral triangle as shown in the figure. If we neglect the charges on the other phase conductors and ground, and assume that spacing between sub-conductors is much larger than their radius, the maximum electric field intensity is experienced at **[2007]**



(A) Point X (B) Point Y
(C) Point Z (D) Point W

5. The total reactance and total susceptance of a lossless overhead EHV line, operating at 50 Hz, are given by 0.045 p.u. and 1.2 p.u., respectively. If the velocity of wave propagation is 3×10^5 km/s, then the approximate length of the line is **[2007]**
(A) 122 km (B) 172 km
(C) 222 km (D) 272 km

6. Out of the following plant categories
(i) Nuclear (ii) Run-of-river
(iii) Pump storage (iv) Diesel
The base load power plants are **[2009]**
(A) (i) and (ii) (B) (ii) and (iii)
(C) (i), (ii) and (iv) (D) (i), (iii) and (iv)

7. For a fixed value of complex power flow in a transmission line having a sending end voltage V, the real power loss will be proportional to **[2009]**
(A) V (B) V^2
(C) $\frac{1}{V^2}$ (D) $1/V$

8. Match the items in List-I with the items in List-II and select the correct answer using the codes given below the lists. **[2009]**

List I To	List II Use
a. Improve power factor	1. Shunt reactor
b. Reduce the current ripples	2. Shunt capacitor
c. Increase the power flow in line	3. Series capacitor
d. Reduce the Ferranti effect	4. Series reactor

(A) a → 2 b → 3 c → 4 d → 1
(B) a → 2 b → 4 c → 3 d → 1
(C) a → 4 b → 3 c → 1 d → 2
(D) a → 4 b → 1 c → 3 d → 2

9. Match the items in List-I with the items in List-II and select the correct answer using the codes given below the lists. **[2009]**

List I Type of transmission line	List II Type of distance relay preferred
a. Short line	1. Ohm relay
b. Medium line	2. Reactance relay
c. Long line	3. Mho relay

(A) a → 2 b → 1 c → 3
(B) a → 3 b → 2 c → 1
(C) a → 1 b → 2 c → 3
(D) a → 1 b → 3 c → 2

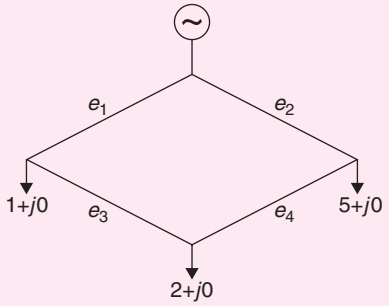
10. Three generators are feeding a load of 100 MW. The details of the generators are **[2009]**

	Rating (MW)	Efficiency (%)	Regulation (p.u.) on 100 MVA base
Generator-1	100	20	0.02
Generator-2	100	30	0.04
Generator-3	100	40	0.03

In the event of increased load power demand, which of the following will happen?
(A) All the generators will share equal power
(B) Generator-3 will share more power compared to Generator-1
(C) Generator-1 will share more power compared to Generator-2
(D) Generator-2 will share more power compared to Generator-3

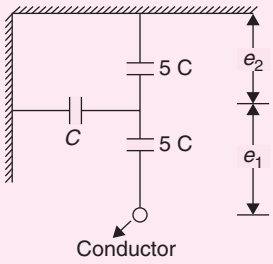
11. An extra high voltage transmission line of length 300 km can be approximated by a lossless line having propagation constant $\beta = 0.00127$ radians per km. Then the percentage ratio of line length to wavelength will be given by **[2008]**
(A) 24.24% (B) 12.12%
(C) 19.05% (D) 6.06%

12. Single line diagram of a 4-bus single source distribution system is shown below, branches e_1, e_2, e_3 and e_4 have equal impedances. The load current values indicated in the figure are in per unit. Distribution company's policy requires radial system operation with minimum loss. This can be achieved by opening of the branch **[2008]**



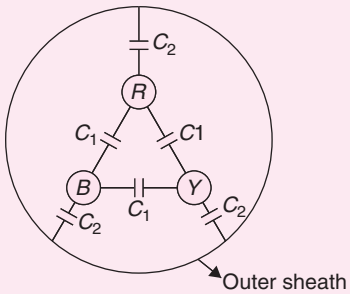
(A) e_1 (B) e_2
(C) e_3 (D) e_4

13. Consider a three-phase, 50 Hz, 11 kV distribution system. Each of the conductors is suspended by an insulator string having two identical porcelain insulators. The self-capacitance of the insulator is 5 times the shunt capacitance between the link and the ground, as shown in the figure. The voltage across the two insulators is **[2010]**



(A) $e_1 = 3.74$ kV, $e_2 = 2.61$ kV
(B) $e_1 = 3.46$ kV, $e_2 = 2.89$ kV
(C) $e_1 = 6.0$ kV, $e_2 = 4.23$ kV
(D) $e_1 = 5.5$ kV, $e_2 = 5.5$ kV

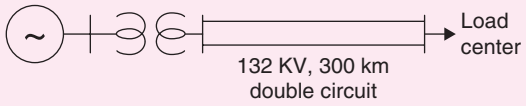
14. Consider a three-core, three-phase, 50 Hz, 11 kV cable whose conductors are denoted as R, Y and B in the figure. The inter-phase capacitance (C_1) between each pair of conductors is 0.2 mF and the capacitance between each line conductor and the sheath is 0.4 mF. The per-phase charging current is **[2010]**

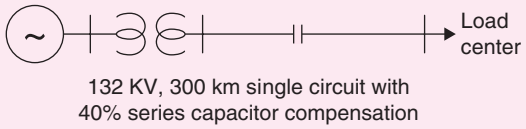


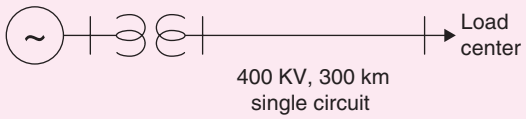
(A) 2.0 A (B) 2.4 A
(C) 2.7 A (D) 3.5 A

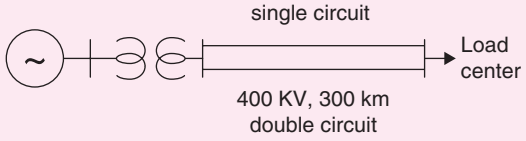
15. For enhancing the power transmission in a long EHV transmission line, the most preferred method is to connect a **[2011]**
(A) Series inductive compensator in the line.
(B) Shunt inductive compensator at the receiving end.
(C) Series capacitive compensator in the line.
(D) Shunt capacitive compensator at the sending end.

16. A nuclear power station of 500 MW capacity is located at 300 km away from a load centre. So the most suitable power evacuation transmission configuration among the following options. **[2011]**

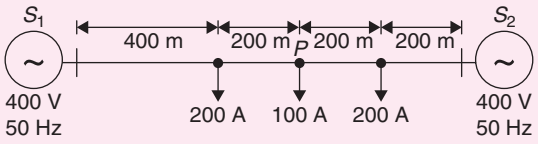
(A) 

(B) 

(C) 

(D) 

17. A distribution feeder of 1 km length having resistance, but negligible reactance, is fed from both the ends by 400 V, 50 Hz balanced sources. Both voltage sources S_1 and S_2 are in phase. The feeder supplies concentrated loads of unity power factor as shown in the figure. **[2014]**



The contributions of S_1 and S_2 in 100 A current supplied at location P, respectively, are

(A) 75 A and 25 A
(B) 50 A and 50 A
(C) 25 A and 75 A
(D) 0 A and 100 A

18. Shunt reactors are sometimes used in high-voltage transmission systems to **[2004]**

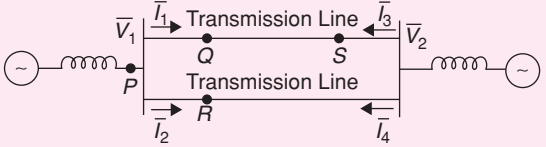
(A) limit the short-circuit current through the line
(B) compensate for the series reactance of the line under heavily loaded condition.
(C) limit overvoltages at the load side under lightly loaded condition.
(D) compensate for the voltage drop in the line under heavily loaded condition.

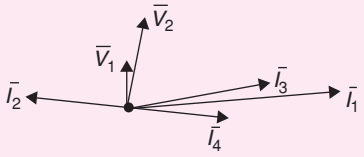
19. Base load power plants are **[2015]**

P: wind farms.
Q: run-of-river plants.
R: nuclear power plants.
S: diesel power plants.

(A) P, Q and S only
(B) P, R and S only
(C) P, Q and R only
(D) Q and R only

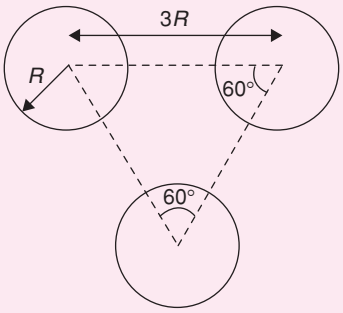
20. A sustained three-phase fault occurs in the power system shown in the figure. The current and voltage phasors during the fault (on a common reference), after the natural transients have died down, are also shown. Where is the fault located? **[2015]**





(A) Location P
(B) Location Q
(C) Location R
(D) Location S

21. A composite conductor consists of three conductors of radius R each. The conductors are arranged as shown below. The geometric mean radius (GMR) (in cm) of the composite conductor is kR. The value of k is _____. **[2015]**



22. A three-phase cable is supplying 800 kW and 600 kVAr to an inductive load. It is intended to supply an additional resistive load of 100 kW through the same cable without increasing the heat dissipation in the cable, by providing a three-phase bank of capacitors connected in star across the load. Given the line voltage is 3.3 kV, 50 Hz, the capacitance per phase of the bank, expressed in microfarads, is _____. **[2016]**

23. Single-phase transmission line has two conductors each of 10 mm radius. These are fixed at a center-to-center distance of 1 m in a horizontal plane. This is now converted to a three-phase transmission line by introducing a third conductor of the same radius. This conductor is fixed at an equal distance D from the two single-phase conductors. The three-phase line is fully transposed. The positive sequence inductance per phase of the three-phase system is to be 5% more than that of the inductance per conductor of the single-phase system. The distance D, in meters, is _____. **[2016]**

24. The inductance and capacitance of a 400kV, three-phases 50HZ lossless transmission line are 1.6mH/km/phase and 10nF/km/phase respectively. The sending end voltage is maintained at 400kV. To maintain a voltage of 400kV at the receiving end, when the line is delivering 300MW load, the shunt compensation required is **[2016]**

(A) Capacitive (B) Inductive
(C) Resistive (D) Zero

25. At no load condition, a 3-phase, 50Hz, lossless power transmission line has sending-end and receiving-end voltages of 400KV and 420KV respectively. Assuming the velocity of travelling wave to be the velocity of light, the length of the line, in km, is _____. **[2016]**

26. The power consumption of an industry is 500kVA, at 0.8 p.f lagging. A synchronous motor is added to raise the power factor of the industry to unity. If the power intake of the motor is 100kW, the p. f. of the motor is _____. **[2016]**

Answer Keys

Exercises

Practice Problems 1

1. C	**2.** B	**3.** C	**4.** A	**5.** C	**6.** A	**7.** B	**8.** C	**9.** D	**10.** C
11. A	**12.** A	**13.** A	**14.** D	**15.** B	**16.** A	**17.** C	**18.** B	**19.** C	**20.** C
21. D	**22.** C	**23.** B	**24.** A	**25.** C	**26.** B	**27.** A			

Practice Problems 2

1. B	**2.** A	**3.** A	**4.** A	**5.** D	**6.** C	**7.** B	**8.** D	**9.** B	**10.** D
11. B	**12.** C	**13.** B	**14.** B	**15.** C	**16.** C	**17.** C	**18.** B	**19.** A	**20.** C
21. B	**22.** B	**23.** C	**24.** A	**25.** D	**26.** D	**27.** C	**28.** A	**29.** C	**30.** C
31. D	**32.** A	**33.** B	**34.** A	**35.** A	**36.** A	**37.** D	**38.** A	**39.** C	**40.** A

Previous Years' Questions

1. B	**2.** D	**3.** C	**4.** B	**5.** C	**6.** A	**7.** C	**8.** B	**9.** A	**10.** C
11. D	**12.** D	**13.** B	**14.** A	**15.** C	**16.** D	**17.** D	**18.** C	**19.** C	**20.** B
21. 1.85 to 1.95		**22.** 48.23	**23.** 1.42	**24.** B	**25.** 294.847Km		**26.** 0.316		

Chapter 2

Fault Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Transients in transmission line
- Symmetrical components
- Sequence networks of synchronous generator
- Negative sequence impedance network
- Zero sequence impedance network
- Delta connected load
- Sequence diagrams for transformers
- Unbalanced fault analysis
- Single line to ground fault
- Double line to ground fault

Introduction

Faults in power systems occur because of various reasons like equipment failure, lightning strikes, falling of branches or trees on the transmission lines, switching surges, insulations failures and other electrical or mechanical causes. All these disturbances are collectively called 'faults' in power systems.

A fault usually causes high current flowing through the lines and if enough protection is not provided it may result in damages for the power apparatus. The determination of the values of such currents enables us to make proper selection of circuit breakers, protective relays and also helps to ensure that the associated apparatus will with stand the forces which arise due to the fault currents during the occurrence of fault prior to the interrupting device during the fault.

Percentage wise, the various causes of faults are

1. Lightning → 6%
2. Equipment failure → 10%
3. Switching to a fault → 10%
4. Sleet, wind → 10%
5. Tree falling, sabotage, etc. → 10%

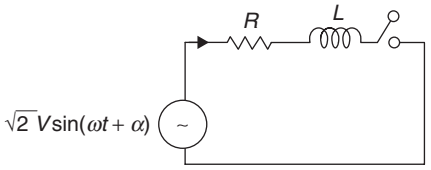
Types of faults and probability of occurrence are given below:

1. Single-phase to ground faults – 70%
2. Phase-to-phase faults – 15%
3. Two-phase to ground faults – 10%
4. Three-phase fault – 5%

Transients in Transmission Line

Assumptions in the analysis

1. Transmission line is fed from a constant voltage source.
2. Short circuit occurred when the line is unloaded.
3. Line capacitance is neglected and the line is represented by a lumped RL series circuit.



The above figure shows the RL circuit supplied by a voltage source according to given assumptions.

The switch is assumed to be closed at $t = 0$ sec. The parameter 'α' indicates the instant on the voltage source when switch is closed. When the switch is closed, the current flowing through the circuit can be written as

$$i = i_s + i_t.$$

where i_s = Steady-state current
i_t = Transient current

$$i_s = \frac{\sqrt{2}V}{|Z|}\sin(\omega t + \alpha - \theta)$$

$$i_t = \frac{\sqrt{2}V}{|Z|}\sin(\theta-\alpha)e^{-\left(\frac{R}{L}\right)t}$$

where

$$Z = \sqrt{R^2+\omega^2 L^2}\angle\tan^{-1}\frac{\omega L}{R};\theta = \tan^{-1}\frac{\omega L}{R}$$

Total current, $i = \frac{\sqrt{2}V}{|Z|}\sin(\omega t + \alpha - \theta) + \frac{\sqrt{2}V}{|Z|}\sin(\theta - \alpha)$ $e^{-\left(\frac{R}{L}\right)t}$

Steady-state sinusoidal current is called 'symmetrical short-circuit current and the unidirectional transient component is called the 'DC off set current'. First peak in this transient current is known as 'maximum momentary short-circuit current' and it is expressed as

$$i_{mm} = \frac{\sqrt{2}V}{|Z|}\sin(\theta-\alpha)+\frac{\sqrt{2}V}{|Z|}$$

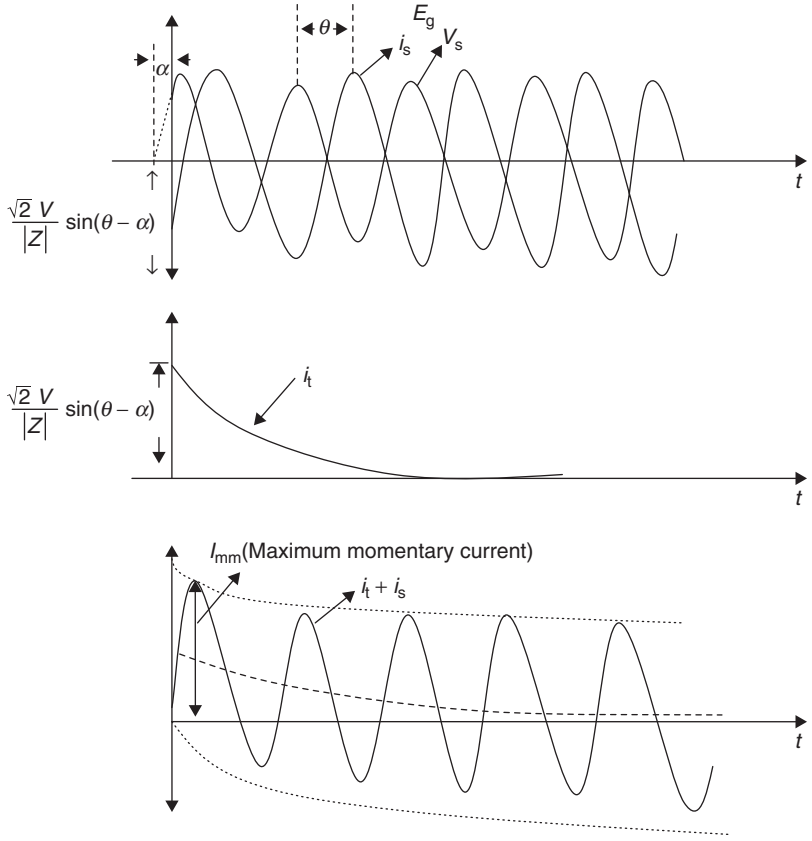
Since the transmission line resistance is small,

$$\theta \cong 90^\circ$$

and i_{mm} has maximum possible value for $\alpha = 0$.

$$i_{mm} = 2\frac{\sqrt{2}V}{|Z|}$$

$= 2 \times$ Maximum symmetrical short-circuit current



Short Circuit of an Unloaded Synchronous Generator

When a three-phase symmetrical short circuit occurs at the terminals of an unloaded synchronous generator, it is assumed that there is no DC off set in the armature current and the magnitude of the current decreases exponentially from a high initial value. The instantaneous expression for the fault current is given by

$$i_f(t) = \sqrt{2}V_t\left[\left(\frac{1}{x_d''}-\frac{1}{x_d'}\right)e^{-t/T_d''}+\left(\frac{1}{x_d'}-\frac{1}{x_d}\right)e^{-t/T_d'}+\frac{1}{x_d}\right]$$
$$\sin\left(\omega t+\alpha-\frac{\pi}{2}\right)$$

where V_t is the magnitude of the terminal voltage.
'α' is the phase angle of the voltage.
X_d'' is the direct axis subtransient reactance.
X_d' is the direct axis transient reactance.
X_d is the direct axis synchronous reactance.

With $X_d'' < X_d' < X_d$. The time constants are.
T_d'' is the direct axis sub transient time constant.
T_d' is the direct axis transient time constant.

If the fault occurs in the system at $t = 0$, the RMS value of the current is given as

$$I_f(0) = I_f'' = \frac{V_t}{X_d''}$$

which is called the 'subtransient fault current'. The duration of the sub transient current is dictated by the time constant T_d''. As the time progresses and $T_d'' < t < T_d'$, the first exponential term of fault current will short decaying and will eventually vanish. However, since 't' is still nearly equal to zero, we have the following RMS value of the current.

$$I_f' = \frac{V_t}{X_d'}$$

This is called the transient fault current. Now as the time progress, further and the second exponential term also decays, we get the following RMS value of the current for the sinusoidal steady state.

$$I_f = \frac{V_t}{X_d}$$

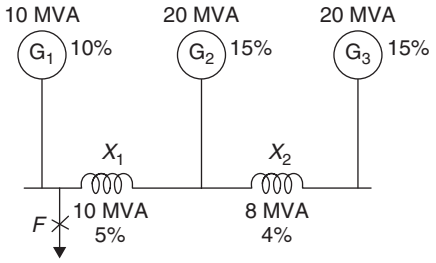
In addition to the AC, the fault currents will also contain the DC offset, the maximum value of the DC off set is given by

$$i_{DC}^{max} = \sqrt{2} I_f'' e^{-t/T_A}$$

where T_A is the armature time constant.

Solved Examples

Example 1: An interconnected generator–reactor system is shown in figure. The base values for a given per cent reactance are the ratings of the individual pieces of equipment. A three-phase short circuit occurs at point 'F'. Determine the fault current and the fault kVA if the busbar line to line voltage is 11 kV and base MVA is 50 MVA.



Solution: Reactance diagram of the system is to be represented on 50 MVA base. New values of reactance on new base are

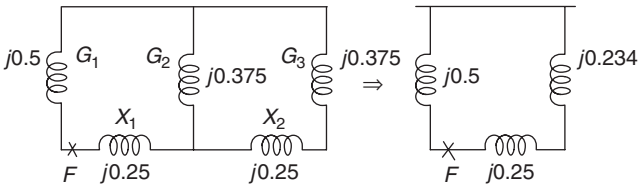
$$X_{G_1} = 0.10 \times \frac{50}{10} = 0.10 \times \frac{\text{New VA}}{\text{Old VA}} = 0.5 \text{ p.u.}$$

$$X_{G_2} = 0.15 \times \frac{50}{20} = 0.375 \text{ p.u.}$$

$$X_{G_3} = 0.15 \times \frac{50}{20} = 0.375 \text{ p.u.}$$

$$X_1 = 0.05 \times \frac{50}{10} = 0.25 \text{ p.u.}$$

$$X_2 = 0.04 \times \frac{50}{8} = 0.25 \text{ p.u.}$$



Total reactance from the fault point to neutral is per unit

$$\text{reactance} = j\frac{0.5(0.2344 + 0.25)}{0.5 + 0.2344 + 0.25}$$

$$= j0.246$$

$$\text{Fault MVA} = \frac{\text{Base MVA}}{Z_{F_{p.u.}}} = \frac{50 \times 10^6}{0.246} = 203.25 \text{ MVA}$$

$$\text{Fault current} = \frac{\text{Fault MVA}}{\sqrt{3}\ \ 11 \text{ kV}} = \frac{203.25 \times 10^6}{\sqrt{3} \times 11 \times 10^3} = 10.66 \text{ kA}$$

Example 2: A 60 Hz alternator is rated 500 MVA, 20 kV, with X_d = 1.0 p.u. and $X_d'' = 0.3$ p.u. It supplies a purely resistive load of 400 MVA at 20 kV. The load is directly connected across the generator terminals when a symmetrical fault occurs at the load terminals. The initial RMS current in the generator circuit is

(A) 4.811 p.u. (B) 11.54 p.u.
(C) 4.16 p.u. (D) 8.33 p.u.

Solution: (C)

Initial fault current depends on sub-transient reactance $X_d'' = 0.3$ p.u.

$$\text{Initial fault current} = \frac{E}{X_{F_{actual}}}$$

$$X_{F_{actual}} = X_{d\ actual}'' = 0.3 \times \frac{V_{base}^2}{VA_{base}}$$

$$= 0.3 \times \frac{20^2}{500} = 0.24\ \Omega$$

Initial fault current

$$= \frac{E}{X_{F_{actual}}} = \frac{\left(\frac{20}{\sqrt{3}}\right) \times 10^3}{0.24} = 48.11 \text{ kA}$$

Base current in load circuit

$$= \frac{\text{Load MVA}}{\sqrt{3} \times \text{kV}} = \frac{400 \times 10^6}{\sqrt{3} \times 20 \times 10^3} = 11.547 \text{ kA}$$

Per-unit value of fault current $I_F = \frac{I_{F\,actual}}{I_{base}}$

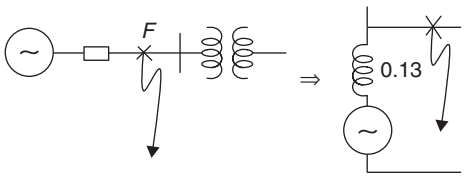
$$= \frac{48.11}{11.547} = 4.16 \text{ p.u.}$$

Example 3: A three-phase generator rated at 110 MVA, 11 kV is connected through circuit breakers to a transformer. The generator is having direct axis sub-transient reactance

$X''_d = 13\%$ and synchronous reactance X_d = 130%. The generator is operating at no-load and rated voltage when a three-phase short-circuit fault occurs between the breaker and the transformer. The magnitude of initial symmetrical RMS current in the breaker will be

(A) 76.9 kA (B) 44.4 kA
(C) 38.45 kA (D) 7.69 kA

Solution: (B)



Fault current $I_f = \dfrac{E}{X''_d} = \dfrac{1}{0.13} = 7.69$ p.u.

$$I_{\text{fault actual}} = I_{\text{base}} \times I_{\text{f p.u.}} = \frac{110 \times 10^6}{\sqrt{3} \times 11 \times 10^3} \times 7.69$$

$$= 44.4 \text{ kA}$$

Example 4: At a 220 kV substation of a power system, it is given that the three-phase fault level is 3500 MVA. Calculate the driving point reactance at the faulted bus

(A) 4.61 Ω (B) 27.65 Ω
(C) 6.91 Ω (D) 13.82 Ω

Solution: (D)

The short-circuit MVA = $3V_{\text{base}} I_{sc}$

Short-circuit current $(I_{sc}) = \dfrac{V_{\text{base}}}{X_f}$

$$\therefore \quad 3.V_{\text{base}} \cdot \frac{V_{\text{base}}}{X_{f_{eq}}} = 3500$$

$$X_{f_{eq}} = \frac{3V_{\text{base}}^2}{3500} = \frac{3 \times \left(\dfrac{220}{\sqrt{3}}\right)^2}{3500}$$

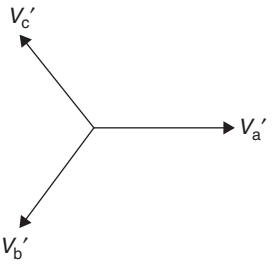
$$X_{f_{eq}} = 13.82\ \Omega$$

Symmetrical Components

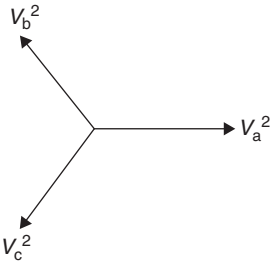
The key idea of symmetrical component analysis is to decompose the unbalanced system into three sequences of balanced networks. The networks are then coupled only at the point of the unbalanced.

The three sequence networks are known as the

1. Positive-sequence components consisting of three phasors equal in magnitude, displaced from each other by 120° in phase, and having the same phase sequence as the original phasors.



2. Negative-sequence components consisting of three phasors equal in magnitude, displaced from each other by 120° in phase, and having the phase sequence opposite to that of the original phasors.



3. Zero-sequence components consisting of three phasors equal in magnitude and with zero-phase displacements from each other.



Assume three unbalanced voltage phasors, V_A, V_B and V_C having a positive sequence (*abc*). Using symmetrical components, it is possible to represent each phasor voltage as

$$V_A = V_A^0 + V_A^1 + V_A^2$$

$$V_B = V_B^0 + V_B^1 + V_B^2$$

$$V_C = V_C^0 + V_C^1 + V_C^2$$

Operator '*a*': Function of the operator '*a*' was introduced, when it is operated upon a phasor that is rotated by +120° without changing the magnitude of it.

$$a = 1\angle 120° = e^{j120°} = -0.5 + j0.866$$

$$a^2 = 1\angle 240° = e^{-j120°} = -0.5 - j0.886$$

$$a^3 = 1\angle 360° = 1\angle 0° = e^{j360°} = 1$$

$$\therefore \quad 1 + a + a^2 = 0$$

$$a^4 = a;\ a^5 = a^2;\ a^6 = 1$$

$$a - a^2 = j\sqrt{3};\ a^2 - a = -j\sqrt{3}$$

Assuming phase '*a*' as reference, the above relation between the unsymmetrical components of phase '*b*' and '*c*' in terms of symmetrical components of phase '*a*' is given by

$$V_a = V_a^0 + V_a^1 + V_a^2$$

$$V_b = V_a^0 + a^2 V_a^1 + a V_a^2$$
$$V_c = V_a^0 + a V_a^1 + a^2 V_a^2$$

Matrix representation of the above relation is given by

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix} = [T] \begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix}$$

Inverse relation is given

$$\begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

The complex power flowing into a three-phase circuit through three lines a, b and c is

$$S_3\phi = P + jQ = V_a I_a^* + V_b I_b^* + V_c I_c^*$$

where V_a, V_b and V_c are the voltages and I_a, I_b and I_c are the currents flowing into the circuit in the three lines.

$$S_{3\phi} = \left(T V_a^{012}\right)^T \left(T I_a^{012}\right)^*$$
$$= V_a^{012^T} T^T T^* I_a^{012^*}$$

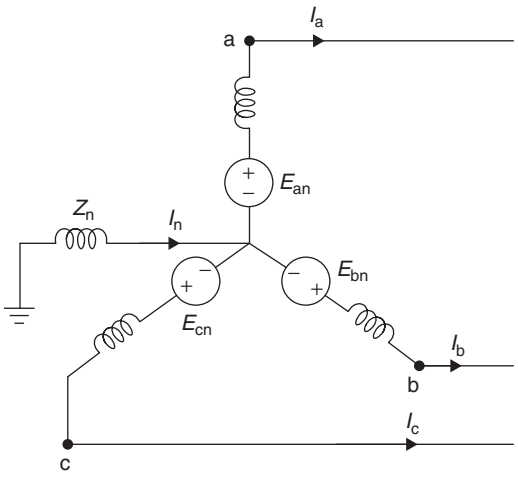
Since $T^T T^* = 3$,

$$\therefore \quad S_3\phi = 3\left(V^{012^T} I^{012^*}\right)$$
$$S_3 = 3V^0 I^{0^*} + 3V' I'^* + 3V^2 I^{2^*}$$

The above equation shows that the total unbalanced three-phase power is equal to the sum of the three symmetrical component powers.

Sequence Networks of Synchronous Generator

A Synchronous generator, grounded through a reactor is shown in figure below.



When the fault takes place at the terminals, the current I_a, I_b and I_c flows in the lines. Whenever the fault involves ground, the total current $I_a + I_b + I_c = I_n$ flows into the neutral of the generator and the line currents can be resolved into symmetrical components.

Positive Sequence Impedance and Network

Synchronous generator induces positive sequence emf's only due to its symmetrical winding designs. When the machine carries positive sequence currents only, this mode of operation is the balanced mode. As the short circuit progresses in time, the machine equivalently offers a direct axis reactance whose value reduces from sub transient reactance (X_d'') to transient reactance (X_d') and finally to steady-state reactance (X_d). If the armature resistance is assumed negligible, the positive sequence impedances of machine is

$$Z_1 = jX_d'' \text{ (sub transient period} \rightarrow 1 \text{ cycle)}$$
$$= jX_d' \text{ (transient period} \rightarrow 3\text{–}4 \text{ cycles)}$$
$$= jX_d \text{ (steady-state value)}$$

Since the neutral current $I_n = 0$, the neutral impedance Z_n does not appear in the model. Since the positive-sequence network is a balanced network, it can be represented by the single-phase network model as shown in figure.

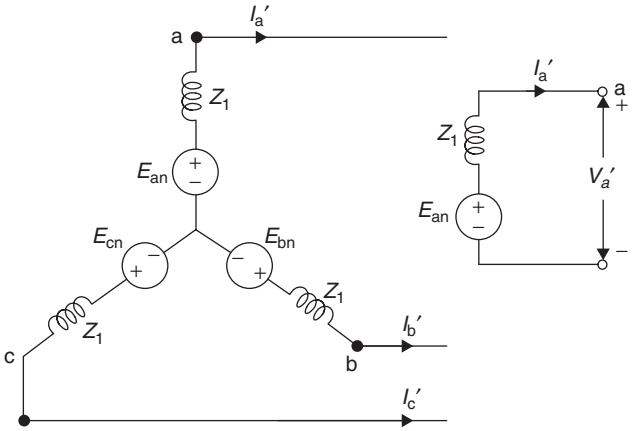


Figure 1 Positive-sequence network and its single-phase equivalent

Negative Sequence Impedance Network

Synchronous machine has no negative sequence induced voltages. With the flow of negative sequence currents in the stator, a rotating field is created which rotates in opposite direction to that of the positive sequence, therefore, at double synchronous speed with respect to rotor. The negative sequence mmf is alternately presented with reluctance of direct and quadrature axes. The negative sequence impedance presented by the machine with consideration given to the damper windings is defined as

$$Z_2 = j\frac{X_q' + X_d''}{2}; |Z_2| < |Z_1|$$

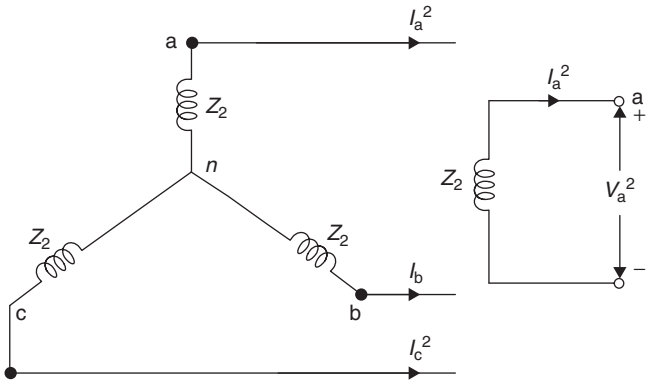


Figure 2 Negative-sequence network and its single-phase equivalent

Zero-sequence Impedance Network

Synchronous machine has no zero-sequence voltages induced, the flow of zero-sequence currents creates three mmf's which are in time phase, but are distributed in space by 120°. The current flowing through the impedance Z_n between neutral to ground is $I_n = 3I_{a_0}$. The zero-sequence voltage drop of zero sequence from point 'a' to ground is

$$V_{a_0} = -(3Z_n + Z_{g_0})I_{a_0}$$

The zero-sequence circuit, which is a single-phase circuit assumed to carry only the zero-sequence current of one phase, must therefore have an impedance of $3Z_n + Z_{g0}$. The total zero-sequence impedance through which I_a^0 flows is

$$Z_0 = 3Z_n + Z_{g_0}$$

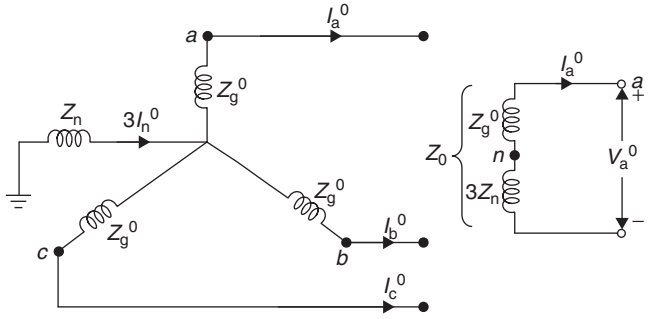
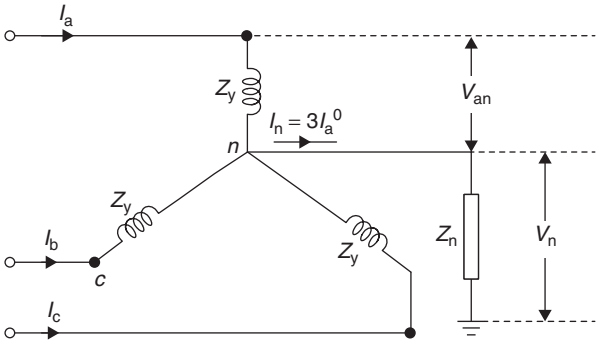


Figure 3 Zero-sequence network and its single-phase equivalent

Sequence Circuit of Y- and Δ-connected Loads

Consider the following Y-connected load.



In the above *Y*-connected load, sum of the line current is equal to the neutral current I_n and it is given by

$$I_n = I_a + I_b + I_c$$

Voltage of phase '*a*' with respect to neutral is given by

$$V_a = V_{an} + V_n,$$

where $V_n = 3I_a^0 Z_n$

We can also write the voltage drops to ground from each of the lines *a*, *b* and *c* as

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} V_{an} \\ V_{bn} \\ V_{cn} \end{bmatrix} + \begin{bmatrix} V_n \\ V_n \\ V_n \end{bmatrix} = Z_Y \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} + 3I_a^0 Z_n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The *a*–*b*–*c* voltages and currents in the above equation can be replaced by their symmetrical components and the symmetrical component relation is given by

$$\begin{bmatrix} V^0 \\ V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} Z_Y + 3Z_n & 0 & 0 \\ 0 & Z_Y & 0 \\ 0 & 0 & Z_Y \end{bmatrix} \begin{bmatrix} I^0 \\ I^1 \\ I^2 \end{bmatrix}$$

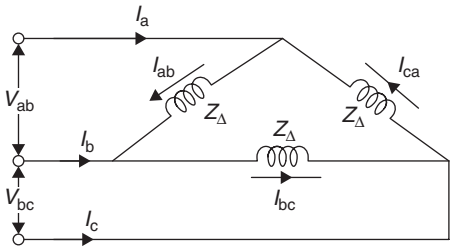
Delta-Connected Load

The delta (Δ) circuit cannot provide a path through neutral. Therefore for a delta-connected load or its equivalent '*Y*' connected cannot contain any zero-sequence component.

$$V_{ab} = Z_\Delta \cdot I_{ab}$$

$$V_{bc} = Z_\Delta \cdot I_{bc}$$

$$V_{ca} = Z_\Delta \cdot I_{ca}$$



The sum of line-to-line voltages or phase currents is always zero.

$$\frac{1}{3}(V_{ab} + V_{bc} + V_{ca}) = V_{abc} = 0$$

$$\frac{1}{3}(I_{ab} + I_{bc} + I_{ca}) = I_{abc} = 0$$

Therefore, for a delta-connected load without sources or mutual coupling there will be no zero-sequence currents at the lines (there are some cases where a circulating currents may circulate inside a delta load and not seen at the terminals of the zero sequence circuit).

System parameters relations are decoupled and written as

$$V^0 = (Z_Y + 3Z_n)I^0$$

$$V^1 = Z_Y I^1$$

$$V^2 = Z_Y I^2$$

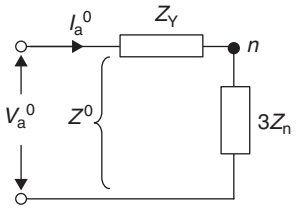


Figure 4 Zero sequence network

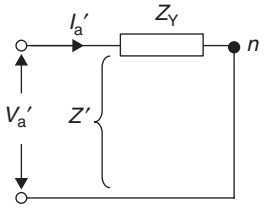


Figure 5 Positive sequence

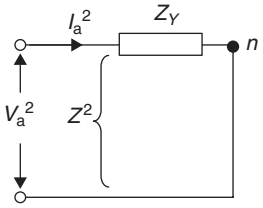
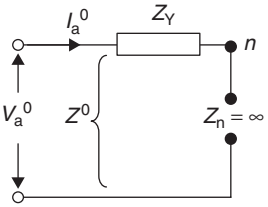


Figure 6 Negative sequence

For ungrounded or Isolated neutral star-connected loads, no zero sequence current can flow and $Z_n = \infty$. Then the positive- and negative-sequence networks remain unchanged but zero-sequence network is as shown in the figure below.



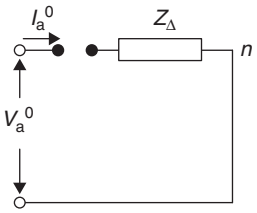


Figure 7 Zero sequence circuit

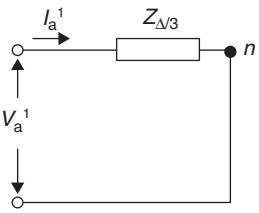


Figure 8 Positive sequence circuit

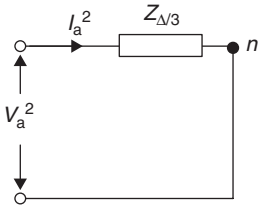
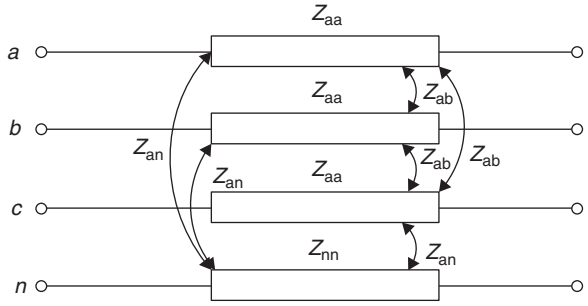


Figure 9 Negative sequence circuit

Sequence Diagrams of Transmission Lines

For transmission lines, assume we have the following, with mutual impedances.



The phase relationships are given by

$$\begin{bmatrix}\Delta V_a\\ \Delta V_b\\ \Delta V_c\end{bmatrix} = \begin{bmatrix}Z_s & Z_m & Z_m\\ Z_m & Z_s & Z_m\\ Z_m & Z_m & Z_s\end{bmatrix}\begin{bmatrix}I_a\\ I_b\\ I_c\end{bmatrix}$$

where Z_s = Self-impedance of the phase = $Z_{aa} + Z_{nn} - 2Z_{an}$

Z_m = Mutual impedance between the phases

$= Z_{ab} + Z_{nn} - 2Z_{an}$

$\therefore \quad \Delta V = ZI$

After converting relationships to sequence representation

$$\Delta V_s = A^{-1} Z A I_s$$

ΔV_s and I_s are symmetrical voltages and currents.

$$A^{-1} ZA = \begin{bmatrix}Z_s - Z_m & 0 & 0\\ 0 & Z_s - Z_m & 0\\ 0 & 0 & Z_s - Z_m\end{bmatrix}$$

$\therefore \quad Z^1 = Z_s - Z_m$

$$Z^2 = Z_s - Z_m$$

$$Z^\circ = Z_s + 2Z_m$$

where $Z_s = Z_{aa} + Z_{nn} - 2Z_{an}$

$$Z_m = Z_{ab} + Z_{nn} - 2Z_{an}$$

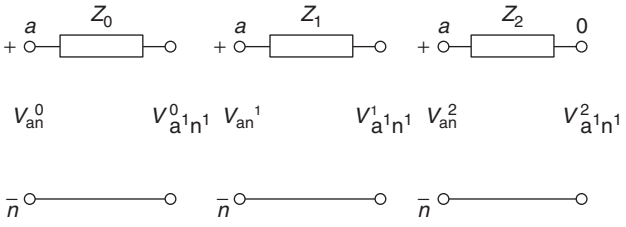


Figure 10 Positive-, negative- and zero-sequence networks of transmission lines

Sequence Diagrams for Transformers

Similar to the transmission lines, the positive and negative sequence diagrams for transformers are same and equal to its leakage impedances.

$$Z^1 = Z^2 = Z_{\text{leakage}}$$

The zero-sequence network of the transformer depends on how the transformer is grounded and its type of connection current flows only through the winding connected in star and the star point in earthed. For delta winding, a star winding or with neutral isolated, the zero-sequence current cannot flow.

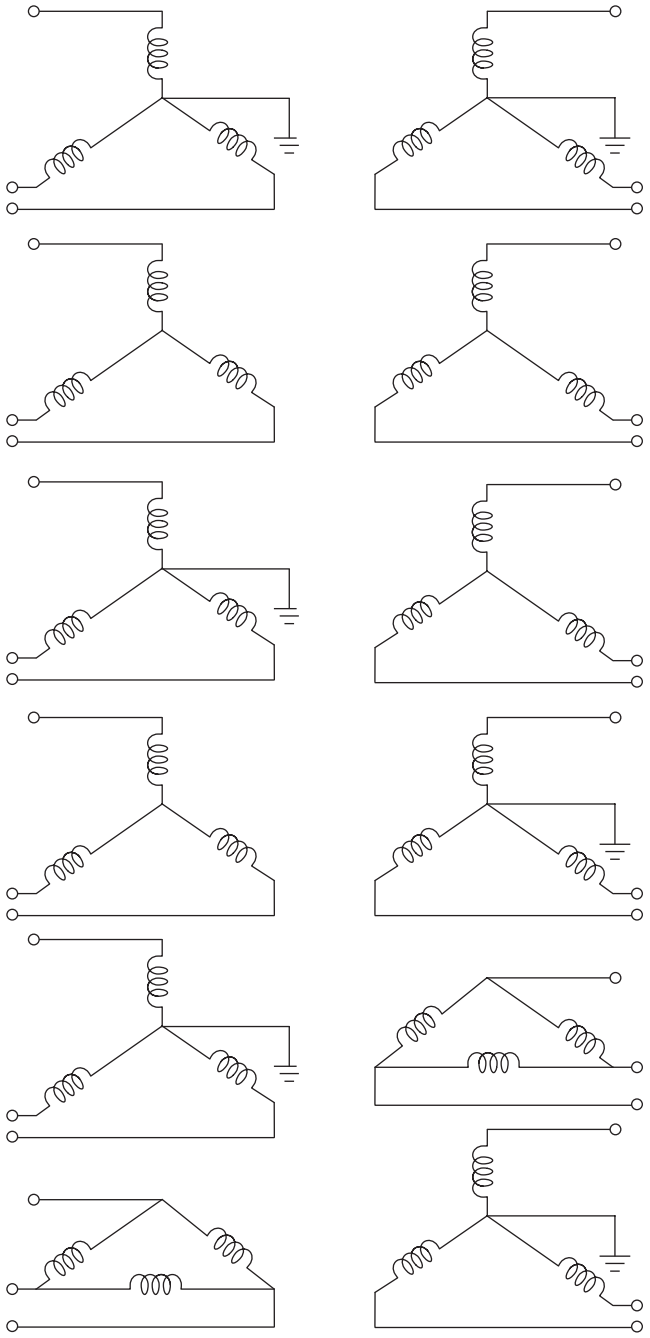


Figure 11 (Continued)

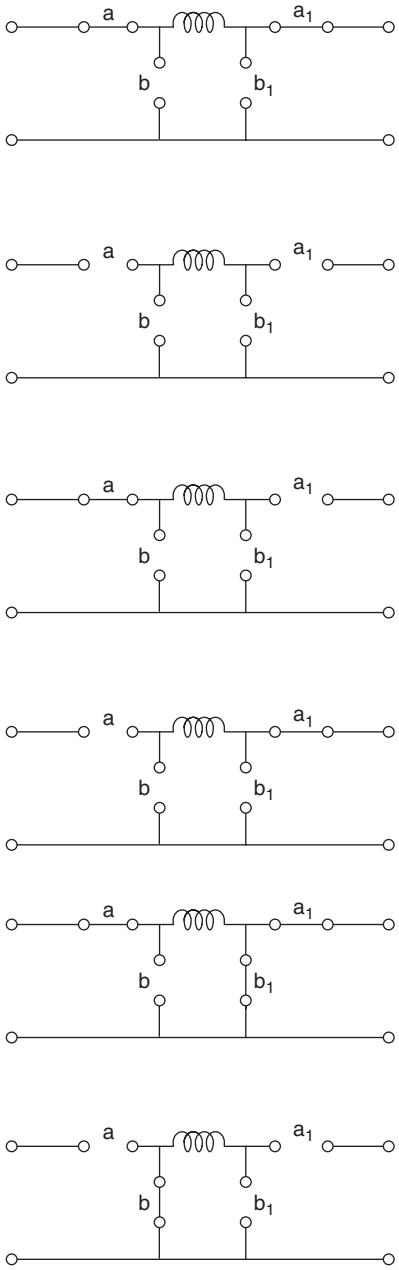


Figure 12 (Continued)

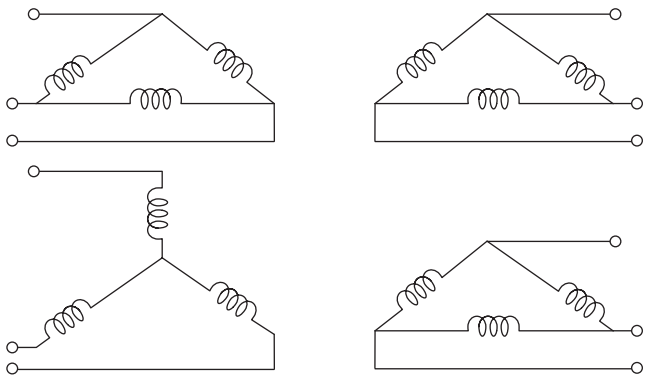


FIgure 11 Type of transformer connection

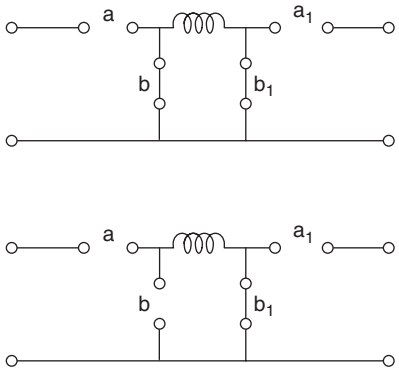


Figure 12 Zero-sequence network

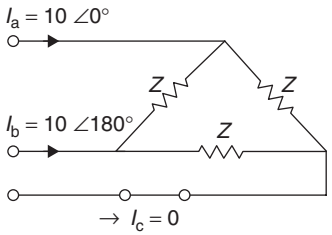
Example 5: One conductor of a 3-ϕ line is open. The current flowing to the Δ-connected load through line 'a' is 10 A with the current in line 'a' as reference and assuming that line 'c' is open, find symmetrical components of line currents.

Solution: Circuit is shown in figure, and the line currents are given by

$$I_a = 10\angle 0^\circ \text{ A}$$

$$I_b = 10\angle 180^\circ \text{ A}$$

$$I_c = 0 \text{ A}$$



$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}$$

$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 10\angle 0 \\ 10\angle 180^\circ \\ 0 \end{bmatrix}$$

$$I_a^0 = 0 \text{ A}$$

$$I_a^1 = 5.78\angle -30^\circ \text{ A}$$

$$I_a^2 = 5.78\angle 30^\circ \text{ A}$$

$$I_b^0 = a^2 I_a^0 = 0 \text{ A}$$

$$I_b^1 = a^2 I_a^1 = 5.78\angle -50^\circ \text{ A}$$

$$I_b^2 = a^2 I_a^2 = 5.78\angle 150^\circ \text{ A}$$

$$I_c^0 = aI_a^0 = 0 \text{ A}$$

$$I_c^1 = aI_a^1 = 5.78\angle 90^\circ \text{ A}$$

$$I_c^2 = aI_a^2 = 5.78\angle -90^\circ \text{ A}$$

Note: Symmetrical components of phase 'c' are non-zero even though there is no current in that phase but the sum of the corresponding phase symmetrical components is equal to zero.

Example 6: The parameters of transposed overhead transmission line are given as

$$\text{Self-reactance } X_s = 0.45 \ \Omega/\text{km}$$

$$\text{Mutual reactance } X_m = 0.12 \ \Omega/\text{km}$$

The positive-sequence reactance X_1 and zero-sequence reactance X_0, respectively, in Ω/km are

(A) 0.69, 0.33 (B) 0.33, 0.69
(C) 0.33, 0.66 (D) 0.66, 0.33

Solution: (B)

Positive-sequence reactance $= X_s - X_m$

$$= 0.45 - 0.12 = 0.33 \ \Omega/\text{km}$$

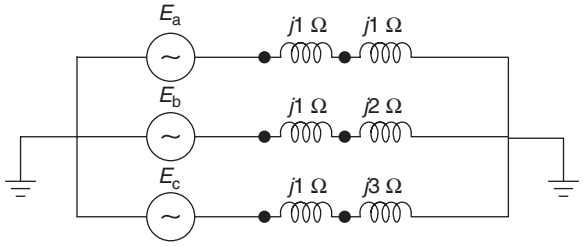
Zero-sequence reactance $= X_s + 2X_m$

$$= 0.45 + 0.24 = 0.69 \ \Omega/\text{km}$$

Example 7: A three-phase alternator generating unbalanced voltages is connected to an unbalanced load through a three-phase transmission line as shown in figure. The neutral of the alternator and the star point of the load are solidly grounded. The phase voltages of the alternator are

$$E_a = 10\angle 0^\circ \text{ V}, \quad E_b = 10\angle -90^\circ \text{ V}, \quad E_c = 10\angle 120^\circ \text{ V}$$

The negative sequence component of the load current is



(A) 0.33 A (B) 5 A (C) 3.33 A (D) 2.5 A

Solution: (A)
Since the circuit is grounded on both the sides, each phase can be treated as a single-phase circuit and the phase currents can be written as

$$I_a = \frac{E_a}{X_a} = \frac{10\angle 0^\circ}{j2} = 5\angle -90^\circ$$

$$I_b = \frac{E_b}{X_b} = \frac{10\angle -90^\circ}{j3} = 3.33\angle -180^\circ$$

$$I_c = \frac{E_c}{X_c} = \frac{10\angle 120^\circ}{j4} = 2.5\angle 30^\circ$$

$$\text{Negative-sequence currents} = I_a^2 = \frac{1}{3}\left(I_a + a^2 I_b + a I_c\right)$$

$$= \frac{1}{3}\left(5\angle -90^\circ + 3.33\angle -300 + 2.5\angle 150^\circ\right)$$

$$= 0.33\angle -120$$

Example 8: A three-phase, 30 MVA, 6.6 kV alternator having 15% reactance is connected through a 30 MVA, 6.6/33 kV delta–star-connected transformer of 5% reactance to a 33 kV transmission line having a negligible resistance and a reactance of 4 Ω. At the receiving end of the transmission line there is a 30 MVA, 33/6.6 kV delta–star-connected transformer of 5% reactance stepping down the voltage at 6.6 kV, both the transformers have their neutral solidly grounded. Draw the one-line diagram and the positive-, negative- and zero-sequence networks of this system.

Solution: One-line diagram of the system is shown in the figure below

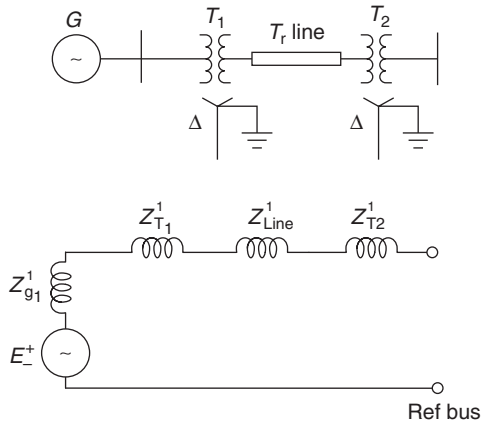


Figure 13 Positive-sequence equivalent network

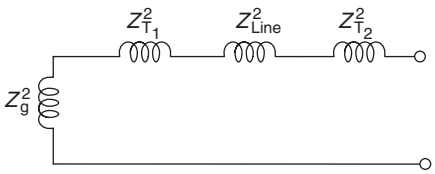


Figure 14 Negative-sequence equivalent network

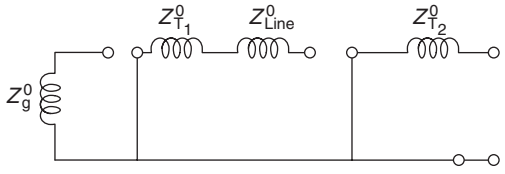
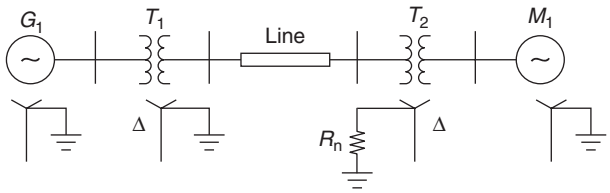
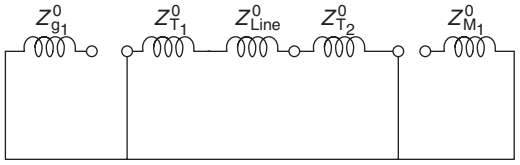


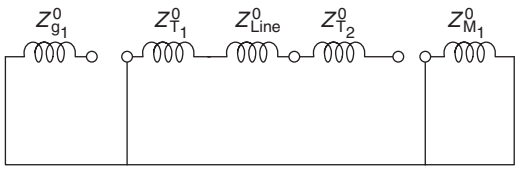
Figure 15 Zero-sequence equivalent network

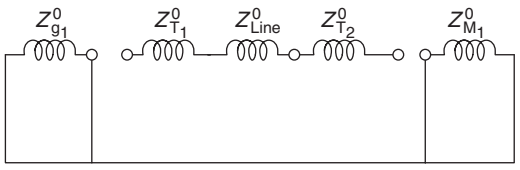
Example 9: Single-line diagram of a power system is given by

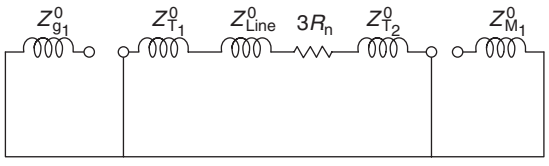


Which one of the following is the zero-sequence equivalent of the above system?

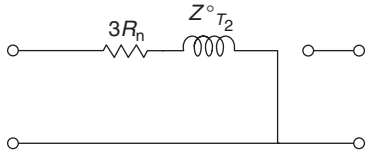
(A)


(B)


(C)


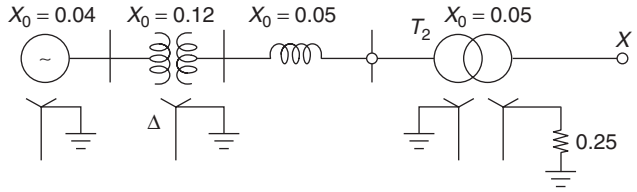
(D)


Solution: (D)
For 3-ϕ transformer 2, equivalent circuit is given by



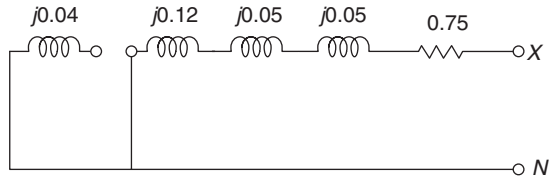
Example 10: A generator is connected to a transformer which feeds another transformer through a short transmission line. The zero-sequence impedances values are expressed in p.u. on a common base and are indicated in

figure. The Thevenin's equivalent zero-sequence impedance at point '*X*' is



(A) $0.75 + j0.22$ (B) $0.75 + j0.26$
(C) $0.25 + j0.26$ (D) $0.25 + j0.22$

Solution: (A)
Zero-sequence equivalent network is given by



Equivalent impedance at point '*X*' is, $Z_{X_n} = j0.22 + 0.75$

Unbalanced Fault Analysis

Various types of unsymmetrical faults that occurs in power systems are

Shunt type faults

1. Single line to ground (LG) fault
2. Line to line (LL) fault
3. Double line to ground (LLG) fault

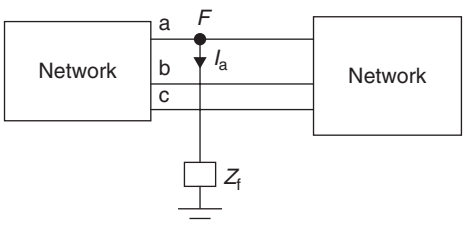
Series type of faults

1. One conductor open fault
2. Two conductor open fault

Symmetrical components are very commonly used in calculating currents arising from unbalanced short circuits.

Single Line to Ground Fault

A single line to ground fault at a point in power system is through a fault impedance as shown in the figure.



At the point of fault occurrence, the currents come out of the network and line to ground voltages are given by

$$I_b = 0$$

$$I_c = 0$$

$$V_a = I_a Z_f$$

Symmetrical components of the fault currents are,

$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I_a \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \Rightarrow I_a^1 = I_a^2 = I_a^0 = \frac{1}{3} I_a \qquad (1)$$

Similarly symmetrical components of the fault voltages are

$$V_a^1 + V_a^2 + V_a^1 = Z_f I_a = 3Z_f I_a^1 \qquad (2)$$

Expression (1) and (2) together describes the sequence networks connection shown in the figure.

Note: Positive-, negative- and zero-sequence components currents of a phase are equal if there is a LG fault in that particular phase.

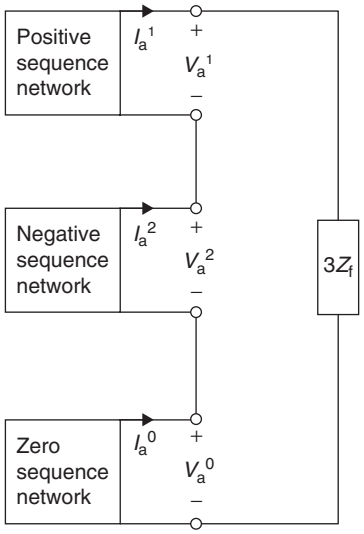
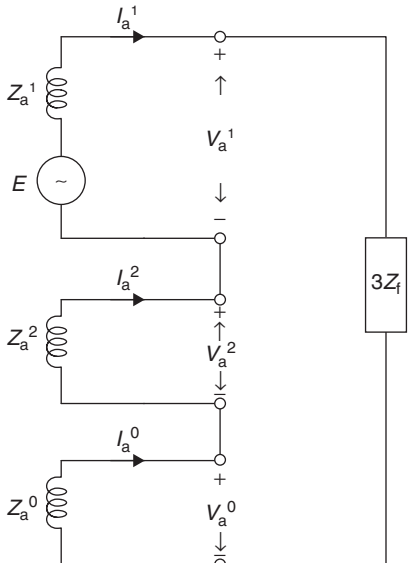


Figure 16 Sequence connection for LG Fault

The Thevenin's equivalent of sequence networks is given by



From the above equivalent reactance diagram,

$$I_a^1 = \frac{E_a}{(Z_1 + Z_2 + Z_0) + 3Z_f}$$

Fault current in phase 'a'

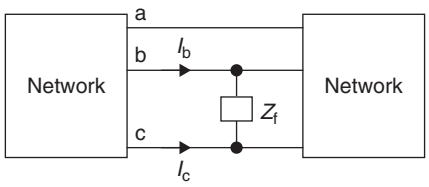
$$I_a = 3I_{a_1} = \frac{3E_a}{(Z_1 + Z_2 + Z_0) + 3Z_f}$$

Phase voltage in faulted phase

$$V_a = I_a Z_f = \frac{3E_a}{(Z_1 + Z_2 + Z_0) + 3Z_f} . Z_f$$

Line-to-Line Fault

A line-to-line fault in a power system on phases 'b' and 'c' through a fault impedance Z_f is shown in the following figure.



The current and voltages at the fault can be expressed as

$$I_a = 0$$

$$I_b = -I_c \text{ and } V_b - V_c = I_b Z_f$$

Symmetrical components of fault currents and voltages are given by

$$I_a^0 = 0 \text{ and } I_a^1 = -I_a^2$$

$$V_a^1 = V_a^2 \text{ and } V_a^1 - V_a^2 = Z_f I_a^1$$

The above relations describe the sequence network connections as shown in the figure.

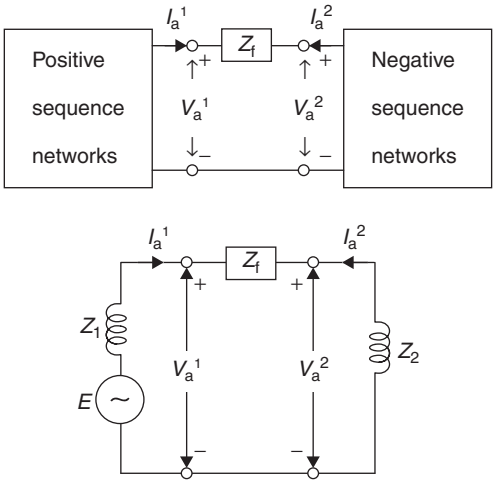


Figure 17 Sequence networks connection and their Thevenin's equivalent network

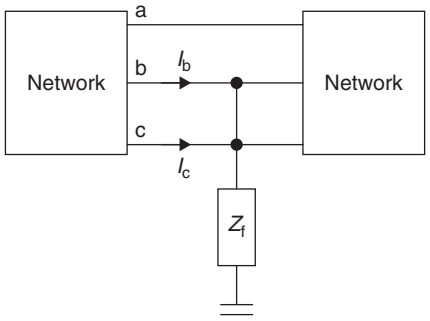
From the equivalent reactance diagram,

$$I_a^1 = \frac{E_a}{Z_1 + Z_2 + Z_f}$$

$$I_f = I_b = -I_c = \frac{-j\sqrt{3}\,E_a}{Z_1 + Z_2 + Z_f}$$

Double Line to Ground Fault

A double line to ground fault in power system is shown in the figure below.



The voltage and current conditions at the fault are expressed as

$$I_a = 0 \text{ or } I_a^1 + I_a^2 + I_a^0 = 0$$

$$V_b = V_c = Z_f(I_b + I_c) = 3Z_f I_a$$

Symmetrical voltages and currents relations are given by

$$V_a^1 = V_a^2 = \frac{1}{3}\left[V_a + \left(a + a^2\right)V_b\right]$$

$$V_a^0 = \frac{1}{3}\left(V_a + 2V_b\right)$$

$$V_a^1 - V_a^0 = \frac{1}{3}\left(2 - a - a^2\right) = V_b = 3Z_f I_{a_0}$$

The above relations describe the sequential network connections as shown in the following figure.

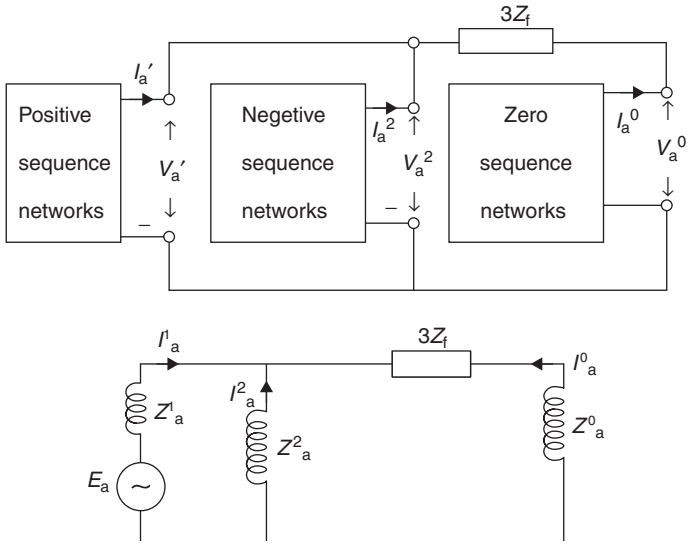


Figure 18 Sequence-network connections and their Thevenin's equivalent network

From the above sequential networks,

$$\Rightarrow \quad I_a^1 = \frac{E_a}{Z_a^1 + \dfrac{Z_a^2\left(Z_a^0 + 3Z_f\right)}{Z_a^0 + Z_a^2 + 3Z_f}}$$

$$I_f = 3I_{a_0} = 3\left[\frac{E_a}{Z_a^2} - \left(\frac{Z_a^1 + Z_a^2}{Z_a^2}\right) I_a^1\right]$$

Example 11: The severity of the line to ground and three-phase fault at the terminals of an unloaded synchronous generator is to be same. If the terminal voltage is 'E' and the positive, negative and zero-sequence impedances are Z_1, Z_2 and Z_0, respectively, then the required inductive reactance for neutral grounding is

(A) $2Z_1 - Z_2 - Z_0$ (B) $\frac{1}{3}\left[2Z_1 - Z_2 - Z_0\right]$

(C) $Z_1 + Z_2 + Z_0$ (D) $\frac{1}{3}\left[Z_1 + Z_2 + Z_3\right]$

Solution: (B)

For 3-ϕ fault, fault current $I_f = \frac{E}{Z_1}$ p.u.

For fault of line to ground

$$I_{f_{Lg}} = \frac{3E}{Z_1 + Z_2 + Z_0 + 3Z_n} \text{ p.u.}$$

If severity of line to ground and 3-ø faults are same

$$\frac{E}{Z_1} = \frac{3E}{Z_1 + Z_2 + Z_0 + 3Z_n}$$

$$Z_1 + Z_2 + Z_0 + 3Z_n = 3Z_1$$

$$Z_n = \frac{2Z_1 - Z_2 - Z_0}{3}$$

$$\Rightarrow \quad Z_n = \frac{1}{3}\left[2Z_1 - Z_2 - Z_0\right]$$

Example 12: The per-unit values of positive-, negative- and zero-sequence reactance of a network at fault are 0.2, 0.17 and 0.12 p.u., respectively. The fault current for double line to ground fault is

(A) 4.718 p.u. (B) 2.04 p.u.

(C) 1.57 p.u. (D) 16.26 p.u.

Solution: (A)

For LLG fault

$$I_f = \frac{3\left(Z_2 + Z_0\right)}{Z_1 Z_2 + Z_1 Z_0 + Z_2 Z_0}$$

$$= \frac{3\left(0.17 + 0.12\right)}{0.2 \times 0.7 + 0.2 \times 0.12 + 0.17 \times 0.12}$$

$$I_f = 4.718 \text{ p.u.}$$

Example 13: Zero-sequence current of fault current is 100 A, then the SLG fault current is

(A) 100 A (B) 33.33 A

(C) 300 A (D) 200 A

Solution: (C)

Single-line to ground fault current $= 3I_{a_0}$

$$= 3 \times 100 \text{ A}$$

$$= 300 \text{ A}$$

Example 14: A 500 MVA, 22 kV generator has $X_d'' = X_1 = X_2 = 15\%$ and $X_0 = 5\%$. Its neutral is grounded through a reactor of 0.3 Ω. The generator is operating at rated voltage, and no load. If it is disconnected from the system when a SLG fault occurs at its terminals, the sub-transient current in the fault phase is

(A) 13.12 kA (B) 31.5 kA

(C) 2.4 kA (D) 10.5 kA

Solution: (B)

Fault current for SLG fault $I_f = 3I_{a1}$

$$I_f = 3I_{a_1} = \frac{3E_a}{Z_1 + Z_2 + Z_0 + 3Z_n}$$

$$= \frac{3}{0.15 + 0.15 + 0.05 + 0.9} = 2.4 \text{ p.u.}$$

$$I_{base} = \frac{S_{base}}{\sqrt{3}V_{base}} = \frac{500}{\sqrt{3} \times 22} = 13.12 \text{ kA}$$

Fault current $= I_{fp.u.} \times I_{base}$

$$= 2.4 \times 13.12 \text{ kA}$$

$$= 31.49 \text{ kA}$$

Example 15: A 25 MVA, 11 kV alternator has positive, negative and zero sequence reactance of 0.3 p.u., 0.35 p.u. and 0.05 p.u., respectively. What value of resistance must be put in the generator neutral so that the fault current for a line to ground fault of zero fault impedance will not exceed the rated line current?

(A) 4.84 Ω (B) 4.7 Ω (C) 0.972 Ω (D) 2.35 Ω

Solution: (B)

Rated current in per unit $I_{rated} = 1$ p.u.

Fault current for SLG fault

$$I_f = \frac{3E_a}{Z_1 + Z_2 + Z_0 + 3Z_n} = 1$$

$$\frac{3 \times 1}{j\left(0.3 + 0.35 + 0.05\right) + 3R_n} = 1$$

$$R_n = \frac{1}{3}\sqrt{3^2 - \left(0.3 + 0.35 + 0.05\right)^2}$$

$$= \frac{1}{3}\sqrt{9 - 0.49} = 0.972 \text{ p.u.}$$

$$Z_{base} = \frac{(KV)_{base}^2}{MVA_{base}} = \frac{11^2}{25} = 4.84\ \Omega$$

Actual $R_n = R_{n\,p.u.} \times Z_{base}$

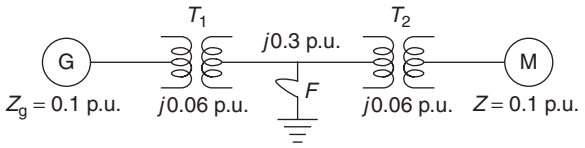
$$= 0.972 \times 4.84 = 4.7\ \Omega$$

EXERCISES

Practice Problems I

Directions for questions 1 to 20: Select the correct alternative from the given choices.

1. A power system is represented by a single line diagram with all the reactance marked in per unit (p.u) on the same base



The system is on no load when three-phase fault occurs at F on the high voltage side of the transformer T_2. The fault current will be

(A) $-j0.726$ p.u. (B) $-j0.84$ p.u.
(C) $-j7.42$ p.u. (D) $-j8.4$ p.u.

2. The 3-ϕ power system operating at 122 kV with 0.6 p.f. Recovery voltage = 0.8 times full line value, Breaking current is symmetrical, frequency of oscillation of restriking voltage is 20 kHz. Assuming the neutral is grounded and the fault does not involve with the ground, the average rate of rise of restriking voltage is

(A) 20.82 kV/ms (B) 30.76 kV/ms
(C) 56.5 V/ms (D) 30.76 V/s

3. Two 11 kV, 30 MVA, 3 f star-connected generators operate in parallel. The positive-, negative- and zero-sequence reactances of each are $j0.2$ p.u., $j0.35$ p.u., $j0.15$ p.u., respectively. Assume that the generator is grounded solidly. Voltage of the healthy phase for a double line to ground fault at the terminals is

(for solid fault, $z_f = 0$)

(A) 0.625 V (B) 0.766 V
(C) 0.52 V (D) 0.25 V

4. A 15 MVA, 13.2 kV alternator has $X_1 = X_2 = 30\%$ and $X_0 = 8\%$ Reactance that must be put in the generator neutral so that the fault current for a line to ground fault of zero fault impedance will not exceed the rated current is

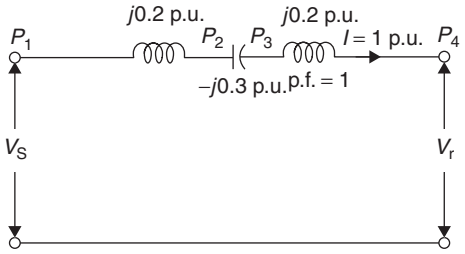
(A) 2.6 W (B) 6.8 W
(C) 9 W (D) 4.8 W

5. In a three-phase, 4-wire system, the line currents under abnormal conditions are given by

$(10 + j6)$, $(12 - j8)$ and $(-14 + j12)$. The positive sequence currents in lines will be

(A) $-2 + 4j$ (B) $-1.26 + 2.7j$
(C) $3.4 - 2.4j$ (D) $1.26 - 2.7j$

6. For the transmission line model with a series capacitor at its midpoint, the maximum voltage on the line is at location



(A) P_1 (B) P_2 (C) P_3 (D) P_4

7. A power system network with the capacity of 150 MVA has a source impedance of 15% at a point. The fault at the point is

(A) 10 MA (B) 150 MVA
(C) 3500 MVA (D) 1000 MVA

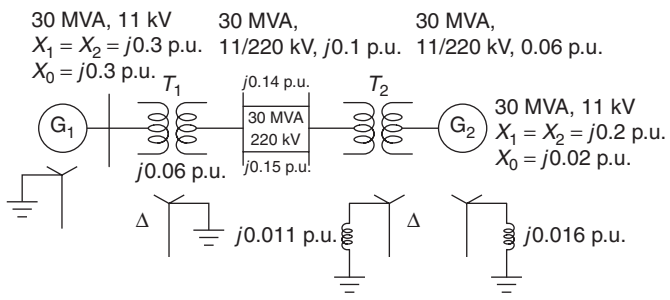
8. The positive-, negative- and zero-sequence impedances of a solidly grounded system under steady-state condition follow the relations

(A) $Z_1 > Z_2 > Z_0$
(B) $Z_1 < Z_2 < Z_0$
(C) $Z_0 < Z_1 < Z_2$
(D) None of these

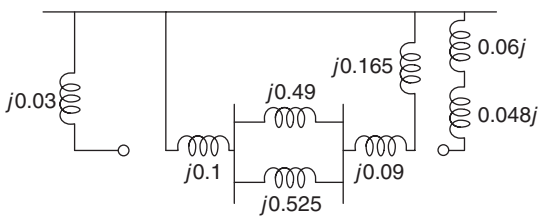
9. A 3-ϕ. 10 MVA, 2.8 kV alternator with a reactance of 6% is connected to a feeder of series impedance of $0.2 + j0.6$ ohm/phase per km. The transformer rated at 5 MVA, 1.8/22 kV has a reactance of 5%. The fault current supplied by the generator operating under no load with a voltage of 3.8 kV, when a 3-ϕ symmetrical fault occurs at a point 10 km along the feeder is

(A) $(1406 \angle 76.288)$ A
(B) $(436 \angle 76.288)$ A
(C) $(15 + j80)$ A
(D) $(400 - j83.4)$ A

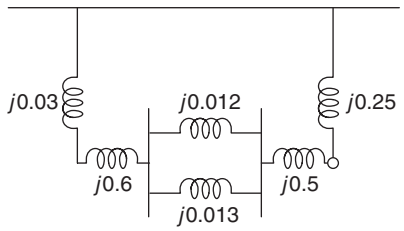
10. The zero-sequence network of the single-line diagram is shown below.



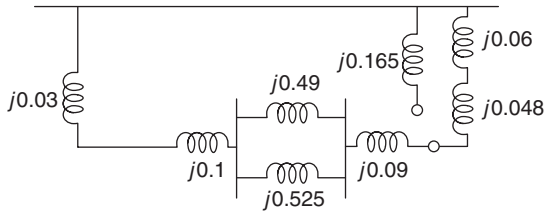
(A)



(B)



(C)



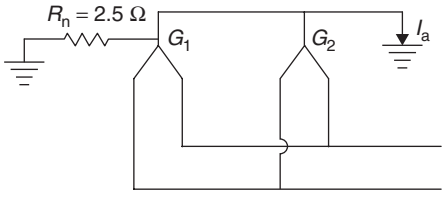
(D) None of these

11. An earth fault occurs on a busbar to which 3, 20 MVA, 3.3 kV, 3-ϕ alternators are connected in parallel. When only one alternator is solidly grounded and others are isolated, the fault current is [Given $X_d'' = 12\%$, $X_2 = 10\%$ and $X_0 = 13.5\%$]
(A) 8.895×10^4 A (B) 6.85×10^4 A
(C) 0.00789 A (D) 5.65 mA

12. A single-phase load of 100 kVA is connected across line bc of a three-phase supply of 3.3 kV. The symmetrical components of line currents are
(A) 17.5, –17.5 and 0 (B) –17.5, 17.5 and 0
(C) 12.5, –12.5 and 0 (D) –12.5, 12.5 and 0

13. A generator is unloaded and operated at rated voltage when a fault occurs. Neglecting resistance identify the fault. The values of currents and voltages are given as follows
$I_{a0} = j2.37$ p.u.
$I_{a1} = -j8.45$ p.u.
$I_{a2} = j6.08$ p.u.
$V_{a0} = V_{a1} = V_{a2} = j0.237$ p.u.
(A) L–G fault (B) L–L fault
(C) L–L–G fault (D) Open conductor fault

Common data for Questions 14–16:
Two 12.8 kV, 25 MVA, three-phase, star-connected generators operate in parallel as shown in the following figure.



The positive, negative and zero sequence reactances are, respectively, $j0.20$, $j0.16$, $j0.12$ p.u. The star point of one of the generator is isolated and that of the other is earthed through a 2.5 Ω resistor. A single line to ground fault occurs at the terminals of one of the generators

14. The fault current will be
(A) 2.55 ∠–15° p.u. (B) 1.0675 ∠–15° p.u.
(C) 2.93 ∠–30° p.u. (D) 3.0675 ∠–30° p.u.

15. The current in the grounding resistor is
(A) 3.0675 ∠–30° p.u (B) 2.55 ∠–15° p.u
(C) 1.0675 ∠–15° p.u (D) 2.93 ∠–30° p.u

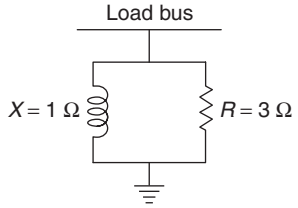
16. The voltage across the grounding resistor is
(A) 3067.5 ∠– 15° p.u. (B) 2.9223 ∠– 15° p.u.
(C) 3.0675 ∠– 15° p.u. (D) 0.9551 ∠– 15° p.u.

17. A three-phase, 20 MVA, 13.2 kV 50 Hz generator with solidly earthed neutral has sub transient reactance X_d'' of 25%; direct-axis reactance X_d' of 20% and synchronous reactance X_d of 50%. Negative – sequence reactance $X_2 = 10\%$ and zero-sequence reactance $X_0 = 6\%$. The generator is operated on open circuit when fault occurs. Take $E_a = 1.0$. If the fault is a three-phase short without an impedance, the value of initial symmetrical sub transient, transient and sustained RMS values of the line current under faulty condition will be
(A) $-j4.0, -j5.0, -j2$ p.u.
(B) $-j1.667, -j4.0, -j5.0$ p.u.
(C) $-j15, -j14, -j11.667$ p.u.
(D) $-j11.667, -j14, -j15$ p.u.

18. Three 10 MVA, 13.8 kV, three-phase star-connected alternators are operating in parallel. Each has $X_d'' = 10\%$, $X_2 = 8\%$ and $X_0 = 3\%$. If an earth fault occurs on one bus bar, the value of fault current when all the three alternators are solidly grounded is
(A) $j25{,}000$ A (B) $-j\ 4062.27$ A
(C) $j5{,}000$ A (D) $j15{,}000$ A

19. A 20 MVA, 6.6 kV alternator with solidly grounded neutral has a sub transient reactance of 0.25 p.u. The negative and zero sequence reactances are 0.30 p.u. and 0.15 p.u., respectively. A single phase to ground fault occurs at the terminals of this unloaded generator. The fault impedance is 0.01 p.u. Then the value of three sequent components of voltages as percentage of rated values will be
(A) 65.75%, –41.1% and –20.55%
(B) –65.75%, 41.1% and 20.55%
(C) 41.75%, –36.2% and –15.4%
(D) –41.75%, 36.2% and 15.4%

20. A load of 25 kVA rating is connected to the bus as shown. If the bus voltage is 400 V, the p.u. value of resistance is



(A) 0.469 (B) 0.0469
(C) 0.187 (D) 0.0187

Practice Problems 2

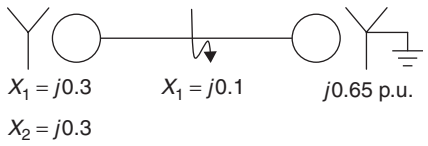
Directions for questions 1 to 17: Select the correct alternative from the given choices.

1. Which of the following statements are true?
1. A balanced three-phase system consists of positive sequence components only.
2. The magnitude of zero-sequence component is one third of current in neutral wire.
3. In a delta-connected load, zero-sequence component is equal to neutral current.
4. The current of a single-phase load drawn from a three-phase system comprises equal positive-, negative- and zero-sequence components.

(A) 1, 2 only (B) 2, 3 only
(C) 2, 3, 4 (D) 1, 2, 3, and 4

2. A generator of negligible resistance having 1.0 per unit voltage behind transient reactance is subjected to a line to line fault resulting in a fault current of 2.68 p.u. The per unit value of zero-sequence reactance when it is subjected to a line to ground fault resulting in 2.68 p.u. fault current is
(A) 0.473 p.u. (B) 0.265 p.u.
(C) 0.98 p.u. (D) 1 p.u.

3. A single line to ground fault occurs in a star-connected three-phase generator. It supplies power to a star-connected inductive load through a transmission line. The star point of the load is grounded and the generator neutral is ungrounded. The single-line diagram is shown below. Fault takes place half way down the line.



Prior to fault, the network is balanced and voltage at fault location is $1\angle 0°$ p.u. The current through the fault path is
(A) $-j2.36$ p.u. (B) $j1.45$ p.u.
(C) $j3.26$ p.u. (D) $j3.4$ p.u.

4. A 100 kVA, 400 V synchronous generator (0.2 p.u. sub-transient reactance) is supplying a 80 kW load at 0.8 lagging power factor. The initial symmetrical RMS current for a three-phase fault at generator terminals is
(A) 0.08 p.u. (B) 0.06 p.u.
(C) 0.04 p.u. (D) 0.02 p.u.

5. Ratio of L-G fault current to three-phase fault
(A) $\dfrac{3X_1}{X_1 + X_2 + X_0}$ (B) $\dfrac{1.732X_1}{X_1 + X_2}$
(C) $\dfrac{X_1}{3\left(X_1 + X_2 + \dfrac{X_0}{3}\right)}$ (D) Both (A) and (C)

6. The generator emf is 0.5 p.u. and the transient reactance is 0.15. The value of transient current is
(A) $2.5\angle -90°$ (B) $5\angle 30°$
(C) $3.33\angle -90°$ (D) $2.5\angle 30°$

7. $E_g = 0.5$ p.u. and $X_d = 0.2$. The value of short-circuit current is
(A) $2.5\angle -90°$ (B) $5\angle -90°$
(C) $2.5\angle 90°$ (D) $5\angle 90°$

8. Which is true for steady state, transient and sub-transient reactances for a synchronous machine?
(A) $X_d > X'_d > X''_d$ (B) $X''_d > X'_d > X_d$
(C) $X'_d > X''_d > X_d$ (D) $X_d > X''_d > X'_d$

9. X_1, X_2 and X_0 values of two generators operated parallely are given as 0.08, 0.24 and 0.10 per unit, respectively. If the generator is solidly grounded, the voltage of healthy phase for a double line to ground fault at terminals of generator is
(A) 0.603 p.u. (B) 0.969 p.u.
(C) 0.152 p.u. (D) 0.846 p.u.

10. The fault which causes a greater electro-magnetic interference between the power line and a nearby communication line
(A) Three-phase fault (B) L–L–G fault
(C) L–L fault (D) L–G fault

11. The zero-sequence impedances of an ideal star–delta-connected transformer (star-grounded)
(A) Looking from star side is zero and looking from delta side is infinite
(B) Looking from star side as well as delta side is infinite
(C) Looking from star side as well as delta side is zero
(D) Looking from star side is infinite and looking from delta side is zero.

12. For a line-to-line fault analysis using symmetrical components
(A) The positive-, negative- and zero-sequence networks at the fault point are connected in series
(B) The positive-, negative- and zero- sequence networks at the fault point are connected in parallel
(C) The positive- and negative-sequence networks at the fault point are connected in series
(D) The positive- and negative-sequence networks at the fault point are connected in parallel

13. The line currents in ampere in phases a, b and c, respectively, are $(300 + j200)$, $(50 - j300)$ and $(-200 + j500)$ referred to the same reference vector. The negative sequence components of currents will be
(A) $138.56\angle 200.14$ (B) $11.82\angle 100.02$
(C) $112.82\angle 200.14$ (D) $98.41\angle 100.02$

14. If a positive sequence current passes through a transformer and its phase shift is 60 degrees, the negative sequence current flowing through the transformer will have a phase shift of

(A) 60 deg (B) –60 deg
(C) 120 deg (D) –120 deg

15. The reason for not considering the load currents in short-circuit calculations are

(i) Short-circuit currents are much larger than load currents

(ii) Short-circuit currents are greatly out of phase with load currents

The correct alternative is
(A) Both (i) and (ii) are correct
(C) Both (i) and (ii) are wrong
(D) (i) is correct and (ii) is wrong
(E) (i) is wrong and (ii) is correct

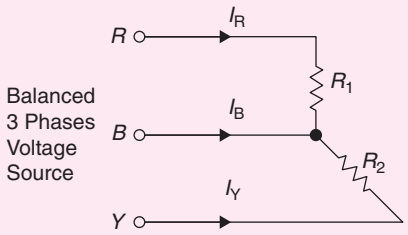
16. For a single line to ground fault, the rise in potential on a healthy phase, if we are considering the voltage as V will be

(A) $\frac{1}{\sqrt{2}}$ V kV
(B) $\sqrt{2}$ V kV
(C) $\sqrt{3}$ V kV
(D) $\frac{1}{\sqrt{3}}$ V kV

17. When a line to ground fault occurs, the current in the phase is 120 A. The zero-sequence current is
(A) 0 A (B) 40 A
(C) 20 A (D) 120 A

Previous Years' Questions

1. For the three-phase circuit shown in Figure, the ratio of the current $I_R : I_y : I_B$ is given by **[2005]**



(A) 1:1: $\sqrt{3}$ (B) 1:1:2
(C) 1:1:0 (D) 1:1: $\sqrt{\frac{3}{2}}$

2. The parameters of a transposed overhead transmission line are given as:
Self-reactance x_s = 0.4 Ω/km and mutual reactance x_m = 0.1 Ω/km
The positive sequence reactance x_1 and zero sequence reactance x_0, respectively, in Ω/km are **[2005]**
(A) 0.3, 0.2 (B) 0.5, 0.2
(C) 0.5, 0.6 (D) 0.3, 0.6

Common Data for Questions 3 and 4:
At a 220 kV substation of a power system, it is given that the three-phase fault level is 4000 MVA and single-line to ground fault level is 5000 MVA. Neglecting the resistance and the shunt susceptances of the system,

3. The positive-sequence driving point reactance at the bus is: **[2005]**
(A) 2.5 Ω (B) 4.033 Ω
(C) 5.5 Ω (D) 12.1 Ω

4. And the zero-sequence driving point reactance at the bus is: **[2005]**
(A) 2.2 Ω (B) 4.84 Ω
(C) 18.18 Ω (D) 22.72 Ω

5. Three identical star-connected resistors of 1.0 p.u. are connected to an unbalanced three-phase supply. The load neutral is isolated. The symmetrical components of the line voltages in p.u. are:
$V_{ab1} = X\angle\theta_1$, $V_{ab2} = Y\angle\theta_2$. If all the p.u. calculations are with the respective base values, the phase-to-neutral sequence voltages are **[2006]**
(A) $V_{an_1} = X\angle(\theta_1 + 30°)$, $V_{an_2} = Y\angle(\theta_2 - 30°)$
(B) $V_{an_1} = X\angle(\theta_1 - 30°)$, $V_{an_2} = Y\angle(\theta_2 + 30°)$
(C) $V_{an_1} = \frac{1}{\sqrt{3}} X\angle(\theta_1 - 30°)$,
$V_{an_2} = \frac{1}{\sqrt{3}} Y\angle(\theta_2 + 30°)$
(D) $V_{an_1} = \frac{1}{\sqrt{3}} X\angle(\theta_1 - 60°)$,
$V_{an_2} = \frac{1}{\sqrt{3}} Y\angle(\theta_2 - 60°)$

Common Data for Questions 6 and 7:
For a power system the admittance and impedance matrices for the fault studies are as follows.

$$Y_{bus} = \begin{bmatrix} -j8.75 & j1.25 & j2.50 \\ j1.25 & -j6.25 & j2.50 \\ j2.50 & j2.50 & -j5.00 \end{bmatrix}$$

$$Z_{bus} = \begin{bmatrix} j0.16 & j0.08 & j0.12 \\ j0.08 & j0.24 & j0.16 \\ j0.12 & j0.16 & j0.34 \end{bmatrix}$$

The pre-fault voltages are 1.0 p.u. at all the buses. The system was unloaded prior to the fault. A solid three-phase fault takes place at bus 2.

6. The post fault voltages at buses 1 and 3 in per unit, respectively, are **[2006]**

(A) 0.24, 0.63 (B) 0.31, 0.76
(C) 0.33, 0.67 (D) 0.67, 0.33

7. The per unit fault feeds from generators connected to buses 1 and 2, respectively, are **[2006]**
(A) 1.20, 2.51 (B) 1.55, 2.61
(C) 1.66, 2.50 (D) 5.00, 2.50

8. A three-phase balanced star-connected voltage source with frequency ω rad/s is connected to a star-connected balanced load which is purely inductive. The instantaneous line currents and phase to neutral voltages are denoted by (i_a, i_b, i_c) and (v_{an}, v_{bn}, v_{cn}), respectively, and their RMS values are denoted by V and I.

If $R = [v_{an}, v_{bn}, v_{cn}] \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$,

then the magnitude of R is **[2007]**
(A) 3VI (B) VI (C) 0.7 VI (D) 0

9. Suppose we define a sequence transformation between '*a-b-c*' and '*p-n-o*' variables as follows:

$\begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix} = k \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} \begin{bmatrix} f_p \\ f_n \\ f_o \end{bmatrix}$, where $\alpha = e^{j\frac{2x}{3}}$ and k is a constant.

Now, if it is given that: $\begin{bmatrix} V_p \\ V_n \\ V_0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} \begin{bmatrix} i_p \\ i_n \\ i_0 \end{bmatrix}$

and $\begin{bmatrix} V_a \\ V_b \\ -V_c \end{bmatrix} = Z \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$ then, **[2007]**

(A) $Z = \begin{bmatrix} 1.0 & 0.5 & 0.75 \\ 0.75 & 1.0 & 0.5 \\ 0.5 & 0.75 & 1.0 \end{bmatrix}$

(B) $Z = \begin{bmatrix} 1.0 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 1.0 \end{bmatrix}$

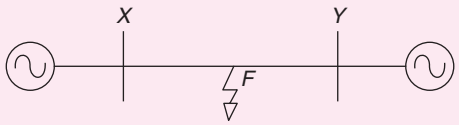
(C) $Z = 3k^2 \begin{bmatrix} 1.0 & 0.75 & 0.5 \\ 0.5 & 1.0 & 0.75 \\ 0.75 & 0.5 & 1.0 \end{bmatrix}$

(D) $Z = \frac{k^2}{3} \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 1.0 & -0.5 \\ -0.5 & -0.5 & 1.0 \end{bmatrix}$

10. A 230 V (phase), 50 Hz, three-phase, 4-wire system has a phase sequence ABC. A unity power-factor load of 4 kW is connected between phase A and neutral N. It is desired to achieve zero neutral current through the use of a pure inductor and a pure capacitor in the other two phases. The value of inductor and capacitor is **[2007]**
(A) 73.95 mH in phase C and 139.02 mF in phase B
(B) 72.95 mH in phase B and 139.02 mF in phase C
(C) 42.12 mH in phase C and 240.79 mF in phase B
(D) 42.12 mH in phase B and 240.79 mF in phase C

11. A two-machine power system is shown below. Transmission line XY has positive-sequence impedance of Z_1 Ω and zero-sequence impedance of Z_0 Ω.

An '*a*' phase to ground fault with zero fault impedance occurs at the centre of the transmission line. Bus voltage at X and line current from X to F for the phase '*a*' are given by V_a V and I_a A, respectively. Then, the impedance measured by the ground distance relay located at the terminal X of line XY will be given by **[2008]**

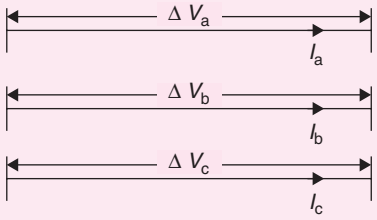


(A) $\frac{Z_1}{2}$ Ω (B) $\frac{Z_0}{2}$ Ω
(C) $\frac{(Z_0 + Z_1)}{2}$ Ω (D) $\frac{V_a}{I_a}$ Ω

12. A three-phase transmission line is shown in the figure voltage drop across the transmission line is given by the following equation

$\begin{bmatrix} \Delta V_a \\ \Delta V_b \\ \Delta V_c \end{bmatrix} = \begin{bmatrix} Z_s & Z_m & Z_m \\ Z_m & Z_s & Z_m \\ Z_m & Z_m & Z_s \end{bmatrix} \begin{bmatrix} I_s \\ I_b \\ I_c \end{bmatrix}$

Shunt capacitance of the line can be neglected. If the line has positive-sequence impedance of 15 Ω and zero-sequence impedance of 48 Ω, then the values of Z_s and Z_m will be **[2008]**

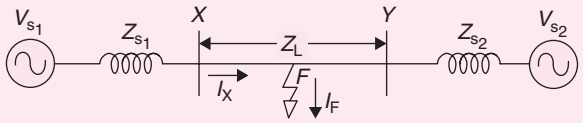


(A) $Z_s = 31.5\ \Omega$; $Z_m = 16.5\ \Omega$
(B) $Z_s = 26\ \Omega$; $Z_m = 11\ \Omega$
(C) $Z_s = 16.5\ \Omega$; $Z_m = 31.5\ \Omega$
(D) $Z_s = 11\ \Omega$; $Z_m = 26\ \Omega$

13. A lossless transmission line having surge impedance loading (SIL) of 2280 MW is provided with a uniformly distributed series capacitive compensation of 30%. Then, SIL of the compensated transmission line will be **[2008]**

(A) 1835 MW (B) 2280 MW
(C) 2725 MW (D) 3257 MW

Common Data for Questions 14 to 16:
Consider a power system shown below:



Given that:

$$V_{s1} = V_{s2} = 1.0 + j0.0 \text{ p.u.};$$

The positive-sequence impedances are $Z_{s1} = Z_{s2} = 0.001 + j0.01$ p.u. and $Z_L = 0.006 + j0.06$ p.u.

Three-phase base MVA = 100

Voltage base = 400 kV (Line to line)

Nominal system frequency = 50 Hz

The reference voltage for phase 'a' is defined as $v(t) = V_m\cos(\omega t)$.

A symmetrical three-phase fault occurs at centre of the line, i.e. point 'F' at time t_0. The positive sequence impedance from source S_1 to point 'F' equals $0.004 + j0.04$ p.u. The waveform corresponding to phase 'a' fault current from bus X reveals that decaying DC offset current is negative and in magnitude at its maximum initial value. Assume that the negative-sequence impedances are equal to positive-sequence impedances, and the zero-sequence impedances are three times positive-sequence impedances.

14. The instant (t_0) of the fault will be **[2008]**

(A) 4.682 ms (B) 9.667 ms
(C) 14.667 ms (D) 19.667 ms

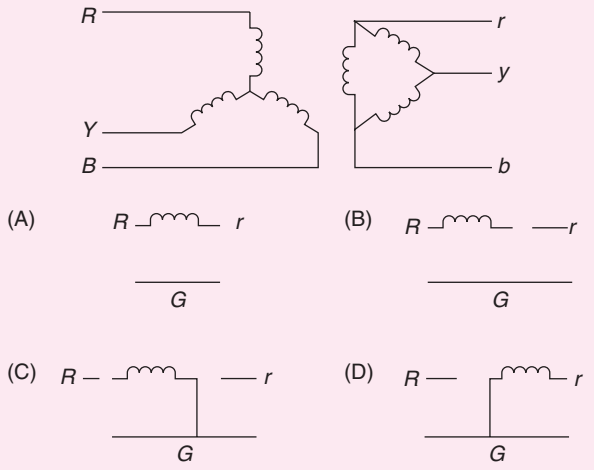
15. The RMS value of the AC component of fault current (I_x) will be **[2008]**

(A) 3.59 KA
(B) 5.07 KA
(C) 7.18 KA
(D) 10.15 KA

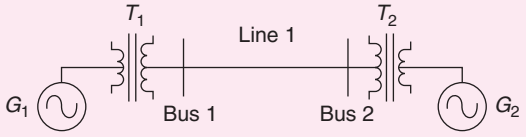
16. Instead of the three-phase fault, if a single line to ground fault occurs on phase 'a' at point 'F' with zero fault impedance, then the RMS value of the AC component of fault current (I_x) for phase 'a' will be **[2008]**

(A) 4.97 p.u.
(B) 7.0 p.u.
(C) 14.93 p.u.
(D) 29.85 p.u.

17. The zero-sequence circuit of the three-phase transformer shown in the figure is **[2010]**

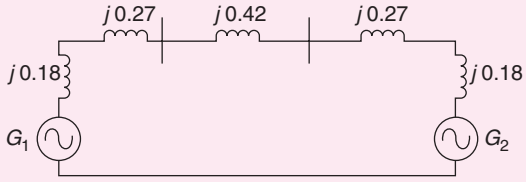


18. For the power system shown in the figure below, the specifications of the components are the following:
G_1: 25 kV, 100 MVA, $X = 9\%$
G_2: 25 kV, 100 MVA, $X = 9\%$
T_1: 25 kV/220 kV, 90 MVA, $X = 12\%$
T_2: 220 kV/ 25 kV, 90 MVA, $X = 12\%$
Line1: 220 kV, $X = 150\ \Omega$

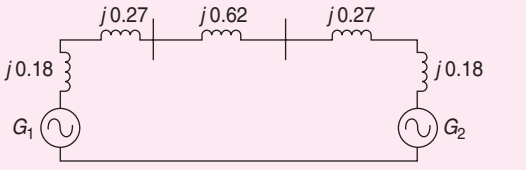


Choose 25 kV as the base voltage at the generator G_1, and 200 MVA as the MVA base. The impedance diagram is **[2010]**

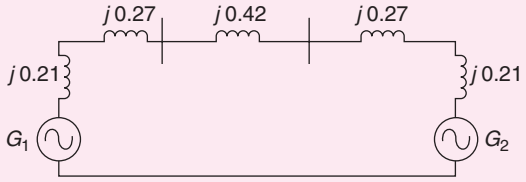
(A)



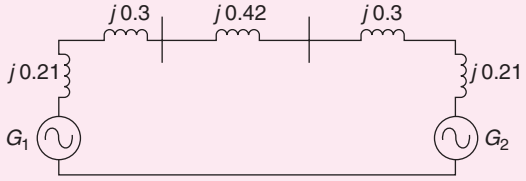
(B)



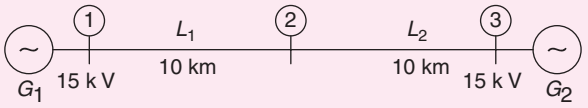
(C)



(D)



Common Data for Questions 19 and 20:
Two generator units G_1 and G_2 are connected by 15 kV line with a bus at the mid-point as shown below

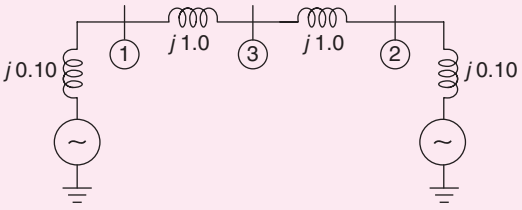


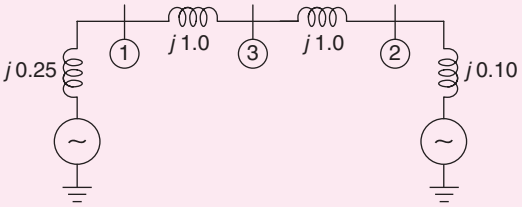
G_1 = 250 MVA, 15 kV, positive-sequence reactance $X = 25\%$ on its own base

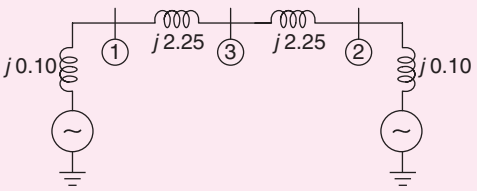
G_2 = 100 MVA, 15 kV, positive-sequence reactance $X = 10\%$ on its own base

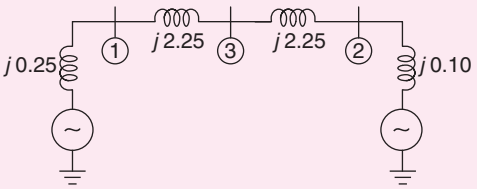
L_1 and L_2 = 10 km, positive-sequence reactance $X = 0.225\ \Omega$/km

19. For the above system the positive sequence diagram with the p.u. values on the 100 MVA, common base is **[2011]**

(A)


(B)


(C)


(D)


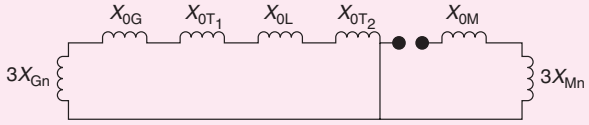
20. In the above system, the three-phase fault MVA at the bus 3 is **[2011]**
(A) 82.55 MVA
(B) 85.11 MVA
(C) 170.91 MVA
(D) 181.82 MVA

21. The sequence components of the fault current are as follows: $I_{positive} = j1.5$ p.u., $I_{negative} = -j0.5$ p.u., $I_{zero} = -j1$ p.u. The type of fault in the system is **[2012]**
(A) LG
(B) LL
(C) LLG
(D) LLLG

22. A 2-bus system corresponding zero-sequence network is shown in the figure.



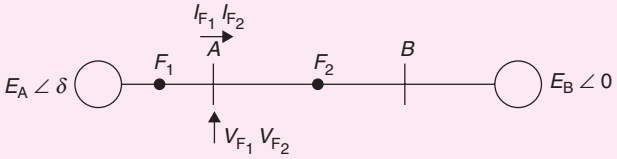
(a)



(b)

The transformers T_1 and T_2 are connected as **[2014]**
(A) Y Y and Y Δ
(B) Y Y and Y Δ
(C) Y Δ and Δ Y
(D) Δ Y and Y Y

23. Three-phase to ground fault takes place at locations F_1 and F_2 in the system shown in the following figure.



If the fault takes place at location F_1, then the voltage and the current at bus A are V_{F_1} and I_{F_1}, respectively. If the fault takes place at location F_2, then the voltage and the current at bus A are V_{F_2} and I_{F_2}, respectively. The correct statement about voltages and currents during faults at F_1 and F_2 is **[2014]**
(A) V_{F_1} leads I_{F_1} and V_{F_2} leads I_{F_2}
(B) V_{F_1} leads I_{F_1} and V_{F_2} lags I_{F_2}
(C) V_{F_1} lags I_{F_1} and V_{F_2} leads I_{F_2}
(D) V_{F_1} lags I_{F_1} and V_{F_2} lags I_{F_2}

24. In an unbalanced three-phase system, phase current $I_a = 1\angle(-90°)$ p.u., negative sequence current $I_{b2} = 4\angle(-150°)$ p.u., zero sequence current $I_{c0} = 3\angle 90°$ p.u. The magnitude of phase current I_b in p.u. is **[2014]**
(A) 1.00
(B) 7.81
(C) 11.53
(D) 13.00

25. A three-phase star-connected load is drawing power at a voltage of 0.9 p.u. and 0.8 power factor lagging. The three-phase base power and base current are

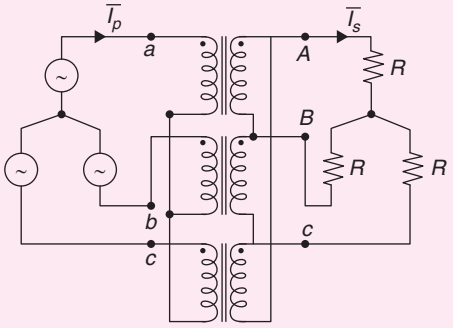
100 MVA and 437.38 A, respectively. The line-to-line load voltage in kV is _____.

[2014]

26. At three-phase, 100 MVA, 25 kV generator has solidly grounded neutral. The positive, negative, and the zero sequence reactances of the generator are 0.2 p.u., 0.2 p.u., and 0.05 p.u., respectively, at the machine base quantities. If a bolted single phase to ground fault occurs at the terminal of the unloaded generator, the fault current in amperes immediately after the fault is _____.

[2014]

27. A balanced (positive sequence) three-phase AC voltage source is connected to a balanced, star connected load through a star-delta transformer as shown in the figure. The line-to-



line voltage rating is 230 V on the star side, and 115 V on the delta side. If the magnetizing current is neglected and $\bar{I}_s = 100\angle 0°$ A, then what is the value of $\bar{I}_p$ in Ampere? **[2015]**

(A) $50\angle 30°$ (B) $50\angle -30°$

(C) $50\sqrt{3}\angle 30°$ (D) $200\angle 30°$

28. The magnitude of three-phase fault currents at buses A and B of a power system are 10 pu and 8 pu, respectively. Neglect all resistances in the system and consider the pre-fault system to be unloaded. The pre-fault voltage at all buses in the system is 1.0 pu. The voltage magnitude at bus B during a three-phase fault at bus A is 0.8 pu. The voltage magnitude at bus A during a three-phase fault at bus B, in pu, is ____.

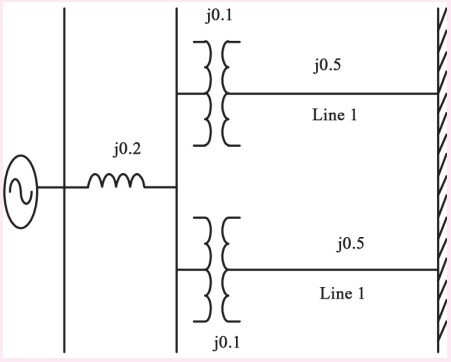
[2016]

29. A 30 MVA, 3-phase, 50 Hz, 13.8 kV, star-connected synchronous generator has positive, negative and zero sequence reactances, 15%, 15% and 5% respectively. A reactance (X_n) is connected between the neutral of the generator and ground. A double line to ground fault takes place involving phases 'b' and 'c', with a fault impedance of j0.1 p.u. The value of X_n (in p.u.) that will limit the positive sequence generator current to 4270 A is **[2016]**

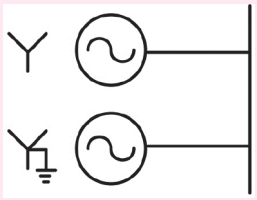
30. A 50MVA, 10KV, 50HZ, star-connected, unloaded three- phase alternator has a synchronous reactance of 1p.u and a sub-transient reactance of 0.2p.u. If a 3-phase short circuit occurs close to the generator terminals, the ratio of initial and final values of the sinusoidal component of the short circuit current is _____. **[2016]**

31. The single line diagram of a balanced power system is shown in the figure. The voltage magnitude at the generator internal bus is constant and 1.0p.u. the p.u reactances of different components in the system are also shown in the figure. The infinite bus voltage magnitude is 1.0p.u. A three phase fault occurs at the middle of line 2.

The ratio of the maximum real power that can be transferred during the pre - fault condition to the maximum real power that can be transferred under the faulted condition is _____



32. Two identical unloaded generators are connected in parallel as shown in the figure. Both the generators are having positive, negative and zero sequence impedances of *j*0.4pu., *j*0.3pu., and *j*0.15pu., respectively. If the pre- fault voltage is 1pu, for a line - to - ground (L – G) fault at the terminals of the generators, the fault current, in pu., is _____. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. D	**2.** B	**3.** B	**4.** C	**5.** B	**6.** C	**7.** D	**8.** A	**9.** A	**10.** A
11. A	**12.** A	**13.** C	**14.** A	**15.** B	**16.** D	**17.** A	**18.** B	**19.** A	**20.** A

Practice Problems 2

1. D	**2.** A	**3.** A	**4.** C	**5.** A	**6.** C	**7.** A	**8.** A	**9.** A	**10.** D
11. A	**12.** D	**13.** C	**14.** B	**15.** A	**16.** C	**17.** B			

Previous Years' Questions

1. A	**2.** D	**3.** D	**4.** B	**5.** C	**6.** D	**7.** C	**8.** A	**9.** B	**10.** B
11. D	**12.** B	**13.** C	**14.** A	**15.** A	**16.** C	**17.** C	**18.** B	**19.** A	**20.** D
21. C	**22.** B	**23.** C	**24.** C	**25.** 118.8	**26.** 15396	**27.** A	**28.** 0.832	**29.** 1.08	**30.** 5
31. 2.3	**32.** 6								

Chapter 3

Load Flow Analysis and Protection

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Bus admittance matrix
- Sparsity in Y_{bus} matrix
- Load flow analysis
- Formation of power flow equations
- Load bus
- Generator bus
- Load flow algorithm
- Optimal operation of generators
- Power system protection
- Protective zones
- Time multiplier setting

Bus Admittance Matrix

Nodal admittance matrix or admittance matrix or Y matrix or Y_{bus} is an $n \times n$ matrix describing a power system with 'n' buses. It represents the nodal admittance of the buses in a power system.

In realistic systems which contain thousands of buses, the Y_b matrix is quite sparse. Each bus in a real power system is usually connected to only a few other buses through the transmission lines. The Y matrix is also one of the data requirements needed to formulate a power flow study

$$Y_{bus} = \begin{bmatrix} y_{11} & y_{12} & \cdots\cdots & y_{1n} \\ y_{21} & y_{22} & \cdots\cdots & y_{2n} \\ . & & & . \\ y_{n1} & y_{n2} & \cdots\cdots & y_{nn} \end{bmatrix}$$

Consider a four-bus power system shown in figure below.

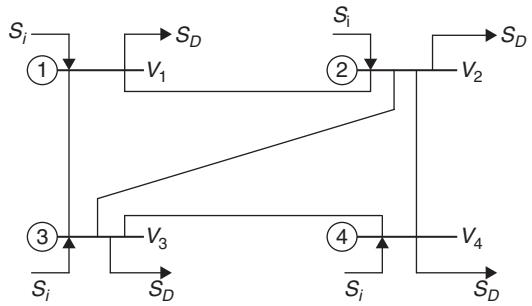


Figure 1 Single-line diagram of a four bus network

Equivalent network of the above single-line diagram is given by

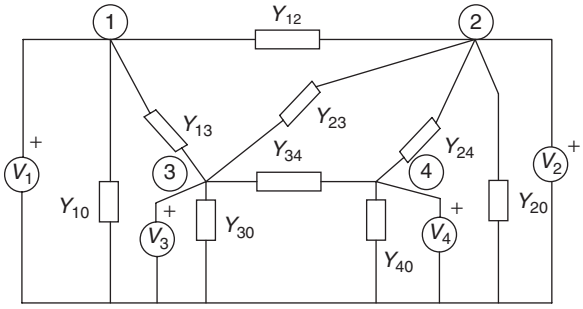


Figure 2 Equivalent circuit of the single-line diagram shown above

Apply Kirchhoff's current law for the above network,

$$I_1 = V_1 y_{10} + (V_1 - V_2) y_{12} + (V_1 - V_3) y_{13}$$

$$I_2 = V_2 y_{20} + (V_2 - V_1) y_{12} + (V_2 - V_3) y_{23} + (V_2 - V_4) y_{24}$$

$$I_3 = V_3 y_{30} + (V_3 - V_1) y_{13} + (V_3 - V_2) y_{23} + (V_3 - V_4) y_{34}$$

$$I_4 = V_4 y_{40} + (V_4 - V_2) y_{24} + (V_4 - V_3) y_{34}$$

Rearrange the equations and rewrite them in matrix form

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} y_{10}+y_{12}+y_{13} & -y_{12} & -Y_{13} & 0 \\ -y_{12} & y_{20}+y_{12}+y_{23}+y_{24} & -y_{23} & -y_{24} \\ -y_{13} & -y_{23} & y_{30}+y_{13}+y_{23}+y_{24} & -y_{34} \\ 0 & -y_{24} & -y_{34} & y_{40}+y_{24}+y_{34} \end{bmatrix} \begin{bmatrix} v \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Above matrix form can be represented as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

where $Y_{11} = y_{10} + y_{12} + y_{13}$

$$Y_{22} = y_{20} + y_{12} + y_{23} + y_{24}$$
$$Y_{33} = y_{30} + y_{13} + y_{23} + y_{34}$$
$$Y_{44} = y_{40} + y_{24} + y_{34}$$
$$Y_{12} = Y_{21} = -y_{12}$$
$$Y_{13} = Y_{31} - y_{13}$$
$$Y_{14} = Y_{41} = -y_{14} = 0$$
$$Y_{23} = Y_{32} = -y_{23}$$
$$Y_{24} = Y_{42} = -y_{24}$$
$$Y_{34} = Y_{43} = -y_{34}$$

Each admittance Y_{ii} is called the self-admittance of node 'i' and equal to the algebraic sum of the admittances terminating on the node.

$$Y_{ii} = y_i + \sum_{\substack{K=1 \\ K \neq i}}^{N} y_{iK}$$

Each off diagonal term Y_{ij} is called the mutual admittance (or transfer admittance) between nodes 'i' and 'j' and is equal to the negative of the sum off all admittances connected directly between those nodes.

$$Y_{ij} = -y_{ij}$$

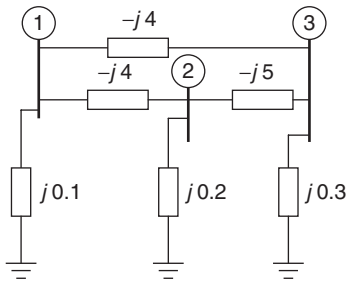
The admittance matrix is symmetric $Y_{ij} = Y_{ij}$.

Relation between bus impedance matrix and admittance matrix is given by

$$[Z_{bus}] = [Y_{bus}]^{1}$$

Solved Examples

Example 1: Three-bus admittance network of a power system is given by



(A) $\begin{bmatrix} -j7.9 & -j4 & -j4 \\ -j4 & -j8.8 & -j5 \\ -j4 & -j5 & -j8.7 \end{bmatrix}$

(B) $\begin{bmatrix} -j7.9 & +j4 & j4 \\ j4 & -j8.8 & j5 \\ j4 & j5 & -j8.7 \end{bmatrix}$

(C) $\begin{bmatrix} -j0.1 & -j4 & -j4 \\ -j4 & -j0.2 & -j5 \\ -j4 & -j5 & -j0.3 \end{bmatrix}$

(D) $\begin{bmatrix} j0.1 & -j4 & -j4 \\ -j4 & j0.2 & -j5 \\ -j4 & -j5 & j0.3 \end{bmatrix}$

Solution: (B)

Admittance matrix order is 3×3 elements of the admittance matrix are

$$Y_{11} = y_{10} + y_{12} + y_{13} = j0.1 - j4 - j4$$
$$= -7.9.$$
$$Y_{22} = y_{20} + y_{21} + y_{23} = j0.2 - j4 - j5 = -j8.8$$
$$Y_{33} = y_{30} + y_{31} + y_{32} = j0.3 - j5 - j4$$
$$= -j8.7$$
$$Y_{12} = y_{21} = j4$$
$$Y_{13} = y_{31} = j4$$
$$Y_{23} = y_{32} = j5$$
$$Y_{bus} = \begin{bmatrix} -j7.9 & j4 & j4 \\ j4 & -j8.8 & j5 \\ j4 & j5 & -j8.7 \end{bmatrix}$$

Sparsity in Y_{bus} Matrix

An element in Y_{bus} has a non-zero value, if the two buses are directly connected through a transmission line or a transformer. The non–existence of a direct connection between

two buses means that the off-diagonal transfer admittances of the Y_{bus} matrix will be zero. Normally in a large interconnected power systems, each bus may be connected with a few neighbouring buses. The non-existence of a direct connection between many buses means that the vast majority of a real world Y_{bus} matrix will be zero. Thus, for a large power systems y_{bus} is highly sparse. In large systems, sparsity may be as high as 97%.

Example 2: For the Y-bus matrix of a 4-bus system given in per unit, the buses having shunt elements are

$$Y_{bus} = j\begin{bmatrix} -7 & 2 & 3 & 0 \\ 2 & -12 & 4 & 2.5 \\ 3 & 4 & -9 & 2 \\ 0 & 2.5 & 2 & -4.5 \end{bmatrix}$$

(A) 1 and 2 (B) 2 and 3
(C) 3 and 4 (D) 1, 2, and 3

Solution: (A)
In an admittance matrix, sum of elements in a row is non-zero if the corresponding node is having shunt branch.

$$\text{For bus 1} = \sum_{j=1}^{4} y_{1j} = y_{11} + y_{12} + y_{13} + y_{14}$$
$$= -2 \neq 0$$

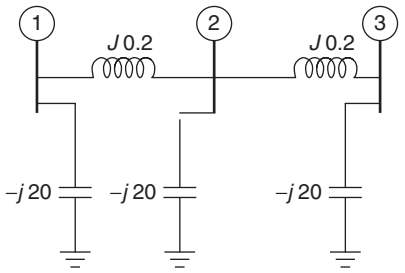
$$\text{For bus 2} = \sum_{j=1}^{4} y_{2j} = y_{21} + y_{22} + y_{23} + y_{24}$$
$$= -3.5 \neq 0$$

$$\text{For bus 3} = \sum_{j=1}^{4} y_{3j} = 0$$

$$\text{For bus 4} = \sum_{j=1}^{4} y_{4j} = 0$$

Bus 1 and 2 has shunt elements.

Example 3: Impedance diagram of a network is given in the figure in per unit. The diagonal elements Y_{22} of the bus admittance matrix Y_{bus} of the network is



(A) $-j9.95$ (B) $-j19.6$ (C) $j0.4$ (D) $j20$

Solution: (A)
Diagonal element $Y_{22} = y_{20} + y_{21} + y_{23}$

$$= \frac{-1}{j20} + \frac{1}{j0.2} + \frac{1}{j0.2} = -9.95j$$

Example 4: Sparsity of a 100 bus network is given by 95%. Number of elements in the network is given by
(A) 500 (B) 4700 (C) 9500 (D) 200

Solution: (D)
Number of elements in a network

$$= \frac{N-n}{2}$$

where N = Number of non-zero elements in an admittance matrix
= 5% of total elements are non-zeros.

$$= \frac{5}{100} \times 10,000 = 500$$

n = order of the Y_{bus}

$$\text{Number of element} = \frac{500-100}{2} = 200 \text{ elements}$$

Bus Impedance Matrix

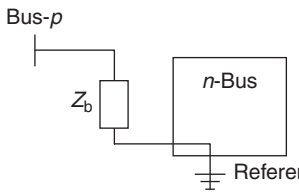
The process of building the bus impedance matrix proceeds by adding impedance one at a time till all the impedances have been included. With each addition of impedance, a new matrix is produced. The order of the matrix may or may not increase depending upon whether the addition of the impedance creates a new bus or not.

Assume that for a partial power system network with 'n' buses, a reference node 'O' and the bus impedance matrix is $Z_{original}$ is known. When an element with impedance Z_b is added to the partial network, four types of modifications are possible.

1. Addition of a tree branch, with impedance Z_b from a new bus to the reference bus.
2. Addition of a tree branch with impedance Z_b from a new bus to an existing bus.
3. Adding a co-tree link with impedance Z_b between two existing buses.
4. Adding a co-tree link with impedance Z_b between an existing bus to the reference bus.

Modification of an Existing Bus Impedance Matrix

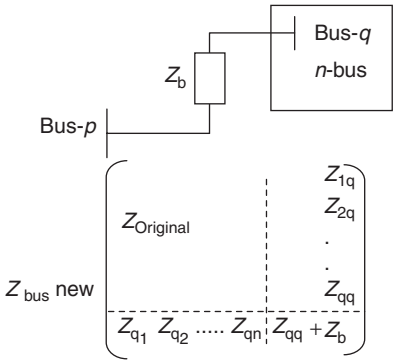
Case I: Adding Z_b from a new bus-p to reference bus



$$Z_{bus\ new} = \begin{bmatrix} Z_{Original} & \begin{matrix} 0 \\ 0 \\ . \\ . \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & Z_b \end{bmatrix}$$

In this case, the elements of $(n+1)^{th}$ column and rows are all zeros except the diagonal. The diagonal element is added to branch impedance Z_b. The elements of the original Z_{bus} matrix are not altered.

Case II: Adding Z_b from a new bus-p to existing bus-q



The elements of the $(n+1)^{th}$ column are the elements of the q^{th} column and the elements of the $(n+1)^{th}$ row are the elements of the q^{th} row. The diagonal element is given by the sum of Z_{qq} and Z_b.

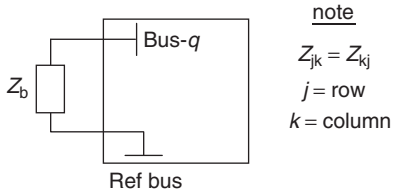
The elements of original Z_{bus} matrix are not altered.

Case III: Adding Z_b from an existing bus-q to reference bus.

It will be an addition as that of case II. The new impedance matrix of order $(n+1)$ can be formed as that of case II.

So the bus impedance matrix has to be modified by eliminating $(n+1)^{th}$ row and $(n+1)^{th}$ column. This reduced bus impedance matrix is the actual new bus impedance matrix. Every element of actual new bus impedance matrix can be determined by using the equation

$$Z_{jk\ actual} = Z_{jk} - \frac{Z_{j(n+1)} Z_{(n+1)k}}{Z_{(n+1)(n+1)}}$$



Case IV: Adding Z_b between two existing buses 'h' and 'q'.

Hence the elements of the $(n+1)^{th}$ row is the difference between the elements of row-h and row-q.

Here the elements of $(n+1)^{th}$ column is the difference between the elements of column-h and column-q. The elements of $(n+1)^{th}$ row is the difference between the elements of row-h and row-q.

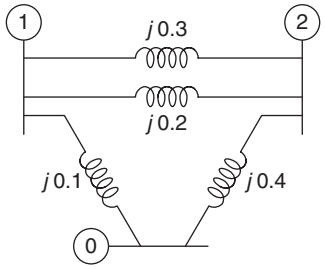
$$Z\ \text{bus new} = \left(\begin{array}{c|c} & Z_{1h} - Z_{1q} \\ & Z_{2h} - Z_{2q} \\ Z_{original} & \\ & \\ & Z_{nh} - Z_{nq} \\ \hline Z_{h1} - Z_{q1} - Z_{hn} - \ldots & Z_{qn} - Z_{(n+1)(n-1)} \end{array}\right)$$

where $Z_{(n+1)(n+1)} = Z_b + Z_{hh} + Z_{qq} - 2Z_{hq}$

Since the modification does not involve addition of new bus, the order of new bus impedance matrix has to be reduced to $n \times n$ by eliminating the $(n+1)^{th}$ column and $(n+1)^{th}$ row. This reduced bus impedance matrix is the actual bus impedance matrix.

$$Z_{jk\ actual} = Z_{jk} - \frac{Z_{j(n+1)} Z_{(n+1)k}}{Z_{(n+1)(n+1)}}$$

Example 5: From the bus impedance matrix for the network shown in the figure.



Solution: Consider a reference bus as 'O', In this case buses 1 and 2 are connected to the reference bus. Connect bus 1 to the reference bus.

$$Z^1_{bus} = j0.1$$

Add bus 2 to the bus 1 (with $j0.2$ as impedance)

$$Z^2_{bus} = \begin{array}{c} \\ ① \\ ② \end{array} \begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ \begin{bmatrix} j0.1 & j0.1 \\ j0.1 & j0.3 \end{bmatrix} \end{array}$$

Add a link between bus (2) and reference bus

$$Z^{new}_{bus} = Z^2_{bus} - \frac{1}{Z_{22} + Z_{new}} \begin{bmatrix} j0.2 \\ j0.3 \end{bmatrix} \begin{bmatrix} j0.2 & j0.3 \end{bmatrix}$$

$$= \begin{bmatrix} j0.1 & j0.1 \\ j0.1 & j0.3 \end{bmatrix} + \frac{1}{j0.3 + j0.4} \begin{bmatrix} 0.04 & 0.06 \\ 0.06 & 0.09 \end{bmatrix}$$

$$= j \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.3 \end{bmatrix} - j \begin{bmatrix} 0.057 & 0.085 \\ 0.085 & 0.128 \end{bmatrix}$$

$$Z^{new}_{bus} = j \begin{bmatrix} 0.043 & 0.015 \\ 0.015 & 0.172 \end{bmatrix}$$

Add a link between bus (1) and (2)

$$Z^{new}_{bus} = Z^{old}_{bus} - \frac{1}{Z_{11} + Z_{22} - 2Z_{12} + z_{new}} \begin{bmatrix} j0.0285 \\ -j0.157 \end{bmatrix} \begin{bmatrix} j0.08 - j0.157 \end{bmatrix}$$

$$= Z^{old}_{bus} - j \frac{1}{0.043 + 0.172 - 0.03 + 0.3} \begin{bmatrix} 7.8 \times 10^{-4} & -4.4 \times 10^{-3} \\ -4.4 \times 10^{-3} & 0.0246 \end{bmatrix}$$

$$= Z^{\text{old}}{}_{\text{bus}} - j\begin{bmatrix} 0.0016 & -0.009 \\ -0.009 & 0.0506 \end{bmatrix}$$

$$= j\begin{bmatrix} 0.043 & 0.015 \\ 0.015 & 0.172 \end{bmatrix} - j\begin{bmatrix} 0.0016 & -0.009 \\ -0.009 & 0.0506 \end{bmatrix}$$

$$Z^{\text{final}}{}_{\text{bus}} = j\begin{bmatrix} 0.0414 & 0.024 \\ 0.024 & 0.1214 \end{bmatrix}$$

Load Flow Analysis

A three-phase electric power system under normal steady state conditions of operation is expected to deliver the demanded real and reactive power to the consumer terminals at the rated voltage and frequency, within the specified limits of tolerance. To obtain a complete description of the behaviour of the power system, it is essential to know the voltages at various node points or buses and the power flowing through the elements of power system.

Power flow studies provide the required information regarding bus voltage and power flowing through transmission lines, transformers, and other elements of power systems for a specified load demand subjected to the regulating capabilities of generators, condensers, tap changing transformers, phase shifting transformers as well as specified net interchange of power with adjoining power systems.

In the power flow analysis, the load powers are assumed as known constants. A given set of loads on the buses can be served from a given set of generators in an infinite number of power flow configurations. Power flow analysis concerns itself not only with the actual physical mechanism which controls the power flow in the network meshes but also with how a best or optimum flow configuration from among the available possibilities is selected.

1. The sum of real power injected at the generating bus must equal the sum of total system load demand plus system losses at each instant of time.
2. The power transfer capability of a transmission line is limited by the thermal loading limit and the stability limit.
3. Keep the voltage levels of certain buses with in close tolerance.
4. The power system must fulfil contractual scheduled inter change of power for neighbouring systems, via its tie lines.

Formation of Power Flow Equations

The general form of network equations for n-bus systems in the bus frame of reference in admittance form is

$$I_{\text{bus}} = Y_{\text{bus}}\, V_{\text{bus}}$$

where I_{bus} is the bus current vector
V_{bus} is the bus voltage vector
Y_{bus} is the bus admittance matrix

In general injected current at any bus '*I*' takes the form

$$I_i = Y_{i1}V_1 + Y_{i2}V_2 + Y_{i3}V_3 + \ldots + Y_{in}V_n \text{ for } i = 1, 2, 3, \ldots, n.$$

$$I_i = \sum_{m=1}^{n} Y_{\text{IN}}\, V_m$$

In power systems, complex power injected at bus 'i'

$$S_i = P_i + jQ_i = V_i I_i^* \text{ for } i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad S_i^* = P_i - JQ_i = V_i^* I_i$$

$$S_i^* = (Y_{i1}V_i + Y_{i2}V_2 + Y_{i3}V_3 + \ldots + Y_{in}V_n)V_i^*$$

$$\therefore \quad S_i^* = P_i - jQ_i = \sum_{m=1}^{n} Y_{im}V_m V_i^* \text{ for } i = 1, 2, 3, \ldots, n$$

Above equation gives the general form for static power flow equations.

Let the bus voltages and admittance be written in rectangular and polar form

$$V_i = e_i + jf_i = |V_i|\angle\delta_i$$

$$|V_i| = \sqrt{e_i^2 + f_i^2}\ ;\ \delta_i = \tan^{-1}\frac{f_i}{e_i}$$

$$Y_{im} = G_{im} + jB_{im} = |Y_{im}|\angle\theta_{im}$$

$$|Y_{im}| = \sqrt{G_{im}{}^2 + B_{im}{}^2}\ ;\ \theta_{im} = \tan^{-1}\frac{B_{im}}{G_{im}}$$

Substitute V_i and Y_{im} in S_i^* equation

$$P_i - jQ_i = \sum_{m=1}^{n}(G_{im} + jB_{im})(e_m + jf_m)(e_i - jf_i)$$

Equating the real and imaginary parts, we get

$$P_i = e_i\sum_{m=1}^{n}(G_{im}e_m - B_{im}f_m) + f_i\sum_{m=1}^{n}(G_{im}f_m + B_{im}e_m)$$

$$Q_i = e_i\sum_{m=1}^{n}(G_{im}f_m - B_{im}e_m) + f_i\sum_{m=1}^{n}(G_{im}e_m + B_{im}f_m)$$

for $I = 1, 2, 3, \ldots, n$.

If the polar form of bus voltage V_i and admittance Y_{im} are substituted in S_i^* expression.

$$P_i - jQ_i = |V_i|e^{-j\delta}{}_i\sum_{m=1}^{n}|Y_{im}|e^{j\theta}{}_{im}|V_m|e^{jS}{}_m$$

for $i = 1, 2, 3, \ldots n$

$$P_i = |V_i|\sum_{m=1}^{n}|Y_{im}|\,|V_m|\cos(\theta_{im} - \delta_i + \delta_m)$$

$$Q_i = |V_i|\sum_{m=1}^{n}|Y_{im}|\,|Y_m|\sin(\theta_{im} - \delta_i + \delta_m)$$

for $i = 1, 2, ,3 \ldots n$.

A typical bus of a power network is show in the figure. Each bus '*i*' is associated with four variables.

- Voltage magnitude $|V_i|$
- Phase angle δ_i
- Net injected real power P_i
- Net injected reactive power Q_i

For 'n' bus system there are total number of $4n$ variables. Above mentioned '$2n$' equations in polar or rectangular form are solved for $2n$ variables provided the other '$2n$' variables are specified as input data.

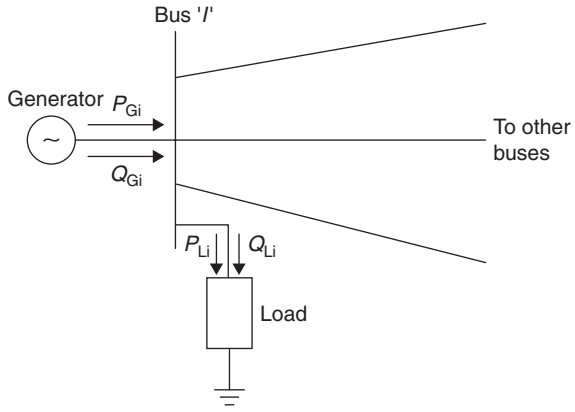


Figure 3 A typical bus of power systems

The study of various methods of solution to power system network is referred as load flow study. The solution provides the voltages at various buses, power flowing in various lines and line losses.

A load flow study of a power system generally requires the following steps

1. Representation of the system by single-line diagram.
2. Obtaining the impedance diagram using the information in single-line diagram.
3. Formulation of network equations
4. Solution of network equations

Types of Buses

The buses of power system can be classified into following three types.

1. Load bus (or PQ-bus)
2. Generator bus (or voltage controlled bus or PV bus)
3. Slack bus (or swing bus or reference bus)

Bus type	Quantities specified	Quantities to be obtained
Load bus	P, Q	$\|V\|$, d
Generator bus	P, $\|V\|$	Q, d
Slack bus	$\|V\|$, d	P, Q

Load Bus

The bus is called load bus, when real and reactive components of power are specified for the bus.

The load flow equations can be solved to find the magnitude and phase of bus voltage. In case of load bus, the voltage is allowed to vary within permissible limits.

Generator Bus

The bus is called generator bus, when real power and magnitude of bus voltage are specified for the bus. The load flow equation can be solved to find the reactive power and phase of bus voltage.

Slack Bus

The bus is called slack bus, if the magnitude and phase of bus voltage are specified for the bus.

The slack bus is the reference bus for load flow equation and usually one of the generator bus is selected as the slack bus.

Gauss–Seidel (G–S) Iterative Technique

In this Gauss–Seidel load flow, the load buses and generator busses are treated differently. In both these type of buses we use the complex power equation for updating voltages

$$V_i = \frac{1}{Y_{ii}}\left[\frac{P_i - jQ_i}{V_i^*} - \sum_{m=1}^{n} Y_{im}\, V_m\right]$$

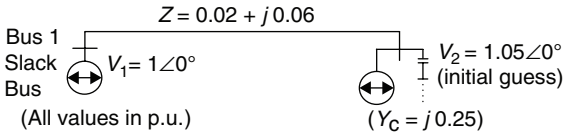
For PV buses, even though the real power is specified, its reactive power is unknown. But to update the voltage of PV buses, we must first estimate the reactive power of this bus. Reactive power expression is given by

$$Q_{i,\,inj} = -\text{Img}\left[V_i^* \sum_{m=1}^{n} Y_{im}\, V_m\right]$$

Q_i value for K^{th} iteration is given by

$$Q_i^K = -\text{Img}\left[V_i^{*(K-1)} \begin{Bmatrix} Y_{i1} V_i + Y_{i2} V_2^K + Y_{i3} V_3^K + \ldots + \\ Y_{ii} V_i^{(K-1)} + \ldots + Y_{IN} V_n^{(K-1)} \end{Bmatrix}\right]$$

Example for Gauss–Seidel Method



using Gauss–Seidel method, what should be next iteration value of V_2?

$$V_2^{(n+1)} = \frac{1}{Y_{22}}\left(\frac{S_2^{n4}}{V_2^{n4}} - \sum_{\substack{K=1 \\ K\#2}}^{n} Y_{2K} V_K^n\right)$$

$$= \frac{1}{Y_{22}}\left(\frac{S_2^{n4}}{V_2^{n4}} - Y_{21} V_1^n\right)$$

$$V_2^{(1)} = \frac{1}{Y_{22}}\left(\frac{S_2^{04}}{V_2^{o4}} - Y_{21} V_1^{(0)}\right) = 1.045\angle -0.83°$$

Advantages of Gauss–Seidel (G.S) Method

1. Calculations are simple, so the programming task is easier.
2. Memory requirement is less.
3. Useful for small size systems.

Disadvantages of G.S Method

1. Requires large number of iterations to reach convergence.
2. Not suitable for large systems.
3. Converge time increases with size of the system.

Newton–Raphson Method

Let us assume that an n-bus power system contains a total of n_p number of P–Q buses while the number of P–V (Generator) buses be n_g such that $n = n_p + n_g + 1$. Bus 1 is assumed to be the stack bus. We shall further use the mismatch equation ΔP_i and ΔQ_i as specified below.

$$\Delta P_i = P_{i\,\text{specified}} - P_{i\,\text{calculated}}$$

$$\Delta Q_i = Q_{i\,\text{specified}} - Q_{i\,\text{calculated}}$$

The approach to Newton–Raphson load flow is similar to that of solving a system of non-linear equations using 'Newton–Raphson' method. At each iteration, we have to form a Jacobian matrix and solve for the correction factor.

$$J\begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ . \\ . \\ \Delta\delta_n \\ \Delta|V_2|/V_2 \\ . \\ . \\ \Delta|V_i + n_p|/|V|_{1+np} \end{bmatrix} = \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ . \\ . \\ \Delta P_n \\ \Delta Q_2 \\ . \\ . \\ \Delta Q_1 + n_p \end{bmatrix}$$

where 'J' is the Jacobian matrix and it is divided into sub-matrices as

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Size of the Jacobian matrix is $(n + n_{p-1}) \times (n + n_{p-1})$

The dimensions of the submatrices are as follows.

$$J_{11} \to (n-1) \times (n-1)$$

$$J_{12} \to (n-1) \times n_p$$

$$J_{21} \to n_p \times (n-1)$$

$$J_{22} \to n_p \times n_p$$

Sub matrices are given by

$$J_{11} = \begin{bmatrix} \partial P_2/\partial\delta_2 & . & . & \partial P_2/\partial\delta_n \\ . & . & . & . \\ . & . & . & . \\ \partial P_n/\partial\delta_2 & . & . & \partial P_n/\partial\delta_n \end{bmatrix} = \begin{bmatrix} L_{22} & . & . & L_{2n} \\ . & . & . & . \\ . & . & .. & . \\ L_{n2} & . & . & L_{nn} \end{bmatrix}$$

where

$$L_{ik} = \frac{\partial P_i}{\partial \delta_k} = -|Y_{ik}\, V_i\, V_k| \sin(\theta_{ik} + \delta_k - \delta_i) \text{ for } i \neq K$$

$$L_{ii} = \frac{\partial P_i}{\partial \delta_i} = \sum_{\substack{k \\ k \neq 1}}^{n} |Y_{ik}\, V_i\, V_k| \sin(\theta_{ik} + \delta_k - \delta_i)$$

$$J_{12} = \begin{bmatrix} |V_2|\dfrac{\partial P_2}{\partial |v_2|} & |V_{1+np}|\dfrac{\partial P_2}{\partial |V_{1+np}|} \\ |V_2|\dfrac{\partial P_n}{\partial |V_2|} & |V_{1+np}|\dfrac{\partial P_n}{\partial |V_{1+np}|} \end{bmatrix} = \begin{bmatrix} N_{22} & N_{2np+1} \\ N_{n2} & N_{nnp+1} \end{bmatrix}$$

where

$$N_{ik} = |V_K| \cdot \frac{\partial P_i}{\partial |V_k|} = |Y_{ik}\, V_i\, V_{ik}| \cos(\theta_{ik} + S_k - S_i)$$

for $i \neq K$

$$N_{ii} = |V_i| \frac{\partial P_i}{\partial |V_i|} = 2|V_i|^2 G_{ii} + \sum_{\substack{k=1 \\ k \neq 1}}^{n} |Y_{ik}\, V_i\, V_k| \cos(\theta_{ik} + \delta_k - \delta_i)$$

$$J_{21} = \begin{bmatrix} \partial Q_2/\partial\delta_2 & . & . & \partial Q_2/\partial\delta_n \\ . & . & . & . \\ . & . & . & . \\ \partial Q_n/\partial\delta_2 & . & . & \partial Q_n/\partial\delta_n \end{bmatrix} = \begin{bmatrix} M_{22} & . & . & M_{2n} \\ . & . & . & . \\ . & . & .. & . \\ M_{n2} & . & . & M_{nn} \end{bmatrix}$$

where

$$M_{ik} = \frac{\partial Q_i}{\partial \delta_k} = -|Y_{ik}\, V_i\, V_k| \cos(\theta_{ik} + \delta_k - \delta_i).$$

$$M_{ii} = \frac{\partial Q_i}{\partial \delta_i} = \sum_{\substack{k=1 \\ k \neq 1}}^{n} |Y_{ik}\, V_i\, V_k| \cos(\theta_{ik} + \delta_k - \delta_i) = P_i - |V_i|^2 Gii$$

$$J_{22} = \begin{bmatrix} |V_2|\dfrac{\partial Q_2}{\partial |V_2|} & |V_{1+np}|\dfrac{\partial Q_2}{\partial |V_{1+np}|} \\ |V_2|\dfrac{\partial Q_{1+np}}{\partial |V_2|} & |V_{1+np}|\dfrac{\partial Q_{1+np}}{\partial |V_{1+np}|} \end{bmatrix} = \begin{bmatrix} O_{22} & Q_{2np} \\ O_{np2} & O_{np\,np} \end{bmatrix}$$

where

$$O_{ik} = |V_K| \cdot \frac{\partial Q_i}{\partial |V_k|} = -|V_i||Y_{ik}\, V_i\, V_{ik}| \sin(\theta_{ik} + \delta_k - \delta_i)$$

for $i \neq K$.

$$Q_{ii} = |V_i| \frac{\partial Q_i}{\partial |V_i|} = 2|V_i|\, 2\, B_{ii} - L_{ii}$$

Load Flow Algorithm

The Newton–Raphson procedure is as follows:

Step 1: Choose initial values for the voltage magnitude $|V|^\circ$ of all n_p load buses and $(n-1)$ angles δ° of the voltages of all the buses except slack bus.

Step 2: Use the estimated voltages and phase angles to calculate $(n-1)$ number of injected real power P°_{calc} and corresponding real power mismatch ΔP° for all buses except slack bus.

Step 3: Use the estimated voltages and phase angles to calculate n_p number of injected reactive power Q°_{calc} and corresponding reactive power mismatch ΔQ° for the PQ buses.

Step 4: Use the estimated $|V|^\circ$ and δ° to formulate Jacobian matrix J°

Step 5: Solve the load flow equation for δ° and $\Delta|V|^\circ/|V|^\circ$

Step 6: Obtain the updates from.

$$\delta' = \delta^\circ + \Delta\delta^\circ$$

$$|V|^1 = |V|^\circ\left[1+\frac{|\Delta V|^\circ}{|V|^\circ}\right]$$

Step 7: Check if all the mismatches are below a small number (tolerance). Terminate process if yes. Otherwise go back to step 1 to start the next iteration with the updates given in the step '6'.

Advantages of Newton–Raphson (N.R) Method

1. The N.R method is faster, more reliable and results are more accurate.
2. Requires less number of iterations for convergence.
3. The number of iterations is independent of the size of system.
4. Suitable for large-size systems.

Disadvantages of Newton–Raphson (N.R) Method

1. The programming logic is more complex than G.S method.
2. The memory requirement is more.
3. Number of calculations per iteration is higher than G.S method.

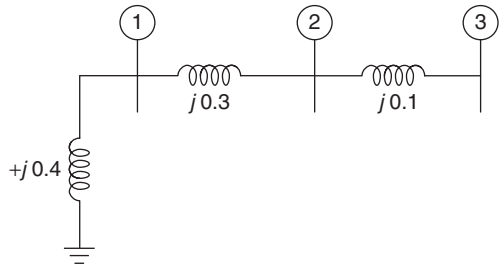
Comparison of Gauss–Seidel and Newton–Raphson Method of Load Flow Solutions

S. No.	Gauss–Seidel method	Newton–Raphson method
1.	Variables are expressed in rectangular co-ordinates.	Variables are expressed in polar co-ordinates.
2.	Computation time per iteration is less.	Computation time per iteration is more.
3.	It has linear convergence characteristics.	It has quadratic convergence characteristics.
4.	The number of iterations required for convergence increases with size of system.	The number of iterations is independent of the size of the system.
5.	The choice of slack bus is critical.	The choice of slack bus is arbitrary.

Example 7: In load flow analysis of power system,
(A) One of the bus is taken as slack bus
(B) Newton–Raphson method is preferred over Gauss–Seidel method
(C) Bus admittance matrix is a sparse matrix
(D) All the above

Solution: (D)

Example 8: Bus impedance matrix of the network given below is



(A) $j\begin{bmatrix}0.4 & 0.3 & 0\\ 0.3 & 0.4 & 0.1\\ 0 & 0.1 & 0.1\end{bmatrix}$ (B) $j\begin{bmatrix}0.4 & 0.4 & 0.4\\ 0.4 & 0.7 & 0.7\\ 0.4 & 0.7 & 0.8\end{bmatrix}$

(C) $j\begin{bmatrix}0.4 & 0 & 0\\ 0 & 0.7 & 0\\ 0 & 0 & 0.8\end{bmatrix}$ (D) $j\begin{bmatrix}0.4 & 0.4 & 0.4\\ 0.4 & 0.7 & 0.4\\ 0.4 & 0.4 & 0.8\end{bmatrix}$

Solution: (B)

Formation of bus 1

$$Z^1_{bus} = \begin{matrix}\\ ①\end{matrix}\begin{matrix}\overset{①}{\\}\end{matrix}\begin{bmatrix}j0.4\end{bmatrix}$$

Formation of bus 2

$$Z^2_{bus} = \begin{matrix}① \\ ②\end{matrix}\begin{matrix}\overset{①}{\\}\end{matrix}\begin{matrix}\overset{②}{\\}\end{matrix}\begin{bmatrix}J0.4 & j0.4\\ j0.4 & j0.7\end{bmatrix}$$

Formation of bus 3

$$Z^3_{bus} = \begin{matrix}① \\ ② \\ ③\end{matrix}\begin{matrix}\overset{①}{\\}\end{matrix}\begin{matrix}\overset{②}{\\}\end{matrix}\begin{matrix}\overset{③}{\\}\end{matrix}\begin{bmatrix}j0.4 & j0.4 & j0.4\\ j0.4 & j0.7 & j0.7\\ j0.4 & j0.7 & j0.8\end{bmatrix}$$

Example 9: A power system consist of 40 bus with 9 voltage controlled buses, the size of the Jacobian matrix is
(A) 70×70 (B) 80×80
(C) 62×62 (D) 79×79

Solution: (A)
Total number of buses (n) = 40
Voltage-controlled buses or generator buses = 9
Order of the Jacobian matrix

$$= (2 \times \text{no. of load buses}) + (\text{Number of generator buses} - 1)$$

$$= 2(40 - 9) + (9 - 1)$$

$$= 70 \times 70$$

Example 10: Voltage of a bus can be controlled by controlling
(A) Active power of the bus
(B) Reactive power of the bus
(C) Both A and B
(D) Phase angle
Solution: (B)

Example 11: The bus impedance matrix of a 4-bus power system is given by

$$Z_{bus} = \begin{bmatrix} j0.3435 & j0.2860 & j0.2723 & j0.2277 \\ j0.2860 & j0.3408 & j0.2586 & j0.2414 \\ j0.2723 & j0.2586 & j0.2791 & j0.2209 \\ j0.2277 & j0.2414 & j0.2209 & j0.2791 \end{bmatrix}$$

A branch having an impedance of $j0.2\ \Omega$ is connected between bus 2 and the reference. Then the values of Z_{22new} and Z_{23new} of the bus impedance matrix of the modified network are, respectively?

Solution: $Z_{Bus}(\text{new}) = Z_{Bus}(\text{old})$

$$= \frac{1}{Z_{jj} + Z_b} \begin{bmatrix} Z_{1j} \\ . \\ . \\ Z_{nj} \end{bmatrix} \begin{bmatrix} Z_{j1 \ldots\ldots Zjn} \end{bmatrix}$$

New elements (Z_b) is connected between j^{th} and reference bus
Here $j = 2$, $n = 4$

$$= \frac{1}{j0.3408 + j0.2} \begin{bmatrix} j0.2860 \\ j0.3408 \\ j0.2586 \\ j0.2414 \end{bmatrix}$$

$$\times [j0.2860\ j0.3408\ j0.2586\ j0.2414]$$

We require only changes in Z_{22} and Z_{23}.

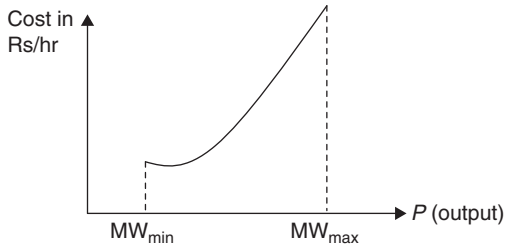
$$= \begin{bmatrix} - & - & - & - \\ - & j0.2147 & j0.16 & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$$

$$Z_{22}\ \text{new} = Z_{22}\ (\text{old}) - j0.2147 = j0.1260$$

$$Z_{23}\ \text{new} = Z_{23}\ (\text{old}) - j0.16296 = j0.956$$

Optimal Operation of Generators

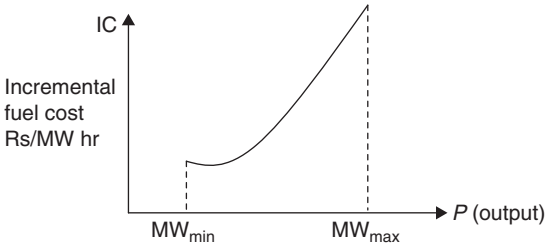
Generator operating cost: The major component of generator operating cost is the fuel input/hour, while the maintenance contributes only to a small extent. The fuel cost is most significant in case of thermal and nuclear stations. Typical cost-power output characteristics of a steam power plant are given below.



In the above characteristics, MW_{min} is the minimum loading limit below which it is uneconomical to operate the unit and MW_{max} is the maximum output limit. An analytical expression for operating cost can be written as

$$C_i = \frac{1}{2} a_i P_{Gi}^2 + b_i P_{Gi} + C_i\ \text{Rs/hour}$$

The slop of the curve, i.e., $\frac{dC_i}{dP_{Gi}}$ is called the 'incremental fuel cost' (IC) and is expressed in units of rupees per megawatt hour (Rs/MW hr). A typical plot of incremental fuel cost versus power output is given below.



When there are 'n' no. of generators present in the power system, total cost of generation is expressed as

$$C = \sum_{i=1}^{n} C_i (P_{Gi})$$

Total cost 'C' is to be minimized under the equality constraint that $\sum_{i=1}^{n} P_{Gi} - P_{Demand} = 0$

The problem can be solved by Lagrange multipliers.

$$L = \sum_{i=1}^{n} C_1(P_{Gi}) - \lambda\left[\sum_{i=1}^{n} P_{Gi} - P_D\right]$$

where λ is the Lagrangian multiplier

Minimization of 'C' is achieved by the condition

$$\frac{\partial L}{\partial P_{Gi}} = 0 \Rightarrow \frac{\partial C_i}{\partial P_{Gi}} = \lambda \text{ for } i = 1, 2, 3....n$$

where $\frac{\partial C_i}{\partial P_{Gi}}$ is the incremental cost of the i^{th} generator

$$\therefore \quad \frac{\partial C_i}{\partial P_{Gi}} = \frac{\partial C_2}{\partial P_{G2}} = \frac{\partial C_3}{\partial P_{G3}} = ... = \frac{\partial C_n}{\partial P_{Gn}} = \lambda$$

$\therefore$ Optimal operation of generators corresponds to the equal incremental cost point of all generators.

If a particular generator loading P_{Gi} reaches the limit $P_{Gi_{max}}$ or $P_{Gi_{min}}$, its loading from now on is held fixed at this and the remaining load is then shared between the remaining generators on equal incremental cost basis.

When line losses are accounted for optimal scheduling, the equality constraint of meeting the load demand is modified as

$$\sum_{i=1}^{n} P_{Gi} - P_D - P_L = 0$$

where P_L = Total system transmission loss

Lagrangian is written as

$$L = \sum_{i=1}^{n} C_i(P_{Gi}) - \lambda\left[\sum_{i=1}^{n} P_{Gi} - P_D - P_L\right]$$

where the transmission losses are function of active power generation at all buses.

$$P_L = P_L(P_{G1}, P_{G2}, P_{G3}, ... P_{Gn})$$

For optimal power dispatch

$$\frac{\partial L}{\partial P_{Gi}} = \frac{\partial C_i}{\partial P_{Gi}} - \lambda + \lambda\frac{\partial P_L}{\partial P_{Gi}} = 0, i = 1, 2.....k$$

$$\frac{\partial C_i}{\partial P_{Gi}} = \lambda\left(1 - \frac{\partial P_L}{\partial P_{Gi}}\right)$$

$$\frac{\partial C_i/\partial P_{Gi}}{\left(1 - \partial P_L/\partial P_{Gi}\right)} = \lambda \quad \text{For } i = 1, 2, 3, \ldots, n$$

where $L_i = \dfrac{1}{\left(1 - \dfrac{\partial P_L}{\partial P_{Gi}}\right)}$ is called penalty factor of i^{th} plant.

The partial derivative $\frac{\partial P_L}{\partial P_{Gi}}$ is referred as the incremental transmission loss. $(ITL)_i$, associated with i^{th} generator. So, the condition for optimal generation can be given by

$$IC_i = \lambda[1 - ITL_i] \text{ for } i = 1, 2, 3.... n$$

So far optimal scheduling of plants, it is necessary to compute ITL of each plant, the functional dependence of transmission losses on real power of generating plants is given by

$$P_L = \sum_{p=1}^{n}\sum_{q=1}^{n} P_{Gp} B_{pq} P_{Gq}$$

where P_{Gp}, P_{Gq} = Real power generation at p^{th} and q^{th} plants

B_{mn} = Loss coefficients which are constants under certain assumed operating conditions.

Example 12: The fuel costs of a two-unit plant are given by

$$C_1 = 100 + 2P_1 + 0.005P_1^2$$

$$C_2 = 200 + 2P_2 + 0.01P_1^2$$

where P_1 and P_2 are in MW. The plant supplies a load of 400 MW. The economic load scheduling of two units will be?

Solution: $\dfrac{dC_1}{dP_1} = 2 + 0.005 \times 2P_1$

$$= 2 + 0.01P_1$$

$$\frac{dC_2}{dP_2} = 2 + 0.02P_2$$

For optimum load division, the two incremental costs should be equal.

$$\frac{dC_1}{dP_1} = \frac{dC_2}{dP_2}$$

$$2 + 0.01P_1 = 2 + 0.02P_2$$

$$P_1 = 2P_2$$

Total load is 400 MW

i.e., $P_1 + P_2 = 400$ MW

$$2P_2 + P_2 = 400 \text{ MW}$$

$$3P_2 = 400 \text{ MW}$$

$\therefore$ $P_2 = 133.4$ MW

$P_1 = 2P_2 = 266.6$ MW

Example 13: The power system has two generating plants and the power is being dispatched economically with P_1 = 125 MW, P_2 = 250 MW. The cost coefficients are

$$B_{11} = 0.1 \times 10^{-2} \text{ MW}^{-1}$$

$$B_{12} = -0.01 \times 10^{-2} \text{ MW}^{-1}$$

$$B_{22} = 0.13 \times 10^{-2} \text{ MW}^{-1}$$

to raise the total load on the system by 1 MW will cost an additional Rs 200 per hour. The penalty factor is _____?

Solution: $P_L = P_1^2 B_{11} + 2P_1P_2B_{12} + P_2^2 B_{22}$

$$\frac{\partial P_L}{\partial P_1} = 2P_1B_{11} + 2P_2B_{12}$$

$$= 2 \times 125 \times 0.1 \times 10^{-2} + 2 \times 250 \times (-0.01 \times 10^{-2}) = 0.2$$

Penalty factor for plant 1

$$L_1 = \frac{1}{1-\frac{\partial P_L}{\partial P_1}} = \frac{1}{1-0.2} = 1.25$$

Example 14: A generating station has two units. The incremental operating costs of the two units are as follows, G_1: $\lambda_1 = 0.1P_1 + 20$, G_2: $\lambda_2 = 0.2P_2 + 10$, where P_1, P_2 are in MW. Total loading is 200 MW. The economic loading will be

(A) $P_1 = 150, P_2 = 50$ (B) $P_1 = 100, P_2 = 100$

(C) $P_1 = 175, P_2 = 25$ (D) $P_1 = 50, P_2 = 150$

Solution: (B)

Given $P_1 + P_2 = 200$ (1)

and we know that $l_1 = l_2$, for economic load scheduling

$$0.1P_1 + 20 = 0.2P_2 + 20$$

$$\Rightarrow \quad 0.1P_1 - 0.2P_2 = -10 \quad (2)$$

Solving (1) and (2), we get

$$P_1 = 100 \text{ MW}, P_2 = 100 \text{ MW}$$

Example 15: A power system has two generating plants and the power is being dispatched economically with P_1 = 100 MW and P_2 = 200 MW. The loss coefficients are $B_{11} = 0.2 \times 10^{-2}$ MW^{-1}, $B_{12} = -0.02 \times 10^{-2}$ MW^{-1}, $B_{22} = 0.09 \times 10^{-2}$ MW^{-1}. The penalty factor for plant-1 will be

(A) 1.35 (B) 1.21 (C) 1.47 (D) 1.63

Solution: (C)

$$P_L = P_1^2 B_{11} + 2P_1P_2 B_{12} + P_2^2 B_{22}$$

$$\frac{\partial P_L}{\partial P_1} = 2P_1B_{11} + 2P_2B_{12}$$

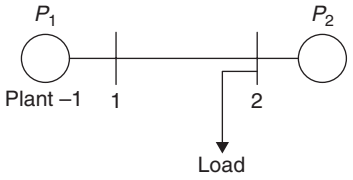
$$= [2(100)(0.2) \times 10^{-2}] + [2 \times 200 \times (-0.02) \times 10^{-2}]$$

$$= 0.4 - 0.08 = 0.32$$

Penalty factor, $L_1 = \frac{1}{1-\frac{\partial P_L}{\partial P_1}}$

$$= \frac{1}{1-0.32} = 1.47$$

Example 16: A system consists of two plants connected by a transmission line as shown in the fig. The load is at plant-2. If a load of 200 MW is transmitted from plant-1 to the load, there is a loss of 20 MW. Determine the loss factor B_{11}.



(A) 5×10^{-4} MW^{-1} (B) 10×10^{-4} MW^{-1}

(C) 15×10^{-4} MW^{-1} (D) 20×15^{-4} MW^{-1}

Solution: (A)

$$P_L = P_1^2 B_{11} + 2P_1P_2B_{12} + P_2^2B_{22}$$

∴ The load is at plant-2 bus

$$B_{12} = B_{21} = 0 \text{ and } B_{22} = 0$$

$$\therefore \quad P_L = P_1^2 B_{11}$$

$$\Rightarrow \quad B_{11} = \frac{P_L}{P_1^2} = \frac{20}{(200)^2}$$

$$\therefore \quad B_{11} = 5 \times 10^{-4} \text{ MW}^{-1}$$

Example 17: A power system has two generating plants and the power is being dispatched economically with P_1 = 100 MW and P_2 = 200 MW. The loss coefficients are $B_{11} = 0.5 \times 10^{-2}$ MW^{-1}, $B_{12} = -0.02 \times 10^{-2}$ MW^{-1}, $B_{22} = 0.15 \times 10^{-2}$ MW^{-1}. To raise the total load on the system by 1 MW will cost an additional Rs 300 per hour. The penalty factor for plant-1 will be

(A) 10 (B) 11.5 (C) 12.5 (D) 15

Solution: (C)

$$P_L = P_1^2B_{11} + 2P_1P_2B_{12} + P_2^2B_{22}$$

$$\frac{\partial P_L}{\partial P_1} = 2P_1B_{11} + 2P_2B_{12}$$

$$= 2(100)(0.5 \times 10^{-2}) + 2(200)(-0.02 \times 10^{-2})$$

$$= 1 - 0.08 = 0.92$$

Penalty factor $L_1 = \frac{1}{1-\frac{\partial P_L}{\partial P_1}}$

$$= \frac{1}{1-0.92} = 12.5$$

Example 18: The fuel costs of two unit plants are given by

$$C_1 = 50 + 5P_1 + 0.5P_1^2,$$

$$C_2 = 10 + 15P_2 + P_2^2$$

where P_1, P_2 are in MW. The plant supplies load of 100 MW. The economic load scheduling of the two units will be

(A) $P_1 = 70, P_2 = 30$ (B) $P_1 = 30, P_2 = 70$

(C) $P_1 = 50, P_2 = 50$ (D) $P_1 = 60, P_2 = 40$

Solution: (A)

$$\frac{dC_1}{dP_1} = \frac{dC}{dP_2}$$

$$5 + P_1 = 15 + 2P_2 \quad (3)$$

$$P_1 - 2P_2 = 10 \quad (3)$$

Given, $P_1 + P_2 = 100$ (4)

Solving (3) and (4), we get

$$P_1 = 70 \text{ MW}, P_2 = 30 \text{ MW}$$

Power System Protection

Undesirable effects of faults in power system are given by

1. Power system components carrying abnormal currents get overheated.
2. Operating voltages can go above or below their acceptable values.
3. Power flow is restricted or completely blocked due to short circuit.
4. Power system areas can lose synchronism.
5. Arcing faults can vaporize equipment in the vicinity leading to possibility of fire and explosion.
6. Open-circuit faults will lead to abnormal system operation and danger to personnel.

So, the faults should be instantly detected from the faulty section. Isolated from the rest of the system in the shortest possible time. Faults are detected automatically be means of relay and the fault section isolated by means of circuit breakers. The functional characteristics of the protection system are enumerated as follows.

1. **Reliability:** Reliability of a relay is a basic requirement. It must operate when it is required. Every component and circuit which is involved in operation of relay must be found in healthy operating condition when called up on to act.
2. **Speed:** A protective relay must operate at the required speed. It should neither be too slow which may results in damage to the equipment nor should it be too fast which may result in undesired operation during transient faults. The time is in the order of 30–100 ms depending upon the fault level of the section involved
3. **Selectivity:** Relaying equipment must clearly discriminate between normal and abnormal system conditions, so it never operates un necessarily
4. **Sensitivity:** A relay should be sufficiently sensitive so that it operates reliably when required under the actual conditions in the system which produces the least tendency for operation.

Basic Definitions in Protection

Pick-up Level

The value of the actuating quantity which is on the threshold above which the relay operates.

Reset Level

The value of the actuating quantity below which a relay opens its contacts and comes to original position.

Reset Time

The time which elapses between the instant when the actuating quantity becomes less than the reset value to the instant when the relay contact return to its normal position.

Operating Time

The time which elapses between the instant when the actuating quantity exceeds the pick-up value to the instant when the relay contacts close.

Primary Relays

The relay which are directly connected in the circuit to be protected.

Secondary Relays

The relays which are connected in the circuit to be protected through current and potential transformers.

Auxiliary Relays

Relays which operate in response to the opening or closing of its operating circuit to assist another relay.

Reach

A distance relay operates whenever the impendence seen by the relay is less than a pre-specified value. The impedance or the corresponding distance is known as the reach of the relay.

Over Reach

The tendency of the relay to operate at impedance more than its setting.

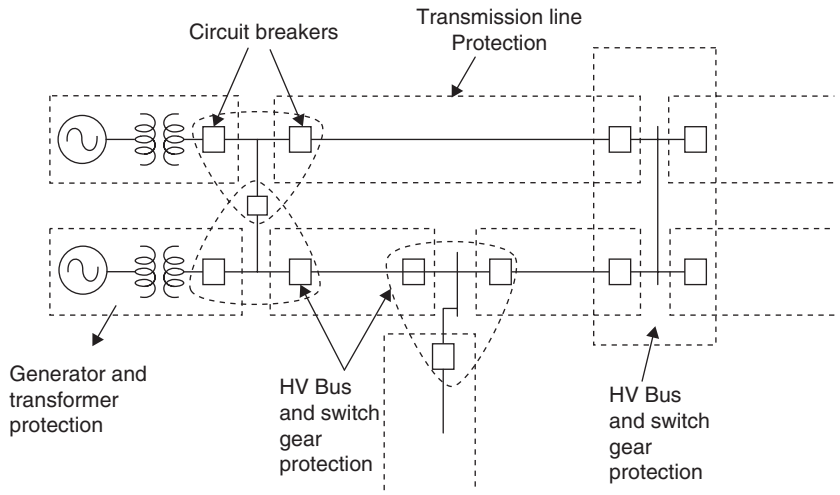
Under Reach

The tendency of the relay to operate at impedance less than its setting.

Protective Zones

In order to limit the number of elements disconnected by the protective system during a fault, the protective system is divided into number of zones. Protective zones of a power system are as shown in the figure below.

Each protective zone has the primary responsibility to disconnect the element or elements in the zone in the event of fault. The protection provided by each zone to its elements is known as ‘primary protection’ in the event of primary protection failure, backup protection is provided.



Classification of Relays

Based on the Time of Operation

1. Instantaneous relay
2. Definite minimum time relay
3. Inverse time relay
4. Inverse definite minimum time relay.

Based on the Construction

1. Electromagnetic relay
2. Induction relay
3. Static relay

Based upon Application

1. Overcurrent relay
2. Overvoltage relay
3. Differential relay
4. Distance relay

Essential Characteristics of Relay

1. Sensitivity
2. Selectivity
3. Speed
4. Reliability
5. Economy
6. Simplicity

Pick-up Value

The minimum value of operating quantity at which the relay is at wedge of operation is called 'pickup value', i.e. ready to operate.

Reset Value

The maximum value of operating quantity at which the relay is at wedge of reset is called 'reset value'.

Time Multiplier Setting (TMS)

$$\text{TMS} = \frac{\text{Actual time of operation required}}{\text{Required time of operation when TMS} = 1}$$

Plug Setting Multiplier (PSM)

It is the ratio of CT secondary fault current to relay operating current.

$$\text{Relay operating current} = \text{Current setting} \times \text{CT secondary rated current}$$

$$\text{PSM} = \frac{\text{CT secondary faulty current}}{\text{Current setting}\ \ \text{CT secondary rated current}}$$

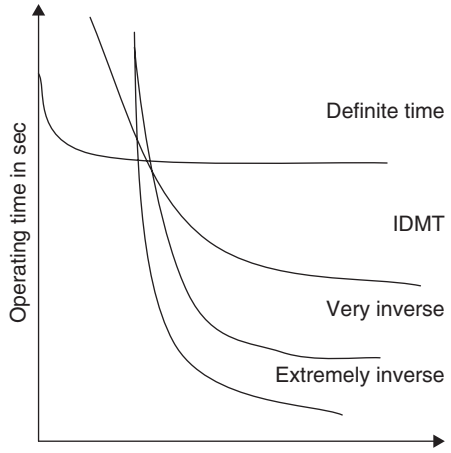
(or)

$$\text{PSM} = \frac{\text{Fault current (primary)}}{\text{Current setting} \times \text{CT secondary current} \times \text{CT ratio}}$$

Overcurrent Relays

Depending upon the time of operation, the relays are categorized as follows.

Characteristics of Various Overcurrent Relays



Instantaneous Overcurrent Relay

- In which no intentional time delay is provided for the operation.
- The time of operation of such relay is approximately 0.1 s.

Inverse Time–current Relay

- In which the operating time reduces as the activating quantity increases in magnitude.

Inverse Definite Minimum Time Current Relay (IDMT)

- In which the operating time is approximately inversely proportional to the fault current near pick-up value and becomes substantially constant slightly above the pickup value of the relay.

Very Inverse Relay

- In which the saturation of the core occurs at a later stage.

Extremely Inverse Relay

- In which the saturation of the core occurs at a still later stage.

Overcurrent (OC) Relay

The relay operates when current through it satisfies the condition

$$Q = K_1 |I|^2 - K_4 > 0$$

$$I > \sqrt{\frac{K_4}{K_1}} = |I_p|$$

where I_p = pick up value of the relay, K_1 and K_4 are constants.

The above condition for relay operation can also be written as.

$$|I| > |I_p| \text{ trip (Relay trips CB)}$$

$$|I| < |I_p| \text{ block (Relay does not trip CB)}$$

The value of the pickup current can be easily adjusted by altering the constant K_1 by means of plug setting.

For co-ordination of overcurrent relays, it is essential that these must have inverse-time characteristics where operating time decreases as relay current increases these relays are called as 'Inverse definite minimum time lag relays'. Characteristics of an IDMTL relay are expressed mathematically as

$$T_{op} = f(|I_{p.u.}| - 1) \text{ for } |I_{p.u.}| > 1$$

$$|I_{p.u.}| = {}^{|I|}/_{|P|}$$

- The inverse time current relays are non-directional relays and are used for the protection of feeders, transmission lines, transformers, machines and other numerous applications.

Directional Relays

At a certain points in power system, current is flowing in one direction (specified direction), if due to any changes in the circuit condition, the current flows in opposite direction, the relay will develop positive torque and will operate.

In a directional relay

$$Q = K_3 |V| |I| \cos(\theta - \tau) - K_4$$

where θ = Angle by which $|I$ leads $|V|$
τ = Adjustable angle parameter

The relay operator when

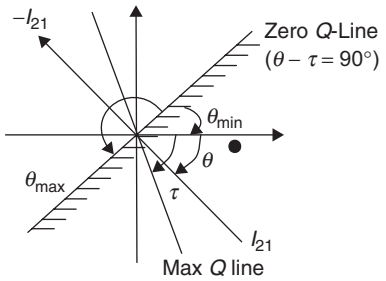
$$Q = K_3 |V| |I| \cos(\theta - \tau) > 0$$

or

$$\cos(\theta - \tau) > 0 \text{ trip}$$

$$\cos(\theta - \tau) < 0 \text{ block}$$

The operating characteristics of directional relay are shown in figure below.



In a directional relay, V is taken as the reference phasor, under fault, it undergoes only a small change in phase angle, This phase is called as the 'polarizing quantity' of the relay. The relay characteristics also can be expressed as

$$\theta_{min} > \theta > \theta_{max} \text{ trip}$$

$$\theta_{min} < \theta < \theta_{max} \text{ block}$$

Impedance Relay

From the general relay equation we have a special case

$$Q = K_1 |I|^2 + K_2 |I|^2 - K_4$$

If effect of K_4 is neglected, the relay operates when

$$K_1 |I|^2 + K_2 |V|^2 > 0$$

Or

$$\frac{|V|}{|I|} = |Z| < \left(\frac{K_1}{-K_2}\right)^{1/2} = |Z_{rs}|$$

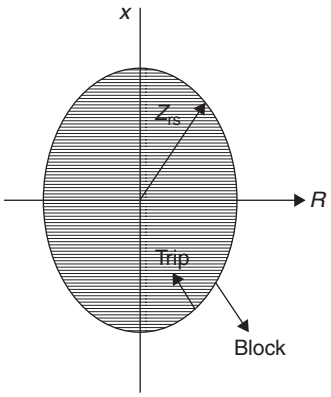
If K_2 is negative, such a relay senses impendence magnitude, and operates if the magnitude of impendence seen from its location is less than a specified value. Since $|V|$ prevents relay operation and $|I|$ tries to operate it, voltage in the restraining quantity and current is operating quantity of the relay.

The characteristics impendence can be generally expressed as

$$|Z| < |Z_{rs}| \text{ trip}$$

$$|Z| > |Z_{rs}| \text{ Block}$$

where $|Z_{rs}|$ = impendence setting of the relay. *RX*-plane of the impendence characteristics is given by



Mho Relay

A modified impendence relay called 'Mho relay' in case of Mho relay

$$Q = K_3 |V| |I| \cos(\theta - \tau) - (-K_1) |V|^2 - K_4$$

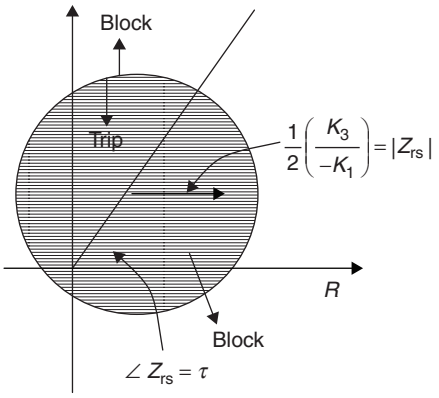
For relay operation neglecting K_4

$$K_3 |V| |I| \cos(\theta - \tau) - (-K_1) |V|^2 > 0$$

$$\frac{|V|}{|I|} = |Z| < \frac{K_3}{-K_1} \cos(\theta - \tau)$$

$$|Z - Z_{rs}| < |Z_{rs}| \text{ trip}$$

$$|Z - Z_{rs}| > |Z_{rs}| \text{ block}$$



Rx–diagram of a Mho relay

Reactance Relay

A reactance relay is an overcurrent relay with directional restraint the direction element is arranged to yield maximum *Q*-contribution when its current lags its voltage by 90°

$$Q = K_1 |I| \cos(\theta - \tau) - (-K_1) |V|^2 - K_4$$

For relay operation neglecting K_4

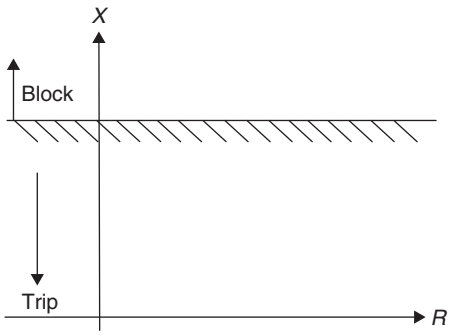
$$K_1 |I|^2 - (K_2) |V| |I| \sin\theta > 0$$

$$\frac{|V|}{|I|} \sin\theta < \frac{K_1}{-K_2}$$

$$|Z| \sin\theta < \frac{K_1}{-K_2}$$

$$x < \frac{K_1}{-K_2}$$

These characteristics are represented in *R*-*x*-plane as shown below.



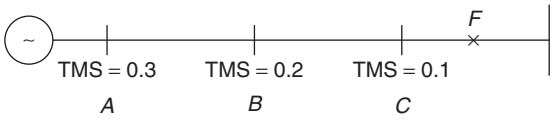
Protection of Feeders

Radial Feeders

Overcurrent protection for radial feeder is done by the following methods.

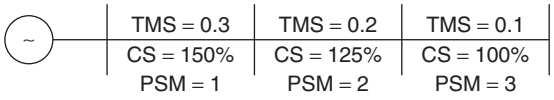
Time-graded System

The time of operation of the relays at various locations is so adjusted that the relay farthest from the source will have minimum time of operation and as it is approached towards the source the operating time increases.



Current Graded Method

In this method for the relay farthest from the source the current setting (cs) is kept minimum. As we go towards the source, the current setting gradually increases.



$$\text{Current setting} \propto \frac{1}{\text{PSM}}$$

$$\text{PSM} \propto \frac{1}{\text{TMS}}$$

Time Current Graded System

According to this system the TMS and current setting for the relay farther from the source is kept minimum and as we go towards the source both these values increase.

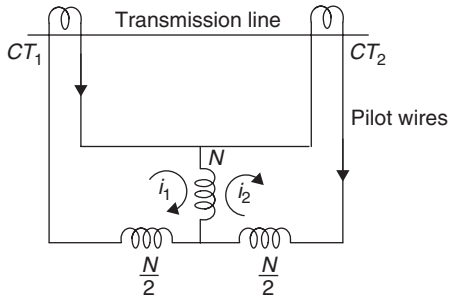
Parallel Feeders

- In this protection scheme, directional as well as non-directional overcurrent relays are used.
- Non directional relays are provided near the source and directional relays are provided near the load.

Ring Main Feeders

- The ring main considered two radial feeders.
- Non-directional relays must be placed near the source.
- Directional relays are placed near the load.

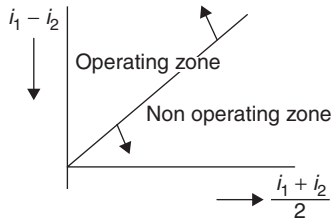
Differential Relays



$$\text{Operating coil} \propto (i_1 - i_2)\, N$$

$$\text{Restraining force} \propto \frac{i_1 N}{2} + \frac{i_1 N}{2}$$

$$\propto \frac{(i_1 + i_2)\, N}{2}$$



- Operating zone and non-operating zone is fixed by percentage differential relay.
- A differential protection using percentage differential relay is called as merz-price circulating current coil.
- A differential relay will not operate for inter turn fault in the generator winding. For inter turn fault protection of generator winding, split phase relay is used.
- For short circuits, normally percentage differential protection is recommended for transformers rated more than 1 MVA. For low rating, overcurrent relaying is used.

Neutral Grounding

The process of connecting neutral point of three-phase system to Earth (i.e. soil), either directly or through some circuit element (E.g. resistance, reactance, etc.) is called neutral grounding.

Methods of Neutral Grounding

Solid or Effective Grounding

Neutral point of a 3-ϕ system is directly connected to earth through a wire of negligible resistance and reactance.

This type of grounding is used for voltages up to 33 kV with total power capacity not exceeding 5000 kVA.

Resistance Grounding

When the neutral point of a three-phase system is connected to earth through a resistor, it is called resistance grounding.

It is used for voltages between 2.2 kV and 33 kV when power capacity is more than 5000 kVA.

Reactance Grounding

- A reactance is inserted between the neutral and ground.
- High transient voltage appears under fault condition.

Resonant Grounding

Capacitance currents are responsible for producing arcing grounds. These capacitive currents flow because capacitance exists between each line and Earth.

When the value of 'L' for arc suppression coil is such that the fault current, I_f exactly balances the capacitive current I_c, it is called resonant grounding (or Peterson coil grounding).

For resonant grounding,

$$I_L = I_C$$

$$\frac{V_{Ph}}{X_L} = \frac{3V_{Ph}}{X_C}$$

$$X_L = \frac{X_C}{3}$$

$$\omega L = \frac{1}{3\omega C}$$

$$\therefore \quad L = \frac{1}{3\omega^2 C}$$

Solid State Relay

A solid state relay (SSR) is an electronic switching device in which a small control signal controls a larger load current or voltage. It comprises a voltage or current sensor which responds to appropriate input (control signal), a solid state electronic switching device of some kind which switches power to the load circuitry either on or off, and some coupling mechanism to control signal to activate this switch without mechanical parts. It serves same functions as an electromechanical relay, but has no moving parts.

Many SSRs use optical coupling. The control voltage energizes an LED which illuminates and switches on a photo sensitive diode (photo-voltaic). The diode current turns on a back to back thyristor, silicon control rectifier, or MOSFET to switch the load.

Advantages of SSRs

- SSRs are faster
- Increased life time
- Clean, bounce less operation
- No sparking, silent operation
- Identically smaller than electromagnetic relay

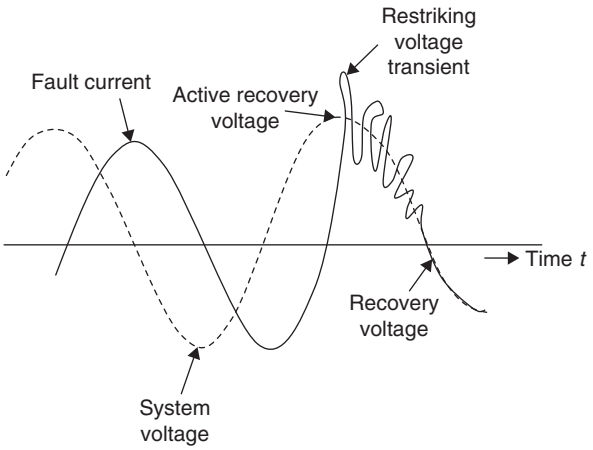
Disadvantages of SSRs

- Isolated bias supply is required for gate charge circuit.
- SSRs have higher transient reverse recovery time due to the presence of diode.

Circuit Breaker

Circuit breakers can be used for opening and closing of electrical circuit under normal and abnormal conditions. Generally circuit breaker consist of both moving and fixed contacts, when the circuit under normal (or) healthy conditions the contacts are closed and when the circuit under abnormal (or) fault condition the contacts are separated.

Consider a circuit operated under normal conditions and after some time interval fault occurs on the system. Then the variation of voltage and currents is as shown below.



When the circuit is interrupted by a circuit breaker under fault condition an arc is formed between the two contacts of the circuit breaker. This arc can be extinguished by two methods.

1. Low resistance (or) current zero method
2. High resistance method

Arc Voltage

The voltage appeared across the contacts of the circuit breaker during arcing period is known as arc voltage.

Restriking Voltage

The transient voltage appears across the circuit breaker contacts at current zero point is known as restriking voltage.

Average Rate of Rise of Restriking Voltage (RRRV)

It is the ratio of peak value of restriking voltage to the time taken to reach peak value voltage.

The average RRRV

$$= \frac{\text{Peak value of restriking voltage}}{\text{Time taken in attaining peak value}}$$

$$= \frac{2V_{max}}{\pi\sqrt{LC}}$$

Insertion P_6

Ratings of Circuit Breaker

The circuit Breaker performance is analysed by the following terms.

1. Breaking capacity
2. Making capacity
3. Short time capacity

Breaking Capacity

It is the product of rated breaking current in terms of kA and rated voltage in terms of kV.

For three-phase, the Breaking capacity = $\sqrt{3}VI$

Making Capacity

The making capacity of circuit breaker is the ability of the circuit breaker to withstand the electromagnetic force under short-circuit condition. Generally making capacity is 2.55 times of breaking capacity.

Making capacity = 2.55 × Breaking capacity.

Short Time Capacity

It is the capacity of the circuit Breaker which can safely applied to the circuit Breaker under its normal condition for 3 seconds time period.

Types of Circuit Breakers

1. Air-break circuit breakers
2. Oil circuit breakers (OCB)
3. Vacuum circuit breakers (VCB)
4. Air-blast circuit breakers (ABCB)
5. SF_6 circuit Breakers

Air Break Circuit Breakers

Air Break circuit Breakers are generally indoor type. These are used for DC circuit up to 12 kV and for medium- and low-voltage AC circuits up to 6.6 kV, 400–200 A.

The constructional details of an ABCB are shown in figure. The arc is lengthened in the breakers by the magnetic

field and are runners and is finally extinguished by are splitters. There are two sets of contacts:

- Main contacts
- Arcing contacts

The main contacts are first to dislodge, while the arcing contacts are still closed under spring pressure. Thus the min contacts do not open any current and have long life. Arcing contacts made of hard copper alloy are easily replicable.

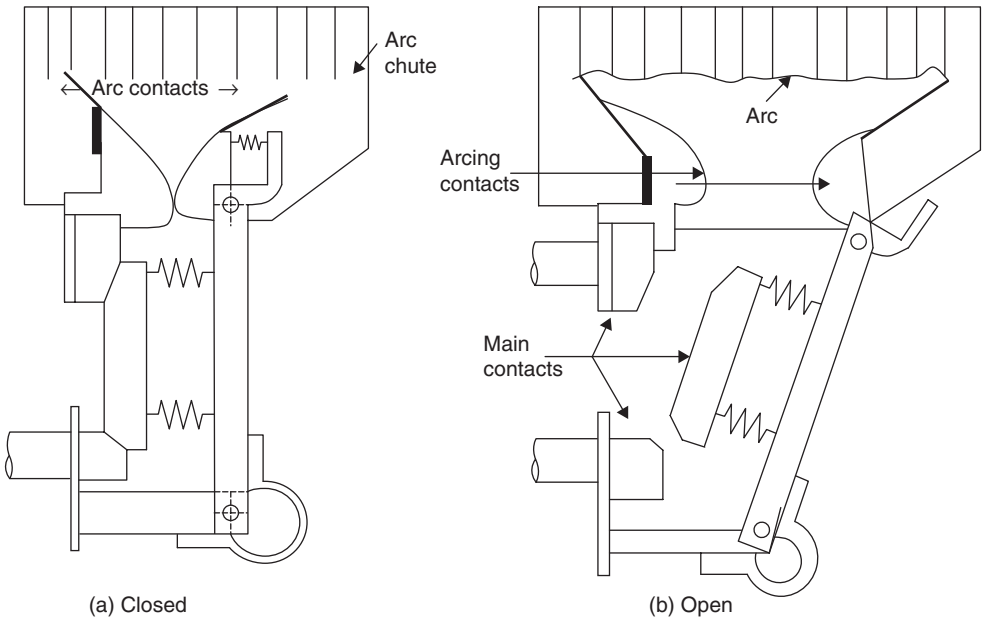
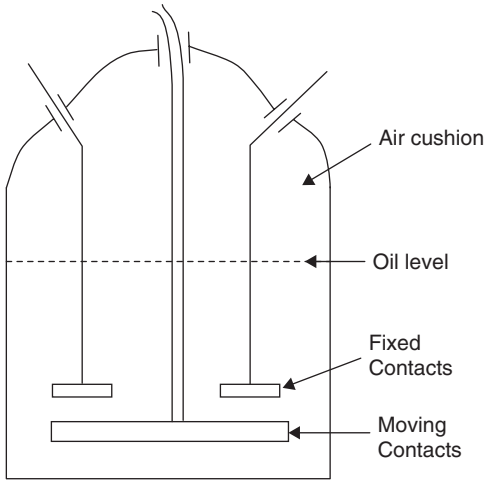


Figure 4 Air break circuit breakers

Oil Circuit Breakers (OCB)

Oil is used as the arc-quenching medium. These CB's have been developed for the voltage levels of 3.6 kV, 7.2 kV, 12 kV, 36 kV, 72.5 kV, 145 kV and 245 kV. The arc control devices are based on axial-flow and /or cross-flow principle. For higher current ratings, cross flow principle is preferred. The technique of arc quenching in oil circuit breakers is shown below.



In oil circuit breakers, as the contacts separate, the heat of the arc causes the oil to decompose in to hydrogen (70%) acetylene, etc. The gaseous formation causes the pressure inside the arc control device to rise and as a result, the arc is pushed across the side vents, there by elongating it. As the contacts move farther apart lengthening of arc occurs and hence the arc will be extinguished. For a specific design and speed of contracts, the gas pressure generated is a function of arc current and arcing time.

Disadvantages of OCB:

1. Arc products are inflammable.
2. Oil is hygroscopic, and must be sealed air-tight in chamber.
3. Dielectric strength of oil is reduced by carbonization during the arcing process.

Vacuum Circuit Breakers

The separation of current carrying contacts comes the metal vapour to the released from the contacts giving rise to plasma – electrons and positive arc. The vapour density depends up on the current in the arc. In the decreasing phase of the current the rate of release of metal vapour quickly disperses and the medium regains its dielectric strength the arc is thus extinguished in just half a cycle. The contact separation needed is of the order of a few millimetres—least among all circuit Breaker types. VCBs have been employed at voltages up to 72.5 kV.

Air Blast Circuit Breakers (ABCB)

Blast of air at high speeds directed at the arc is very effective in cooing it, and in removing the products of ionization after current zero with consequent arc extinction with in a cycle. The high speed pressure is produced by externally generated pressure. The breaker in designated to direct a jet of air derived from the high pressure source to the contact space at the instant of contact separation. Various possible alternative arrangements for achieving this are shown below.

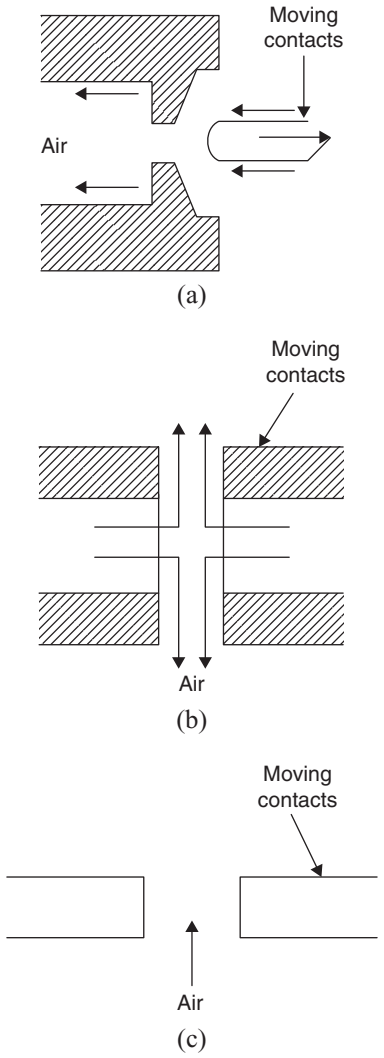


Figure 5 (a) Axial blast with hollow fixed contact; (b) Radial blast; (c) Cross blast

Advantages

- Reliable because an external source is used for arc extinction.
- clean, non-compassable, and non-inflammable.
- Fresh medium is used every time.
- Faster contact travel.
- Small contact travel is involved.

Disadvantages

- High air noise while operating
- Chopping possibility
- Independent pressure system is needed for ABCB.

Sulphur Hexafluoride (SF_6)CB

The attachment of the electron with the natural gas molecules may occur in two ways

$$SF_6 + e^- \rightarrow SF_6^-$$

$$SF_6 + e^- \rightarrow SF_5 + f$$

The negative ions formed are relatively heavier than free electrons and therefore under a given field the ions do not attain sufficient energy to lead cumulative ionization in the gas. This property gives rise to very high dielectric strength of SF_6. The property of SF_6 makes it very effective in quenching arcs.

Advantages

- Current-chopping tendency is minimized.
- There is no exhaust of high-pressure gas to atmosphere.
- There is short time arc, low-contact erosion and no contact replacement.
- No carbon deposition takes place.
- The smaller size of conductors and clearness lead to small overall breaker size.
- They are non-inflammable.

Example 19: Which of the following protection relay is inherently directional?

(A) Reactance relay (B) Impedance relay
(C) OHM relay (D) MHO relay

Solution: (D)

Example 20: Purpose of harmonic restraint in protection transformers is to protect against

(A) Lightning
(B) Unbalanced operation
(C) Switching overvoltages
(D) Magnetizing inrush current

Solution: (D)

Example 21: Reactance relay is mainly used for

(A) Earth faults (B) Open-circuit faults
(C) Phase faults (D) None of the above

Solution: (A)

Example 22: A negative sequence relay is commonly used to protect

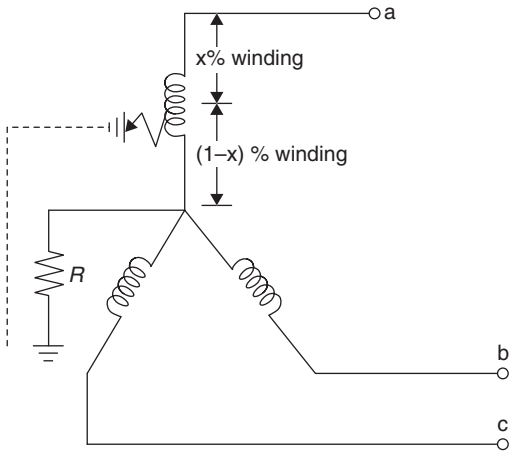
(A) A transformer (B) An alternator
(C) A transmission line (D) A bus bar

Solution: (B)

Example 23: The neutral of 10 MVA, 11 kV, alternator is earthed through a resistance of 5 Ω. The earth fault relay is set to operate at 0.65 A. The CTs have a ratio of 1000/5. What percentage of the alternator winding is protected?

(A) 89.7% (B) 98.3%
(C) 10.3% (D) 1.7%

Solution: Fault current $(I_f) = (1 - x)\frac{V_{ph}}{R}$



$\therefore$ Primary fault setting current $(I_p) = (1 - x)\frac{V_{ph}}{R}$

Earthing resistor current setting $(I_s) = \frac{V_{ph}}{R}$

$\therefore \quad I_p = (1 - X)I_s$

$\Rightarrow \quad x = \left(1 - \frac{I_{ph}}{I_s}\right)$

$\therefore$ Given that the earth fault relay set current = 0.65 A.

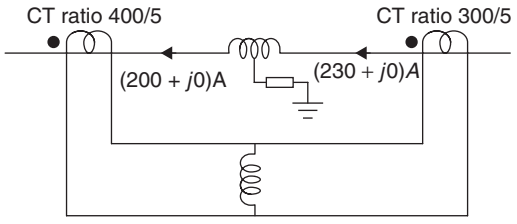
Primary fault setting current $(I_p) = 0.65 \times \frac{1000}{5} = 130$ A

Earth resistor current setting I_R

$$= \frac{V_{ph}}{R} = \frac{11000}{\sqrt{3}} \times \frac{1}{5} = 1270.17 \text{ A}$$

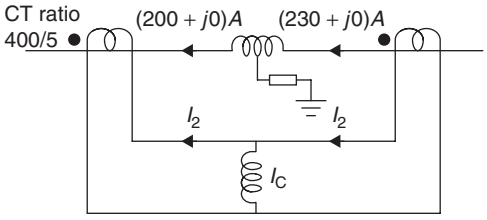
% of winding protected $= = \left(1 - \frac{130}{1270.17}\right) \times 100 = 89.7\%$

Example 24: Consider a stator winding of an alternator with an internal high resistance ground fault. The currents under the fault condition are shown in the figure the winding is protected using differential current scheme with current transformer of ratio 300/5 A as shown. The current through the operating coil is



(A) 3.83 A
(B) 3.33 A
(C) 1.33 A
(D) 0.369 A

Solution: (C)



$$I_1 = 200 \times \frac{5}{400} = 2.5 \text{ A}$$

$$I_2 = 230 \times \frac{5}{300} = 3.83$$

$$I_c = I_2 - I_1 = 1.33 \text{ A}$$

Example 25: A 3-ϕ, 33 kV oil-circuit breaker is rated 1100 A, 2000 MVA, 2.5 s. The symmetrical braking current is

(A) 35 kA
(B) 3600 A
(C) 1200 A
(D) 105 A

Solution: (A)

Symmetrical breaking current $= \frac{P}{\sqrt{3}V_L}$

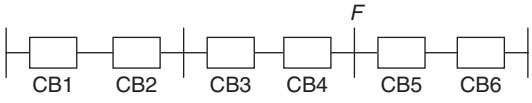
$$= \frac{2000 \times 10^6}{\sqrt{3} \times 33 \times 10^3} = 35 \text{ kA}$$

Example 26: The interrupting time of a circuit breaker is the time period between

(A) Initiation of short circuit and the parting of primary arc contacts.
(B) Energizing of the trip circuit and the parting of primary arc contacts.
(C) Energizing of the trip circuit and the arc extinction on an opening operation.
(D) Initiation of short circuit and the arc extinction on an opening operation.

Solution: (C)

Example 27: Three sections of a feeder are provided with circuit breaker: CB1, CB2, CB3, CB4, CB5, CB6, for a fault 'F' as indicated in the following figure.



(A) CB5 must be set to trip before CB1, CB2, CB3 and CB4 trip.
(B) CB5 must be set to trip after CB1 trip
(C) CB5 must be set to trip after CB2 trips
(D) CB5 must be set to trip after CB2, and CB4 trips

Solution: (A)

Example 28: Resistance switching is normally employed in
(A) Bulk oil breakers
(B) Minimum oil breakers
(C) Air blast circuit breakers
(D) All breakers

Solution: (D)

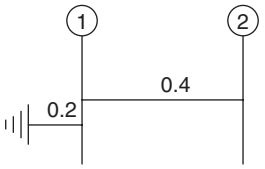
EXERCISES

Practice Problems I

Directions for questions 1 to 22: Select the correct alternative from the given choices.

1. In power flow analysis, a load connected at a bus acts as
(A) Constant impedance connected at the bus
(B) Voltage and frequency dependent source at the bus.
(C) Constant real and reactive power drawn from the bus
(D) Constant current drawn from the bus

2. A system has two buses marked 1 and 2. The per unit impedances are also marked. The bus admittance matrix is



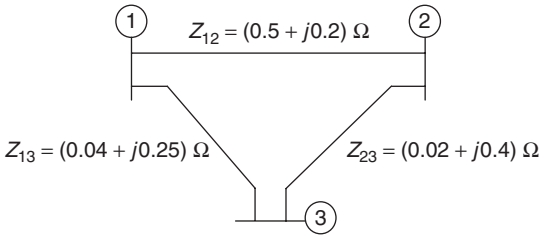
(A) $\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}$ (B) $\begin{bmatrix} 7.5 & 2.5 \\ 2.5 & 2.5 \end{bmatrix}$

(C) $\begin{bmatrix} 0.6 & -0.4 \\ -0.4 & 0.4 \end{bmatrix}$ (D) $\begin{bmatrix} 7.5 & -2.5 \\ -2.5 & 7.5 \end{bmatrix}$

3. The load flow studies is performed using Y bus matrix rather than Z bus matrix. This is because
(A) Y bus is a square matrix
(B) Y bus is a sparse matrix
(C) Z bus is a sparse matrix
(D) Z bus is a singular matrix

4. The number of iterations required for solving load flow problems in Gauss Seidel and Newton–Raphson methods is
(A) Both directly proportional to number of buses
(B) Directly proportional to number of buses and independent of number of buses, respectively
(C) Independent of number of buses and directly proportional to number of buses, respectively
(D) both independent of the number of system

5. The value of $Y11$ of the bus admittance matrix of the network is



(A) $1.8 - j8.6$ (B) $1.3 - j7.2$
(C) $1.8 + j8.6$ (D) $1.3 + j7.2$

6. Two alternators in parallel supplying a total load of 75 MW have the following data:
Machine 1: 50 MVA with 4% speed regulation
Machine 2: 70 MVA with 5% speed regulation
The load shared between machines 1 and 2 is
(A) $P_1 = 50$ MW, $P_2 = 25$ MW
(B) $P_1 = 25$ MW, $P_2 = 50$ MW
(C) $P_1 = 40$ MW, $P_2 = 35$ MW
(D) $P_1 = 35$ MW, $P_2 = 40$ MW

7. Two generators delivering a total of 220 MW have their incremental cost characteristics as follows

$$\frac{dF_1}{dP_1} = 1 + 0.15P_1$$

$$\frac{dF_2}{dP_2} = 4.0 + 0.1P_2$$

For economic operation P_1 and P_2 should be
(A) $P_1 = 120$ MW, $P_2 = 100$ MW
(B) $P_1 = 100$ MW, $P_2 = 120$ MW
(C) $P_1 = 80$ MW, $P_2 = 140$ MW
(D) $P_1 = 140$ MW, $P_2 = 80$ MW

8. The most inappropriate among the disadvantages of Gauss–Seidel method of load flow solution is
(A) Slow convergence
(B) Unreliable convergence
(C) A good initial guess is essential for convergence.
(D) Choice of slack bus affects convergence

9. For the Y bus matrix of a 4-bus system given in per unit, the buses having shunt elements are,

$$Y_{BUS} = j\begin{bmatrix} -5 & 2 & 2.5 & 0 \\ 2 & -10 & 2.5 & 4 \\ 2.5 & 2.5 & -9 & 4 \\ 0 & 4 & 4 & -8 \end{bmatrix}$$

(A) 3 and 4 (B) 2 and 3
(C) 1 and 2 (D) 1, 2 and 4

10. If the equation $x^2 - 500 = 0$ is to be solved using Newton–Raphson method, the iterative steps involved is

(A) $x_{R+1} = x_R + \frac{500}{x_R}$ (B) $x_{R+1} = \frac{1}{2}\left(x_R + \frac{500}{x_R}\right)$

(C) $x_{R+1} = \frac{1}{2}\left(x_R - \frac{500}{x_R}\right)$ (D) $x_{R+1} = x_R - \frac{500}{x_R}$

11. The equation $e^x - 1 = 0$ is to be solved using Newton–Raphson method with initial guess $x_0 = 1$ After one step, estimate x of solution is given by
(A) 0.36788 (B) 0.71828
(C) 0.20587 (D) 0.0000

12. The specified variables in a slack bus are
(A) Active power demand and reactive power demand at the bus
(B) Active power demand and magnitude of voltage at the bus
(C) Reactive power demand and phase angle of the voltage
(D) Magnitude of the voltage and phase angle of the voltage

Common Data for Questions 13 and 14:
The fuel cost of a system having two plants are given by

$$C_1 = (100 + 2P_1 + 0.005P_1^2) \text{ Rupees}$$

$$C_2 = (100 + 2P_2 + 0.01P_2^2) \text{ Rupees}$$

where P_1 and P_2 are in MW.

The loss coefficients are

$$B_{11} = 0.1 \times 10^{-2} \text{ MW}^{-1}$$

$$B_{12} = -0.09 \times 10^{-2} \text{ MW}^{-1}$$

$$B_{22} = 0.09 \times 10^{-2} \text{ MW}^{-1}$$

13. If the system supplies a load of 600 MW, the scheduling of two plants will be
(A) $P_1 = 266.7$ MW, $P_2 = 333.3$ MW
(B) $P_1 = 333.3$ MW, $P_2 = 266.7$ MW
(C) $P_1 = 400$ MW, $P_2 = 200$ MW
(D) $P_1 = 200$ MW, $P_2 = 400$ MW

14. The penalty factor of plant 1 is
(A) 1.79 (B) 1.92 (C) 1.25 (D) 1.36

15. The instantaneous value of current interrupted is 8 A. If the inductance and capacitance of a system are 2.0 H and 0.025 μF, voltage across the breaker contacts will be
(A) 71.55 V (B) 71.55 kV
(C) 5.155 kV (D) 715.5 kV

16. A10 A, 3 second overcurrent relay having a current setting of 125% and a time setting multiplier of 0.4 is connected to supply circuit through a 400/10 current transformer. When the circuit carries a fault current of 4000 A, operating time is?
Given

PSM	4	6	8	1	0
TSM	2.8	3.2	3.5	1.2	0.8

(A) 1.4 s (B) 2.2 s (C) 3.6 s (D) 4.25 s

17. A percentage differential relay is employed for protecting a 80 MVA, 110/66 kV, Δ/Y, three-phase power transformer.

The ratios of primary and secondary currents in the CT's located in delta and wye side of power transformer are 400/10 A and 1100/10 A, respectively. The relay current at full load is
(A) −0.5219 A (B) 0.12 A
(C) 0.68 A (D) 6.5 A

18. The bus bars of a power station are in two sections P and Q separated by a reactor. Section P has a 20 MVA generator of 10% reactance and Q has two generators; one is 25MVA with 13% reactance and other 10 MVA with 8% reactance. Feeders are connected to bus bars through transformers, each rated 10 MVA and 6% reactance. Maximum short-circuit MVA with which the circuit breakers on the outgoing side of the transformers have to deal [The reactor is rated at 15MVA and 16% reactance] is
(A) 150 MVA (B) 163 MVA
(C) 180 MVA (D) 100 MVA

19. For a 132 kV system, the inductive reactance and capacitance up to the location of circuit breaker is 10 Ω and 0.06 μF.
A resistance of 300 Ω is connected across the contacts. The value of resistance which will give damped frequency of oscillation of 1/4th natural frequency
(A) 128.65 Ω (B) 363.37 Ω
(C) 280.56 Ω (D) 500.08 Ω

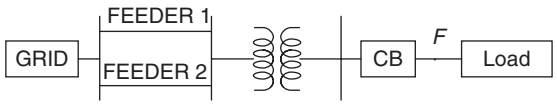
20. A 50 Hz three-phase alternator with grounded neutral has an inductance of 1.4 mH per phase and is connected to the bus bars through a circuit breaker. The capacitance to earth of the circuit between the alternator and CB is 0.00165 μF per phase. Time to attain maximum RRRV is
(A) 1.85 (B) 2.38 μs
(C) 1.75 μs (D) 3.55 μs

21. Consider grid as infinite bus. Choose 10 MVA as base
Transformer: Three-phase 66/33 kV, 10 MVA,

$0.02 + j0.06$ p.u. impedance

Load: Three-phase, 11 kV, 6000 kVA, 0.8 lag, $j0.6$ p.u. impedance.
Impedance of each feeder $= 8 + j16$ Ω.

The required MVA rating of CB is



(A) 35.714 MVA (B) 25.65 MVA
(C) 10.5 MVA (D) 8.82 MVA

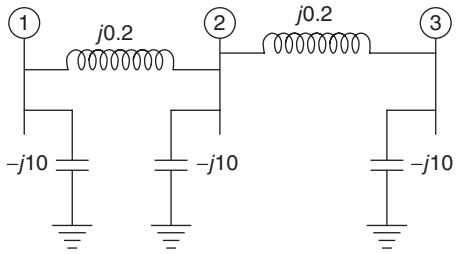
22. In a short-circuit test on a circuit breaker, time to reach the peak restriking voltage is 85 ms and the peak restriking voltage is 150 kV. The average rate of rise of restriking voltage is
(A) 1.76 kV/μS (B) 2.08 kV/μS
(C) 0.82 kV/μS (D) 5.04 kV/μS

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. The power generated by two plants are $P_1 = 80$ MW and $P_2 = 100$ MW
If the loss coefficients are $B_{11} = 0.002$, $B_{22} = 0.004$ and $B_{12} = -0.0008$
The power loss is
(A) 12.0 MW (B) 16.4 MW
(C) 15.4 MW (D) 10.0 MW

2. The network shown in the figure has impedances in p.u. as indicated. The diagonal element Y_{22} of the bus admittance matrix Y bus of the network is

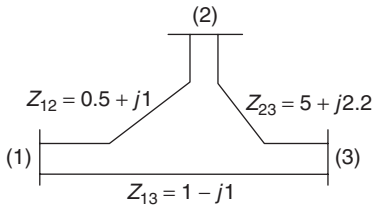


(A) $-j9.6$ (B) $+j0.104$
(C) $+j9.9$ (D) $-j9.9$

3. If X is the reactance of a line and R is the resistance of the line, the power transferred is maximum when
(A) $X = R$ (B) $X = 2R$
(C) $X = \sqrt{3}R$ (D) $X = \sqrt{2}R$

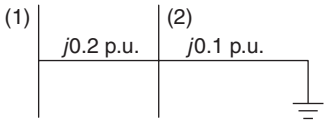
4. The fuel cost of two or more plants operating in parallel is minimum when
(A) The penalty factor of each plant is the same.
(B) Incremental fuel cost of each plant is the same.
(C) The incremental fuel cost multiplied by its penalty factor for each plant is the same.
(D) Ratio of incremental fuel cost to penalty factor of each plant is the same.

5. The value of Y_{11} of the bus admittance matrix for the network shown is



(A) $0.9 - j0.3$ (B) $0.1 - j0.3$
(C) $1 - j0.3$ (D) $0.9 - j0.8$

6. The bus admittance matrix for the network shown below is



(A) $\begin{bmatrix} j5 & j5 \\ j5 & -j13 \end{bmatrix}^{-1}$ (B) $\begin{bmatrix} j5 & j5 \\ j5 & -j15 \end{bmatrix}$

(C) $\begin{bmatrix} j0.2 & j0.1 \\ -j0.2 & -j0.1 \end{bmatrix}$ (D) $\begin{bmatrix} j10 & -j15 \\ j5 & j15 \end{bmatrix}$

7. For a generator rated at 30 MVA, 22 kV, 3-cycle circuit breaker is employed. The interrupting current is [Take $X_d' = 30\%$, $E_g = 1$ p.u.]
(A) 4.8 kA (B) 3.15 kA
(C) 5.6 A (D) None of these

8. For a relay, the operating speed depends upon,
(A) Spring tension
(B) Armature core air gap
(C) Rate of flux build-up
(D) All of these

9. For a system, the normal line current is 120 A. The circuit is designed to operate at 125% of normal line current and the trip coil is connected through a CT of ratio of 100:1. The trip mechanism should be set to operate at
(A) 1.5 A (B) 2.5 A
(C) 1.2 A (D) 2 A

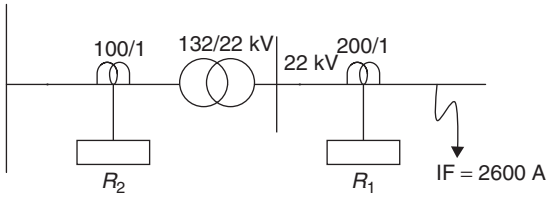
10. For universal relays, the torque equation is
(A) $K_3 VI\cos(\theta - \delta)$ (B) $K_1 I^2 - K_2 V^2$
(C) $\frac{V}{I} = \sqrt{\frac{K_1}{K_2}}$ (D) None of these

11. The relay which is operated depending on the ratio of applied voltage to current is
(A) Distance relay
(B) Impedance relay
(C) Induction type overcurrent relay
(D) Both (B) and (A)

12. The shunt resistance connected across the breaker contacts
(1) Modifies the oscillatory restriking voltage to a periodic wave.
(2) Should have a value of $\frac{1}{2}\sqrt{\frac{L}{C}}$ for critical damping.
(3) RRRV will be increased
(4) Results in increase in rupturing capacity
Which of the statements is correct?
(A) 1, 2 (B) 1, 2 and 3
(C) 1, 2 and 4 (D) 1, 2, 3 and 4

13. Maximum value of restriking voltage for a 132 kV system with 3 Ω reactance is
(A) 126.5 V (B) 180.5 kV
(C) 215.55 kV (D) 208.56 kV

14. Fault current level at 22 kV side is 2600 A; CT ratio at 22 kV side is 200:1 and 132 kV side is 100:1. Two relays shown in the figure are set at 125% plug setting. The time of operation of relay R_1 when the time grading margin is 0.6 s and TMs for relay 1150.15 is



Given: relay characteristic curve

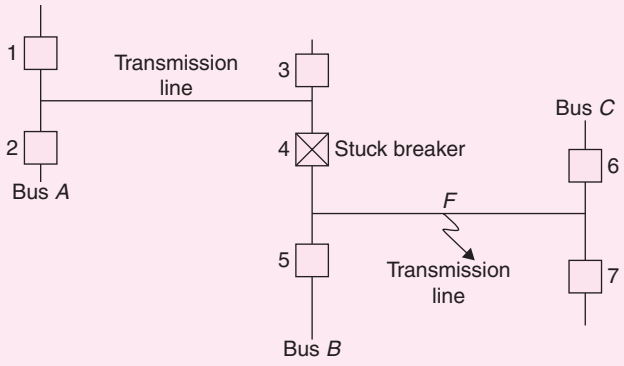
PSM:	1.3	2	4	8	10	20
TSM:	30	10	5	3.3	3	2.2

(A) 0.68 s (B) 0.25 s
(C) 1.8 s (D) 0.39 s

15. Which of the following are correct?
(1) Relay time is the interval between the occurrence of the fault and up to the activated signal given to circuit breaker.
(2) Time interval between closure of trip circuit and final arc extinction is called breaker time.
(3) Time elapsed between the instant of occurrence of fault and final arc extinction is called fault clearing time.
(4) Time during which stored energy is dissipated after the characteristic quantity has been suddenly restored is reset time
(A) 1, 2
(B) 1, 2 and 3
(C) 1 and 3
(D) 1, 2 and 4

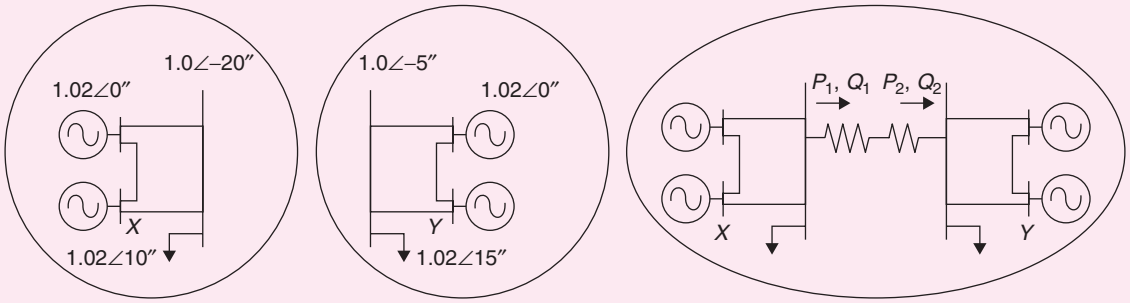
Previous Years' Questions

1. Consider the protection system shown in the figure below. The circuit breakers, numbered from 1 to 7 are of identical type. A single line to ground fault with zero fault impedance occurs at the midpoint of the line (at point F), but circuit breaker 4 fails to operate ('stuck breaker'). If the relays are coordinated correctly, a valid sequence of circuit breaker operations is **[2007]**



(A) 1, 2, 6, 7, 3, 5
(B) 1, 2, 5, 6, 7, 3
(C) 5, 6, 7, 3, 1, 2
(D) 5, 1, 2, 3, 6, 7

2. Consider the two power systems shown in figure A below, which are initially not interconnected, and are operating in steady state at the same frequency. Separate load flow solutions are computed individually for the two systems, corresponding to the scenario. The bus voltage phasors so obtained are indicated on figure A. These two isolated systems are now interconnected by a short transmission line as shown in figure B, and it is found that $P_1 = P_2 = Q_1 = Q_2 = 0$: **[2007]**



The bus voltage phase angular difference between generator bus X and generator bus Y after the interconnection is

(A) 10° (B) 25° (C) –30° (D) 30°

3. A lossless power system has to serve a load of 250 MW. There are two generators (G_1 and G_2) in the system with cost curves C_1 and C_2, respectively, defined as follows

$C_1(P_{G1}) = P_{G1} + 0.055 \times P_{G1}^2$

$C_2(P_{G2}) = 3P_{G2} + 0.03 \times P_{G2}^2$

where P_{G1} and P_{G2} are the MW injections from generator G_1 and G_2, respectively. Then the minimum cost dispatch will be **[2008]**

(A) $P_{G1} = 250$ MW; $P_{G2} = 0$ MW
(B) $P_{G1} = 150$ MW; $P_{G2} = 100$ MW
(C) $P_{G1} = 100$ MW; $P_{G2} = 150$ MW
(D) $P_{G1} = 0$ MW; $P_{G2} = 250$ MW

4. For the Y-bus matrix of a 4-bus system given in per unit, the buses having shunt elements are $Y_{BUS} =$

$$j\begin{bmatrix} -5 & 2 & 2.5 & 0 \\ 2 & -10 & 2.5 & 4 \\ 2.5 & 2.5 & -9 & 4 \\ 0 & 4 & 4 & -8 \end{bmatrix}$$ **[2009]**

(A) 3 and 4 (B) 2 and 3
(C) 1 and 2 (D) 1,2 and 4

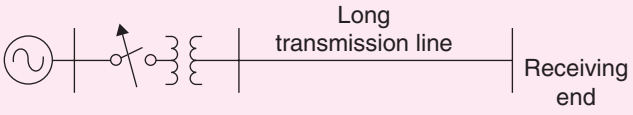
5. A 500 MW, 21 kV, 50 Hz, three-phase, 2-pole synchronous generator having a rated p.f. = 0.9, has a moment of inertia of 27.5×10^3 kg-m^2. The inertia constant (H) will be **[2009]**

(A) 2.44 s (B) 2.71 s
(C) 4.88 s (D) 5.42 s

6. Voltage phasors at the two terminals of a transmission line of length 70 km have a magnitude of 1.0 per unit but are 180 degrees out of phase. Assuming that the maximum load current in the line is $\frac{1}{5}$ th of minimum three-phase fault current, which one of the following transmission line protection schemes will NOT pick up for this condition? **[2008]**

(A) Distance protection using mho relays with zone-1 set to 80% of the line impedance
(B) Directional overcurrent protection set to pick up at 1.25 times the maximum load current
(C) Pilot relaying system with directional comparison scheme
(D) Pilot relaying system with segregated phase comparison scheme

7. A 50 Hz synchronous generator is initially connected to a long lossless transmission line which is open-circuited at the receiving end. With the field voltage held constant, the generator is disconnected from the transmission line. Which of the following may be said about the steady state terminal voltage and field current of the generator? **[2010]**



(A) The magnitude of terminal voltage decreases, and the field current does not change.
(B) The magnitude of terminal voltage increases, and the field current does not change.
(C) The magnitude of terminal voltage increases, and the field current increases.
(D) The magnitude of terminal voltage does not change, and the field current decreases.

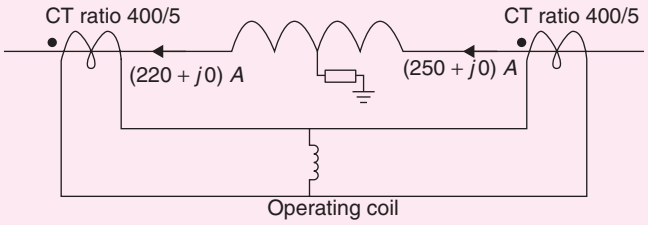
8. Consider two buses connected by an impedance of $(0 + j5)\ \Omega$. The bus 1 voltage is $100\angle 30°$ V, and bus 2 voltage is $100\angle 0°$ V. The real and reactive powers supplied by bus 1, respectively, are **[2010]**

(A) 1000 W, 268 VAr (B) –1000 W, –134 Var
(C) 276.9 W, –56.7 Var (D) –276.9 W, 56.7 Var

9. A three-phase, 33 kV oil-circuit breaker is rated 1200 A, 2000 MVA, 3 s. The symmetrical breaking current is **[2010]**

(A) 1200 A (B) 3600 A
(C) 35 kA (D) 104.8 kA

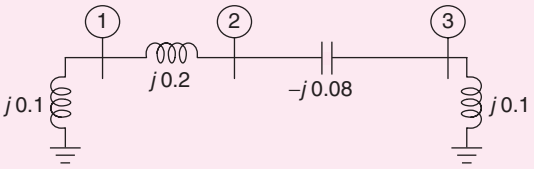
10. Consider a stator winding of an alternator with an internal high-resistance ground fault. The currents under the fault condition are as shown in the figure. The winding is protected using a differential current scheme with current transformers of ratio 400/5 A as shown. The current through the operating coil is **[2010]**



(A) 0.17875 A (B) 0.2 A
(C) 0.375 A (D) 60 KA

11. A negative sequence relay is commonly used to protect **[2011]**
(A) An alternator (B) A transformer
(C) A transmission line (D) A busbar

12. A three-bus network is shown in the figure below indicating the p.u. impedances of each element. **[2011]**



The bus admittance matrix, *Y*-bus of the network is

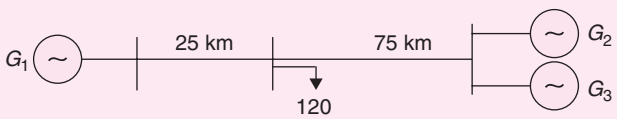
(A) $j\begin{bmatrix} 0.3 & 0.2 & 0 \\ -0.2 & 0.12 & 0.08 \\ 0 & 0.08 & 0.02 \end{bmatrix}$

(B) $j\begin{bmatrix} -15 & 5 & 0 \\ 5 & 7.5 & -12.5 \\ 0 & 12.5 & 2.5 \end{bmatrix}$

(C) $j\begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.12 & -0.08 \\ 0 & -0.08 & 0.10 \end{bmatrix}$

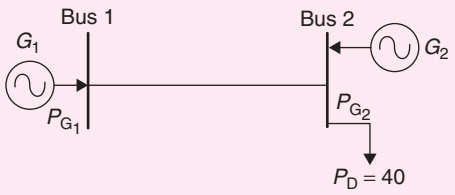
(D) $j\begin{bmatrix} 10 & 5 & 0 \\ 5 & 7.5 & 12.5 \\ 0 & 12.5 & -10 \end{bmatrix}$

13. A load centre of 120 MW derives power from two power stations connected by 220 kV transmission lines of 25 km and 75 km as shown in the figure below. The three generators G_1, G_2 and G_3 are of 100 MW capacity each and have identical fuel cost characteristics. The minimum loss generation schedule for supplying the 120 MW load is **[2011]**



(A) $P_1 = 80$ MW + losses
$P_2 = 20$ MW
$P_3 = 20$ MW
(B) $P_1 = 60$ MW
$P_2 = 30$ MW + losses
$P_3 = 30$ MW
(C) $P_1 = 40$ MW
$P_2 = 40$ MW
$P_3 = 40$ MW + losses
(D) $P_1 = 30$ MW + losses
$P_2 = 45$ MW
$P_3 = 45$ MW

14. The figure shows a two-generator system supplying a load of $P_D = 40$ MW, connected at bus 2.



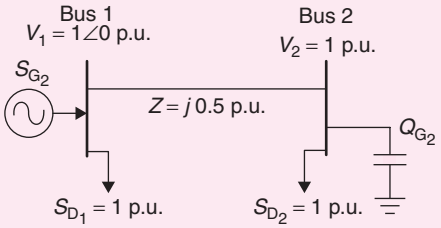
The fuel cost of generators G_1 and G_2 are:
$C_1(P_{G1}) = 10{,}000$ Rs/MWh and $C_2(P_{G2}) = 12{,}500$ Rs/MWh and the loss in the line is $P_{loss(p.u.)} = 0.5\ P^2_{G1(p.u.)}$, where the loss coefficient is specified in p.u. on a 100 MVA base. The most economic power generation schedule in MW is **[2012]**
(A) $P_{G1} = 20, P_{G2} = 22$
(B) $P_{G1} = 22, P_{G2} = 20$
(C) $P_{G1} = 20, P_{G2} = 20$
(D) $P_{G1} = 0, P_{G2} = 40$

15. The bus admittance matrix of a three-bus-line system is

$$Y = j\begin{bmatrix} -13 & 10 & 5 \\ 10 & -18 & 10 \\ 5 & 10 & -13 \end{bmatrix}$$

If each transmission line between the two buses is represented by an equivalent π-network, the magnitude of the shunt susceptance of the line connecting bus 1 and 2 is **[2012]**
(A) 4 (B) 2
(C) 1 (D) 0

16. For the system shown below, S_{D1} and S_{D2} are complex power demands at bus 1 and bus 2, respectively. If $|V_2| = 1$ p.u., the VAR rating of the capacitor (Q_{G2}) connected at bus 2 is **[2012]**



(A) 0.2 p.u. (B) 0.268 p.u.
(C) 0.312 p.u. (D) 0.4 p.u.

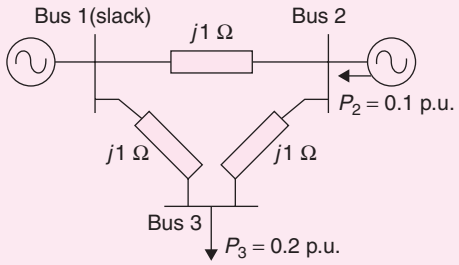
17. For a power system network with n nodes, Z_{33} of its bus impedance matrix is $j0.5$ per unit. The voltage at node 3 is 1.3 $\angle -10^{o}$ per unit. If a capacitor having reactance of $-j3.5$ per unit is now added to the network between node 3 and the reference node, the current drawn by the capacitor per unit is **[2013]**

(A) 0.325 $\angle -100^{o}$ (B) 0.325$\angle 80^{o}$

(C) 0.371 $\angle -100^{o}$ (D) 0.433$\angle 80^{o}$

Common Data for Questions 18 and 19:

In the following network, the voltage magnitudes at all buses are equal to 1 p.u., the voltage phase angles are very small, and the line resistance are negligible. All the line reactances are equal to $j1\ \Omega$.



18. The voltage-phase angles in radians at buses 2 and 3 are **[2013]**

(A) $\theta_2 = -0.1,\ \theta_3 = -0.2$ (B) $\theta_2 = 0,\ \theta_3 = -0.1$

(C) $\theta_2 = 0.1,\ \theta_3 = 0.1$ (D) $\theta_2 = 0.1,\ \theta_3 = 0.2$

19. If the base impedance and the line-to-line base voltage are 100 Ω and 100 kV, respectively, then the real power in MW delivered by the generator connected at the slack bus is **[2013]**

(A) −10 (B) 0 (C) 10 (D) 20

20. The fuel cost functions of two power plants are

Plant P_1: $C_1 = 0.05\,P_{g_1}^2 + AP_{g1} + B$

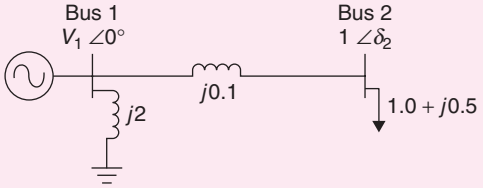
Plant P_2: $C_2 = 0.10\,P_{g_2}^2 + 3AP_{g2} + 2B$

where P_{g1} and P_{g2} are the generated powers of two plants, and A and B are the constants. If the two plants optimally share 1000 MW load at incremental fuel cost of 100 Rs/MWh, the ratio of load shared by plants P_1 and P_2 is **[2014]**

(A) 1:4 (B) 2:3

(C) 3:2 (D) 4:1

21. A two-bus power system shown in the figure supplies load of $1.0 + j0.5$ p.u.

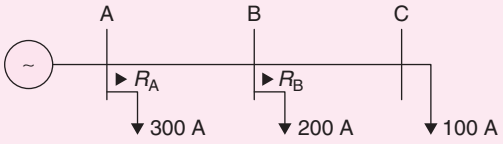


The values of V_1 in p.u. and δ_2, respectively, are **[2014]**

(A) 0.95 and 6.00° (B) 1.05 and −5.44°

(C) 1.1 and −6.00° (D) 1.1 and −27.12°

22. The overcurrent relays for the line protection and loads connected at the buses are shown in the figure.



The relays are IDMT in nature having the characteristic

$$t_{op} = \frac{0.14 \times \text{Time Multiplier Setting}}{\left(\text{Plug Setting Multiplier}\right)^{0.02} - 1}$$

The maximum and minimum fault currents at bus B are 2000 A and 500 A, respectively. Assuming the time multiplier setting and plug setting for relay R_B to be 0.1 and 5 A, respectively, the operating time of R_B (in seconds) is_____ **[2014]**

23. A 183-bus power system has 150 PQ buses and 32 PV buses. In the general case, to obtain the load flow solution using Newton–Raphson method in polar coordinates, the minimum number of simultaneous equations to be solved is _____. **[2014]**

24. Consider the economic dispatch problem for a power plant having two generating units. The fuel costs in Rs/MWh along with the generation limits for the two units are given below: **[2015]**

$C_1(P_1) = 0.01\,P_1^2 + 30P_1 + 10;\ 100\ \text{MW} \le P_1 \le 150$ MW

$C_2(P_2) = 0.05\,P_2^2 + 10P_2 + 10;\ 100\ \text{MW} \le P_2 \le 180$ mW

The incremental cost (in Rs/MWh) of the power plant when it supplies 200 MW is _____.

24. Determine the correctness or otherwise of the following Assertion [*a*] and the Reason [*r*].

Assertion: Fast decouples load flow method gives approximate load flow solution because it uses several assumptions.

Reason: Accuracy depends on the power mismatch vector tolerance. **[2015]**

(A) Both [*a*] and [*r*] are true and [*r*] is the correct reason for [*a*].

(B) Both [*a*] and [*r*] are true but [*r*] is not the correct reason for [*a*].

(C) Both [*a*] and [*r*] are false.

(D) [*a*] is false and [*r*] is true.

26. A 3-bus power system network consists of 3 transmission lines. The bus admittance matrix of the uncompensated system is

$$\begin{bmatrix} -j6 & j3 & j4 \\ j3 & -j7 & j5 \\ j4 & j5 & -j8 \end{bmatrix} \text{pu.}$$

If the shunt capacitance of all transmission lines is 50% compensated, the imaginary part of the 3rd row 3rd column element (in pu) of the bus admittance matrix after compensation is **[2015]**

(A) $-j7.0$ (B) $-j8.5$
(C) $-j7.5$ (D) $-j9.0$

27. The incremental costs (in Rupees/MWh) of operating two generating units are functions of their positive powers P_1 and P_2 in MW, and are given by

$$\frac{dC_1}{dP_1} = 0.2P_1 + 50$$

$$\frac{dC_2}{dP_2} = 0.24P_2 + 40$$

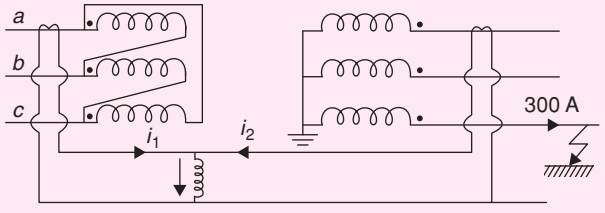
Where

$20 \text{ MW} \le P_1 \le 150 \text{ MW}$

$20 \text{ MW} \le P_2 \le 150 \text{ MW}.$

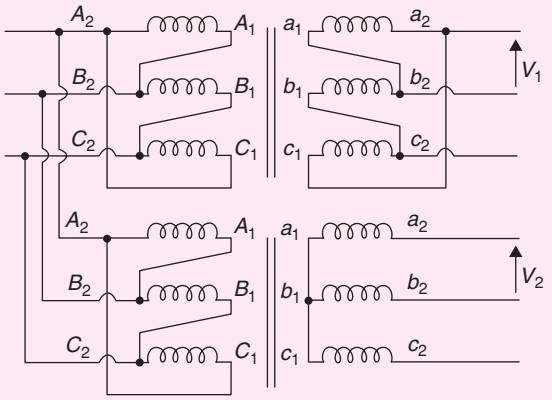
For a certain load demand, P_1 and P_2 have been chosen such that $dC_1/dP_1 = 76$ Rs/MWh and $dC_2/dP_2 = 68.8$ ₹/MWh. If the generations are rescheduled to minimize the total cost, then P_2 is _____. **[2015]**

28. A 3-kphase transformer rated for 33 kV/11 kV is connected in delta/star as shown in figure. The current transformers (CTs) on low and high voltage sides have a ratio of 500/5. Find the currents i_1 and i_2, if the fault current is 300 A as shown in figure. **[2015]**



(A) $i_1 = 1/\sqrt{3}$ A, $i_2 = 0$ A
(B) $i_1 = 0$ A, $i_2 = 0$ A
(C) $i_1 = 0$ A, $i_2 = 1/\sqrt{3}$ A
(D) $i_1 = 1\sqrt{3}$ A, $i_2 = 1/\sqrt{3}$ A

29. Two three-phase transformers are realized using single-phase transformers as shown in the figure.

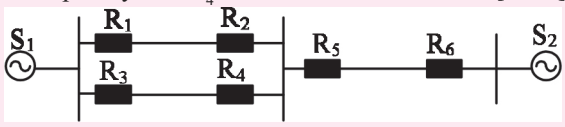


The phase difference (in degree) between voltages V_1 and V_2 is _____. **[2015]**

30. In a 100 bus power system, there are 10 generators. In a particular iteration of Newton Raphson load flow technique (in polar coordinates), two of the PV buses are converted to PQ type. In this iteration, **[2016]**

(A) the number of unknown voltage angles increases by two and the number of unknown voltage magnitudes increases by two.
(B) the number of unknown voltage angles remains unchanged and the number of unknown voltage magnitudes increases by two.
(C) the number of unknown voltage angles increases by two and the number of unknown voltage magnitudes decreases by two.
(D) the number of unknown voltage angles remains unchanged and the number of unknown voltage magnitudes decreases by two.

31. A power system with two generators is shown in the figure below. The system (generators, buses and transmission lines) is protected by six over current relays R_1 to R_6. Assuming a mix of directional and non directional relays at appropriate locations, the remote backup relays for R_4 are **[2016]**



(A) R_1, R_2 (B) R_2, R_6
(C) R_2, R_5 (D) R_1, R_6

32. A power system has 100 buses including 10 generator buses, for the load flow analysis using Newton – Raphson, method in polar coordinates, the size of the Jacobian is **[2016]**

(A) 189×189 (B) 100×100
(C) 90×90 (D) 180×180

Answer Keys

Exercises

Practice Problems 1

1. C	**2.** D	**3.** B	**4.** B	**5.** A	**6.** B	**7.** B	**8.** C	**9.** C	**10.** B
11. A	**12.** D	**13.** C	**14.** A	**15.** B	**16.** A	**17.** A	**18.** B	**19.** B	**20.** B
21. A	**22.** A								

Practice Problems 2

1. D	**2.** D	**3.** C	**4.** C	**5.** A	**6.** B	**7.** B	**8.** D	**9.** A	**10.** A
11. D	**12.** C	**13.** C	**14.** C	**15.** B					

Previous Years' Questions

1. C	**2.** A	**3.** C	**4.** C	**5.** A	**6.** A	**7.** B	**8.** A	**9.** C	**10.** C
11. A	**12.** B	**13.** A	**14.** A	**15.** B	**16.** B	**17.** D	**18.** B	**19.** C	**20.** D
21. B	**22.** 0.23	**23.** 332	**24.** 20	**25** D	**26.** B	**27.** 135 to 137		**28.** A	**29.** 30
30. B	**31.** D	**32.** A							

Chapter 4

Power Systems Stability

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Steady state stability
- The swing equation
- Multi-machine systems
- Steady state stability analysis
- Transient stability analysis
- Equal area criterion
- Principal type of transient disturbances
- Sudden change in mechanical input
- Switching operation
- 3-phase

Introduction

The stability of a system refers to the ability of a system to return back to its steady state when subjected to a disturbance. Power is generated by synchronous generators that operate in synchronism with the rest of the system. A generator is synchronized with a bus when both of them have same frequency, voltage and phase sequence. So, the stability is also defined as the ability of the power system to return to steady state without losing synchronism. The power system stability is categorized as follows:

1. Steady-state stability
2. Transient stability
3. Dynamic stability

Steady-state Stability

Steady-state stability is restricted to small and gradual changes in the system operating conditions. This stability basically concentrates on restricting the bus voltage close to their nominal values. Steady-state stability ensures that phase angle between two buses are not too large and check for the overloading of the power equipment and transmission lines.

Transient Stability

Transient stability involves study of the power system following a major disturbance. Following a large disturbance the synchronous alternator power angle changes due to sudden acceleration of the rotor shaft.

Dynamic Stability

The ability of a power system to maintain stability under continuous small disturbances is investigate under the name of dynamic stability. It is also known as small signal stability.

Power-angle Relationship

Consider the single machine infinite bus system shown in the figure. In this the reactance 'X' includes the reactance of the transmission line and the synchronous reactance or the transient reactance of the generator. The sending end voltage is then the internal emf of the generator. Let the sending end and receiving end voltages be given by

$$V_s = V_1 \angle \delta,\ V_R = V_2 \angle 0°$$

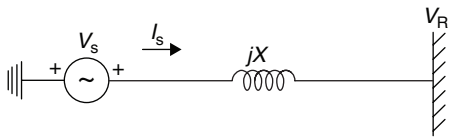


Figure 1 An SMIB system

We then have

$$I_s = \frac{V_1 \angle \delta - V_2}{jX} = \frac{V_1 \cos\delta - V_2 + jV_1 \sin\delta}{jX}$$

The sending end real power and reactive power are given by

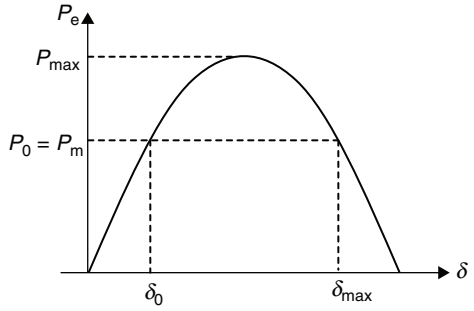
$$P_s + jQ_s = V_s I_s^* = V_1(\cos\delta + j\sin\delta)\frac{V_1\cos\delta - V_2 + jV_1\sin\delta}{jX}$$

$$P_s + jQ_s = \frac{V_1V_2\sin\delta + j\left(V_1^2 - V_1V_2\cos\delta\right)}{X}$$

Since the line is lossless, the real power dispatched from the sending end is equal to the real power received at the receiving end. We can therefore write.

$$P_e = P_s = P_R = \frac{V_1V_2}{X}\sin\delta = P_{max}\sin\delta$$

where $P_{max} = \frac{V_1V_2}{X}$ is the maximum power that can be transmitted over the transmission line. The power angle curve is shown in figure below



From the figure, we can see that for a given power 'P_0' there are two possible values of the angle $\delta \to \delta_0$ and δ_{max}.

The angles are given by

$$\delta_0 = \sin^{-1}\left(\frac{P_0}{P_{max}}\right)$$

$$\delta_{max} = 180° - \delta_0$$

The Swing Equation

Figure shows the torque, speed and flow of mechanical and electrical powers in a synchronous machine. It is assumed that the windage, friction and iron-loss are neglected. The differential equation governing the rotor dynamics can be written as

$$J\cdot\frac{d^2\theta_m}{dt^2} = T_m - T_e$$

where θ_m = Angle in rad (mechanical)

T_m = Turbine torque in Nm

T_e = Electromagnetic torque developed in Nm

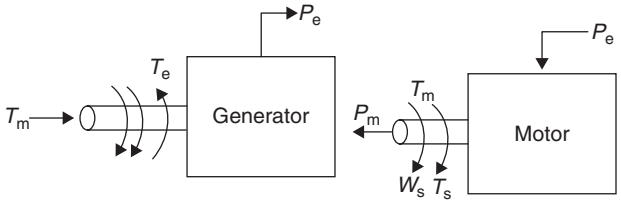


Figure 2 Flow of mechanical and electrical power in a synchronous machine

$$J\,\omega_{sm}\cdot\frac{d^2\theta_m}{dt^2} = P_m - P_e$$

ω_{sm} = Synchronous speed

P_m = Mechanical power input in MW

P_e = Electrical power output in MW

$$J\left(\frac{Z}{P}\right)^2\times\omega_s\cdot\frac{d^2\theta_e}{dt^2} = P_m - P_e$$

$$M\cdot\frac{d^2\theta_e}{dt^2} = P_m - P_e$$

where M = Moment of inertia $= J\cdot\left(\frac{2}{P}\right)^2\omega_s$

If the angular position of the rotor is with respect to the synchronous rotating reference frame

Torque angle $(\delta) = \theta_e - \omega_s t$

$$\frac{d^2\delta}{dt^2} = \frac{d^2\theta_e}{dt^2}$$

$$\therefore \quad M.\frac{d^2\delta}{dt^2} = P_m - P_e$$

$$\frac{GH}{\pi f}.\frac{d^2\delta}{dt^2} = P_m - P_e$$

$\frac{H}{\pi f}.\frac{d^2\delta}{dt^2} = P_m - P_e$ in p.u. of machine base ratings

$$M.\frac{d^2\delta}{dt^2} = P_m - P_e,$$

where $M_{p.u.} = \frac{H}{\pi f}$

The above equation is called the swing equation.

Multi-machine Systems

In a stability study of a power system with many synchronous machines only one MVA base common to all parts of the system can be chosen. This is accomplished by converting 'H' for each machine based on its own individual rating to a value determined by the system base

$$H_{system} = H_{mech}\cdot\frac{S_{mech}}{S_{system}}$$

Consider a power plant with two generators connected to the same bus which is electrically remote from the network disturbances. The swing equations on the common base are

$$\frac{2H_1}{\omega_s}.\frac{d^2\delta_1}{dt^2} = P_{m_1} - P_{e_1} \text{ per unit}$$

$$\frac{2H_2}{\omega_s}.\frac{d^2\delta_2}{dt^2} = P_{m_2} - P_{e_2} \text{ per unit}$$

If both the machines swing together, adding the two equations and denoting δ_1 and δ_2 by 'δ'

$$\frac{2H}{\omega_s}.\frac{d^2\delta}{dt^2} = P_m - P_e \text{ per unit}$$

where $H = (H_1 + H_2)$, $P_m = P_{m_1} + P_{m_2}$,

$$P_e = P_{e_1} + P_{e_2}$$

For any pair of non-coherent machines in a system, the resultant two machine swing equation can be written as

$$\frac{d^2\delta_1}{dt^2} - \frac{d^2\delta_2}{dt^2} = \frac{\omega_s}{2}\left(\frac{P_{m_1} - P_e}{H_1} - \frac{P_{m_2} - P_{e_2}}{H_2}\right)$$

$$\frac{2}{\omega_s}\left(\frac{H_1 H_2}{H_1 + H_2}\right)\frac{d^2(\delta_1 - \delta_2)}{dt^2} = \frac{P_{m_1} H_2 - P_{m_2} H_1}{H_1 + H_2} - \frac{P_{e_1} H_2 - P_{e_2} H_1}{H_1 + H_2}$$

which can be written as

$$\frac{2}{\omega_s}.H_{12}.\frac{d^2\delta_{12}}{dt^2} = P_{m_{12}} - P_{e_{12}}$$

where $\delta_{12} = \delta_1 - \delta_2$

$$H_{12} = \frac{H_1 H_2}{H_1 + H_2}$$

$$P_{m_{12}} = \frac{P_{m_1} H_2 - P_{m_2} H_1}{H_1 + H_2}$$

$$P_{e_{12}} = \frac{P_{e_1} H_2 - P_{e_2} H_1}{H_1 + H_2}$$

Steady-State Stability Analysis

Assumptions made in steady-state stability analysis:

1. The study considers small amplitude, long duration disturbances.
2. The damping term in the characteristicequation is absent, because of assumption of a loss less system and neglecting the effect of damper windings.
3. Non linearities are ignored, and hence the linearized form of the swing equation can be used.
4. The response of the governor and the exciter are ignored. This results in the mechanical power and electrical power to be constant throughout the transient period.

In a single machine infinite bus system, dynamics of a synchronous machine are described by the swing equation.

$$M.\frac{d^2\delta}{dt^2} = P_m - P_e$$

where $M.\frac{H}{\pi f}$ and $P_e = \frac{|E||V|}{X_{12}}\sin\delta = P_{max}\sin\delta$

Let the system be operating initially at equilibrium. Then the steady-state power transfer $P_{e_0} = P_m$ with losses neglected and a rotor angle of δ_0. Assume a small increment ΔP in the electric power output with the mechanical input P_m as fixed. This change causes the torque angle to change to $(\delta_0 + \Delta\delta)$. From the small disturbance, linearized equation can be written as

$$\Delta P_e = \left(\frac{\partial P_e}{\partial \delta_0}\right)\Delta\delta$$

Swing equation can be written as

$$M.\frac{d^2\Delta\delta}{dt^2} = P_m - (P_{e_0} + \Delta P_e) = -\Delta P_e$$

$$M\frac{d^2\Delta\delta}{dt^2} + \left[\frac{\partial P_e}{\partial \delta}\right]\Delta\delta = 0$$

Characteristic equation is given by

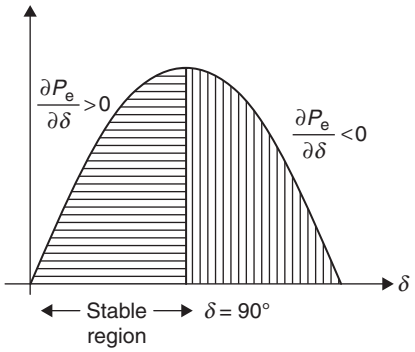
$$MD^2 + \left(\frac{\partial P_e}{\partial \delta_0}\right) = 0$$

Roots of the characteristic equation are

$$D = \pm\left[\frac{-\left(\frac{\partial P_e}{\partial \delta}\right)_0}{M}\right]^{\frac{1}{2}}$$

When $\frac{\partial P_e}{\partial \delta}$ is positive, the roots of the characteristic equation lie on the $j'\omega'$(imaginary) axis, and the system rotor angle behaviour is oscillating about 'δ_0'. The line resistance and damper winding of machine which have been ignored in the above modelling cause the system oscillations to decrease. The system is therefore be stable for a small increment in electrical power.

If $\frac{\partial P_e}{\partial \delta}$ is negative, the roots be real with one of them being positive and other negative but of equal magnitude which is 'unstable and causes rotor angle to increase without bound upon occurrence of a small power increment.



The term $\left(\frac{\partial P_e}{\partial \delta}\right)$ is known as 'synchronizing power coefficient'. This is also called stiffness of the synchronous machine.

The system is unstable if $\frac{\partial P_e}{\partial \delta} < 0$ which implies $\delta > 90°$, so the system is unstable if $\delta > 90°$. The maximum power

that can be transferred without loss of stability (steady state) occurs for $\delta = 90°$ and it is given by

$$P_{max} = \frac{EV}{X_{12}}$$

P_{max} is known as steady-state stability limit.

Methods for Improving Steady-state Stability Limit

1. Reduce the reactance of the transmission lines.
   - By adding parallel lines which also increases the reliability of the system.
   - Series capacitors which also provides better voltage regulation.
2. Increasing either or both $|E|$ and $|V|$ by means of excitation control.

Solved Examples

Example 1: A sending end bus transfer power P_0 through a transmission line of p.u. impedance 0.1 If the steady-state stability margin is 40% and the bus voltages at both the ends is 1.0 p.u. The operating power angle and magnitude of P_0 are

(A) 36.87°, 6 p.u. (B) 36.87°, 4 p.u.
(C) 23.57°, 0.6 p.u. (D) 23.57°, 0.4 p.u.

Solution: (A)
Steady-state stability margin

$$= \frac{P_{max} - P_0}{P_{max}} \times 100$$

$$40 = \left(1 - \frac{P_0}{P_{max}}\right) \times 100$$

$$0.4P_{max} - P_{max} = -P_0$$

$$P_0 = 0.6P_{max} = P_{max} \sin\delta$$

$$\delta = \sin^{-1}(0.6) = 36.87°$$

Magnitude of $P_0 = \frac{V_1 V_2}{X_{12}} \sin\delta_0$

$$= \frac{1}{0.1} \sin\delta_0 = 6 \text{ p.u.}$$

Example 2: The sending end and receiving end voltages of a transmission line at 100 MW load are equal at 230 kV, per phase line impedance is $(4 + j8)$. Steady-state stability limit of the system is

(A) 1090 MW/ph (B) 363.3 MW/ph
(C) 3,269.8 MW/ph (D) 1,887.8 MW/ph

Solution: (A)
When the line resistance is considered

$$P_{max} = \frac{V_S V_R}{|Z|} - \frac{R|V_R^2|}{|Z|^2}$$

where $V_S = V_R = \frac{230}{\sqrt{3}} = 132.8 \text{ kV}$

$$P_{max} = \frac{(132.8)^2}{\sqrt{4^2+8^2}} - \frac{4 \times (132.8)^2}{\left(\sqrt{8^2+4^2}\right)^2}$$

$$= 1971.74 - 881.8$$

$$= 1089.948 \cong 1090 \text{ MW/Ph}$$

Example 3: A 500 MVA, 11 kV, 50 Hz, 4-pole turbo-generator has an inertia constant of $H = 7.5$ MJ/MVA. If the mechanical power input is 552 MW and the electrical power output of 400 MW and stator copper loss is assumed to be negligible, then the angular acceleration is

(A) 182.4 rpm/s^2
(B) 364.8 rpm/s^2
(C) 25.33 rpm/s^2
(D) 50.66 rpm/s^2

Solution: (C)
From the swing equation

$$\frac{H}{\pi f} \cdot \frac{d^2\delta}{dt^2} = P_m - P_e \text{ in p.u.}$$

$$\frac{7.5}{180 \times 50} \frac{d^2\delta}{dt^2} = \frac{552 - 400}{500}$$

$$\frac{d^2\delta}{dt^2} = 364.8 \text{ elec degree/s}^2$$

For a four pole machine

$$\frac{d^2\delta}{dt^2} = \frac{364.8}{2} = 182.4 \text{ mech.dec/s}^2$$

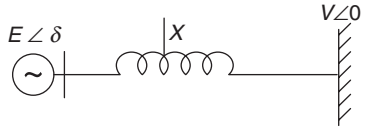
$$\text{Angular acceleration} = \frac{182.4}{360} \times 50$$

$$= 25.33 \text{ rpm/s}^2$$

Example 4: An alternator having a reactance of 1.3 p.u. is connected to an infinite bus as shown in the figure. It delivers 1.0 p.u. current at 0.8 p.f. lagging at $V = 1.0$ p.u. The steady-state stability limit of the system is

(A) 0.58 (B) 1.584
(C) 1.3 (D) 1.7

Solution: (B)
System configuration given in the problem is given by



For the given power factor of 0.8, current delivered from the source is $I = (0.8 - j0.6) = 1\angle -36.86$

Sending end voltage $E = V + jIX_d$

$$= 1.0 + j(0.8 - j0.6) \times 1.3$$

$$= 2.06 \angle 30.3$$

Steady-state stability limit $P_{max} = \frac{EV}{X_d} = \frac{2.06 \times 1}{1.3}$

$$P_{max} = 1.584 \text{ p.u.}$$

Example 5: A 250 MVA synchronous generator having inertia constant 5 MJ/MVA is connected in parallel with a 200 MVA generator having inertia constant of 4 MJ/MVA. Both the generators are swinging coherently. The equivalent inertia constant of the machine on a base of 100 MVA is

(A) 20.5 MJ/MVA (B) 12.5 MJ/MVA
(C) 8 MJ/MVA (D) 9 MJ/MVA

Solution: (A)

Both the inertia constants are to be represented with same base

$$H_{p.u.} = \frac{H_{p.u._{old}}}{P_{base_{new}}} \times P_{base_{new}}$$

$$H_{p.u.} = \frac{5 \times 250}{100} = 12.5 \text{ MJ/MVA}$$

$$H_{2_{p.u.}} = \frac{4 \times 200}{100} = 8 \text{ MJ/MVA}$$

$$H_{eq} = H_{1_{p.u.}} + H_{2_{p.u.}} = 12.5 + 8$$

$$= 20.5 \text{ MJ/MVA}$$

Example 6: A synchronous generator is connected to a 11 kV infinite bus through a transmission line. The reactances of the generator and transmission line are 1.1 and 0.7 , respectively. The terminal voltage of the synchronous generator is 15 kV. If the generator delivers 75 MW power to the infinitely bus, the load angle (δ) for stable operation is

(A) 55° (B) 125°
(C) Both A and B (D) 63.4°

Solution: (A)

Power angle equation is given by

$$P_e = \frac{|E||V|}{X_{12}} \sin\delta$$

$$75 = \frac{11 \times 15}{(1.1 + 0.7)} \sin\delta$$

$$\delta = 55°$$

Transient Stability Analysis

If the disturbance is large, changes in angular difference may be well enough to cause the loss of synchronism of the machines. The types of disturbances are mostly the faults, sudden. Large load changes may also cause instability of the machines. This type of instability is known as 'transient instability'.

Assumptions in Transient Stability Analysis

1. Resistance of the transmission lines and synchronous machine are neglected.
2. Synchronous machine damper winding is ignored which contributes damping term.
3. Rotor speed is assumed to be synchronous even though it varies insignificantly during the stability transients.
4. Mechanical input to machine is assumed to remain constant the transients.
5. Voltage behind transient reactance is assumed to remain constant.
6. Shunt capacitors ignored in stability study which does not cause any significant error.
7. Loads are modelled as constant admittances.

Equal Area Criterion

To examine the stability of a two machine system without solving the swing equation, a direct approach is given as follows.

Consider a swing equation.

$$H \frac{d^2\delta}{dt^2} = P_m - P_e = P_a = \text{Accelerating power}$$

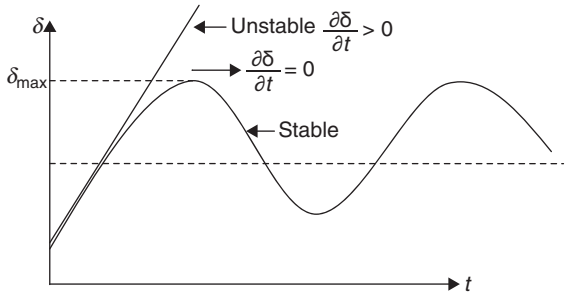
If the system is stable, $\delta(t)$ performs oscillations, whose aptitude decreases in actual practice because of damping terms (Which are neglected in the swing equation) on the other hand, if the system is unstable 'δ' continues to increase indefinitely with time and the machine loses synchronism. So when the system is stable, then $\delta(t)$ will go to maximum and start to reduce. This fact can be stated as stability criterion, that the system is stable if at some time

$$\frac{d\delta}{dt} = 0$$

and system is unstable if

$$\frac{\partial \delta}{\partial t} > 0 \text{ for sufficiently long time.}$$

The stability criterion stated above can be converted into a simple and easily applicable form for a single machine infinite bus system.



Multiply both sides of swing equation by $\left(2.\frac{d\delta}{dt}\right)$, we get

$$2.\frac{d\delta}{dt}.\frac{d^2\delta}{dt^2}=\frac{2P_a}{M}.\frac{d\delta}{dt}$$

Integrate on both sides,

$$\frac{d\delta^2}{dt}=\frac{2}{M}\int_{\delta_0}^{\delta}P_a.d\delta$$

$$\frac{d\delta}{dt}=\left(\frac{2}{M}\int_{\delta_0}^{\delta}P_a.d\delta\right)^{1/2}$$

where 'δ_0' is the initial rotor angle before it begins to swing due to disturbance.

The condition for stability can be written as

$$\left(\frac{2}{M}\int_{\delta_0}^{\delta}P_a.d\delta\right)^{1/2}=0$$

$$\Rightarrow \quad \int_{\delta_0}^{\delta}P_a.d\delta=0$$

The condition for stability is stated as:

- The system is stable if the area under accelerating power (P_a) – δ curve reduces to zero at some value of δ.

(or)

- Positive area under P_a – δ curve must equal the negative area and hence the name 'equal area' criterion of stability.

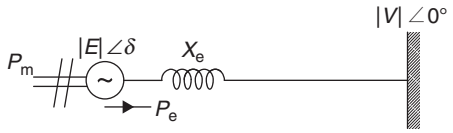
Principal Type of Transient Disturbances

For transient stability studies we have to consider the effect of

1. Sudden change in mechanical input
2. Switching operations
3. Fault with subsequent circuit isolation of the system

Sudden Change in Mechanical Input

Let us consider a synchronous machine connected to infinite bus as shown in the following figure.



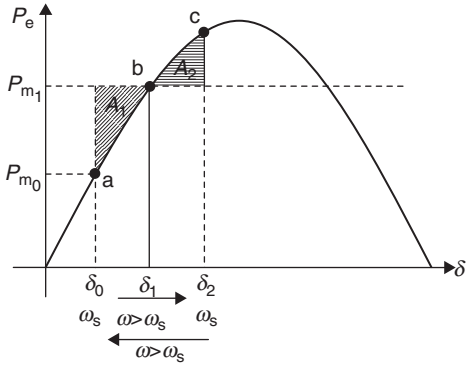
Electric power transmitted is given by

$$Pe=\frac{|E||V|}{X_e}\sin\delta=P_{max}\sin\delta$$

Under steady operating conditions

$$P_m=P_{eo}=P_{max}\sin\delta_0.$$

Let the mechanical input to the rotor be suddenly increased to P_{m1} by increasing steam input. The accelerating power $P_a=P_{m1}-P_e$ causes the rotor speed to increase and so does the rotor angle. At angle δ_1, $P_{m1}-P_e=P_a=0$ (At point b). but the rotor angle continues to increase as $\omega>\omega_s$. P_a now becomes negative, the rotor speed begins to reduce but the angle continues to increase till at angle δ_2, $\omega=\omega_s$ once again (at point 'c'), the decelerating area A_2 equals the accelerating area A_2 i.e., $\int_{\delta_0}^{\delta_2}P_a d\delta=0$.



As the oscillation decay out because of damping, the system settles to the new steady state when

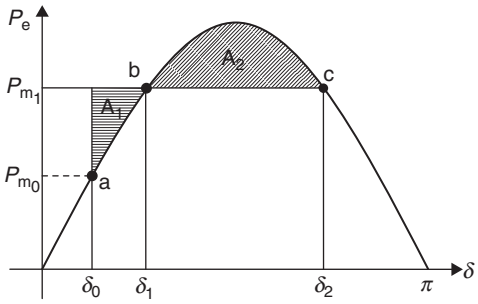
$$P_{m_1}=P_e=P_{max}\sin\delta_1,$$

From the power angle diagram, the expression for A_1 and A_2 can be written as.

$$A_1=\int_{\delta_0}^{\delta_1}\left(P_{m_1}-P_e\right)d\delta$$

$$A_2=\int_{\delta_1}^{\delta_2}\left(P_e-P_{m_1}\right)d\delta$$

For the system to be stable, it should be possible to find 'δ_2' such that $A_1=A_2$. As P_{m_1} increases, a limiting condition is finally reached when A_1 equals the area above the P_{m_1} line as shown in the figure below.

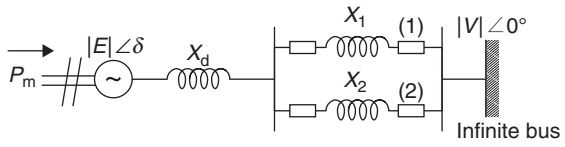


Under these conditions, δ_2 acquired the maximum value such that

$$\delta_2=\delta_{max}=\pi-\delta_1=\pi-\sin^{-1}\left(\frac{P_{m_1}}{P_{max}}\right)$$

Switching Operation

Consider a single machine connected to infinite bus through two parallel lines as shown in the figure.



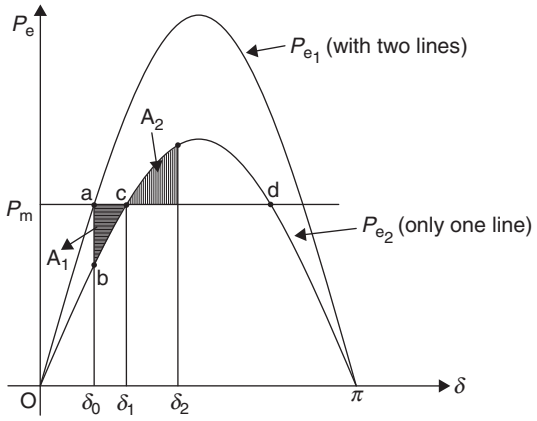
For stability analysis, consider the case when one of the lines is suddenly switched off with the system operating at a steady load. Before switching off the line, power transfer expression is given by

$$P_{e_1} = \frac{|E||V|}{X_d + (X_1 // X_2)} \cdot \sin\delta = P_{max} \sin\delta.$$

Immediately after the line '2' is switched off

$$P_{e_2} = \frac{|E||V|}{X_d + X_1} \sin\delta = P_{max_2} \sin\delta.$$

Since $(X_d + X_1) > (X_d + (X_1 // X_2)$, the relation between the maximum powers in two cases becomes $P_{max_2} < P_{max_1}$. The system is operating initially with a steady power transfer $P_e = P_m$ at a torque angle δ_0 on curve *I*.



Immediately after switching off line 2, operating point on curve 1 shifts to curve 2 (From point '*a*' to point '*b*') thereafter, operation of the system will be exactly same as sudden change in mechanical input case.

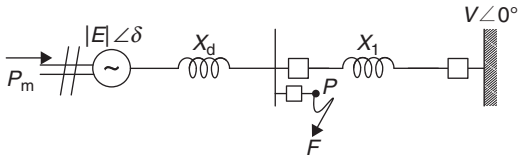
When an area A_2 corresponding to the decelerating energy is equal to an area A_1 corresponding to accelerating area, the system will be stable and finally operates at '*C*' corresponding to a new rotor angle $\delta_1 > \delta_0$. This is because a single line after larger reactance and larger rotor angle is needed to transfer the same steady power.

For the limiting case of stability, δ_1 has a maximum value given by

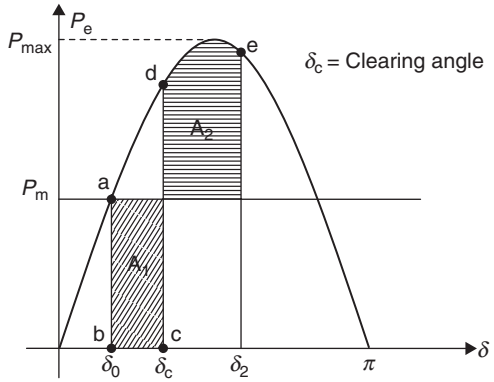
$$\delta_1 = \delta_{max} = \pi - \delta_c$$

Three-phase Fault on Radial Line

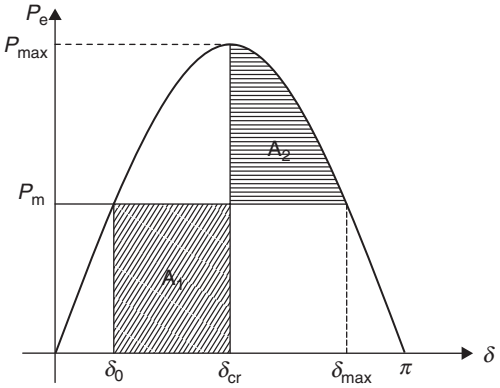
Consider a system operating in steady state and configured as shown in figure below.



If the fault is at point '*p*' on the radial line, the electrical output of the generator (power transferred to the load) will reduce to zero. (state point is '*b*') The rotor angle of the system starts increasing with the accelerating area and state point moves along path BC. If the fault is cleared by opening circuit breakers at time t_c corresponding to angle δ_c. Value of t_c and δ_c are known as clearing time and clearing angle, respectively. The system once again becomes healthy and transmits $P_0 = P_{max} \sin\delta$. The rotor now decelerates along de. If an angle δ_1 can be found such that $A_1 = A_2$, the system is found to be stable. The system finally settles down to the steady operating point '*a*' in an oscillatory manner because of inherent damping.



As the clearing of the fault line is delayed, A_1 increases and does δ_1 to find $A_2 = A_1$ till $\delta_1 = \delta_{max}$ as shown in figure below. For a clearing time larger than this value, the system would be unstable as $A_2 < A_1$ the maximum allowable value of clearing time angle for the system to remain stable are, respectively, known as critical clearing time and angle.



From the above figure $\delta_{max} = \pi - \delta_0$

For a system to be stable $A_1 = A_2$

$$\Rightarrow \quad \int_{\delta_o}^{\delta_{cr}} (P_m - 0)\, d\delta = \int_{\delta_{cr}}^{\delta_{max}} (P_{max} \sin\delta - P_m)\, d\delta$$

$$\Rightarrow \qquad \delta_{cr} = \cos^{-1}\left[(\pi - 2\delta_0)\sin\delta_0 - \cos\delta_0\right]$$

where δ_{cr} = critical clearing angle.

From the swing equation

$$\frac{d^2\delta}{dt^2} = \frac{\pi f}{H} P_m \left[\because P_e = 0\right]$$

Integrating twice, we get

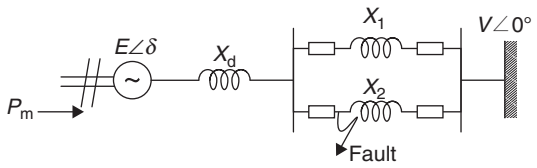
$$\delta_{cr} = \frac{\pi f}{2H} P_m t_{cr}^2 + \delta_0$$

$$t_{cr} = \sqrt{\frac{2H(\delta_{cr} - \delta_0)}{\pi f P_m}},$$

t_{cr} = Critical clearing time

Fault at One End of Parallel Lines

If fault occurs at one of the parallel lines connected between synchronous machine and infinite bus.



Before occurrence of a fault, power transfer expression is given by

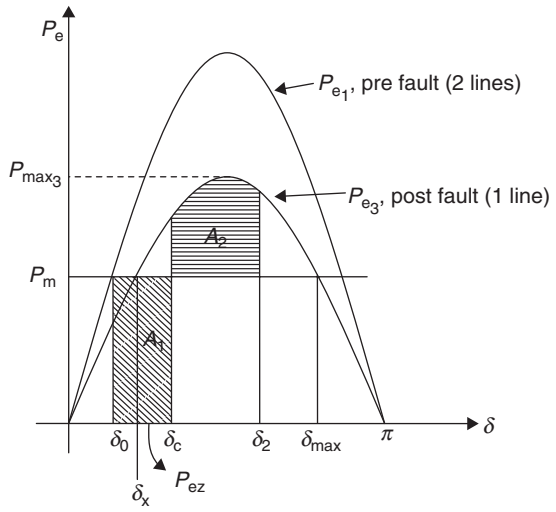
$$P_{e_1} = \frac{|E||V|}{X_d + (X_1 \parallel X_2)} \cdot \sin\delta = P_{max_1} \sin\delta$$

During the period of fault, $P_{e_2} = 0$

After fault is cleared,

$$P_{e_3} = \frac{|E||V|}{X_d + X_1} \sin\delta = P_{max\,3} \sin\delta.$$

Since $[X_d + (X_1//X_2)] < [X_d + X_1,]$ the relation between the maximum power transfer in both the cases is related as $P_{max_3} < P_{max_1}$



For stable operation

$$\delta_{max} = \pi - \delta_x = \pi - \sin^{-1}\left(\frac{P_m}{P_{max_3}}\right)$$

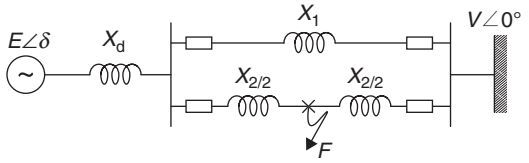
$$A_1 = A_2$$

$$\int_{\delta_0}^{\delta_2} (P_m - 0)\,d\delta = \int_{\delta_C}^{\delta_2} \left(P_{max_3}\sin\delta - P_m\right)\sin\delta$$

For stable operation of the system, clearing time must be less than critical clearing time at which δ_2 is equal to δ_{max}.

Fault Away From One of the Parallel Lines

When the fault occurs away from line end, there is some power flow during the fault though considerably reduces, as different from previous case, where $P_e = 0$. Circuit model of the system during fault is shown below.



Power angle curve before fault

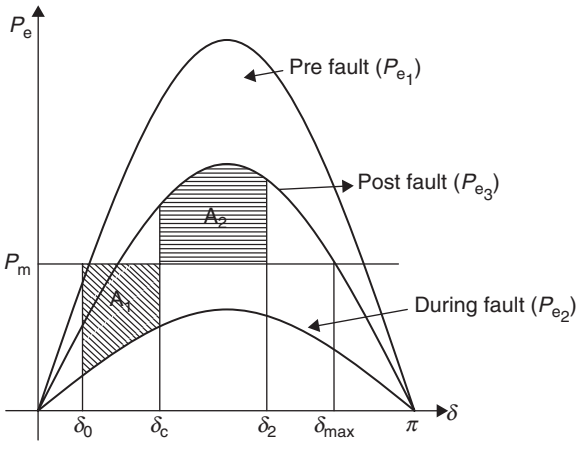
$$P_{e_1} = \frac{|E||V|}{X_d + (X_1/X_2)} \sin\delta$$

Power angle curve during fault

$$P_{e_2} = \frac{|E||V|}{X_F} \sin\delta.$$

If the fault is cleared by removing faulted line after fault is cleared,

$$P_{e_3} = \frac{|E||V|}{X_d + X_1} \sin\delta$$



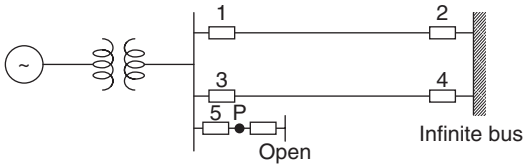
Critical clearing angle

$$\cos\delta_{cr} = \frac{\frac{\pi}{180} P_m (\delta_{max} - \delta_0) - P_{max_2}\cos\delta_0 + P_{max_3}\cos\delta_{max}}{P_{max_3} - P_{max_2}}$$

Design methods for improving transient stability

1. Improved steady-state stability; achieved by:
   (a) Higher system voltage
   (b) Additional transmission lines
   (c) Smaller reactances in transmission lines and transformers
   (d) Series capacitive compensation
   (e) Using FACTS
2. High-speed fault clearing
3. High-speed reclosure of circuit breakers
4. Larger machine inertia
5. Fast valving

Example 7: For single-line diagram of three-phase power system given below, the generator is delivering 1.0 p.u. power to infinite bus. The pre fault power angle equation is $P_e = 2.1 \sin\delta$.



Calculate the critical clearing angle and critical clearing time when the system is subjected to a 3-F fault at point 'P'. The fault is cleared by breaker 5. The inertia constant $H = 50$ MJ/MVA.

Solution: Before the fault occurrence

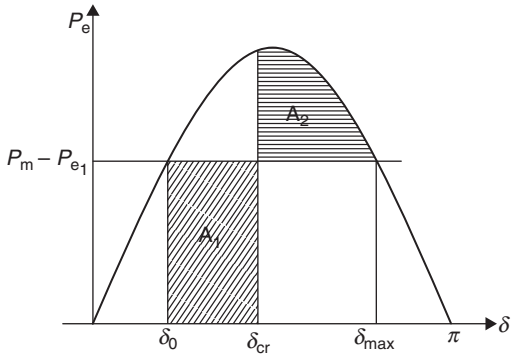
$$P_e = P_{max} \sin\delta = 2.1 \sin\delta$$

Power transferred $P_e = P_m = 1 = 2.1 \sin\delta$

$$\delta_0 = 28.43°$$

$$\therefore \quad \delta_{max} = \pi - \delta_0 = 151.56°$$

When the fault occurs, power angle curve is as shown in the figure.



For the system to be stable at critical clearing angle,

$$A_1 = A_2$$

$$\Rightarrow \quad \int_{\delta_o}^{\delta_{cr}} P_m . d\delta = \int_{\delta_{cr}}^{\delta_{max}} (P_{max} \sin\delta - P_m) d\delta$$

$$\int_{\delta_o}^{\delta_{cr}} P_m . d\delta = \int_{\delta_{cr}}^{\delta_{max}} (2.1 \sin\delta - P_m) d\delta$$

$$P_m(\delta_{cr} - \delta_0) = 2.1(\cos\delta_{cr} - \cos\delta_{max}) - P_m(\delta_{max} - \delta_{cr})$$

$$\Rightarrow \quad \cos\delta_{cr} = 0.144$$

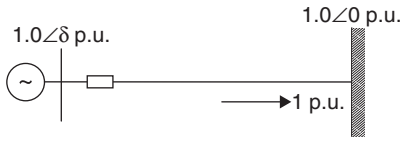
Critical clearing $\delta_{cr} = 81.72°$

$$t_{cr} = \left[\frac{2H(\delta_{cr} - \delta_0)}{\pi f P_m}\right]^{\frac{1}{4}} \quad t_{cr} = 0.243 \text{ sec}$$

Example 8: The transient stability of the power system can be effectively improved by
(A) Excitation improved
(B) Phase-shifting transformer
(C) Single-pole switching of circuit breakers
(D) Increase the turbine value opening

Solution: (C)

Example 9: A loss less single machine infinite bus power systems is shown below.



The synchronous generator transfers 1.0 p.u. of power to the infinite bus. The critical clearing time of the circuit breaker is 0.4 sec. If another identical generator is connected in parallel to the existing generator and each generator is scheduled to supply 0.5 p.u. of power, then the critical clearing time of the circuit breaker will be
(A) Reduce to 0.2 s
(B) Reduce but will be more than 0.2 s
(C) Remains constant at 0.4 s
(D) Increase beyond 0.4 s

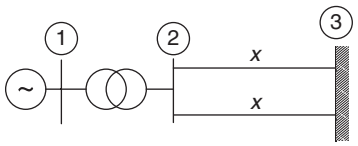
Solution: (D)

Critical clearing time $t_{cr} = \sqrt{\frac{2H}{\pi f}\frac{(\delta_{cr} - \delta_0)}{P_i}}$

$$\therefore \quad t_{cr} \propto \sqrt{H}$$

When two identical generators are added in parallel, resultant inertia will be $2H$. Then critical clearing time will increase.

Example 10: A generator with constant 1.0 p.u. terminal voltage supplies power through a step-up transformer of 0.14 p.u. reactance and a double circuit line to an infinite bus bar as shown in the figure. The infinite bus voltage is maintained at 1.0 p.u. Neglecting the resistance and susceptance of the system, the steady-state stability power limit of the system is 6.5 p.u. If one of the double-circuit is tripped, then resulting steady-state stability power limit in p.u. will be



(A) 6.5 p.u.
(B) 3.25 p.u.
(C) 5.96 p.u.
(D) 2.98 p.u.

Solution: (C)

$$\text{Steady-state stability limit} = \frac{V_1 V_2}{X_{12}} = 6.5$$

$$\frac{1.0 \times 1.0}{\left(0.14 + \frac{X}{2}\right)} = 6.5$$

$$X = 2\left(\frac{1}{6.5} - 0.14\right)$$

$$\Rightarrow \quad X = 0.0277.$$

$$\text{New steady-state stability limit} = \frac{V_1 V_2}{0.14 + X}$$

$$= \frac{1}{0.14 + 0.0277} = \frac{1}{0.1677} = 5.96 \text{ p.u.}$$

Example 11: A synchronous generator delivers 0.5 p.u. power in the steady state to an infinite bus through a transmission line of reactance 0.5 p.u. The generator no load voltage is 1.5 p.u. and infinite bus voltage is 1 p.u. The inertial constant of the generator is 5 MW-s/MVA and the generator reactance is 1 p.u. The critical clearing angle, in degrees, for a 3-Ø dead short-circuit fault at the generator terminal is

(A) 79.4° (B) 30°
(C) 150° (D) 100.6°

Solution: (A)

Power transferred before fault

$$P_e = \frac{V_1 V_2}{X_{12}} \sin\delta = 0.5 \text{ p.u.}$$

Maximum power can be transferred

$$= \frac{V_1 V_2}{X_{12}} = \frac{1 \times 1.5}{1 + 0.5} = 1 \text{ p.u.}$$

$$\sin\delta_0 = \frac{0.5}{1} = 0.5$$

$$\Rightarrow \quad \delta_0 = \sin^{-1}(0.5) = 30°$$

$$\delta_{max} = \pi - 30° = 150°.$$

Critical clearing angle (δ_c)

$$= \cos^{-1}\left[\frac{P_e(\delta_{max} - \delta_0) + P_{m_3}\cos\delta_{max}}{P_{m_3}}\right]$$

$$= 79.45°$$

EXERCISES

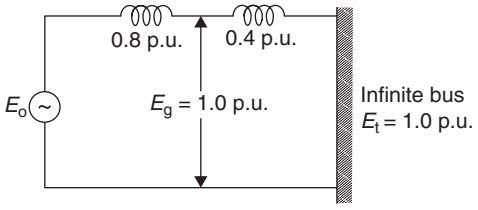
Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. A 60 MW, 11 kV, 0.85 lag p.f. water wheel generator has an inertia constant of 4 MJ/MVA. The energy stored in the rotor at synchronous speed is
(A) 36.5 MJ
(B) 180 kJ
(C) 282.3 MJ
(D) 282.3 kJ

2. Which of the following statement is true
(A) Steady-state stability limit can be increased by increasing its reactance
(B) Steady-state stability limit is equal to transient stability limit
(C) Steady-state stability of a power system is improved by increasing generator inertia
(D) Steady-state stability can be improved by using double circuit line instead of single circuit line

3. A generator of equivalent reactance of 0.8 p.u. is connected to an infinite bus through a series reactance of 0.4 p.u. The terminal voltage of the generator (E_g) is 1.0 p.u. and voltage of infinite bus is 1.0 p.u. The steady-state stability limit is



(A) 2.17 p.u. (B) 1.6 p.u.
(C) 3.98 p.u. (D) 1.0 p.u.

4. If the mechanical input of a 4-pole, 50 Hz, 20 MVA turbo generator is suddenly raised to 75 MW for an electrical load of 30 MW, the rotor acceleration is (Neglect mechanical and electrical losses and take Inertia constant as 6.0 MJ/MVA)
(A) 4.82 electrical degrees/s^2
(B) 337.5 electrical degrees/s^2
(C) 3375 electrical degrees/s^2
(D) 280 electrical/rad/s^2

5. A 60 MW, 0.85 lag p.f. synchronous generator operates on full load at a frequency of 50 Hz. The load is suddenly reduced to 30 MW. The steam valve begins to close after 0.3 s due to time lag in governor system. The change in frequency that occurs in this time is [Given $H = 4$ MJ/MVA]
(A) 50.79 Hz (B) 0.79 Hz
(C) 0.69 Hz (D) 0.59 Hz

6. To an infinite bus operating at a voltage of 1 p.u., a 50 Hz generator with a synchronous reactance of 1.2 p.u. is connected. The generator no load voltage is 1.0 p.u. and the inertia constant is 4 MJ/MVA. The frequency of the resulting natural oscillations of the generator when it is suddenly loaded to 60 per cent of its maximum power limit is
(A) 1.03 Hz (B) 2.89 Hz
(C) 60.42 Hz (D) 50 Hz

7. A 100 MVA synchronous machine has $H_1 = 4.2$ MJ/MVA and a 1200 MVA machine has $H_2 = 3.2$ MJ/MVA. The two machines operate in parallel. The equivalent H constant for the two in 200 MVA base is
(A) 2.4 MJ/MVA (B) 11.6 MJ/MVA
(C) 21.3 MJ/MVA (D) None of these

8. A 50 Hz, 4-pole turbo generator is rated 200 MVA, 2.2 kV and has an inertia constant of 7.5. Assume the generator is synchronized with a large power system and has a zero accelerating power while delivering a power of 450 MW. Suddenly its input power is changed to 475 MW. The speed of generator in rpm at the end of a period of 10 cycles is
(A) 1400 pm (B) 1500 rpm
(C) 1600 rpm (D) 1508.178 rpm

9. For an HVDC valve, the value of output voltage, when the firing angle is 30° and maximum value of line voltage is 200 V is
(A) 233.83 V (B) 150 V
(C) 100 V (D) 220 V

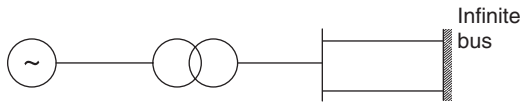
10. The maximum additional load that can be suddenly applied on a transmission line inter connector carrying 100 MW if the power angle diagram is given by $P = 120 \sin \alpha$
$P \rightarrow$ Power transmitted in MW
$\alpha \Rightarrow$ Displacement between voltage phasors at the two ends
(A) 4.87 MW (B) 3.87 MW
(C) 2.87 MW (D) 1.87 MW

11. It is preferred to transmit bulk power over long distances using high-voltage DC system. This is due to
(A) Reduced harmonics
(B) Protection system being simple
(C) Low cost of HVDC terminals
(D) Minimum line power losses

12. A 250 MVA, 50 Hz 3-ϕ turbo alternator produces power at 11 kV. The alternator is Y-connected and the neutral is solidly grounded. The sequence reactance are $X_0 = 0.05$ p.u., $X_1 = 0.2$ p.u. $X_2 = 0.2$ p.u. The system is running on no load at rated voltage. The magnitude of positive sequence line current for a single line to ground fault at generator terminal would be
(A) 2.5 p.u. (B) 2.22 p.u.
(C) 5 p.u. (D) 4.44 p.u.

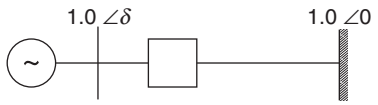
13. A 400 kV transmission line has a maximum power transfer capability P_{max} at 400 kV. If the line voltage is increased to 800 kV, with series reactance unchanged, the maximum power transfer capability is approximately
(A) $4P$ (B) $2P$ (C) P (D) $P/4$

14. An alternator with terminal voltage of 1 p.u. supplies power through a transformer of 0.10 p.u. reactance and a double circuit line to an infinite bus bar. The infinite bus voltage is maintained at 1 p.u. Susceptances and resistance are neglected. The steady-state stability power limit of the system is 8 p.u. If one of the lines of the double circuit is tripped, the resulting the steady-state stability power limit will be



(A) 8 p.u. (B) 3.33 p.u.
(C) 6.67 p.u. (D) 4 p.u.

15. A single machine infinite bus power system is as shown



The alternator transfers 1 p.u. of power to infinite bus. The critical clearing time of circuit breaker is 0.4 s. If another alternator is connected in parallel to the existing alternator and each alternator is scheduled to supply 0.5 p.u. of power, the critical clearing time of breaker
(A) Reduces to 0.2 s
(B) Reduces but will be greater than 0.2 seconds
(C) Remains at 0.4 s
(D) Increases beyond 0.4 s

***Directions for questions 16 to 19*:** Consists of two statements: one is Assertion (A) and the other is Reason (R). You have to examine these two statements and select the answer using the code given below.
(A) Both A and R are individually correct and R is the correct explanation of A
(B) Both A and R are individually correct, but R is not the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

16. A: The most extensively used form of differential relay is the percentage differential or biased beam relay.
R: Percentage differential relay is more sensitive in comparison to differentially connected over current relay.

17. A: In comparison to the making capacity of a circuit breaker, breaking capacity is normally higher.
R: The breaking capacity of a circuit breaker is expressed as $\sqrt{3} \times V \times I \times 10^{-6}$ MVA, where V is the rated service. Voltage in volts and I is the RMS value of symmetrical breaking current in amperes.

18. A: Extinction of DC arc is much difficult than that of an AC arc.
R: In an AC circuit, current wave passes through zero point twice during each cycle but in dc circuit full current has to be broken.

19. A: It is not desirable to operate a high voltage 3-ϕ system of considerable capacitance with an isolated neutral.
R: There is no zero-sequence current in an isolated neutral power system.

Practice Problems 2

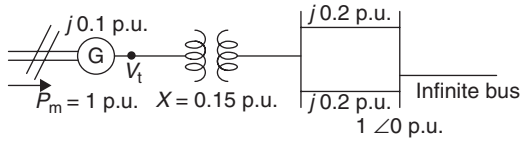
Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Swing equation relates
(A) Power with load angle
(B) Load angle and time
(C) Relative motion of rotor with respect to the stator field as a function of time
(D) Both (B) and (C)

2. A 4-pole 50 Hz, 40 MVA, 13.2 kV turbo alternator has an inertia constant of $H = 6$ kW sec/kVA. The kinetic energy stored in the rotor is
(A) 240 MJ (B) 380 kJ
(C) 220 MJ (D) 260 kJ

3. An alternator is connected to an infinite bus. It delivers 1.6 p.u. current at 0.9 p.f. lag and a voltage of 1.0 p.u. The reactance is 1.1 p.u. Keeping the active power fixed, the excitation is reduced. The critical excitation emf corresponding to operation at stability limit is
(A) 1.089 p.u. (B) 2.45 p.u.
(C) 1 p.u. (D) None of these

4. The maximum power that can be transferred from the generator connected to infinite bus is
[Given $V_c = 1.2$ p.u.]



(A) 1 p.u. (B) 5 p.u.
(C) 0.5 p.u. (D) 3 p.u.

5. For a 2-pole 50 Hz, 40 MVA turbo alternator, inertia constant is given by 4 MJ/MVA. The moment of inertia is
(A) 3.24 kg-m^2 (B) 0.5095 kg-m^2
(C) 286.5×10^3 kg-m^2 (D) 2×10^3 kg/m^3

6. The inertia constants of two groups of machines which do not swing together are M_1 and M_2. The equivalent inertial constant of the system is:
(A) $M_1 + M_2$ (B) $M_1 - M_2$ if $M_1 > M_2$
(C) $\dfrac{M_1 + M_2}{M_1 M_2}$ (D) $\dfrac{M_1 M_2}{M_1 + M_2}$

7. The inertia constant of a 500 MVA alternator is 1 p.u. The value corresponding to 1000 MVA is
(A) 2 p.u. (B) 1 p.u.
(C) 0.5 p.u. (D) 0.25 p.u.

8. Two groups of machines swinging together have their inertia constants M_1 and M_2. The inertia constant of the system is
(A) $M_1 + M_2$ (B) $\dfrac{M_1 + M_2}{M_1 M_2}$
(C) $M_1 - M_2$, $M_1 > M_2$ (D) $\dfrac{M_1 M_2}{M_1 + M_2}$

9. A high-voltage DC transmission system, reactive power is needed for rectifier at sending end and inverter at receiving end. During the operation of such a DC link,
(A) Inverter supplies leading reactive power and rectifier receives lagging reactive power.
(B) Inverter supplies lagging reactive power and rectifier receives leading reactive power.
(C) Inverter supplies lagging reactive power and the rectifier receives lagging reactive power.
(D) Inverter supplies leading reactive power and rectifier receives leading reactive power.

Data for Linked Questions

I. Question Nos.: 10 and 11

A 100 km, 3-phase, 50 Hz transmission line has the line constants $A = D = 0.853\angle 1.8°$, $B = 126.5\angle 50.4$, $C = 0.002\angle 90°$. The sending end voltage is 400 kV.

10. The receiving end voltage when the load is disconnected will be
(A) $2 + j60$ kV (B) $210\angle 90°$ kV
(C) $1.51\angle 0°$ kV (D) $270.7\angle -1.8°$ kV

11. The sending end current is
(A) $541.4\angle 88.2$ A (B) $541.4\angle 60$ A
(C) $17.006 + j541.13$ A (D) Both (A) and (C)

II. Question Nos.: 12 and 13

A 50 Hz, 4-pole turbo generator of rating 30 MVA, 13.2 kV has inertia constant of 8 kW sec/kVA.

12. Angular momentum of rotor in MJ-sec/electrical deg is

(A) $\frac{1}{150}$ (B) $\frac{4}{150}$ (C) $\frac{4}{6}$ (D) $\frac{40}{600}$

13. If the input rotational losses is 25,100 HP and the electric power developed is 10 MW, the acceleration developed will be

(A) 2.8 rad/s^2 (B) 5.72 rad/s^2
(C) 3.4 rad/s^2 (D) 4 rad/s^2

III. Question Nos.: 14 and 15

A 50 Hz, 4-pole 100 MVA, 11 kV turbo alternator has an inertia constant of 4.5 kW s/kVA. The input to the alternator is 134102 HP and the electrical power developed is 96 MW.

14. The angular momentum of the rotor is MJ-s/elect deg is

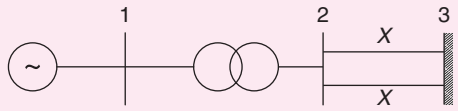
(A) $\frac{1}{50}$ (B) $\frac{1}{20}$ (C) $\frac{1}{32}$ (D) $\frac{1}{48}$

15. The acceleration of the rotor is

(A) 1.39 rad/s^2 (B) 0.39 rad/s^2
(C) 2.39 rad/s^2 (D) 3.39 rad/s^2

Previous Years' Questions

1. A generator with constant 1.0 p.u. terminal voltage supplies power through a step-up transformer of 0.12 p.u. reactance and a double-circuit line to an infinite bus bas as shown in Figure. The infinite bus voltage is maintained at 1.0 p.u. Neglecting the resistances and susceptances of the system, the steady-state stability power limit of the system is 6.25 p.u. If one of the double-circuit is tripped, the resulting steady-state stability power limit in p.u. will be **[2005]**



(A) 12.5 p.u. (B) 3.125 p.u.
(C) 10.0 p.u. (D) 5.0 p.u.

Common Data for Questions 2 and 3:

A generator feeds power to an infinite bus through a double circuit transmission line. A three-phase fault occurs at the middle point of one of the lines. The infinite bus voltage is 1 p.u., the transient internal voltage of the generator is 1.1 p.u. and the equivalent transfer admittance during fault is 0.8 p.u. The 100 MVA generator has an inertia constant of 5 MJ/MVA and it was delivering 1.0 p.u. power prior of the fault with rotor power angle of 30°. The system frequency is 50 Hz.

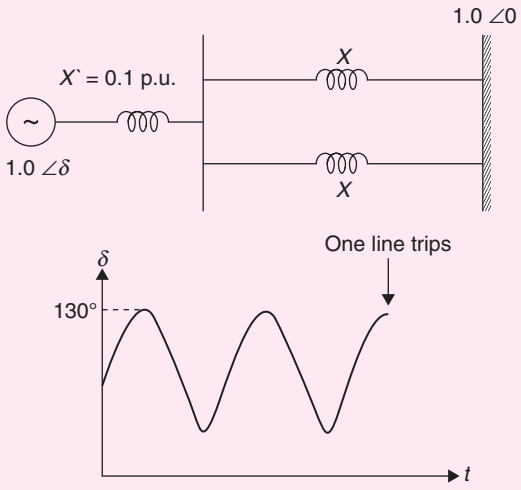
2. The initial accelerating power (in p.u.) will be **[2006]**

(A) 1.0 (B) 0.6 (C) 0.56 (D) 0.4

3. If the initial accelerating power is X p.u., the initial acceleration in elect deg/s^2, and the inertia constant in MJ-s/elect deg, respectively, will be **[2006]**

(A) 31.4X, 18 (B) 1800X, 0.056
(C) X/1800, 0.056 (D) X/31.4, 18

4. Consider a synchronous generator connected to an infinite bus by two identical parallel transmission lines. The transient reactance x of the generator is 0.1 p.u. and the mechanical power input to it is constant at 1.0 p.u. Due to some previous disturbance, the rotor angle (d) is undergoing an undamped oscillation, with the maximum value of $d(t)$ equal to 130°. One of the parallel lines trips due to relay mal-operation at an instant when $d(t)$ = 130° as shown in the figure. The maximum value of the per unit line reactance, x, such that the system does not lose synchronism subsequent to this tripping is **[2007]**



(A) 0.87 (B) 0.74 (C) 0.67 (D) 0.54

5. An isolated 50 Hz synchronous generator is rated at 15 MW which is also the maximum continuous power limit of its prime mover. It is equipped with a speed governor with 5% droop. Initially, the generator is feeding three loads of 4 MW each at 50 Hz. One of these loads is programmed to trip permanently if the frequency falls below 48 Hz. If an additional load of 3.5 MW is connected then the frequency will settle down to **[2007]**

(A) 49.417 Hz (B) 49.917 Hz
(C) 50.083 Hz (D) 50.583 Hz

6. A lossless single machine infinite bus power system is shown below



The synchronous generator transfers 1.0 per unit of power to the infinite bus. Critical clearing time of circuit breaker is 0.28 s. If another identical synchronous generator is connected in parallel to the existing generator and each generator is scheduled to supply 0.5 per unit of power. Then the critical clearing time of the circuit breaker will **[2008]**

(A) Reduce to 0.14 s
(B) Reduce but will be more than 0.14 s
(C) Remain constant at 0.28 s
(D) Increase beyond 0.28 s

7. A cylindrical rotor generator delivers 0.5 p.u. power in the steady state to an infinite bus through a transmission line of reactance 0.5 p.u. The generator no-load voltage is 1.5 p.u. and the infinite bus voltage is 1 p.u. The inertia constant of the generator is 5 MW-s/MVA and the generator reactance is 1 p.u. The critical clearing angle, in degrees, for a three-phase dead short-circuit fault at the generator terminal is **[2012]**

(A) 53.5 (B) 60.2 (C) 70.8 (D) 79.6

8. The angle δ in the swing equation of a synchronous generator is the **[2013]**

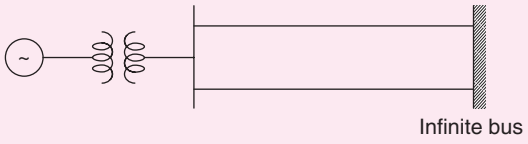
(A) Angle between stator voltage and current
(B) Angular displacement of the rotor with respect to the stator.
(C) Angular displacement of the stator mmf with respect to a synchronously rotating axis.
(D) Angular displacement of an axis fixed to the rotor with respect to a synchronously rotating axis.

9. A synchronous generator is connected to an infinite bus with excitation voltage $E_f = 1.3$ p.u. The generator has a synchronous reactance of 1.1 p.u. and is delivering real power (P) of 0.6 p.u. to the bus. Assume the infinite bus voltage to be 1.0 p.u. Neglect stator resistance. The reactive power (Q) in p.u. supplied by the generator to the bus under this condition is _____. **[2014]**

10. There are two generators in a power system. No-load frequencies of the generators are 51.5 Hz and 51 Hz, respectively, and both are having droop constant of 1 Hz/MW. Total load in the system is 2.5 MW. Assuming that the generators are operating under their respective droop characteristics, the frequency of the power system in Hz in the steady state is_____. **[2014]**

11. A non-salient pole synchronous generator having synchronous reactance of 0.8 p.u. is supplying 1 p.u. power to a unity power factor load at a terminal voltage of 1.1 p.u. Neglecting the armature resistance, the angle of the voltage behind the synchronous reactance with respect to the angle of the terminal voltage in degrees is _____. **[2014]**

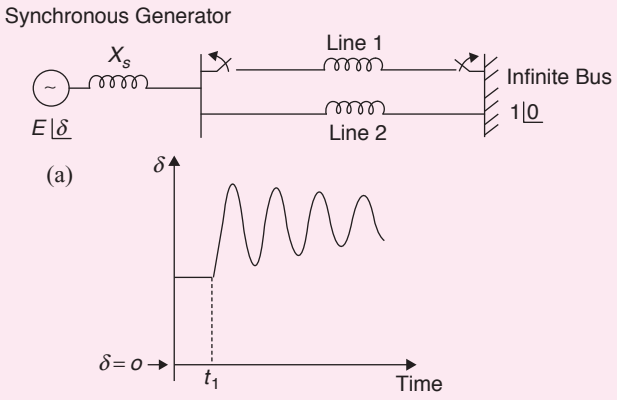
12. The figure shows the single line diagram of a single machine infinite bus system.



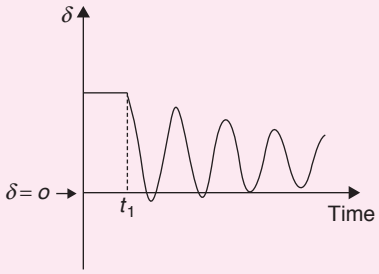
The inertia constant of the synchronous generator $H = 5$ MW-s/MVA. Frequency is 50 Hz. Mechanical power is 1 p.u. The system is operating at the stable equilibrium point with rotor angle δ equal to 30°. A three-phase short circuit fault occurs at a certain location on one of the circuits of the double-circuit transmission line. During fault, electrical power in p.u. is $P_{max} \sin\delta$. If the values of δ and $d\delta/dt$ at the instant of fault clearing are 45° and 3.762 radian/s, respectively, then P_{max} (in p.u.) is_____. **[2014]**

13. A 50 Hz generating unit has H-constant of 2 MJ/MVA. The machine is initially operating in steady state at synchronous speed, and producing 1 pu of real power. The initial value of the rotor angle δ is 5°, when a bolted three phase to ground short circuit fault occurs at the terminal of the generator. Assuming the input mechanical power to remain at 1 pu, the value of δ in degrees, 0.02 second after the fault is _____. **[2015]**

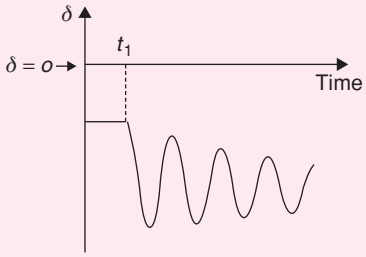
14. The synchronous generator shown in the figure is supplying active power to an infinite bus via two short, lossless transmission lines, and is initially is steady state. The mechanical power input to the generator and the voltage magnitude E are constant. If one line is tripped at time t_1 by opening the circuit breakers at the two ends (although there is no fault), then it is seen that the generator undergoes a stable transient. Which one of the following waveforms of the rotor angle δ shows the transient correctly? **[2015]**



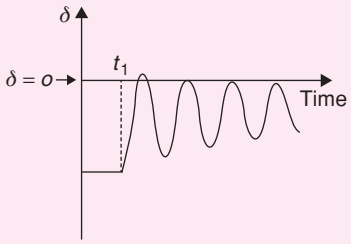
(b)



(c)



(d)



Answer Keys

Exercises

Practice Problems 1

1. C	**2.** D	**3.** A	**4.** C	**5.** B	**6.** A	**7.** C	**8.** D	**9.** A	**10.** D
11. D	**12.** B	**13.** A	**14.** C	**15.** B	**16.** C	**17.** D	**18.** A	**19.** B	

Practice Problems 2

1. D	**2.** A	**3.** A	**4.** B	**5.** A	**6.** D	**7.** C	**8.** A	**9.** A	**10.** D
11. D	**12.** B	**13.** B	**14.** B	**15.** A					

Previous Years' Questions

1. D	**2.** C	**3.** B	**4.** C	**5.** A	**6.** D	**7.** D	**8.** D	**9.** **0.**109	**10.** 50
11. 33.61	**12.** 0.24	**13.** 5.90	**14.** D						

Chapter 5

HVDC and FACTS

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- HVDC
- Monopolar link
- Bipolar link
- Homopolar link
- FACTS
- Series connected controllers
- Shunt connected controllers
- Combined shunt and series connected controllers
- Other controllers

High-voltage DC Transmission

The limitations for use of high-voltage AC transmission of bulk power over long distance leads to high-voltage DC transmission (HVDC). The advantages of HVDC transmission are.

1. Power transmitted per conductor is more for DC and DC line is cheaper as it requires two conductors instead of three and hence costs on insulation and towers are less.
2. The line construction is simpler and ground return is possibly by using single conductor in HVDC transmission.
3. There is no charging current in DC transmission system, No dielectric losses.
4. Skin effect is low in DC systems and hence current density is also high.
5. Corona loss and radio interference are low as compared to AC systems.
6. HVDC does not have any stability problem and hence asynchronous operation of transmission link among the connected machines is possible.
7. Line losses are less in HVDC transmission.
8. Voltage regulation problems are less serious in HVDC since the reactance drop does not exist in DC systems.
9. HVDC system is economical for transmitting bulk power over long distances say above 550 km.
10. Low short-circuit current is required on HVDC lines.
11. Grid control of the converters can drastically reduce the fault current in HVDC.
12. Compensation of lines does not require in DC transmission.

However, HVDC transmission system has a few disadvantages as follows.

1. Application of HVDC is restricted to the distance above 500 km due to its high initial cost which is uneconomical for distances below 500 km.
2. Circuit breaking for protection of HVDC is difficult and expensive.
3. Convertor stations require considerable reactive power.
4. Harmonic generation and subsequent filtration are problems in HVDC.
5. Reactive power required by the load is to be supplied locally as no reactive power can be transmitted through HVDC lines.
6. Overload capacity of HVDC converters is low.
7. Maintenance of insulators in HVDC transmission is low.

Principle of HVDC System Operation

A typical HVDC transmission system consists of one rectifier station at the sending end and one inverter station at the receiving end. The two stations are interconnected by a DC transmission line. The rectifier station converts AC to DC while the inverter

station converts DC to AC. By varying the firing angle of the thyristor in the convertor, the DC output voltage magnitude is controlled.

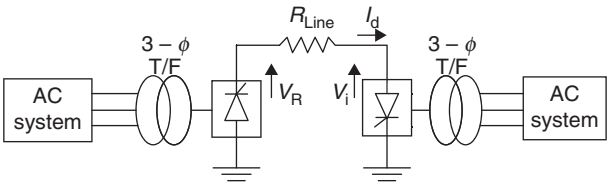


Figure 1 Schematic of a HVDC system

Rectified firing angle is $0° < \alpha < 90°$

Inverter firing angle is $90° < \alpha < 180°$

The current flows from higher voltage to lower voltage by proper setting of the rectifier and inverter. In the DC transmission system the line current (I_d) is given by

$$I_d = \frac{V_R - V_i}{R}$$

where V_R is the DC output voltage at the rectifier and V_i is the DC output voltage at the inverter end.

$$V_R = \left[\left(\frac{3\sqrt{2}}{\pi}\right)V_{AC_r}\cos\alpha - \frac{3X_{cr}}{\pi}I_d\right]$$

$$V_i = \left[\left(\frac{3\sqrt{2}}{\pi}V_{AC_i}\right)\cos\gamma - \frac{3X_{ci}}{\pi}I_d\right]$$

where V_{AC_r} = AC side, line-to-line RMS voltage at rectifier side

V_{AC_i} = AC side, line to line RMS voltage at inverter side

α = Firing angle of rectifier

γ = Extinction angle of inverter

X_{cr} or X_{ci} = Communication reactance at the rectifier and inverter.

The power transfer is given by

$$P = V_i I_d = \left(\frac{V_R - V_i}{R}\right)V_i \text{ W}$$

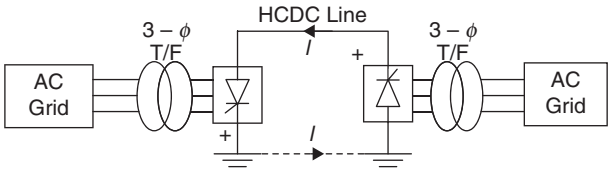
Types of DC Links

HVDC links are classified as

1. Monopolar link
2. Bipolar link
3. Homopolar link

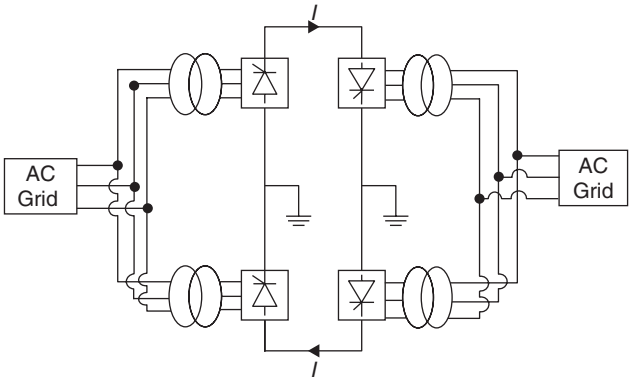
Monopolar Link

This configuration uses only one conductor and ground or sea water is used as return conductor. Negative polarity is used as the transmission conductor due to comparatively lesser ratio interference.



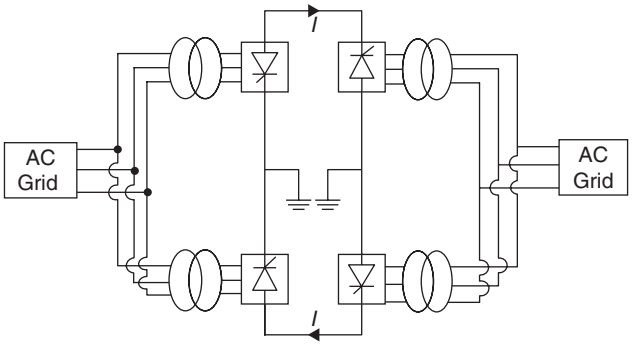
Bipolar Link

This configuration uses two conductors, one positive and another negative. In each terminal, two converters of equal rated voltages are connected in series, neutral points being grounded. Two poles can operate independently when both the terminals are grounded. When the currents in the two conductors are equal, the ground current is zero. In case of fault in one conductor, the other conductor can supply 50% of the rated load along with the ground return.



Homopolar Link

This configuration has two conductors. But having same polarity, ground is used as return path. In case of fault in one conductor, the other conductor can supply 50% of the rated load along with the ground return.



FACTS

Flexible alternating current transmission systems (FACTS) is a technology which provides a methodology for the utilities to effectively utilize their assets, enhance transmission capability by loading lines to their full transmission capability and therefore, minimize the gap between the stability and

the thermal limits, and improve grid reliability. The FACTS technology is based on the use of reliable high-speed power electronics, advanced control technology, advanced micro computers, and powerful analytical tools.

Consider a transmission line of reactance X_{pu} (negligible resistance) connecting two buses in a power system. The system voltage magnitudes in per unit and phase angles are indicated in the diagram.

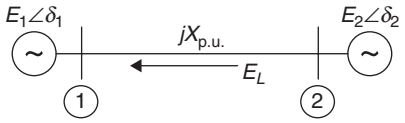


Figure 2 Single-line diagram of AC transmission connecting two buses

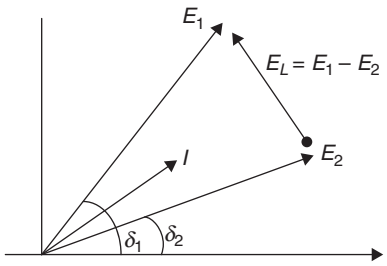


Figure 3 Phasor diagram of the two bus power systems

Line current lags the line voltage drop $E_L = (E_1 - E_2)$ by 90° and is given by

$$I = \frac{E_L}{X} = \frac{E_1 - E_2}{X}$$

Active power flow along the line is given by

$$P = \frac{|E_1||E_2|}{X}\sin\delta$$

where $\delta = \delta_1 - \delta_2$

Reactive power flows Q_1 and Q_2 are expressed as

$$Q_1 = \frac{1}{X}\left(|E_1|^2 - |E_1||E_2|\cos\delta\right)$$

$$Q_2 = \frac{1}{X}\left(|E_2|^2 - |E_1||E_2|\cos\delta\right)$$

From the expression of active and reactive power it can be observed that the active and reactive power on transmission line are depended on line reactance X, voltage magnitudes $|E_1|$ and $|E_2|$, line voltage drop E_L and phase angle difference between the two voltages $\delta = \delta_1 - \delta_2$.

FACTS technology utilizes thyristor-controlled fast-acting controllers to control the following:

1. Line reactance
2. Magnitude of terminal voltages
3. Magnitude of voltage drop (E_L)
4. Phase angle difference

Control action	Parameter affected
(1) Line reactance control	Line current and active power control
(2) Phase angle difference control	Line current and active power control
(3) Voltage injection in series with the line and perpendicular to the current	Real power control through the control of line current
(4) Voltage injection in series with the line and any phase angle with respect to the driving voltage	Accurate control of real and reactive power flow in the line

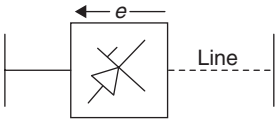
Basic Types of FACTS Controllers

FACTS devices are power electronic based systems and other static equipment that provide control of one or more transmission system parameters. System of FACTS controller is given by

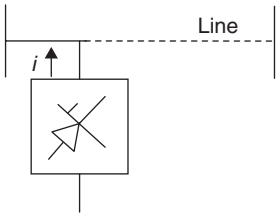


The four basic types of FACTS controllers are given by

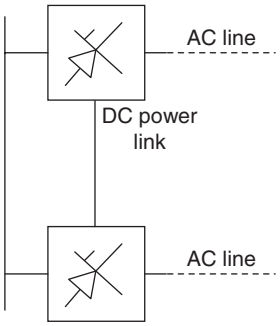
1. Series controllers



2. Shunt controllers



3. Combined series–series controllers



4. Combined series-shunt controllers

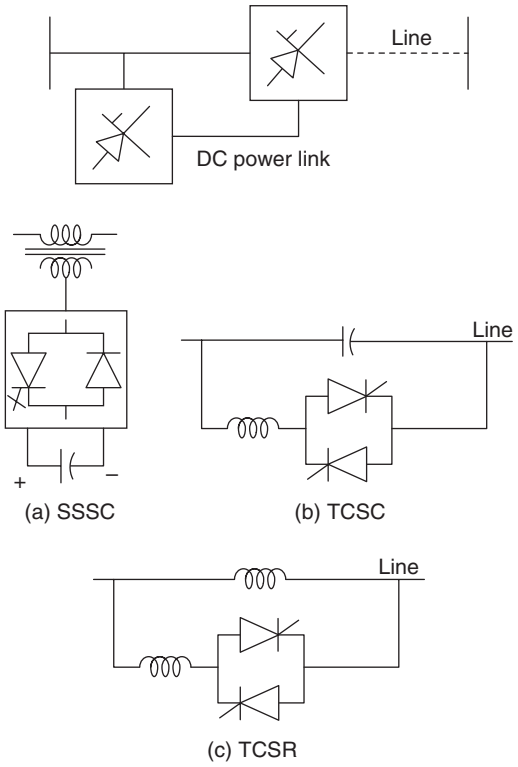


Figure 4 Series-connected FACTS controllers

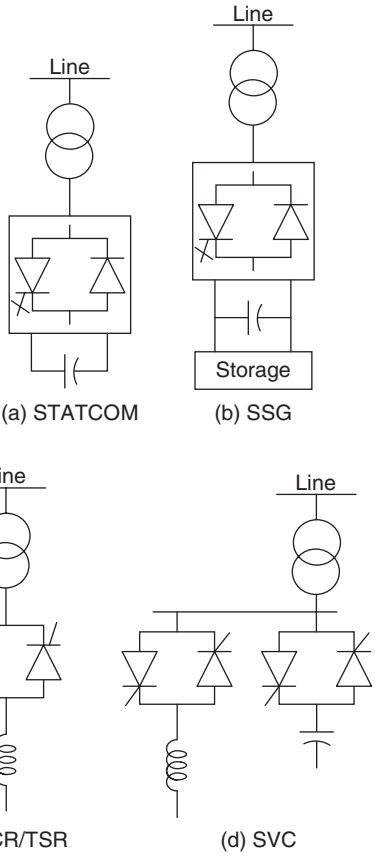


Figure 5 Shunt FACTS controllers

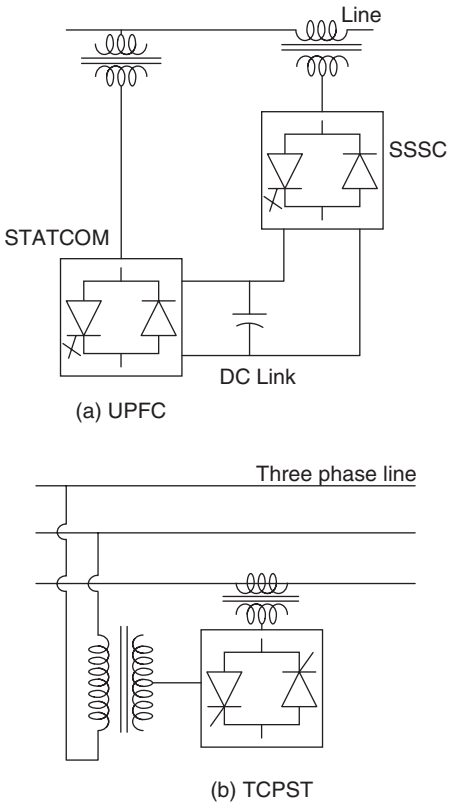


Figure 6 Combined series-shunt controllers

Series-connected Controllers

Static Synchronous Series Compensator (SSSC)

A static synchronous operates as a series compensator whose output voltage is in quadrature with, and controllable independent of, the line current for the purpose of increasing the overall reactive voltage drop across the line and there by controlling the transmitted electrical power.

Inter Power Flow Controller (IPFC)

It is a combination of SSSC's, which are controlled to provide independent reactive compensation for the Adjustment of real power flow in each line.

Thyristor-controlled Services Capacitor (TCSC)

This is a capacitive reactive compensator which consists of a series capacitor bank shunted by a thyristor-controlled reactor in order to provide a smoothly variable series capacitive reactance.

Thyristor-controlled Series Reactor (TCSR)

It consists of a series reactor shunted by a thyristor-controlled reactor in order to provide a smoothly variable series inductive reactance.

Shunt-connected Controllers

Static Synchronous Compensator (STATCOM)

It is a shunt-connected static VAR compensator whose capacitive or inductive output current can be controlled independent of the AC system voltage.

Static Synchronous Generator (SSG)

It produces a set of adjustable multiphase output voltages, which may be coupled to an AC power system for the purpose of exchanging independently controllable real and reactive power.

Static VAR Compensator (SVC)

Its output is adjusted to exchange capacitive or inductive current so as to maintain or control specific parameters of the electrical power system.

Thyristor-controlled Reactor (TCR)

Its effective reactance is varied in a continuous manner by partial conduction control of the thyristor value.

Combined Shunt- and Series-connected Controllers

Unified Power Flow Controller (UPFC)

It is a combination of STATCOM and SSSC to provide real and reactive series line compensation. The UPFC may also provide independently controllable shunt reactive compensation.

Thyristor-controlled Phase-Shifting Transformer (TCPST)

It is adjusted by thyristor switches to provide a rapidly variable phase angle

Interphase Power Controller (IPC)

The active and reactive power can be set independently by adjusting the phase shifts and/or the branch impedances, using mechanical or electrical switches.

Other Controllers

Thyristor-controlled Voltage Limiter (TCVL)

A thyristor-switched metal-oxide varistor (MOV) used to limit the voltage across its terminals during transient conditions.

Thyristor-controlled Voltage Regulator (TCVR)

Thyristor-controlled transformer is one which can provide variable in-phase voltage with continuous control.

- When the inverter fundamental output voltage is higher than the system line voltage, the STATCOM works as a capacitor, however when the inverter voltage is lower than the system line voltage, the STATCOM acts as an inductor.

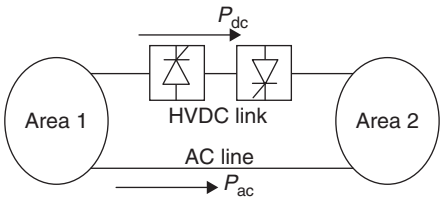
Advantages of STATCOM are as follows:

1. Steady-state load ability of lines is improved.
2. The voltage rise due to capacitance switching is substantially reduced both in magnitude and duration.
3. Voltage variation due to customer's loading is reduced.
   - In SVC devices, the current injected into the system depends upon the system voltage.
   - TCSC provides dynamic control of the series compensated lines, which would increase the transfer capability.
   - A thyristor control phase angle regulator (TCPAR) is equivalent to a mechanically phase shifting transformer but, unlike a UPFC, it does not provide a controlled reactive power generation.
   - STATCOM mainly provides dynamic reactive power to the system but it does not directly controls the flow of real power on a transmission line.

EXERCISES

Practice Problems I

1. In a two area power system both the areas are interconnected by an AC line and a HVDC link as shown in figure. Which one of the following statement is true in steady state.



(A) The total power flow between the regions (P_{AC} + P_{DC}) can be changed by controlling the HVDC converters alone.
(B) Both regions need not have the same frequency.
(C) The directions of power flow in the HVDC link (P_{DC}) can be reversed.
(D) The power sharing between the AC line and the HVDC link can be changed by controlling the HVDC converters alone.

2. Which one of the following is an advantage of HVDC transmission over EHV-AC?
(A) System stability can be improved
(B) VAR compensation is not required
(C) Harmonic problem is avoided
(D) HVDC terminal equipment inexpensive

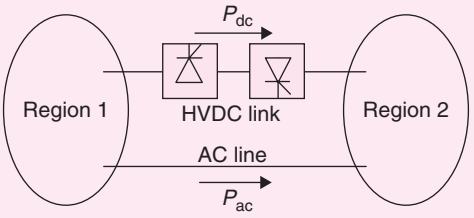
3. Main application of HVDC transmission is
(A) Bulk power transmission over long distances.
(B) Interconnecting two systems with the same nominal frequency.
(C) Minimizing harmonics at the converter station.
(D) Eliminating reactive power requirement in the operation.

Previous Years' Questions

1. HVDC transmission is mainly used for **[2005]**
(A) Bulk power transmission over very long distances
(B) Inter-connecting two systems with the same nominal frequency
(C) Eliminating reactive power requirement in the operation
(D) Minimizing harmonics at the converter stations

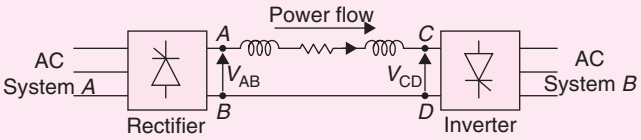
2. An HVDC link consists of rectifier, inverter transmission line and other equipments. Which one of the following is true for this link? **[2006]**
(A) The transmission line produces/supplies reactive power
(B) The rectifier consumes reactive power and the inverter supplies reactive power from/to the respective connected AC systems
(C) Rectifier supplies reactive power and the inverted consumes reactive power to/from the respective connected AC systems
(D) Both the converters (rectifier and inverter) consume reactive power from the respective connected AC systems

3. Two regional systems, each having several synchronous generators and loads are interconnected by an AC line and a HVDC link as shown in the figure. Which of the following statements is true in the steady state?



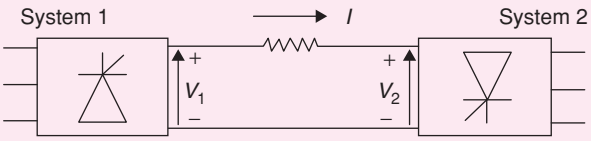
(A) Both regions need not have the same frequency **[2007]**
(B) The total power flow between the regions (P_{AC} + P_{DC}) can be changed by controlling the HVDC converters alone
(C) The power sharing between the AC line and the HVDC link can be changed by controlling the HVDC converters alone.
(D) The direction of power flow in the HVDC link (P_{DC}) cannot be reversed.

4. Power is transferred from system A to system B by an HVDC link as shown in the figure. If the voltages V_{AB} and V_{CD} are as indicated in the figure, and $I > 0$, then **[2010]**



(A) $V_{AB} < 0, V_{CD} < 0, V_{AB} > V_{CD}$
(B) $V_{AB} > 0, V_{CD} > 0, V_{AB} < V_{CD}$
(C) $V_{AB} > 0, V_{CD} > 0, V_{AB} > V_{CD}$
(D) $V_{AB} > 0, V_{CD} < 0$

5. Consider a HVDC link which uses thyristor based line-commutated converters as shown in the figure. For a power flow of 750 MW from system 1 to System 2, the voltages at the two ends, and the current, are given by: $V_1 = 500$ kV, $V_2 = 485$ kV and $I = 1.5$ kA. If the direction of power flow is to be reversed (that is, from System 2 to System 1) without changing the electrical connections, then which one of the following combinations is feasible? **[2015]**



(A) $V_1 = -500$ kV, $V_2 = -485$ kV and $I = 1.5$ kA
(B) $V_1 = -485$ kV, $V_2 = -500$ kV and $I = 1.5$ kA
(C) $V_1 = 500$ kV, $V_2 = 485$ kV and $I = -1.5$ kA
(D) $V_1 = -500$ kV, $V_2 = -485$ kV and $I = -1.5$ kA

Answer Keys

Exercises

Practice Problems I

1. D **2.** A **3.** A

Previous Years' Questions

1. A **2.** B **3.** C **4.** C **5.** B

TEST
Power Systems

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. A Non conventional source of energy is
(A) Water
(B) Radio active substances
(C) Fossil fuels
(D) Geothermal, ocean tides and waves

2. The function of steel wire in an ACSR conductor is
(A) Reducing conductivity
(B) Compensating skin effect
(C) Providing additional mechanical strength
(D) Purely of economical consideration

3. The string efficiency of insulators can be increased by
(A) Changing the orientation of strings
(B) Increasing the number of strings
(C) Correct grading of insulators of various capacitances
(D) Reducing the number of strings

4. In a DC transmission line
(A) It is necessary for the sending end and receiving end to be operated in synchronism
(B) There are no effects due to inductive and capacitive reactances
(C) Power transfer capability is limited by stability considerations
(D) The effects of inductive and capacitive reactance are greater

5. RRRV depends upon the
(A) Inductance of the system only
(B) Capacitance of the system only
(C) Inductance and capacitance of the system
(D) Type of circuit breaker

6. The largest size of thermal generating unit in India
(A) 265 MW (B) 310 MW
(C) 660 MW (D) 500 MW

7. Base Impedance of a power system is given as
(A) $(\text{Base KV})^2/\text{Base MVA}$
(B) $\text{Base KV}/(\text{Base MVA})^2$
(C) Base KV/Base MVA
(D) $(\text{Base KV})^2/(\text{Base MVA})^2$

8. The values of current and voltage for an unloaded synchronous generator which has a fault occurring at the terminals is given as I_{a_0}, I_{a_1} and I_{a_2} as $j1.37$ p.u., $j2.05$ p.u. and $j0.68$ p.u. respectively $V_{a_0} = V_{a_1} = V_{a_2} = j1.37$ p.u. The fault that has occurred is
(A) *L–G* fault (B) *L–L–G* fault
(C) *L–L* fault (D) None of the above

9. An overhead transmission line having surge impedance 'Z_1' is terminated to an underground cable of surge impedance 'Z_2'. The reflection coefficient for the travelling wave at the junction of the line and cable is
(A) $\dfrac{Z_2}{Z_1 + Z_2}$ (B) $\dfrac{-Z_2}{Z_1 + Z_2}$
(C) $\dfrac{Z_2 - Z_1}{Z_1 + Z_2}$ (D) $\dfrac{Z_1 - Z_2}{Z_1 + Z_2}$

10. The insulation resistance of a cable of length 20 km is 1 MW, its resistance for 100 km length will be
(A) 0.2 MW (B) 1 MW
(C) 5 MW (D) None of the above

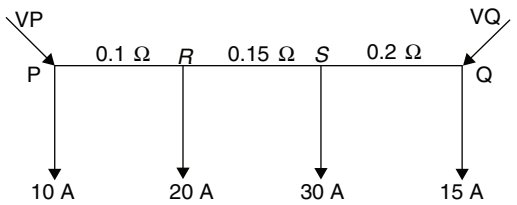
11. If a generator of 250 MVA has an inertia constant of 20 MJ/MVA, its inertia constant on 500 MVA base is
(A) 5 MJ/MVA (B) 10 MJ/MVA
(C) 15 MJ/MVA (D) 20 MJ/MVA

12. A power station is to supply three regions of loads whose Peak values are 40 MW, 25 MW and 50 MW. The diversity factor and annual load factor is given as 2.2 and 0.25 respectively. The maximum demand on the station and average load respectively will be
(A) 52.27, 13.07
(B) 45.08, 15.08
(C) 72.28, 16.08
(D) 60.04, 10.02

13. A transmission line conductor having a diameter of 18.2 mm, weights 0.80 kg/m. The span is 270 m. The wind pressure is 35 kg/m^2 of projected area with ice coating of 12 mm. The ultimate strength of the conductor is 6500 kg. If the factor of safety is 2 and ice weights 890 kg/m^3; the value of maximum sag will be
(A) 6.95 m (B) 5.36 m
(C) 4.82 m (D) 5.20 m

14. A 2 km of a three phase metal sheathed belted cable gave a measured capacitance of 1.2 μF between one conductor and the other two conductors joined together with the earth sheath and 1.5 μF measured between all the three conductors joined and sheath. The value of charging current when the cable is connected at 11 KV, 50 Hz supply is
(A) 1.42 A/ph (B) 3.09 A/ph
(C) 1.59 A/ph (D) 0.68 A/ph

15. In the DC distribution system shown below, the two ends of the feeder are fed by voltage source such that $V_p - V_q = 3$ V. The value of the voltage V_p for a minimum voltage of 220 V at any point along the feeder.



(A) 220 V (B) 228.58 V
(C) 225.89 V (D) 222.89 V

16. The velocity of wave propagation of a lossless overhead EHV line operating at 50 Hz is 3×10^5 Km/s. If the total reactance and susceptance are given by 0.035 p.u. and 1.3 p.u. respectively, approximate length of the line is
(A) 158 Km
(B) 140 Km
(C) 180 Km
(D) 175 Km

17. A power system network consists of three elements 0–1, 1–2, and 2–0, of per unit impedances 0.2, 0.4 and 0.4 respectively. Its bus impedance matrix is given by _____.

(A) $\begin{matrix} & 1 & 2 \\ 1 & 7.5 & -2.5 \\ 2 & -2.5 & 5.0 \end{matrix}$ (B) $\begin{matrix} & 1 & 2 \\ 1 & 0.16 & 0.08 \\ 2 & 0.08 & 0.24 \end{matrix}$

(C) $\begin{matrix} & 1 & 2 \\ 1 & 0.16 & -0.18 \\ 2 & -0.08 & 0.24 \end{matrix}$ (D) $\begin{matrix} & 1 & 2 \\ 1 & 0.6 & 0.4 \\ 2 & 0.4 & 0.8 \end{matrix}$

18. The incremental fuel cost of a power plant consisting of three generating units are

$IC_1 = 20 + 0.3P_1$

$IC_2 = 30 + 0.4P_2$

$IC_3 = 30$

where P_i is the power in MW generated by unit i, for i = 1, 2, and 3. Assume that all three units are operating all the time. Minimum and maximum load on each unit are 50 MW and 300 MW respectively. If the plant supplies a total power demand of 700 MW power generated by each unit is
(A) $P_1 = 157.14$ MW $P_2 = 242.86$ MW $P_3 = 300$ MW
(B) $P_1 = 242.86$ MW $P_2 = 157.14$ MW $P_3 = 300$ MW
(C) $P_1 = 300$ MW $P_2 = 300$ MW $P_3 = 100$ MW
(D) $P_1 = 233.3$ MW $P_2 = 233.3$ MW $P_3 = 233.4$ MW

19. The leakage reactance referred to lv side for a 3 kVA 11/0.4 KV single phase transformer is given as 0.90 Ω. Then the leakage reactance in per unit is
(A) 0.001 p.u. (B) 0.017 p.u.
(C) 0.584 p.u. (D) 1.23 p.u.

20. The maximum allowable tension in an overhead line having a span of 150 m is given as 1200 kg. If the line conductor weights 0.5 kg per meter, then the maximum sag will be
(A) 1.17 (B) 3.5
(C) 0.15 (D) 4.8

21. For a step up transformer the line to ground voltage on phases a, b and c are given as 80 KV, 33 KV and 42 KV respectively. The voltage relations are given like phase a lags that of phase b by 150° and leads that of phase c by 80°. Then the positive and negative components of phase voltage are
(A) (13.51 – 6.2 1i), (46.91 + 14.49j)
(B) (13.51 + 15.78j), (46.91 – 14.49j)
(C) (13.51 – 15.78j), (15.63 + 14.49j)
(D) (13.51 + 5.26j), (15.63 – 14.49j)

22. The losses and other mechanical load for a synchronous motor which improves the p.f. from 0.5 to 0.7 lagging is given as (150 + 100) kW. If the load is given as 1200 kW, then the value of the input to the synchronous motor in kVA
(A) 649 (B) 550
(C) 700 (D) 585

23. For a DC three wire system the same supply and load voltages are given to neutral and balanced conditions. If this system is converted into a 3-phase four wire AC system which of the following expression is correct
(A) $P_2 = P_1$ (B) $P_2 = 1.5\ P_1$
(C) $P_1 = -P_2$ (D) $P_1 = 1.5\ P_2$

Statement for Linked Answer Questions 24 and 25:
The maximum demand at a power station is 60 MW, at a load factor of 0.7 and plant capacity factor of 0.6.

24. The annual energy generated in kWh is
(A) 367.920×10^6 (B) 183.96×10^6
(C) 735.84×10^6 (D) 147.17×10^6

25. The reserve capacity of the plant is
(A) 5 MW (B) 10 MW
(C) 15 MW (D) 20 MW

Answer Keys

1. D	**2.** C	**3.** C	**4.** B	**5.** C	**6.** C	**7.** A	**8.** B	**9.** C	**10.** A
11. B	**12.** A	**13.** A	**14.** B	**15.** B	**16.** A	**17.** D	**18.** B	**19.** B	**20.** A
21. A	**22.** A	**23.** B	**24.** A	**25.** B					

Control Systems

UNIT VIII

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Chapter 1

Introduction to Control Systems

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Basic definitions
- Classification of control systems
- Transfer function
- Poles and zeros of transfer function
- Effect of feedback on disturbance
- Block diagram
- Block diagram reduction techniques
- Signal-flow graph
- Signal-flow graph algebra
- Mason's gain formula

Basic Definitions

System

A set of components/elements connected in a proper sequence to perform a specific task.

Controller

Controller is an element/subsystem inside or outside the system which regulates the operating condition/response of the system.

Disturbance

Disturbance is a signal that tends to adversely affect the value of the output of the system. If a disturbance is generated within the system, it is called internal disturbance, while an external disturbance is generated outside the system.

Control System

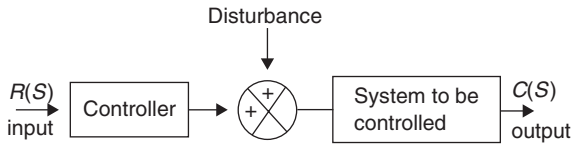
A set of components connected in a proper sequence to form a system which provides the desired response.
Control system is mainly classified into two types:

1. Open loop control system.
2. Closed loop control system.

Open Loop Control System (OLCS)

Any physical system in which the output is controlled directly by a controller/actuator without help of feedback is known as open loop control system. The input of the OLCS is independent of the output.

The output of an open loop control system is affected not only by input but also by disturbance in it. Open loop system does not automatically correct the disturbances.



Advantages

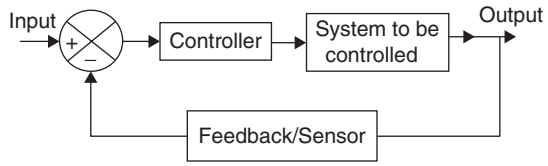
- Simple and easy to design.
- Cheap due to less number of components.
- Generally open loop systems are stable.

Disadvantages

- More sensitive to disturbances and unable to correct the disturbances.
- Inaccurate and Unreliable.

Closed Loop Control System (CLCS)

A Control System in which input is a function of output in order to maintain the desired value of output is called closed loop control system.



Since the feedback facilitates the system to automatically correct the system input to meet the desired response, it is also called as 'Automatic Control System.'

Advantages

- Accurate and reliable.
- Less sensitive to disturbances.
- Accurate even with presence of non-linearities.

Disadvantages

- Design is complex and costly gain of the system.
- Feedback presence reduces the overall.
- Feedback may lead to oscillatory response.
- Improper design of the Controller may cause the system to become unstable.

Comparison Between Open Loop and Closed Loop System

	Open loop control system		Closed loop control system
1.	Input is independent of output	1.	Input is dependent on output.
2.	Design is easy and cheap.	2.	Design is complex and costly.
3.	Inaccurate and Unreliable	3.	Accurate and reliable
4.	More sensitive to disturbances	4.	Less sensitive to disturbances
5.	Feedback does not exist.	5.	Feedback exists and reduces the gain of the system. It may also cause oscillations.
6.	Generally stable in operation	6.	Improper design of the controller may cause unstable operation of the system.

Other Classification of Control Systems

Depending on the nature of the system

- Linear and non-linear control systems.
- Time-variant and time-invariant systems.

Depending on the type of signals present at various parts of a feedback control system,

- Continuous and Discrete line control systems.

Linear and Non-Linear Control Systems

A system which obeys superposition and homogeneity principle is said to be a linear system.

Let $x_1(t)$ and $x_2(t)$ be two inputs to a system and $y_1(t)$ and $y_2(t)$ be the corresponding outputs. For arbitrary real constants k_1 and k_2, for input $k_1x_1(t) + k_2x_2(t)$, if the output of the system is given by $k_1y_1(t) + k_2y_2(t)$, then the system is said to be linear.

Any system which does not obey superposition and homogeneity principle is said to be 'non-linear'.

Physical systems are in general non-linear and analysis of such systems is very complicated. Hence these systems are usually linearized and analysed using linear techniques.

Time-variant and Time-invariant Control Systems

A system is said to be 'time variant' if its characteristics explicitly depend upon time.

A 'time-invariant' system is one whose output does not depend explicitly on time.

If an input signal $x(t)$ produces an output $y(t)$, then any time shifted input $x(t + \delta)$, results in a time shifted output $y(t + \delta)$, then system is time-invariant.

Continuous Time and Discrete Time Systems

If the signals in all parts of a control system are functions of time, the system is said to be continuous time control system.

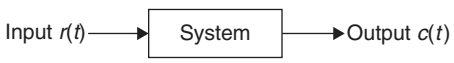
A system in which one or more parts of control systems signals are of the pulse form, it is said to be discrete time control system.

Transfer Function

Transfer function of a linear time-invariant system is defined to be the ratio of the Laplace transform of the output variables to the Laplace transform of the input variables under the assumption that all initial conditions are zero.

(or)

Transfer function is defined as the Laplace transform function of an impulse response of the system when all initial conditions are assumed to be zero.



Transfer function $= \frac{L[c(t)]}{L[r(t)]} = \frac{C(s)}{R(s)}$ with zero initial conditions

Transfer Function = $L[C(t)]$ zero initial conditions and impulse input

Note 1: Transfer function gives mathematical model of all components and individual analysis of various components.

Note 2: Transfer function is independent of input and output of the system.

Note 3: Transfer function is useful in determining response of the system for any type of input applied.

Note 4: Transfer function is helpful to obtain differential equations related to the system.

Note 5: Transfer function is helpful in stability analysis.

Limitations of Transfer Function

1. TF is applicable only for linear time-invariant systems.
2. Does not provide any information about the physical structure of the system.
3. Initial condition's effects are completely neglected. So, initial conditions lose their importance.

Solved Examples

Example 1: The transfer function of the system described by $5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 2\frac{du}{dt} + 4u$, with u as input and y as output is

(A) $\frac{2s+4}{(5s^2+4s)}$ (B) $\frac{2s+4}{5s+4}$

(C) $\frac{2s+2}{5s^2+4s}$ (D) $\frac{2s}{5s^2+4}$

Solution: (A)

Apply Laplace transform on both sides for the given differential equation with zero initial condition,

$$\Rightarrow \quad 5s^2\,Y(s) + 4s.Y(s) = 2s\,U(s) + 4U(s)$$

$$(5s^2 + 4s)\,Y(s) = (2s + 4)\,U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{2s+4}{(5s^2+4s)}$$

Example 2: The impulse response of the system is given as $c(t) = -4e^{-2t} + 6e^{-4t}$. The step response of the same system for $t \geq 0$ is equal to

(A) $\frac{1}{2}\left[-1+e^{-2t}\right]$ (B) $\frac{1}{2}\left[1-e^{-2t}\right]$

(C) $\frac{1}{4}\left[1-e^{-2t}+2e^{-4t}\right]$ (D) $\frac{1}{2}\left[-1+4.e^{-2t}-3.e^{-4t}\right]$

Solution: (D)

Transfer function of the system = L {Impulse response} = $\frac{-4}{s+2} + \frac{6}{s+4}$

Step response of the system $C(s) = \text{T.F} \times R(S) = \frac{2s-4}{(s+2)(s+4)} \times \frac{1}{s}$

$$c(t) = L^{-1}\left\{\frac{2s-4}{s(s+2)(s+4)}\right\}$$

$$L^{-1}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}\right\}$$

$$8A = -4$$

$$\Rightarrow \quad A = \frac{-1}{2};$$

$$B = C(s) \times (s+2)$$

at $S = -2$

$$B = 2 \text{ and } C = C(S) \times (S+4)$$

at $S = -4$

$$C = \frac{-3}{2}$$

$$C(s) = L^{-1}\left\{\frac{-1}{2s} + \frac{2}{s+2} - \frac{3}{2(s+4)}\right\}$$

$$C(t) = \frac{-1}{2} + 2.e^{-2t} - \frac{3}{2}.e^{-4t}$$

Example 3: A control system is defined by the following differential mathematical relationship

$$9\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 5x = 12\,(1 - e^{-2t})$$

The response of the system as $t \to \infty$ is

(A) $x = 16$ (B) $x = 5$

(C) $x = 2.4$ (D) $x = -9$

Solution: (C)

Taking Laplace transform on both sides with zero initial conditions

$$X(s)\,[9s^2 + 16s + 5] = 12\left[\frac{1}{s} - \frac{1}{s+2}\right] = 12\left[\frac{2}{s(s+2)}\right]$$

$$\Rightarrow \quad X(s) = \frac{24}{s(s+2)(9s^2+16s+5)}$$

Response of the system $x(t)$ at $t \to \infty = \underset{t\to\infty}{\text{Lt}}\; x(t)$

$$= \underset{s\to 0}{\text{Lt}}\; sX(s) \qquad [\because \text{ Final value theorem}]$$

$$= \underset{s\to 0}{\text{Lt}}\; s\left[\frac{24}{s(s+2)(9s^2+16s+5)}\right] = \frac{12}{5} = 2.4$$

Response of the system at $t \to \infty = 2.4$

Example 4: The impulse response of an initially relaxed linear system is $e^{-3t}u(t)$. To produce a response of $te^{-3t}u(t)$, the input must be equal to

(A) $3e^{-t}u(t)$ (B) $\frac{1}{3}e^{-3t}u(t)$

(C) $e^{-3t}u(t)$ (D) $te^{-3t}u(t)$

Solution: (C)
Impulse response = $e^{-3t}u(t)$
Transfer function = L [Impulse Response]

$$= L\left[e^{-3t}u(t)\right] = \frac{1}{s+3}$$

Response of the system

$$[C(s)] = L[t\,e^{-3t}u(t)]$$

$$= \frac{1}{(s+3)^2}$$

$$\text{Input } R(s) = \frac{C(s)}{\text{Transfer function}}$$

$$= \frac{\frac{1}{(s+3)^2}}{\frac{1}{s+3}} = \frac{1}{s+3}$$

Input $r(t) = L^{-1}[R(s)] = e^{-3t}u(t)$

Example 5: A linear time-invariant system has an impulse response e^{2t}, $t > 0$. If the initial conditions are zero and input is e^{8t}, then output for $t > 0$ is

(A) $\frac{1}{6}\left[e^{8t} - e^{2t}\right]$ (B) e^{5t}

(C) $e^{8t} + e^{2t}$ (D) None

Solution: (A)
Transfer Function = L [Impulse response]

$$= L\left[e^{2t}\right] = \frac{1}{s-2}$$

Input applied $R(s) = L[e^{8t}] = \frac{1}{s-8}$

Response of the system = T.F × R(s)

$$= \frac{1}{s-2} \times \frac{1}{s-8}$$

Output of the System

$$C(t) = L^{-1}\left[\frac{1}{(s-2)(s-8)}\right] = L^{-1}\left[\frac{\frac{1}{6}}{s-8} - \frac{\frac{1}{6}}{s-2}\right]$$

$$C(t) = \left(\frac{e^{8t} - e^{2t}}{6}\right)$$

Example 6: Let $x(t)$ be the input to a linear, time-invariant system. The required output is $8x(t-4)$. The transfer function of the system should be

(A) $8e^{j8\pi f}$ (B) $4e^{-j8\pi f}$

(C) $8e^{-j8\pi f}$ (D) $4e^{j8\pi f}$

Solution: (C)
Required output of the System

$$y(t) = 8x(t-4)$$

$$\Rightarrow \quad Y(s) = 8e^{-4s}X(s)$$

Transfer Function $\frac{Y(s)}{X(s)} = 8e^{-4s}$

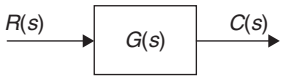
$$= 8e^{-4\omega j} \quad [\because s = j\omega]$$

Transfer Function = $8e^{-j8\pi f}$ $[\because\ \omega = 2\pi f]$

Poles and Zeros of Transfer Function

Transfer Function

It is defined as the ratio of Laplace transform of the response to the Laplace transform of the excitation or input, with all initial conditions are zero.



$$G(s) = \frac{C(s)}{R(s)}$$

The impulse response of the system

$$H(s) = C(s) = G(s)$$

The transfer function of a linear control system can be expressed as

$$G(s) = \frac{K(s+b_0)(s+b_1)(s+b_2)\ldots}{(s+a_0)(s+a_1)(s+a_2)\ldots}$$

Characteristic equation $1 + G(s)\,H(s) = 0$

$$(s+a_0)(s+a_1)(s+a_2)\ \ldots + k(s+b_0)(s+b_1)\,(s+b_2)\ \ldots = 0$$

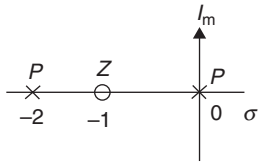
If $K = 0$
Poles of the system are $S = -a_0, -a_1, -a_2 \ldots$ etc.
If $K = \infty$
Zeros of the system is

$$s = -b_0, -b_1, -b_2 \ \ldots \text{ etc.}$$

where k is gain factor of the transfer function.

Example 7: Consider the unity feedback open loop system transfer function $G(s) = \frac{s+1}{s(s+2)}$. Draw the p–z location and explain about the stability.

Solution: Poles are at $S = 0, -2$
Zero at $S = -1$



Impulse response of the system $H(s) = \frac{s+1}{s(s+2)}$

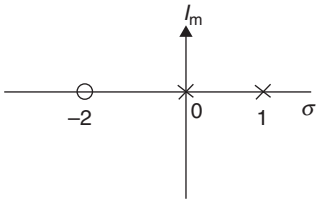
$$h(t) = L^{-1}\left\{\frac{A}{S} + \frac{B}{S+2}\right\}$$

$$\therefore \quad h(t) = \{A + B.e^{-2t}\}u(t)$$

It is a stable system [exponentially decaying function]

Example 8: Consider the impulse response of the system is $H(s)=\frac{(s+2)}{s(s-1)}$ Explain about the stability of system.

Solution:



$$h(t) = \{A + Be^{t}\}u(t)$$

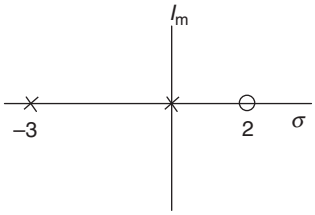
If $t \to \infty$

$$h(t) = \infty \text{ \{unbounded for bounded input\}}$$

So, the system is unstable. [exponentially increasing function]

Example 9: If $H(s)=\frac{(s-2)}{s(s+3)}$ then the system stability is _____

Solution: P–Z location:



$$h(t) = (A + B.e^{-3t})\, u(t)$$

It is a exponentially decaying function.

If $t \to \infty;\ h(t) = A$

For bounded input bounded output,
So, the system is stable.

Note:
System stability depends on the poles location but not zeros.

If system poles located only in the LHS of *S*-plane (non-repeated at origin and Imaginary axis), then the system is called stable system otherwise it is unstable.

Sensitivity Analysis

Sensitivity: Sensitivity is the ratio of the percentage change in the function to the percentage change in the Parameter.

$$\text{Sensitivity} = \frac{\text{Percentage change in } F(s)}{\text{Percentage change in } P(s)}$$

where $F(s)$ is the function and $P(s)$ is the parameter.

Transfer Function Sensitivity with Respect to Parameter Variation

Transfer function = $T(s)$.

Forward path gain = $G(s)$

Transfer function sensitivity with respect to parameter variation = S_G^T

$$S_G^T = \frac{\frac{\partial T}{T}\times 100}{\frac{\partial G}{G}\times 100} = \frac{\partial T}{\partial G}\times\frac{G}{T}$$

For open loop system: $S_G^T = 1$

For closed loop system: $S_G^T = \frac{1}{1+GH}$ (Negative feedback)

Note 1: The sensitivity of closed loop system with respect to variation in G is reduced by a factor $(1+ GH)$ as compared to that of an open loop system.

Sensitivity of *T* with Feedback Parameters

Feedback gain = H

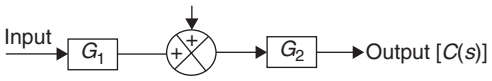
Sensitivity of T with feedback parameters = S_H^T

$$S_H^T = \frac{\partial T}{\partial H}\times\frac{H}{T} = \frac{-GH}{1+GH}.$$

Note 2: Closed loop system is more sensitive to the feedback parameters variation than that of the forward path parameters variation.

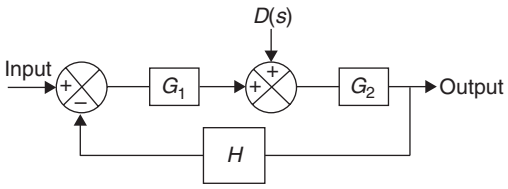
Effect of Feedback on Disturbance

Open loop system:



Output due to disturbance = $G_2D(s)$.

Closed loop system:



Output due to disturbance = $\frac{G_2}{1+G_1G_2H}.\ D(s)$.

Note: Negative feedback reduces the effect of noise on output by a factor of $1+G_1G_2H$ as compared to that of open loop systems.

Note: For positive feedback

Output due to disturbance = $\frac{G_2}{1-G_1G_2H}.\ D(s)$.

Effect of disturbance on output of the system increases as compared to open loop and negative feedback closed loop control systems.

Example 10: A negative feedback system has an amplifier of gain 10 with ±1% tolerance in the forward path, and an alternator of a value $\frac{9}{10}$ in the feedback path. The overall system gain is approximately.

(A) 10 ±1% (B) 9±1%

(C) 1±0.1% (D) 9±0.1%

Solution: (C)

Overall gain of the system without tolerance = $\frac{10}{1+\frac{90}{10}} = \frac{1}{10}$

Sensitivity of gain with tolerance $\left(S_G^T\right) = \frac{1}{1+GH}$ (For negative feedback) = $\frac{1}{1+\frac{90}{10}} = \frac{1}{10}$

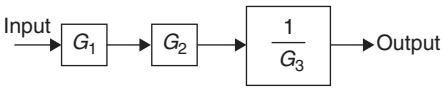
$$\left(S_G^T\right) = \frac{\%\text{ Change in gain}}{\%\text{ Change in } G}$$

$$\%\text{ Change in gain} = \frac{1}{10} \times \text{Change in G}$$

$$= \frac{1}{10} \times 1\% = 0.1\%$$

Overall gain with tolerance = 1±0.1%

Example 11: Errors associated with each respective subsystems G_1, G_2 and G_3 are 4, 5 and 8. The error associated with the output is



(A) 160 (B) 17 (C) 1 (D) $\frac{5}{2}$

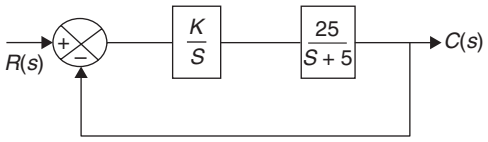
Solution: (C)

Total error = $4 \times S_{G_1}^T + 5 \times S_{G_2}^T + 8 \times S_{G_3}^T$

$= 4 + 5 - 8 \quad \left[\because\ S_{G_1}^T = 1,\ S_{G_2}^T 1,\ S_{G_3}^T = -1\right]$

$= 1$

Example 12: The sensitivity of transfer function $\frac{C(s)}{R(s)}$ to variation in parameter K if system operating frequency $\omega = 5$ rad/sec and $K = 1$ is



(A) 1.41 (B) 2.82 (C) 0 (D) $\frac{1}{5}$

Solution: (A)

Transfer function T.F = $\frac{C(s)}{R(s)} = \frac{25K}{5^2 + 5s + 25K}$

Sensitivity of T.F with $K\left(S_K^T\right) = \frac{\partial T}{\partial K} \times \frac{K}{T}$

$$= \frac{s(s+5)}{s^2 + 5s + 25K}$$

If $K = 1$ and $W = 5$ rad/sec

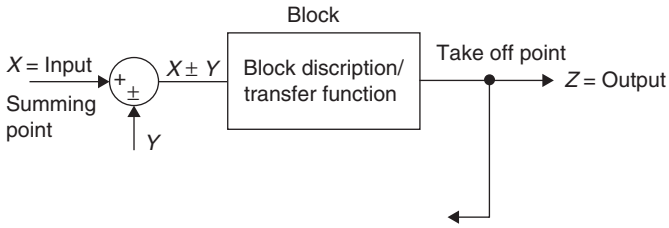
$$\left|S_K^T\right| = 1.41.$$

Block Diagram

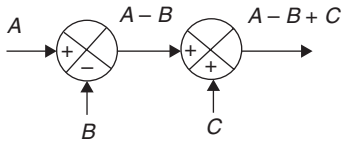
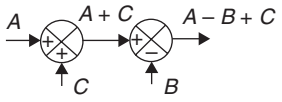
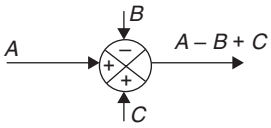
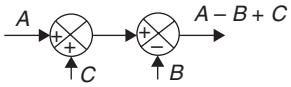
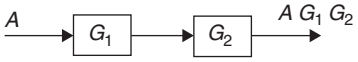
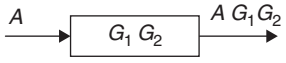
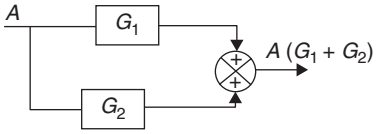
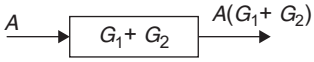
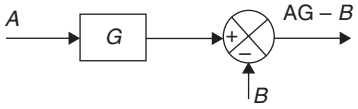
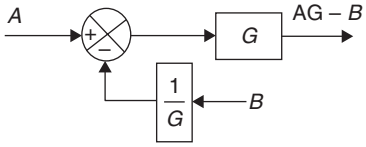
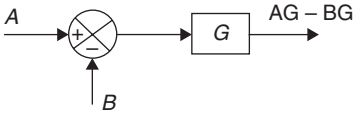
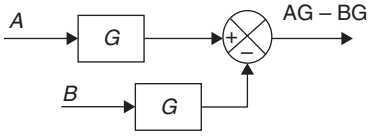
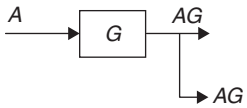
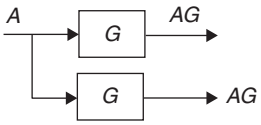
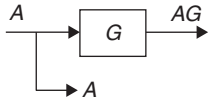
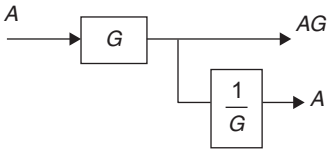
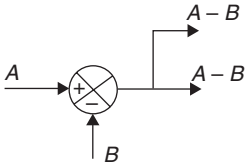
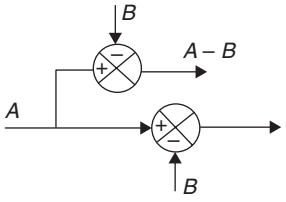
Block diagram is a pictorial representation of system between input and output. Block diagram together with transfer function is used to describe the cause- and -effect relationship throughout the system.

Different elements in a block diagram are:

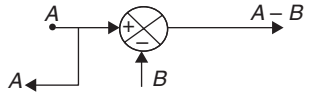
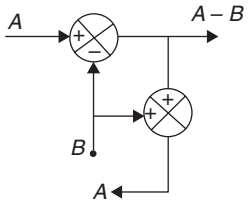
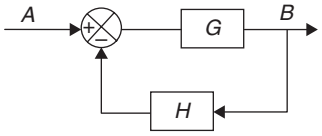
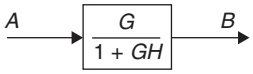
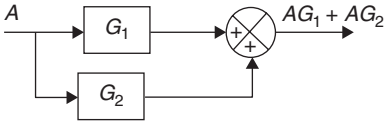
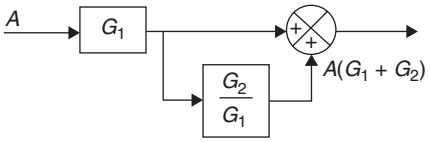
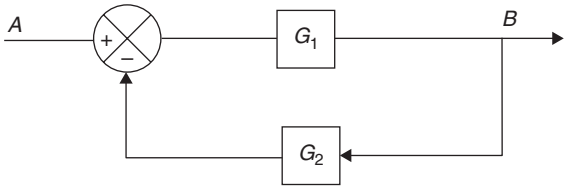
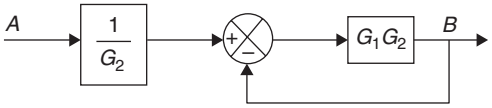
1. Block: Physical description or transfer function of a subsystem.
2. Summing point: Addition or subtraction of all incoming signals.
3. Take-off point: Measurement or sensing a signal.
4. Line with arrow: Represents unidirectional signal-flow and connectivity between subsystem.



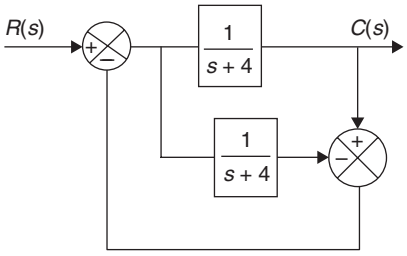
Block Diagram Reduction Techniques

Original diagram	Equivalent diagram
1. Associative law	
	
	
2. Blocks in series (or) combining blocks in cascade.	
	
3. Blocks in parallel	
	
4. Shifting summing point behind the block	
	
5. Shifting summing point beyond the block	
	
6. Shifting a take-off point behind the block	
	
7. Shifting a take-off point beyond the block	
	
8. Shifting take-off point after summing point	
	

(*Continued*)

Original diagram	Equivalent diagram
9. Shifting take-off point before summing point 	
10. Removing minor feedback loop 	
11. Removing block in forward path 	
12. Removing block in feedback path 	

Example 13: The transfer function $\frac{C(s)}{R(s)}$ of the system shown is



(A) $\frac{1}{s+4}$ (B) $\frac{1}{s+8}$ (C) $\frac{1}{s+3}$ (D) $\frac{2}{s+8}$

Solution: (A)

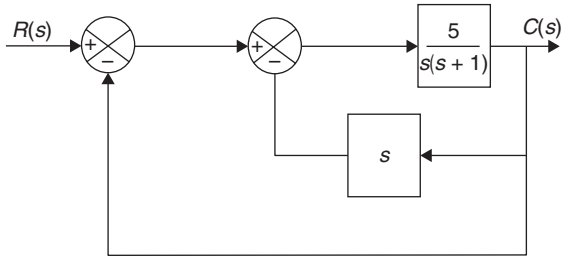
$$\text{Forward path} = \frac{1}{s+4}$$

$$\text{Loops} = \frac{1}{s+4}, -\frac{1}{s+4}$$

$$\text{Transfer function} = \frac{P_1}{1-L_1-L_2}$$

$$= \frac{\frac{1}{s+4}}{1-\frac{1}{s+4}+\frac{1}{s+4}} = \frac{1}{s+4}$$

Example 14: For the system shown in figure the transfer function $\frac{C(s)}{R(s)}$ is equal to



(A) $\frac{5}{s^2+4s+5}$ (B) $\frac{5}{s^2-4s+5}$

(C) $\frac{s^2}{s^2+6s+5}$ (D) $\frac{5}{s^2+6s+5}$

Solution: (D)

$$\text{Forward path} = \frac{5}{s(s+1)}$$

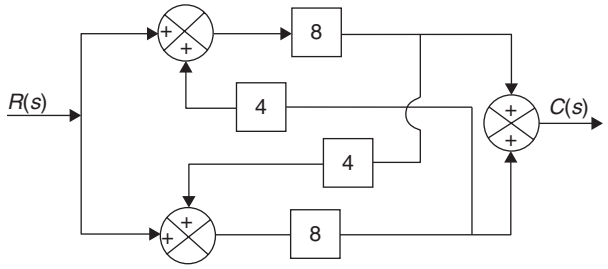
$$\text{Loops} = \frac{-5s}{s(s+1)}, \frac{-5}{s(s+1)}$$

$$\text{Transfer function} = \frac{\frac{5}{s(s+1)}}{1+\frac{5s}{s(s+1)}+\frac{5}{s(s+1)}}$$

$$= \frac{5}{s^2+s+5s+5}$$

$$\text{Transfer function} = \frac{5}{s^2+6s+5}$$

Example 15: The overall transfer function of the system in figure is



(A) $\frac{31}{16}$ (B) $\frac{16}{31}$

(C) $\frac{-16}{31}$ (D) $-\frac{32}{15}$

Solution: (C)

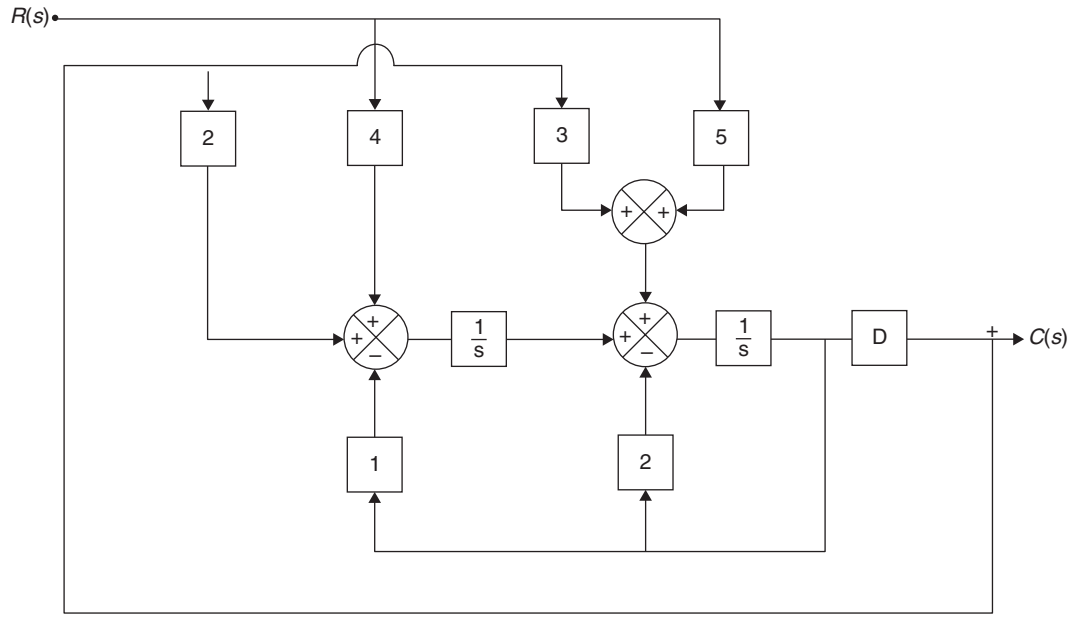
Forward paths $\rightarrow$ 8, 8 – 4 – 8, 8 – 4 – 8, 8

$$\text{Transfer function} = \frac{8+(8\times4\times8)+8+(8\times4\times8)}{1-(8\times4\times8\times4)}$$

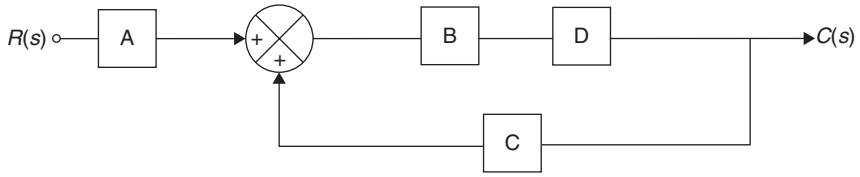
$$= \frac{16(1+32)}{1-32\times32}$$

$$\text{Transfer function} = \frac{16}{1-32} = \frac{-16}{31}$$

Example 16: The system shown in the figure below.



Can be reduced to the form



(A) $A = 2 + 3s, B = \frac{1}{s^2+2s+1}, C = 4s + 5$

(B) $A = 4s + 5, B = \frac{1}{s^2+2s+1}, C = 2 + 3s$

(C) $A = 2 + 3s, B = \frac{4s+5}{s^2+2s+5}, C = 4s + 5$

(D) $A = 4s + 5, B = 2 + 3s, C = 1/s^2 + 2s + 5$

Solution: (B)

$$\text{Forward paths} = 4-\frac{1}{s}-\frac{1}{s}-D\left[\frac{4D}{s^2}\right]$$

$$= 5-\frac{1}{s}-D\left[\frac{5D}{s}\right]$$

$$\text{Loops} = \frac{1}{s}-2\ [-2/s]$$

$$\frac{1}{s}-1-\frac{1}{s}\ [-1/s^2]$$

$$\frac{1}{s}-D-2-\frac{1}{s}\ [2D/s^2]$$

$$\frac{1}{s}-D-3\ [3D/s]$$

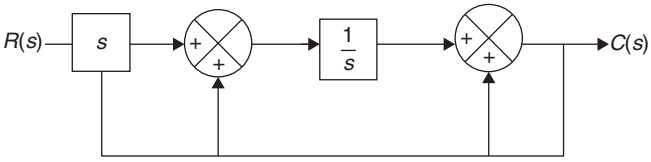
$$\text{Transfer function} = \frac{\frac{4D}{s^2}+\frac{5D}{s}}{1+\frac{2}{s}+\frac{1}{s^2}-\frac{2D}{s^2}-\frac{3D}{s}}$$

$$= \frac{(4s+5)D}{(s^2+2s-1)-(2D+3Ds)}$$

$$B\ \frac{AD}{1-BCD} = \frac{1}{s^2+2s+1}\cdot\frac{(4s+5)D}{1-\frac{(2+3s)}{s^2+2s+1}.D}$$

$$\therefore\ A=(4s+5),\ B=\frac{1}{s^2+2s+1},\ C=(2+3s)$$

Example 17: For the block diagram shown in figure, the transfer function is equal to



(A) $\frac{s}{2s+1}$ (B) $s+\frac{1}{s}$

(C) $\frac{s}{s^2+1}$ (D) $\frac{2s+1}{s}$

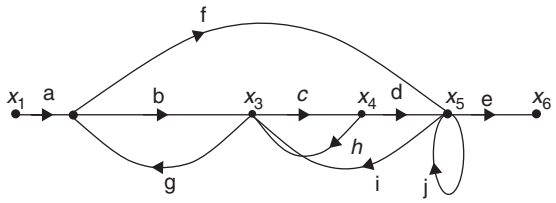
Forward paths = $1, \frac{1}{s}, 1$

$$\text{Transfer function} = 1+\frac{1}{s}+1 = \frac{2s+1}{s}$$

Solution: (D)

Signal-flow Graph

Signal-flow graph is a graphical representation of simultaneous algebraic equations.



Node: A node is a point representing a variable or signal.

Transmittance: This is a real gain or complex gain between two nodes.

Branch: A branch is a directed line segment joining two nodes.

Input node or source: A source is a node that has only outgoing variables. (node x_1)

Output node or sink: A sink is a node that has only incoming branches. (node x_6)

Mixed node: A mixed node has both incoming and outgoing branches. (nodes x_2, x_3, x_4, x_5)

Forward path: A forward path is a path from an input node (source) to an output node (sink) that does not cross any nodes or branch more than once. ($x_1-x_2-x_3-x_4-x_5-x_6$ and $x_1-x_2-x_5-x_6$)

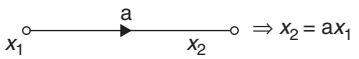
Loop: A loop is a closed path with no node or branch repeated more than once. ($x_2-x_3-x_2$, $-x_3-x_4-x_3$, $x_3-x_4-x_5-x_3$, x_5-x_5) and ($x_2-x_5-x_2$)

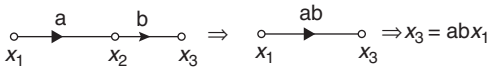
Loop gain: The loop gain is the product of the branch transmittances of a loop. (bg, ch, cdi,f, fig)

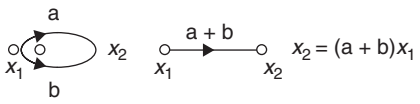
Non touching loops: Loops are non-touching if they do not possess any common nodes.

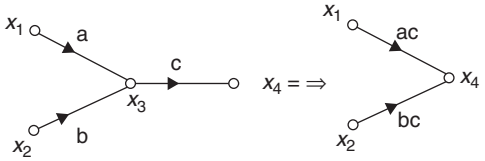
Signal-flow Graph Algebra

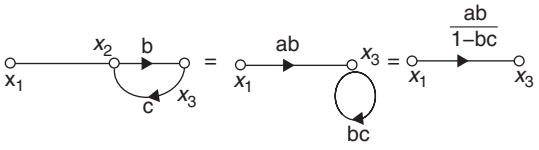
Signal-flow graph for a system can be reduced to obtain the transfer function of the system using the following rules.

1. 

2. 

3. 

4. 

5. 

Mason's Gain Formula

The transfer function of a signal-flow graph can be found from mason's gain formula as follows.

$$M = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

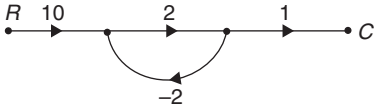
M = Overall gain of the system

P_k = Gain of the k^{th} forward path

Δ = 1 – (Sum of all individual loop gains) + (Sum of gain products of all possible combinations of two non-touching loops) – (Sum of gain products of all possible combinations of three non-touching loops) + …

Δ_k = Same as Δ but formed by loops not touching the k^{th} forward path

Example 18: In the signal-flow graph of figure $\frac{C}{R}$ is equals



(A) $\frac{-20}{3}$ (B) 4

(C) 2 (D) 18

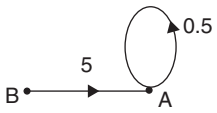
Solution: (B)

Forward paths = 10 – 2 – 1 (20)

Loops = 2 – –2 (–4)

Transfer function = $\frac{20}{1+4} = 4$

Example 19: In the signal-flow graph shown in the figure, if $A = TB$, where T is equal to



(A) 2.5 (B) 5.5

(C) 5 (D) 10

Solution: (D)

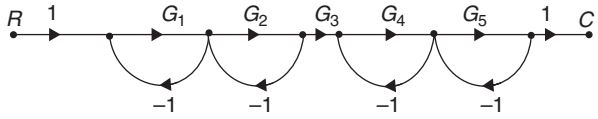
Forward path = 5,

Loop = 0.5

Transfer function = $\frac{A}{B} = \frac{5}{1-0.5} = 10$

$\Rightarrow \quad A = 10\,B$

Example 20: The $\frac{C}{R}$ for the signal-flow graph in figure is



(A) $\frac{G_1\,G_2\,G_3\,G_4\,G_5}{1+G_1+G_2+G_3+G_4}$

(B) $\frac{G_1\,G_2\,G_3\,G_4\,G_5}{\left(1+G_1+G_2+G_3+G_4+G_1\,G_2\,G_4\,G_5\right)}$

(C) $\frac{G_1\,G_2\,G_3\,G_4\,G_5}{\left(1+G_1+G_2\right)\left(1+G_4+G_5\right)}$

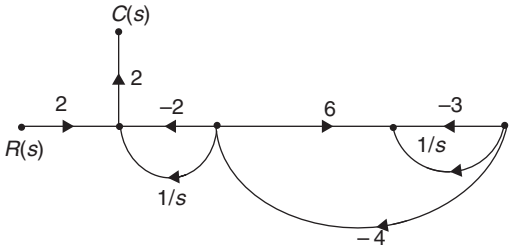
(D) $\frac{G_1\,G_2\,G_4\,G_5}{\left(1+G_1\right)\left(1+G_2\right)\left(1+G_4\right)\left(1+G_5\right)}$

Solution: (C)

Given signal-flow graph can be considered as two series signal-flow graphs

Transfer function = $\frac{G_1\,G_2\,G_3\,G_4\,G_5}{\left(1+G_1+G_2\right)\left(1+G_4+G_5\right)}$

Example 21: The signal flow graph of a system is shown in figure. The transfer function $\frac{C(s)}{R(s)}$ of the system is



(A) $\frac{s+27}{s^2+29s+6}$ (B) $\frac{6s}{s^2+29s+6}$

(C) $\frac{27s}{s^2+6s+6}$ (D) $\frac{4(s+27)}{s^2+29s+6}$

Solution: (D)

Forward path = 2 – 2 (4)

Loops = $-2-\frac{1}{s}\left(-\frac{2}{s}\right)$

$6-\frac{1}{s}--4\left(\frac{-24}{s}\right) = \frac{1}{s}--3\left(\frac{-3}{s}\right)$

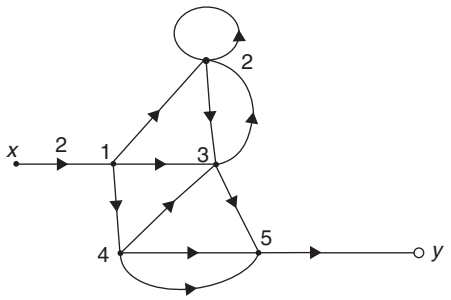
Non-touching loops pair = $\left(\frac{-2}{s}, \frac{-3}{s}\right)$

Non-touching loops to forward path = $\frac{-24}{s}, \frac{-3}{s}$

Transfer function = $\frac{4\left(1+\frac{24}{s}+\frac{3}{s}\right)}{1+\frac{2}{s}+\frac{24}{s}+\frac{3}{s}+\frac{6}{s^2}}$

$= \frac{4(s+27)}{\left(s^2+29s+6\right)}$

Example 22: The signal-flow graph is shown in the figure, has—forward paths and—self-loops



(A) 4, 4 (B) 4, 1 (C) 3, 3 (D) 3, 1

Solution: (B)

Forward path = $X-1-2-3-5-y$

$X-1-3-5-y$

$X-1-4-3-5-y$

$X-1-4-5-y$

Total = 4

Self-loop = 2 – 2

Total = 1

EXERCISES

Practice Problems I

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Which of the following statements are true?
(i) In a closed loop system the effect of non-linearities is reduced.
(ii) Feedback in closed loop may lead to oscillatory response.
(iii) Feedback cannot control dynamics of the system.
(iv) Open loop systems are stable.
(A) i, ii and iv (B) i, ii and iii
(C) ii, iii and iv (D) i and iii

2. The transfer function of a linear system is
(A) Ratio of two quantities which have the same units.
(B) Ratio of the output to the input.
(C) Ratio of the Laplace transform of the output to that of the input.
(D) Ratio of the Laplace transform of the output to that of the input with all initial conditions zeros.

3. Which of the following is not a characteristic of negative feedback system?
(A) Rejection of disturbance signal
(B) High sensitivity to parameter variations
(C) Reduction in gain
(D) Accuracy in tracking steady-state value

4. Which one of the following is not a closed loop system?
(A) Respiratory system of an animal
(B) Execution of a program by a computer
(C) Air-conditioning system
(D) Driving a car

5. If $\frac{dy}{dx} = x$ represents as the equation of an integrator then which of the following is true
(A) The system is stable
(B) The system is unstable
(C) The system is marginally stable
(D) Cannot be determined

6. The speed of response of the given three systems will be in the order

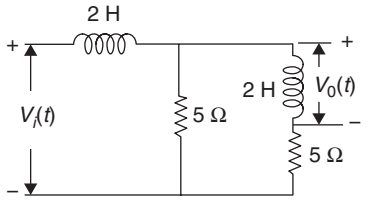
Given $G_1(s) = \frac{5}{0.5s+1}$

$G_2(s) = \frac{5}{2s+1}$

$G_3(s) = \frac{5}{5s+1}$

(A) $G_1(s) > G_2(s) > G_3(s)$
(B) $G_1(s) = G_2(s) = G_3(s)$
(C) $G_1(s) < G_2(s) < G_3(s)$
(D) Cannot be determined

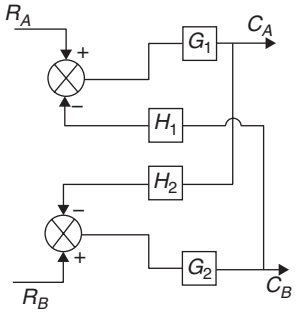
7. Derive the transfer function of the network shown



(A) $\frac{s}{4s^2+30s+25}$ (B) $\frac{10s}{4s^2+30s+25}$

(C) $\frac{1}{s(s^2+6s+5)}$ (D) $\frac{10}{s^2+3s+5}$

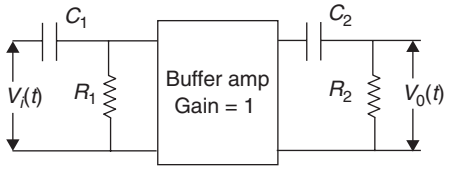
8. Find $\frac{C_A}{R_B}$ of the given system if $G_1 = \frac{20}{S}$, $G_2 = \frac{S}{S+1}$, $H_1 = 50s + 1$ and $H_2 = 0.5s + 1$.



(A) $\frac{20s+1}{25s^2+100s+2}$ (B) $\frac{5s+1}{25s^2+40s+2}$

(C) $\frac{20}{25s^2+100s+10}$ (D) $\frac{(5s+1)}{s^2+4s+25}$

9. Find the transfer function:



Given $R_1 = R_2 = 10\,\Omega$

$C_1 = C_2 = 0.1\ \mu F$

(A) $\frac{s}{s^2+2s+1}$ (B) $= 1$

(C) $\frac{s^2}{(s+1)^2}$ (D) $\frac{(1s)^2}{(s+10)^2}$

10. The Laplace transform of a function is given by $\frac{s}{(s+1)^2}$. Find its impulse response

(A) t^{-t} e cos t (B) $\frac{te^{-t}}{1-t}$

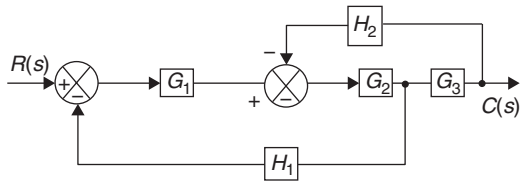
(C) $t\,e^{-t}$ (D) $e^{-t}(1-t)$

11. A system is described by $3\frac{dc(t)}{dt} + c(t) = r(t-3)$, where $r(t)$ and $c(t)$ are the input (ramp) and output, respectively. The transfer function of the system is

(A) $\frac{1+3s}{s^2(1+6s)}$ (B) $\frac{1-3s}{(1+3s)}$

(C) $\frac{1-3s}{s^2(1-6s)}$ (D) $\frac{1-3s}{s(1+3s)}$

12. The transfer function of the system whose block diagram is shown in fig is given by



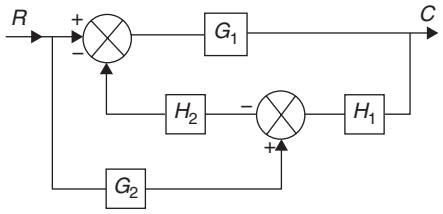
(A) $\frac{1+G_1G_2G_3}{1+G_1G_2H_1+G_2G_3H_2}$

(B) $\frac{G_1G_2G_3}{1+G_1G_2H_1+G_2G_3H_2}$

(C) $\frac{G_1G_2G_3}{1+G_1G_2G_3H_1H_2}$

(D) $\frac{G_1G_2G_3}{1+G_1G_3H_1+G_1G_2H_2}$

13. Using the block diagram given below, find the transfer function:



(A) $\frac{G_2}{H_2}\left[\frac{G_1}{(1+G_1H_1)}\right]$

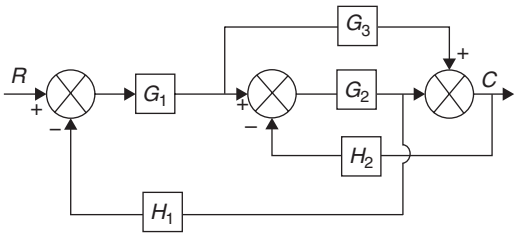
(B) $\frac{G_1[1+G_2]}{1+G_1H_1H_2}$

(C) $\frac{[1+G_2H_2]G_1}{1+G_1H_1H_2}$

(D) $\frac{G_1G_2}{H_2+G_1H_1}$

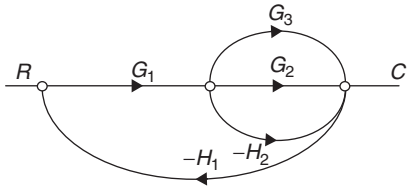
Common Data for Questions 14 and 15:

Given the block diagram representation of a closed loop control system

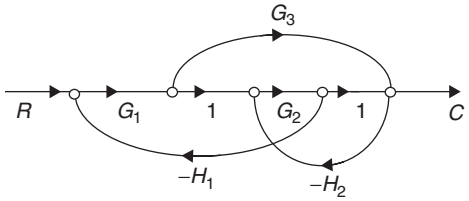


14. Draw the signal-flow graph of the given system

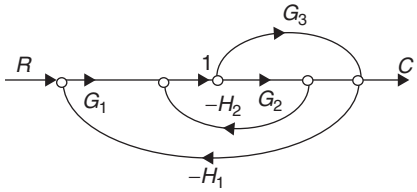
I.



II.



III.



(A) I and II are true
(B) All true
(C) II only
(D) I only

15. Obtain the transfer function representation of the above block diagram.

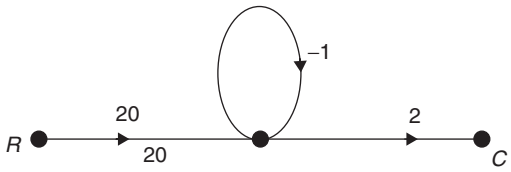
(A) $\dfrac{G_1G_2+G_1G_3}{1+G_1G_2H_2H_3+G_2H_2}$

(B) $\dfrac{G_1G_2+G_1G_3}{1+G_2H_2+G_1G_2H_1+G_3H_1}$

(C) $\dfrac{G_1G_2+G_1G_3}{1+G_2H_2+G_1G_2H_1}$

(D) $\dfrac{G_1G_2+G_1G_3}{G_2H_2+G_1G_2H_1+G_2G_3H_2}$

16. Transfer function $\dfrac{C}{R}$ of the given signal-flow graph is



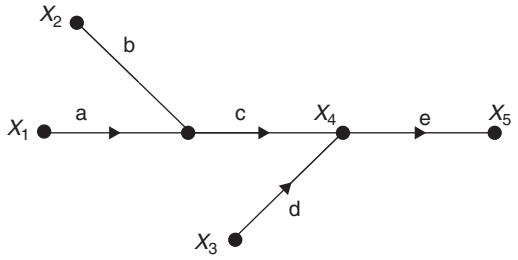
(A) 40 (B) 20
(C) 10 (D) 2

17. The dynamics equation related to a system is given by $\dfrac{d^2c(t)}{dt^2}+5\dfrac{dc(t)}{dt}+8c(t)=7r(t)$ where $r(t)$ is input and $c(t)$ is output. Transfer function of the system is given by

(A) $\dfrac{8}{7s^2+5s+1}$ (B) $\dfrac{7}{s^2+5s+8}$

(C) $\dfrac{5}{s^2+7s+8}$ (D) $\dfrac{7}{7s^2+5s+8}$

18. The expression for the following signal-flow graph is



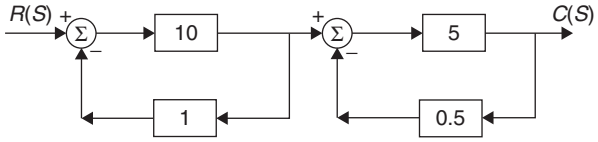
(A) $\left[(acX_1+bcX_2)+dX_3\right]e$

(B) $\left[(aX_1+bX_2)cdX_3\right]e$

(C) $\left[(aX_1+bX_2)+dX_3\right]e$

(D) $\left[acX_1+bcX_2+dX_3\right]e$

19. The transfer function $\dfrac{C(s)}{R(s)}$ of the block diagram given bellow is



(A) 5 (B) 10 (C) 15 (D) 50

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. In a time-variant system,
(A) The system parameters are independent of time.
(B) The system parameters are functions of time.
(C) The input and output are functions of time.
(D) The system parameters depend on time varying input and output.

2. The transfer function of a tachometer has
(A) A pole at origin, and a zero anywhere in the real axis
(B) A zero at origin, pole anywhere in the real axis
(C) Only a zero at origin
(D) Only a pole at origin

3. Match the following

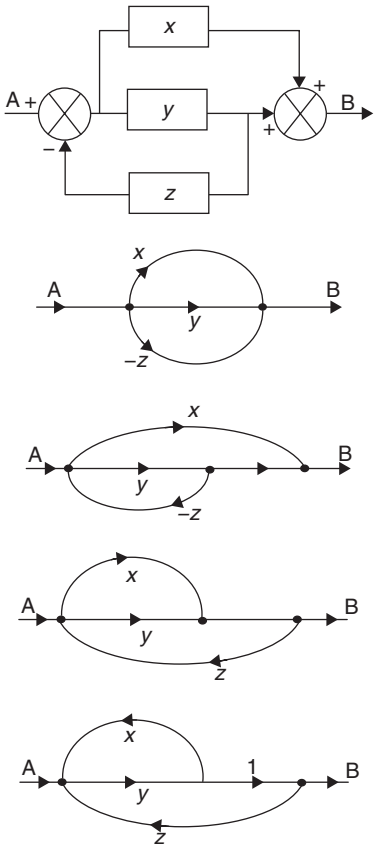
	Type of roots		Nature of response term
1.	Single root at $s=\sigma$	I	$A\sin(\omega t+\beta)$
2.	Roots of multiplicity K at the origin	II	$(A_1+a_2t+\ldots)e^{\sigma t}$
3.	Roots of multiplicity K, at $s=\sigma$	III	$Ae^{\sigma t}$
4.	Single complex conjugate root pair on the $j\omega$ axis	IV	$A+A_2t+\ldots+A_Kt^{K-1}$

(A) I – III, 2 – II, 3 – IV, 4 – I
(B) 1 – III, 2 – IV, 3 – II, 4 – I
(C) I – III, 2 – I, 3 – IV, 4 – II
(D) 1 – I, 2 – III, 3 – II, 4 – IV

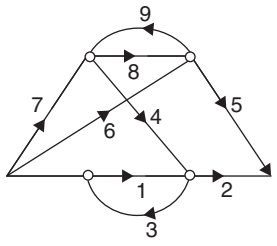
4. A system $Y = f(x)$ is said to be linear if it satisfies the following properties.
(A) $f(x+y) = f(x) + f(y)$
(B) $f(\alpha x) = \alpha f(x)$
(C) $f(ax_1 + bx_2) = af(x_1) + bf(x_2)$
(D) None

5. The error transfer function of system is given by
(A) $E(s) = \dfrac{G(s)}{1+G(s)\,H(s)} - R(s)$
(B) $E(s) = \dfrac{R(s)}{1+G(s)\,H(s)}$
(C) $E(s) = \dfrac{1}{1+G(s)\,H(s)}$
(D) $E(s) = R(s) - G(s)\,H(s)$

6. The signal-flow graph for the given system is



Common Data for Questions 7 and 8:
SFD of a system is shown below. Study the system to give answers to the following questions.

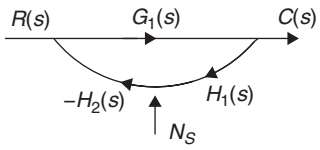


7. Find the number of forward paths in the given SFD.
(A) 5 (B) 4 (C) 6 (D) 7

8. Find the number of independent loops in the system.
(A) 4 (B) 3 (C) 2 (D) 5

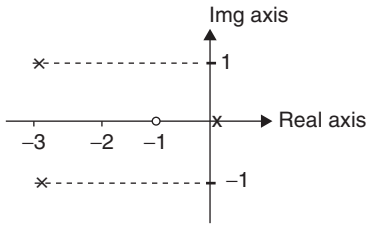
9. The impulse response of a system is given by $g(t) = e^{-2t}(1-\cos 2t)$. Find the transfer function of the system.
(A) $\dfrac{1}{s+2}$ (B) $\dfrac{4}{(s+2)(s^2+4s+8)}$
(C) $\dfrac{8}{(s+2)(s^2+4s+8)}$ (D) $\dfrac{1}{(s^2+4s+8)}$

10. For the system shown if gain is very high, the transfer function $\dfrac{C(s)}{R(s)}$ will be



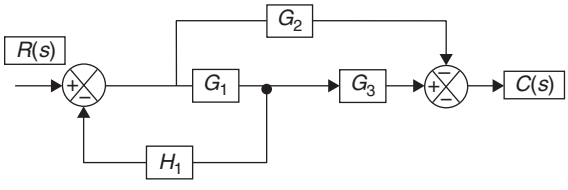
(A) $\dfrac{G_1(s)}{1+G_1(s)\,H_1(s)\,H_2(s)}$
(B) $\dfrac{G_1(s)}{1-H_1(s)\,H_2(s)\,N(s)}$
(C) $\dfrac{1}{H_1(s)\,H_2(s)}$
(D) $\dfrac{1}{1-H_1(s)\,H_2(s)}$

11. A transfer function $G(s)$ has the pole zero plot as shown in the figure. Given the steady-state gain as 10, find the transfer function.



(A) $\dfrac{10(s+1)}{s^2+6s+10}$ (B) $\dfrac{10(s-1)}{(s^2+6s+10)s}$
(C) $\dfrac{10(s+1)}{s^2(s^2+6s+10)}$ (D) $\dfrac{10(s+1)s}{s^2+6s+10}$

12.



The transfer function of the block diagram shown in above figure is

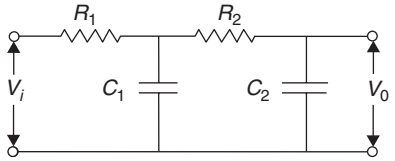
(A) $\frac{G_1G_2 - G_3}{1 + G_1H}$

(B) $\frac{G_1G_3 + G_2}{1 + G_1H}$

(C) $\frac{G_1G_2}{1 + G_1H} - G_3$

(D) $\frac{G_1G_3 - G_2}{1 + G_1H}$

13. The transfer function of the electrical network shown in the figure is



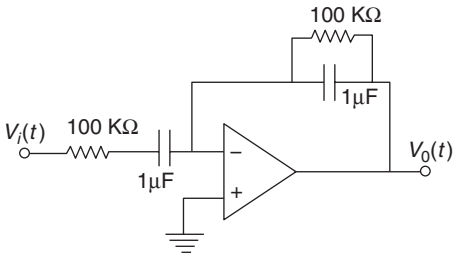
(A) $\frac{1}{1 + s[R_1C_1 + R_2C_2 + R_1C_2] + s^2R_1R_2C_1C_2}$

(B) $\frac{1}{1 + s^2[R_1C_1 + R_2C_2 + R_2C_1] + s^4R_1R_2C_1C_2}$

(C) $\frac{1}{1 + s^2[R_1C_1 + R_2C_2 + R_1C_2] + s^4R_1R_2C_1C_2}$

(D) $\frac{1}{1 + s[R_1C_1 + R_2C_2 + R_1C_2] + s^3R_1R_2C_1C_2}$

14. For the system given, find the transfer function:



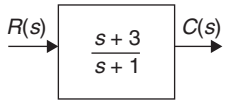
(A) $\frac{(0.1s + 2)s}{(0.1s + 1)^2}$

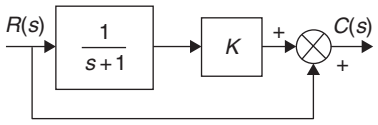
(B) $\frac{0.1s}{(0.1s + 2)}$

(C) $\frac{-0.1s}{(0.1s + 1)^2}$

(D) None of these

15. For what value of k, are the two block diagrams shown below equivalent?





(A) 2

(B) 1

(C) $s + 1$

(D) $s + 2$

Previous Years' Questions

1. Two systems with impulse responses $h_1(t)$ and $h_2(t)$ are connected in cascade. Then the overall impulse response of the cascaded system is given by **[2013]**

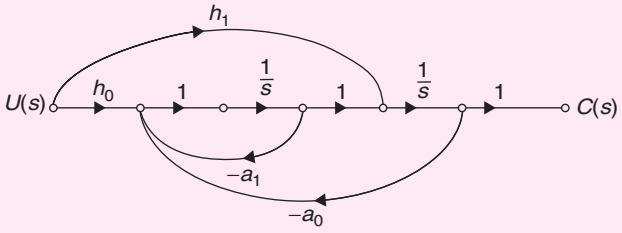
(A) Product of $h_1(t)$ and $h_2(t)$

(B) Sum of $h_1(t)$ and $h_2(t)$

(C) Convolution of $h_1(t)$ and $h_2(t)$

(D) Subtraction of $h_2(t)$ from $h_1(t)$

2. The signal-flow graph of a system is shown below. $U(s)$ is the input and $C(s)$ is the output. **[2014]**



Assuming $h_1 = b_1$ and $h_0 = b_0 - b_1 a_1$, the input-output transfer function, $G(s) = \frac{C(s)}{U(s)}$ of the system is given by

(A) $G(s) = \frac{b_0s + b_1}{s^2 + a_0s + a_1}$

(B) $G(s) = \frac{a_1s + a_0}{s^2 + b_1s + b_0}$

(C) $G(s) = \frac{b_1s + b_0}{s^2 + a_1s + a_0}$

(D) $G(s) = \frac{a_0s + a_1}{s^2 + b_0s + b_1}$

3. A single-input single-output feedback system has forward transfer function $G(s)$ and feedback transfer

function $H(s)$. It is given that $|G(s)\,H(s)| < 1$. Which of the following is true about the stability of the system? **[2014]**

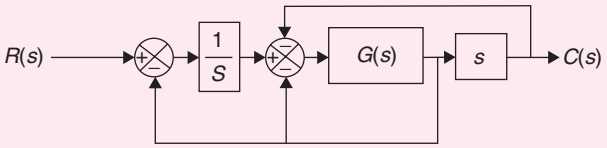
(A) The system is always stable

(B) The system is stable if all zeros of $G(s)H(s)$ are in left half of the S-plane.

(C) The system is stable if all poles of $G(s)H(s)$ are in left half of the S-plane.

(D) It is not possible to say whether or not the system is stable from the information given.

4. The block diagram of a system is shown in the figure **[2014]**

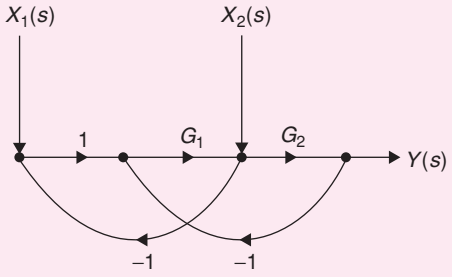


If the desired transfer function of the system is $\frac{C(s)}{R(s)} = \frac{s}{s^2+s+1}$ then $G(s)$ is

(A) 1 (B) s

(C) $1/s$ (D) $\frac{-s}{s^3+s^2-s-2}$

5. For the signal-flow graph shown in the figure, which one of the following expressions is equal to the transfer function $\left.\frac{Y(s)}{X_2(s)}\right|_{X_1(s)=0}$? **[2015]**



(A) $\frac{G_1}{1+G_2(1+G_1)}$ (B) $\frac{G_2}{1+G_1(1+G_2)}$

(C) $\frac{G_1}{1+G_1G_2}$ (D) $\frac{G_2}{1+G_1G_2}$

6. In the signal flow diagram given in the figure, u_1 and u_2 are possible inputs whereas y_1 and y_2 are possible outputs. When would the SISO system derived from this diagram be controllable and observable? **[2015]**

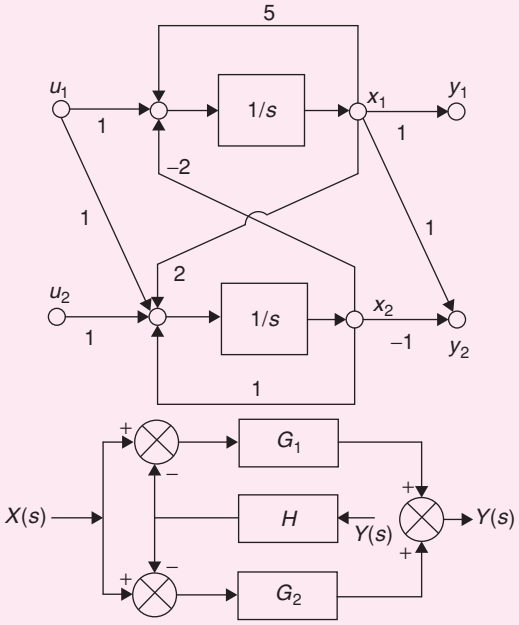
(A) When u_1 is the only input and y_1 is the only output.

(B) When u_2 is the only input and y_1 is the only output.

(C) When u_1 is the only input and y_2 is the only output.

(D) When u_2 is the only input and y_2 is the only output.

7. Find the transfer function $\frac{Y(s)}{X(s)}$ of the system given below. **[2015]**



(A) $\frac{G_1}{1-HG_1}+\frac{G_2}{1-HG_2}$

(B) $\frac{G_1}{1+HG_1}+\frac{G_2}{1+HG_2}$

(C) $\frac{G_1+G_2}{1+H(G_1+G_2)}$

(D) $\frac{G_1+G_2}{1-H(G_1+G_2)}$

8. An open loop control system results in a response of $e^{-2t}(\sin 5t + \cos 5t)$ for a unit impulse input. The DC gain of the control system is _____. **[2015]**

9. For linear time invariant systems, that are Bounded Input Bounded Output stable, which one of the following statements is TRUE? **[2015]**

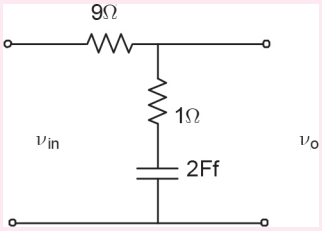
(A) The impulse response will be integrable, but may not be absolutely integrable.

(B) The unit impulse response will have finite support.

(C) The unit step response will be absolutely integrable.

(D) The unit step response will be bounded.

10. For the network shown in the figure below, the frequency (in rad/s) at which the maximum phase lag occurs, is _____. **[2016]**



11. A second-order real system has the following properties: **[2016]**

(a) the damping ratio $\delta = 0.5$ and undamped natural frequency $\omega_n = 10$rad/s

(b) the steady state value of the output, to a unit step input, is 1.02.

The transfer function of the system is

(A) $\frac{1.02}{S^2+5S+100}$ (B) $\frac{102}{S^2+10S+100}$

(C) $\frac{100}{S^2+10S+100}$ (D) $\frac{102}{S2+5S+100}$

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** D	**3.** B	**4.** B	**5.** C	**6.** A	**7.** B	**8.** C	**9.** C	**10.** D
11. B	**12.** B	**13.** C	**14.** C	**15.** C	**16.** B	**17.** B	**18.** A	**19.** D	

Practice Problems 2

1. B	**2.** C	**3.** B	**4.** C	**5.** C	**6.** B	**7.** A	**8.** A	**9.** B	**10.** C
11. C	**12.** B	**13.** A	**14.** C	**15.** A					

Previous Years' Questions

1. C	**2.** C	**3.** A	**4.** B	**5.** B	**6.** B	**7.** C	**8.** 0.23 to 0.25	**9.** D
10. 0.316	**11.** B							

Chapter 2

Time Response Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Time response
- Steady state response
- Standard test signals
- Type of the system
- Order of the system
- Unit step response
- Delay time
- Rise time
- Settling time
- Steady state error analysis
- Parabolic input

TIME RESPONSE

The response given by the system which is function of time for the applied excitation is called 'time response'.

Time response can be obtained by solving the differential equations governing the system (or) from the transfer function of the system and input given to the system.

$$C(t) = C_t(t) + C_{ss}(t)$$

The time response of a control system consists of two parts:

1. Transient response $C_t(t)$
2. Steady-state response $C_{ss}(t)$

Transient Response

The output variation during the time it takes to achieve its final value is called as 'transient response'.

The time required to achieve the final values is called 'transient period'.

For a stable operating system, transient response $C_t(t)$ can be written as

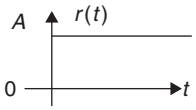
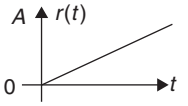
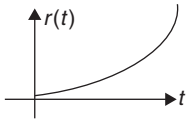
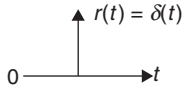
$$\lim_{t \to \infty} C_t(t) = 0$$

Steady-state Response

The response which remains after complete transient response vanishes (transient period). Steady-state response indicates the accuracy of a system.

Standard Test Signals

The information of input signals is required to pre-estimate the response of a system. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and a constant acceleration. So the equivalent test signals are used as input signals to predict the performance of the system.

Name of the signal	Waveform	Physical interpretation	Mathematical representation	Laplace transform
Step		A sudden change/ constant position	$r(t) = A\ t \geq 0 = 0\ t < 0$	$\frac{A}{s}$
Ramp		A constant velocity	$r(t) = \text{At}\ t \geq 0 = 0\ t < 0$	$\frac{A}{s^2}$
Parabolic		A constant acceleration	$r(t) = \frac{At^2}{2}\ t \geq 0 = 0\ t < 0$	$\frac{A}{s^3}$
Impulse		A sudden shock	$r(t) = \infty;\ t = 0 = 0\ t \neq 0$	1

Type and Order or of the System

Type of the System

Type of a system is defined as the number of open loop poles at origin.

Steady-state behaviour of the system depends on the type of the system.

Example:

$$G(s)\,H(s) = \frac{K\left(1+T_{z_1}s\right)\left(1+T_{z_2}s\right)\left(1+T_{z_3}s\right)\ldots}{s^n\left(1+T_{p_1}s\right)\left(1+T_{p_2}s\right)\left(1+T_{p_3}s\right)\ldots}$$

If $n = 0 \Rightarrow$ type '0' system
If $n = 1 \Rightarrow$ type '1' system
$\therefore$ n = type of the system

Order of the System

The highest power of 's' in the denominator of the closed loop transfer function is defined as the order of the system.

Example:

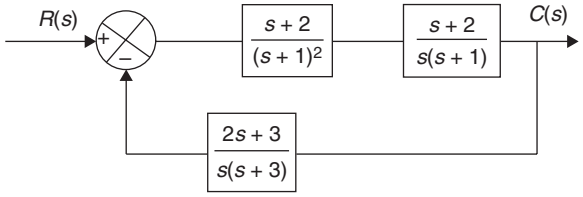
$$\text{Transfer function} = \frac{C(S)}{R(S)} = \frac{a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_n}{b_0s^m + b_1s^{m-1} + b_2s^{m-2} + \ldots + b_m}$$

In the above behaviour of the system depends on the order of the system.

Transient behaviour of the system depends on the order of the system.

Solved Examples

Example 1: The feedback control system shown in the figure is



(A) Type '0' system (B) Type '1' system
(C) Type '2' system (D) Type '3' system

Solution: (C)
Number of integrators in the open loop transfer function are '2'. So the type of the system is '2'.

Example 2: A system has the following transfer function

$$G(s) = \frac{10(s+5)(s+50)}{s^4(s+1)(s^2+3s+10)}$$

The type and order of the system are, respectively,
(A) 4 and 9 (B) 4 and 7
(C) 5 and 7 (D) 7 and 5

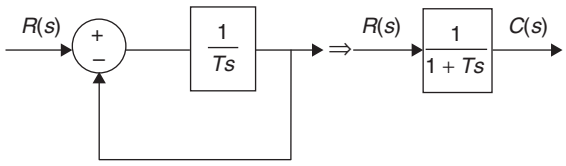
Solution: (B)
Number of integrators = 4 = type of system
Highest power of 's' in denominator = order of the system = 7

Example 3: Type of a system depends on the
(A) No. of poles
(B) Difference between no. of poles and zeros
(C) No. of real poles
(D) No. of poles it has at the origin
Solution: (D)

Time Response of a First-order System

A system with highest power of 's' in the denominator of its transfer function equal to 1 is known as 'first-order' system

Consider a first order unity feedback control system shown in the figure



$$\text{Transfer function of the system} = \frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$

Unit Step Response

Unit step input $r(t) = u(t)$

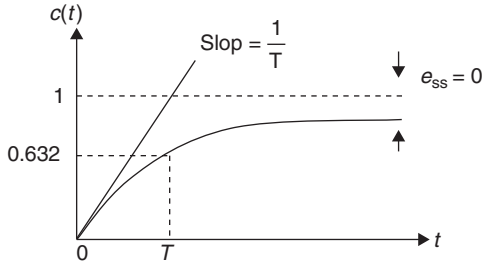
$$R(t) = \frac{1}{s}$$

Unit step response in 's' domain $[C(s)] = \frac{1}{1+Ts} * \frac{1}{s}$

Unit step response in time domain $[C(t)] = 1 - e^{-t/T}$

Steady-state value of the response $= \underset{t\to\infty}{\text{Lt}}\, C(t) = 1$

Transient response of the system $= e^{-t/\tau}$



Time Response for Unit Step Input

Time Constant

Time takes for the step response to raise to 63% of its final value.

Rise Time (T_r)

Time taken by the step response to go from 10% to 90% of its final value.

$$T_r = 2.31T - 0.11T = 2.22T$$

Setting Time (T_s)

Time taken by the step response to reach and stay within 2% or 5% of its final value.

$$T_s = 4T \text{ for } \pm 2\% \text{ tolerance}$$

$$= 3T \text{ for } \pm 5\% \text{ tolerance}$$

Unit Ramp Response

Unit ramp input $r(t) = tu(t)$

$$R(s) = \frac{1}{s^2}$$

Unit ramp response in 's' domain $= \frac{1}{1+T_s} \times \frac{1}{s^2}$

Unit ramp response in time domain $c(t) = L^{-1}\left\{\frac{1}{s^2(1+T_s)}\right\}$

$$e(t) = (t - \tau + \tau e^{-t/\tau})$$

Error for unit ramp input $e(t) = r(t) - c(t) = (\tau - \tau e^{-t/\tau})$

The steady-state error for unit ramp input $= \underset{t\to\infty}{\text{Lt}}\, e(t)$

$$= \underset{t\to\infty}{\text{Lt}} \left(\tau - \tau e^{-t/\tau}\right)$$

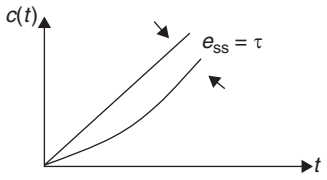


Figure 1 Unit ramp response of first-order system

Unit Impulse Response

Unit impulse input $r(t) = \delta(t)$

$$R(s) = 1$$

Unit impulse response in 's' domain $c(s) = \frac{1}{1+T_s}$

Unit impulse response in time domain $c(t) = \frac{1}{T} e^{-t/T}$ for $t \geq 0$

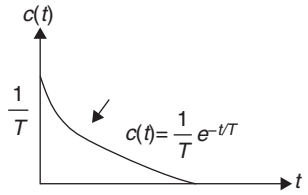


Figure 2 Unit impulse response of first-order system

Example 4: Transfer function of a system is given by

$$G(s) = \frac{20}{(s+20)}$$

rise time of the system is

(A) 0.33 s (B) 44 s

(C) 0.11 s (D) 0.2 s

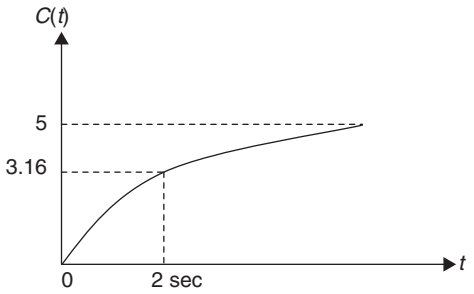
Solution: (C)

Transfer function $= \frac{20}{s+20} = \frac{1}{(0.05s+1)}$

Time constant = 0.05 s

Rise time = 2.2 × Time constant = 0.11 s

Example 5: Time response of the system $G(s) = \frac{K}{s+a}$ is given by



Then the values of 'K' and 'a' are

(A) $K = 5, a = 2$ (B) $K = 5, a = 0.5$

(C) $K = 2.5, a = 0.5$ (D) $K = 2.5, a = 2$

Solution: (C)
Time constant = 2 s
Steady-state value = 5

$$\text{Transfer function} = \frac{5}{2s+1} = \frac{2.5}{(s+0.5)}$$

$$\therefore \quad K = 2.5$$
$$a = 0.5$$

Example 6: Transfer function of the system is given by

$$G(s) = \frac{2}{(s+2)}$$

Time required for the system unit step response to reach 95% of its final value is
(A) 0.5 s (B) 1 s (C) 1.5 s (D) 2 s

Solution: (C)

$$\text{Transfer function} = \frac{2}{s+2} = \frac{1}{1+0.5\,s}$$

Time constant = 0.5

Time required for unit step response to reach 95% of its final value = 3 × Time constant = 1.5 s

Time Response of a Second-order System

A system with highest power of 's' in the denominator of its transfer function equal to '2' is known as 'second-order' system.

Closed loop transfer function for a standard second-order system is given by

$$\text{Transfer function} = \frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}} = \frac{N(s)}{D(s)}$$

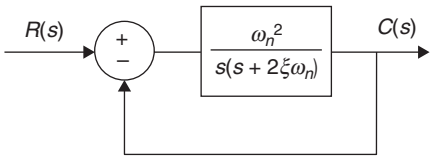


Figure 3 Block diagram of standard second-order system

where ω_n = Undamped natural frequency
ξ = Damping ratio constant

Tendency to oppose the oscillatory behaviour of the system is called as 'damping', it is denoted by 'ξ'. As 'ξ' is increased, the response becomes progressively less oscillatory till it becomes critically damped ($\xi = 1$) and becomes over damped for $\xi > 1$.

The denominator polynomial of the transfer function $D(s)$ is called as 'characteristic equation'. The characteristic equation of the second-order system is

$$s^2 + 2\xi\omega_n s + \omega_n^{\,2} = 0$$

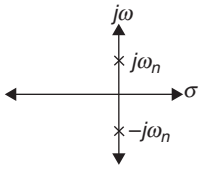
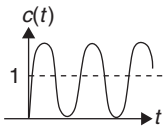
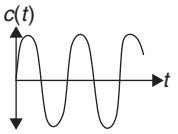
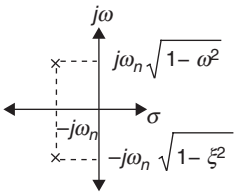
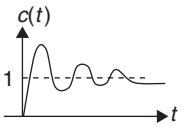
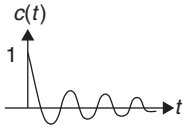
The roots of the characteristic equation are given by

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

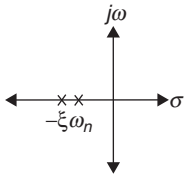
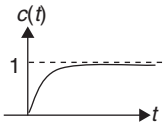
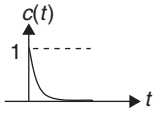
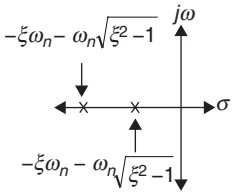
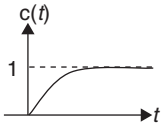
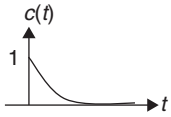
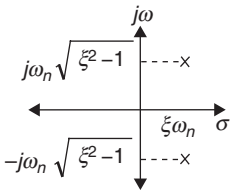
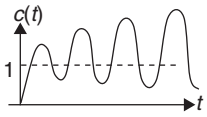
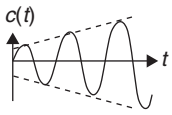
where $\omega_d = \omega_n\sqrt{1-\xi^2} \Rightarrow$ Damping natural frequency.

The time response of any system is characterized by the roots of the denominator polynomial (poles of the system), which depends on the damping ratio (ξ) of the system. Detailed summary of the system response characterization with variation is ξ/poles is given in the table.

Summary of Second-order System Response

System characterization	Range of ξ	Poles of second-order system	Poles on the *S*-plane	Nature of the response	Unit step response	Impulse response
Undamped	$\xi = 0$	$\pm j\omega_n$		Constant frequency and magnitude oscillations		
Under damped	$0 < \xi < 1$	$-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$		Damped oscillations		

(*Continued*)

System characterization	Range of ξ	Poles of second-order system	Poles on the *S*-plane	Nature of the response	Unit step response	Impulse response
Critically damped	$\xi = 1$	$-\xi\omega_n$		Pure exponential		
Over damped	$\xi > 1$	$-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$		Slow exponential		
Negative damped	$\xi < 0$	$\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$		Increasing oscillations		

Transient Response Specifications of Second-order System

The common model for physical problems is under damped second-order system due to its speed in reaching steady state. The specifications of the under damped second-order system are shown in figure.

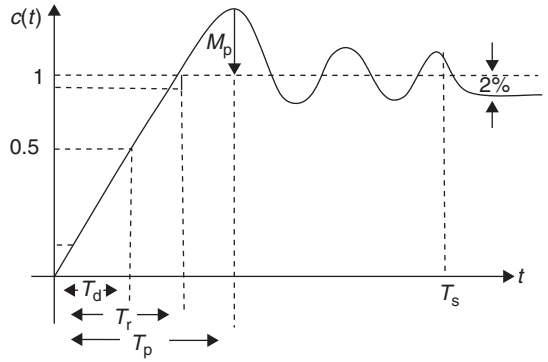


Figure 4 Unit step response of the under damped second-order system

Delay Time, t_d

The delay time is the time required for the response to reach half the final value for the very first time.

$$t_d = \frac{1 + 0.7\xi}{\omega_n} s$$

Rise Time, t_r

It is the time required for the response to rise from 0 to 100% of its final value for undamped case. For underdamped case and overdamped case, it is 10% to 90% of its final value.

i.e., $$t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

where $\beta = \cos^{-1}\xi$

Peak Time, t_p

The peak time is the time required for the response to reach the first peak of the overshoot.

i.e., $$t_p = \frac{\pi}{\omega_d}$$

The peak time t_p corresponds to one half-cycle of the frequency of damped oscillation.

Peak Overshoot, M_p

The maximum overshoot is the maximum peak value of the response curve measured from unity.

$$M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\% \text{ peak overshoot} = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

Settling Time (t_s)

This is the time required for the response to reach and remain within a specified tolerance band of its final value.

The settling time corresponding to a ±2% or ±5% tolerance band may be measured in terms of the time constant $T = \dfrac{1}{\xi\omega_n}$

$$t_s = 4T = \frac{4}{\xi\omega_n} \quad (2\%\text{ band})$$

$$t_s = 3T = \frac{3}{\xi\omega_n} \quad (5\%\text{ band})$$

Note: Time period of oscillations $T_{osci} = \dfrac{2\pi}{\omega_d}$

Number of oscillations $(N) = \dfrac{t_s}{T_{osci}} = \dfrac{t_s\,\omega_d}{2\pi}$

Example 7: A unity feedback system has open loop transfer function as $G(s) = \dfrac{25}{s(s+25)}$

The natural frequency of the system is

(A) 10 (B) 5 (C) 25 (D) 2.5

Solution: (B)

Given open loop transfer function $= \dfrac{25}{s(s+25)}$

Standard open loop transfer function $= \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

$\therefore \quad \omega_n^2 = 25$

Natural frequency $\omega_n = 5$ rad/s

Example 8: Given a unity feedback with

$$G(s) = \frac{K}{s(s+5)}$$

The value of 'K' for damping ratio of 0.5 is

(A) 1 (B) 25 (C) 5 (D) 2.5

Solution: (B)

Standard second-order OLTF $= \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

Comparing with given OLTE $2\xi\omega_n = 5$

$$2 \times 0.5 \times \omega_n = 5$$

$\Rightarrow \quad \omega_n = 5$

$\therefore \quad K = \omega_n^2 = 25$

Example 9: The unit impulse response of a second-order system is $\dfrac{1}{6}e^{-0.8t}\sin(0.6t)$. Then the natural frequency and damping ratio of the system are, respectively,

(A) 2 and 0.3 (B) 1 and 0.6

(C) 1 and 0.8 (D) 2 and 0.8

Solution: (C)

Transfer function = $L\{\text{Impulse response}\}$

$$= \frac{1}{6}\cdot\frac{0.6}{(s+0.8)^2 + (0.6)^2}$$

Transfer function $= \dfrac{0.1}{s^2 + 1.6s + 1}$

After comparing with standard second-order transfer function

$$\omega_n = 1,\ 2\xi\omega_n = 1.6$$

$$\xi = 0.8$$

Example 10: A second-order system has

$$M(j\omega) = \frac{100}{100 - \omega^2 + j20\omega}$$

If M_p (peak magnitude) is

(A) 100 (B) 1 (C) 0.5 (D) 2

Solution: (B)

Comparing with standard second-order system,

$$\xi = 1,\ \omega_n = 10$$

System is critically damped $\Rightarrow$ Peak magnitude = 1

Example 11: A unity feedback control system has a forward path transfer function equal to $G(s) = \dfrac{30.25}{s(s+5.5)}$

The unit step response of the system starting form rest will have its maximum value at a time equal to

(A) 0.5 s (B) 0.66 s

(C) 0 s (D) Infinity

Solution: (B)

Standard form of second-order OLTF $= \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

$\therefore \quad \omega_n = 5.5,\ \xi = 0.5$

Peak time $(t_p) = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.66$ s

Example 12: A plant has the following transfer function

$$G(s) = \frac{1}{s^2 + 0.4s + 1}$$

For a step input, it is required that the response value settles to within 2% of its final value. The plant setting time is

(A) 10 s (B) 20 s

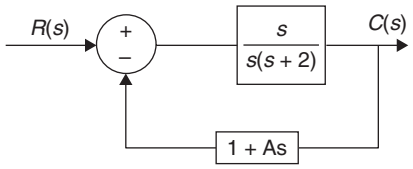
(C) 30 s (D) Infinity

Solution: (B)

For the given system transfer function $\omega_n = 1,\ \xi = 0.2$

Setting time of 2% tolerance $(t_s) = \dfrac{4}{\xi\omega_n} = \dfrac{4}{0.2} = 20$ s

Example 13: The block diagram of a closed loop control system is given in figure. The value of 'K' and 'A' such that the system has a damping ratio of 0.8 and undamped natural frequency ω_n of 5 rad/s, are respectively equal to



(A) 4 and 0.8 (B) 5 and 0.24
(C) 25 and 0.24 (D) 20 and 0.8

Solution: (C)
Transfer function of the given system

$$= \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2+(2+AK)s+(K)} \quad (1)$$

$$\text{Required transfer function} = \frac{25}{s^2+8s+25} \quad (2)$$

Comparing equations (1) and (2),we have

$$K = 25$$

$$2 + AK = 8$$

$$25A = 6$$

$$\Rightarrow \quad A = \frac{6}{25} = 0.24$$

Steady-state Error Analysis

Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$.

A generalized expression for steady-state error is obtained by using feedback system shown in the following figure.

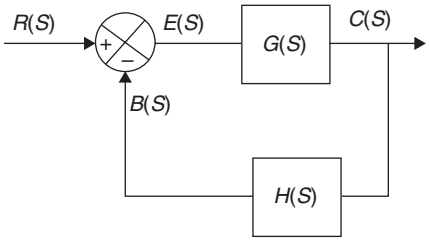


Figure 5 Feedback control system

From the above block diagram

$$E(s) = R(s) - B(s) = R(s) - C(s)\,H(s)$$

$$E(s) = R(s) - E(s)\,G(s)\,H(s)$$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

The steady-state error using final value theorem can be written as

$$e_{ss} = \underset{t\to\infty}{\text{Lt}}\, e_{ss}(t) = \underset{s\to0}{\text{Lt}}\, s\,E(S) = \underset{s\to0}{\text{Lt}}\, s.\frac{R(s)}{1+G(s)H(s)}$$

From the above expression is evident that steady-state error depends on input $R(s)$ and the open loop transfer function $G(s)H(s)$.

The expressions for steady-state errors for various types of standard test signals are given below.

(i) **Unit step input:**

$$\text{Steady-state error} = e_{ss} = \underset{s\to0}{\text{Lt}} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \underset{s\to0}{\text{Lt}} \frac{s.1/s}{1+G(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\underset{s\to0}{\text{Lt}}\, G(s)H(s)} = \frac{1}{1+K_P}$$

$K_P = \underset{s\to0}{\text{Lt}}\, G(s)H(s)$ = Position error constant

(ii) **Unit ramp input:**

$$\text{Steady-state error} = e_{ss} = \underset{s\to0}{\text{Lt}} \frac{s.1/s^2}{1+G(s)H(s)}$$

$$= \underset{s\to0}{\text{Lt}} \frac{1}{s+sG(s)H(s)}$$

$$= \frac{1}{\underset{s\to0}{\text{Lt}}\, G(s)H(s)} = \frac{1}{K_V}$$

$K_V = \underset{s\to0}{\text{Lt}}\, sG(s)H(s)$ = Velocity error constant

(iii) **Unit parabolic input:**

$$\text{Steady-state error } e_{ss} = \underset{s\to0}{\text{Lt}} \frac{s.1/s^3}{1+G(s)H(s)}$$

$$= \frac{1}{\underset{s\to0}{\text{Lt}}\, s^2G(s)H(s)} = \frac{1}{K_a}$$

$K_a = \underset{s\to0}{\text{Lt}}\, s^2G(s)H(s)$ = Acceleration error constant.

Steady-state Errors for Different Types of System

Type 0 System

Open loop transfer function $G(S)H(s)$

$$= \frac{K(T_{z1}S+1)(T_{z2}S+1).....}{(T_{p1}S+1)(T_{p2}S+1).....}$$

For step input: $e_{ss} = \dfrac{1}{1+K_P}$

Position error constant $(K_P) = \underset{s\to0}{\text{Lt}}\, G(S)\,H(s) = K$

Steady-state error $(e_{ss}) = \dfrac{1}{1+K}$

For Ramp Input

Velocity error constant $(K_v) = \underset{s\to 0}{\text{Lt}}\ s\ G(S)\ H(s) = 0$

Steady-state error $(e_{ss}) = \infty$

For Parabolic Input

Acceleration error constant $(K_a) = \underset{s\to 0}{\text{Lt}}\ s^2\ G(S)H(s) = 0$

Steady-state error $(e_{ss}) = \infty$

Type 1 System

Open loop transfer function

$$G(S)H(s) = \frac{K\left(T_{z1}s+1\right)\left(T_{z2}s+1\right)\left(T_{z3}s+1\right)\ldots}{S\left(T_{p1}s+1\right)\left(T_{p2}s+1\right)\left(T_{p3}s+1\right)\ldots}$$

Step Input

Position error constant $(K_p) = \underset{s\to 0}{\text{Lt}}\ G(S)\ H(s) = \infty$

$$\text{Steady-state error } (e_{ss}) = \frac{1}{1+K_p} = 0$$

Parabolic Input

Acceleration constant $(K_a) = \underset{s\to 0}{\text{Lt}}\ S^2\ G(S)\ H(s) = 0$

$$\text{Steady-state error } (e_{ss}) = \frac{1}{K_a} = \infty$$

Type 2 System

Open loop transfer function

$$G(S)H(s) = \frac{K\left(T_{z1}s+1\right)\left(T_{z2}s+1\right).....}{s^2\left(T_{p1}s+1\right)\left(T_{p2}s+1\right).....}$$

Step Input

Position error constant $(K_p) = \underset{s\to 0}{\text{Lt}}\ G(S)\ H(s) = \infty$

$$\text{Steady-state error } (e_{ss}) = \frac{1}{1+K_p} = 0$$

Ramp Input

Velocity error constant $(K_v) = \underset{s\to 0}{\text{Lt}}\ s\ G(S)\ H(s) = \infty$

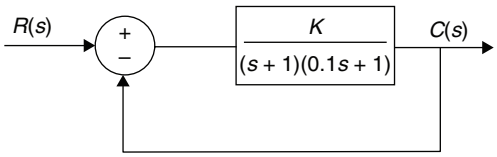
$$\text{Steady-state error } (e_{ss}) = \frac{1}{K_v} = 0$$

Parabolic Input

Acceleration error constant $(K_a) = \underset{s\to 0}{\text{Lt}}\ s\ G(S)H(s) = K$

$$\text{Steady-state error } (e_{ss}) = \frac{1}{K_a} = \frac{1}{K}$$

Example 14: The system shown in the figure has a unit step input. In order that the steady-state error is 0.1, the value of 'K' required is



(A) 0.1 (B) 9
(C) 0.9 (D) 1.0

Solution: (B)

Steady-state error $= \underset{s\to 0}{\text{Lt}}\ s.\frac{R(s)}{1+G(s)H(s)}$

Input is unit step

$$e_{ss} = \frac{1}{1+\underset{s\to 0}{\text{Lt}}\ G(s)H(s)} = \frac{1}{1+K_p}$$

$$K_p = \underset{s\to 0}{\text{Lt}}\ G(s)H(s) = \underset{s\to 0}{\text{Lt}}\ \frac{K}{(s+1)(0.1s+1)} = K$$

From the given problem,

$$e_{ss} = \frac{1}{1+K} = 0.1$$

$$K = 9$$

Example 15: If the time response of the system is given by thefollowing expression

$$y(t) = 5 + 4\sin(\omega t + \delta_1) + e^{-8t}\sin(\omega t + \delta_2) + e^{-4t},$$

then the steady-state part of the above response is given by

(A) $5 + e^{-4t}$

(B) $5 + 4(\sin(\omega t + \delta_1)$

(C) $5 + 4\sin(\omega t + \delta_1) + e^{-8t}(\omega t + \delta_2)$

(D) 5

Solution: (B)

Steady-state part $= \underset{t\to\infty}{\text{Lt}}\ y(t)$

$$= 5 + 4\sin(\omega t + \delta_1)$$

Example 16: Which of the following equation gives the steady-state error for a unity feedback system excited by?

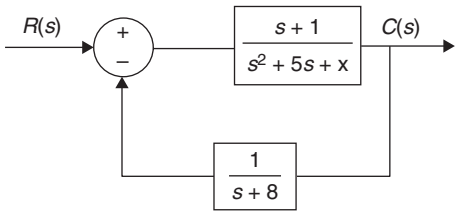
$$u_s(t) + tu_s(t) + t^2\frac{u_s(t)}{2}$$

(A) $\frac{1}{1+K_p} + \frac{1}{K_v} + \frac{1}{K_a}$ (B) $\frac{1}{1+K_p} + \frac{1}{K_v} + \frac{2}{K_a}$

(C) $\frac{1}{K_p} + \frac{1}{K_v} + \frac{1}{K_a}$ (D) $\frac{1}{K_p} + \frac{1}{K_v} + \frac{2}{K_a}$

Solution: (A)

Example 17: For what values of x, does the system shown in figure have a zero steady-state error (timed) for a step input?



(A) $x = 5$
(B) $x > 1$
(C) $x = 0$
(D) For no value of x

Solution: (C).
When step input is applied, the steady-state error is zero when type of the system is greater than zero (1).

For $X = 0 \Rightarrow$ type of the system = 1

Example 18: The system $G(S) = \dfrac{0.4}{s^2 + s - 2}$ is excited by a unit step input. The steady-state output is

(A) 0 (B) 0.83
(C) 1.25 (D) Unbounded

Solution: (D)
Given system is unstable, so the error is unbounded.

Example 19: The transfer function of a system is given by $\dfrac{(s+2)}{s(2s^3 + 3s^2 + s)}$

The steady-state error of the system with input $r(t) = (8t + 4t^2)\,u(t)$ is

(A) 4 (B) 12 (C) 8 (D) 2

Solution: (A)
Given system is type '2' system. Steady-state error for $8t(u(t)$ is zero.

Steady-state error for $4t^2$

$$= \frac{A}{K_a}\left[\because \frac{At^2}{2} = \frac{8}{2}t^2\right]$$

$$= \frac{8}{\underset{s\to 0}{\text{Lt}}\, s^2 G(s) H(s)} = \frac{8}{2} = 4$$

Example 20: Which of the following is the steady-state error for a step input applied to a unity feedback system with open loop? transfer function $G(S) = \dfrac{5}{s^2 + 14s + 25}$

(A) $e_{ss} = 0.83$ (B) $e_{ss} = 0$
(C) $e_{ss} = 1$ (D) $e_{ss} = \infty$

Solution: (A)
Steady-state error for step input $= \dfrac{A}{1 + K_p}$

$$= \frac{1}{1 + \underset{s\to 0}{\text{Lt}}\, G(s) H(s)}.$$

$$= \frac{1}{1 + 0.2}\, e^4 = 0.833$$

EXERCISES

Practice Problems I

Directions for questions 1 to 25: Select the correct alternative from the given choices.

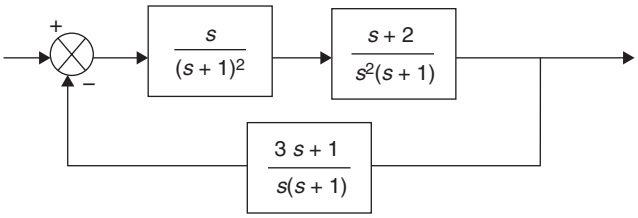
1. The closed loop transfer function of a second-order system is

$$\frac{C(s)}{R(s)} = \frac{100}{s^2 + 16s + 100}$$

The type of damping in the system is
(A) Underdamped.
(B) Overdamped.
(C) Critically damped.
(D) Undamped

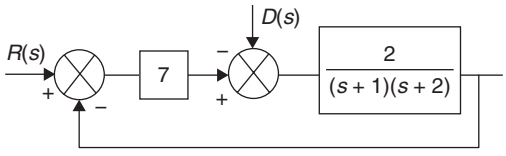
2. The unit impulse response of a second-order system is $\dfrac{1}{6} e^{-0.8t} \sin(0.6t)$. The natural frequency and damped frequency oscillations are, respectively,
(A) 1 and 0.8. (B) 1 and 0.6.
(C) 0.8 and 1. (D) 1.6 and 1.

3. The feedback control system shown in the figure represents a



(A) Type 1 system (B) Type 2 system
(C) Type 3 system (D) None

4. The steady-state error due to unit ramp disturbance input $D(s)$ is given by



(A) 0 (B) 0.012 (C) 0.021 (D) 0.025

5. A unity feedback control system is represented by the open loop transfer function $G(S) = \dfrac{K}{s(s+2)}$. The range values of K, so that the system remain under damped.

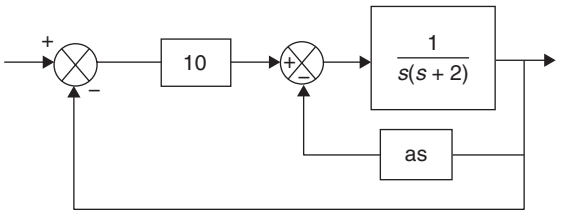
(A) $K < 1$ (B) $K > 1$
(C) $K = 1$ (D) None of these

6. For a proportional control action, which among the following is true?
(i) As gain K increases, ξ decreases.
(ii) The system tend to have large overshoots as gain increases.
(iii) As gain increases, the system will be over damped.
(iv) As gain increases, settling time will be more.
(A) All true (B) i, ii, iv are true
(C) i, ii true (D) ii, iii, iv true

7. The unit impulse response of a system is given by $C(t) = 4e^{-3t}$. Find the step response of the system for $t > 0$.
(A) $1 - e^{-3t}$ (B) $3e^{-3t}$
(C) $\frac{4}{3}(1 - e^{-3t})$ (D) e^{-3t}

8. If the system's transfer function is given by $T(s) = \frac{6}{s(s+2)(s+3)}$, then its time constant is
(A) $\frac{1}{6}$ (B) 6 (C) $\frac{1}{2}$ (D) $\frac{1}{3}$

Common Data for Questions 9 and 10:
A feedback system employing output-rate damping is shown in the following figure.



9. In the absence of derivative feedback, the damping factor and natural frequency of the system are
(A) 0.1 and 10 (B) 1 and 10
(C) $\frac{1}{\sqrt{10}}$ and $\sqrt{10}$ (D) $\frac{1}{\sqrt{5}}$ and $\sqrt{5}$

10. In the absence of derivative feedback the steady-state error resulting from unit ramp input is
(A) 5 (B) 0.2 (C) 2 (D) 0.02

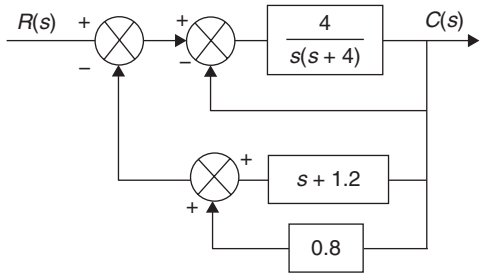
11. A unity feedback system is characterized by an open loop transfer function $G(s) = \frac{K}{s(s+20)}$. The values of 'K' so that the system will have a damping ratio of 0.5 and the peak time are
(A) 200 and 0.157 s (B) 400 and 0.1813 s
(C) 100 and 0.157 s (D) $\sqrt{200}$ and 0.181 s

12. A unity feedback has a loop transfer function of $G(S) = \frac{20(s+5)}{s(s+1)(s+2)}$
The steady-state error for unit ramp input is
(A) 50 (B) Zero
(C) Infinite (D) 0.02

13. A second-order control system has
$$M(j\omega) = \frac{64}{64 - \omega^2 + 8\sqrt{2}j\omega}$$
Its M_p (Peak magnitude) is
(A) 0.5 (B) 1.5 (C) $\sqrt{2}$ (D) 1

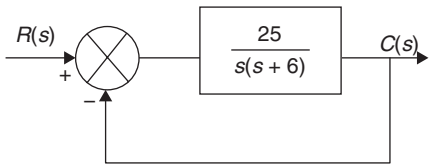
14. A servo system is given by the equation
$$C(t) = 1 - 2e^{-5t} + e^{-10t}$$
Find the natural frequency of the system, if step input is applied to the system.
(A) 17.52 (B) 50.01
(C) 7.07 (D) 2.04

15. Obtain the damping factor of the following system



(A) 0.8 (B) 4 (C) 4.3 (D) 3.46

16. The transfer function of second-order closed loop control system is shown below. The value of its response is maximum when $t = t_{max}$. Then t_{max} = ?



(A) $\pi/8$ (B) $\pi/4$ (C) $\pi/2$ (D) π

Common Data for Questions 17 and 18:
A unity feedback system is characterized by the open loop transfer function $G(S) = \frac{100}{s(s+10)}$

17. The static error constants (K_p, K_v, K_a) for the system are
(A) 0, 10 and ∞ (B) ∞, 10 and 0
(C) ∞, 0.1 and 0 (D) 0, 10 and 0

18. The steady-state error of the system when subjected to an input given by
$$r(t) = 3 + 4t + \frac{7}{2}t^2$$
(A) 0.4 (B) 0
(C) ∞ (D) 0.35

Common Data for Questions 19 and 20:
Open loop transfer function of a system is given by

$$G(s)H(s)=\frac{15}{5s^3+2s^2+3s}$$

19. The steady-state error of the system when it is subjected to an input $r(t)=5t$

(A) 1 (B) 3
(C) 5 (D) 15

20. The steady-state error of the system when it is subjected to an input $r(t) = 5 + 8t + 3t^2$

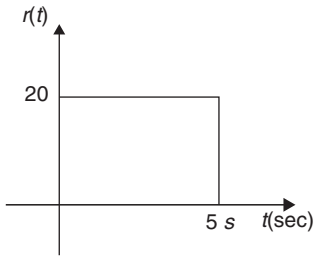
(A) 0 (B) 1
(C) 3 (D) ∞

21. Match the following

Transfer function	Type of damping
1. $\frac{5}{s^2+10s+5}$	P. Undamped
2. $\frac{25}{s^2+10s+25}$	Q. Underdamped
3. $\frac{25}{s^2+s+25}$	R. Critically damped
4. $\frac{5}{s^2+5}$	S. Overdamped

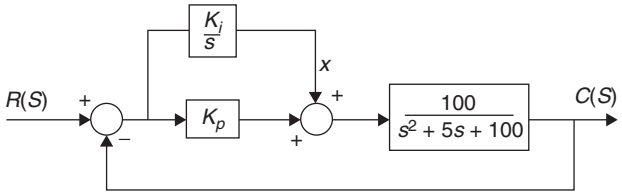
Code:	P	Q	R	S
(A)	4	3	2	1
(B)	3	2	4	1
(C)	4	2	3	1
(D)	3	4	1	2

22. The steady-state error of a unity feedback linear system for unit step input is 0.08. The steady-state error of the same system for a pulse input $r(t)$ having a magnitude of 20 and duration of 5 s as shown in the following figure.



(A) 0 (B) 08 (C) 1.6 (D) 0.16

23. Consider the feedback system shown below which is subjected to a unit input. The system is stable and has the controller parameters as $K_p = 5$, $K_i = 10$. The steady-state value of 'X' is



(A) 100 (B) 0 (C) 1 (D) 0.1

24. The roots of the characteristic equation are symmetric about origin and are on the real axis of the S-plane, then which one of the following is true?

(A) One row of zeros present in the RH table.
(B) System is unstable.
(C) One sign change present in the first column of the RH table.
(D) All the above

25. If the imaginary part of second-order underdamped closed loop control system is increased and real part remains same

(A) ξ increases and ω_n decreases.
(B) ξ decreases and ω_n increases.
(C) ξ and ω_n decrease
(D) ξ and ω_n increase

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Consider the unity feedback control system with open loop transfer function

$$G(s)=\frac{k}{s(s+a)}$$

The steady-state error of the system due to a unit step input is

(A) Always zero.
(B) Depends on the value of 'k'.
(C) Depends on 'a'.
(D) Depends on both 'k' and 'a'.

2. Given that

$$e(t)=C_0r(t)+C_1\dot{r}(t)+\frac{C_2}{2!}\ddot{r}(t)+\frac{C_3}{3!}\dddot{r}(t)+.....+\frac{Cn}{n!}r^n(t)$$

where $e(t)$ and $r(t)$ represents error and input signal, respectively, and K_p, K_v and K_a represent static error constants then which of the following statements is true?

(i) Generalized error series gives error signal as a function of time.
(ii) Generalized error constants C_0, C_1, C_2 … C_n are functions of time.
(iii) Using generalized error constants steady-state error can be determined for any type of input.
(iv) Using static error constants the steady-state error can be determined for any type of input.

(A) i, ii and iv (B) i, ii and iii
(C) i, iii and iv (D) i and iii

3. If $y(t) = A + X\sin(\omega t + \theta_1) + e^{-bt}\sin(\omega t + \theta_1)$ represents the equation of a system, then the steady-state part of the above response is
(A) A (B) $A + X\sin(\omega t + \theta_1)$
(C) $A + e^{-bt}(\sin\omega t + \theta_1)$ (D) $X\sin(\omega t + \theta_1)$

4. The steady-state error of first-order system with ramp input is
(A) 0 (B) T(C) 1 (D) $1 - T$

5. Which among the following are true
(i) An overdamped system gives sluggish response.
(ii) An integral controller improves the steady-state response of the system.
(iii) The pole of an over damped system is in the real axis.
(iv) When $\xi = 0$, the system is an undamped one.
(A) i, iii are true (B) i, ii, iv are true
(C) i, ii are true (D) All true

6. The position and velocity error constants for the following system are

$$G(S)\,H(s) = \frac{100}{(s+2)\,(s+5)}$$

(A) 10, 0 (B) 10, infinity
(C) .1, 0 (D) 0, 10

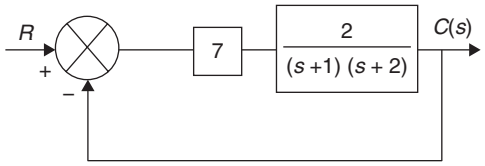
7. $\frac{K}{s(\tau s+2)}$ represents the open loop transfer function of a unity feedback system. Keeping gain constant, what should be the change in time constant in order to reduce the damping factor from .8 to .4
(A) $T_2 = 4T_1$ (B) $T_2 = 2T_1$
(C) $T_2 = \frac{T_1}{4}$ (D) $T_2 = \frac{T_1}{2}$

Common Data for Questions 8 and 9:
The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(1+sT)}$, where T and K are constants having positive values. By what factor should the amplifier gain be reduced so that

8. the peak overshoot of unit response of the systems are reduced from 75% to 50%?
(A) 5.6 (B) 10 (C) 2.5 (D) 20

9. the damping ratio increases from 0.1 to 0.5?
(A) 25 (B) 5 (C) $\sqrt{5}$ (D) $\frac{1}{\sqrt{5}}$

10. The maximum undershoot of the system given in the figure below will be



(A) 0.14 (B) 0.28
(C) 0.079 (D) 0.092

11. The open loop transfer function of a unity feedback system is given as $\frac{10}{s(s+5)}$. The system is subjected to an input $r(t) = 1 + 5t$. Find the steady-state error.
(A) 2 (B) α (C) 2.5 (D) 1

12. A unity feedback system is characterized by the open loop transfer function

$$G(S) = \frac{100}{s(5s+10)(2s+10)}.$$

The steady-state errors for unit-step, unit-ramp and unit-acceleration inputs are
(A) 1, 1, ∞ (B) 0, 1, 1
(C) 0, 1, ∞ (D) 1, 1, ∞

13. A servo system is given by the equation

$$C(t) = 1 - 2e^{-5t} + e^{-10t}$$

Find the natural frequency of the system, if step input is applied to the system.
(A) 17.52 (B) 50.01
(C) 7.07 (D) 2.04

14. A unity feedback control system has its open loop transfer function $G(S) = \frac{8s+1}{16s^2}$. Determine an expression for time response when it is subjected to a unit impulse input
(A) $\frac{e^{-0.5t}}{20}[1+t]$ (B) $\frac{e^{-0.25t}}{20}[1+t]$
(C) $\frac{e^{-0.25t} - 4te^{-0.25t}}{20}$ (D) $\frac{te^{-.25t}+1}{20}$

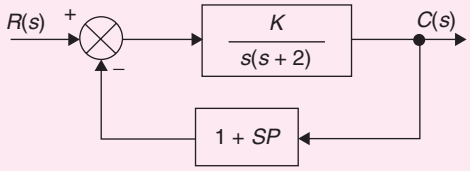
15. The steady-state error of the system with loop transfer function $\frac{K}{(s+1)\,(s+2)}$ is given as 0.1. Find the value of gain K, if system is subjected to step input
(A) 10 (B) 16
(C) 18 (D) 20

Previous Years' Questions

1. Consider the function $F(s) = \dfrac{5}{s\left(s^2+3s+2\right)}$, where $F(s)$ is the Laplace transform of the function $f(t)$. The initial value of $f(t)$ is equal to **[2004]**

(A) 5 (B) $\frac{5}{2}$ (C) $\frac{5}{3}$ (D) 0

2. The block diagram of a closed loop control system is given by figure. The values of K and P such that the system has a damping ratio of 0.7 and an undamped natural frequency ω_n of 5 rad/s are, respectively, equal to **[2004]**



(A) 20 and 0.3 (B) 20 and 0.2
(C) 25 and 0.3 (D) 25 and 0.2

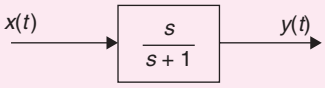
3. The unit impulse response of a second-order under-damped system starting from rest is given by

$$c(t) = 12.5e^{-6t} \sin 8t,\ t \geq 0$$

The steady-state value of the unit step response of the system is equal to **[2004]**

(A) 0 (B) 0.25 (C) 0.5 (D) 1.0

4. In the system shown in figure, the input $x(t) = \sin t$. In the steady state, the response $y(t)$ will be **[2004]**



(A) $\frac{1}{\sqrt{2}}\sin(t-45°)$ (B) $\frac{1}{\sqrt{2}}\sin(t+45°)$

(C) $\sin(t-45°)$ (D) $\sin(t+45°)$

5. The open loop transfer function of a unity feedback control system is given as $G(S) = \dfrac{as+1}{s^2}$

The value of 'a' to give a phase margin of 45° is equal to **[2004]**

(A) 0.141 (B) 0.441
(C) 0.841 (D) 1.141

6. The Laplace transform of a function $f(t)$ is $F(s) = \dfrac{5s^2+23s+6}{s\left(s^2+2s+2\right)}$. As $t \to \infty$, $f(t)$ approaches **[2005]**

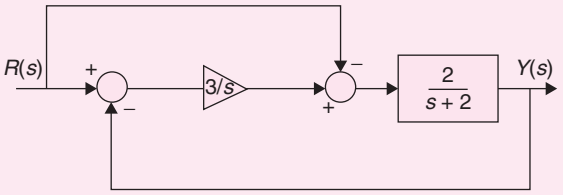
(A) 3 (B) 5 (C) $\frac{17}{2}$ (D) ∞

7. A system with zero initial conditions has the closed loop transfer function

$T(s) = \dfrac{s^2+4}{(s+1)(s+4)}$. The system output is zero at the frequency **[2005]**

(A) 0.5 rad/s (B) 1 rad/s
(C) 2 rad/s (D) 4 rad/s

8. When subjected to a unit step input, the closed loop control system shown in the following figure will have a steady-state error of **[2005]**



(A) –1.0 (B) –0.5 (C) 0 (D) 0.5

9. If the loop gain K of a negative feedback system having a loop transfer function $\dfrac{K(s+3)}{(s+8)^2}$ is to be adjusted to induce a sustained oscillation then **[2007]**

(A) The frequency of this oscillation must be $\frac{4}{\sqrt{3}}$ rad/s.
(B) The frequency of this oscillation must be 4 rad/s.
(C) The frequency of this oscillation must be $\frac{4}{\sqrt{3}}$ rad/s.
(D) Such a K does not exist.

10. If $u(t)$, $r(t)$ denote the unit step and unit ramp functions respectively and $u(t) * r(t)$ their convolution, then the function $u(t+1) * r(t-2)$ is given by **[2007]**

(A) $(1/2)(t-1)(t-2)$ (B) $(1/2)(t-1)(t-2)$
(C) $(1/2)(t-1)^2 u(t-1)$ (D) None of these

11. A function $y(t)$ satisfies the following differential equation

$$\frac{dy(t)}{dt} + y(t) = \delta(t)$$

where $\delta(t)$ is the delta function. Assuming zero initial condition, and denoting the unit step function by $u(t)$, $y(t)$ can be of the form **[2008]**

(A) e^t (B) e^{-t}
(C) $e^t u(t)$ (D) $e^{-t} u(t)$

12. A system with input $x(t)$ and output $y(t)$ is defined by the input-output relation

$$y(t) = \int_{-\infty}^{-2t} x(t)\,dt$$

The system will be **[2008]**

(A) Causal, time-invariant and unstable
(B) Causal, time-invariant and stable
(C) Non-causal, time-invariant and unstable
(D) Non-causal, time-variant and unstable

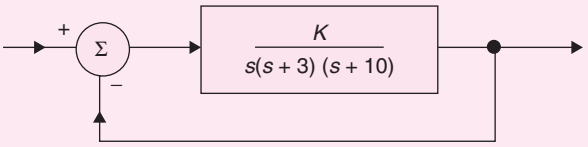
13. The transfer function of a linear time invariant system is given as

$$G(S) = \frac{1}{s^2 + 3s + 2}$$

The steady-state value of the output of this system for a unit impulse input applied at time instant $t = 1$ will be **[2008]**

(A) 0 (B) 0.5 (C) 1 (D) 2

14. Figure shows a feedback system where $K > 0$. The range of K for which the system is stable will be given by **[2008]**



(A) $0 < K < 30$ (B) $0 < K < 39$
(C) $0 < K < 390$ (D) $K > 390$

15. The transfer function of a system is given as $\frac{100}{s^2 + 20s + 100}$ This system is **[2008]**

(A) An overdamped system
(B) An underdamped system
(C) A critically damped system
(D) An unstable system

16. The open loop transfer function of a unity feedback system is given by $G(S) = (e^{-0.1s})/s$. The gain margin of this system is **[2009]**

(A) 11.95 dB (B) 17.67 dB
(C) 21.33 dB (D) 23.9 dB

17. Consider an LTI system with transfer function $H(s) = \frac{1}{S(S+4)}$ If the input to the system is $\cos(3t)$ and the steady-state output is $A \sin(3t + \alpha)$, then the value of A is **[2014]**

(A) 1/30 (B) 1/15 (C) 3/4 (D) 4/3

18. Consider an LTI system with impulse response $h(t) = e^{-5t} u(t)$. If the output of the system is $y(t) = e^{-3t} u(t) - e^{-5t} u(t)$ then the input, $x(t)$, is given by **[2014]**

(A) $e^{-3t} u(t)$ (B) $2 e^{-3t} u(t)$
(C) $e^{-5t} u(t)$ (D) $2 e^{-5t} u(t)$

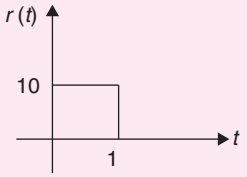
19. The characteristic equation of a closed-loop system is $s(s + 1)(s + 3) + k(s + 2) = 0$, $k > 0$. Which of the following statements is true? **[2010]**

(A) Its roots are always real.
(B) It cannot have a breakaway point in the range $-1 < \text{Re}[s] < 0$.
(C) Two of its roots tend to infinity along the asymptotes $\text{Re}[s] = -1$.
(D) It may have complex roots in the right half plane.

20. For the system $\frac{2}{(s+1)}$, the approximate time taken for a step response to reach 98% of its final value is **[2010]**

(A) $1s$ (B) $2s$ (C) $4s$ (D) $8s$

21. The steady-state error of a unity feedback linear system for a unit step input is 0.1. The steady-state error of the same system, for a pulse input $r(t)$ having a magnitude of 10 and a duration of one second, as shown in the figure is **[2011]**



(A) 0 (B) 0.1 (C) 1 (D) 10

22. The response $h(t)$ of a linear time invariant system to an impulse $\delta(t)$, under initially relax condition is $h(t) = e^{-t} + e^{-2t}$. The response of this system for a unit step input $u(t)$ is **[2011]**

(A) $u(t) + e^{-t} + e^{-2t}$ (B) $(e^{-t} + e^{-2t})\, u(t)$
(C) $(1.5 - e^{-t} - 0.5e^{-2t})$ (D) $e^{-t}\, \delta(t) + e^{-2t}\, u(t)$

23. A system with transfer function

$$G(S) = \frac{(s^2 + 9)(s + 2)}{(s+1)(s+3)(s+4)}$$

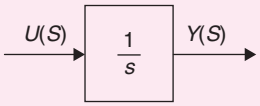
is excited by $\sin(\omega t)$. The steady-state output of the system is zero at **[2012]**

(A) $\omega = 1$ rad/s (B) $\omega = 2$ rad/s
(C) $\omega = 3$ rad/s (D) $\omega = 4$ rad/s

24. The impulse response of a continuous time system is given by $h(t) = \delta(t - 1) + \delta(t - 3)$. The value of the step response at $t = 2$ is **[2013]**

(A) 0 (B) 1 (C) 2 (D) 3

25. Assuming zero initial condition, the response $y(t)$ of the system given below to a unit step input $u(t)$ is **[2013]**



(A) $u(t)$ (B) $tu(t)$
(C) $\frac{t^2}{2} u(t)$ (D) $e^{-t} u(t)$

26. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t - 1)$, the output is **[2013]**

(A) $\frac{t^2}{2} u(t)$ (B) $\frac{t(t-1)}{2} u(t-1)$
(C) $\frac{(t-1)^2}{2} u(t-1)$ (D) $\frac{t^2 - 1}{2} u(t-1)$

27. Which one of the following statements is NOT TRUE for a continuous time causal and stable LTI system? **[2013]**

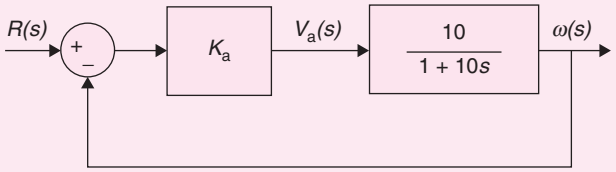
(A) All the poles of the system must lie on the left side of the $j\omega$ axis.

(B) Zeros of the system can lie anywhere in the S-plane.

(C) All the poles must lie within $|s| = 1$.

(D) All the roots of the characteristic equation must be located on the left side of the $j\omega$ axis.

28. The open-loop transfer function of a DC motor is given as $\frac{\omega(s)}{V_a(s)} = \frac{10}{1+10s}$ When connected in feedback as shown below, the approximate value of K_a that will reduce the time constant of the closed loop system by one hundred times as compared to that of the open-loop system is **[2013]**



(A) 1 (B) 5 (C) 10 (D) 100

Answer Keys

Exercises

Practice Problems 1

1. A	**2.** B	**3.** B	**4.** A	**5.** B	**6.** C	**7.** C	**8.** C	**9.** C	**10.** B
11. B	**12.** D	**13.** D	**14.** C	**15.** B	**16.** A	**17.** B	**18.** C	**19.** A	**20.** D
21. A	**22.** A	**23.** B	**24.** D	**25.** B					

Practice Problems 2

1. A	**2.** B	**3.** B	**4.** B	**5.** D	**6.** A	**7.** A	**8.** A	**9.** A	**10.** C
11. C	**12.** C	**13.** C	**14.** B	**15.** C					

Previous Years' Questions

1. D	**2.** D	**3.** D	**4.** B	**5.** C	**6.** A	**7.** C	**8.** C	**9.** B	**10.** C
11. D	**12.** D	**13.** A	**14.** C	**15.** C	**16.** D	**17.** B	**18.** B	**19.** C	**20.** C
21. A	**22.** C	**23.** C	**24.** B	**25.** B	**26.** C	**27.** C	**28.** C		

Chapter 3

Stability Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Stability
- Routh–Hurwitz criterion
- Auxiliary equation
- Root locus
- Angle and magnitude conditions
- Break away point
- Frequency response analysis
- Frequency-Domine specifications
- Resonant peak
- Polar plot

Introduction to Stability

Stability

A linear time-invariant system is stable if the output of the system is bounded for a bounded input and the output of the system tends towards zero in the absence of the input.

Stability is classified as follows:

1. Absolute stability
2. Conditional stability
3. Marginal stability
4. Unstable

Absolute Stability

A system is absolutely stable with respect to a parameter, if the system is stable for all values of that parameter.

Conditional Stability

A system in conditionally stable with respect to a parameter, if the system is stable for only certain bounded ranges of values of this parameter.

Marginal Stability

A system is marginally stable if the natural response of the system neither decays nor grows but remains constant or oscillates as time approaches infinity.

Unstable: A system is unstable if its response is unbounded with a bounded input applied.

Stability and Poles

The system poles that are in the left half plane yield either pure exponential decay or damped sinusoidal natural response, which is the necessary condition for a system to be stable.

Note 1: Stable system have closed loop transfer function with poles only in the left half plane.

Note 2: Unstable systems have loop transfer function with at least one pole in the right half plane or poles of multiplicity greater than 1 on the imaginary axis.

Note 3: Marginally stable system have closed loop transfer function with only imaginary poles of multiplicity 1 and poles in the left half plane.

Necessary Conditions for Stability

1. Positiveness of the coefficients of characteristic equation is necessary as well as sufficient condition for stability of first and second-order system.
2. Positiveness and existence of the all coefficients of the characteristic equation is necessary condition for stability of the system.

Note: Roots with negative real part indicates all positive coefficients in characteristic equation but all positive coefficients does not indicate proofs with negative real part in the characteristic equation.

Routh–Hurwitz Criterion

Routh–Hurwitz Criterion gives the necessary and sufficient condition for all roots of polynomial to lie in the left half of the *S*-plane, without actually solving for the roots of the equation.

The characteristic equation of the *n*th-order system is

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

Routh Array

S^n	a_0	a_2	a_4	a_6	.	.	.	.
S^{n-1}	a_1	a_3	a_5	a_7	.	.	.	.
S^{n-2}	b_1	b_2	b_3	b_4				
S^{n-3}	c_1	c_2	c_3					
S^{n-4}	d_1	d_2						
.								
.								
S^2	e_1	a_n						
S^1	f_1							
S^0	a_n							

The coefficients b_1, b_2 … are evaluated as follows

$$b_1 = (a_1 a_2 - a_0 a_3)/a_1;$$

$$b_2 = (a_1 a_4 - a_0 a_5)/a_1;$$

$$b_3 = (a_1 a_6 - a_0 a_7)/a_1 \ldots$$

This process will continue till we get a zero as the least coefficient in the third row. Similarly the coefficients of the other rows are also evaluated.

The roots of the characteristic equation are all in the left half of *S*-plane if all the Coefficients of the first column of the Routh's tabulation are of the same sign.

The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right half of *S*-plane.

Special case 1: When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term.

In this case, if zero appears as the first element of a row, the elements in the next row will all becomes infinite, to overcome this problem we replace the zero element by an arbitrary small positive number 'ε' and then proceed with Routh's tabulation.

Finally substitute the value of $\varepsilon = 0$ and find the values of the elements of the array which are functions of 'ε'. The resultant Routh's array is analysed as usual.

Note: If there is a single element zero in *s*' row, it is considered as row of all zeros.

Special case 2: When all the elements in one row of Routh's tabulation are zeros before the tabulation is properly terminated, it indicates the following:

(i) There are symmetrically located roots in *S*-plane
(ii) Pair of real roots with opposite signs and/or pair of conjugate roots on the imaginary axis and/or complex conjugate roots forming quadrates in the *S*-plane.

Auxiliary Equation

The polynomial formed by the coefficients of the row just above the row of zeros in the Routh array is called auxiliary equation [$A(s) = 0$].

Note 1: The order of the auxiliary equation is always even
Note 2: The roots of the auxiliary equation also satisfy the original characteristic equation.
Note 3: Break down in the Routh table due to zero row is overcome by replacing the row of zeros with first derivative of auxiliary equation $\left(\frac{dA(s)}{ds}\right)$ with respect to '*s*'.

Solved Examples

Example 1: If a system transfer function has some poles lying on the imaginary axis, it is
(A) Unconditionally stable
(B) Conditionally stable
(C) Unstable
(D) Marginally stable

Solution: (D)
When the poles are on imaginary axis, system is marginally stable.

Example 2: System has some roots with real parts equal to zero, but none with positive real part is
(A) Absolutely unstable (B) Absolutely stable
(C) Relatively stable (D) Marginally stable

Solution: (D)
Marginally stable

Example 3: Closed loop stability implies that $1 + G(s)H(s)$ has only _____ in the left half of the *S*-plane
(A) Poles (B) Zeros
(C) Poles and zeros (D) Poles or zeros

Solution: (B)
Zeros of characteristic equation are poles of the transfer function

Example 4: None of the poles of a linear control system lie in the right half of *S*-plane. For a bounded input the output of this system
(A) Could be bounded (B) Always tends to zero
(C) Is always bounded (D) None of the above

Solution: (D)
Poles are not on the right half indicates they can be on imaginary axis, so stability cannot be justified

Example 5: For the equation $s^3 - 4s^2 + s + 5 = 0$, the number of roots in the left half of S-plane will be
(A) Zero (B) One
(C) Two (D) Three

Solution: (B)
Routh array for $s^3 - 4s^2 + s + 5 = 0$

s^3	1	1
s^2	-4	5
s^1	2.25	
s°	5	

Sign changes in first column of Routh array are $2(1 \to -4 \to 1)$ poles on left half = 3 – 2 = 1

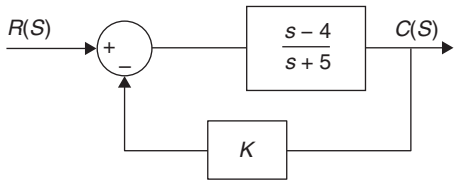
Example 6: The number of roots of the equation $2s^4 + s^3 + 5s + 6 = 0$ that lie in the right half of S-plane is
(A) Zero (B) One
(C) Two (D) Four

Solution: (C)
Routh array for $2s^4 + s^3 + 3s^2 + 5s + 6 = 0$

s^4	2	3	6
s^3	1	5	
s^2	-7	6	
s^1	$\frac{41}{7}$		
s°	6		

No. of sign changes in Routh array are $2\ (1 \to -7 \to \frac{41}{7})$
No. of poles on the right half = 2

Example 7: For what range of K is the following system is asymptotically stable; assume $K \geq 0$



(A) $0 \leq K < \frac{5}{4}$ (B) $0 \leq K < \frac{4}{5}$
(C) $K > -\frac{4}{5}$ (D) $K > -\frac{5}{4}$

Solution: (A)
Given system transfer function

$$= \frac{K(s-4)}{(1+K)s+(5-4K)}$$

Characteristic equation of the system is

$$(1 + K)\, s + (5 - 4K) = 0$$

For the system to be stable, all the coefficients of 's' in the characteristic equation must be positive

$$1 + K > 0 \qquad 5 - 4K > 0$$
$$K > -1 \qquad -4K > -5$$
$$K < \frac{5}{4}$$

Actual ranges of 'K' is $-1 < K < \frac{5}{4}$

Given $K \geq 0;\ 0 \leq K < \frac{5}{4}$.

Example 8: The open loop transfer function of a unity feedback system is given below

$$G(s) = \frac{K(s+4)}{(s+1)(s+2)}$$

The range of positive values of 'K' for which the closed loop system will remain stable is
(A) $2 < K < 3$ (B) $\frac{2}{4} < K < 3$
(C) $0 < K < \infty$ (D) $\frac{2}{4} < K < \infty$

Solution: (C)
Closed loop transfer function

$$= \frac{G(s)}{1+G(s)} = \frac{K(s+4)}{s^2+(3+K)s+(2+4K)}$$

Characteristic equation of the system $s^2 + (3 + K)s + (2 + 4K) = 0$

Condition for stability is that all coefficients of 's' must be greater than zero in characteristic equation

$$3 + K > 0 \qquad 2 + 4K > 0$$
$$K > -3 \qquad 4K > -2$$
$$K > -\frac{2}{4}$$

$\therefore$ System is stable for all value of $K > -\frac{2}{4}$

$\therefore$ Range of positive values of 'k' for stability is $0 < k < \infty$

Example 9: A certain closed loop system with unity feedback has the following transfer function given by $G(s) = \frac{k}{s(s+2)(s+4)}$ with the gain set at the ultimate value, the system will oscillate at an angular frequency of
(A) 2 rad/sec (B) 4 rad/sec
(C) 8 rad/sec (D) $2\sqrt{2}$ rad/sec

Solution: (D)
Characteristic equation of the system is

$$s^3 + 6s^2 + 8s + K = 0$$

System will oscillate when it is marginally stable/ from Routh array

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	
s^0	K	

System is marginally stable if $48 - K = 0 \Rightarrow K = 48$

Then Auxiliary equation is $6s^2 + 48 = 0$

$$s^2 = -8$$

$$\Rightarrow \quad s = j2\sqrt{2}$$

Oscillation frequency = $2\sqrt{2}$ rad/sec

Root Locus

The root locus is basically the technique of finding the locus of roots as a single gain is changed, by solving for the roots of the characteristic equation, at each gain.

The gain that is to be varied will be open loop gain. Note this does not mean the gain of the open loop system that is typically fixed: this refers to cascading a controller in the forward path. Using the root locus method the control system engineer can predict the effect of varying gain on the open loop poles or what effect will be caused by adding open loop poles or open loop zeros.

Angle and Magnitude Conditions

Consider the following general system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation of the system is obtained by setting the denominator of the closed loop system to zero as follows.

$$1 + G(s)\,H(s) = 0$$

$$\therefore \quad G(s)\,H(s) = -1$$

Since complex variable has both an angle and a magnitude, we can split the above equation into two separate equations as follows.

$$\angle G(s)\,H(s) = \pm 180^\circ\,(2K+1) \quad (K = 0, 1, 2, \ldots)$$

Angle condition

$$\left|G(s)H(s)\right| = 1$$

Magnitude condition

The values of 's' that satisfy the angle and magnitude conditions are the roots of the characteristic equation (The closed loop poles). ONLY these values will be the roots. As we vary the gain, these values of 's' that satisfy both conditions will change. The resulting collection of point in S-plane are called root locus.

Note: Open loop gain 'K' corresponding to any point on root locus can be calculated using the equation.

$$K = \frac{\text{Product of lengh of vectors from open loop poles to the point}}{\text{Product of lenght of vectors form open loop zeros to the point}}$$

Rules for Construction of Root Locus

1. The root locus is symmetric about origin
2. Number of branches in a root locus is equal to either the number of poles (n) or the number of zeros (m) whichever is greater. Each branch of root locus starts form open poles (Assuming number of poles is greater than zero) corresponding to $K = 0$ and terminates at either a finite open loop zero or infinity corresponding to $K = \infty$. 'n' number of branches will terminate to finite open loop zeros and remaining branches of root locus ($n - m$) will terminate to infinity.
   $B = P$ if $P > Z \Rightarrow P - Z$ branches will terminate at ∞
   $B = Z$ if $Z > P \Rightarrow Z - P$ branches will terminate at ∞
   P = Number of poles, Z = number of zeros
   B = Number of branches of root locus
3. A section of real axis lies on root locus if the total number of open loops poles plus zeros to the right of that section is odd.
4. The angle of asymptotes and centroid:
   If $P > Z$, $P - Z$ number of branches will terminate at ∞ along straight line (asymptotes) making angle with real axis given by

   $$\phi_A = \frac{180(2q+1)}{P-Z};\ (q = 0, 1, 2, 3, \ldots (P - Z - 1))$$

   If $Z > P \Rightarrow \phi_A = \frac{180(2q+1)}{Z-P};\ (q = 0, 1, 2, 3, \ldots (Z - P - 1))$

   The point of intersection of the asymptotes with the real axis is called centroid denoted by 'σ'

   $$\text{Centroid } (\sigma) = \frac{\text{Sum of real part of pole} - \text{Sum of real part of zeros}}{P-Z}$$

5. Breakaway/in point:
   A point on root locus where multiple poles/zeros exist is known as breakaway/ in point.
   The breakaway or breakin point is given by the roots of the equation $\frac{dK}{ds} = 0$, where 'K' is obtained form
   $1 + KG(S)\,H(S) = 0$
   Note 1: Breakaway point exists if there is a root locus on real axis between two adjacent poles.
   Note 2: Breakin point exists if there is a root locus on real axis between two adjacent zeros.
   Note 3: Breakin point exists if there is a zero on real axis and left to that there is no root loci or poles or zeros.

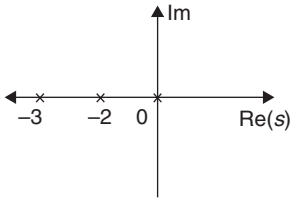
6. The angle of departure/arrival:
   The angle of departure or angle of arrival is given by
   Angle of departure = $180 - \phi$
   Angle of arrival = $180 + \phi$
   where $\phi =$

$$\left[\begin{array}{ll} \text{Sum of angles of vectors} & \text{Sum of angles of vectors} \\ \text{to the complex pole/zero from} & - \text{ to the complex pole/zero from} \\ \text{other poles} & \text{other zeros.} \end{array}\right]$$

7. The intersection of root locus with the imaginary axis can be determined by the use of Routh criterion [finding poles on imaginary axis]

Example 10: Plot the root locus for a transfer function $G(s) = \dfrac{K}{s(s+2)\,(s+3)}$

Solution: The number of poles = 3
The poles are at $s = 0$, $s = -2$ and $s = -3$



Break-away Point

$$\frac{d}{ds}(s^3 + 5s^2 + 6s) = 0$$

$$3s^2 + 10s + 6 = 0$$

$$s = -0.784 \text{ and } s = -2.549$$

$s = -2.549$ does not lie on the root locus

Asymptotes

$$\theta_1 = \pm\frac{180}{3} = \pm 60°$$

$$\theta_3 = \pm\frac{3\times 180}{3} = \pm 180°$$

Centroid

$$s = \frac{(0-2-3)-(0)}{3-0} = -1.667$$

Imaginary Axis Cross-over

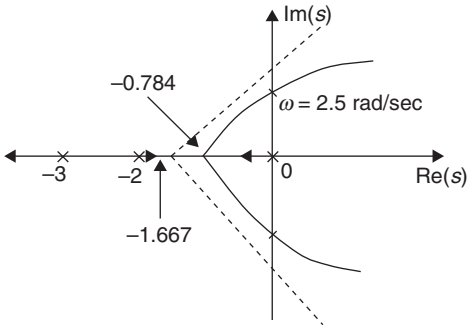
$$G(j\omega) = \frac{K}{j\omega(j\omega+2)\,(j\omega+3)}$$

$$G(j\omega) = K\left[\frac{-5\omega^2}{25\omega^4 + (6\omega - \omega^3)^2} - j\,\frac{(6\omega - \omega^3)}{25\omega^4 + (6\omega - \omega^3)^2}\right]$$

Equating the imaginary part to zero

$$\omega = \pm 2.5 \text{ rad/sec.}$$

The root locus is drawn as shown in the figure



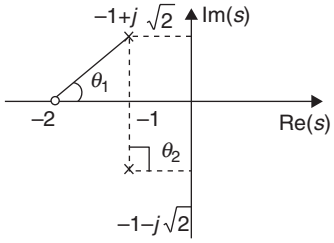
Example 11: $G(s) = \dfrac{K(s+2)}{s^2+2s+3}$, $H(s) = 1$

Sketch the root locus

Solution: No of branches of root locus = 2

The poles are at $s = -1 \pm j\sqrt{2}$

The zero is at $s = -2$



The root locus starts from the conjugate poles and break in on the real axis between –2 and –∞. One root locus ends in $s = -2$, the other ends at $s = -\infty$.

Asymptote

$$\theta_1 = \pm\frac{180}{2-1} = \pm 180$$

Angle of Departure

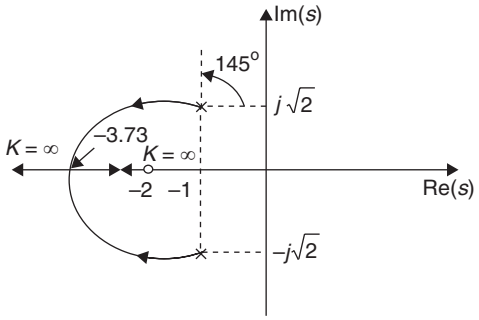
$$= 180 - \left(90 - \tan\frac{\sqrt{2}}{1}\right) = 145°$$

Break-away Point

$$\frac{dG(s)}{ds} = 0$$

$$s = -3.73$$

The root locus is drawn in the following figure.



1. $N - P = Z$
   Here $P = 1$
   If $K > 1$, $N = 1$,
   $Z = 0$, then the closed loop system is stable
2. If $K < 1$, $N = 0$
   $Z \neq 0$, then the closed loop system is unstable.

Example 13: Given $G(s)\,H(s) = \dfrac{K}{s(s+2)(s+5)}$, the point of intersection of the asymptotes of the root locus with the real axis is

(A) 0 (B) – 2 (C) – 2.3 (D) – 3.5

Solution: (C)
No. of poles (P) = 3 (0, – 2, – 5)
No. of zeros (Z) = 0
No. of asymptotes = 3
Centroid (Intersection of the asymptotes)

$$= \frac{\Sigma \text{ Real part of all poles } - \Sigma \text{ Real part of all zero}}{P - Z}$$

$$= \frac{0-2-5-0}{3} = \frac{-7}{3} = -2.33.$$

Example 14: The open loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K(s+2)}{s(s^2+2s+1)}$$

The centroid and angles of root locus are, respectively,

(A) $-\dfrac{2}{3}$ and +60°, –60° (B) –2 and +90°, –90°
(C) Zero and +90°, –90° (D) – 2 and +60°, –60°

Solution: (C)
No. of poles = 3 (0, – 1, – 1)
No. of zeros = 1 (– 2)
No. of asymptotes = 2
Angle of asymptotes = +90° and –90°

$$\text{Centroid} = \frac{\Sigma \text{Real part of all poles} - \Sigma \text{Real part of all zero}}{2}$$

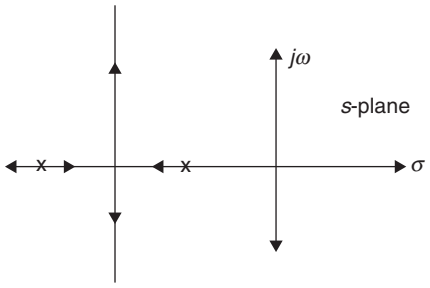
$$= \frac{(0-1-1)-(-2)}{2} = \frac{-2+2}{2} = 0.$$

Example 15: Figure shown below gives root locus of the open loop transfer function $G(s)\,H(s)$ of a system.

Consider the following inference drawn from the figure.

(1) It has no zero.
(2) It is a stable system.
(3) It is a second-order system.

Which of these inferences are correct?



(A) 1, 2, and 3 (B) 1 and 2
(C) 2 and 3 (D) 1 and 3

Two poles are terminated to infinity indicates that there are no zeros.

Two poles indicates the order of the system as '2'.

Example 16: The characteristic equation of a unity-feedback control system is given by $S^3 + AS^2 + S + B = 0$.

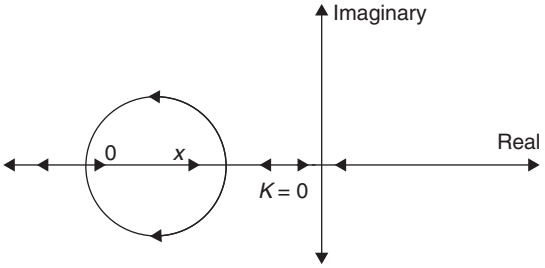
Consider the following statements in this regard.

1. For a given value of B, all the root-locus branches will terminate at infinity for the variable 'A' in the positive direction.
2. For a given value of B, only one root locus branch will terminate at infinity for the variable K, in the positive direction.
3. For a given value, of A, all the root locus branches will terminate at infinity for the variable 'B' in the positive direction.

Of these statement
(A) 1 and 3 are correct.
(B) 2 and 3 are correct.
(C) Only 2 is correct.
(D) Only 1 is correct.

Solution: (B).

Example 17: The root locus of a unity feedback system is shown in the following figure. The open loop transfer function is given by



(A) $\dfrac{K}{s(s+2)(s+4)}$ (B) $\dfrac{K(s+4)}{s(s+3)(s+5)}$

(C) $\dfrac{K(s+4)}{s(s+3)}$ (D) $\dfrac{Ks}{(s+3)(s+5)}$

Solution: (C)
The given root locus indicates that the open loop transfer function has 2 poles and one zero.

One pole is at origin and another pole location is on the right side to the zero.

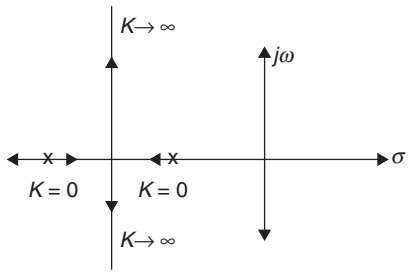
So option 'C' is correct.

Example 18: The closed loop transfer function of a feedback system is given by

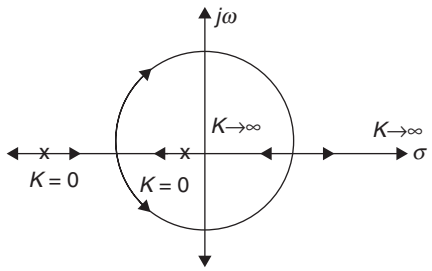
$$\frac{C(S)}{R(S)} = \frac{K}{s^2 + (4-K)s + 3}$$

Which of the following diagrams represents a root locus of the system for $K > 0$?

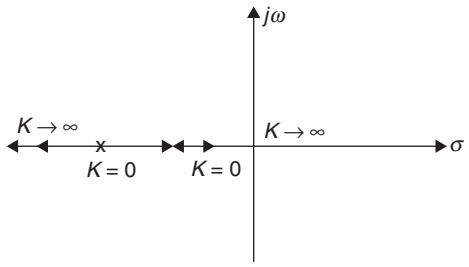
(A)



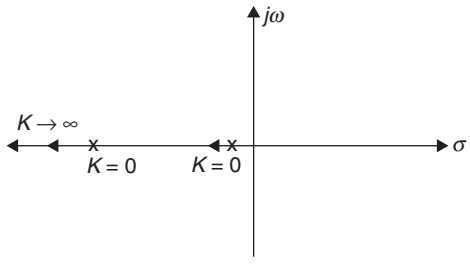
(B)



(C)



(D)



Solution: (B)
When the value of 'K' is increasing the location of poles will tend to right-hand side of the S-plane when $K > 4$

→ When $K = 4$ characteristic equation $s^2 + 3 = 0$
$s = \pm j\sqrt{3} =$
(Poles are on the imaginary axis)

→ When $K = 10$ characteristic equation $s^2 - 6s + 3 = 0$
$(s-3)^2 = 0$
$s = +3, +3$
(Poles are on RHS and equal)

→ When $K > 10$ poles are on RHS but not equal
∴ Option 'B' is the suitable root locus.

Example 19: A control system has

$$G(s)\,H(s) = \frac{K(s+5)}{(s+2)(s+3)}$$

The break away and break in points are located, respectively, at

(A) –2 and –1 (B) –1.589 and –7.5
(C) –2.55 and –7.5 (D) –1.5 and –6.89

Solution: (C)
Characteristic equation

$$1 + G(s)\,H(s) = 0$$

$$\Rightarrow \quad K = \frac{-(s+2)(s+3)}{(s+5)} = \frac{s^2+5s+6}{s+5}$$

Breakaway or breakin points are roots of $\dfrac{dK}{ds} = 0$

$$\frac{dK}{ds} = \frac{(2s+5)(s+5) - (s^2+5s+6)}{(s+5)^2} = 0$$

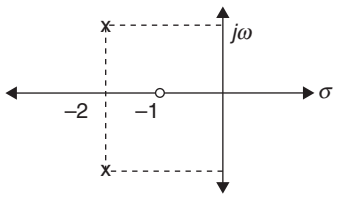
$$2s^2 + 15s + 25 - s^2 - 5s - 6 = 0$$

$$s^2 + 10s - 19 = 0$$

$$s = -2.55, -7.449$$

Breakaway point is –2.55, breakin point –7.449.

Example 20: A transfer function $G(s)$ has type pole zero plot as shown in figure. Given that the steady-state gain is 3, the transfer function $G(s)$ will be



(A) $\dfrac{2(s+1)}{s^2+4s+5}$ (B) $\dfrac{5(s+1)}{s^2+4s+5}$

(C) $\dfrac{15(s+1)}{s^2+4s+5}$ (D) $\dfrac{15(s+1)}{(s+2)^2}$

Solution: (C)

Form the given pole zero plot transfer function has

1. Zero is at (–1, 0)
2. Complex poles

Only for option 'C' steady-state gain is '3'.

$$\underset{s\to 0}{\text{Lt}}\, G(s) = \underset{s\to 0}{\text{Lt}} \frac{15(s+1)}{s^2+4s+5} = \frac{15}{5} = 3.$$

Frequency Response Analysis

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed as frequency response. In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal

when the input of a linear time-invariant system is sinusoidal with amplitude A and frequency ω_0.

$$r(t) = A\sin\omega_0 t.$$

The steady-state output of a system $y(t)$ will be a sinusoidal with the same frequency ω_0 but possibly with different amplitude and phase

$$y(t) = B\sin(\omega_0 t + \phi)$$

$$\frac{B}{A} = |\text{Transfer function}| = \frac{|G(j\omega)|}{1+G(j\omega)\,H(j\omega)} = |M(j\omega)|$$

$\phi = \angle\text{Transfer function} = \angle G(j\omega) - \angle[1 + G(j\omega)\,H(j\omega)] = \angle M(j\omega)$

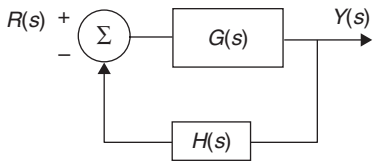
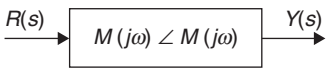


Figure 1 Closed loop control system



The ease and accuracy of measurements are some of the advantages of the frequency response method. Extraction of transfer function is easy from frequency response test than step response test (time response). The design and parameter adjustment of the open-loop transfer function of a system for specific. Closed loop performance is carried out more easily in frequency domain than in time domain. The effect of noise disturbance and parameter variation are relatively easy to visualize and access through frequency response. Nyquist criterion is a powerful frequency domain method of extracting. The information regarding stability as well as relative stability of a system without the needs to evaluate roots of the characteristic equation.

Frequency-domain Specifications

Resonant Peak (M_r)

The resonant peak M_r is the maximum value of $|M(j\omega)|$. The magnitude M_r gives indication on the relative stability of a stable closed loop system.

For second-order system,

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \text{ for } \xi \le \frac{1}{\sqrt{2}}$$

$$M_r = 1 \text{ for } \xi > \frac{1}{\sqrt{2}}$$

Note: A large M_r corresponds to a large maximum over short of the step response.

Resonant Frequency (ω_r)

The resonant frequency ω_r is the frequency at which the peak resonance M_r occurs.

For second-order system, $\omega_r = \omega_n\sqrt{1-2\xi^2}$ for $\xi \le \frac{1}{\sqrt{2}}$

$$\omega_r = 0 \quad \text{for} \quad \xi > \frac{1}{\sqrt{2}}$$

Bandwidth (BW)

The bandwidth (BW) is the frequency at which $|M(j\omega)|$ drops to 70.7% of or 3 dB down from its zero frequency value.

For second-order system

$$\text{BW} = \omega_n\left[\left(1-2\xi^2\right)+\sqrt{4\xi^4-4\xi^2+2}\right]^{1/2}$$

Bandwidth gives an indication of the transient response of a control system, noise filtering characteristics and robustness of the system.

Gain Margin (GM)

Gain margin is the amount of gain in decibel (dB) that can be added to the open loop before the closed loop system becomes unstable.

$$\text{Gain margin} = \text{GM} = 20\log_{10}\frac{1}{\left[M(j\omega_{pc})\right]}$$

$$= -20\log\left|M(j\omega_{pc})\right|\text{dB}$$

The phase crossover frequency (ω_{pc}) is the frequency at which phase angle becomes –180°.

Phase Margin

Phase margin (PM) is defined as the angle m degrees through which the $M(j\omega)$ plot must be rotated about the origin so that the gain cross over passes through the (–1. j0) point.

$$\text{Phase margin} = \text{PM} = \angle M(j\omega_{gc}) - 180°$$

Gain crossover frequency (ωgc) is the frequency at which $M(j\omega)$ becomes 1 or decibel magnitude of

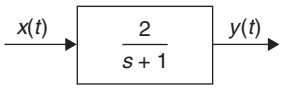
$M(j\omega)$ becomes zero.

Relation Between Time Domain and Frequency Domain Characteristics

1. The resonant peak M_r of the closed loop frequency response depends on ξ only. when $\xi = 0$, $M_r = \infty$ when ξ is negative, the system is unstable, and the value of M_r ceases to have any meaning. As ξ increases, M_r decreases.
   In comparison to time response, maximum peak overshoot also depends on 'ξ' only. The peak overshoot is zero if $\xi \geq 1$.
2. Bandwidth is directly proportional to ω_n.
   $\rightarrow B\omega$ increases linearly with ω_n.
   $\rightarrow B\omega$ decreases with increase in ξ for a fixed ω_n.
   For time response, rise time increases as ω_n decreases.
   $\therefore$ Bandwidth $\alpha \dfrac{1}{\text{Rise time}}$.
3. Band width (BW) and M_r are proportional to each other for $0 \leq \xi \leq \dfrac{1}{\sqrt{2}}$.

Example 21: For the system shown in figure, the input $x(t) = \sin t$.

In the steady-state, the response $y(t)$ will be



(A) $\dfrac{1}{\sqrt{2}}\sin(t - 45°)$ (B) $\sqrt{2}\sin(t - 45°)$

(C) $\dfrac{1}{\sqrt{2}}\sin(t - 45°)$ (D) $\sqrt{2}\sin(t - 45°)$

Solution: (D)

Transfer function $(T) = \dfrac{2}{s+1} = \dfrac{2}{j\omega + 1}$

Input = $\sin t$ [$\because \omega = 1$]

$$|T| \angle\theta = \frac{2}{\sqrt{1+1}} \angle -\tan^{-1}\left(\frac{1}{1}\right)$$

$$= \sqrt{2} \angle -45°$$

$$y(t) = 1 \times \sqrt{2} \sin(t - 45°)$$

$$y(t) = \sqrt{2} \sin(t - 45°)$$

Example 22: A system with zero initial condition has the closed loop transfer function

$$T(s) = \frac{s^2 + 16}{(s+2)(s+3)}$$

The system output is zero at the frequency.

(A) 1 rad/s (B) 2 rad/s

(C) 3 rad/s (D) 4 rad/s

Solution: (D)

Magnitude of transfer function

$$= \frac{|-\omega^2 + 16|}{|(j\omega + 2)(j\omega + 3)|}$$

Magnitude of transfer function will affect the magnitude of the system output. output becomes zero when transfer function magnitude is zero

$$\frac{|-\omega^2 + 16|}{|(j\omega + 2)(j\omega + 3)|} = 0$$

$$-\omega^2 + 16 = 0$$

$$\omega = 4 \text{ rad/sec}$$

Example 23: The gain margin of a unity feedback control system with the open loop transfer function $G(s) = \dfrac{s+4}{s^2}$ is

(A) 0 (B) $\dfrac{1}{\sqrt{4}}$

(C) $\sqrt{4}$ (D) ∞

Solution: (A)

Phase crossover frequency $\angle G(s) = -180°$

$$\tan^{-1}\left(\frac{\omega_{pc}}{4}\right) - 180° = -180°$$

$$\tan^{-1}\left(\frac{\omega_{pc}}{4}\right) = 0$$

$$\omega_{pc} = 0$$

Magnitude of transfer function at phase crossover frequency

$$\left[\frac{j\omega_{pc} + 4}{(j\omega_{pc})^2}\right]\omega_{pc=0} = \infty$$

Gain margin = $\dfrac{1}{|G(j\omega_{pc})|} = \dfrac{1}{\infty} = 0$

Example 24: The open loop transfer function of a unit feedback control system B given as $G(s) = \dfrac{sx+1}{s^2}$. The value of 'x' to give a phase margin of $\dfrac{\pi}{4}$ is equal to

(A) 0.441 (B) 0.141

(C) 1.141 (D) 0.841

Solution: (D)

Phase margin = $180° \angle G(j\omega) = 45°$

$$180° + \tan^{-1}\left(\frac{x\omega_{gc}}{1}\right) - 180° = 45°$$

$$x\omega_{gc} = 1$$

Magnitude phase margin $\left|G\left(j\omega_{pc}\right)\right|=1$

$$\left|\frac{j\omega_{gc}+1}{\left(j\omega_{gc}\right)^2}\right|=1$$

$$_{gc}=12^{1/4}$$

$$x\omega_{gc}=1$$

$$x=\frac{1}{2^{1/4}}=0.841$$

Example 25: In the $G\,H(s)$ plane, the Nyquist plot of the loop transfer function $G(s)\,H(s)=\dfrac{\pi.e^{-s}}{s}$ passes through the negative real axis at the point

(A) $(-1, j0)$ (B) $(-0.5, j0)$

(C) $(-3, j0)$ (D) $(-2, j0)$

Solution: (D)

At the point of intersection with negative real axis,

$$\angle G(s)\,H(s)=-\pi$$

$$-\left(\omega_{pc}+\frac{\pi}{2}\right)=-\pi$$

$$\omega_{pc}=\frac{\pi}{2}$$

Magnitude of the $G(s)\,H(s)$ at $\pi=\omega_{pc}$ in the intersection point with negative real axis

$$\left|G(s)\,H(s)\right|w=w_{pc}=\left|\frac{\pi e^{-s}}{s}\right|=\left|\frac{\pi e^{-j\omega_{pc}}}{j\omega_{pc}}\right|$$

$$=\frac{\pi}{\omega_{pc}}=\frac{\pi}{\frac{\pi}{2}}=2$$

$\therefore$ Nyquist plot passes through (–2, 0).

Bode Plot

Bode plot is a graph of the transfer function of a linear, time-invariant system frequency plotted with a log-frequency axis, to show the system's frequency response. It is usually a combination of a Bode magnitude plot, expressing the magnitude of the frequency response gain, and a Bode phase plot expressing the frequency response phase shift.

The standard logarithmic magnitude of open loop transfer function of $G(j\omega)$ is $20\log_{10}\left|G\left(j\omega\right)\right|$. The units used in this representation of the magnitude are the decibel, usually denoted as dB.

Generally, a transfer function can be expressed in terms of factors of its poles and zeros. The advantage of the logarithmic plot is the conversion of these multiplicative factors to additive terms.

Consider the general open loop transfer function.

$$G(s)=\frac{K\left(1+sT_{z_1}\right)\left(1+sT_{z_2}\right)\ldots\left(1+sT_{zm}\right)}{s^{P}\left(1+sT_{P_1}\right)\left(1+sT_{p_2}\right)\ldots\left(1+sT_{pn}\right)}$$

In this example, the transfer function includes 'm' number of zeros, 'p' number of poles at origin and in the mentioned part 'n' number of poles. Let $m=1$, $n=2$, $p=1$.

$$\Rightarrow\qquad G(s)=\frac{K\left(1+sT_{z1}\right)}{s\left(1+sT_{p1}\right)\left(1+sT_{p2}\right)}$$

$$G(j\omega)=\frac{K\left(1+j\omega T_{z1}\right)}{j\omega\left(1+j\omega T_{p1}\right)\left(1+j\omega T_{p2}\right)}$$

$$\text{Magnitude of } G(j\omega)=\frac{K\sqrt{1+\omega^2T_{z1}{}^2}}{\omega\sqrt{1+\omega^2\,T_{p_1}{}^2}\sqrt{1+\omega^2\,T_{p_2}{}^2}}$$

Magnitude of $G(j\omega)$ is decibels is

$$\left|G\left(j\omega\right)\right|\text{ in dB}=20\log\left|G(j\omega)\right|$$

$$=20\log K+20\log\sqrt{1+\omega^2\,T_{p1}^2}\;-20\log\omega$$

$$-20\log\sqrt{1+\omega^2\;T_{p_1}{}^2}-20\log\sqrt{1+\omega^2T_{p_2}{}^2}$$

The phase angle of $G(j\omega)=\angle G(j\omega)=\tan^{-1}\omega T_{z1}-90°-\tan^{-1}\omega T_{P1}-\tan^{-1}\omega T_{P2}$

Note: From the above analysis, it is clear that, when the magnitude is expressed in dB, the multiplication is converted to addition.

Therefore to sketch the magnitude plot, knowledge of the magnitude variation of individual factors of the open loop transfer function is essential. The various factors of open loop transfer function are

1. Constant gain, K
2. Poles (or zeros) at origin, $\dfrac{1}{\left(j\omega\right)^n}$ or $\left(j\omega\right)^m$
3. First-order factor, $\dfrac{1}{1+j\omega T_p}$ or $1+j\omega T_z$
4. Quadratic factor, $\dfrac{1}{1+2\xi\left(j\omega/\omega_n\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2}$

   or $1+2\xi\left(\dfrac{j\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2$

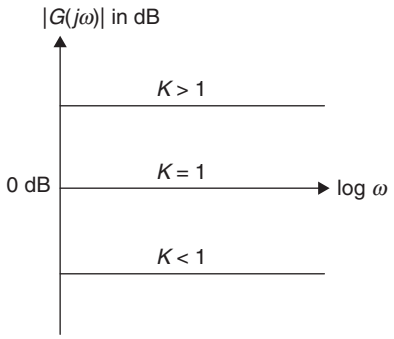
Constant Gain: *K*

Let $G(s)=K$

$\therefore\qquad G(j\omega)=K\angle0°$

$$\left|G\left(j\omega\right)\right|\text{ in dB}=20\log K$$

$$\phi=\angle G(j\omega)=\tan^{-1}\left(\frac{0}{k}\right)=0°$$



Note 1: The magnitude plot and phase plot of a constant '*K*' is independent of frequency and straight line.

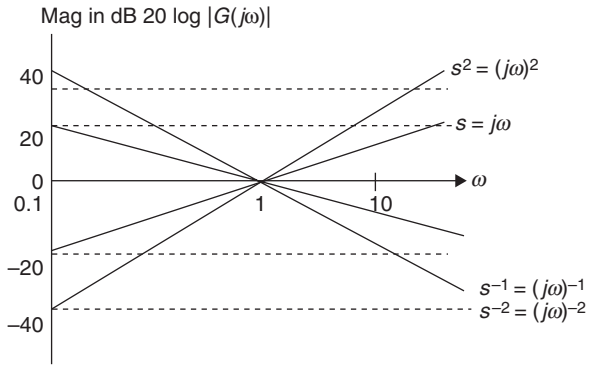
Note 2: A constant (*K*) greater than unity has a positive value in decibels, while a number smaller than unity has a negative value in decibels.

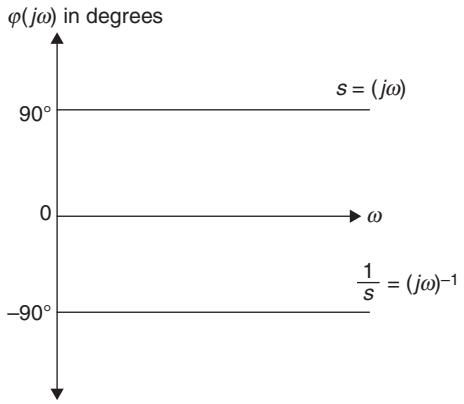
Note 3: The change in the value of gain (*K*) of transfer function is increase or decrease in the magnitude plot.

Poles (zeros) at Origin $(j\omega)^{\pm n}$

Pole: Open loop transfer function $[G(s)] = \frac{1}{s^n}$

Phase angle $\phi = -n.\tan^{-1}\left(\frac{\omega}{0}\right) = -n \times 90°$





Zeros: Open loop transfer function $[G(S)] = s^n$

Log magnitude = 20 log$|(j\omega)^n|$ = 20n logω

$$\text{Phase angle } (\phi) = n \times \tan^{-1}\left(\frac{\omega}{0}\right) = n \times 90°$$

Note 1: Magnitude plot of $S^{\pm n}$ is a straight line with slop of $\pm 20 \times n$ dB/decade that passes through the point [0 dB, 1 rad/s]

Note 2: Phase angle plot of $S^{\pm n}$ is independent of frequency and it is constant angle of value $\pm 90n$ degrees.

First-order Factor $(1 + j\omega T)^{\pm 1}$

Pole: open loop transfer function $G(s) = \frac{1}{1+sTp}$

$$\text{Log magnitude} = 20\log\left|\frac{1}{1+j\omega T_p}\right| = -20\log\sqrt{1+\omega^2 T_p^2}$$

For $\omega << \frac{1}{T_p}$; the asymptote is 20 log1 = 0 dB

For $\omega >> \frac{1}{T_p}$; the asymptote is –20 log ωT_p: It is a straight line with slope of –20 dB/decade. This asymptote intersect 0 dB at the break frequency $\omega_c = 1/T_p$, which is known as corner frequency.

$$\text{Phase angle } \phi = -\tan^{-1}\left(\frac{\omega T_p}{1}\right) = -\tan^{-1}\left(\omega T_p\right)$$

At corner frequency $\phi = -\tan^{-1}(\omega T_p)$

$$= -\tan^{-1}1 = 45°$$

The phase angle of the factor $(1 + sT_p)^{-1}$ varies from 0 to –90° as 'ω' is varied from 0 to infinity. The phase angle plot crosses –45° at $\omega = \omega_c = \frac{1}{T_p}$

Zero: open loop transfer function

$$G(s) = (1 + sT_z)$$

$$\text{Log magnitude} = 20\log\sqrt{1+\omega^2 T_z^2}$$

For $\omega << \frac{1}{T_z}$; the asymptote is 20 log 1 = 0 dB

For $\omega >> \frac{1}{T_z}$; the asymptote is 20 logωT_z: it is a straight line with slope of +20 dB/decade. This asymptote intersect 0 dB at the break frequency $\omega_c = \frac{1}{T_z}$, which is known as corner frequency.

$$\text{Phase angle } \phi = \tan^{-1}\omega T_z$$

The phase angle of the factor $(1 + sT_z)$ varies from 0 to 90° as 'ω' is varied from zero to infinity. The phase angle plot crosses 45° at $\omega = \omega_c = \frac{1}{T_z}$.

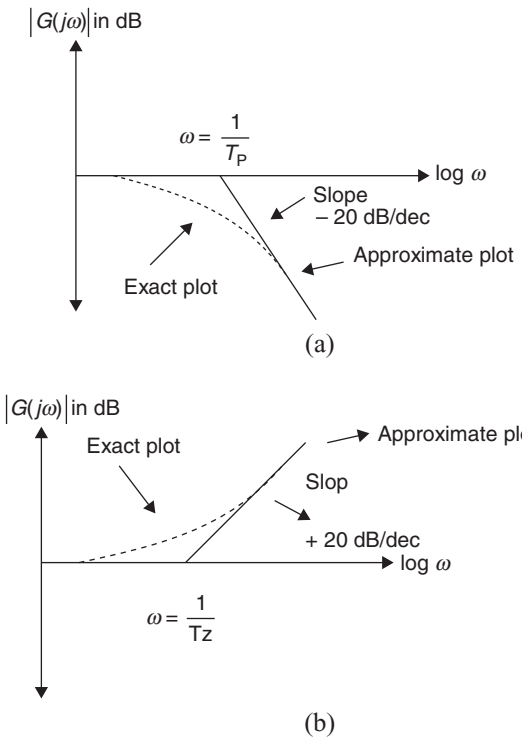


Figure 2 Magnitude plot for first-order pole (a) and first-order zero (b)

Quadratic Factor

$$\left[1+2\xi\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]^{\pm1} = \text{Open loop transfer function} = \left\{1+2\xi\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right\}^{\pm1}$$

The magnitude in decibels is

$$=\pm 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}$$

For $\omega << \omega_n$, the log magnitude is asymptotic to a straight line of constant gain 0 dB and phase angle approaches to 0 deg.

For $\omega >> \omega_n$, the log magnitude approaches $\pm 40\log\left(\frac{\omega}{\omega_n}\right)$; a straight line with slop of ± 40 dB/dec.

Asymptote intersect 0 dB at corner frequency $\omega = \omega_n$.

Note: The resonant frequency is given by

$$\omega_r = \omega_n\sqrt{1-2\xi^2} \quad \text{for } \xi < \frac{1}{\sqrt{2}}$$

The maximum magnitude is

$$\text{Mp} = \left|G(j\omega_r)\right| = \frac{1}{2\xi\sqrt{1-\xi^2}} \text{ for } \xi < \frac{1}{\sqrt{2}}$$

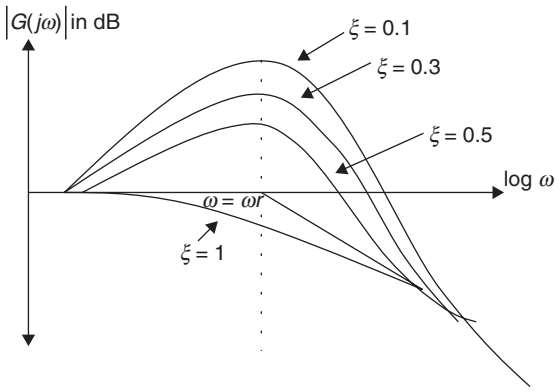


Figure 3 Bode plot for quadratic factor in denominator

Example 26: Draw the Bode plot for a system having

$$G(s)H(s) = \frac{100}{s(s+1)(s+2)}$$

Find
(A) Gain margin
(B) Phase margin
(C) Gain crossover frequency
(D) Phase crossover frequency

Solution: $G(j\omega)H(j\omega) = \dfrac{50}{j\omega(1+j\omega)(1+0.5j\omega)}$

The corner frequencies are

$$\omega = 1 \text{ rad/s and } \omega = 2 \text{ rad/s}$$

For $\omega \le 1$ rad/s

$$G(j\omega)H(j\omega) = \frac{50}{j\omega}$$

Slope = –20 dB/decade

$$|G(j\omega)H(j\omega)|_{\text{dB}} = 20\log 50 - 20\log\omega$$

at $\omega = 0.1$

$$|G(j\omega)H(j\omega)|_{\text{dB}} = 20\log 50 - 20\log(0.1) = 53.98 \text{ dB}$$

At $\omega = 1$

$$|G(j\omega)H(j\omega)| = 20\log 50 = 33.98 \text{ dB}$$

For $1 < \omega \le 2$

$$G(j\omega)H(j\omega) = \frac{50}{j\omega(1+j\omega)}$$

Slope = –20 – 20

= –40 dB/decade

As ω increases from 1 to 2, the reduction in gain

$$= 40\log\left(\frac{2}{1}\right) = 12.04 \text{ dB}$$

At $\omega = 2$

$$|G(j\omega)H(j\omega)|_{\text{dB}} = 21.94 \text{ dB}$$

For $\omega > 2$

$$G(j\omega)H(j\omega) = \frac{50}{j\omega(1 + j\omega)(1 + 0.5j\omega)}$$

Slope = –40 – 20 = –60 dB/decade

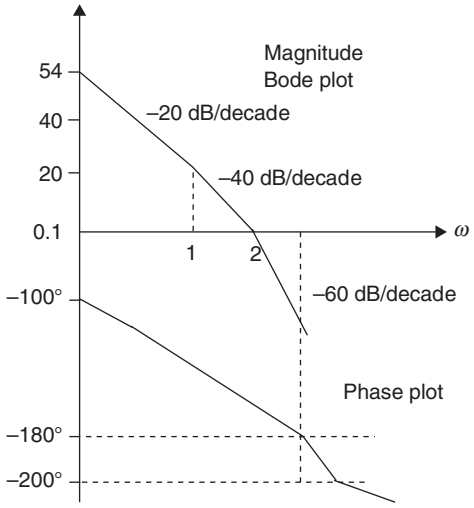
As ω increases from 2 to 10, the reduction in gain = $60 \log\left(\frac{10}{2}\right) = 41.94$ dB

At $\omega = 10$,

$$|G(j\omega)H(j\omega)|_{dB} = -19.99 \text{ dB}$$

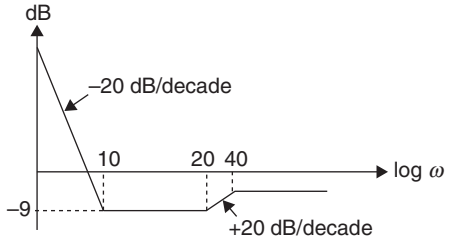
$$\angle G(j\omega)H(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}(0.5\omega)$$

ω	$\angle G(j\omega)H(j\omega)$
0	–90
0.1	–98.6
0.2	–107
0.5	–130.6
1	–161.6
1.3	–175.5
1.4	–179.5
1.5	–183.2
2	–198.4



Gain crossover frequency = 4.45 rad/s
Phase crossover frequency = 1.40 rad/s
Gain margin = 27 dB
Phase margin = 53°

Example 27: Find the transfer function of the system whose asymptotic Bode plot is shown in the following figure.



Solution: The line with a slope of –20 dB/decade does not pass through $\omega = 1$ rad/s, i.e. there is a term $\frac{K}{s}$

$$20 \log K = -9$$

$$K = 0.35$$

At $\omega = 1$ rad/sec, slope changes to 0 dB/dec indicating a zero at $\omega = 1$ rad/sec. The term is $(1 + s)$

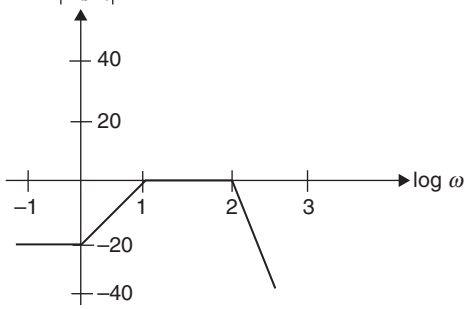
At $\omega = 20$ rad/sec, the slope changes to +20 dB/decade, indicating a term $\left(1 + \frac{s}{20}\right)$ or $(1 + 0.05\,s)$

At $\omega = 40$ rad/sec, the slope changes to 0 dB/dec indicating a term $\left(1 + \frac{s}{40}\right)$ in the denominator.

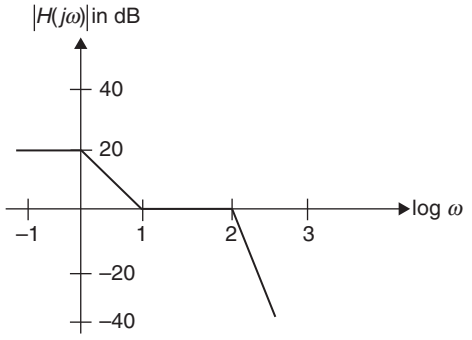
i.e., $$G(s) = \frac{0.35\,(1+s)(1+0.05s)}{s(1 + 0.025\,s)}$$

Example 28: The Bode magnitude plot of $H(s) = \frac{10^4(1+s)}{(10+s)(100+s)^2}$

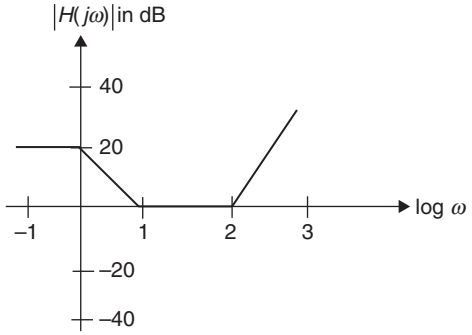
(A)



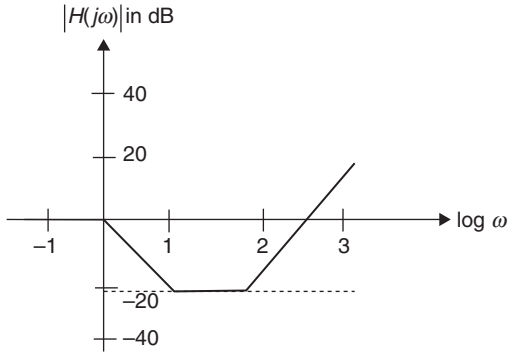
(B)



(C)



(D)



Solution: (A)

Given function $H(s) = \dfrac{10^4(1+s)}{(10+s)(10+s)^2}$

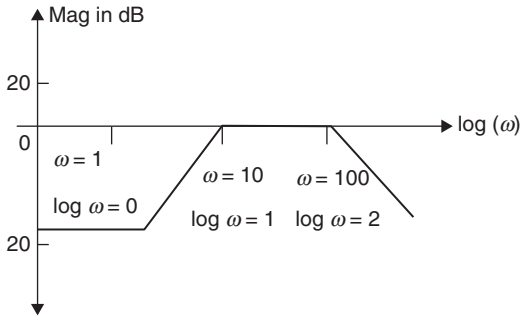
$$H(s) = \frac{10^4(1+s)}{(1+0.1s)(1+0.01s)^2 \times 10 \times 100^2}$$

$$H(s) = \frac{0.1(1+s)}{(1+0.1s)(1+0.01s)^2}$$

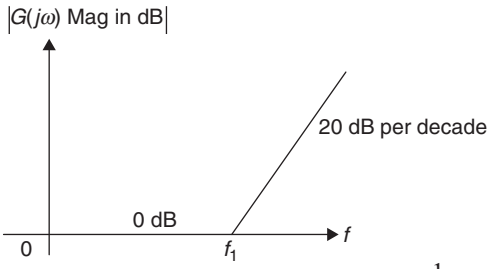
Corner frequencies are 1, 10 and 100

Initial magnitude = 20 log 0.1 = –20 dB

Magnitude starts increasing with slop of +20 dB/dec at ω_c = 1 rad, constant at ω = 10 rad and decays with a slope of 20 dB/dec at ω = 100 rad.



Example 29: The function corresponding to the Bode plot of figure is



(A) $G = jf/f_1$

(B) $G = \dfrac{1}{\left(1 + j\dfrac{f}{f_1}\right)}$

(C) $G = \dfrac{1}{\left(1 - j\dfrac{f}{f_1}\right)}$

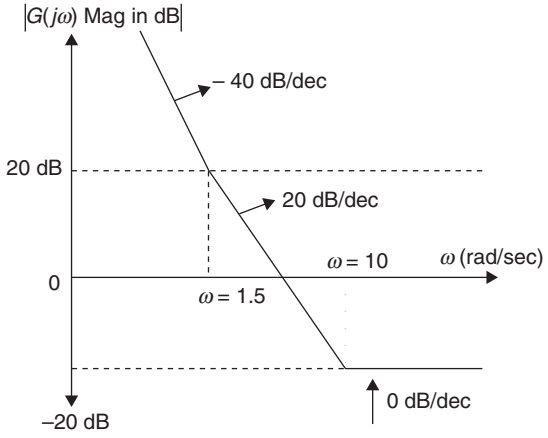
(D) $G = 1 + jf/f_1$

Solution: Magnitude plot slop change at frequency 'f_1' and its increasing. This indicates there is a zero at $f = f_1$.

$$G = (1 + sT_1) = (1 + j\omega T_1)$$

$$= \left(1 + j\frac{2\pi f}{2\pi f_1}\right)\left[\begin{matrix}\omega = 2\pi f \\ T_1 = \dfrac{1}{\omega_c} = \dfrac{1}{2\pi f_1}\end{matrix}\right] \Rightarrow G = 1 + j\frac{f}{f_1}$$

Example 30: The asymptotic Bode magnitude plot of a minimum phase transfer function is shown in figure.



This transfer function has

(A) Two poles and one zero

(B) Two poles and two zeros

(C) One pole and two zeros

(D) Three poles and one zero

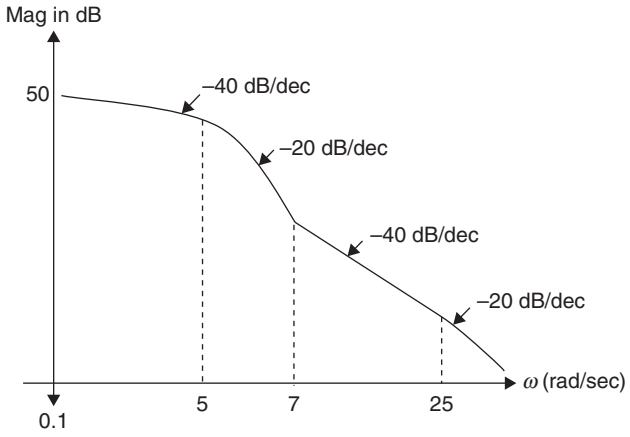
Solution: Initial slop of the magnitude plot is –40 dB/dec, indicates 2 poles of the system are at origin

Reduction is slop by 20 dB/dec at ω = 1.5 indicates a zero.

Reduction in slop by 20 dB/dec at ω =10 indicates another zero.

∴ Total 2 poles and 2 zeros.

Example 31: The asymptotic approximation of the log-magnitude versus frequency plot of a minimum phase system with real poles and one zero is shown in figure. Its transfer function is



(A) $\dfrac{10(s+5)(s+25)}{s^2(s+7)}$ (B) $\dfrac{4.4(s+7)(s+5)}{s^2(s+25)}$

(C) $\dfrac{10(s+7)}{s^2(s+5)(s+25)}$ (D) $\dfrac{4.4(s+5)(s+25)}{s^2(s+7)}$

Solution: Transfer function has corner frequencies 5, 7 and 25

Zeros are at $\omega = 5$ and $\omega = 25$

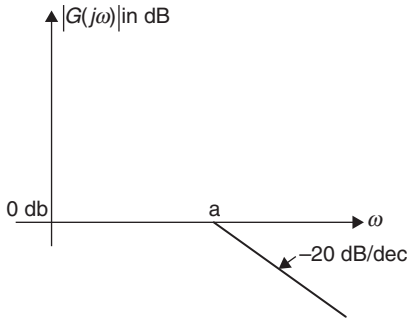
Poles are at $\omega = 7$

$$\text{T. F} = \frac{K\left(1+\frac{s}{5}\right)\left(1+\frac{s}{25}\right)}{s^2(1+s/7)} = \frac{7\times 5}{25}\frac{K(s+5)(s+25)}{s^2(s+7)}$$

Initial magnitude = 20 log K = 10 $\Rightarrow K = 3.16$

$$\text{Transfer function} = \frac{4.4(s+5)(s+25)}{s^2(s+7)}$$

Example 32: The asymptotic Bode magnitude plot of a transfer function $\dfrac{1}{1+s/a}$ is show in figure. The error in dB gain at a frequency of $\omega = 0.5a$ is



(A) 3 dB
(B) 9 dB
(C) 0.97 dB
(D) 5.4 dB

Solution: Actual magnitude of given transfer function = $-20\log\left(1+\dfrac{j\omega}{a}\right)$

At $\omega = 0.5a$

$$\Rightarrow \quad |G(j\omega)| = -20\log\left(1+j\frac{0.5a}{a}\right)$$

$$= -20\log\sqrt{1+(0.5)^2}$$

$$|G(j\omega)| = -0.969 = -0.97 \text{ dB}$$

Approximated magnitude in given plot at $\omega = 0.5a$ is 0 dB

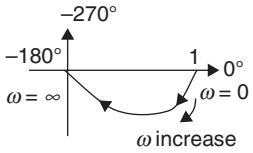
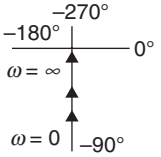
Error = 0 dB – (0.97 dB) = 0.97 dB.

Polar Plot

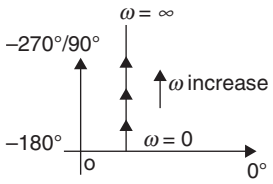
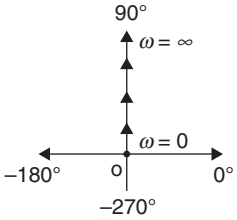
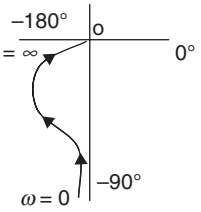
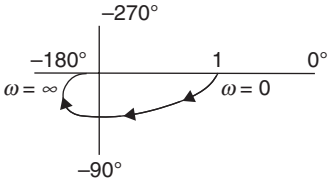
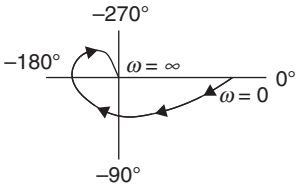
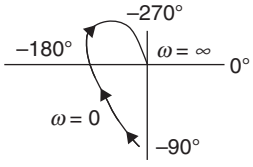
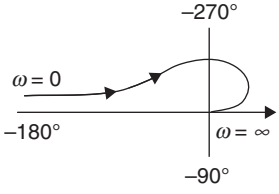
The transfer function $G(s)$ is a complex function and it is given by

$$G(s) = G(j\omega) = |G(j\omega)|\ \angle G(j\omega) = M\angle\phi$$

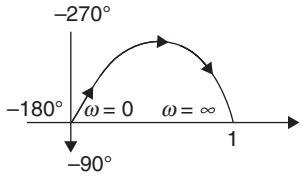
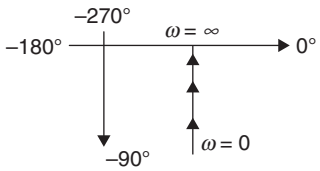
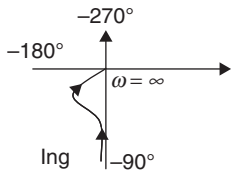
As the input frequency is varied from 0 to ∞, the magnitude M and phase angle 'ϕ' change, the locus traced by the tip of the phasor $G(j\omega)$ is known as polar plot.

$G(s)$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$\lvert G(j\omega)\rvert\angle G(j\omega)$ value		Polar plot
			$\omega \to 0$	$\omega \to \infty$	
$\dfrac{1}{1+sT}$	$\dfrac{1}{\sqrt{1+\omega^2T^2}}$	$-\tan^{-1}\omega T$	$1\angle 0$	$0\angle -90°$	
$\dfrac{1}{s}$	$\dfrac{1}{\omega}$	$-90°$	$\infty\angle -90°$	$0\angle -90°$	

(Continued)

$G(s)$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$\lvert G(j\omega)\rvert \angle G(j\omega)$ value		Polar plot
			$\omega \to 0$	$\omega \to \infty$	
$1 + sT$	$1 + \omega T$	$90°$	$(1+0)$ $\angle 90°$	$(1+\infty)\angle$ $90°$	
s	ω	$90°$	$0 \angle 90°$	$\infty \angle 90°$	
$\frac{1}{s(1+sT)}$	$\frac{1}{\omega\sqrt{1+\omega^2T^2}}$	$-90° - \tan^{-1}\omega T$	$\infty \angle -90°$	0 $\angle -180°$	
$\frac{1}{(1+sT_1)(1+sT_2)}$	$\frac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$	$-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$	$1 \angle 0°$	0 $\angle -180°$	
$\frac{1}{s(1+sT_1)(1+sT_2)}$	$\frac{1}{\omega\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$	$-90° - \tan^{-1}\omega T_1 - \tan^{-1}\frac{\omega}{T_2}$	$\infty \angle -90°$	$0 \angle -270°$	
$\frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$	$\frac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$	$-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-}\omega T_3$	$1 \angle 0°$	$0 \angle -270°$	
$\frac{1}{s^2(1+sT_1)(1+sT_2)}$	$\frac{1}{\omega^2\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$	$-180° - \tan^{1}\omega T_1 - \tan^{-1}\omega T_2$	$\infty \angle -180°$	$0 \angle -360°$	

(*Continued*)

$G(s)$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$\lvert G(j\omega)\rvert \angle G(j\omega)$ value		Polar plot
			$\omega \to 0$	$\omega \to \infty$	
$\frac{s}{1+sT}$	$\frac{\omega}{\sqrt{1+\omega^2T^2}}$	$90° - \tan^{-1}\omega T$	$0\angle 90°$	$1\angle 0°$	
$\frac{1+sT}{s}$	$\frac{\sqrt{1+\omega^2T^2}}{\omega}$	$-90° + \tan^{-1}\omega T$	$\infty\angle -90°$	$1\angle 0°$	
$\frac{1+sT_1}{s(1+sT_2)(1+sT_3)}$	$\frac{\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T^2}\sqrt{1+\omega^2T_3^2}}$	$\tan^{-1}\omega T_1 - 90° - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$	$\infty \angle -90°$	$0 \angle -180°$	

Note 1: Addition of a non-zero pole to a transfer function results in further rotation of the polar plot through an angle of –90° as $\omega \to \infty$ (head of the polar plot shifts).

Note 2: Addition of a pole at origin to a transfer function results in rotation of the polar plot at zero and infinite frequency (head and tail of polar plot) by further angle of –90°.

Note 3: The effect of addition of a zero to a transfer function is to rotate the high frequency portion of the polar plot by 90° in anti-clockwise direction.

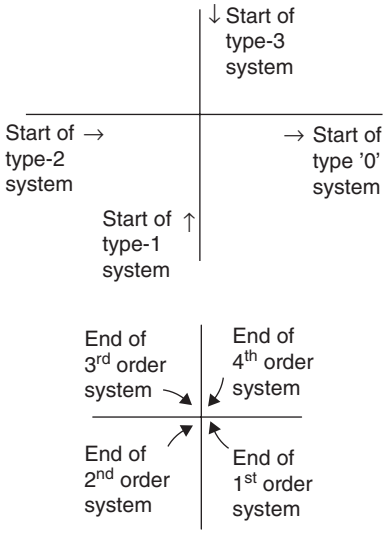


Figure 4 Start point and end point of polar plot for different system types and orders

Nyquist Criterion

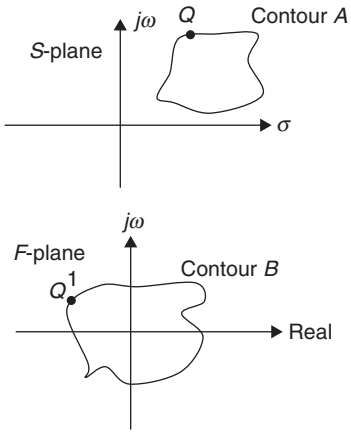
The Nyquist criterion relates the stability of a closed-loop-system to the open-loop frequency response and open-loop pole location. This criterion can tell us how many closed-loop poles are in the right half of S-plane.

The Nyquist criterion used the following concepts for the establishment of criterion.

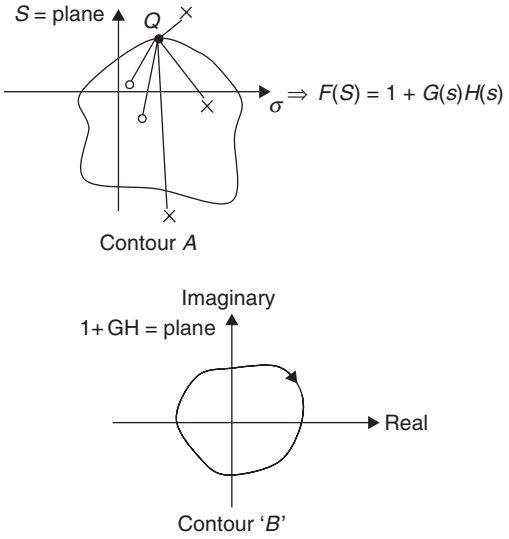
1. The poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$ are same.
2. The zero of the $1 + G(s)H(s)$ are the poles of the closed loop transfer function $T(s)$ of the system.
3. Mapping: Consider a complex number on the S-plane and substitute it into a function $F(s)$, another complex number results. This process is called 'mapping'.
   Example: Substituting $s = 4 + j3$ into function $F(s) = s^2 + 2s + 1$, results in $16 + j30$. We say that $4 + j3$ maps into $16 + j30$ through the function $(s^2 + 2s + 1)$.
5. Mapping contours:
   Consider the collection of points, called a contour, shown in figure as contour A. Also assume that

$$F(s) = \frac{(s-Z_1)(s-Z_2)(s-Z_3)\ldots}{(s-P_1)(s-P_2)(s-P_3)\ldots}$$

Contour 'A' can be mapped through $F(s)$ into contour 'B' by substituting each point of contour A into the function $F(s)$ and plotting the resulting complex numbers. For example, point 'Q' in S-plane maps into point 'Q' through the function $F(s)$.



Let us first assume that $F(s) = 1 + G(s)H(s)$, with the picture of the poles and zeros of $1 + G(s)H(s)$ as shown in figure below. As each point Q of the contour 'A' is substituted into $1 + G(s)H(s)$, a mapped point results on a contour 'B'. As we move round contour 'A' in a clockwise direction, each vector of $F(s)$ that lies inside contour A will appear to undergo a complete rotation or a change in angle of 360°. On the other hand, each vector drawn from the poles and zeros of $1 + G(s)H(s)$ that exist outside contour A will appear to oscillate and return to its previous position, undergoing a net angular change of 0°.



Number of anti-clockwise rotations of contour 'B' about origin $(N) = P - Z$

where, P = Number of poles of $1 + G(s)H(s)$ inside contour A.

Z = Number of zeros of $1 + G(s)H(s)$ inside contour A.

Note: Since the poles of $1 + G(s)H(s)$ are the poles of $G(s)H(s)$ and zeros of $1 + G(s)H(s)$ are poles of closed loop system,

P = Number of open poles enclosed

Z = Number of closed loop poles enclosed

$N = Z - P$ = Number of closed loop poles inside the contour.

If we extend the contour to include the entire right half of S-plane, we can count the number of right-half-plane closed loop poles inside contour 'A' and determine a system's stability.

Note: When we map the entire right half of S-plane through $G(s)H(s)$ instead of $1 + G(s)H(s)$, the resulting contour is same as mapping through $1 + G(s)H(s)$, except that it is translated one unit to the left. So we count rotations about $-1 + j0$ instead of rotations about the origin.

Statement of the Nyquist Stability Criterion

If a contour 'A' that encircles the entire right half-plane is mapped through $G(s)H(s)$, then the number of the closed loop poles (Z) in the right half-plane equals the number of open loop poles (P) that are in the right half-plane minus the number of contour clockwise revolutions (N) around '–1' of the mapping (i.e., $Z = P - N$). The mapping is called the Nyquist diagram/Nyquist plot of $G(s)H(s)$.

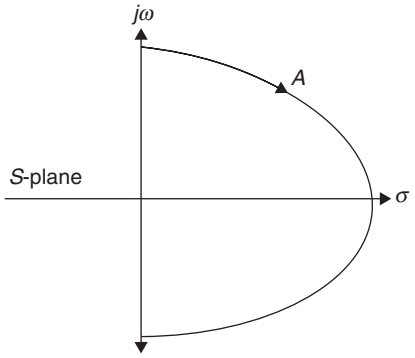


Figure 5 Contour enclosing right half of S-plane of determine stability

Note 1: If the contour 'A' of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the S-plane encircles the point $(-1 + j0)$ in the anti-clockwise direction as many times as the number of right half S-plane poles $G(s)H(s)$, the closed loop system is stable.

Note 2: No encirclement of $-1 + j0$ implies that the system is stable if there are no poles of $G(s)H(s)$ in the right half of 's' plane; otherwise the system is unstable.

Note 3: Clockwise encirclements in the Nyquist plane indicate that the system is unstable.

If $G(s)H(s)$ has any poles on $j\omega$ axis, the Nyquist contour defined earlier cannot be used as such. Also, the S-plane contour should not pass through a singularity of $1 + G(s)H(s)$. The stability in such cases is studied with modified Nyquist contour which bypasses any $j\omega$-axis poles. This is accomplished by indenting the Nyquist contour around the '$j\omega$' poles along a semicircle of radius 'ε', where $\varepsilon \to 0$.

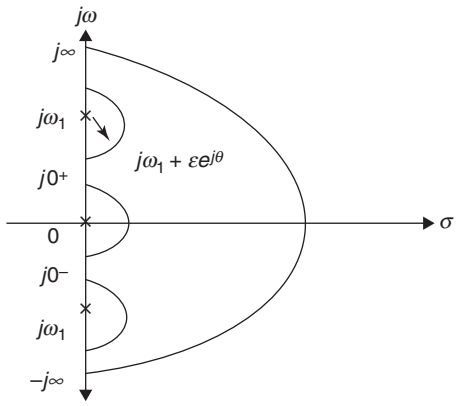


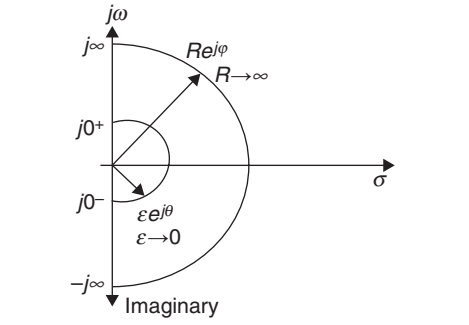
Figure 6 Indented Nyquist Contour for $j\omega$-axis open loop poles

Example 33: Consider a system with an open loop transfer function

$$G(s)H(s)=\frac{(4s+1)}{s^2(s+1)(2s+1)}$$

Find the stability of the system using Nyquist plot.

Solution: The given open loop transfer function has a double pole at origin. The Nyquist contour is intended to bypass the origin. The mapping of Nyquist contour is obtained as follows.



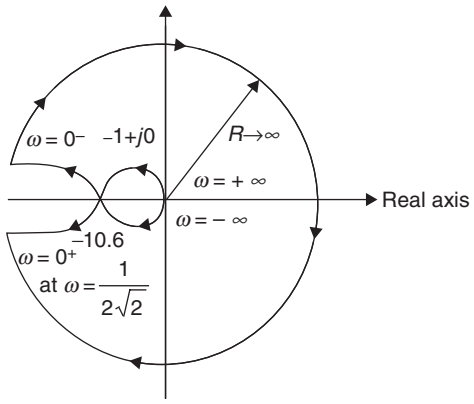


Figure 7 Nyquist contour and corresponding Nyquist plot

1. Semicircular indent represented by $s=\lim_{\varepsilon\to 0}\varepsilon e^{j\theta}$ (where 'θ' varied from –90° through 0° to 90°) is mapped into

$$\lim_{\varepsilon\to 0}\left[\frac{4\varepsilon e^{j\theta}}{\varepsilon^2 e^{j2\theta}(\varepsilon e^{j\theta}+1)(2\varepsilon e^{j\theta}+1)}\right]=\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2 e^{j2\theta}}\right)=\infty e^{-j2\theta}$$

$$=\infty\ (\angle 180^\circ\to 0^\circ\to\angle -180^\circ).$$

This part of map is an infinite circle.

2. Mapping of positive imaginary axis

$$G(j\omega)H(j\omega)=\frac{(1+j4\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

For various values of 'ω', $G(j\omega)H(j\omega)$ is calculated and plotted using polar plots.

The $G(j\omega)H(j\omega)$ – locus intersects the real axis at a point where

$\angle G(j\omega)\,H(j\omega)=-180^\circ$

$-180^\circ-\tan^{-1}\omega-\tan^{-1}2\omega+\tan^{-1}4\omega=-180^\circ$

$$\therefore\ \omega=\frac{1}{2\sqrt{2}}=0.354\ \text{rad/sec}$$

$$\therefore\left|G(j\omega)H(j\omega)\right|_{\omega}=\frac{1}{2\sqrt{2}}=10.6$$

Further as $\omega\to+j\infty$

$$\Rightarrow\left|G(j\omega)H(j\omega)\right|\angle G(j\omega)H(j\omega)\Rightarrow 0\angle -270^\circ$$

as $\omega\to 0^+\Rightarrow|G(j\omega)\,H(j\omega)|\Rightarrow\infty\angle -180^\circ$

3. The infinite semicircle of the Nyquist contour represented by $s=\lim_{R\to\infty}\mathrm{Re}^{j\phi}$ (ϕ varies from + 90° through 0° to + 90°) is mapped to

$$\lim_{R\to\infty}\frac{(1+4\,\mathrm{Re}^{j\phi})}{R^2e^{j2\phi}(1+\mathrm{Re}^{j\phi})(1+2\,\mathrm{Re}^{j\phi})}=0e^{-j3\phi}$$

$=0(\angle -270^\circ\to\angle 0^\circ\to\angle +270^\circ)$

Number of counter clockwise encirclements to origin are '–2'.

Number of right half poles of open loop is zero.

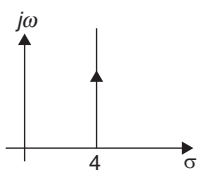
$$Z=P-N=0-(-2)=+2$$

∴ Number of poles on right half plane for closed loop transfer function is '2'.

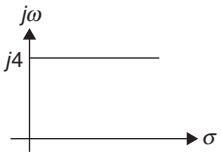
∴ System is unstable.

Example 34: Nyquist plot for the transfer function $G(s)$ = $(4+s)$ for positive frequencies has the form

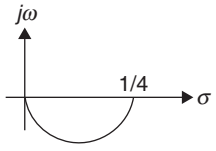
(A)



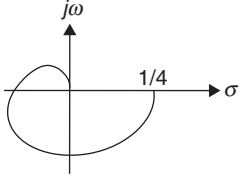
(B)



(C)



(D)



Solution: (A)
Given transfer function $G(s) = 4 + s = 4 + j\omega$

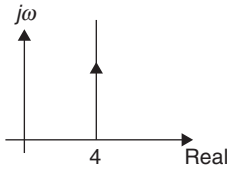
At $\omega = 0 \Rightarrow 4 + j.0$

$\omega = 10 \Rightarrow 4 + j10$

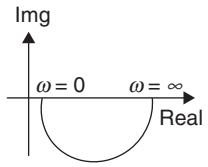
$\omega = 100 \Rightarrow 4 + j100$

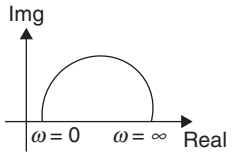
$\omega = \infty \Rightarrow 4 + j\infty$

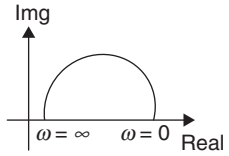
$\therefore$ Nyquist plot is parallel to imaginary axis

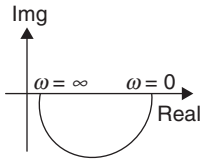


Example 35: Which one of the following polar diagram corresponds to a lag network?

(A) 

(B) 

(C) 

(D) 

Solution: (D)
Lag network offers only negative phase angles and. Let us consider a lag network example.

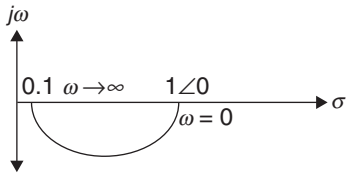
$$G(j\omega) = \frac{s+1}{10s+2}$$

$$|G(j\omega)| \sqrt{\frac{\omega^2+1}{100\omega^2+1}}; \angle G(j\omega) = \tan^{-1}\frac{\omega}{1} \tan^{-1} 10\,\omega.$$

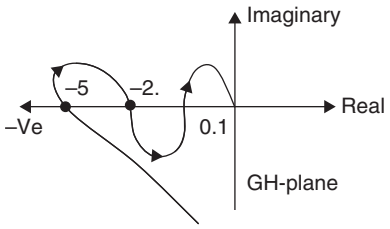
At $\omega = 0 \Rightarrow 1\angle 0$

At $\omega = 5 \Rightarrow 0.103\angle -10.16$

At $\omega = \infty \Rightarrow 0.1\ \angle 0^{o}$



Example 36: The polar plot of a conditionally stable system for open loop gain $K = 1$ is shown in the figure. The open loop transfer function of the system is known to be stable. The closed loop system is stable for



(A) $0.5 < K < 10$ (B) $K < \frac{1}{5}$

(C) Both A and B (D) $0.5 > K > 10$

Solution: (C)
System gain 'K' should be adjusted such that the point $(-1 + j0)$ lies in the 0.1 to 0.2 region, because no. of encirclements in this case is zero which results in stable operation of the system.

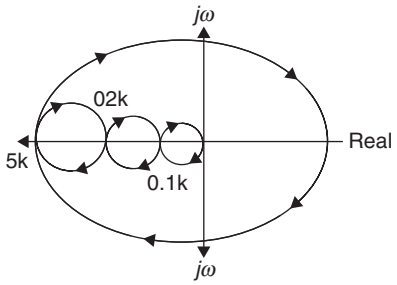
$\therefore$ $0.1k < 1 \Rightarrow K < 10$

$2K > 1 \Rightarrow K > 0.5$

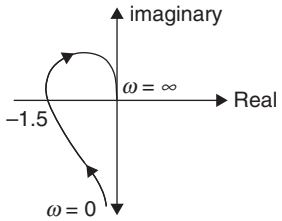
Range of K is $0.5 < K < 10$

System is also stable if $5K < 1$ [no. of encirclements will be zero in this case]

$$K < \frac{1}{5}$$



Example 37: The polar plot of an open loop stable system as shown below the closed loop system is



(A) Marginally stable
(B) Always stable
(C) Unstable with one pole on the RH S-plane
(D) Unstable with two poles on the RH S-plane

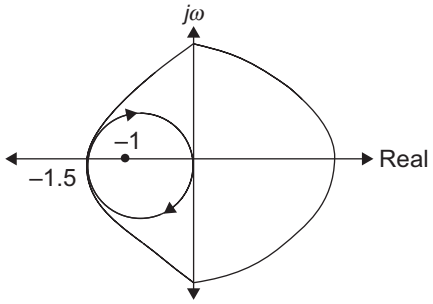
Solution: (D)
Complete polar plot of the given system is given in the figure No. of encirclements of $(-1 + j.0)$ are '–2'

$\therefore$ No. of open loop poles the right hand side = 0

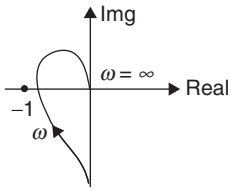
$$N = P - Z \Rightarrow Z = P - N$$

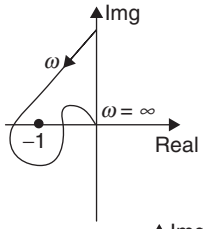
$$Z = 0 - (-2) = 2$$

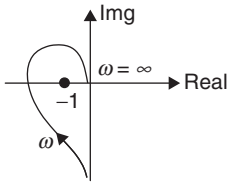
∴ RHS-plane poles of closed loop system are '2' ⇒ Unstable

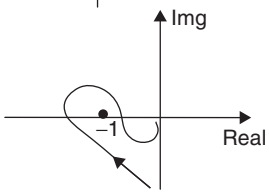


Example 38: Consider the following Nyquist plots of loop transfer function over positive frequencies. Which of the plots represents an unstable system?

1.


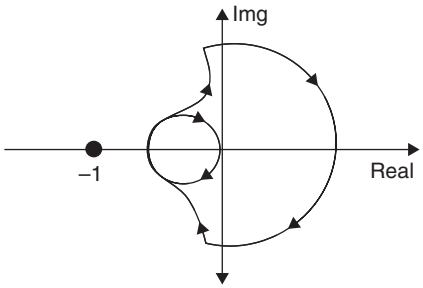
2.


3.


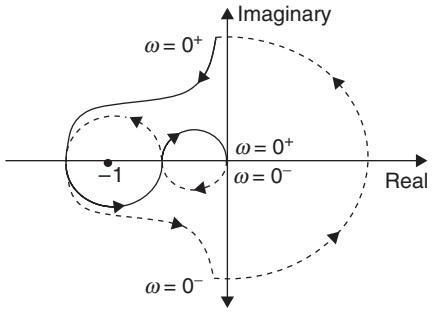
4.


(A) (1), (2) and (3)
(B) (2), (3) and (4)
(C) (1), (2) and (4)
(D) (3) and (4) only

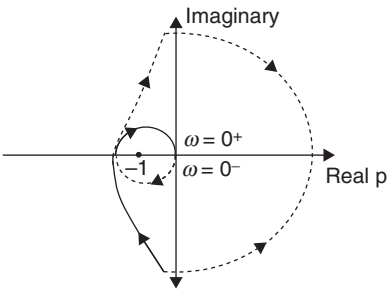
Solution: Plot (1) Nyquist plot is



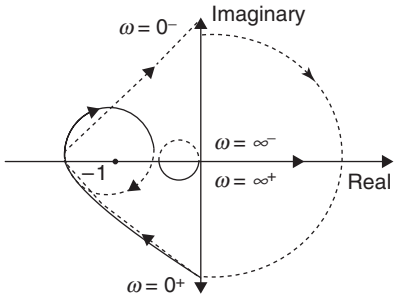
No. of encirclements (N) = 0
If the open loop poles on RHS = 0
∴ System is stable
Plot (2) complete Nyquist plot is



No. of anti-clockwise encirclements (N) = 2.
No. of anti-clockwise encirclements (N) = $P - Z = 2$
No. of open loop poles on RHS side (p) = 0
No. of poles of closed loop system = 2
∴ System is unstable.
Plot (3) complete Nyquist plot is



No. of counter clockwise encirclements (N) = –2
No. of poles of closed loop system on RHS = 2
∴ Unstable system
Option (4) complete Nyquist plot is



Solution: (B)
No. of counter clockwise encirclements (N) = – 2
No. of RHS poles of closed loop control system (Z) = $P - N = 2$
∴ System is unstable.

Example 39: A unity feedback system has the open loop transfer function

$$G(s) = \frac{1}{(s-1)(s+2)(s+3)}$$

The Nyquist plot of 'G' encircles the origin
(A) Once
(B) Twice
(C) Thrice
(D) Never

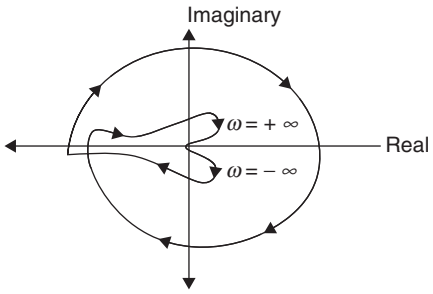
Solution: (A)
No. of encirclements equals the difference between no. of right hand side poles of $G(s)$ and zeros.

$$N = P_{OLTF} - Z_{OLTF}$$

$$P_{OLTF} = 1 \text{ and } Z_{OLTF} = 0$$

$$N = 1.$$

Example 40: Which of the following is the transfer function of a system having the Nyquist plot shown in the following figure?



(A) $\frac{K}{s^2(s+4)(s+8)}$ (B) $\frac{K}{s^2(s+4)^2(s+5)}$

(C) $\frac{K(s+2)}{s^2(s+4)(s+8)}$ (D) $\frac{K(s+2)(s+2)}{s^2(s+4)(s+5)}$

Solution: (A)

Nyquist plot started at –180° angle. It indicates that the open loop system has two poles at origin.

Magnitude and phase angle at $\omega \to 0$

$$\Rightarrow \infty\angle -180^\circ$$

Magnitude and phase angle at $\omega \to \infty$

$$\Rightarrow 0\angle -360^\circ$$

Angle at the termination of NP is – 360°

Angle of termination –360° indicated system order is '4'.

∴ System is type 2 and order '4' system with no zeros.

Example 41: In the GH-plane, the Nyquist plot of the loop transfer function $G(s)H(s) = \frac{2\pi e^{-5s}}{s}$ pass through the negative real axis at the, point

(A) $(-5, j0)$ (B) $(-2, j0)$

(C) $(-10, j0)$ (D) $(-20, j0)$

Solution: (D)

At the point of intersection of Nyquist plot with real axis phase angle $\angle G(s)\,H(s) = -180^\circ = -\pi$

$$\angle \frac{2\pi e^{-5j\omega}}{j\omega} = -\pi$$

$$-5\omega - \tan^{-1}\frac{\omega}{0} = -\pi$$

$$-5\omega - \frac{\pi}{2} = -\pi$$

$$5\omega = \frac{\pi}{2} \Rightarrow \omega = \frac{\pi}{10}$$

$$\left|G(s)H(s)\right|_{\omega=\frac{\pi}{10}} = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{10}} = 20$$

∴ Nyquist plot passes through $(-20, j0)$

Exercises

Practice Problems I

***Directions for questions 1 to 22*:** Select the correct alternative from the given choices.

1. The characteristic equation of a system is given by $s^6 + 3s^5 + 8s^4 + 18s^3 + 37s^2 + 75s + 50 = 0$; the system is

(A) Stable. (B) Unstable.

(C) Marginally stable. (D) Conditionally stable.

2. How many roots of the characteristic equation

$$s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

lie in the left half of S-plane?

(A) 4 (B) 5 (C) 1 (D) 6

3. A system described by the transfer function $H(s) = \frac{1}{s^3 + \alpha s^2 + k s + 2}$ is stable. The constraints on α and k are

(A) $\alpha > 0,\ \alpha k > 2$ (B) $\alpha > 0,\ \alpha k < 2$

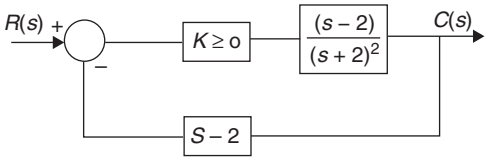
(C) $\alpha > 0,\ \alpha k > 0$ (D) $\alpha < 0,\ \alpha k < 0$

4. The characteristic equation of a system is given by $s(s^2 + 2s + 2) + K(s + 3) = 0$. The range of k for which the system is stable is

(A) $0 < k < 30$. (B) $K > 3$.

(C) $0 < k < 4$. (D) $3 < K < 29$.

5. The feedback control system is fig is stable



(A) for all $K \geq 0$ (B) only if $K \geq 1$

(C) only if $0 \leq k < 1$ (D) only if $0 \leq k \leq 1$

6. Consider the points $S_1 = -3 + j4$ and $S_2 = -3 - j2$ in the S–plane. Then for a system with the open-loop transfer function $G(s)\,H(s) = \frac{k}{(s+1)^4}$ is

(A) S_1 is on the root locus, but not S_2.

(B) Both S_1 and S_2 are on the root locus.

(C) S_2 is on the root locus, but not S_1.

(D) Neither S_1 nor S_2 on the root locus.

7. The gain margin (in dB) of a system having the open loop transfer function.

$$G(s)\,H(s) = \frac{\sqrt{2}}{s(s+1)} \text{ is}$$

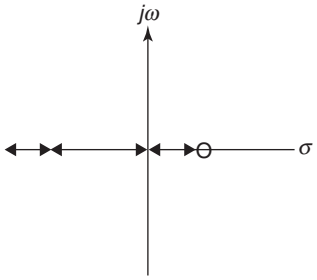
(A) 0 (B) 3.01 (C) –3.01 (D) ∞

8. The characteristic equation of a feedback control system is given by $s^3 + 5s^2 + (K+6)s + K = 0$ In the root loci diagram the asymptotes of the root loci for large 'K' meet at a point in the s-plane whose coordinates are
(A) (2, 0) (B) (–1, 0)
(C) (–2, 0) (D) (–3, 0)

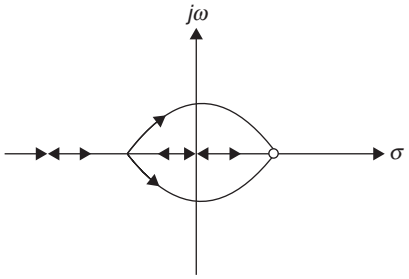
9. The open-loop transfer function of a system is given by $G(s) = \dfrac{k}{s(s+1)(s+2)}$ the value of k which will cause sustained oscillations in the closed-loop unity feed book system is
(A) 4 (B) 6 (C) 5 (D) 3

10. A unity feedback system is given as $G(s) = \dfrac{k(1-s)}{s(s+3)}$ Indicate the correct root Locus diagram.

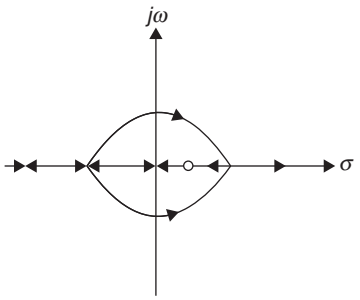
(A)



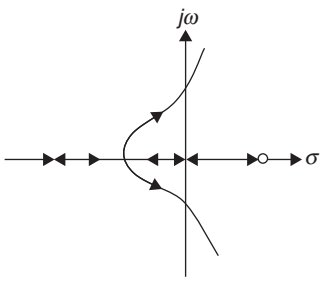
(B)



(C)

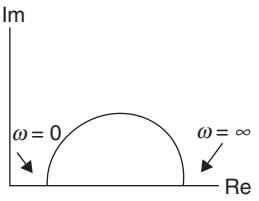


(D)

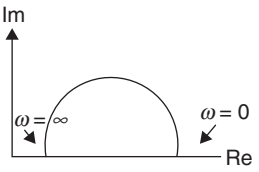


11. Which one of the following polar diagram corresponds to a lag network?

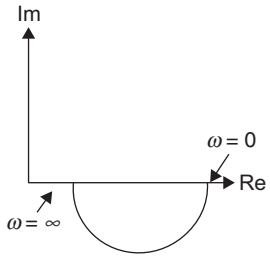
(A)



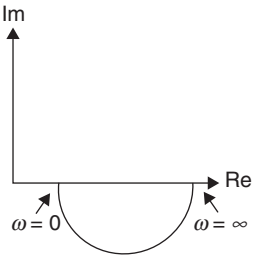
(B)



(C)



(D)



Common Data for Questions 12 and 13:
The open-loop transfer function of a unity feedback system is given by $G(S) = \dfrac{3e^{-2s}}{s(s+2)}$

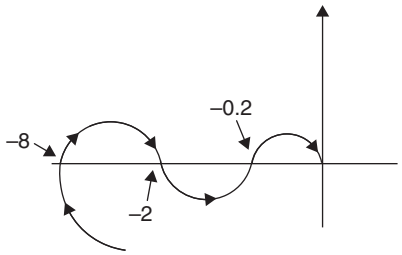
12. The gain and phase cross – over frequencies in rad/s, respectively,
(A) 0.485 and 0.632. (B) 1.26 and 0.632.
(C) 0.632 and 1.26. (D) 0.632 and 0.485.

13. Based on the above results, the gain and phase margins of the system will be
(A) –7.09 dB and 87.5°.
(B) 7.09 dB and 87.5°.
(C) 7.09 dB and –87.5°.
(D) –7.09 dB and –87.5°.

14. The loop transfer function of a closed-loop control system is given as $G(S)H(S) = \dfrac{k(s+1)}{s(s+2)(s+3)}$. The centroid of the asymptotes is
(A) (–4, 0) (B) (–1, 0)
(C) (–2, 0) (D) (–3, 0)

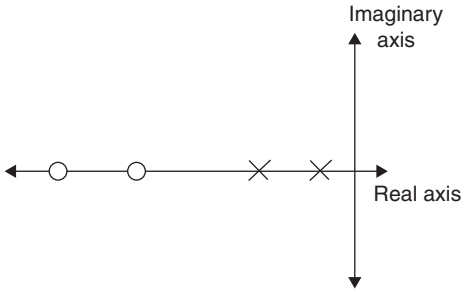
15. A system has 10 poles and 2 zeroes. The slope of its highest frequency asymptote in its magnitude plot is
(A) –100 dB/dec (B) –120 dB/dec
(C) –160 dB/dec (D) –240 dB/dec

16. The polar diagram of a conditionally stable system for open loop gain $K=1$ is shown in figure. The open loop transfer function of the system is known to be stable. The closed loop system is stable for



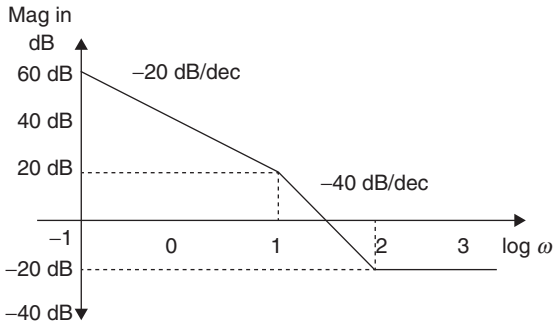
(A) $K<5$ and $\frac{1}{2}<K<\frac{1}{8}$ (B) $K<\frac{1}{8}$ and $\frac{1}{2}<K<5$
(C) $K<\frac{1}{8}$ and $5<K$ (D) $K>\frac{1}{8}$ and $K<5$

17. Pole zero plot of a loop transfer function is shown in figure below, the breakaway/ breakin points in the root locus diagram is



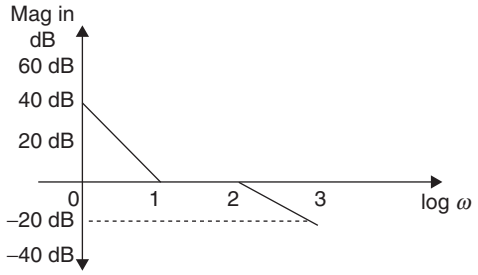
(A) 1 (B) 2 (C) 0 (D) 3

18. Loop transfer function $G(s)H(s)$ of the magnitude plot shown in the figure



(A) $\frac{100(S+2)^2}{S(S+1)}$ (B) $\frac{10(.01s+1)}{s(.1s+1)}$
(C) $\frac{100(.01S+1)^2}{S(.1S+1)}$ (D) $\frac{(.01S+1)^2}{S(.1S+1)}$

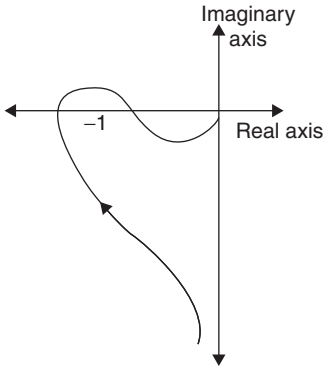
19. Loop transfer function $G(s)H(s)$ of the magnitude plot shown in the figure



(A) $\frac{100(0.1s+1)^2}{s^2(0.01s+1)}$ (B) $\frac{100(s+10)^2}{(s+100)(s+1)^2}$
(C) Both A and B (D) None of the above

20. closed loop control system with transfer function $\frac{G(s)}{1+G(s)H(s)}$ is stable when
(A) Poles of the transfer function are on the left hand side of the *S*-plane.
(B) Zeros of the characteristic equation are on the left half of the *S*-plane.
(C) Poles of the characteristic equation are on the left half of the *S*-plane.
(D) Both A and B.

21. Polar plot of an open-loop stable system is shown in the figure. The system is



(A) System is stable.
(B) System is unstable with one pole on the right-hand side of *S*-plane.
(C) System is unstable with two poles on the right-hand side of *S*-plane.
(D) System is marginally stable.

22. Which of the flowing are effects of PD controller on system?
1. Reduces peak overshoot.
2. Reduces raise time.
3. Improves damping.
4. Reduces steady-state error.

(A) 1, 2, 3 (B) 2, 3, 4
(C) 2, 3, 4 (D) 1, 3, 4

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Which of the following statements are 'true'?

(i) Root Locus is a frequency response plot.

(ii) The roots of characteristic equation are not a function of open Loop gain K.

(iii) Root Locus technique is a tool for adjusting the location of closed loop poles to achieve the desired system performance.

(iv) The exact root– locus is sketched by trial-and-error procedure.

(A) i and ii
(B) ii and iii
(C) iii and iv
(D) ii, iii, and iv

2. The following statements refer to the equation $P(s) + KQ(s) = 0$, where $P(s)$ and $Q(s)$ are polynomials of s with constant coefficients. Identify the statements which are 'true'.

(i) The intersection of the asymptotes must always be on the real axis.

(ii) The breakaway points of the root loci must always be on the real axis.

(iii) Given the equation $1 + KG_1(s)H_1(s) = 0$, where $G_1(s)H_1(s)$ is a rational function of s and does not contain K, the roots of $\frac{dG_1(s)H_1(s)}{ds}$ are all break away points on the root loci $(-\infty < K < \infty)$

(iv) At the break away points on the root loci the root sensitivity is infinite.

(A) i and iv
(B) i, ii, and iv
(C) ii and iii
(D) ii, iii and iv

3. Which of the following statements are true?

(i) Adding a zero to the function $G(s)H(s)$ tends to push the root loci to the left.

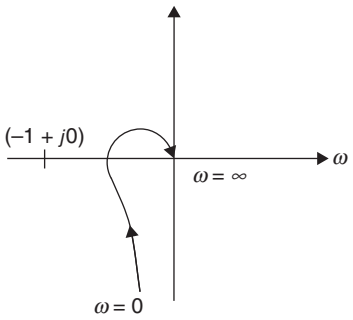
(ii) Adding a zero to the forward-path transfer function will generally improve the system damping, and thus always reduce the maximum over shoot of the system.

(iii) Adding a pole to $G(s)H(s)$ has the effect of pushing the root loci to the right.

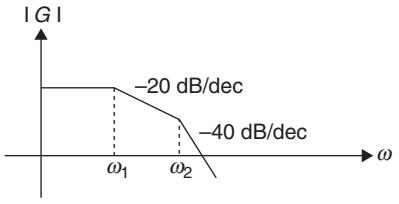
(iv) Complementary root locus (CRL) refers to root loci with positive 'k'.

(A) i, ii, and iii
(B) i, ii and iv
(C) ii, iii and iv
(D) ii and iv

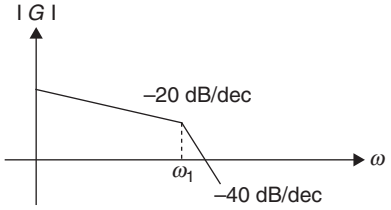
4. The Nyquist plot for a control system is shown in figure. The bode plot for the same system will be



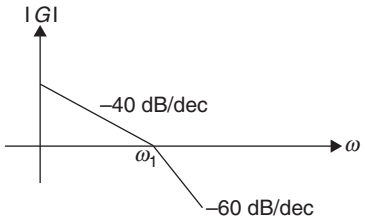
(A)



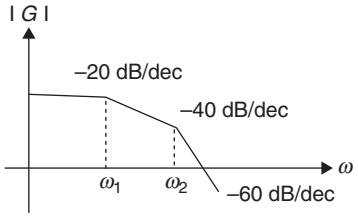
(B)



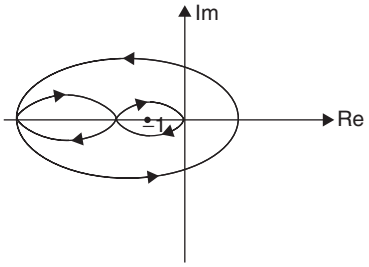
(C)



(D)



5. The Nyquist plot for the open- loop transfer function $G(s)$ of a unity negative feedback system is shown in the figure, if $G(s)$ has no pole in the right half of S-plane, the number of roots of the system characteristic equation in the right-half of S-plane is



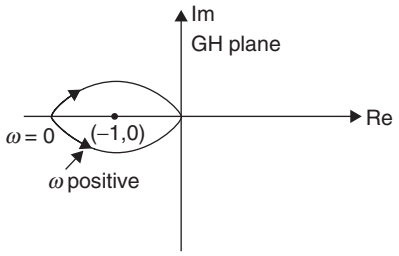
(A) 0 (B) 1 (C) 2 (D) 3

6. Which of the following points is NOT on the root locus of a system with the open-loop transfer function?

$$G(s)H(s)=\frac{K}{s(s+1)(s+3)}$$

(A) $s=-j\sqrt{3}$ (B) $s=-1.5$
(C) $s=-3$ (D) $s=-\infty$

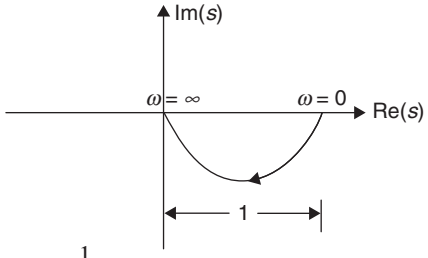
7. The figure shows the Nyquist plot of the open-loop transfer function $G(s)H(s)$ of a system; If $G(s)H(s)$ has one right-hand pole, the closed loop system is



(A) Always stable
(B) Unstable with one closed loop right-hand pole
(C) Unstable with two closed loop right-hand poles
(D) Unstable with three closed-loop right-hand poles

8. Given $G(s)H(s)=\frac{K}{s(s+1)(s+3)}$, the point of intersection of the asymptotes of the root loci with the real axis is
(A) –4 (B) 1.33 (C) –1.33 (D) 4

9. The polar plot shown in the figure represents the transfer function:



(A) $G(s)=\frac{1}{s}$
(B) $G(s)=\frac{1}{s(1+sT)}$
(C) $G(s)=\frac{1}{1+sT}$
(D) $G(s)=\frac{1}{(1+sT_1)(1+sT_2)}$

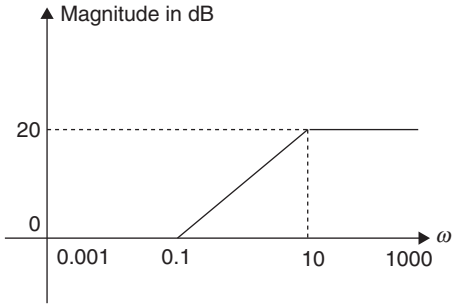
10. The open loop transfer function of a unity gain feedback control system is given by

$$G(s)=\frac{K}{(s+1)(s+3)}$$

The gain margin of the system is dB is given by
(A) ∞ (B) 1 (C) 20 (D) 0

11. If the closed loop transfer function of a control system is given by $T(s)=\frac{s-5}{(s+2)(s+3)}$, then it is
(A) An unstable system
(B) An uncontrollable system
(C) A minimum-phase system
(D) A non-minimum phase system

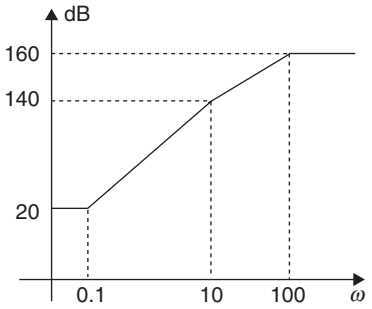
12. For the asymptotic Bode magnitude plot shown in the following figure, the system transfer function can be



(A) $\frac{10s+1}{0.1s+1}$ (B) $\frac{100s+1}{0.1s+1}$
(C) $\frac{100s}{10s+1}$ (D) $\frac{0.1s+1}{10s+1}$

13. The root locus of the system $G(s)H(s)=\frac{K}{s(s+2)(s+3)}$ has the break-away point located at
(A) (–0.5, 0) (B) (–2.548, 0)
(C) (–4, 0) (D) (–0.784, 0)

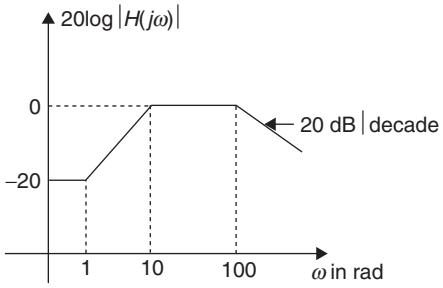
14.



The approximate Bode-Magnitude plot of a minimum-phase system is shown in the figure. The transfer function of the system is

(A) $10^8\frac{(s+0.1)^3}{(s+10)^2(s+100)}$
(B) $10^7\frac{(s+0.1)^3}{(s+10)(s+100)}$
(C) $10^8\frac{(s+0.1)^2}{(s+10)^2(s+100)}$
(D) $10^9\frac{(s+0.1)^3}{(s+10)(s+100)^2}$

15. Consider the Bode magnitude plot shown in the figure. The transfer function $H(s)$ is



(A) $\dfrac{(s+10)}{(s+1)(s+100)}$

(B) $\dfrac{10(s+1)}{(s+10)(s+100)}$

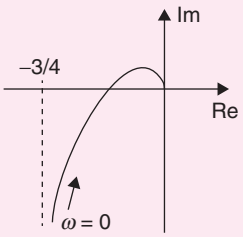
(C) $\dfrac{10^2(s+1)}{(s+10)(s+100)}$

(D) $\dfrac{10^3(s+100)}{(s+1)(s+10)}$

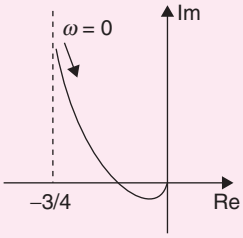
Previous Years' Questions

1. The frequency response of $G(s) = 1/[s\,(s+1)\,(s+2)]$ plotted in the complex $G(j\omega)$ plane (for $0 < \omega < \infty$) is **[2010]**

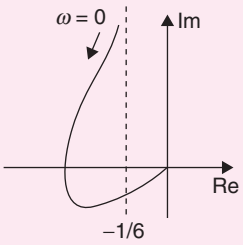
(A)



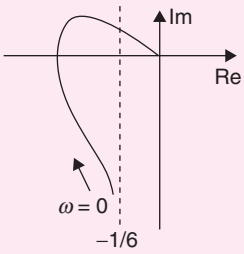
(B)



(C)



(D)



2. The frequency response of a linear system $G(j\omega)$ is provided in the tabular form below

$\lvert G(j\omega)\rvert$	1.3	1.2	1.0	0.8	0.5	0.3
$\angle\, G(j\omega)$	–130°	–140°	–150°	–160°	–180°	–200°

The gain margin and phase margin of the system are **[2011]**

(A) 6 dB and 30°
(B) 6 dB and –30°
(C) –6 dB and 30°
(D) –6 dB and –30°

3. The open loop transfer function $G(s)$ of a unity feedback control system is given as, $G(s) = \dfrac{k\left(s+\frac{2}{3}\right)}{s^2(s+2)}$

From the root locus, it can be inferred that when k tends to positive infinity. **[2011]**

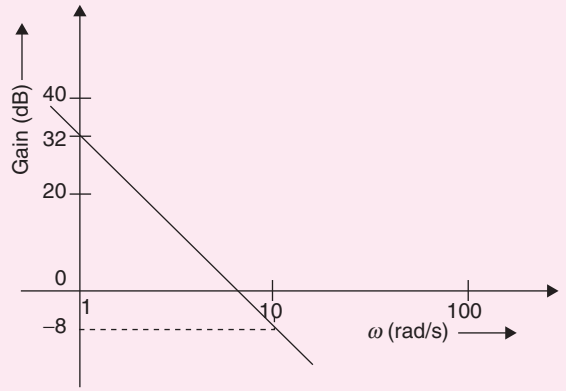
(A) Three roots with nearly equal real parts exist on the left half of the S-plane.

(B) One real root is found on the right half of the S-plane.

(C) The root loci cross the $j\omega$ axis for a finite value of k; $k \neq 0$.

(D) Three real roots are found on the right half of the S-plane.

4. The Bode plot of a transfer function $G(s)$ is shown in the below figure. The gain ($20 \log|G(s)|$) is 32 dB and –8 dB at 1 rad/s and 10 rad/s, respectively. The phase is negative for all ω. The $G(s)$ is **[2013]**

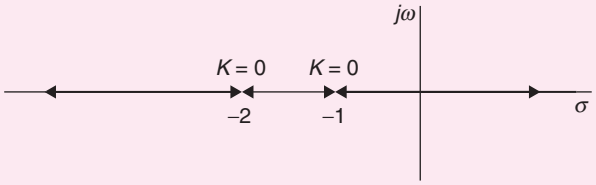


(A) $\frac{39.8}{s}$ (B) $\frac{39.8}{s^2}$ (C) $\frac{32}{s}$ (D) $\frac{32}{s^2}$

5. In the formation of Routh–Hurwitz array for a polynomial all the elements of a row have zero values. This premature termination of the array indicates the presence of **[2014]**

(A) Only one root at the origin
(B) Imaginary roots
(C) Only positive real roots
(D) Only negative real roots

6. The root locus of a unity feedback system is shown in the figure **[2014]**



The closed loop transfer function of the system is

(A) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)}$

(B) $\frac{C(s)}{R(s)} = \frac{-K}{(s+1)(s+2)+K}$

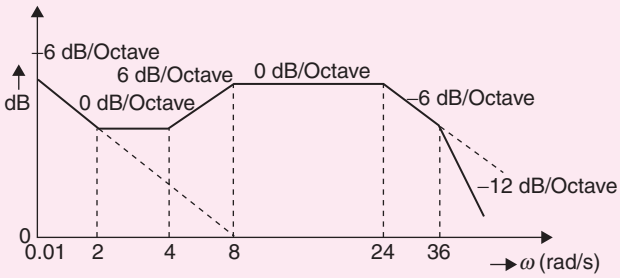
(C) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)-K}$

(D) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)+K}$

7. For the given system, it is desired that the system be stable. The minimum value of α for this condition is _____. **[2014]**

8. The Bode magnitude plot of the transfer function $G(s) = \frac{K(1+0.5s)(1+as)}{s\left(1+\frac{s}{8}\right)(1+bs)\left(1+\frac{s}{36}\right)}$ is shown below:

Note that-6 dB/octave = –20 dB/decade. The value of $\frac{a}{bK}$ is _____. **[2014]**



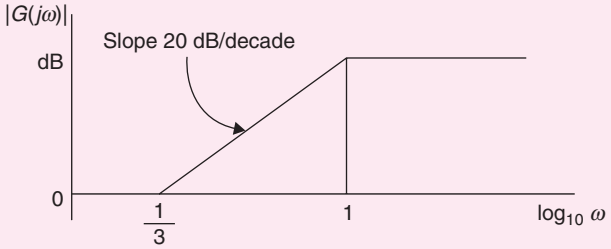
9. A system with the open loop transfer function $G(s) = \frac{K}{s(s+2)(s^2+2s+2)}$ is connected in a negative feedback configuration with a feedback again of unity. For the closed loop system to be marginally stable, the value of K is _____ **[2014]**

10. For the transfer function $G(s) = \frac{5(s+4)}{s(s+0.25)(s^2+4s+25)}$

The values of the constant gain term and the highest corner frequency of the Bode plot, respectively, are **[2014]**

(A) 3.2, 5.0 (B) 16.0, 4.0
(C) 3.2, 4.0 (D) 16.0, 5.0

11. The magnitude Bode plot of a network is shown in the figure **[2014]**

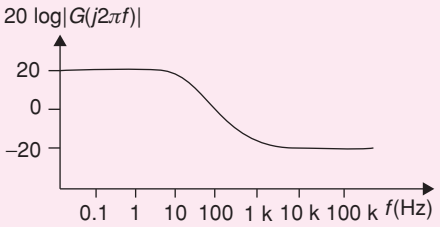


The maximum phase angle Φ_m and the corresponding gain G_m, respectively, are

(A) -30^o and 1.73 dB
(B) -30^o and 4.77 dB

(C) +30° and 4.77 dB
(D) +30° and 1.73 dB

12. A Bode magnitude plot for the transfer function $G(s)$ of a plant is shown in the figure. Which one of the following transfer functions best describes the plant? **[2015]**



(A) $\frac{1000(s+10)}{s+1000}$

(B) $\frac{10(s+10)}{s(s+1000)}$

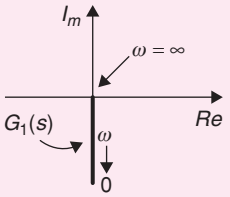
(C) $\frac{s+1000}{10s(s+10)}$

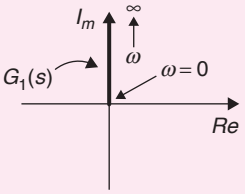
(D) $\frac{s+1000}{10(s+10)}$

13. The transfer function of a second order real system with a perfectly flat magnitude response of unity has a pole at $(2-j3)$. List all the poles and zeroes. **[2015]**
(A) Poles at $(2 \pm j3)$, no zeroes.
(B) Poles at $(\pm 2-j3)$, one zero at origin.
(C) Poles at $(2-j3)$, $(-2+j3)$, zeroes at $(-2-j3)$, $(2+j3)$
(D) Poles at $(2 \pm j3)$, zeroes at $(-2 \pm j3)$

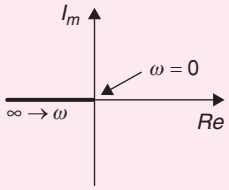
14. The open loop poles of a third order unity feedback system are at 0, –1, –2. Let the frequency corresponding to the point where the root locus of the system transits to unstable region be K. Now suppose we introduce a zero in the open loop transfer function at –3, while keeping all the earlier open loop poles intact. Which one of the following is TRUE about the point where the root locus of the modified system transits to unstable region? **[2015]**
(A) It corresponds to a frequency greater than K
(B) It corresponds to a frequency less than K
(C) It corresponds to a frequency K
(D) Root locus of modified system never transits to unstable region.

15. Nyquist plots of two functions $G_1(s)$ and $G_2(s)$ are shown in figure. **[2015]**

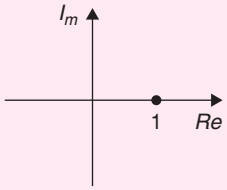




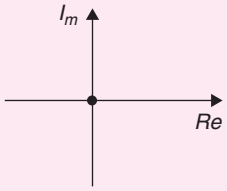
(a)



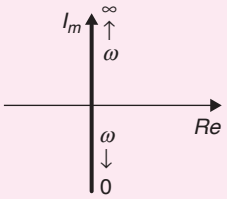
(b)



(c)



(d)



16. An open loop transfer function $G(s)$ of a system is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

For a unity feedback system, the breakaway point of the root loci on the real axis occurs at, **[2015]**
(A) –0.42
(B) –1.58
(C) –0.42 and –1.58
(D) none of the above

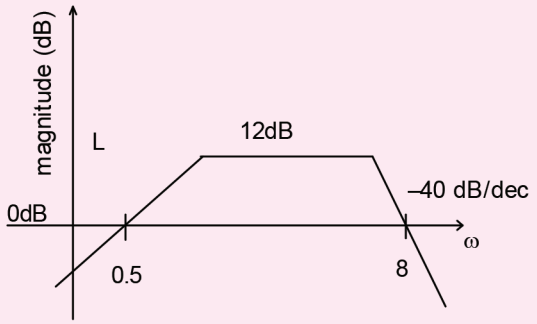
17. The transfer function of a system is $\frac{Y(s)}{R(s)} = \frac{s}{S+2}$ the steady state output $y(t)$ is $A\cos(2t+\phi)$ for the input $\cos(2t)$the values of A and ϕ respectively are **[2016]**

(A) $\frac{1}{\sqrt{2}}$,–45° (B) $\frac{1}{\sqrt{2}}$,+ 45°

(C) $\sqrt{2}$,–45° (D) $\sqrt{2}$,+45°

18. The phase cross-over frequency of the transfer function $G(S) = \frac{100}{(S+1)^3}$ in rad/s is **[2016]**

19. Consider the following asymptotic Bode magnitude plot (ω is in rad/s).



Which one of the following transfer functions is best represented by the above Bode magnitude plot? **[2016]**

(A) $\frac{2S}{(1+0.5S)(1+0.25S)^2}$

(B) $\frac{4(1+0.5S)}{S(1+0.25S)}$

(C) $\frac{2S}{(1+2S)(1+4S)}$

(D) $\frac{4S}{(1+2S)(1+4S)^2}$

20. Loop transfer function of a feedback system is $G(s)$ $H(s) = \frac{s+3}{s^3(s-3)}$ Take the Nyquist contour in the clockwise direction. Then, the Nyquist plot of $G(s)$ $H(s)$ encircles – 1 + j 0. **[2016]**

(A) once in clockwise direction
(B) twice in clockwise direction
(C) once in anticlockwise direction
(D) twice in anticlockwise direction

21. Given the following polynomial equation

$S^3 + 5.5S^2 + 8.5^S + 3 = 0$, the number of roots of the polynomial, which have real parts strictly less than –1 is _____. **[2016]**

22. Consider a linear time - invariant system with transfer function

$$H(S) = \frac{1}{(S+1)}$$

If the input is Cos(t) and the steady state output is $A\text{Cos}(t + \alpha)$, then the value of A is _____. **[2016]**

23. The open loop transfer function of a unity feedback control system is given by

$$G(S) = \frac{K(S+1)}{S(1+Ts)(1+2S)}, K > 0, T > 0$$

The closed loop system will be stable if, **[2016]**

(A) $0 < T < \frac{4(K+1)}{K-1}$

(B) $0 < K < \frac{4(T+2)}{T-2}$

(C) $0 < K < \frac{T+2}{T-2}$

(D) $0 < T < \frac{8(K+1)}{K-1}$

24. The gain at the breakaway point of the root locus of a unity feedback system with open loop transfer function $G(S) = \frac{KS}{(S-1)(S-4)}$ is **[2016]**

(A) 1 (B) 2

(C) 5 (D) 9

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** B	**3.** A	**4.** C	**5.** C	**6.** C	**7.** D	**8.** C	**9.** B	**10.** C
11. D	**12.** B	**13.** D	**14.** C	**15.** C	**16.** B	**17.** B	**18.** C	**19.** A	**20.** C
21. A									

Practice Problems 2

1. C	**2.** A	**3.** A	**4.** D	**5.** A	**6.** B	**7.** A	**8.** C	**9.** C	**10.** A
11. D	**12.** A	**13.** D	**14.** A	**15.** C					

Previous Years' Questions

1. A	**2.** A	**3.** A	**4.** B	**5.** B	**6.** C	**7.** 0.618	**8.** 0.75	**9.** 5	**10.** A
11. C	**12.** D	**13.** D	**14.** D	**15.** B	**16.** A	**17.** B	**18.** A	**19.** A	**20.** A
21. 2	**22.** 0.707	**23.** C	**24.** A						

Chapter 4

Controllers and Compensators

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Compensators and controllers
- Compensators
- Lag compensator
- Lead-compensator
- Lead–Lag compensator
- Controllers
- Proportional controller
- Proportional plus integral controller
- Proportional plus derivative controller
- Proportional plus integral plus derivative controller

Compensators and Controllers

Every control system is designed for a specific application and to meet certain performance parameters or specifications. System specifications in time domain and/or in frequency domain such as peak overshoot, settling time, gain margin, phase margin. A device inserted into the system for the purpose of satisfying the specification is called a compensator.

There are two types of compensation schemes.

1. Series compensation
2. Feedback or Parallel compensation

Series Compensation

In series compensation, a compensator is introduced in series with the plant to change the system behaviour and to meet the desired specifications.

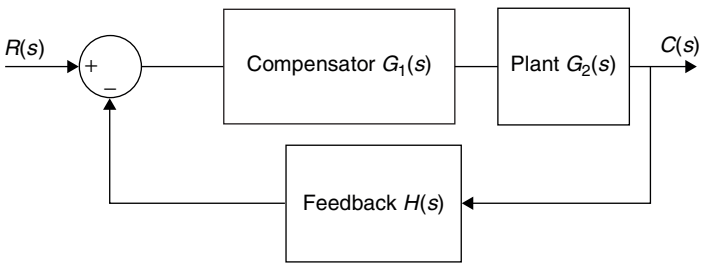


Figure 1 Block diagram for series compensation

Feedback Compensation

In feedback compensation, a compensator is introduced in the feedback path to meet the desired specifications.

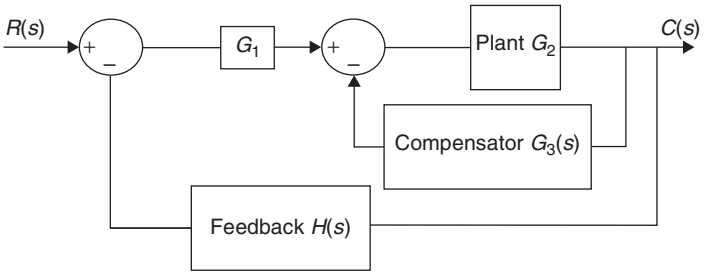


Figure 2 Block diagram of feedback compensation

Compensators

Different types of compensators are

1. Lag compensator
2. Lead compensator
3. Lead–lag compensator

Lag-Compensator

The low-pass filter is known as phase-lag compensator. The ideas to filter and phase shift are useful if designs are carried in the frequency domain.

The transfer function of a simple lag compensator is given by $\Rightarrow G_c(s) = \frac{1+Ts}{1+aTs}$; $a > 1$

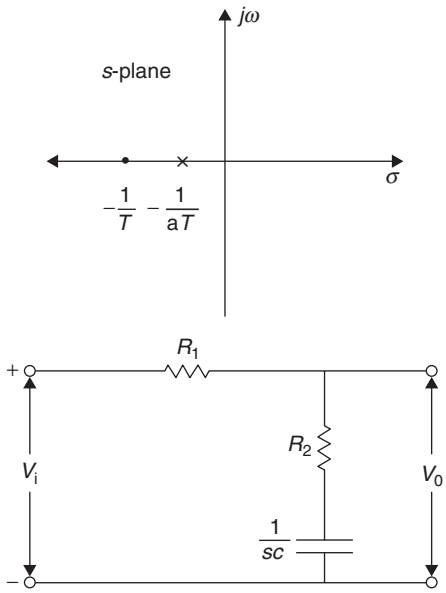


Figure 3 Pole zero configuration of electrical lag compensator

Lag compensator improves the steady-state behaviour of a system, while transient behaviour remains unchanged.

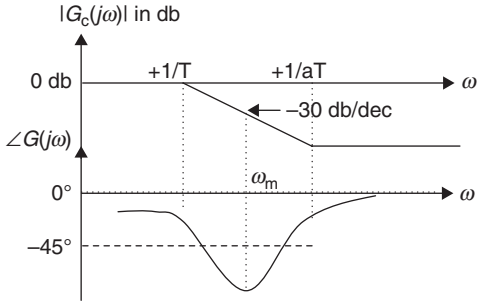


Figure 4 Bode plot of the lag compensator

The value of phase angle is maximum at a frequency of

$$\omega_m = \sqrt{\omega_{c1} \times \omega_{c2}} = \sqrt{\frac{1}{aT} \times \frac{1}{T}} = \frac{1}{T\sqrt{a}}$$

$\therefore$ Maximum phase angle

$$\angle G(j\omega)\big|_{\omega=\omega_m} = \phi_m = \tan^{-1}\left(\frac{1-a}{2\sqrt{a}}\right)$$

$$\angle G(j\omega)\big|_{\omega=\omega_m} = \phi_m = \sin^{-1}\left(\frac{1-a}{1+a}\right)$$

Stability of the system relatively reduces with addition of Lag-compensator. The gain crossover frequency of the system decreases and thus the bandwidth of the system is reduced. The raise time and settling time of the system are usually longer, because the bandwidth is usually decreased. System is more sensitive to parameter variations.

Lead Compensator

The high-pass filter is known as phase-lead compensator. These ideas to filter and phase shift are useful if designs are carried in the frequency domain. The transfer function of a simple lead compensator is given by

$$G_c(s) = \frac{1+\zeta s}{1+a\zeta s} ; \; a < 1$$

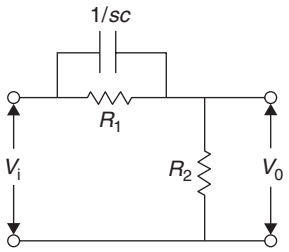


Figure 5 Electrical lead compensator

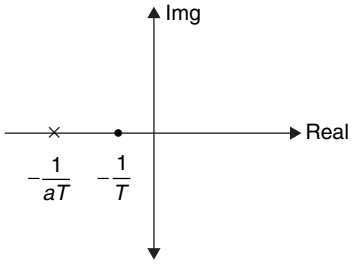


Figure 6 Pole zero configuration of lead compensator

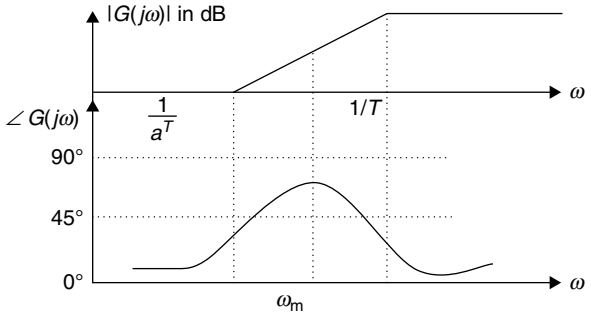


Figure 7 Bode plot of lead compensator

The value of phase angle is maximum at frequency of $\omega_m = \sqrt{\omega_{c1} \times \omega_{c2}} = \frac{1}{T\sqrt{a}}$

Maximum phase angle $\phi_m = \tan^{-1}\left(\frac{1-a}{2\sqrt{a}}\right)$.

The lead compensator affects the transient response of the system. It adds damping to the system and thus the rise time and settling time are reduced. The gain cross over frequency is increased and improves the phase margin of the closed loop system. The relative stability of the system is improved with improvement in gain and phase margins.

The bandwidth of the closed loop system is increased and results in fast response.

The steady-state error of the system is not affected.

Lead–Lag Compensator

The combination of lag and lead compensators is used to utilize the advantages of both the schemes. The transfer function of a simple lead–lag compensator is given by

$$G_c(s)=\left(\frac{1+a_1T_1s}{1+T_1s}\right)\left(\frac{1+a_2T_2s}{1+T_2s}\right)(a_1>1, a_2<1)$$

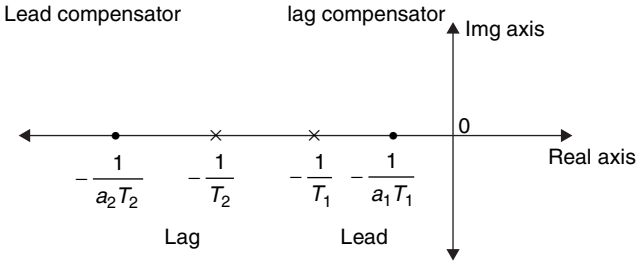


Figure 8 Pole-zero configuration of Lead-Lag compensator

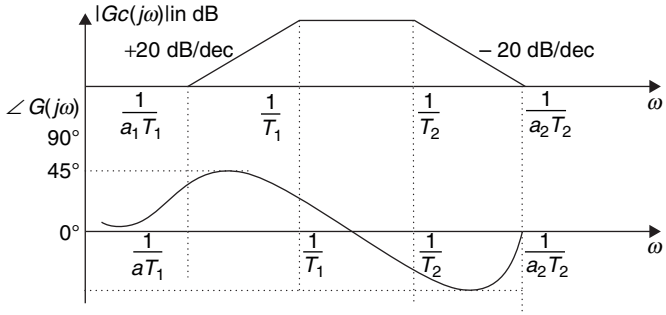
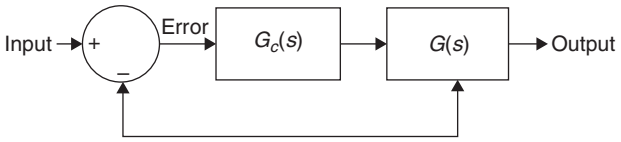


Figure 9 Bode plot of lead-lag compensator

CONTROLLERS

The cascaded controllers are used to modify the transient and the steady-state response of the system.



The different types of controllers available in control system are given as follows:

1. Proportional controller
2. Proportional plus integral controller (PI-Controller)
3. Proportional plus derivative controller (PD-Controller)
4. Proportional plus derivative plus integral controller (PID-Controller)

Proportional Controller

A controller that produces output proportional to the input signal (error $e(t)$) is called a proportional controller.

$$\therefore \quad u(t) = K_P.e(t)$$

where

$u(t)$ = Input to the plant
$e(t)$ = Input to the controller
K_P = Gain of the proportional controller

The proportional controller improves the steady-state accuracy, disturbance signal rejection and relative stability.

The proportional controller decreases the sensitivity of the system to parameter variation.

The proportional controller is not used alone because it produces a constant steady-state error.

Proportional Plus Integral Controller (PI)

A controller that produces output signal $u(t)$ proportional to the input signal (error $e(t)$) and to the integral of input signal.

$$\therefore \quad u(t) = K_Pe(t) + K_I\int e.dt$$

K_P = Proportional controller gain
K_I = Integral controller gain

The PI controller increases the order and type of the system. It acts as a low-pass filter. The PI controller reduces the steady-state error.

Proportional Plus Derivative Controller (PD)

A controller that produces output signal $u(t)$ proportional to the input signal (error $e(t)$ and derivative of the input signal $\left(\frac{de(t)}{dt}\right)$.

$$u(t) = K_Pe(t)+K_D\frac{de(t)}{dt}$$

The PD controller increases damping of the system which results in reducing the peak overshoot.

The PD controller acts as a high-pass filter. It improves gain margin and phase margin.

The PD controller increases bandwidth, reduces rise time and settling time system stability is improved.

Proportional Plus Integral Plus Derivative Controller

This is a combination of proportional, Integral and derivative controllers.

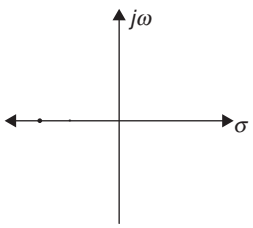
$$u(t)=K_Pe(t)+K_I\int e(t)dt+K_D\frac{de(t)}{dt}$$

The PID controller decreases the steady-state error and increases stability.

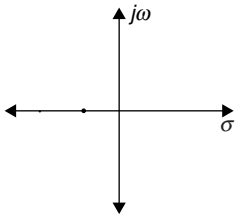
Solved Examples

Example 1: The pole zero configuration of a phase Lag Compensator is given by

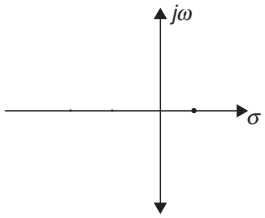
(A)



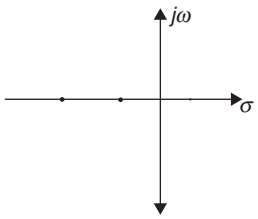
(B)



(C)



(D)



Solution: (A)
Lag compensator has a dominant pole with a single pole zero configuration.

Example 2: A lead compensator used for a closed loop controller has the following transfer function

$$\frac{k(1+as)}{(1+bs)}$$

For such a lead compensator
(A) $a < b$ (B) $a > b$
(C) $a > kb$ (D) $a < kb$

Solution: (B)
Pole zero values of given compensator are

$$-\frac{1}{a} \text{ and } \frac{-1}{b}$$

For a lead compensator, pole is dominating; so

$$-\frac{1}{a} > \frac{-1}{b} \Rightarrow \frac{1}{a} < \frac{1}{b} \Rightarrow a > b.$$

Hence, the correct option is (B).

Example 3: The transfer function of PID controller is given by

(A) $\dfrac{s}{K_I s^2 + K_D s + K_p}$

(B) $\dfrac{K_D s^2 + K_p s + K_I}{s}$

(C) $\dfrac{s}{K_D s^2 + K_p s + K_I}$

(D) $\dfrac{K_I s^2 + K_D s + K_P}{s}$

Solution: (B)
The relation between input and output of a PID controller is given by

$$u(t) = K_P e(t) + K_I \int e(t) + K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides

$$u(s) = \left(K_P + \frac{K_I}{S} + K_D s \right) E(s)$$

$$\frac{u(s)}{E(s)} = \frac{K_D s^2 + K_P s + K_I}{S}.$$

Example 4: Maximum phase lead of the compensator $D(s)$ is

$$D(s) = \frac{0.4s+1}{0.04s+1}$$

(A) 50° (B) 55°
(C) 60° (D) None of the above

Solution: (B)

Compensator $D(s) = \dfrac{0.4s+1}{0.04s+1} = \dfrac{1+aTs}{1+Ts}$

$aT = 0.4;\ 0.04 = T$

$\therefore \quad a = 10$

Maximum phase angle $\phi_m = \sin^{-1}\left(\dfrac{a-1}{a+1}\right) = 55°$

Example 5: Phase angle of the PID controller at high frequencies is (as frequency tends to infinity)
(A) ∞ (B) 90°
(C) –90° (D) 180°

Solution: (B)
Transfer function of the PID controller

$$= \frac{K_D s^2 + K_P s + K_I}{s}$$

Phase angle of

$$\text{controller} = \tan^{-1}\left(\frac{\omega K_P}{\left(K_I - \omega^2 K_D\right)}\right) - \tan^{-1}\left(\frac{\omega}{\infty}\right)$$

$$\phi = 180^\circ - 90^\circ = 90^\circ \text{ (as } \omega \to \infty)$$

Example 6: The transfer function of two compensators are given by

$$C_1 = \frac{20(1+s)}{(20+s)}, C_2 = \frac{s+10}{10(s+1)}$$

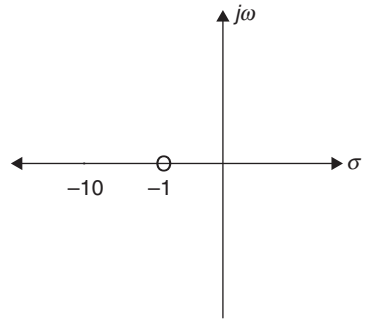
Which of the following statement is correct?

(A) C_1 is a lag compensator and C_2 is a lead compensator.
(B) C_1 is a lead compensator and C_2 is a lag compensator.
(C) Both C_1 and C_2 are lead compensators.
(D) Both C_1 and C_2 are lag compensators.

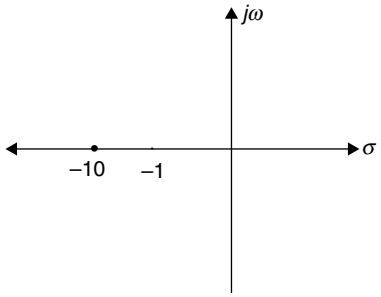
Solution: (B)

Compensator $C_1 = \frac{10(s+1)}{(s+10)}$

From pole zero configuration, C_1 is lead compensator.



Compensator $C_2 = \frac{s+10}{10(s+1)}$



From the above pole zero,

Configuration C_2 is lag compensator.

Example 7: The system $\frac{900}{s(s+1)(s+9)}$ is to be such that its gain crossover frequency becomes same as its uncompensated phase crossover frequency and provides 45° phase margin. To accomplish this, one may use

(A) A lag–lead compensator that provides an amplification of 20 dB and a phase lead of 45° at the frequency of $\sqrt{3}$ rad/sec.
(B) A lag–Lead compensator that lprovides an attenuation of 20 dB and phase lead of 45° at a frequency of 3 rad/s.
(C) A lag compensator that provides an attenuation of 20 dB and a phase angle of 45° at the frequency of $3\sqrt{3}$ rad/s.
(D) A lead compensator that provides an amplification of 20 dB and a phase lead of 45° at the frequency of 3 rad/s.

Solution: (B)

Phase crossover frequency of the system can be calculated as

$$\angle G(j\omega)\big|_{\omega=\omega_{pc}} = -180$$

$$\omega_{pc} = \frac{1}{\sqrt{T_1 T_2}} = \frac{1}{\sqrt{1\times\frac{1}{9}}} = 3 \text{ rad/sec}$$

Gain margin of the system (GM)

$$= 20\log\frac{1}{\left|G(j\omega_{pc})H(j\omega_{pc})\right|}$$

$$\left|G(j\omega_{pc})H(j\omega_{pc})\right| = \left|\frac{900}{s(s+1)(s+9)}\right|$$

$$\left|\frac{900}{\omega\sqrt{(\omega^2+1)}\sqrt{\omega^2+9^2}}\right| = 10$$

$$\text{Gain margin (GM)} = 20\log\frac{1}{\left|G(j\omega)H(j\omega)\right|} = 20 \text{ dB}$$

From the given data, phase crossover frequency is equal to gain crossover frequency. For this we need to make the magnitude of the system at $\omega = \omega_{pc}$ equal to zero. A lag compensator is used to reduce the gain of the system by 20 dB at $\omega = \omega_{pc}$.

To provide phase margin of 45°, we need to increase the phase angle of the system by 45° which is 0° at $\omega = \omega_{pc}$ 3 rad/s. A Lead compensator is used to obtain 45° of phase margin at $\omega = \omega_{pc} = 3$ rad/s.

EXERCISES

Practice Problems I

Directions for questions 1 to 9: Select the correct alternative from the given choices.

1. Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I	List-II
(A) Phase lag controller	(1) Improvement in transient response
(B) Addition of zero at origin	(2) Reduction in steady-state error
(C) Derivative output compensation	(3) Reduction in settling time
(D) Derivative error compensation	(4) Increase in damping constant

Codes:

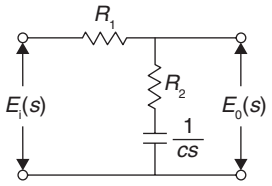
	A	B	C	D
(a)	4	3	1	2
(b)	2	1	3	4
(c)	4	1	3	2
(d)	2	3	1	4

2. The effect of phase lead compensator on gain cross over frequency (ω_{gc}) and on bandwidth (ω_b) is that
(A) both increases.
(B) ω_{gc} increases but ω_b decreases.
(C) ω_{gc} decreases but ω_b increases.
(D) both decreases.

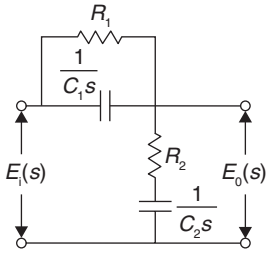
Match List-I (circuits) with List-II and select the correct answer using the codes given below the lists for questions (3) and (4).

3.

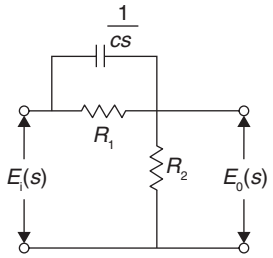
A.



B.



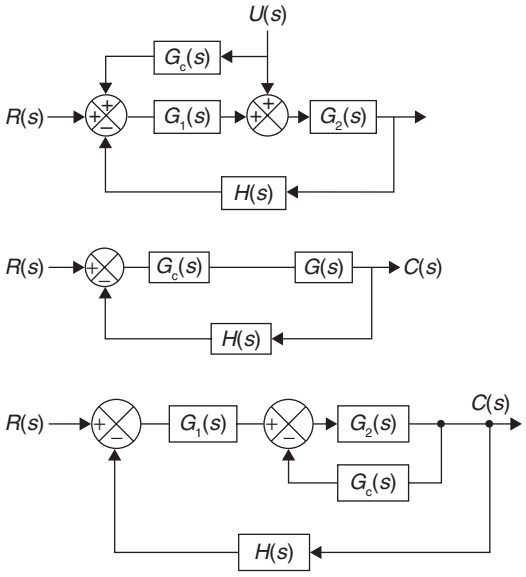
C.



List-II
1. Lag network
2. Lead network
3. Lag–lead network

	A	B	C
(A)	1	2	3
(B)	1	3	2
(C)	2	3	1
(D)	2	1	3

4.



List-II
1. Cascade compensation
2. Feedback compensation
3. Feedforward compensation

	A	B	C
(A)	1	2	3
(B)	2	1	3
(C)	2	3	1
(D)	3	1	2

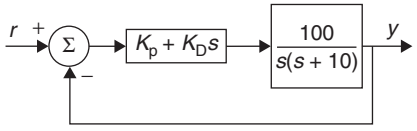
5. Which of the following statements are true?
(i) Adding a zero to the loop transfer function will always decrease the bandwidth of the closed loop system.
(ii) Adding a pole to the loop transfer function makes the system less stable while decreasing the bandwidth.
(iii) The resonant peak 'M_r' indicates the relative stability of a stable closed loop system.
(iv) The slope of a magnitude curve of the Bode plot at the gain crossover indicates the relative stability of the closed loop system.
(A) i and ii (B) ii, iii and iv
(C) i and iii (D) ii and iv

6. Which of the following are effects of phase–lead compensation?
(i) Improves the phase margin of the closed loop system.
(ii) Steady-state error is reduced.
(iii) Improves the relative stability.
(iv) The rise and settling times are increased.
(A) i, and ii (B) ii and iii
(C) i, iii and iv (D) i, and iii

7. The compensator has the transfer function $G_c(s) = \dfrac{10(1+0.04s)}{(1+0.01s)}$ the maximum phase – angle lead provided by this compensator will occur at a frequency ω_n equal to
(A) 25 rad/s
(B) 50 rad/s
(C) 4 rad/s
(D) 10 rad/s

8. Indicate which one of the following transfer functions represent phase lead compensator?
(A) $\dfrac{1+0.5s}{1+s}$ (B) $\dfrac{s+1}{s+2}$
(C) $\dfrac{6s+3}{6s+2}$ (D) $\dfrac{s+5}{3s+2}$

9. A control system with a PD controller is shown in the figure. If the velocity error constant $K_v = 1000$ and the damping ratio $\zeta = 0.5$, then the value of K_P and K_D are



(A) $K_P = 100$, $K_D = 0.09$.
(B) $K_P = 100$, $K_D = 0.9$.
(C) $K_P = 10$, $K_D = 0.09$.
(D) $K_P = 10$, $K_D = 0.9$.

Directions for question 10: The following questions consists of two statements one Assertion and the other reasoning. Select your answer to these questions using the codes given below.

Codes:
(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

10. Assertion(A): Stability of a system deteriorates when integral control is incorporated in it.

Reason (R): With integral control action order of the system increases, and higher the order of the system the more the system tends to become unstable**.**

Directions for questions 11 to 18: Select the correct alternative from the given choices.

11. The transfer function $\dfrac{1+0.4s}{1+s}$ represents a
(A) Lag network (B) Lead network
(C) Lead–lag network (D) None of these

12. The transfer function of a phase lead compensator is given by $\dfrac{1+aTs}{1+Ts}$. When $a > 1$ and $T > 0$, the maximum phase shift provided by such a compensator is
(A) $\tan^{-1}\left(\dfrac{a-1}{a+1}\right)$ (B) $\tan^{-1}\left(\dfrac{a+1}{a-1}\right)$
(C) $\sin^{-1}\left(\dfrac{a-1}{a+1}\right)$ (D) $\sin^{-1}\left(\dfrac{a+1}{a-1}\right)$

13. Which one of the following is the correct expression for the transfer function of an electrical RC phase-lead compensating network?
(A) $\dfrac{(1+sT)}{\alpha(1+s\alpha T)};\alpha<1$ (B) $\dfrac{\alpha(1+sT)}{(1+s\alpha T)};\alpha<1$
(C) $\dfrac{\beta(1+sT)}{(1+s\beta T)};\beta<1$ (D) $\dfrac{(1+sT)}{(1+sT\beta)};\beta>1$

14. Maximum phase lead of the compensation $D(s) = \dfrac{(0.5s+1)}{(0.05s+1)}$ is
(A) 55° at 12 rad/s
(B) 35° at 10 rad/s
(C) 45° at 4 rad/s
(D) 55° at 6.3 rad/s

15. The transfer function of a phase lag compensator is found to be of the form $\dfrac{s+Z_1}{s+P_1}$ and that of a lead compensator to be of the form $\dfrac{s+Z_2}{s+P_2}$. Which of the following conditions must be satisfied?

(A) $Z_1 > P_1$ and $Z_2 > P_2$ (B) $Z_1 > P_1$ and $Z_2 < P_2$
(C) $Z_1 < P_1$ and $Z_2 > P_2$ (D) $Z_1 < P_1$ and $Z_2 < P_2$

16. A system gain crossover frequency is less than its phase crossover frequency then
(A) Lag–lead compensator which will decrease the gain and increase the phase angle can be used to stabilize the system.
(B) Lead–lag compensator which will increase the gain and decrease the phase angle can be used to stabilize the system.
(C) Lead compensator which will increase the gain can stabilize the system.
(D) None of the above.

17. Match the following

Transfer Function	Type of Damping
1. PI Controller	P. Improves damping of the system
2. PD controller	Q. Increases bandwidth
3. Lead compensator	R. Decreases bandwidth
4. Lag compensator	S. Decreases steady-state error

Code:

	P	Q	R	S
(A)	2	4	1	3
(B)	3	2	4	1
(C)	2	3	4	1
(D)	3	4	1	2

18. Transfer function of a compensator is given by $\frac{10(s+.001)}{(s+0.1)}$. The compensator offers maximum frequency at
(A) 0.01 rad/s
(B) 1 rad/s
(C) 10 rad/s
(D) 100 rad/s

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Which of the following are effects of PD control?
(i) Increases rise time and setting time.
(ii) Improves GM, PM, and M_r.
(iii) Decreases bandwidth.
(iv) Improves damping and reduces maximum overshoot.
(A) i, ii and iii (B) ii and iv
(C) i, iii and iv (D) i, and iv

2. Which of the following are effects of PI control?
(i) Filters out high-frequency noise.
(ii) Increases bandwidth.
(iii) Improves damping and reduces maximum overshoot.
(iv) By proper design PI control can improve transient and the steady-state performances.
(A) i, ii and iii (B) ii and iii
(C) iii and iv (D) i, iii and iv

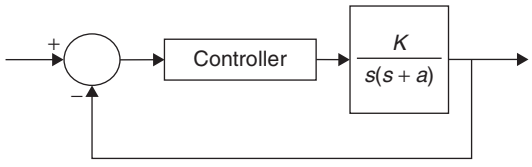
3. Phase-lag compensation results in
(i) Increase in gain-crossover frequency.
(ii) Reduction of bandwidth.
(iii) More sensitivity.
(iv) Improvement of the relative stability.
(A) i, ii and iv (B) ii and iv
(C) ii, iii and iv (D) iii and iv

4. A double integrator plant $G(s) = \frac{k}{s^2}$, $H(s) = 1$ is to be compensated to achieve the damping ratio $\zeta = 0.5$ and an undamped natural frequency $\omega_n = 5$ rad/s. Which one of the following compensator $G_C(s)$ will be suitable?

(A) $\frac{(s+3)}{(s+9.9)}$ (B) $\frac{(s+9.9)}{(s+3)}$

(C) $\frac{(s-6)}{(s+8.33)}$ (D) $\frac{(s+6)}{(s+9.99)}$

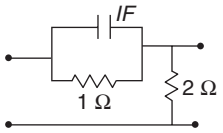
5. In the control system shown in given figure, the controller which can give zero steady-state error to a ramp input, with $k = 9$ is



(A) Proportional type
(B) Integral type
(C) Derivative type
(D) PD type

6. The transfer function of a simple RC network functioning as a controller $G_C(s) = \frac{s+z_1}{s+P_1}$; the condition of RC network to act as a phase lead controller is
(A) $P_1 < z_1$ (B) $P_1 = 0$
(C) $P_1 = z_1$ (D) $P_1 > z_1$

7. For a given phase-lead network, the maximum possible phase-lead is



(A) 90° (B) 45°
(C) 30° (D) 15°

8. Consider the following statements: In a feedback control system, lead compensator

(1) Speeds up the transient response

(2) High-frequency gain increases

(3) Bandwidth increases. of these statements, which one is correct?

(A) (1) and (2)
(B) (1) and (3)
(C) (2) and (3)
(D) (1), (2) and (3)

9. The effect of cascade lag compensation on the transient response of a control system can be neutralized by choosing

(A) A slightly higher value of the static position error constant
(B) A slightly higher value of the static velocity error constant
(C) A slightly higher value of damping ratio.
(D) A slightly higher value of undamped natural frequency

10. Select the statements regarding the properties of phase-lead compensation.

(1) It improves phase margin of the closed loop system.

(2) It increases bandwidth of the closed loop system.

(3) It gives slow response.

Among these, which one is correct?

(A) (1) and (2)
(B) (1) and (3)
(C) (2) and (3)
(D) (1), (2) and (3)

11. Maximum phase lead of $\frac{4(1+0.15s)}{(1+0.05s)}$ is

(A) 45° (B) 60°
(C) 30° (D) 90°

12. The maximum value of phase lead for which a single stage cascade lead compensator should be designed is

(A) 90° (B) 65°
(C) 135° (D) 180°

13. For $G_c(s) = K\frac{1+6s}{1+2s}$.

The minimum phase lead and corresponding frequencies are

(A) 45°, $\frac{1}{2\sqrt{3}}$ (B) 30°, $\frac{1}{2}$

(C) 30°, $\frac{1}{2\sqrt{3}}$ (D) 45°, $\frac{1}{2}$

14. The open loop transfer function of a plant is given as $G(s)=\frac{1}{s^2-1}$. If the plant is operated in a unity feedback configuration, then the lead compensator that can stabilize this control system is

(A) $\frac{10(s-1)}{s+2}$ (B) $\frac{10(s+4)}{s+2}$

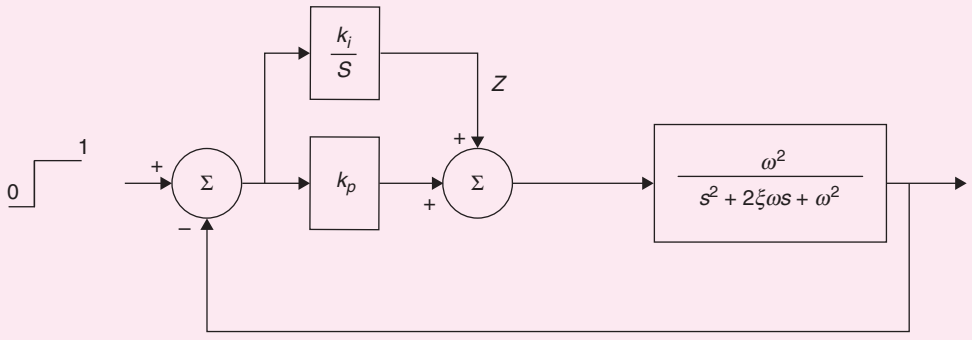
(C) $\frac{10(s+2)}{s+10}$ (D) $\frac{10(s+10)}{s+2}$

15. The transfer function of a phase lead compensator is $\frac{1+3Ts}{1+Ts}$. The maximum value of phase provided by this compensator is

(A) 90° (B) 60°
(C) 45° (D) 30°

Previous Years' Questions

1. Consider the feedback control system shown below which is subjected to a unit step input. The system is stable and has the following parameters $k_p = 4$, $k_i = 10$, $\omega = 500$ and $\xi = 0.7$.



The steady-state value of z is **[2007]**

(A) 1 (B) 0.25

(C) 0.1 (D) 0

2. The system $900/s(s+1)(s+9)$ is to be compensated such that its gain-crossover frequency becomes same as its uncompensated phase-crossover frequency and provides a 45° phase margin. To achieve this, one may use **[2007]**

(A) A lag compensator that provides an attenuation of 20 dB and a phase lag of 45° at the frequency of $3\sqrt{3}$ rad/s.

(B) A lead compensator that provides an amplification of 20 dB and a phase lead of 45° at the frequency of 3 rad/s.

(C) A lag-lead compensator that provides an amplification of 20 dB and a phase lag of 45° at the frequency of $\sqrt{3}$ rad/s.

(D) A lag-lead compensator that provides an attenuation of 20 dB and phase lead of 45° at the frequency of 3 rad/s.

3. The transfer function of two compensators are given below.

$$C_1 = \frac{10(s+1)}{(s+10)}, C_2 = \frac{s+10}{10(s+1)}$$

Which one of the following statements is correct? **[2008]**

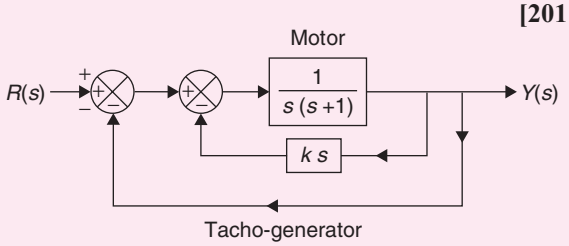
(A) C_1 is a lead compensator and C_2 is a lag compensator.

(B) C_1 is a lag compensator and C_2 is a lead compensator.

(C) Both C_1 and C_2 are lead compensators.

(D) Both C_1 and C_2 are lag compensators.

4. A two-loop position control system is shown below. **[2011]**



The gain k of the **Tacho**-generator influences mainly the

(A) Peak overshoot

(B) Natural frequency of oscillation

(C) Phase shift of the closed loop transfer function at very low frequencies ($\omega \to$ O)

(D) Phase shift of the closed loop transfer function at very high frequencies ($\omega \to \infty$)

5. An open loop system represented by the transfer function $G(s) = \frac{(s-1)}{(s+2)(s+3)}$ is **[2011]**

(A) Stable and of the minimum phase type

(B) Stable and of the non-minimum phase type

(C) Unstable and of the minimum phase type

(D) Unstable and of the non-minimum phase type

Common Data for Questions 6 and 7:

The transfer function of a compensator is given as $G_c(s) = \frac{s+a}{s+b}$

6. $G_c(s)$ is a lead compensator if **[2012]**

(A) $a = 1, b = 2$ (B) $a = 3, b = 2$

(C) $a = -3, b = -1$ (D) $a = 3, b = 1$

7. The phase of the above lead compensator is maximum at **[2012]**

(A) $\sqrt{2}$ rad/s (B) $\sqrt{3}$ rad/s

(C) $\sqrt{6}$ rad/s (D) $1/\sqrt{3}$ rad/s

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** B	**4.** D	**5.** B	**6.** D	**7.** B	**8.** B	**9.** B	**10.** A
11. A	**12.** C	**13.** B	**14.** D	**15.** B	**16.** A	**17.** C	**18.** A		

Practice Problems 2

1. B	**2.** D	**3.** C	**4.** A	**5.** B	**6.** D	**7.** C	**8.** D	**9.** D	**10.** A
11. C	**12.** B	**13.** C	**14.** A	**15.** D					

Previous Years' Questions

1. A	**2.** D	**3.** A	**4.** A	**5.** B	**6.** A	**7.** A

Chapter 5

State Space Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- System state
- State vector
- State equations
- State space representation
- Controllable canonical form
- Observable canonical form
- Diagonal canonical form
- Jordan canonical form
- Observability
- Kalman's test for observability
- Transfer function

Basic Definitions

System state: minimum information needed in order to completely determine the output of the system from a given moment, provided the input is known from that moment.

System variable: Any variable that responds to an input (or) initial conditions in a system.

State Variables: The smallest set of linearly, independent system variables such that the values of the set members at time t_0 along with known forcing function completely determine the value of all system variables for all $t \geq t_0$.

State Vector: Vector whose elements are the state variables.

State space: n-dimensional space whose axes represents the state variables.

State equations: A set of n simultaneous, first-order differential equations with 'n' variables, where 'n' variables to be solved are the state variables.

State Space Representation

A state space representation of an linear time invariant (LTI) system has the general for x.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) - Du(t)$$
$$x(t_0) = X_0 \rightarrow \text{initial conditions}$$

where

$x(t)$: State vector (n-dimensional)
$y(t)$: Output vector (P-dimensional)
$u(t)$: Input or control vector (m-dimensional)
A : Dynamic or system matrix ($n \times n$)
B : Input matrix ($n \times m$)
C : Output matrix ($p \times n$)
D : Feed forward (direct) matrix ($p \times m$)

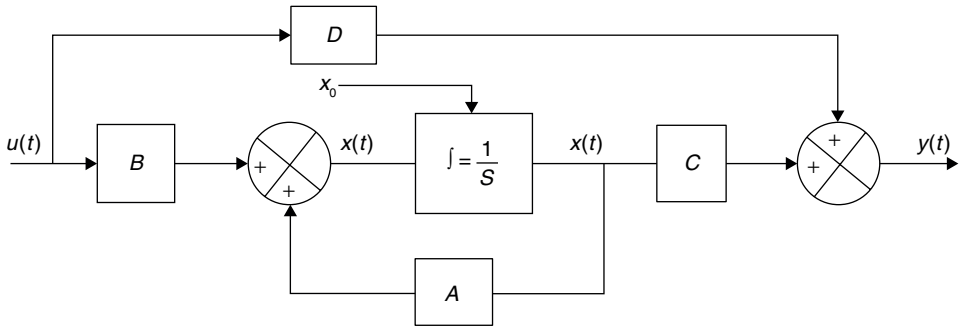


Figure 1 Block diagram representation of state space model

Advantages of State Space Analysis

1. It is applicable to multiple input, multiple output systems.
2. It is applicable to system with non-zero initial conditions.
3. It is applicable to both LTI and Non Linear time varying system.
4. All the internal states can be determined.
5. More accurate than transfer function.
6. It gives the information about the controllability and observability.

State Space Representation

Selecting the state variables must follow the following rules

- A minimum number of state variables must be selected.
- They must be linearly independent.

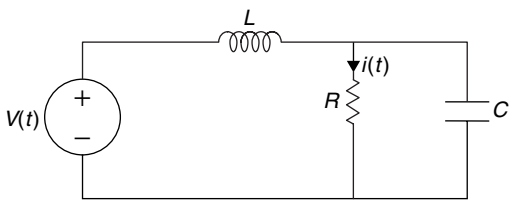
The minimum number of state variables required equals the order of the differential equation describing the system. From the transfer function point of view, the order of the differential equation is the order of the denominator of the T.F. after cancelling common factors in the numerator and denominator.

A practical way to determine the number of state variables is to count the number of independent energy storage elements in the system (Capacitor and inductors in electrical system and masses and springs in mechanical system).
Procedure for state representation of electrical network

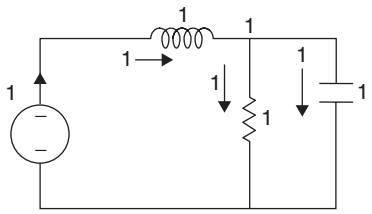
1. Write simple derivative equation for each energy storage element [Node equation for the node at which an inductor is connected and loop equation for the Loop in which capacitor is connected in electrical network].
2. Solve for the each derivate term as a linear combination of system variable and the input.
3. Each differentiated variable is selected as a state variable.
4. All other variables and output are written in terms of the state variables and the input.

Solved Examples

Example 1: Find the state space representation of the system shown in the figure if the output is the current through the resistor



Solution: Step 1: Label all the branch variables in the network



Step 2: Select the state variables: Write the derivative equations for all energy storage elements (L and C)

$$C.\frac{dv_c}{dt} = i_c;\ L.\frac{di_L}{dt} = V_L$$

Choose the differentiated quantities as the state variables (v_C and i_L).

Step 3: Write the each differentiated term as a linear, combination of system variables and the input.

$$C.\frac{dv_c}{dt} = i_C = f_1(v_C, i_C, V(t)) \quad (1)$$

$$L.\frac{di_L}{dt} = \vartheta_C = f_2(v_C, i_L, V(t)) \quad (2)$$

Applying node equation at V_1

$$i_C = -i_R + i_L = -\frac{1}{R}\vartheta_C + i_L \quad (3)$$

Applying loop equation in capacitor loop

$$V_L = -V_C + V(t) \quad (4)$$

From Equations (1), (2), (3) and (*4),

$$\frac{dv_c}{dt} = -\frac{1}{RC}\vartheta_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}\vartheta_C + \frac{1}{L}V(t)$$

Output equation $i_R = \frac{1}{R}\vartheta_C$

Step 4: Obtain state space representation in vector matrix form

$$\begin{bmatrix}\dot{\vartheta}_C \\ i_L\end{bmatrix} = \begin{bmatrix}-\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}\vartheta_C \\ i_L\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{2}\end{bmatrix}v(t)$$

$$[i_R] = \begin{bmatrix}\frac{1}{R} & 0\end{bmatrix}\begin{bmatrix}V_C \\ i_L\end{bmatrix}$$

Transfer Function to State Space Model

This case corresponds to a linear system that can be represented as a n^{th}-order differential equation with constant co-efficient like:

$$\frac{d^n\ y(t)}{dt^{\ n}}+\frac{d^{n-1}\ y(t)}{dt^{n-1}}a_{n-1}+\frac{d^{n-2}\ y(t)}{dt^{\ n-2}}a_{n-2}+....$$
$$+a_1\frac{dy(t)}{dt}+a_0$$
$$=b_0U(t)$$

The classical transfer function (T.F.) representation of the system is obtained by applying the Laplace transform to the differential equation.

$$G(S)=\frac{Y(S)}{U(S)}=\frac{b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+....+a_1s+a_0}$$

If the state space representation has to be obtained, a convenient way to select state variables is to choose the output $y(t)$ and its 'n–1' derivatives as the state variables. This is called phase variables choice.

The state space representation using the phase variable choice of the state variables is said to be in the 'Controllable canonical form' (CCF).

Controller canonical form of the given differential equation is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ .\\ .\\ .\\ .\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0&1&0&0&.&.&.&.&0\\0&0&1&0&.&.&.&.&0\\0&0&0&1&.&.&.&.&0\\0&0&0&0&1&.&.&.&0\\.&&&&&&&&\\.&&&&&&&&\\.&&&&&&&&\\0&0&0&0&0&0&.&.&1\\-a_0&-a_1&-a_2&-a_3&.&.&.&.&-a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\.\\.\\.\\x_{n-1}\\x_n\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\.\\.\\.\\b_0\end{bmatrix}U\quad Y=\begin{bmatrix}1&0&0&.&.&.&.&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\.\\.\\.\\x_n\end{bmatrix}$$

Consider a transfer function with polynomial in numerator

$$y^n+a_1y^{n-1}+a_2y^{n-2}+\ldots\ldots+a_{n-1}\ \dot{y}\ +a_ny=b_0u^n+b_1u^{n-1}+\ldots..+b_{n-1}\dot{u}\ +b_nu$$

where u is the input and y is the output. The transfer function can be written as

$$\frac{Y(S)}{U(S)}=\frac{b_0s^n+b_1s^{n-1}+....b_{n-1}s+b_n}{s^n+a_1s^{n-1}+...+a_{n-1}s+a_n}$$

State space representation of the system in controllable canonical form and observable canonical form are given by

Controllable Canonical Form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ .\\ .\\ .\\ .\\ .\\ .\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0&1&0&.&.&.&.&0\\0&0&1&.&.&.&.&0\\.&.&.&.&.&.&.&.\\.&.&.&.&.&.&.&.\\.&.&.&.&.&.&.&.\\.&.&.&.&.&.&.&.\\.&.&.&.&.&.&.&.\\0&0&0&.&.&.&.&1\\-a_n&-a_{n-1}&-a_{n-2}&.&.&.&.&-a_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\.\\.\\.\\.\\.\\x_{n-1}\\x_n\end{bmatrix}+\begin{bmatrix}0\\0\\.\\.\\.\\.\\.\\0\\1\end{bmatrix}U\quad y=[b_n-a_nb_0:b_{n-1}-a_{n-1}b_0:\ldots\ldots:b_1-a_1b_0]\begin{bmatrix}x_1\\x_2\\.\\.\\.\\.\\.\\x_n\end{bmatrix}+b_0u.$$

The controllable canonical form is important in discussing the pole placement approach to the control system design.

Observable Canonical Form

The following state-space representation is called an observable canonical form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ & & \vdots & & \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

System matrix in observable canonical form is the transpose of system matrix in controllable canonical form.

Diagonal Canonical Form

Transfer function with numerator polynomial can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots b_{n-1} s + b_n}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)}$$

$$= b_0 + \frac{C_1}{s+p_1} + \frac{C_2}{s+P_2} + \dots \frac{C_n}{s+P_n}$$

where $p_1, p_2, \dots p_n$ are the location of poles.

The diagonal canonical form of the state space representation of this system is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \dots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form

When the system involves multiple roots, the diagonal canonical form must be modified into Jordon canonical form. For example, if there are three equal poles ($P_1 = P_2 = P_3$). Then the factored form $Y(s)/U(s)$ becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+P_1)^3} + \frac{C_2}{(s+p_1)^2} + \frac{C_3}{(s+p_1)} + \frac{C_4}{s+P_4} + \dots + \frac{C_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -p_1 & 1 & 1 & 0 & \dots & \\ 0 & 0 & -p_1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & & -p_4 & \dots & 0 \\ & & & & & & \\ 0 & 0 & 0 & & 0 & \dots & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \dots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u.$$

Eigen Values of an $n \times n$ Matrix A

Characteristic equation of a square matrix 'A' is given by

$$|\lambda I - A| = 0$$

The values of λ which satisfies the characteristic equation are called as eigen values of 'A' matrix.

Note:
The poles of the transfer function are given by the $|sI - A| = 0$. This function is same equation as the characteristic equation of 'A'. Therefore, we can conclude that the eigen values of the state model and the poles of the transfer function are the same.

Note:
Stability of SISO system depends on the eigen values of system matrix in its state space model.

Controllability

A state $x(t)$ is said to be controllable at $t = t_0$ if there exist a piecewise continuous input u(t) that will drive the state to any final state $x(t_f)$ for a finite time $(t_f - t_0) \geq 0$.

If every state $x(t_0)$ of the system is controllable in a finite time interval, the system is said to be completely controllable or simply, controllable.

Kalman's Test for Controllability

Consider a state space model

$$\dot{x} = Ax + Bu$$

Then controllability matrix

$$Q_C = [B:AB:A^2B: \ldots.: A^{n-1}B]$$

Note:
A system is said to be controllable if the rank of Q_C is equal to the order of matrix 'A'.

i.e., $$|Q_C| \neq 0.$$

Note:
Number of uncontrollable states is computed as the difference between order of A matrix (n) and rank of Q_C matrix (P).

Observability

Linear time invariant system is said to be observable if given any input $u(t)$, there exist a finite time $t_f \geq t_0$ such that the knowledge of $u(t)$ for $t_0 \leq t, < f_f$, matrix A, B, C and D and the output $y(t)$ for $t_0 \leq t < t_f$ are sufficient to determined $x(t_0)$. If a every state of the system is observable for a finite t_f, we say that the system is completely observable or simply observable.

Kalman's Test for Observability

Consider a state space model

$$\dot{x} = Ax + Bu \text{ and } y = Cx + Du$$

Then observability matrix

$$Q_0 = [C\ CA\ CA^2\ \ldots\ldots CA^{n-1}]^T$$

Note:
The system is said to be observable if the rank of Q_0 is equal to the order of matrix A.

i.e., $$|Q_c| \neq 0.$$

Note:
Number of unobservable states is computed as the difference between order of A matrix (n) and rank of Q_0 matrix (q).

Solution of State Equations

The state equations of a linear system is

$$\dot{x} = Ax + Bu;\ x(t_0) = x(0)$$

Solution of above state equation is

$$x(t) = \underbrace{e^{At}\, x(0)}_{\text{Homogeneous Solution}} + \underbrace{\int_0^t e^{-A(t-\tau)} Bu(\tau)\, .d\tau}_{\text{Forced Solution}}$$

In the absence of the input to the system, response of the system or solution of state equations, with initial conditions alone is given by

$$X(t)\ e^{At}\, x(0) = e^{At} x_0$$

It is observed that the initial state x_0 at $t = 0$ is driven to a state $x(t)$ at time 't'. This transition in state is carried out by the matrix exponential e^{At}. Because of this property, e^{At} is known as 'State transition matrix' and is denoted by $\phi(t)$.

Properties of State Transition Matrix $\phi(t)$

1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$
3. $\phi(t_1 + t_2) = e^{A(t_1 + t_2)} = e^{At_1} \cdot e^{At_2}$
   $= \phi(t_1).\ \phi(t_2) = \phi(t_2).\ \phi(t_1)$
4. $[\phi(t)]^n = [e^{At}]^n = e^{An} + = \phi(nt)$
5. $\phi(t_1 - t)\ .\ \phi(t - t_2) = e^{A(t1 - t)} .\ e^{A(t - t2)}$
   $= e^{A(t1 - t2)} = \phi(t_1 - t_2)$

Transfer Function

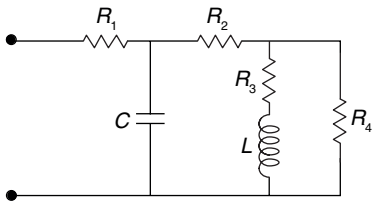
Given the state space model of SISO system as

$$\dot{x} = Ax + Bu$$
$$Y = Cx + Du$$

The transfer function of the system is

$$\text{T. F.} = C[sI - A]^{-1} B + D$$

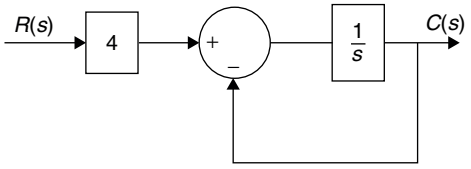
Example 2: The maximum number of states required to describe the network shown in figure is



(A) 1 (B) 2
(C) 3 (D) 4

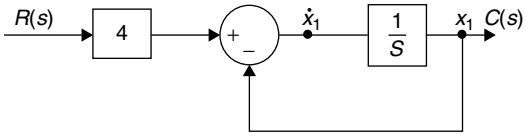
Solution: (B)
No. of energy storage elements (L, C) = 2
No. of states required to analysis = 2

Example 3: The matrix of any state space equation for the transfer function $\frac{C(s)}{R(s)}$ of the system shown in figure is



(A) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (B) $[-1]$
(C) $[4]$ (D) $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

Solution: (B)
Output of integrator is considered as state and no. of integrators is equal to the no. of states.
No. of states = 1



$\dot{x}_1 = -x_1 + 4r(t)$
$[x^\circ_1] = [-1]\,[x_1] + [4]\,[r(t)]$
Matrix of state space equation is $[-1]$

Example 4: A system is described by the state equations. $\dot{x} = Ax + Bu$. The output is given by $y = Cx$

where $A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

The transfer function $G(S)$ of the system is

(A) $\frac{s}{s^2+5s+7}$ (B) $\frac{s+7}{s^2+5s+7}$
(C) $\frac{1}{s^2+5s+7}$ (D) $\frac{s}{s^2+s+5}$

Solution: (B)
Transfer function of the state space model is

$$\text{T. F} = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

$$\text{T. F} = -C\,[sI - A]^{-1}B = \frac{[0 \quad 1]\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{[0 \quad 1]\begin{bmatrix} s+1-1 \\ 3+s+4 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{s+7}{s^2+5s+7}.$$

Example 5: Given that homogeneous state space equations $\dot{x} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}x$, the steady-state value of $x_{ss} = \lim_{t\to\infty} x(t)$, given the initial state value of $x[0] = [10 - 10]^T$ is

(A) $x_{ss} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ (B) $x_{ss} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$
(C) $x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (D) $x_{ss} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$

Solution: (C)
Solution of homogenous equation $A\dot{x} = Ax$

$$x(t) = e^{At}x(0)$$

$$e^{At} = L\left[[sI - A]^{-1}\right]$$

$$sI - A = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{-1}{(s+4)(s+3)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+4)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right] = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 10e^{-4t} + 10e^{-3t} + 10e^{-4t} \\ -10e^{-3t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 20e^{-4t} - 10e^{-3t} \\ -10^{-3t} \end{bmatrix}$$

$$x_{ss} = \lim_{t\to\infty} x(t) = \begin{bmatrix} \lim_{t\to\infty}\left(20e^{-4t} - 10e^{-3t}\right) \\ \lim_{t\to\infty} -10e^{-3t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 6: A second-order system starts with an initial condition of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ without any external input. The state transition matrix for the system is given by $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$

The state of the system at end of 2 seconds is given by

(A) $10^{-3} \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (B) $10^{-3} \times \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (D) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Solution: (B)

Solution of state equations without external input is

$$x(t) = e^{At} \times (0)$$

$$= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 2e^{-3t} \\ 3e^{-4t} \end{bmatrix}$$

At $t = 2$ sec, $x(2) = \begin{bmatrix} 2e^{-3\times 2} \\ 3e^{-4\times 2} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.10^{-3}$

Example 7: For a system with transfer function $G(S) = \dfrac{2s+4}{s^3+4s^2+9s+4}$

The matrix 'A' in the state space form $\dot{x} = Ax + Bu$ is equal to

(A) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -9 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 & 0 \\ -2 & -4 & 1 \\ -4 & -9 & -4 \end{bmatrix}$

Solution: (A)

From the standard controllable canonical form of the transfer function

$$\text{T.F.} = \frac{2s+4}{s^3+4s^2+9s+4}$$

State space representation

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t).$$

Example 8: For the system $\dot{x} = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, which of the following statement about the system is true?

(A) Controllable and stable
(B) Uncontrollable and stable
(C) Controllable and unstable
(D) Uncontrollable and unstable

Solution: (D)

For stability analysis, location of poles are the roots of characteristic equation.

$$|sI - A| = \begin{vmatrix} s-3 & -4 \\ 0 & s-5 \end{vmatrix} = (s-3)(s-5) = 0$$

$s = 3$ and 5

$\therefore$ Poles are located on RHS plane, system is unstable.

Controllability matrix $Q_C = [B \quad AB]$

$$[A \quad B] = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow |Q_C| = 0$$

$\therefore$ System is uncontrollable.

EXERCISES

Practice Problems I

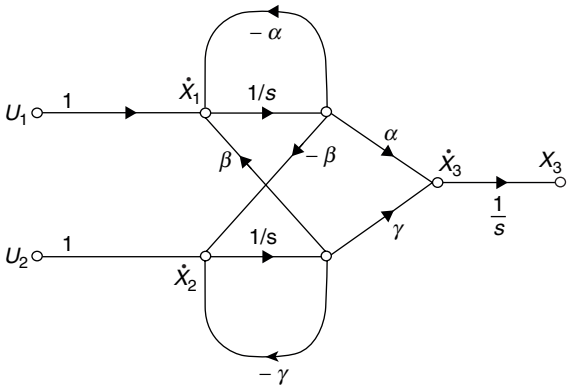
Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Given that $\dot{X} = AX$ for the system described by the differential equation $\ddot{y} + 2\dot{y} + 3y = 0$. The matrix 'A' is

(A) $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ -1 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$

2. A signal flow graph of a system is given below



If the state equation that corresponds to the above signal flow graph is $\dot{X} = AX + BU$, the matrices 'A' and 'B' are

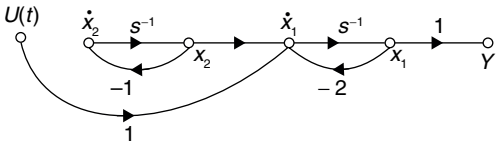
(A) $\begin{bmatrix} \beta & -\gamma & 0 \\ \gamma & \alpha & 0 \\ -\alpha & -\beta & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & \alpha & \gamma \\ 0 & -\alpha & -\gamma \\ 0 & \beta & -\beta \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} -\alpha & 0 & \beta \\ \gamma & 0 & \alpha \\ -\beta & 0 & -\alpha \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.



In the state diagram of a system shown in the above figure, which variables are controllable?

(A) $x_1(t)$
(B) $x_2(t)$
(C) Both $x_1(t)$ and $x_2(t)$
(D) Neither $x_1(t)$ nor $x_2(t)$

4. The state equations of a linear time-invariant system are represented by $\frac{dx(t)}{dt} = AX(t) + BU(t)$.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transition matrix $\phi(t)$ is

(A) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t/3} & 0 \\ 0 & e^{t/3} \end{bmatrix}$ (D) $\begin{bmatrix} e^{t/3} & 0 \\ 0 & e^{-t/3} \end{bmatrix}$

5. A particular control system is described by the following state equations

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \text{ and}$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transfer function of this system is

(A) $\frac{Y(s)}{U(s)} = \frac{2}{s^2+2s+5}$ (B) $\frac{Y(s)}{U(s)} = \frac{4}{s^2+2s+5}$

(C) $\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+5}$ (D) $\frac{Y(s)}{U(s)} = \frac{3}{s^2+2s+5}$

Common Data for Questions 6 and 7: A system is characterized by the following state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U, (t > 0)$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

6. The transfer function of the system is

(A) $\frac{s}{(s+2)(s+1)}$ (B) $\frac{1}{s(s+2)(s+1)}$

(C) $\frac{s}{(s-2)(s+1)}$ (D) $\frac{1}{(s+2)(s+1)}$

7. The state transition matrix of the system is

(A) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}+e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

(B) $\begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(D) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

8. Match List-I (Matrix) with List-II (dimensions) for the state equations:
$\dot{X}(t) = AX(t) + BU(t)$ and $Y(t) = CX(t) + DU(t)$ and select the correct answer using the codes given below the lists:

List-I	List-II
A	(1) $n \times p$
B	(2) $q \times n$
C	(3) $n \times n$
D	(4) $q \times p$

Codes:

	A	B	C	D
(A)	1	3	4	2
(B)	1	3	2	4
(C)	3	1	4	2
(D)	3	1	2	4

9. Consider the single-input, single-output system with its state variable representation:

$$\dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} U,\ Y = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} X$$

The system is
(A) Neither controllable nor observable.
(B) Controllable but not observable.
(C) Uncontrollable but observable.
(D) Both controllable and observable.

10. Consider the state transition matrix: $\phi(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

The eigen values of the matrix when $t = 0$ are
(A) 1, 2 (B) 2, 1
(C) 1, 1 (D) 1, 3

11. A linear system is described by the following state equation $-\dot{X} = AX(t) + BU(t)$.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The state transition matrix of the system is

(A) $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

(B) $\begin{bmatrix} -\cos t & \sin t \\ -\sin t & -\cos t \end{bmatrix}$

(C) $\begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix}$

(D) $\begin{bmatrix} \cos t & -\sin t \\ \cos t & \sin t \end{bmatrix}$

12. Consider the system $\frac{dx}{dt} = AX + BU$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix}$, where p and q are arbitrary real numbers.
Which of the following statements about the controllability of the system is true?
(A) The system is completely controllable for any non-zero values of p and q.
(B) Only $p = 0$ and $q = 0$ result in controllability.
(C) The system is uncontrollable for all values of p and q.
(D) We cannot conclude about controllability from the given data.

13. The state transition matrix of the system whose state equation are $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(A) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

(B) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$

(C) $\begin{bmatrix} \sin t & -\cos t \\ +\cos t & \sin t \end{bmatrix}$

(D) $\begin{bmatrix} \sin\sqrt{2}t & \cos\sqrt{2}t \\ -\cos\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

14. If the eigen values of a 3×3 matrix A are 1, –3 and 4. The eigen values of $P^{-1}AP$ (Q P is a linear transformation) are
(A) 1, $\frac{-1}{3}$ and $\frac{1}{4}$
(B) 1, – 3 and 4
(C) 1, 9 and 16
(D) –1, 3 and – 4

15. The state equations of a system are $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. The closed loop poles of the system are
(A) ± 2 (B) $-1, -1$
(C) $+1, -2$ (D) $+2, -1$

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. The state equation of a linear system is given by $\dot{X} = AX + BU$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The state transition matrix of the system is

(A) $\begin{bmatrix} e^{-t} & e^{-t} + e^{-2t} \\ 0 & e^{2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} 0 & e^{-2t} \\ e^{-t} & e^{-t} - e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} 0 & e^{-t} - e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix}$

2. The eigen value and eigen vector pairs (λ_i, V_i) for the system are

(A) $\left[-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

(B) $\left[-2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[-1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

(C) $\left[-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

(D) $\left[2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

3. The system matrix A is:

(A) $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

4. Let $X = \dot{X} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$, $Y = \begin{bmatrix} b & 0 \end{bmatrix} X$

where 'b' is an unknown constant. This system is

(A) Observable for all values of b.
(B) Unobservable for all values of b.
(C) Observable for all non-zero values of b.
(D) Unobservable for all non-zero values of b.

5. A state variable representation of a system is given by the expression

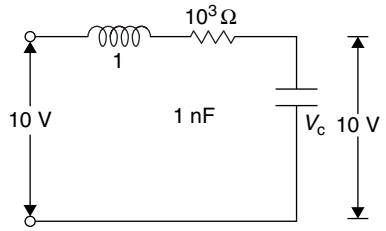
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t),\ Y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transfer function of the system is

(A) $\frac{2}{s+1}$ (B) $\frac{2s}{(s-1)(s+1)}$

(C) $\frac{2}{(s-1)(s+1)}$ (D) $\frac{1}{(s+1)}$

6. The state space model for an electrical network shown in fig where the current I and voltage across the capacitor V_c are the state variables.



(A) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 10^6 & 10^3 \\ 10^9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10^3 \end{bmatrix} \times 10$

(B) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 10^9 & 0 \\ 10^6 & 10^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10^3 \end{bmatrix} \times 10$

(C) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -10^6 & -10^3 \\ 10^9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} \times 10$

(D) None of these

7. The poles and zero of the following system:

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$y = [-17 \ -5]\, x + [1]\, u$ are

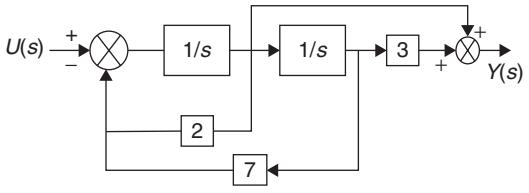
(A) Poles: – 4, – 5 and zeros: – 1, – 3
(B) Poles: – 3, – 4, and zeros: – 3, – 2
(C) Poles: – 2, – 3 and zeros: – 2, – 4
(D) None of these

8. The second-order system $\dot{x} = Ax$ has $A = \begin{bmatrix} -1 & -2 \\ +1 & -1 \end{bmatrix}$.

The values of its damping factor and natural frequency ω_n are, respectively,

(A) 1.732, 0.577 (B) 1.414, 0.6
(C) 0.577, 1.732 (D) 0.6, 1.414

Common Data for Questions 9 and 10:
Consider the following block diagram



9. In the above figure, the state space representation in vector matrix from is

(A) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [3\ 1]\, x$

(B) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [3\ 1]\, x$

(C) $\dot{x} = \begin{bmatrix} 1 & 0 \\ -2 & -7 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$Y = [1\ 3]\, x$

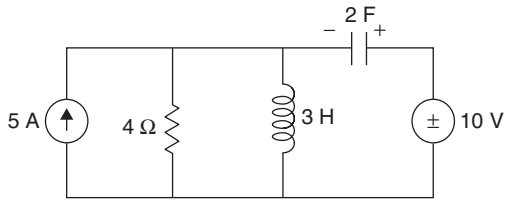
(D) None of these

10. From the question number (9), the transfer function of the system is

(A) $\dfrac{s}{s^2+2s+7}$ (B) $\dfrac{s+3}{s^2+2s+7}$

(C) $\dfrac{3}{s^2+2s+7}$ (D) $\dfrac{2s+3}{s^2+2s+7}$

Common Data for Questions 11 and 12:
The circuit diagram is shown in the following figure.



11. The state equations of the above circuit are

(A) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{3} \\ \frac{-1}{2} & \frac{-1}{8} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{8} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(B) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{3} \\ \frac{1}{2} & \frac{-1}{8} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{8} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(C) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & 0 \\ \frac{1}{8} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{8} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(D) None of these

12. From the Q. No. (11), the eigen values are

(A) $-0.06 \pm j0.06$
(B) $-0.06 \pm j0.403$
(C) $-0.403 \pm j0.06$
(D) $-0.12 \pm j0.12$

13. A continuous time, linear time invariant system is described by $\dfrac{d^2y}{dt} + 4\dfrac{dy}{dt} + 3y(t) = 2\dfrac{dx}{dt} + 4x$.
Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}\,u(t)$ is given by

(A) $(e^{-t} - e^{-3t})\,u(t)$
(B) $(e^{-t} + e^{-3t})\,u(t)$
(C) $(e^{t} - e^{+3t})\,u(t)$
(D) $(e^{t} + e^{3t})\,u(t)$

14. The system with the state equation

$\dot{x} = Ax + Bu$ and $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(A) System is controllable.
(B) System is uncontrollable.
(C) System is controllable and observable.
(D) None of these

15. Consider the state transition matrix:

$$\phi(t) = L^{-1}[\phi(s)] = L^{-1}\begin{bmatrix} \dfrac{s+1}{s^2+3s+2} & \dfrac{-1}{s^2+3s+2} \\ 0 & \dfrac{s+2}{s^2+3s+2} \end{bmatrix}$$

The eigen values of the system are

(A) 0 and – 2
(B) – 1 and 2
(C) 1 and 2
(D) – 1 and – 2

Previous Years' Questions

Data for Question 1:
A state variable system

$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$, with the initial condition

$X(0) = [-1\ 3]^T$ and the unit step input $u(t)$ has

1. And the state transition equation **[2005]**

(A) $X(t) = \begin{bmatrix} t - e^{-t} \\ e^{-t} \end{bmatrix}$ (B) $X(t) = \begin{bmatrix} t - e^{-t} \\ 3e^{-3t} \end{bmatrix}$

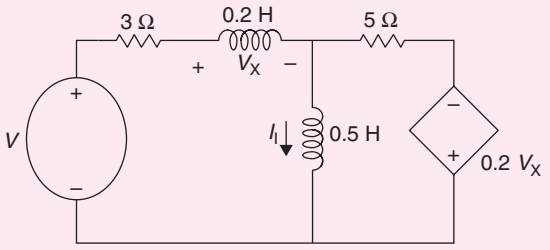
(C) $X(t) = \begin{bmatrix} t - e^{-3t} \\ 3e^{-3t} \end{bmatrix}$ (D) $X(t) = \begin{bmatrix} t - e^{-3t} \\ e^{-t} \end{bmatrix}$

2. For a system with the transfer function $H(s) = \frac{3(s-2)}{4s^2-2s+1}$, the matrix A in the state space form $x = Ax + Bu$ is equal to **[2006]**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 1 \\ 1 & -2 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix}$

3. The state equation for the current I_1 shown in the network shown below in terms of the voltage V_x and the independent source V, is given by **[2007]**



(A) $\frac{dI_1}{dt} = -1.4V_x - 3.75I_1 + \frac{5}{4}V$

(B) $\frac{dI_1}{dt} = -1.4V_x - 3.75I_1 - \frac{5}{4}V$

(C) $\frac{dI_1}{dt} = -1.4V_x + 3.75I_1 + \frac{5}{4}V$

(D) $\frac{dI_1}{dt} = -1.4V_x + 3.75I_1 - \frac{5}{4}V$

Common Data for Questions 4 and 5:
A system is described by the following state and output equations.

$$\frac{dx_1(t)}{dt} = -3x_1(t) + x_2(t) + 2u(t)$$
$$\frac{dx_2(t)}{dt} = -2x_2 + u(t)$$
$$y(t) = x_1(t)$$

where $u(t)$ is the input and $y(t)$ is the output

4. The system transfer function is **[2009]**

(A) $\frac{s+2}{s^2+5s-6}$ (B) $\frac{s+3}{s^2+5s+6}$

(C) $\frac{2s+5}{s^2+5s+6}$ (D) $\frac{2s-5}{s^2+5s-6}$

5. The state transition matrix of the above system is **[2009]**

(A) $\begin{bmatrix} e^{-3t} & 0 \\ e^{-2t}+e^{-3t} & e^{-2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-3t} & e^{-2t}+e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} e^{3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

6. The second-order dynamic system

$$\frac{dX}{dt} = PX + Qu$$
$$y = RX$$

has the matrices P, Q and R as follows:

$$P = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} R = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The system has the following controllability and observability properties: **[2014]**

(A) Controllable and observable
(B) Not controllable but observable
(C) Controllable but not observable
(D) Not controllable and not observable

7. Consider the system described by the following state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If u is unit step input, then the steady-state error of the system is **[2014]**

(A) 0 (B) 1/2 (C) 2/3 (D) 1

8. The state variable description of an LTI system is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where y is the output and u is the input. The system is controllable for **[2012]**

(A) $a_1 \neq 0, a_2 = 0, a_3 \neq 0$

(B) $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$

(C) $a_1 = 0, a_2 = 0, a_3 = 0$

(D) $a_1 \neq 0, a_2 \neq 0, a_3 = 0$

Common Data for Questions 9 and 10:
The state variable formulation of a system is given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,\ x_1(0) = 0,\ x_2(0) = 0$$

and $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

9. The system is **[2013]**
(A) Controllable but not observable
(B) Not controllable but observable
(C) Both controllable and observable
(D) Both not controllable and observable

10. The response $y(t)$ to a unit step input is **[2013]**
(A) $\frac{1}{2} - \frac{1}{2}e^{-2t}$ (B) $1 - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$
(C) $e^{-2t} - e^{-t}$ (D) $1 - e^{-t}$

11. For the system governed by the set of equations:
$dx_1/dt = 2x_1 + x_2 + u$
$dx_2/dt = -2x_1 + u$
$y = 3x_1$
the transfer function $Y(s)/U(s)$ is given by **[2015]**
(A) $3(s + 1)/(s^2 - 2s + 2)$
(B) $3(2s + 1)/(s^2 - 2s + 1)$
(C) $(s + 1)/(s^2 - 2s + 1)$
(D) $3(2s + 1)/(s^2 - 2s + 2)$

12. Consider the following state-space representation of a linear time-invariant system. **[2016]**

$$x(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t),\ y(t) = c^T x(t),\ c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ and

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ the value of $y(t)$ for $t = \log_e 2$ is _____.

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** C	**3.** A	**4.** B	**5.** C	**6.** D	**7.** B	**8.** C	**9.** A	**10.** C
11. A	**12.** C	**13.** B	**14.** B	**15.** B					

Practice Problems 2

1. B	**2.** A	**3.** D	**4.** C	**5.** D	**6.** C	**7.** A	**8.** C	**9.** B	**10.** B
11. B	**12.** B	**13.** A	**14.** B	**15.** D					

Previous Years' Questions

1. C	**2.** B	**3.** A	**4.** C	**5.** B	**6.** C	**7.** A	**8.** D	**9.** A	**10.** A
11. A	**12.** 6								

TEST
Control Systems

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The time response of a first-order control system subjected to unit impulse input is

(A) $C_{(t)} = \frac{1}{T}e^{-t/T}$ (B) $C_{(t)} = e^{-t/T}$

(C) $C_{(t)} = 1 - e^{-t/T}$ (D) $C_{(t)} = T - e^{-t/T}$

2. For a second-order system with the closed-loop transfer function $T(s) = \frac{16}{s^2 + 4s + 16}$. The damped frequency of oscillations is

(A) 4 rad/sec (B) $4\sqrt{3}$ rad/sec

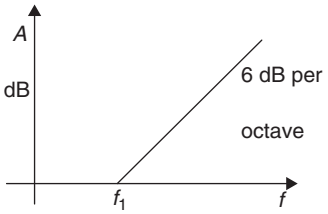
(C) $2\sqrt{3}$ rad/sec (D) 2 rad/sec

3. For a unity feedback closed loop system $G(s) = \frac{K}{s(s+4)(s+3)}$. The range of K for which the system is stable will be given by

(A) $K > 84$ (B) $0 < K < 84$

(C) $0 > K > 84$ (D) $0 < K > 84$

4. The function corresponding to the Bode plot of the figure is



(A) $A = \frac{jf}{f_1}$ (B) $A = \frac{1}{(1 - jf_1/f_1)}$

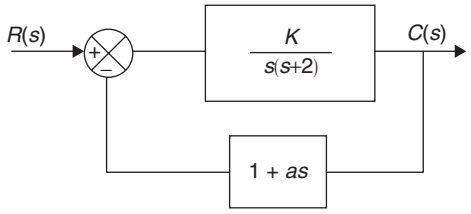
(C) $A = \frac{1}{(1 + jf_1/f)}$ (D) $A = 1 + jf/f_1$

5. Given a unity feedback system with $G(s) = \frac{K}{s(s+4)}$, the value of K for damping ratio of 0.5 is

(A) 1 (B) 16

(C) 4 (D) 2

6. For the system shown in the figure, with a damping ratio ξ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec, the values of K and 'a' are



(A) $K = 4, a = 0.35$ (B) $K = 8, a = 0.455$

(C) $K = 16, a = 0.225$ (D) $K = 64, a = 0.9$

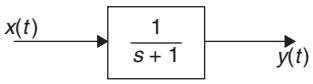
7. For an RLC series circuit, match list I with list II and select the correct answer using the codes given below the lists:

List – I (condition)	List – II (Transient current response)
(P) $R = 0$	(1) Undamped oscillations
(Q) $R < 2\sqrt{\frac{L}{C}}$	(2) Damped oscillations
(R) $R = 2\sqrt{\frac{L}{C}}$	(3) Critically damped response
(S) $R > 2\sqrt{\frac{L}{C}}$	(4) Non – oscillatory response

Codes:

	P	Q	R	S
(A)	1	2	3	4
(B)	1	4	3	2
(C)	3	2	1	4
(D)	3	4	1	2

8. In the system shown below, $x(t) = \sin t\, u(t)$. In steady-state, the response $y(t)$ will be



(A) $\frac{1}{\sqrt{2}}\sin\left[t - \frac{\pi}{4}\right]$ (B) $\frac{1}{\sqrt{2}}\sin\left[t + \frac{\pi}{4}\right]$

(C) $\frac{1}{\sqrt{2}}e^{-t}\sin t$ (D) $\sin t - \cos t$

9. A differentiator has a transfer function whose

(A) Phase increases linearly with frequency

(B) Amplitude remains constant

(C) Amplitude increases linearly with frequency

(D) Amplitude decreases linearly with frequency

10. If $f(s) = L[f(t)] = \frac{3(s+2)}{s^2 + 6s + 12}$, then the initial and final values of $f(t)$ are respectively

(A) 0, 3 (B) 3, 0

(C) 0, 1/2 (D) $\frac{1}{2}$, 0

11. The transfer function of a system is given by

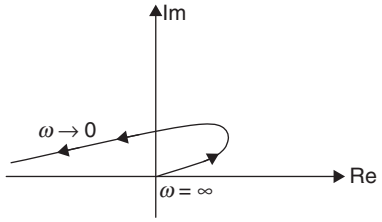
$$\frac{Y_{(s)}}{X_{(s)}} = \frac{1}{s^3 + 8s^2 + 19s + 12}$$

Obtain the pole – zero form representation of matrix A

(A) $A=\begin{bmatrix}0&1&0\\0&0&1\\-12&-19&-8\end{bmatrix}$ (B) $A=\begin{bmatrix}-8&-19&-12\\0&1&0\\0&0&1\end{bmatrix}$

(C) $A=\begin{bmatrix}-4&1&0\\0&-3&1\\0&0&-1\end{bmatrix}$ (D) $A=\begin{bmatrix}-1&0&0\\0&-3&0\\0&0&-6\end{bmatrix}$

12. The Nyquist plot of a control system is shown below. For this system, $G(s)H(s)$ is equal to



(A) $\frac{K}{(1+sT_1)}$ (B) $\frac{K}{s^2(1+sT_1)}$

(C) $\frac{K}{s^3(1+sT_1)}$ (D) $\frac{K}{s^2(1+sT_1)(1+sT_2)}$

13. A system described by the transfer function $H(S) = \frac{1}{s^3+\alpha s^2+Ks+4}$ is stable. The constraints on a and K are

(A) $\alpha>0,\ \alpha K<4$ (B) $\alpha>0,\ \alpha K>4$

(C) $\alpha>0,\ \alpha K>0$ (D) $\alpha<0,\ \alpha K<0$

14. A system has an open-loop transfer function of $\frac{10}{s(1+0.5s)(1+0.2s)}$ and is has a feedback whose transfer function is $H(s) = 1 + Ts$. What should be the value of T so that the closed-loop system operates in a stable manner?

(A) $T \geq 23.33$ (B) $T \geq 2.333$

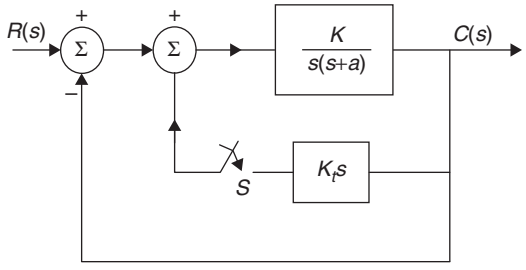
(C) $T \geq 0.42$ (D) $T \geq 0.042$

15. The number of roots of the equation $2s^4 + s^3 + 3s^2 + 5s + 7 = 0$ that lie in the right half of S plane is

(A) Zero (B) One

(C) Two (D) Three

16. The system shown in the given figure, is subjected to a unit ramp input



By closing the switch 'S'

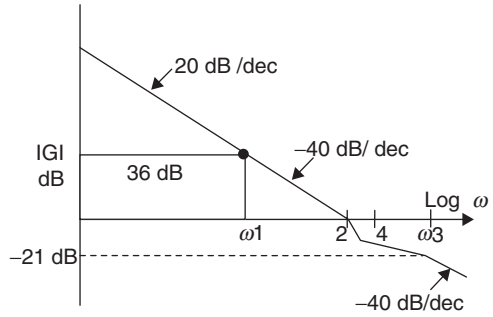
(A) Steady-state error will increase and damping coefficient 'ζ' will decrease

(B) Both steady-state error and damping coefficient 'ζ' will increase

(C) Both steady-state error and damping coefficient 'ζ' will decrease

(D) Steady-state error will decrease and damping coefficient 'ζ' will increase

17. The asymptotic magnitude Bode plot of a system is given in the figure. The system is a minimal phase system. The transfer function of the system is



(A) $\frac{K\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$ (B) $\frac{K\left(1+\frac{s}{\omega_1}\right)}{s\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$

(C) $\frac{K\left(1+\frac{s}{\omega_2}\right)}{s\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$ (D) $\frac{K\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$

18. The open loop transfer function of a feed back control system is given by

$$G(s)\,H(s) = \frac{K(s+5)}{s(s+6)(s^2-6s+10)}$$

In the root locus diagram, the asymptotes of the root loci for large values of K meet at

(A) $\left(\frac{4}{3},\ 0\right)$ (B) $\left(\frac{-5}{3},\ 0\right)$

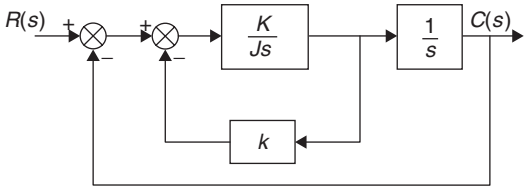
(C) $\left(\frac{5}{3},\ 0\right)$ (D) $\left(\frac{-10}{3},\ 0\right)$

19. A servo mechanism represented by the equation $\frac{d^2\theta}{dt^2} + 10\frac{d\theta}{dt} = 150\,E$, where $E = (r-\theta)$ is the actuating signal. The value of damped frequency of oscillation is

(A) 11.17 rad/sec (B) 12.25 rad/sec

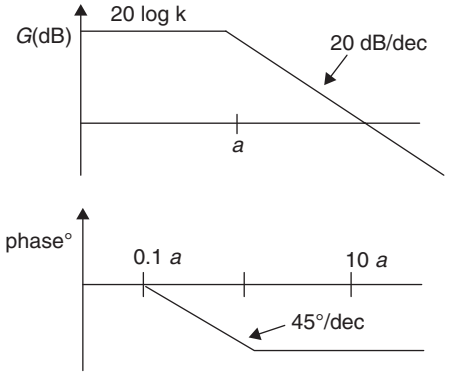
(C) 13.62 rad/sec (D) 9.81 rad/sec

20. A closed loop control system is shown in the figure. The maximum overshoot in the unit step response is to be limited to 25% and the peak time is 2 sec. Assuming the inertia constant $J = 1$ kg–m^2, the value of 'K' and 'k' are respectively



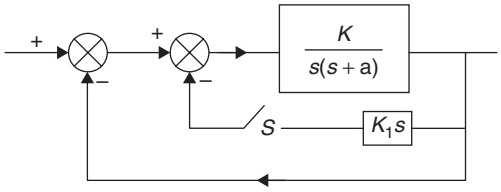
(A) 4.62, 0.471 (B) 2.95, 0.471
(C) 4.62, 0.961 (D) 2.95, 0.961

21. The asymptotic Bode plot of the transfer function $\frac{K}{1+\frac{s}{a}}$ is given in the figure. The error in the phase angle and dB gain at a frequency of $\omega = 0.5a$ are respectively



(A) 4.06°, 0.97 dB (B) 5.7°, 3 dB
(C) 4.9°, 3dB (D) 5.7°, 0.97 dB

22. The system shown in the given figure, is subjected to a unit ramp input on closing the switch 's'



(A) Steady-state error will increase and damping co-efficient 'ζ' will decrease
(B) Both steady-state error and damping coefficient 'ζ' will increase
(C) Both steady-state error and damping coefficient 'ζ' will decrease
(D) Steady-state error will decrease and damping co-efficient 'ζ' will increase

23. A system is described by

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U,\ Y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

The transfer functions is

(A) $\frac{1}{s^2+2s+3}$ (B) $\frac{1}{2s^2+3s+1}$
(C) $\frac{s+3}{s^2+3s+2}$ (D) $\frac{1}{3s^2+2s+1}$

Common Data for Questions 24 and 25:
A system is described by the following state and output equations

$$\frac{dx_1(t)}{dt} = -3x_1(t) + x_2(t) + 2u(t)$$

$$\frac{dx_2(t)}{dt} = -2x_2(t) + u(t)$$

$$y(t) = x_1(t)$$

where $u(t)$ is the input and $y(t)$ is the output.

24. The system transfer function is _____.

(A) $\frac{s+2}{s^2+5s-6}$ (B) $\frac{s+3}{s^2+5s+6}$
(C) $\frac{2s+5}{s^2+5s+6}$ (D) $\frac{2s-5}{s^2+5s-6}$

25. The state-transition matrix of the system is _____.

(A) $\begin{bmatrix} e^{-3t} & 0 \\ e^{-2t}+e^{-3t} & e^{-2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{3t} & e^{-2t}-e^{3t} \\ 0 & e^{2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-3t} & e^{-2t}+e^{3t} \\ 0 & e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} e^{-3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

Answer Keys

1. A	**2.** C	**3.** B	**4.** D	**5.** B	**6.** C	**7.** A	**8.** A	**9.** C	**10.** B
11. C	**12.** D	**13.** B	**14.** D	**15.** C	**16.** B	**17.** C	**18.** C	**19.** A	**20.** B
21. A	**22.** B	**23.** C	**24.** C	**25.** D					